Born-Infeld gravity and cosmological singulartities

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In 2010, Banados and Ferreira (BF) constructed a variant of Born-Infeld (BI) gravity with a simple matter coupling and demonstrated how the standard background FRW cosmology could become free of curvature singularities (big-bang). Further investigations revealed many interesting consequences of this BF version of BI gravity. For a toy 3D version, we show a simple analytical solution exhibiting the removal of the big-bang singularity. Thereafter, we look at BI gravity coupled to scalar BI matter, where we are able to find non-singular (loitering and bounce types) background solutions with late as well as early time acceleration. Finally, we elevate the BI gravity parameter to a spacetime dependent field (in a novel Brans-Dicke like way) and demonstrate how cosmologies without singularities and with late as well as early-time acceleration can indeed arise quite naturally.

1. The singularity problem

The fact that curvature singularities inevitably appear in many solutions of Einstein's field equations of General Relativity (GR) is well-known. Very general theorems on singularities (largely defined using geodesic incompleteness) with minimal assumptions (causal structure and energy conditions) were proven long ago by Penrose, Hawking and others. In electrodynamics too, a singular behaviour of the electric field at the location of the charge is known. It is not any huge embarrasment in electrodynamics. On the other hand a spacetime singularity (be it in the sense of diverging curvature scalars or geodesic incompleteness or both) is indeed problematic because one cannot extend the spacetime beyond that location. Apart from the fact that via Einstein equations, spacetime singularities corrrespond to infinities in energy density of matter, it is the inextendability of spacetime beyond such special points where some pathologies arise and/or curvature invariants diverge, which makes the theory unacceptable at those points.

It is believed that quantum gravity would cure classical GR of the singularity problem. And, indeed there are indications in various such attempts – i.e. in string theory and loop quantum gravity. On the other hand, it may be asked if any *other* classical theory of gravity can admit non-singular solutions. This is a valid question and it is along these lines we direct our presentation below.

2. Born-Infeld structures and Born-Infeld gravity

The appearance of a singularity at the location of the electric charge, is well known. A way out was sought in the work of Born and Infeld (BI)[1], which had links with earlier work due to Eddington[2]. BI proposed a nonlinear electrodynamics with an action given as:

$$S_{BI} = \int \frac{1}{\kappa^2} \left\{ 1 - \sqrt{-Det[\eta_{ij} + \kappa F_{ij}]} \right\} d^4x. \tag{1}$$

The field equations when solved for a point source lead to a resolution of the self-energy singularity in the electric field. κ plays the role of a new parameter which yields an upper bound on the electric field of a point charge.

In 1998, Deser and Gibbons[3] first proposed a gravity action (in a metric formulation) building on BI electrodynamics and Eddington's old ideas (notably the *square root* and the *determinant* in the action). Later, in 2004, Vollick[4] worked out a Palatini formulation of the Deser-Gibbons theory, incorporating matter in a rather artificial way. More recently, Banados and Ferreira (BF)[5] have come up with a theory where the matter coupling is quite simple. The action proposed by BF is given as:

$$S_{BF}(g, \Gamma, \Psi) = \frac{1}{\kappa} \int d^4x \left[\sqrt{-|g_{ij} + \kappa R_{ij}|} - \lambda\sqrt{-g} \right] + S_M(g, \Psi), \tag{2}$$

where $|g_{ij}+\kappa R_{ij}| = Det[g_{ij}+\kappa R_{ij}]$ and $g = Det[g_{ij}]$. Note that the matter coupling here is standard, using the g_{ij} and matter fields Ψ. For κR small we get back the Einstein-Hilbert action ($\Lambda = \frac{\lambda-1}{\kappa}$). When κR is large we recover the Eddington action[2]. Without matter, BF theory is equivalent to GR.

Variation w.r.t. g_{ij} gives us the field equation:

$$\sqrt{-q}q^{ij} = \lambda\sqrt{-g}g^{ij} - \kappa\sqrt{-g}T^{ij}. \tag{3}$$

On varying w.r.t Γ^i_{jk} we find that the q_{ij} as obtained from solving the equation,

$$q_{ij} = g_{ij} + \kappa R_{ij}(q) \tag{4}$$

is a metric with a Christofel connection. The usual conservation law (covariant derivative using g_{ij}), $\nabla_j T^{ij} = 0$ holds. One may also consider the BF action as a bimetric gravity action where the two metrics are q_{ij} and g_{ij}. g_{ij} is the physical one while q_{ij} is auxiliary.

The main problem in finding analytical solutions is in the inversion of the equation $q_{ij} = g_{ij}+R_{ij}(q)$. Since these equations cannot be *solved*, one needs to postulate g_{ij} and q_{ij} separately (with different unknown functions) and write down the field equations. Then only one may be able to find solutions.We shall now provide examples in cosmology. For more details on Born-Infeld inspired gravity theories and their consequences see a recent review[6].

2.1. *Cosmology*

We assume line elements of the form:

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{5}$$

$$ds_q^2 = -U(t)dt^2 + a^2(t)V(t)\left(dx^2 + dy^2 + dz^2\right), \tag{6}$$

where $U(t)$, $V(t)$, $a(t)$ are three unknowns. The energy-momentum tensor T^{ij} is that of a perfect fluid, i.e. $T^{00} = \rho$, $T^{\alpha\alpha} = p$ $(\alpha = 1, 2, 3)$.

With these assumptions one writes down all the field equations stated earlier, for this case. Assuming $p = \frac{\rho}{3}$ one gets the Friedmann equation as:

$$3H^2(\bar{\rho}) = \frac{1}{\kappa}\left[\bar{\rho} - 1 + \frac{1}{3\sqrt{3}}\sqrt{(1+\bar{\rho})(3-\bar{\rho})^3}\right] \times \frac{(1+\bar{\rho})(3-\bar{\rho})^2}{(3+\bar{\rho}^2)^2} \tag{7}$$

where $\bar{\rho} = \kappa\rho$. For small $\bar{\rho}$ one gets get $H^2 \sim \frac{\rho}{3}$, which is the Friedmann equation in GR. Note that H^2 has zeros at $\bar{\rho} = 3$ and $\bar{\rho} = 0$ for $\kappa > 0$. For $\kappa < 0$ zeros are at $\bar{\rho} = 0$ and $\bar{\rho} = -1$. Note the difference with the usual FRW model where $3H^2 = \rho$ and the only zero is at $\rho = 0$.

One can find numerical solutions of the scale factor for $\kappa < 0$, $\kappa > 0$ (Fig. 1):

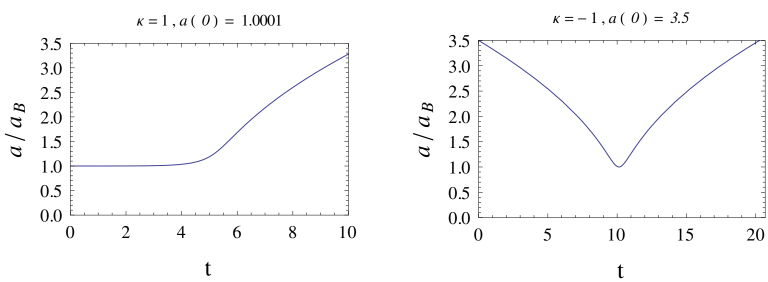


Fig. 1. Plot of the scale factor $a(t)$ for $\kappa = \pm 1$. a_B is the nonzero minimum value of $a(t)$ corresponding to maximum density $\bar{\rho}_B$ and t is an arbitrary time scale. $8\pi G = 1$, $c = 1$.

Note the nonzero minimum value of $a(t)$ which corresponds to the upper limit on ρ. There is no big-bang singularity or any singularity in the future. The absence of the singularity is largely controlled by the BI parameter κ and the structure of BI gravity. In a toy $2+1$ dimensional model it is possible to find simple analytical solutions which exhibit non-singular features[7]. For example, in a $2+1$ version of the BF theory, the scale factor in the $2+1$ cosmology (with $\kappa < 0$ and $p = \frac{\rho}{2}$), has the form: $a(t) = \sqrt{\frac{t^2}{|\kappa|} + 1}$, which never vanishes and yields a toy, nonsingular universe.

3. BI gravity + BI matter: cosmologies

A useful question to ask is: what happens if there are BI structures in both the gravity and matter sectors in BF theory? We assume a simple scalar BI structure which has been studied extensively as *tachyon matter* about a decade or so ago. The action we work with is $S = S_{BI}(g, \Gamma, \Psi) + S_M$, where

$$S_{BI} = \frac{c^3}{8\pi G\kappa}\int d^4x\left[\sqrt{-|g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma)|} - \lambda\sqrt{-g}\right],$$

where $\lambda = \kappa\Lambda + 1$, and

$$S_M = -\frac{1}{c}\int \sqrt{-g}\alpha_T^2\mathcal{V}(\phi)\sqrt{1 + \alpha_T^{-2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}d^4x,$$

where $\mathcal{V}(\phi)$ is the potential for the scalar field and α_T is the constant parameter. Analytical solutions are possible. More importantly, we ask: what are the features of the cosmological solutions? Here, in Fig. 2, the scale factor a (for a constant negative pressure solution with the equivalent pressure and energy densities of the scalar field, $p_\phi = -\alpha_T^2 C_2 c^2$ and $\rho_\phi = C_2(a^{-3} + \alpha_T^2))$ is plotted as a function of the cosmological time (τ). We choose $8\pi G = 1$, $c = 1$ and $\kappa = 0.5, \alpha_T^2 = 5.0, C_2 = 0.001$. Initial value of $a(\tau)$ is choosen as $a = 0.06$ at $\tau = 0.06$. Initial loitering is followed by deceleration and late-time acceleration. The zoomed version in Fig. 3 shows clearly the loitering phase and the transition into deceleration. The loitering phase also includes an acceleration, where the scale factor has an exponential growth ($a \sim a_0\exp(2\sqrt{2}c\tau/\sqrt{3\kappa})$). For negative κ we obtain a bounce solution (not shown).

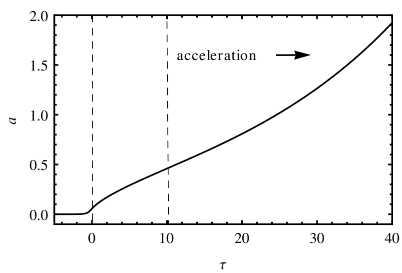


Fig. 2. Plot of scale factor $a(\tau)$.

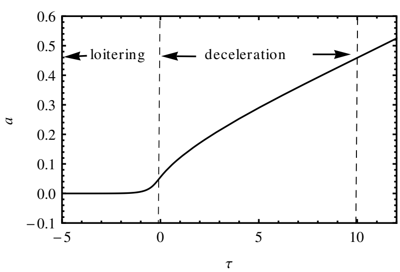


Fig. 3. Zoomed $a(\tau)$ near big-bang

We have fitted the late-time acceleration found in our solution, using Type Ia supernovae (Union2.1 compilation) data. The best fit values of the cosmological parameters for the positive κ solution are given as:

$$q_0 = -0.605^{+0.026}_{-0.054}\ ;\ \Omega_{DM0} = 0.255^{+0.051}_{-0.021}\ ;\ \Omega_{DE0} = 0.745^{+0.016}_{-0.051}$$

Note that there is no separate dark matter or dark energy. We use a result due to Roy Chowdhury and Padmanabhan[9] to split the BI matter into a dark matter part and a dark energy part. This is characteristic of BI matter like the scalar BI used here. In summary, the BI matter coupled to BI gravity does produce a nonsingular universe with a late-time acceleration. More details are available in[8].

4. BI gravity with a Brans-Dicke scalar: cosmologies

Let us finally turn to another important issue which is related to the BI gravity parameter κ. From various proposals for tests of the BF theory one obtains various bounds on κ. This is rather problematic and we wish to avoid this in a Brans-Dicke like way. Recall that the original BD idea was to replace G by a smooth field $\phi(x, y, z, t)$. Here we replace κ with a smooth field $\kappa(x, y, z, t)$. The action is taken as:

$$S_{BI\kappa} = \int \left[\frac{1}{\kappa} \left(\sqrt{-|g_{\alpha\beta} + \kappa R_{\alpha\beta}(\Gamma)|} - \sqrt{-g} \right) - \sqrt{-g}\tilde{\omega}(\kappa) g^{\mu\nu} \partial_\mu \kappa \partial_\nu \kappa \right] d^4x + S_M(g, \Psi) . \quad (8)$$

Here, $\tilde{\omega}(\kappa)$ is a coupling function (like in scalar-tensor theories).

The three field equations (Γ, g and κ variation) are:

$$q_{\alpha\beta} = g_{\alpha\beta} + \kappa R_{\alpha\beta}(q), \quad (9)$$

$$\sqrt{-q} q^{\alpha\beta} - \sqrt{-g} g^{\alpha\beta} = -\kappa \sqrt{-g}\, T^{\alpha\beta}_{eff}, \quad (10)$$

where

$$T^{\alpha\beta}_{eff} = T^{\alpha\beta} - \tilde{\omega} g^{\alpha\beta} g^{\mu\nu} \partial_\mu \kappa \partial_\nu \kappa + 2\tilde{\omega} g^{\mu\alpha} g^{\nu\beta} \partial_\mu \kappa \partial_\nu \kappa, \quad (11)$$

$T^{\alpha\beta}$ is the usual stress-energy tensor, and finally,

$$2\kappa\tilde{\omega}(\kappa) \nabla_\mu \nabla^\mu \kappa + \kappa \tilde{\omega}'(\kappa) \nabla_\mu \kappa \nabla^\mu \kappa + \frac{1}{\kappa} + \frac{\sqrt{-q}}{\sqrt{-g}} \left(\frac{1}{2} q^{\alpha\beta} R_{\alpha\beta}(q) - \frac{1}{\kappa} \right) = 0. \quad (12)$$

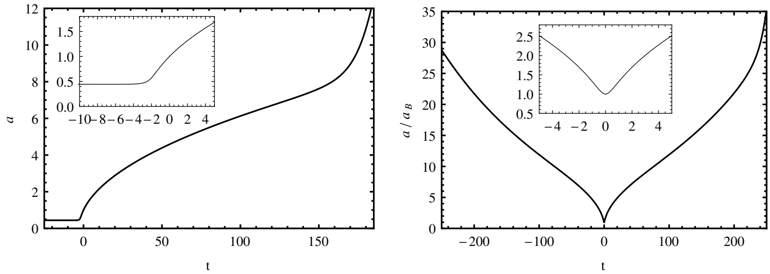
We now turn to FRW cosmologies. Assuming $\kappa(t) = \kappa_0 + \epsilon\, e^{\mu\, t}$ with κ_0, ϵ, and μ as parameters, we consider several cases: $\rho, p = 0$ (vacuum); $p = 0$ (pressureless dust); $p = \frac{\rho}{3}$ (radiation). We investigate the nature of the scale factors for various ranges of κ_0, ϵ and μ.

Let us illustrate this for the case of radiation: (a) $p = \frac{\rho}{3}$: $\kappa_0 > 0$, $\mu > 0$, $\epsilon < 0$ (Fig. 4(a)), (b) $p = \frac{\rho}{3}$: $\kappa_0 < 0$, $\mu > 0$, $\epsilon < 0$ (Fig. 4(b)).

Generic features for $\kappa > 0$ (non-singular beginning), and $\kappa < 0$ (bounce) are retained in Figs. 4(a) and 4(b). Further details on this work is available in[10].

5. Concluding remarks

A recently proposed novel theory of gravity based on a Born–Infeld structure which matches with GR in vacuum is discussed with emphasis on its ability to resolve the cosmological singuarity problem. We have first shown how the original BF version yields a nonsingular FRW universe. Incorporating BI structures in both the gravity and matter sectors we have analytically obtained non-singular FRW cosmologies with initial loitering and late time acceleration. We have also fitted the late-time cosmology with available Supernova data and obtained best-fit values for the cosmological parameters. Further, solutions with an initial bounce and late acceleration are found.



(a) (b)

Fig. 4. (a) plot of scale factor $a(t)$ for $\kappa_0 > 0$, $\mu > 0$, $\epsilon < 0$. The parameters used are $\kappa_0 = 1$, $\mu = 0.1$, and $\rho_0 = 0.1$ (in $\rho = \rho_0/a^4$). We choose $a(0) = 1$, $\kappa(0) = 0.999$ for the numerical solution. (b) plot of scale factor $a(t)$ for $\kappa_0 < 0$, $\mu > 0$, $\epsilon < 0$. The parameters used are $\kappa_0 = -1$, $\mu = 0.1$, and $\rho_0 = 0.01$. We choose $a(0) = (-\kappa_0\rho_0)^{1/4} = a_B$, $\kappa(0) = -1.00001$ for the numerical solution.

In the final part of this article, we have proposed a Brans-Dicke like modification of the BI gravity theory where the BI parameter is elevated to a scalar field. Analysing cosmological solutions we have observed that a nonsingular FRW universe is possible here too for $\kappa > 0$, while for $\kappa < 0$, we have an initial bounce.

Are these solutions useful as background cosmologies? Can we study cosmological perturbations, structure formation, early universe phenomenology, inflation, dark matter, CMB .... in the models we have discussed here? Unless that is done and more links with present-day cosmological observations are established, these models and solutions will merely remain elegant mathematical curiosities.

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