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On the time-reversal symmetry in pseudo-Hermitian systems

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In a recent paper [M. Sato, K. Hasebe, K. Esaki, and M. Kohmoto, Prog. Theor. Phys. **127**, 937 (2012)] Sato and his collaborators established a generalization of the Kramers degeneracy structure to pseudo-Hermitian Hamiltonian systems, admitting even time-reversal symmetry, $T^2 = 1$. This extension is achieved using the mathematical structure of split-quaternions instead of quaternions, usually adopted in the case of Hermitian Hamiltonians with odd time-reversal symmetry, $T^2 = -1$. Here we find that the metric operator for the pseudo-Hermitian Hamiltonian H that allows the realization of the generalized Kramers degeneracy is necessarily indefinite. We show that such H with real spectrum also possesses odd antilinear symmetry induced from the existing odd time-reversal symmetry of its Hermitian counterpart h, so that the generalized Kramers degeneracy of H is in fact crypto-Hermitian Kramers degeneracy. We study in greater detail a new example of the pseudo-Hermitian split-quaternionic four-level Hamiltonian system, which admits an indefinite metric operator and time-reversal symmetry and, as a consequence, a generalized Kramers degeneracy structure. We provide a complete solution of the eigenvalue problem, construct pseudo-Hermitian ladder operators closing the normal and abnormal pseudo-fermionic algebras, and show that this system fulfills a crypto-Hermitian degeneracy.

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1. Introduction

The study of pseudo-Hermitian and PT-symmetric Hamiltonians has attracted a great deal of interest over the last decade [2–9]. In this framework, Sato and his collaborators [1] have extended the Kramers theorem to the case that time-reversal symmetry is even ($T^2 = 1$). They have shown that η-pseudo-hermiticity is consistent with the time-reversal symmetry if the "metric operator" η commutes or anticommutes with the time-reversal operator T. The metric that ensures the generalized Kramers degeneracy of the eigenvalues of pseudo-Hermitian Hamiltonians corresponds to the anticommutative case.

In the usual Hermitian quantum mechanics the even time-reversal symmetry ($T^2 = 1$) is appropriate to the boson systems or systems with an even number of fermions, in which there is no Kramers degeneracy structure. However, as established in Ref. [1], the situation is different in the case of the pseudo-Hermitian quantum mechanics in which a Kramers-degeneracy-like structure exists for systems with even time-reversal symmetry. This Kramers degeneracy is achieved by the use of the split-quaternions[1] in lieu of the quaternions used traditionally in the odd time-reversal symmetry ($T^2 = -1$).

The aim of the present paper is to investigate new relevant features of this kind of pseudo-Hermitian system and to provide a new example.

The paper is organized as follows. In Sect. 2, after having reviewed the main results on the Kramers degeneracy for even time-reversal invariant pseudo-Hermitian systems (called generalized Kramers degeneracy [1]), we highlight the indefiniteness feature of the metric operator. We establish in Sect. 3 that the Kramers degeneracy in a system with pseudo-Hermitian Hamiltonian H with real spectrum is in fact cryptic Kramers degeneracy in the related Hermitian system, i.e. it is crypto-Hermitian degeneracy. Performing the diagonalization of the Hermitian counterpart h of such pseudo-Hermitian H, we construct the odd time-reversal symmetry of h and establish that it induces an odd antilinear symmetry of the original H. For the purposes of illustration, we study in Sect. 4 an eight-parametric pseudo-Hermitian split-quaternionic four-level system that is invariant under the even time-reversal symmetry and reveals Kramers degeneracy. We construct the complete solution of the eigenvalue problem for this system. In the case of the real spectrum of our parametric pseudo-Hermitian Hamiltonian H, we construct the corresponding positive-defined metric operator η_+ and perform the explicit diagonalization of the related Hermitian Hamiltonian h, revealing its odd time-reversal symmetry, and the induced odd antilinear symmetry of H. In the last subsection we construct the creation and annihilation operators of our system that satisfy the normal and abnormal pseudo-fermionic algebras in a consequence of its pseudo-hermiticity. The paper ends with concluding remarks.

2. Generalized Kramers degeneracy

2.1. *Kramers degeneracy and pseudo-hermiticity*

We begin this section by reviewing the main results introduced in Ref. [1] on the Kramers degeneracy for even time-reversal invariant pseudo-Hermitian systems ($T^2 = 1$). We consider a diagonalizable pseudo-Hermitian Hamiltonian H with a discrete spectrum. Then a complete biorthonormal eigenbasis $|\psi_n\rangle$, $|\phi_n\rangle$ exists, i.e. a basis such that [6,9–11]

$$H\big|\psi_n\big\rangle = E_n\big|\psi_n\big\rangle,\ H^\dagger\big|\phi_n\big\rangle = E_n^*\big|\phi_n\big\rangle, \tag{1}$$

$$\big\langle\psi_n\big|\phi_m\big\rangle = \big\langle\phi_n\big|\psi_m\big\rangle = \delta_{nm}, \tag{2}$$

$$\sum_n\big|\phi_n\big\rangle\big\langle\psi_n\big| = \sum_n\big|\psi_n\big\rangle\big\langle\phi_n\big| = \mathbf{1}. \tag{3}$$

The eigenvalues of H are either real or come in complex-conjugate pairs. If both real and complex eigenvalues are present, then we restrict n to labeling the real eigenvalues a_n only, ν to labeling the complex eigenvalues α_ν with positive imaginary part, and $-\nu$ to labeling the complex eigenvalues $\alpha_{-\nu} = \alpha_\nu^*$ with negative imaginary part. This yields the following spectral representation of H and metric operator η [6,9]:

$$H = \sum_{n=1}^{N_0}\big|\psi_n\big\rangle\big\langle\phi_n\big| + \sum_{\nu=1}^{N_1}\big(\alpha_\nu\big|\psi_\nu\big\rangle\big\langle\phi_\nu\big| + \alpha_\nu^*\big|\psi_{-\nu}\big\rangle\big\langle\phi_{-\nu}\big|\big), \tag{4}$$

[1] The quaternion algebra is generated by the 2×2 unit matrix σ_0 and the pure imaginary complex number i multiplied by the SU(2) Pauli matrices $(i\sigma_1, i\sigma_2, i\sigma_3)$, while the split-quaternion algebra is generated by the 2×2 unit matrix σ_0 and the pure imaginary complex number i multiplied by the SU(1,1) Pauli matrices $(-\sigma_1, -\sigma_2, i\sigma_3)$.

$$\eta = \sum_{n=1}^{N_0} \left|\phi_n\right\rangle \left\langle\phi_n\right| + \sum_{\nu=1}^{N_1} \left(\left|\phi_\nu\right\rangle \left\langle\phi_{-\nu}\right| + \left|\phi_{-\nu}\right\rangle \left\langle\phi_\nu\right|\right), \tag{5}$$

with the properties

$$\left|\phi_n\right\rangle = \eta\left|\psi_n\right\rangle, \left|\psi_n\right\rangle = \eta^{-1}\left|\phi_n\right\rangle, \tag{6}$$

$$\left|\phi_{\pm\nu}\right\rangle = \eta\left|\psi_{\mp\nu}\right\rangle, \left|\psi_{\pm\nu}\right\rangle = \eta^{-1}\left|\phi_{\mp\nu}\right\rangle. \tag{7}$$

Reference [1] suggests a given non-Hermitian Hamiltonian H that satisfies three conditions, namely: (i) H is pseudo-Hermitian, which means that H satisfies the relation $H^\dagger = \eta H \eta^{-1}$, where η is a Hermitian operator, called a metric operator; (ii) H is invariant under the even time-reversal symmetry ($T^2 = 1$): $\left[H, T\right] = 0$, where the time-reversal operator is given by $T = UK$, and U is a unitary matrix that can be chosen as a real matrix with all the diagonal terms equal to the 2×2 Pauli matrix σ_x and all the off-diagonal terms equal to zero, K being a complex-conjugation operator; (iii) the time-reversal operator T anticommutes with the metric operator η. As a consequence of the above conditions [1], the eigenstates of H admit a generalized Kramers degeneracy structure with Kramers pair $\left|\psi_n\right\rangle$ and $T\eta^{-1}\left|\phi_n\right\rangle$.

2.2. *Kramers degeneracy and indefinite metrics*

In this subsection we consider the properties of the metric operator η, establishing that in the above-described scheme the metrics is necessarily indefinite. In this aim we shall show that the η-norm $\left(\text{related to the } \eta\text{-inner product } \left\langle\psi\middle|\phi\right\rangle_\eta = \left\langle\psi\middle|\eta\phi\right\rangle\right)$ of the eigenstates of H is indefinite. We shall consider the cases of the complex and real eigenvalues separately.

For the complex eigenvalue case, we calculate the norms $\left\langle T\eta^{-1}\phi_\nu\middle|T\eta^{-1}\phi_\nu\right\rangle_\eta$ and $\left\langle\psi_\nu\middle|\psi_\nu\right\rangle_\eta$ of the Kramers pairs. We readily find that $\left\langle\psi_\nu\middle|\psi_\nu\right\rangle_\eta$ is vanishing:

$$\left\langle\psi_\nu\middle|\psi_\nu\right\rangle_\eta = \left\langle\psi_\nu\middle|\eta\psi_\nu\right\rangle = \left\langle\psi_\nu\middle|\phi_{-\nu}\right\rangle = 0. \tag{8}$$

For the second norm, we first take into account the anticommutation relation between η and T to obtain

$$\left\langle T\eta^{-1}\phi_\nu\middle|T\eta^{-1}\phi_\nu\right\rangle_\eta = -\left\langle T\eta^{-1}\phi_\nu\middle|T\eta\eta^{-1}\phi_\nu\right\rangle = -\left\langle T\eta^{-1}\phi_\nu\middle|T\phi_\nu\right\rangle. \tag{9}$$

Then we use the antiunitary property of T and $T^2 = 1$, and get

$$\left\langle T\eta^{-1}\phi_\nu\middle|T\eta^{-1}\phi_\nu\right\rangle_\eta = -\left\langle T^2\phi_\nu\middle|T^2\eta^{-1}\phi_\nu\right\rangle = -\left\langle\phi_\nu\middle|\eta^{-1}\phi_\nu\right\rangle, \tag{10}$$

wherefrom, by using (7), we find

$$\left\langle T\eta^{-1}\phi_\nu\middle|T\eta^{-1}\phi_\nu\right\rangle_\eta = -\left\langle\phi_\nu\middle|\psi_{-\nu}\right\rangle = 0. \tag{11}$$

Thus the η-norms of eigenstates of H with complex eigenvalues are all vanishing. This proves that the η-metric is indefinite in the complex eigenvalue case.

For the real eigenvalue case, we first note that $|\psi_n\rangle$ and $|\phi_n\rangle$ are related as $\left|\phi_n\right\rangle = \eta|\psi_n\rangle$ and $|\psi_n\rangle = \eta^{-1}\left|\phi_n\right\rangle$, wherefrom we find that the Kramers pair is $|\psi_n\rangle$ and $T|\psi_n\rangle$. Thus we have to calculate the norms $\langle\psi_n|\psi_n\rangle_\eta$ and $\langle T\psi_n|T\psi_n\rangle_\eta$. We have

$$\langle\psi_n|\psi_n\rangle_\eta = \langle\psi_n|\eta\psi_n\rangle = \langle\psi_n|\phi_n\rangle = 1, \tag{12}$$

and

$$\langle T\psi_n|T\psi_n\rangle_\eta = \langle T\psi_n|\eta T\psi_n\rangle. \tag{13}$$

By using again the anticommutation relation between η and T and the antiunitary property of T and $T^2 = 1$, we obtain

$$\langle T\psi_n|T\psi_n\rangle_\eta = -\langle T\psi_n|T\eta\psi_n\rangle = -\langle T^2\eta\psi_n|T^2\psi_n\rangle = -\langle\eta\psi_n|\psi_n\rangle, \tag{14}$$

wherefrom, by using the hermiticity property of η and Eqs. (6) and (12), we derive

$$\langle T\psi_n|T\psi_n\rangle_\eta = -\langle\psi_n|\eta\psi_n\rangle = -\langle\psi_n|\psi_n\rangle_\eta = -1. \tag{15}$$

We see that the norm $\langle\psi_n|\psi_n\rangle_\eta$ is positive while the norm of the Kramers partner $\langle T\psi_n|T\psi_n\rangle_\eta$ is negative. This means that the operator η is not positive definite, i.e. it is a pseudo-metric (or indefinite metric) operator [13–16]. Thus we have shown that the operator η is indefinite in both cases of the complex and real eigenvalues.

3. Crypto-Hermitian degeneracy

Consider the case of the pseudo-Hermitian H with real eigenvalue and Kramers degeneracy structure. For such H there exists a positive definite metric operator η_+ (in the form of the first sum in (5)) and the similarity transform $\rho H\rho^{-1} \equiv h$, $\rho = \sqrt{\eta_+}$, is a Hermitian operator (therefore H is also called crypto-Hermitian or quasi-Hermitian [17,18]). This h inherits the same energy-level degeneracy, and being Hermitian, it should possess odd time-reversal symmetry. In order to construct the related odd time-reversal operator, it is convenient to consider the diagonal form h_d of h:

$$UhU^{-1} = \mathrm{diag}\{E_1, E_2, \ldots, E_N\} \equiv h_d, \tag{16}$$

where the operator U is unitary. The total diagonalizing operator D, $DHD^{-1} = h_d$ takes the product form, $D = U\rho$. One can verify that $\eta_+ = D^\dagger D$ [19].

The diagonal form h_d enables us to find easily the odd time-reversal operator $T_0 = S_0 K$. For example, if N is even and $E_{2k} = E_{2k-1}$, $k = 1, 2, \ldots, N/2$, then the $N \times N$ odd involution matrix S_0 that commutes with h_d takes the block-diagonal form (σ_2 being the second Pauli matrix)

$$S_0 = \mathrm{diag}\{i\sigma_2, i\sigma_2, \ldots, i\sigma_2\}. \tag{17}$$

One can verify that T_0 is antilinear and antiunitary, $T_0^2 = -1$ and $[T_0, h_d] = 0$. Such an odd time-reversal operator was introduced in Ref. [20]. Using this, we construct the odd time-reversal operator

$$T_1 = U^{-1}T_0U = U^{-1}S_0KU, \tag{18}$$

which (is antilinear, antiunitary, and) satisfies $T_1^2 = -1$ and $[T_1, h] = 0$, i.e. it is an odd time-reversal symmetry of h. Further similar construction produces

$$T_2 = \rho^{-1}T_1\rho = \rho^{-1}U^{-1}T_0U\rho = D^{-1}T_0D, \tag{19}$$

which is an antilinear odd involution, satisfying $T_2^2 = -1$ and $[T_2, H] = 0$, but its antiunitarity is not guaranteed in view of the nonunitarity of D (due to the nonunitarity of ρ). So T_2 represents an odd antilinear (but not antiunitary) symmetry of H.

Similarly, one can verify that, if T is an even time-reversal operator, commuting with crypto-Hermitian H, then the operator

$$\tilde{T} = \rho T \rho^{-1}, \quad \rho = \sqrt{\eta_+}, \tag{20}$$

is an antilinear involution $\left(\tilde{T}^2 = 1\right)$, commuting with Hermitian h. However, its antiunitarity is not guaranteed in view of the nonunitarity of ρ. So $\tilde{T}$ represents an even antilinear (but not antiunitary) symmetry of Hermitian h.

In conclusion to this section we note that, since, in the case of H with real spectrum, the same energy-level degeneracy occurs in the spectrum of the Hermitian counterpart h of H, which can be described by means of an odd time-reversal symmetry, the generalized Kramers degeneracy of the pseudo-Hermitian H can be viewed as induced from the Kramers degeneracy of Hermitian h. Also, in analogy to the name "crypto-Hermitian" of the pseudo-Hermitian H with real spectrum [17,18], one could say that the generalized Kramers degeneracy of the spectrum of such H is "crypto-Hermitian degeneracy".

4. Illustration

In this section we illustrate the above general results with an example. We consider the split-quaternionic four-level model, described by the following Hamiltonian:

$$H = \begin{pmatrix} \delta a & \alpha b \\ \beta c & -\delta a \end{pmatrix}, \tag{21}$$

where α, β, and δ are real parameters, $b = b_0\sigma_0 - b_1\sigma_1 - b_2\sigma_2 + ib_3\sigma_3$ and $c = b_0\sigma_0 + b_1\sigma_1 + b_2\sigma_2 - ib_3\sigma_3$ are real split-quaternions, and $a = a_0\sigma_0$ is the real split-quaternion proportional to the identity; σ_i $(i = 1, 2, 3)$ are the Pauli matrices. By setting $A_\pm = b_1 \pm ib_2$ and $B_\pm = b_0 \pm ib_3$, this Hamiltonian H can also be written as a four-level Hamiltonian as follows:

$$H = \begin{pmatrix} \delta a_0 & 0 & \alpha B_- & \alpha A_- \\ 0 & \delta a_0 & \alpha A_+ & \alpha B_+ \\ \beta B_+ & -\beta A_- & -\delta a_0 & 0 \\ -\beta A_+ & \beta B_- & 0 & -\delta a_0 \end{pmatrix}. \tag{22}$$

We remark that, in the particular case in which $\alpha = \beta = \delta = 1$, our Hamiltonian (21), (22) recovers the SO(3,2)-type Hamiltonian studied in Ref. [1]. So we can consider the Hamiltonian (22) as an extension of SO(3,2)-type Hamiltonians.

4.1. *The eigenvalue problem and time-reversal symmetry*

The Hamiltonian H in (22) is pseudo-Hermitian with indefinite metric operator η,

$$H^\dagger = \eta H \eta^{-1}, \tag{23}$$

with η, which anticommutes with the time-reversal operator T,

$$\eta = \begin{pmatrix} \beta\sigma_z & 0 \\ 0 & \alpha\sigma_z \end{pmatrix}. \tag{24}$$

One can easily verify that the following even time-reversal operator T [1],

$$T = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} K, \tag{25}$$

commutes with H and anticommutes with the pseudo metric operator (24): $[H, T] = 0$, $\{\eta, T\} = 0$.

The eigenvalues of H are

$$E = \pm\Omega, \quad \text{with} \quad \Omega = \sqrt{\delta^2 a_0^2 + \alpha\beta(b_0^2 + b_3^2 - b_1^2 - b_2^2)}. \tag{26}$$

These eigenvalues are twofold degenerate as a consequence of the generalized Kramers degeneracy.

Here we consider the case of real eigenvalues, i.e.

$$\Omega^2 = \delta^2 a_0^2 + \alpha\beta(b_0^2 + b_3^2 - b_1^2 - b_2^2) > 0. \tag{27}$$

The eigenstates $\left(|\psi_{-+}\rangle, |\psi_{--}\rangle\right)$ and $\left(|\psi_{++}\rangle, |\psi_{+-}\rangle\right)$ associated with the negative and positive energy are given respectively as follows:

For the negative-energy $E = -\Omega$,

$$|\psi_{-+}\rangle = k \begin{pmatrix} -\alpha B_- \\ -\alpha A_+ \\ (\Omega + \delta a_0) \\ 0 \end{pmatrix}, \quad |\psi_{--}\rangle = T|\psi_{-+}\rangle = k \begin{pmatrix} -\alpha A_- \\ -\alpha B_+ \\ 0 \\ (\Omega + \delta a_0) \end{pmatrix}. \tag{28}$$

For the positive-energy $E = \Omega$,

$$|\psi_{++}\rangle = k' \begin{pmatrix} (\Omega + \delta a_0) \\ 0 \\ \beta B_+ \\ -\beta A_+ \end{pmatrix}, \quad |\psi_{+-}\rangle = T|\psi_{++}\rangle = k' \begin{pmatrix} 0 \\ (\Omega + \delta a_0) \\ -\beta A_- \\ \beta B_- \end{pmatrix}. \tag{29}$$

The eigenstates $\left(|\phi_{-+}\rangle, |\phi_{--}\rangle\right)$ and $\left(|\phi_{++}\rangle, |\phi_{+-}\rangle\right)$ associated with $H^\dagger$ are obtained by the action of η to the eigenstates of H, which are given as follows:

For the negative-energy $E = -\Omega$,

$$|\phi_{-+}\rangle = \alpha k \begin{pmatrix} -\beta B_- \\ \beta A_+ \\ (\Omega + \delta a_0) \\ 0 \end{pmatrix}, \quad |\phi_{--}\rangle = \alpha k \begin{pmatrix} -\beta A_- \\ \beta B_+ \\ 0 \\ -(\Omega + \delta a_0) \end{pmatrix}. \tag{30}$$

For the positive-energy $E = \Omega$,

$$|\phi_{++}\rangle = \beta k' \begin{pmatrix} (\Omega + \delta a_0) \\ 0 \\ \alpha B_+ \\ \alpha A_+ \end{pmatrix}, \quad |\phi_{+-}\rangle = \beta k' \begin{pmatrix} 0 \\ -(\Omega + \delta a_0) \\ -\alpha A_- \\ -\alpha B_- \end{pmatrix}, \tag{31}$$

where k and k' are normalization constants, fixed by the requirement that:

$$2\Omega\alpha(\Omega + \delta a_0)k^2 = 1, \quad 2\Omega\beta(\Omega + \delta a_0)k'^2 = 1. \tag{32}$$

These states satisfy the abnormal relations that are a consequence of the indefinite metric η given in (24),

$$\begin{gathered} \langle\psi_{++}|\phi_{++}\rangle = 1,\ \langle\psi_{+-}|\phi_{+-}\rangle = -1,\ \langle\psi_{-+}|\phi_{-+}\rangle = 1,\ \langle\psi_{--}|\phi_{--}\rangle = -1, \\ \langle\psi_{ms}|\phi_{\overline{m}s}\rangle = 0,\ \langle\psi_{ms}|\phi_{\overline{ms}}\rangle = 0,\ \langle\psi_{ms}|\phi_{m\bar{s}}\rangle = 0, \end{gathered} \tag{33}$$

where $s = \pm$, $m = \pm$, $\overline{m}$, and $\bar{s}$ are the opposite signs of m and s respectively. These states also satisfy the relations

$$|\psi_{++}\rangle\langle\phi_{++}| - |\psi_{+-}\rangle\langle\phi_{+-}| + |\psi_{-+}\rangle\langle\phi_{-+}| - |\psi_{--}\rangle\langle\phi_{--}| = \mathbf{1}. \tag{34}$$

The (indefinite) η-norms of the Kramers partners are given by

$$\langle\psi_{++}|\psi_{++}\rangle_\eta = \langle\psi_{++}|\phi_{++}\rangle = 1, \quad \langle\psi_{+-}|\psi_{+-}\rangle_\eta = \langle\psi_{+-}|\phi_{+-}\rangle = -1, \tag{35}$$

$$\left\langle\psi_{-+}\middle|\psi_{-+}\right\rangle_\eta = \left\langle\psi_{-+}\middle|\phi_{-+}\right\rangle = 1, \quad \left\langle\psi_{--}\middle|\psi_{--}\right\rangle_\eta = \left\langle\psi_{--}\middle|\phi_{--}\right\rangle = -1. \tag{36}$$

4.2. *Crypto-Hermitian degeneracy for the split-quaternionic four-level model*

Our Hamiltonian (22) in the case of the real spectrum (27) also admits a positive definite metric operator η_+ of the form (5). Using the above-obtained explicit solutions for the eigenstates, we find

$$\eta_+ = \frac{1}{\Omega}\begin{pmatrix} M/2 & -\dfrac{\alpha\beta^2 A_- B_-}{\Omega+\delta a_0} & 0 & \alpha\beta A_- \\ -\dfrac{\alpha\beta^2 A_+ B_+}{\Omega+\delta a_0} & M/2 & \alpha\beta A_+ & 0 \\ 0 & \alpha\beta A_- & \dfrac{\alpha}{2\beta}M & \dfrac{\alpha^2\beta A_- B_+}{\Omega+\delta a_0} \\ \alpha\beta A_+ & 0 & \dfrac{\alpha^2\beta A_+ B_-}{\Omega+\delta a_0} & \dfrac{\alpha}{2\beta}M \end{pmatrix}. \tag{37}$$

Using the relation $\eta_+ = D^\dagger D$, we obtain the diagonalizing operator D^{-1} in the form

$$D^{-1} = \begin{pmatrix} k'(\Omega+\delta a_0)\sigma_0 & -k\alpha b \\ k'\beta c & k(\Omega+\delta a_0)\sigma_0 \end{pmatrix}, \tag{38}$$

where b and c are split-quaternions, defined after Eq. (21), and

$$M = \beta(\Omega+\delta a_0) + \frac{\alpha\beta^2}{\Omega+\delta a_0}(B_+B_- + A_+A_-). \tag{39}$$

The action of the positive definite metric operator η_+ to the eigenstates of H generates new eigenstates of $H^\dagger$ denoted by $|\chi_{ms}\rangle$, which are related to the eigenstates $|\phi_{ms}\rangle$ given previously in Eqs. (30) and (31) by the relation $|\chi_{ms}\rangle = s|\phi_{ms}\rangle$. The states $|\psi_{ms}\rangle$ and $|\chi_{ms}\rangle$ form a complete biorthonormal set (to be compared with the set of $|\psi_{ms}\rangle$ and $|\phi_{ms}\rangle$), i.e.

$$\left\langle\psi_{ms}\middle|\chi_{m's'}\right\rangle = \delta_{mm'}\delta_{ss'}, \tag{40}$$

$$|\psi_{++}\rangle\langle\chi_{++}| + |\psi_{+-}\rangle\langle\chi_{+-}| + |\psi_{-+}\rangle\langle\chi_{-+}| + |\psi_{--}\rangle\langle\chi_{--}| = \mathbf{1}. \tag{41}$$

The Hermitian counterpart in the diagonal form h_d of the pseudo-Hermitian Hamiltonian H and the related matrix S_0 of the odd time-reversal operator $T_0 = S_0 K$ read

$$h_d = \Omega\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \qquad S_0 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}. \tag{42}$$

The odd time-reversal operators S_0K with S_0 in the forms (17) and (42) have been introduced in Refs. [20,21]. We see that the Kramers degeneracy of our Hamiltonian (22) in the case of the real spectrum is in fact a "crypto-Hermitian degeneracy", since its diagonal Hermitian counterpart h_d is invariant under the odd time-reversal transformation S_0K, realizing the "Hermitian" Kramers degeneracy.

4.3. *Pseudo-fermionic algebras*

Fermionic algebra in quantum mechanics has been extended to the case of pseudo-Hermitian quantum mechanics by Mostafazadeh [12], showing that there exist two types of pseudo-fermionic algebra, which depend on whether the associated metric operator is definite or indefinite.

In the first one, which corresponds to the positive definite metric operator η_+, the defining algebra is the pseudo-Hermitian generalization of the usual fermion algebra, namely [12]:

$$Y^2 = Y^{\#2} = 0, \quad YY^{\#} + Y^{\#}Y = 1, \tag{43}$$

where $Y^{\#} = \eta_+^{-1}Y^{\dagger}\eta_+$ is the pseudo-adjoint of Y. The operators $Y^{\#}$ and Y are respectively the creation and annihilation operators of what is called the pseudo-fermion or simply phermion [12].

The second type of pseudo-fermionic algebra corresponds to the indefinite metric operator η. This algebra is spanned by two operators Z, $Z^{\#} = \eta^{-1}Z^{\dagger}\eta$, which satisfy the relations [12]:

$$Z^2 = Z^{\#2} = 0, \quad ZZ^{\#} + Z^{\#}Z = -1. \tag{44}$$

The object that is "created" and "annihilated" by the operators $Z^{\#}$, Z is called an abnormal pseudo-fermion or abnormal phermion [12].

In our case, we firstly introduce the annihilation operator Y associated with the Hamiltonian H given in Eq. (22),

$$Y = \sum_{m=+,-} |\psi_{-,m}\rangle\langle\chi_{+,m}| = |\psi_{-+}\rangle\langle\chi_{++}| + |\psi_{--}\rangle\langle\chi_{+-}|. \tag{45}$$

The creation operator, which is the η_+-pseudo-Hermitian adjoint of Y, is given by

$$Y^{\#} = \eta_+^{-1}Y^{\dagger}\eta_+ = \sum_{m=+,-} |\psi_{+,m}\rangle\langle\chi_{-,m}| = |\psi_{++}\rangle\langle\chi_{-+}| + |\psi_{+-}\rangle\langle\chi_{--}|. \tag{46}$$

Y and $Y^{\#}$ act on the eigenvectors of H as follows:

$$Y|\psi_{-,m}\rangle = 0, \quad Y|\psi_{+,m}\rangle = |\psi_{-,m}\rangle, \tag{47}$$

$$Y^{\#}|\psi_{+,m}\rangle = 0, \quad Y^{\#}|\psi_{-,m}\rangle = |\psi_{+,m}\rangle. \tag{48}$$

Operator Y annihilates the lowest eigenstates $|\psi_{-,m}\rangle$, $Y^{\#}$ brings this states onto the upper eigenstates $|\psi_{+,m}\rangle$.

After calculation, we find that $Y^{\#}$ and Y satisfy the normal pseudo-fermionic algebra, namely

$$Y^2 = Y^{\#2} = 0, \quad YY^{\#} + Y^{\#}Y = \mathbf{1}. \tag{49}$$

Secondly, we introduce the abnormal "annihilation" operator Z associated with the Hamiltonian H given in (22),

$$Z = |\psi_{-+}\rangle\langle\phi_{+-}| + |\psi_{--}\rangle\langle\phi_{++}|. \tag{50}$$

The abnormal "creation" operator, which is the η-pseudo-Hermitian adjoint of Z, is given by

$$Z^{\#} = \eta^{-1}Z^{\dagger}\eta = |\psi_{++}\rangle\langle\phi_{--}| + |\psi_{+-}\rangle\langle\phi_{-+}|. \tag{51}$$

The operators Z and $Z^{\#}$ act on the eigenstates of H as follows:

$$Z|\psi_{-,m}\rangle = 0, \quad Z|\psi_{+,m}\rangle = m|\psi_{-,\overline{m}}\rangle, \tag{52}$$

$$Z^{\#}|\psi_{+,m}\rangle = 0, \quad Z^{\#}|\psi_{-,m}\rangle = m|\psi_{+,\overline{m}}\rangle, \tag{53}$$

where $m = \pm$ and $\overline{m}$ is the opposite sign of m. The operator Z annihilates the lowest eigenstates $|\psi_{-,m}\rangle$, and $Z^{\#}$ brings these states onto the upper eigenstates $m|\psi_{+,\overline{m}}\rangle$. After calculation, we find that $Z^{\#}$ and Z satisfy the abnormal pseudo-fermionic algebra, namely

$$Z^2 = Z^{\#2} = 0, \quad ZZ^{\#} + Z^{\#}Z = -\mathbf{1}. \tag{54}$$

It is worth pointing out that this abnormal pseudo-fermionic algebra does not admit the fermionic algebra in the usual Hermitian quantum mechanics as a particular case, corresponding to $\eta = 1$.

5. Concluding remarks

In this article, we have highlighted new relevant properties for pseudo-Hermitian Hamiltonians H that are invariant under the even time-reversal symmetry. We have shown that the metric operator that allows the realization of the generalized Kramers degeneracy is necessarily indefinite. For H with real spectrum, the related Hermitian counterpart h admits the same Kramers degeneracy, this time generated by an odd time-reversal symmetry. Moreover, using the relations between H and h, we found that h is also invariant under an even antilinear symmetry, induced from H, and H also commutes with an odd antilinear operator, induced from h. In view of all that, we can say that the Kramers degeneracy in pseudo-Hermitian systems with real spectrum is crypto-Hermitian degeneracy.

We have illustrated the above general results by explicit calculations in the interesting model of the split-quaternionic four-level pseudo-Hermitian Hamiltonian H, which is regarded as a generalization of the SO(3,2) model of Ref. [1]. For this more general Hamiltonian system, we have constructed a complete solution of the eigenvalue problem and, performing the diagonalization of H and its Hermitian counterpart h, have revealed the crypto-Hermitian character of the Kramers degeneracy of H. For our system, we have also considered the pseudo-Hermitian generalizations of the usual (normal) fermion algebra [12], constructing the creation and annihilation operators that obey the normal and abnormal pseudo-fermionic algebras, the latter being a consequence of the indefinite metrics.

References

[1] M. Sato, K. Hasebe, K. Esaki, and M. Kohmoto, Prog. Theor. Phys. **127**, 937 (2012).
[2] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[3] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002).
[4] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
[5] F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne, Ann. Phys. **213**, 74 (1992).
[6] A. Mostafazadeh, J. Math. Phys. **43**, 205 (2002).
[7] A. Mostafazadeh, J. Math. Phys. **43**, 2814 (2002).
[8] A. Mostafazadeh, J. Math. Phys. **43**, 3944 (2002).
[9] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. **7**, 1191 (2010).
[10] J. Wong, J. Math. Phys. **8**, 2039 (1967).
[11] F. H. M. Faisal and J. V. Moloney, J. Phys. B **14**, 3603 (1981).
[12] A. Mostafazadeh, J. Phys. A **37**, 10193 (2004).
[13] A. Mostafazadeh, Czech J. Phys. **53**, 1079 (2003).
[14] E. C. G. Sudarshan, Phys. Rev. **123**, 2183 (1961).
[15] W. Pauli, Rev. Mod. Phys. **15**, 175 (1943).
[16] T. D. Lee and G. C. Wick, Nucl. Phys. B **9**, 209 (1969).
[17] F. Bagarello and M. Znojil, J. Phys. A **44**, 415305 (2011).
[18] M. Znojil, SIGMA **4**, 001 (2008).
[19] Z. Ahmed, J. Phys. A **36**, 9711 (2003).
[20] K. Jones-Smith and H. Mathur, Phys. Rev. A **82**, 042101 (2010).
[21] O. Cherbal and D. A. Trifonov, Phys. Rev. A **85**, 052123 (2012).