FUNDAMENTAL PROPERTIES OF NON-LINEAR FOCUSING

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(Presented by V. N. Melekhin)

1. INTRODUCTION

This study deals with a simple model of non-linear focusing having no immediate practical value, but, due to its simplicity permitting to carry out a fairiy complete analysis of all the fundamental properties of non-linear focusing. What is meant it is not merely the calculation of quantitative characteristics which depend greatly upon the choice of a magnetic field shape, but rather the corroboration of fundamental qualitative motions and the proof of the existence of main effects.

First of all it is the corroboration of existence of steady motion, taking into account all the non-linear in z terms in equations of motion, as well as the corroboration of estimation according to which the regions of tolerable values of z and tolerable spread of Δr amplitudes of γ-oscillations, generally speaking, are of the same order:

$$z \sim \Delta\gamma \, | \, I \, |$$

In order to have in the first approximation of the disturbance theory at least one r-z-resonance in the middle of the stability region. We choose a non-linear H_z field of the fifth power.

Stability of motion was investigated mainly numerically. In the region of main resonance on the coupling of oscillations an analytical stable solution is obtained as well.

Then it is the corroboration of existence of effect of external autophasing at overlapping disturbances of the type $H_z = h \cdot \cos \omega \varphi$, where frequency ω coincides with a certain value of frequency of oscillations within the stability region. It should be shown by means of numerical calculations on a model that autophasing does lead to conservation of motion stability, when including adiabatic damping, and that small z oscillations do not account for this effect.

And finally, the corroboration of existence of mutual autophasing of r-z-oscillations, appearing also in the absence of external disturbance in the resonance region of oscillations (3). This interesting effect also leads to conservation of motion stability in the case of damping. The simplicity of the model permits to carry out a rigorous analytical evidence of existence of the effect which is apparently impossible in more complicated cases. The calculations, which were presented in original version of this report, are amitted here because of lack of space.

2. THE FIRST REGION OF STABILITY

The dependence of the magnetic field $H_z(r)$ in the plane in the region of non-linear γ-oscillations is shown in Fig. 1. Here H_o is the equilibrium field with respect to which the field H_r-H_o is symmetrical, R is the equilibrium radius so that $cp = eH_oR$. The notation used is (see Fig. 1)

$$\rho = r/b, \xi = z/B, H_{\tilde{\sigma}} = H_c, H_{\xi(xi)} = (H_z - H_o) H_e \quad [1]$$

$$H_\sigma = \frac{5^{5/4}}{4} \cdot (H_{zmax} - H_o), \theta = \varphi \sqrt{\frac{R}{B}} \cdot \frac{H_c}{H_o}, \gamma \equiv \frac{dr}{d\theta} \quad [2]$$

is the azimuthal variable equal to 2 on one revolution. In this notation the components of the field chosen are of the form

$$H_{\xi(xi)}(\rho, \xi) = \rho - 5\rho\xi^4 + 10\rho^3\xi^2 - \rho^5$$

$$H_\xi(\rho, \xi) = \xi - 5\rho^4\xi + 10\rho^2\xi^3 - \xi^5 \quad [3]$$

The equations of motion in the same approximation as in (1) can be written in the form

$$\rho'' + \alpha\rho' + \rho - \rho^5 = 5\rho\xi^4 - 10\rho^2\xi^2 \quad [4]$$

$$\xi'' + \alpha\xi' + (5\rho^4 - 1)\xi = 10\rho^2\xi^3 - \xi^5 \quad [5]$$

where $\alpha \equiv p'/p$ in our case describes adiabatic damping. At $\alpha = 0$ system [4], [5] has a conserved Hamiltonian

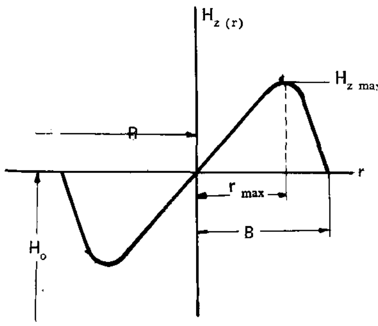

Fig. 1

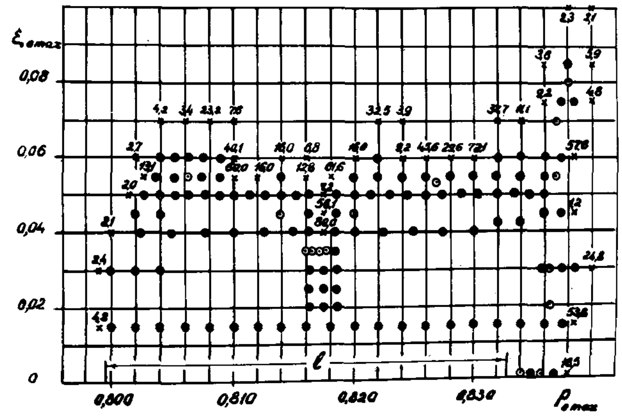

Fig. 2

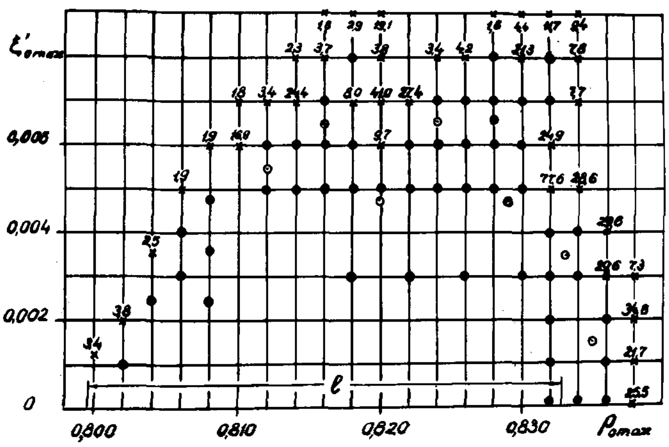

Fig. 3

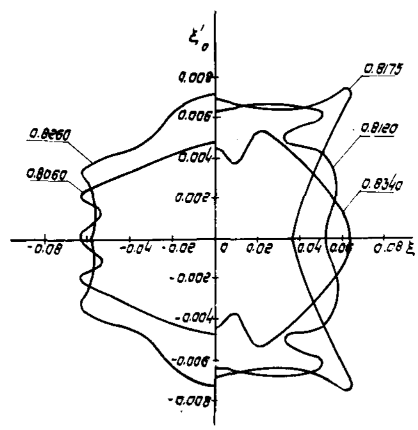

Fig. 4

$$G = \frac{\rho'^2 + \xi'^2}{2} + \frac{\rho^2 - \xi^2}{2} - \frac{\rho^6 - \xi^6}{6} + \frac{5}{2}\rho^2 \xi^2 (\rho^2 - \xi^2) \qquad [6]$$

If the right-hand side parts of [4] and [5] are discarded, the solution of the remaining equations are readily obtained analytically. In this approximation all the values of ξ appear to be permissible, if the amplitudes of ρ-oscillations are within the limits of $0.7987 \leq \rho_{max} \leq 0,8323$ the first region of stability (the other regions have not been studied); oscillations are of a strictly periodic character with the period of $\Phi\rho$, and ξ is written in the form

$$\xi = cf(2\nu\theta) \exp(2i\mu\nu\theta) + KC \qquad [7]$$

where C is the arbitrary constant, $\nu = \pi/\Phi\rho$, μ is the number of ξ oscillations per one half-period of ρ-oscillations.

At $\mu = 0$; 1/2; parametric resonances occur in the left-hand side of equation [5] which corresponds to the left and right boundaries of the linear region of stability (see Figs. 2 and 3). In linear ξ approximation the motion ρ is described by two frequencies ν and μ, and the motion ρ by one frequency ν. In an accurate solution of complete system [4], [5] the boundaries of resonances depend upon initial conditions for ξ, besides there arise additional combination resonances. The motion along ρ loses its strict periodicity which effects the stability of ξ oscillations. Due to these effects only a limited phase volume of ξ oscillations is involved in the stable motion conditions. This is evident from Figs. 2

and 3 which differ from each other by the fact that Fig. 2 shows initial conditions containing $\xi'_0 = 0$; $\rho'_0 = 0$ while in Fig. 3 $\xi_0 = 0$; $\rho'_0 = 0$. The dots and incircled dots indicate initial conditions under which the particles in motion remain within the boundaries $-0{,}3 \leq \xi \leq 0{,}3$ during 100 or 800 periods of ρ-oscillations, respectively; the crosses denote the initial conditions under which the particles break trough these boundaries beyond the number of periods of ρ-oscillations, shown near the sign. In these Figs. the is the dimension of the stability region in a linear approximation. The counting was tested by checking of G Hamiltonian with the accuracy of $\Delta G/G \leq 5 \cdot 10^{-6}$. An analytical solution of equations [4] and [5] is given for the resonance region $\mu = 1/2$ only (see paragraph 4); there is proved the existence of stable oscillations in this region. It is seen in Fig. 2 that nonlinearity may become beneficial: particles with small initial $\xi_0 \geqslant 0.001$, lying to the right of the linear region of stability, still remain within the stable motion conditions (numerical calculations for the same points, using the linear theory, show that these particles break through the above limits rather rapidly).

Fig. 4 shows the stability region boundaries on the (ξ_0, ξ'_0) place for different ρ_0 and $\rho'_0 = 0$. To conclude this section it should be noted that, generally speaking, the permissible spread of ξ_0 must not considerably differ from that of $\Delta\rho_0$.

3. EXTERNAL AUTHOPHASING

Inclusion of the damping of $\alpha \neq 0$ oscillations leads to loss of stability in « time » $\Delta\theta \sim \Delta\rho/\alpha$, where $\Delta\rho$ is the width of stability region with respect to ρ, $\rho \sim 1$.

Curves 1, 2 in Fig. 5 indicate the motion of points with coordinates $\xi = \rho^1 = 0$ on a plane (ρ_{max}, ξ_{max}) at $\alpha = 10^{-4}$ (see also Fig. 6).

The deviation of amplitude ρ from stability region at damping may be eliminated by means of external harmonic disturbance of the type $\Delta H\xi = h \cdot \cos(\omega\theta + \beta)$. This brings about a resonance for such ρ_1-oscillation whose frequency $\nu = \omega/K$, K is an integer.

In the case of $\xi \neq 0$ ρ oscillations fail to be strictly periodical and the accurate resonance is thus violated. Therefore, the conditions of autophasing should be examined accounting for non-linear ξ terms.

Curve 3 in Fig. 5 shows the motion with initial values $\rho_0 = 0.823$, $\xi_0 = 0.04$ at $\alpha = 10^{-4}$, $h = 3 \cdot 10^{-4}$, $\omega = 0.836$, $\beta = \pi$. This is a typical picture of how phase oscillations take place near resonance values corresponding to $\nu_{\to op} =$, where $\nu_{\to op}$ is the frequency of oscillations, accounting for $\xi \neq 0$ (passages of curve 3 between the « a » points is not shown in the graph. The calculation is finished after 250 oscillations).

Only those points are shown in Fig. 5 for which $\rho^1 = \xi^1 = 0$; the figure at the end of the curve indicates the number of periods of oscillations passed by a particle.

4. MUTUAL AUTHOPHASING

Conservation of stability at damping is possible also without inclusion of a harmonic external force. If, for example, $\mu = 1/2$ so that the frequencies of and oscillations coincide, the ρ oscillations themselves play the role of that harmonic force which autophases ρ-oscillations (3). In this case the inverse effect of oscillations upon

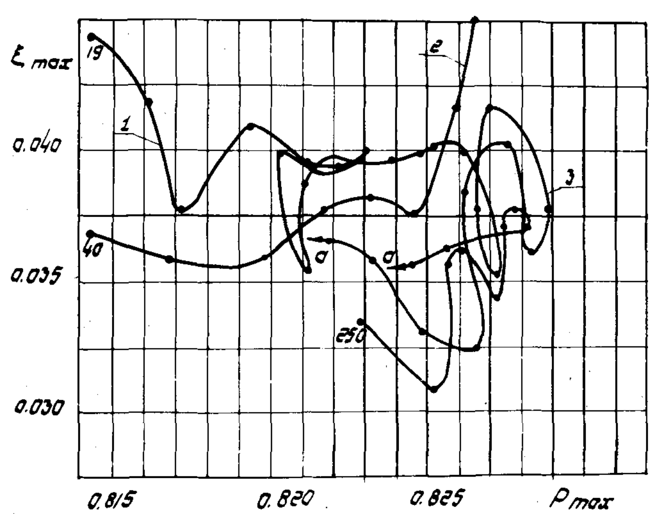

Fig. 5

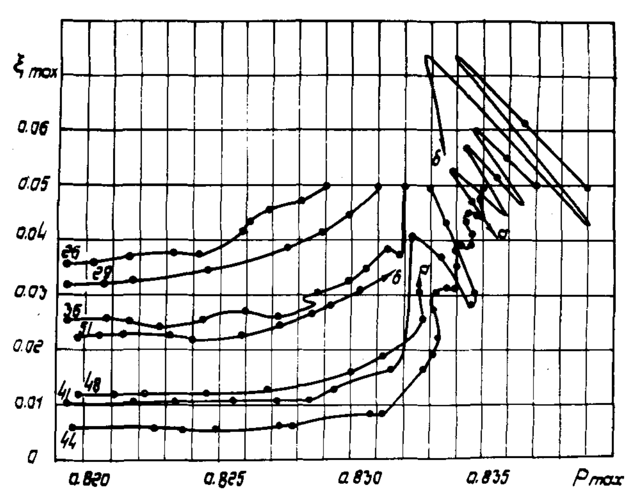

Fig. 6

ξ oscillations is also of considerable significance *so that autophasing is strictly mutual.*

Fig. 6 shows the motion of points $\rho^1 = \xi^1 = 0$ at $\alpha = 10^{-4}$ and the curves of Fig. 4 is repeated here for obviousness; figures near the curves indicate the number of periods of ρ-oscillations, passed by a particle.

It should be noted that in actual more complex fields H_z-H_o and at usually non-symmetric dependance of difference H_z-H_o of r the terms, containning Z^2 in the right-hand part of the equation for r, do not produce resonance in the first *region of stability; Z^4 members produce resonance* only on the right-hand boundary of this region. For the mutual autophasing to be developed within the limits of the first region in the nonsymmetric field, it is necesary that strong terms with Z^{2k}, $K \geqslant 3$ should be present in the magnetic field.

Thus, the calculations of our simple model of non-linear focusing confirm the fundamental principles of the theory.

REFERENCES

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(2) B. V. Chirikov: Doklady Akad. Nauk USSR, *125*, (1959).
(3) Yu. F. Orlov: Problems of Physics of Elementary Particles. (Erevan-Nor-Ambert, 1964) p. 578.

DISCUSSION

WIDERÖE: Was this calculation made for a machine with a straight section?

MELEKHIN: Calculation was made with no straight section.

C. BERNARDINI: Is it that a small electron model could be buildt on these lines?

MELEKHIN: It seems to be possible and usefull.

M. H. BLEWETT: Has a magnet model of this nonlinear type been built to see if field of this kind can be achieved with sufficient accuracy?

MELEKHIN: It seems to me that no magnets of this type have been constructed.

SYMON: One difficulty with non-linear focusing fields is the necessity to calculated orbits for very many resolutions to determine stability. We have cases when orbits uppear stable for 100.000 revolutions an then become unstable. Do you have any way of determining whether the orbits you have calculated for 100 revolutions were really stable for many revolutions?

MELEKHIN: In fact this is the main question in the work. Anyway the instability region can be easily found in such a calculation.