

Dynamics of Loop Quantum Gravity encoded in graphs

Marcin Kisielowski^{1,2}

Based on work done in collaboration with prof. Jerzy Lewandowski and Jacek Puchta.

¹ Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul. Hoża 69, 00-681 Warszawa (Warsaw), Polska (Poland)

² St. Petersburg Department of Steklov Mathematical Institute RAS, 27, Fontanka, 191023 St. Petersburg, Russia

Abstract. We present an approach to spin foam models, where the 2-complex is constructed from more basic building blocks. Those building blocks are graphs which we equip with certain additional structure. We show how the spin foam amplitude can be read directly from this diagrammatic notation.

1. Introduction

One could interpret spin foam models as procedures for constructing discretized path integrals for quantum gravity, and for many other theories. In the case of models of quantum gravity the foams are 2-complexes having one vertex in the center of each 4-simplex, one edge intersecting each tetrahedron and one face intersecting each triangle of simplicial decomposition of a manifold (such foams we call simplicial foams) [1]. A spin foam is a labelled foam obtained by assigning to each face an irreducible representation of a given group (e.g. $SU(2)$ group) and to each edge a pair of intertwiners (or equivalently an operator [2]). In this interpretation spin foams contain all data needed to calculate transition amplitudes between states of quantized discrete gravity.

However, spin foam models of quantum gravity may be also interpreted as histories of states of Loop Quantum Gravity – the states of *full* quantum gravity. In this case, the class of possible foams is broader. An example of such a spin foam model is the Engle-Pereira-Rovelli-Livine model [3] extended in [4] to oriented linear 2-cell complexes. The extended model allows more varied structures of a vertex neighbourhood (see [4]) and full variety of boundary graphs (a boundary of a foam in this class can be an arbitrary graph). We looked for an optimal class of foams compatible with an extension of this type – a class which is just broad enough to have the desired properties discussed above (but not too broad) and does not use auxiliary linear structures which are not compatible with diffeomorphism invariance. The solution we found is naturally expressed in terms of certain diagrams, which we call graph diagrams [5, 6]. We found a coloring of a graph diagram which is equivalent to a coloring of the corresponding foam. A graph diagram together with a coloring is called an operator spin network diagram (OSN diagram). All information needed to calculate the transition amplitudes between Loop Quantum Gravity states (in general – the boundary transition amplitude [7]) is encoded in the OSN diagrams. Therefore, one does not need to reconstruct foams and may use only OSN diagrams. Since it

is easier to imagine the diagrams than the foams, we expect OSN diagrams to become a useful technical tool.

This article is structured as follows: In section 2 we introduce graph diagrams and their coloring, and we comment on the correspondence between OSN diagrams and spin foams. In section 3 we present a formula for the corresponding spin foam operator [2] which involves only the data from the OSN diagram.

2. Graph diagrams and operator spin network diagrams

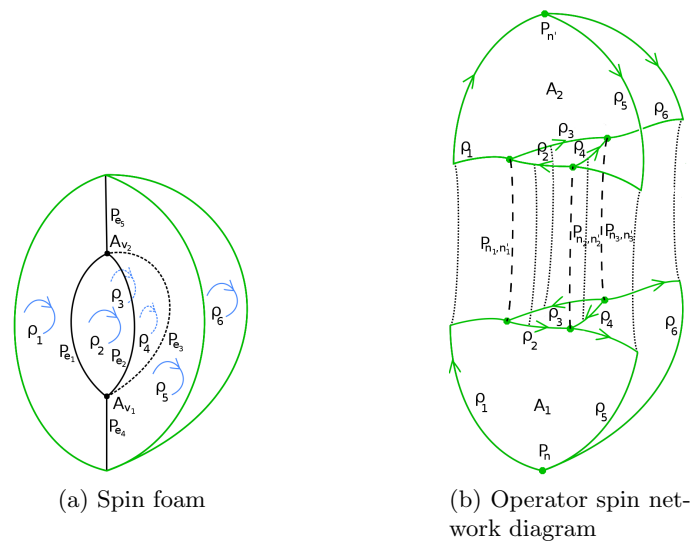


Figure 1: A spin foam and an equivalent operator spin network diagram

A **graph diagram** $(\mathcal{G}, \mathcal{R})$ consists of a set \mathcal{G} of oriented graphs $\{\Gamma_1, \dots, \Gamma_N\}$, and a family \mathcal{R} of relations. One of the relations is called a *node relation*. It is a symmetric relation in the set of nodes of the graphs. Each node n is either in relation with precisely one $n' \neq n$ or it is unrelated (such node is called a boundary node). In order to be related, two nodes need to have equal number of incoming / outgoing links (with possible closed links counted twice). One defines a bijective map between all the incoming / outgoing links of Γ_I at n , with the outgoing / incoming links of $\Gamma_{I'}$ at n' . Two links identified with each other by the bijection are called to be in relation $\mathcal{R}_{\text{link}}^{(n, n')}$ at the pair of nodes n, n' (this relation is symmetric). A link of $\Gamma_I / \Gamma_{I'}$ which intersects n / n' twice, emerges in the relation twice: once as incoming and once as outgoing. A family of relations $\mathcal{R}_{\text{link}}^{(n, n')}$ is called a *link relation*. An example of a graph diagram is depicted on figure 1b (omit a coloring A, ρ and P at this point), node relations are depicted by dashed lines and link relations are depicted by dotted lines.

To each graph in a graph diagram corresponds a foam which is an image of a homotopy of this graph to a point. It has one internal vertex and its boundary is the input graph. This foam may be considered to be an elementary building block of the 2-complex corresponding to the graph diagram. Node and link relations code the way in which we glue together these building blocks. Detailed construction of a foam from a graph diagram is given in [6]. Each graph in the graph diagram is a boundary of a suitable neighbourhood of a vertex in the corresponding 2-complex – the so called vertex graph [4]. In this sense graph diagrams define a natural class of spin foams compatible with the extension of the EPRL model presented in [4].

Graph diagrams can be colored in a way which corresponds to a coloring of the corresponding foam. A colored diagram is called an **operator spin network diagram**

$(\mathcal{G} = \{\Gamma_1, \dots, \Gamma_n\}, R, \rho, P, A)$. The coloring is the following.

- Links are colored with representations ρ acting in a Hilbert space \mathcal{H}_ρ . If two links are in link relation the corresponding representations are equivalent.
- For each node define a Hilbert space $\mathcal{H}_n = \text{Inv} \left(\bigotimes_i \mathcal{H}_{\rho_i}^* \otimes \bigotimes_j \mathcal{H}_{\rho_j} \right) \subset \left(\bigotimes_i \mathcal{H}_{\rho_i}^* \otimes \bigotimes_j \mathcal{H}_{\rho_j} \right)$, where i / j labels the links incoming / outgoing at n . Pairs of nodes $\{n, n'\}$ in the node relation are colored with an operator

$$P_{n,n'} \in \mathcal{H}_n \otimes \mathcal{H}_{n'}.$$

Boundary nodes (those not related with any other node) are colored with an operator

$$P_n \in \mathcal{H}_n \otimes \mathcal{H}_n^*.$$

- Each graph Γ_I is colored with a tensor

$$A_{\Gamma_I} \in \left(\bigotimes_n \mathcal{H}_n \right)^* \quad (1)$$

where n runs through the nodes of Γ_I . We call it a contractor.

Figure 1 may clarify the correspondence between colorings of foams and diagrams. For every link, there is a face colored with the representation equivalent to the one assigned to a link. For every pair of nodes in relation there is an edge with both endpoints being internal vertexes of the corresponding foam, and for every boundary node there is an edge which has one endpoint being internal vertex and one being boundary vertex. The coloring with contractors is a new idea we introduced to include the SU(2) spin foam model constructed from EPRL vertex [8, 9] (and possible other models). The contractor is a generalization of a standard vertex contraction [4, 2], i.e. it generalizes the contractor A^{Tr} implicitly used in operator spin foam formalism [2]:

$$A_{\Gamma}^{\text{Tr}} = \bigotimes_l \text{Tr}_l \in \bigotimes_l \mathcal{H}_{\rho_l}^* \otimes \mathcal{H}_{\rho_l} \equiv \left(\bigotimes_n \mathcal{H}_n \right)^*,$$

where l runs through links of the graph Γ .

3. The spin foam operator

There is a natural contraction

$$\left(\bigotimes_{\Gamma_I} A_{\Gamma_I} \right) \lrcorner \left(\bigotimes_n P_n \right), \quad (2)$$

where Γ_I ranges the set of graphs and n ranges the set of boundary nodes and pairs of nodes related by the node relation (for more details on the contraction see [6]).

Operator (2) is a spin foam operator of the spin foam corresponding to the graph diagram. In (2) every face amplitude and boundary edge amplitude is equal 1. We now generalize this formula to include general boundary edge and face amplitudes.

We redefine the operators P_n corresponding to boundary nodes. Since boundary nodes correspond to boundary vertexes and links intersecting these nodes correspond to boundary edges (the second correspondence is 2-1), the redefinition of P_n allows us to include general boundary edge amplitudes. We define

$$A_n = \sqrt{\prod_l A_l}, \quad (3)$$

where l ranges the set of links intersecting the node n . The terms A_l become boundary edge amplitudes when the spin foam is reconstructed. The square root in formula (3) comes from the fact that each link is counted twice. We redefine operators P_n by setting:

- $\tilde{P}_n = A_n P_n$ for boundary nodes,
- $\tilde{P}_n = P_n$ when $n = \{n_1, n_2\}$.

The node relation and the link relation lead to an equivalence relation in the set of links of the graphs. The resulting equivalence relation, which we call face relation, carries information about faces of the corresponding 2-complex, and allows to introduce face amplitude without direct reference to that 2-complex. The face relation \sim is described as follows:

- Each link is in relation with itself.
- Two different links l and l' are in relation when they are in the link relation.
- It may happen that a link is in the link relation with two other (different) links (because every link has two ends). In general there may be a sequence of links (l_1, l_2, \dots, l_m) such that l_i is in the link relation with l_{i+1} , $i \in \{1, \dots, m-1\}$ and it is natural to assign a single amplitude to a set $\{l_1, \dots, l_m\}$. As a result we define l and l' to be in a face relation when there exists l'' such that l is in the link relation with l'' , and l'' is in the link relation with l' .

The face relation induces division of the set of links into equivalence classes. We assign an amplitude (a complex number) to f and denote it by A_f . It becomes a face amplitude in the reconstructed spin foam. Denote by \mathcal{L}/\sim a set of those equivalence classes. The spin foam operator is:

$$P = \left(\prod_{f \in \mathcal{L}/\sim} A_f \right) \left(\bigotimes_I A_I \right) \lrcorner \left(\bigotimes_n \tilde{P}_n \right). \quad (4)$$

It is equal to an operator assigned to the corresponding spin foam [6]. As anticipated in the introduction, this formula involves only objects defined by an operator spin network diagram and no reference to the resulting spin foam is needed.

Summary. Graph diagrams describe a class of foams which we suggest to use in spin foam models of quantum gravity [6]. The data from the operator spin network diagram is sufficient to calculate the spin foam operator – no reference to the corresponding spin foam is needed. Therefore this formalism is independent from the 2-complex approach. We expect it to be a useful technical tool, as the diagrams are easier to imagine than the 2-complexes.

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