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Quantum Retarded Field Engine

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Abstract: Recent efforts to conceptually design a technologically meaningful electromagnetic retarded engine indicated that this can only be done using the immense charge and current densities which exist in the atomic scale. However, this scale cannot be described by Newtonian physics, and only a quantum description will suffice to describe the dynamics of an electron on this scale properly. Here we study the retarded field quantum engine and highlight the differences between the quantum and the classical retarded engines. It is emphasized that the constituents of the retarded engine studied in the current paper do not move in relativistic speeds, hence they are analyzed using un-relativistic classical mechanics and un-relativistic quantum mechanics (Schrödinger's equations). The retardation effect is due to the finite propagation speed of the field and not the relativistic motion of the particles.

Keywords: retardation; quantum mechanics

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1. Introduction

Special relativity, originating from Einstein's seminal 1905 paper "On the Electrodynamics of Moving Bodies" [1], is a theory that fundamentally reshapes our understanding of space and time. Its development was spurred by empirical observations and Maxwell's equations of electromagnetism [2–5], which were refined by Oliver Heaviside in the 19th century [6]. Maxwell's equations notably imply that electromagnetic signals propagate at the constant speed of light c in vacuum, suggesting that light behaves as an electromagnetic wave. Building on this, Albert Einstein [1,4,5] formulated special relativity, positing that c represents the maximum achievable velocity in the universe. According to this theory, nothing—whether an object, message, signal (even non-electromagnetic), or field—can surpass the speed of light in a vacuum. This principle introduces the concept of retardation: any change made by someone at a distance R from an observer will take at least a retardation time of $\frac{R}{c}$ for the observer to become aware of it. Consequently, actions and their reactions cannot occur simultaneously due to the finite propagation speed of signals.

Newton's laws of motion comprise three fundamental principles that form the cornerstone of classical mechanics. These laws delineate the connection between a body, the forces acting upon it, and its resulting motion while introducing the concept of force into physics. Initially formulated by Isaac Newton in his work "Philosophiae Naturalis Principia Mathematica" (Mathematical Principles of Natural Philosophy) [7,8], first published in 1687, these laws have enduring significance in physics. Of particular interest here is Newton's third law [9,10], which asserts that when one body exerts a force on the other body, the other body **concurrently** exerts a force of equal magnitude but opposite direction on the first body. This concurrent force cannot be accommodated with the principle of retardation, because a change done in one subsystem cannot be possibly affect another subsystem which is at a distant R before a duration of $\frac{R}{c}$ passes. This was noted by [11] but without emphasizing the importance of retardation in the third law violation.

In the beginning of the second quarter of the twentieth century the "new" quantum mechanics was introduced [12]. Schrödinger suggested his famous quantum wave equation and soon Pauli [13] and Dirac [14] suggested corrections, motivated by experiment and the desire to accommodate quantum and relativistic theories which imply symmetry under the Lorentz transformations. This new formalism of mechanics rejected the concept of force and embraces the concept of potential instead (scalar and vector electromagnetic potentials). The gauge freedom in potential definition was thus connected to the (total) phase freedom of the wave function. Of course the potentials themselves are derived from electromagnetic theory and thus must be retarded.

It was clear that there must a connection between the classical and quantum levels of reality, and indeed such a connection was suggested by Ehrenfest [15] formalism which reintroduce the concept of force as an expectation value (to be discussed below) and also by Bohm [16–18] and the realistic school of quantum mechanics which suggested that one can write a Newton's second law equation for a quantum particle that has a trajectory (regardless if it is measured or not). However, one needs to add a nonlocal quantum force to the electromagnetic forces in the Bohm formalism. As the quantum forces do not obey Newton's third law and moreover they do not depend on the separation distance between particles and allow action at a distance in a very counterintuitive fashion. It follows that the Bohmian approach to quantum mechanics leads to an even stronger violation of Newton's third law than retardation alone. We shall show below that Bohmian mechanics entails a "quantum engine" that can self propel without interacting with environment, such a device can be tested and validate (or invalidate) Bohmian mechanics. Finally we notice that the connections between QED and Bohmian mechanics have been discussed in [19].

Newton's third law stipulates that in a system unaffected by external forces, the total sum of forces is zero. This principle has garnered numerous experimental confirmations, solidifying its status as a fundamental tenet of the physical sciences. However, it's evident that action and reaction cannot occur simultaneously due to the finite speed of signal propagation. Moreover, a quantum potential generates a force that does satisfy the classical "third law". Therefore, while Newton's third law holds true in many practical scenarios owing to the high velocity of signal propagation, and the macroscopic dimensions of classical objects, it cannot be deemed entirely accurate in an exact sense. Consequently, the total sum of forces within a system cannot remain zero at all times. And thus the Newton's first law demanding that a body will continue with the same velocity unless affected by an external force is also not strictly satisfied.

Current locomotive systems typically rely on coupled material components, wherein each component gains momentum equal and opposite to that gained by the other. A classic example is a rocket, which expels gas to propel itself forward. However, relativistic effects suggest an alternative type of propulsion system that involves both matter and field, rather than two distinct material elements. Initially, it may seem that the material body gains momentum, thus violating momentum conservation. However, it can be demonstrated [20] that an equivalent amount of momentum is imparted to the field, ensuring total momentum conservation. This phenomenon arises from Noether's theorem, which asserts that systems possessing translational symmetry conserve momentum. While individual components of the system (matter or field) may not exhibit translational symmetry, the entire physical system composed of both matter and field remains invariant under translations. Feynman [5] elucidates a scenario involving two charges moving orthogonally, seemingly challenging Newton's third law as the forces induced by the charges do not cancel each other. However, this apparent contradiction is resolved by recognizing that the momentum gained by the two-charge system is transferred to the field.

A retarded engine is characterized by a system where the material center of mass is set in motion through the interaction of its constituent material components. These components may either move relative to each other or be fixed within a rigid framework. However, our focus lies solely on the motion of the center of mass. It's important to note that a retarded engine enables motion along all three axes, including vertical movement. Unlike conventional engines, it may lack moving parts and doesn't require traditional fuel consumption, thereby eliminating carbon emissions. Instead, it operates by consuming electromagnetic energy, which can be conveniently supplied by sources like solar panels.

This makes the retarded engine particularly well-suited for space travel, where a significant portion of the spacecraft's volume is typically allocated for fuel storage.

In our present study, we make the assumption that the medium's magnetization and polarization are negligible. Consequently, we do not take into account corrections to the Lorentz force as proposed by Mansuripur [21]. Griffiths and Heald [22] highlighted that Coulomb's and Biot-Savart's laws govern the configurations of electric and magnetic fields exclusively for static sources. To extend the applicability beyond static scenarios, time-dependent generalizations of these laws, as described by Jefimenko, have been utilized. These generalized laws enable the investigation of Coulomb and Biot-Savart formulas in dynamic contexts, stepping beyond the constraints of static conditions. This is true when the sources (charge & current densities) are either classical or quantum.

In a previous study, we employed Jefimenko's equation [4,23], to investigate the force interaction between two current loops [24]. This research was subsequently expanded to explore the forces between a current-carrying loop and a permanent magnet [25]. Since the device operates for a finite duration, it acquires mechanical momentum and energy. Consequently, the question arises whether we must relinquish the principles of momentum and energy conservation. The issue of momentum conservation was addressed in prior research [20]. Additionally, discussions in other studies delved into the exchange of energy between the mechanical components of the retarded engine and the electromagnetic field [26]. Notably, it was demonstrated that the total electromagnetic energy expenditure exceeds the kinetic energy gained by the retarded motor by a factor of six. Furthermore, these studies examined how energy might be radiated from the retarded engine device if the coils are improperly configured.

In previous analyses, the assumption was made that the bodies under study were macroscopically natural, implying an equal number of electrons and ions in every volume element. However, later we relaxed this assumption and considered charged bodies [27]. Therefore, we investigated the implications of charge on the feasibility and behavior of a potential electric retarded engine. This yielded much higher momentum and force than a non charged motor, however, the limitations of dielectric breakdown resulted in a still impractical devices.

The aforementioned limitations have led us to consider utilizing the high charge densities found at the microscopic scale [28], such as those present in ionic crystals. This concept was explored in a prior publication, where we calculated the remarkably high charge densities and current densities at the atomic level. Our findings indicated that an isolated hydrogen atom, whether in a ground or excited state, does not yield momentum for a retarded motor. However, this changes when the atom interacts with other atoms or particles, or when the electron within the atom is in a non-eigenstate state. As a result, we proposed two simplistic forms for a wave function that could potentially lead to advantageous gains for a relativistic engine: a wave packet within a hydrogen atom and an eigenstate within a simple molecule that introduces a static electric field with broken spherical symmetry.

Thus, to obtain a practical retarded engine it is needed to manipulate matter at subatomic levels. In a previous paper [29] we investigated two ways of doing so one that is related to free electrons and the other to confined electrons. While we started with a classical description of the problem we could not and did not ignore the fact that on the atomic level a quantum description is necessary. It was also shown that the quantum effects are more significant for confined electrons in comparison with free electrons. In [29] it was emphasized that the quantum retarded motor is very important for future space transportation (and possibly ground transportation) and thus is both interesting and meaningful.

We note that a single classical accelerating body is known to radiate an electromagnetic field, and slows down at the same time (synchrotron radiation). This demonstrates transfer of linear momenta and energy from the body to the electromagnetic field. However, for a retarded engine effect, a single body (even if accelerating) will not suffice, a two-body system is needed, and the retarded engine effect is concerned with the center of mass motion of this two-body system. This has been demonstrated already ten years ago [24] and was never disputed in the scientific literature. Moreover, it was demonstrated again for charged electromagnetic systems three year ago [27]. As in the single accelerating body system also the two-body system exchanges linear momentum and energy with the electromagnetic field, however, conditions were derived in which in the two-body system the center of mass may accelerate (rather than decelerate as for the damping on a single body system) and thus constitute a retarded engine.

We underline that the electromagnetic field is treated classically (and not by using quantum electrodynamics (QED)), this is the standard method in atomic, molecular and solid-state physics (and thus almost all chemistry and most of physics). The particles are described by a wave function when treated quantum mechanically, while the field is treated classically. Second quantization of wave function and quantization of the electromagnetic field are usually only addressed in high energy physics and the theory of elementary particles, which is not relevant to most physical systems and will not be discussed herer. It is generally accepted that if the electric field does not surpass the Schwinger limit (10^{18} V/m), or the magnetic field is well below 4×10^9 T, QED can be safely ignored as we do in the current work. Notice, however, that QED works also at low energy. A small shift of the hydrogen energy levels (Lamb's shift) is not described by classical electrodynamics only by QED. Moreover, spontaneous emission [30] is believed to be described by QED and not classical electrodynamics. Thus every phenomena described in the current paper does not include those QED effects and any transfer of energy from the material part of the system to the electromagnetic field is to be considered as a classical emission.

Previous papers relied on a classical description of a two body system, in which the two bodies interacted through retarded electromagnetic fields. When quantum effects were considered this was only done in the case of a classical system (the hydrogen nucleus) interacting with a quantum system (the electron), in this case classical formulae were use in which case the electron charge density and current density were derived from its wave function (they were taken to be proportional to the probability density and probability current density without any justification) and plugged into a classical force equation. In the current paper we would like to discuss the case in which the two bodies interacting are both quantum and wether it is justified to compare probability densities and charge densities.

We shall attempt two possible connections of the classical and quantum worlds, one through the Ehrenfest theorem the other through the formalism of Bohm. Finally we shall discuss the conservation of momentum showing that in either case the linear momentum of a closed system will not be conserved, an effect which can be rectified in the macroscopic level only through the concept of field momentum. Thus the total linear momentum of matter and field is conserved but not the linear momentum of each separately.

To be clear the current paper does not present an alternative theory. All theories discussed in the paper are well known. Those include classical electrodynamics, classical mechanics, Schrödinger's quantum mechanics and Bohm's quantum mechanics. The purpose of the paper is to study a retarded engine in the framework of those theories and elucidate its properties. In all parts of the paper the field is described by classical electrodynamics (not QED), however, we compare different theoretical representations of the material part (classical mechanics, Schrödinger's quantum mechanics and Bohm's quantum mechanics) and show the differences and similarities of the predictions those theories. This is done without forming any apriori opinion on the validity (or invalidity) of any of the theories.

2. Retarded Electromagnetic Fields

Electromagnetism is described by a set of the four Maxwell equations (in MKS units):

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1}$$

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}, \qquad \partial_t \equiv \frac{\partial}{\partial t} \tag{2}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{3}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \partial_t \vec{D}, \tag{4}$$

in which $\vec{\nabla}$ has a standard meaning in vector analysis, $\vec{E}$ is the electric field, $\vec{H}$ is the magnetic field, $\vec{D}$ is the displacement field and $\vec{B}$ the magnetic flux density. Also ρ is the charge density, and $\vec{J}$ is the current density. In electromagnetic theory the charge and current densities are assumed to be given and cannot be determined from the theory itself. In the following sections we shall try to derive them from both classical and quantum mechanics.

In vacuum we have simple relations between the electric and displacement fields:

$$\vec{D} = \varepsilon_0 \vec{E}, \tag{5}$$

in which ε_0 is the vacuum susceptibility. And also:

$$\vec{B} = \mu_0 \vec{H}, \tag{6}$$

in which μ_0 is the vacuum permeability. Thus in vacuum we may write the last two equations in a form that depends only on $\vec{E}$ and $\vec{B}$:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{7}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \partial_t \vec{E} = \mu_0 \vec{J} + \frac{1}{c^2} \partial_t \vec{E}, \tag{8}$$

in which the velocity of light in vacuum is defined as: $c \equiv \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$. The first two equations (Equations (1) and (2)) that do not depend on the charge and current densities (or the medium) can be easily solved in terms of scalar Φ and vector $\vec{A}$ potentials:

$$\vec{E} = -\partial_t \vec{A} - \vec{\nabla} \Phi. \tag{9}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{10}$$

We can now solve Equations (7) and (8) for the scalar and vector potentials and obtain the results [4] in terms of the retarded expressions:

$$\Phi(\vec{x}) = k \int d^3x' \frac{\rho(\vec{x}', t_{ret})}{R}, \qquad \vec{R} \equiv \vec{x}' - \vec{x}, \qquad t_{ret} \equiv t - \frac{R}{c}, \qquad k = \frac{1}{4\pi\epsilon_0}. \tag{11}$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}', t_{ret})}{R}. \tag{12}$$

To obtain the above solutions we must demand that the scalar and vector potentials satisfy the Lorentz gauge conditions:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \partial_t \Phi = 0 \tag{13}$$

This is possible since we have gauge freedom in the choice of vector and scalar potentials. The Lorentz gauge condition is guaranteed to be satisfied for all times due to Equations (11) and (12) and since charge is conserved:

$$\vec{\nabla} \cdot \vec{J} + \partial_t \rho = 0 \tag{14}$$

(See appendix A of [27]). The solutions for the electric field and magnetic flux density are thus given by Jefimenko's retarded fields [4,23,27] by inserting Equations (11) and (12) into Equations (9) and (10):

$$\vec{E}(\vec{x}) = -k \int d^3x' \frac{1}{R^2} \left[\left(\rho(\vec{x}', t_{ret}) + \left(\frac{R}{c}\right) \partial_t \rho(\vec{x}', t_{ret}) \right) \frac{\vec{R}}{R} + \left(\frac{R}{c}\right)^2 \frac{\partial_t \vec{J}(\vec{x}', t_{ret})}{R} \right]. \tag{15}$$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{R}}{R^3} \times \left(\vec{J}(\vec{x}', t_{ret}) + \left(\frac{R}{c}\right)\partial_t \vec{J}(\vec{x}', t_{ret})\right). \tag{16}$$

The potentials can also be used to formulate the electromagnetic problem in a variational form in which the action is given as:

$$\mathcal{A}_{EM} = \int L_{EM} dt, \qquad L_{EM} = L_F + L_I, \qquad L_F = \int d^3x' \mathcal{L}_F, \qquad L_I = \int d^3x' \mathcal{L}_I. \tag{17}$$

In the above the Lagrangian density of the field is:

$$\mathcal{L}_F[\vec{A}, \Phi] = \frac{1}{2}\left(\varepsilon_0 E^2 - \frac{B^2}{\mu_0}\right) = \frac{1}{2}\varepsilon_0\left(E^2 - c^2 B^2\right) = \frac{1}{2}\varepsilon_0\left((\partial_t \vec{A} + \vec{\nabla}\Phi)^2 - c^2(\vec{\nabla} \times \vec{A})^2\right). \tag{18}$$

And the Lagrangian density of the interaction is:

$$\mathcal{L}_I[\vec{A}, \Phi] = \vec{J} \cdot \vec{A} - \rho\, \Phi, \tag{19}$$

if ρ and $\vec{J}$ do not depend on the electromagnetic potentials, this is true for a system of classical particles but not in the quantum case to be discussed in the following sections.

We emphasize that electromagnetic theory does not suffice to determine what $\vec{J}$ and ρ are, but is certainly sufficient to determine that the fields must be retarded.

3. The Classical Description

The classical picture of the world, describes it as the arena in which a very large number (say N) of point particles interact. Modern physics recognizes only four possible types of fundamental interactions: electromagnetic, gravitational, strong nuclear and weak nuclear. The nuclear interactions are only important for processes which happen to occur in the nucleus or in high energy processes, hence it is not important for chemistry, biology and most of our daily life. Nevertheless the description of those interactions is Lorentz invariant hence they suffer retardation non the less. We shall not consider this any further in the current paper.

Hence we are left with two possible types of interaction electromagnetic and gravitational, the second is much weaker than the first and needs to be considered only for astronomical bodies, thus we will also neglect gravity. We are left with only the electromagnetic interaction. The effect of an electromagnetic field on the classical particle i is given through the Lorentz force and Newton's second law (we assume that the particle is not relativistic that is its velocity is small with respect to c):

$$m_i \vec{ac}_i = \vec{F}_i = q_i\left(\vec{E}(\vec{x}_i(t), t) + \vec{v}_i \times \vec{B}(\vec{x}_i(t), t)\right), \qquad \vec{ac}_i \equiv \frac{d\vec{v}_i}{dt}, \qquad \vec{v}_i \equiv \frac{d\vec{x}_i}{dt}. \tag{20}$$

In the above m_i is the particle's mass and q_i is the particle's charge. $\vec{x}_i$ designates the location of the particle with respect to the origin of an inertial frame, $\vec{v}_i$ is the particle velocity and $\vec{ac}_i$ is the particles acceleration. The (retarded) fields $\vec{E}$ and $\vec{B}$ are evaluated at the particles momentary location. A variational description of the above system can be derived from the action:

$$\mathcal{A}_p = \int L_p dt, \qquad L_p = L_k + L_I. \tag{21}$$

In the above we have a sum of kinetic Lagrangian:

$$L_k = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2. \tag{22}$$

and an interaction Lagrangian:

$$L_I = \sum_{i=1}^{N} q_i \Big(\vec{A}(\vec{x}_i(t), t) \cdot \vec{v}_i - \Phi(\vec{x}_i(t), t) \Big). \tag{23}$$

Notice, however, that according to Maxwell's theory (Equations (17) and (19)) the interaction Lagrangian should be written as:

$$L_I = \int d^3x' \mathcal{L}_I = \int d^3x' (\vec{J}(\vec{x}') \cdot \vec{A}(\vec{x}') - \rho(\vec{x}')\, \Phi(\vec{x}')) \tag{24}$$

To accommodate simultaneously those two forms, it seems that one should adopt the following charge density and current density definitions:

$$\rho_c(\vec{x}, t) \equiv \sum_{i=1}^{N} q_i \delta^3(\vec{x} - \vec{x}_i(t)) \tag{25}$$

$$\vec{J}_c(\vec{x}, t) \equiv \sum_{i=1}^{N} q_i \vec{v}_i(t) \delta^3(\vec{x} - \vec{x}_i(t)) \tag{26}$$

in which δ^3 is a three dimensional Dirac delta function. Thus the classical world view resolves the sources of the electromagnetic field. However, the price to be paid is introducing unphysical charge and current densities which become infinite at various points of space in which the particles happen to be. This defies physical intuition that demands that every physical quantity must be finite in every point of space. It also shows through Equations (15) and (16) that the interaction of classical particles with each other is not immediate but retarded. We conclude this section by reminding the reader that the total classical action of field and particles is:

$$\begin{aligned} \mathcal{A}_T &= \int L_T dt, \\ L_T &= L_k + L_I + L_F \\ &= \sum_{i=1}^{N} \left[\frac{1}{2} m_i v_i^2 + q_i \Big(\vec{A}(\vec{x}_i(t), t) \cdot \vec{v}_i - \Phi(\vec{x}_i(t), t) \Big) \right] + \frac{1}{2} \varepsilon_0 \int d^3x' \Big(E^2 - c^2 B^2 \Big). \end{aligned} \tag{27}$$

And thus the canonical momentum for each particle:

$$\vec{p}_{can\ i} \equiv \frac{\partial L_i}{\partial \vec{v}_i} = m_i \vec{v}_i + q_i \vec{A}(\vec{x}_i(t), t), \tag{28}$$

and the classical linear momentum for each particle:

$$\vec{p}_{cl\ i} = m_i \vec{v}_i, \tag{29}$$

are not the same unless $\vec{A} = 0$.

4. The Quantum Description

A quantum description of an N particles system involves a complex wave function $\Psi(\vec{x}_1, \vec{x}_2 \ldots, \vec{x}_N, t)$ and solves the Schrödinger equation:

$$i\hbar \partial_t \Psi = \hat{H} \Psi, \tag{30}$$

we will consider as in the previous section low velocity particles without spin. In the above $i \equiv \sqrt{-1}$ is the imaginary number, and $\hbar$ is Planck's constant divided by 2π. Generally speaking Ψ cannot be written as a multiplication of functions of each $\vec{x}_i$ separately (the particles are generically correlated). Moreover, it is known that for an identical set of particles the function must be either symmetric to interchange of particles (bosonic case) or antisymmetric to the interchange of particles (fermionic case). Some authors write the hamiltonian $\hat{H}$ ([17] Equation (7.1.1)) in the form:

$$\hat{H} = -\sum_{i=1}^{N} \frac{\hbar^2}{2\, m_i} \nabla_i^2 + V(\vec{x}_1, \vec{x}_2 \ldots, \vec{x}_N, t). \tag{31}$$

However, this form (which is rather abstract) hides the source of the interaction which as we saw in the previous section is mainly electromagnetic. It also devoids the particle of the possibility to interact with the magnetic field. Thus we shall write the Hamiltonian in the form:

$$\hat{H} = \sum_{i=1}^{N} \hat{H}_i = \sum_{i=1}^{N} \left[-\frac{1}{2\, m_i} \left(\hbar \vec{\nabla}_i - i q_i \vec{A}(\vec{x}_i, t) \right)^2 + q_i \Phi(\vec{x}_i, t) \right] \tag{32}$$

which overcomes those objections.

In terms of the quantum Hamiltonian operator we may write a Lagrangian density in the form:

$$\mathcal{L}_q = \frac{1}{2} i\hbar (\Psi^* \partial_t \Psi - \partial_t \Psi^* \Psi) - \Psi^* \hat{H} \Psi. \tag{33}$$

However, this density is a density in the $3N$ dimensional configuration space (not in the standard three dimensional space). Defining the configuration space coordinate:

$$\vec{X} \equiv \vec{x}_1, \vec{x}_2 \ldots, \vec{x}_N, \tag{34}$$

We may write:

$$\mathcal{A}_q = \int L_q dt, \qquad L_q = \int d^{3N} X \mathcal{L}_q. \tag{35}$$

As we saw in the previous section the key to identifying the charge and current densities is identifying the interaction terms in the Lagrangian, this is also true in the quantum case. For this it will be useful to write the wave function in a "polar" form:

$$\Psi = a e^{i\phi} = a e^{i \frac{S}{\hbar}} \tag{36}$$

Using this form we may write the quantum Lagrangian Equation (35) in the form:

$$\begin{aligned} L_q &= \int d^{3N} X \Bigg\{ -\sum_{i=1}^{N} \Bigg[a^2 \Bigg(\frac{\hbar^2}{2\, m_i} (\vec{\nabla}_i \phi)^2 - \frac{q_i \hbar}{m_i} \vec{\nabla}_i \phi \cdot \vec{A}(\vec{x}_i, t) + \frac{q_i^2}{2\, m_i} A^2(\vec{x}_i, t) + q_i \Phi(\vec{x}_i, t) \Bigg) \\ &+ \frac{\hbar^2}{2\, m_i} (\vec{\nabla}_i a)^2 \Bigg] - \hbar a^2 \partial_t \phi \Bigg\}. \end{aligned} \tag{37}$$

in which we omitted some boundary terms that do not effect the equations. From the above Lagrangian it is easy to see that the interaction term with the scalar potential takes the form:

$$L_{q\Phi} = -\sum_{i=1}^{N} \int d^{3N} X a^2 q_i \Phi(\vec{x}_i, t) \tag{38}$$

Introducing the reduced square amplitude:

$$a_i^2(\vec{x}_i, t) \equiv \int d^{3N-3} X_i a^2, \qquad d^{3N} X = d^{3N-3} X_i d^3 x_i \tag{39}$$

In which $d^{3N-3}X_i$ is a volume element including all coordinates of configuration space except the i^{th} coordinate. The Born propensity rule associates with a^2 a probability density function such that $\int d^{3N} X a^2 = 1$, thus in terminology of random variables theory a_i^2 is a marginal probability density function satisfying also: $\int d^3 x_i a_i^2 = 1$.

Now we may write:

$$L_{q\Phi} = -\sum_{i=1}^{N} \int d^3 x_i q_i a_i^2(\vec{x}_i, t)\Phi(\vec{x}_i, t) = -\sum_{i=1}^{N} \int d^3 x' q_i a_i^2(\vec{x}', t)\Phi(\vec{x}', t) \tag{40}$$

In which the second equation one is due a change of the name of the integration variable. Comparing Equation (40) with Equation (24) it follows that the quantum charge density should be of the form:

$$\rho_q(\vec{x}, t) = \sum_{i=1}^{N} \rho_{qi}(\vec{x}, t), \qquad \rho_{qi} \equiv q_i a_i^2. \tag{41}$$

For a single body this is just $\rho_q = q a^2$ which justifies the quantum charge density used in [28,29]. For a classical body one needs to take the limit:

$$a_i^2(\vec{x}, t) \to \delta^3(\vec{x} - \vec{x}_i(t)) \qquad \Rightarrow \qquad \rho_q(\vec{x}, t) \to \rho_c(\vec{x}, t), \tag{42}$$

the right limit follows only if all particles are classical. We notice that while the quantum charge density is finite (and hence physical), the classical charge density is just a useful mathematical construct that is used to simplify the calculations when one is far away from the support of the reduced amplitude ("the location" of the particle).

The interaction term with the vector potential takes the following form:

$$L_{q\vec{A}} = \sum_{i=1}^{N} \int d^{3N} X a^2 \left[\frac{q_i \hbar}{m_i} \vec{\nabla}_i \phi \cdot \vec{A}(\vec{x}_i, t) - \frac{q_i^2}{2\, m_i} A^2(\vec{x}_i, t) \right]. \tag{43}$$

Now this term is second order in $\vec{A}$ (as are also Lagrangians of classical Eulerian charged fluids [31]). It follows from comparison with Equation (24) that the quantum current density must be dependent on $\vec{A}$. We also recall that the Maxwell field equations are obtained by taking a variational derivative with respect to $\vec{A}$ of the electromagnetic Lagrangian hence we compare the quantum and electromagnetic interaction terms when varied with respect to $\vec{A}$:

$$\delta_{\vec{A}} L_q = \sum_{i=1}^{N} q_i \int d^{3N} X a^2 \delta \vec{A}(\vec{x}_i, t) \cdot \left[\frac{\hbar}{m_i} \vec{\nabla}_i \phi - \frac{q_i}{m_i} \vec{A}(\vec{x}_i, t) \right]. \tag{44}$$

We now define the quantum velocity field:

$$\vec{v}_{qi}(\vec{X}, t) = \frac{\hbar \vec{\nabla}_i \phi - q_i \vec{A}(\vec{x}_i, t)}{m_i} = \frac{\vec{\nabla}_i S - q_i \vec{A}(\vec{x}_i, t)}{m_i}. \tag{45}$$

One should notice that in the generic quantum case this velocity field depends on the entire configuration space and not only on the coordinates of particle i, which is of course connected to the inherent non-locality of quantum mechanics. Nevertheless, Bohm's interpretation of quantum mechanics [16,17] suggests that a quantum particle has a well defined trajectory and the trajectory satisfies the differential equation:

$$\frac{d\vec{x}_i(t)}{dt} = \vec{v}_{qi}(\vec{X}(t), t). \tag{46}$$

for other interpretations of quantum mechanics (Copenhagen) this is just a vector field with units of velocity. We emphasize that in order to obtain the trajectory in configuration space one must evaluate $\vec{v}_{qi}$ along this trajectory. We may thus write the variation of the quantum Lagrangian with respect to $\vec{A}$ as:

$$\delta_{\vec{A}} L_q = \sum_{i=1}^{N} q_i \int d^{3N} X a^2 \delta \vec{A}(\vec{x}_i, t) \cdot \vec{v}_{qi}(\vec{X}, t). \tag{47}$$

We can now define a reduced quantum velocity:

$$a_i^2(\vec{x}_i)\vec{v}_{rqi}(\vec{x}_i,t) = \int d^{3N-3}X_i a^2(\vec{X},t)\vec{v}_{qi}(\vec{X},t). \tag{48}$$

And write the variation as:

$$\delta_{\vec{A}} L_q = \sum_{i=1}^{N} q_i \int d^3x_i a_i^2(\vec{x}_i)\vec{v}_{rqi}(\vec{x}_i,t)\cdot\delta\vec{A}(\vec{x}_i,t) = \int d^3x' \left[\sum_{i=1}^{N} q_i a_i^2(\vec{x}')\vec{v}_{rqi}(\vec{x}',t)\right]\cdot\delta\vec{A}(\vec{x}',t). \tag{49}$$

Maxwell theory demands that:

$$\delta_{\vec{A}} L_I = \int d^3x' \vec{J}(\vec{x}')\cdot\delta\vec{A}(\vec{x}') \tag{50}$$

this form is more general than Equation (24) which assumes implicitly that $\vec{J}$ does not depend on the vector potential which is not the quantum case. Comparing Equations (49) and (50) suggests the following form of the quantum current density:

$$\vec{J}_q(\vec{x},t) = \sum_{i=1}^{N} \vec{J}_{qi}(\vec{x},t), \qquad \vec{J}_{qi} \equiv q_i a_i^2 \vec{v}_{rqi}. \tag{51}$$

This will reduce to the classical case only if the particle is localized $a_i^2(\vec{x},t) \to \delta^3(\vec{x}-\vec{x}_i(t))$ and the reduced quantum velocity takes a classical value. Notice, however, that wether the charge and current densities are classical or quantum their electromagnetic effect on their peers is retarded as follows from Equations (15) and (16). Moreover, we may write a total quantum Lagrangian (describing both particles and field) in the form:

$$L_{Tq} = L_q + L_F. \tag{52}$$

This signifies the fact that the quantum wave function despite its connection to epistemological constructs such as probability (Born propensity rules) is not less real than the electromagnetic field as the field cannot be more real than its sources which are wave function derived quantities. We stress that charge and current densities are not directly related to the particles coordinates even in Bohm's interpretation, and writing classical type of charge and current densities (containing delta functions) even with Bohm's quantum velocity field will not lead to a correct coupling between the wave function and electromagnetic field. The wave function although determined deterministically does not deterministically determine the trajectory of a single quantum particle (only its tendency to move to specific locations) and in this sense quantum mechanics is not deterministic, this is well known result of the two slit experiment [17]. However, according to Bohm [16] this indeterminism could be removed if we knew the location of the particle at any time t_0, in that case Equation (46) will tell us the location of the said particle in any future or past time using our knowledge of the quantum mechanical wave function phase.

5. Fisher Information

The quantum Lagrangian of a single particle can be interpreted through Fisher information as wad demonstrated in [31], however, this also true for a multiple particle system. To see that rewrite L_q given in Equation (37) in terms of the quantum velocity defined in Equation (45):

$$L_q = \int d^{3N}X \left\{ -\sum_{i=1}^{N}\left[a^2\left(\frac{1}{2}m_i v_{qi}^2 + q_i\Phi(\vec{x}_i,t)\right) + \frac{\hbar^2}{2\,m_i}(\vec{\nabla}_i a)^2\right] - \hbar a^2\partial_t\phi\right\}. \tag{53}$$

According to [31,32] the Fisher information of a $3N$ dimensional random variable is:

$$F_I = 4\int d^{3N}X\left(\vec{\nabla}_X a\right)^2 = 4\sum_{i=1}^{N}\int d^{3N}X\left(\vec{\nabla}_i a\right)^2 \tag{54}$$

Hence the Fisher information of the $3N$ dimensional random can be written as sum of components each of the form:

$$F_{Ii} = 4\int d^{3N}X\left(\vec{\nabla}_i a\right)^2 \qquad \Rightarrow \qquad F_I = \sum_{i=1}^{N} F_{Ii} \tag{55}$$

In terms of the Fisher information components we may write the quantum Lagrangian in the form:

$$L_q = -\int d^{3N}X\left\{\sum_{i=1}^{N} a^2\left(\frac{1}{2}m_i v_{qi}^2 + q_i\Phi(\vec{x}_i,t)\right) + \hbar a^2\partial_t\phi\right\} - \sum_{i=1}^{N}\frac{\hbar^2}{8\,m_i}F_{Ii}. \tag{56}$$

In the case that the quantum system is composed of particles of identical mass $m_i = m$ (but not necessarily of identical charge) the above expression can be written simply as:

$$L_q = -\int d^{3N}X\left\{\sum_{i=1}^{N} a^2\left(\frac{1}{2}m v_{qi}^2 + q_i\Phi(\vec{x}_i,t)\right) + \hbar a^2\partial_t\phi\right\} - \frac{\hbar^2}{8\,m}F_I. \tag{57}$$

Although Equation (53) artificially looks like the fluid Lagrangian + Fisher information of [31] (Equation (45)), it is remarkably different due to the inherent quantum correlations encapsulated in the $\int d^{3N}X$ symbol which requires taking an integral over all the $3N$ dimensional configuration space and not over three dimensional space as the fluid analogy might suggest. However, in the special case that the phase can be partitioned to a sum of phases each dependent only on the coordinates of just one particle that is:

$$\phi(\vec{X},t) = \sum_{i=1}^{N}\phi_i(\vec{x}_i,t) \tag{58}$$

which is the case that the system wave function can be written as a multiplication of ("independent") single particle wave functions (but is more general). If Equation (58) holds we may further simplify the Lagrangian. In this case:

$$\vec{v}_{qi}(\vec{X},t) = \vec{v}_{rqi}(\vec{x}_i,t) = \frac{1}{m_i}\left(\hbar\vec{\nabla}_i\phi_i(\vec{x}_i,t) - q_i\vec{A}(\vec{x}_i,t)\right) \tag{59}$$

and we obtain:

$$L_q = -\sum_{i=1}^{N}\left\{\int d^3x_i a_i^2\left[\frac{1}{2}m_i v_{rqi}^2 + q_i\Phi(\vec{x}_i,t) + \hbar\partial_t\phi_i\right] + \frac{\hbar^2}{8\,m_i}F_{Ii}\right\}. \tag{60}$$

The above can be written in terms of quantum three dimensional Lagrangian densities:

$$L_q = \sum_{i=1}^{N}\left\{\int d^3x_i\mathcal{L}_{qi} - \frac{\hbar^2}{8\,m_i}F_{Ii}\right\}, \qquad \mathcal{L}_{qi} \equiv -a_i^2\left[\frac{1}{2}m_i v_{rqi}^2 + q_i\Phi(\vec{x}_i,t) + \hbar\partial_t\phi_i\right]. \tag{61}$$

In which $\mathcal{L}_{qi}$ can be easily seen to be the Lagrangian density of a charged potential flow by introducing the notations for mass density:

$$\hat{\rho}_{mi} = m_i a_i^2 \tag{62}$$

and flow potential:

$$\hat{v}_i = \frac{\hbar \phi_i}{m_i}. \tag{63}$$

Thus we obtain the potential flow Lagrangian density [31] in the form:

$$\mathcal{L}_{qi} = -\left[\partial_t \hat{v}_i + \frac{1}{2}(\vec{\nabla}\hat{v}_i - \frac{q_i}{m_i}\vec{A})^2 + \frac{q_i}{m_i}\Phi\right]\hat{\rho}_i. \tag{64}$$

We stress that in this case every particle is associated with its unique flow (there is no one flow describing the dynamics of all particles). The particles are aware of one another only through the electromagnetic field.

6. Retarded and Quantum Engines

For the purpose the current work an engine is a system that is able to move itself as a whole. This motion will be described by the motion of the systems center of mass which is defined as:

$$\vec{R}_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{x}_i, \qquad M \equiv \sum_{i=1}^{N} m_i. \tag{65}$$

For classical and Bohmian particles it is meaningful to discuss the trajectories of the particles. Hence we may write:

$$\vec{R}_{cm}(t) = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{x}_i(t). \tag{66}$$

The motion of a the classical center of mass can be written in terms of the motion of the particles of the system:

$$\vec{v}_{cm}(t) = \frac{d\vec{R}_{cm}}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{v}_i(t) = \frac{1}{M}\sum_{i=1}^{N} \vec{p}_{cl\ i}(t), \qquad \Rightarrow \vec{P}_T \equiv \sum_{i=1}^{N} \vec{p}_{cl\ i}(t) = M\vec{v}_{cm}(t). \tag{67}$$

We can suppose without loss of generality that at $t = 0$, $\vec{R}_{cm}(0)$ is the origin of axes, that is $\vec{R}_{cm}(0) = \vec{0}$, and that at the same initial time the engine is at rest that is $\vec{v}_{cm}(0) = \vec{0}$. We shall inquire, what are the conditions to put the system in motion. Those are obviously the conditions for the acceleration of the center of mass at the same time $t = 0$ to be different from zero:

$$\overrightarrow{ac}_{cm}(t) = \frac{d\vec{v}_{cm}}{dt}, \qquad \overrightarrow{ac}_{cm}(0) \neq 0. \tag{68}$$

For only in this case can we expect to have at future time a velocity different from zero which will cause the engine to move. However:

$$\overrightarrow{ac}_{cm}(t) = \frac{1}{M}\sum_{i=1}^{N} m_i \overrightarrow{ac}_i(t) = \frac{1}{M}\sum_{i=1}^{N} \vec{F}_i. \tag{69}$$

in which we have used Newton's second law. As the fundamental forces come about through interactions (that is due the effect of one particle on another), it follows that:

$$\vec{F}_i = \sum_{j=1, j\neq i}^{N} \vec{F}_{ji}. \tag{70}$$

Thus:

$$\overrightarrow{ac}_{cm}(t) = \frac{1}{M}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \vec{F}_{ji} = \frac{1}{M}\sum_{\text{all pairs } (i,j)\ j\neq i}^{\frac{N}{2}(N-1)} \left(\vec{F}_{ji} + \vec{F}_{ij}\right). \tag{71}$$

Now according to Newton's third law:

$$\vec{F}_{ji} = -\vec{F}_{ij} \tag{72}$$

And it follows that:

$$\vec{ac}_{cm}(t) = 0. \tag{73}$$

So a closed system cannot move as a system, it needs an external force to cause motion. However, as we require Lorentz invariance of any physical interaction, all interaction must be retarded. And thus Newton third law can only be an approximation as was shown previously [27]. The total force between two subsystems 1 and 2 in an electromagnetic system calculated to second order in $\frac{1}{c}$ ([27] Equation (81)):

$$\vec{F}_T^{[2]} = \vec{F}_{12}^{[2]} + \vec{F}_{21}^{[2]} = \frac{\mu_0}{4\pi}\partial_t \int\int d^3x'_1 d^3x'_2 \left[\frac{1}{2}(\rho_2\partial_t\rho_1 - \rho_1\partial_t\rho_2)\hat{R} - (\rho_1\vec{J}_2 + \rho_2\vec{J}_1)R^{-1}\right]. \tag{74}$$

Let us choose the two systems to be two classical particles with indices 1 and 2. Now according to Equations (25) and (26) those particles are associated with charge and current densities:

$$\begin{aligned} \rho_{c1}(\vec{x},t) &= q_1\delta^3(\vec{x}-\vec{x}_1(t)), \qquad \rho_{c2}(\vec{x},t) = q_2\delta^3(\vec{x}-\vec{x}_2(t)), \\ \vec{J}_{c1}(\vec{x},t) &= q_1\vec{v}_1(t)\delta^3(\vec{x}-\vec{x}_1(t)), \qquad \vec{J}_{c2}(\vec{x},t) = q_2\vec{v}_2(t)\delta^3(\vec{x}-\vec{x}_2(t)). \end{aligned} \tag{75}$$

Plugging the above expression into Equation (74) will yield after some tedious but straightforward computations:

$$\begin{aligned} \vec{F}_T^{[2]} &= \vec{F}_{Ta}^{[2]} + \vec{F}_{Tv}^{[2]} \\ \vec{F}_{Ta}^{[2]} &= -\frac{kq_1q_2}{2c^2R_{12}}[(\vec{ac}_1 + \vec{ac}_2) + ((\vec{ac}_1 + \vec{ac}_2)\cdot\hat{R}_{12})\hat{R}_{12}], \\ \vec{R}_{12}(t) &\equiv \vec{x}_1(t) - \vec{x}_2(t), \qquad R_{12}(t) = |\vec{R}_{12}(t)|, \qquad \hat{R}_{12} = \frac{\vec{R}_{12}}{R_{12}} \\ \vec{F}_{Tv}^{[2]} &= -\frac{kq_1q_2}{c^2R_{12}^2}\left\{\left[v_1^2 - v_2^2 - 3\left((\vec{v}_1\cdot\hat{R}_{12})^2 - (\vec{v}_2\cdot\hat{R}_{12})^2\right)\right]\hat{R}_{12} + 2\hat{R}_{12}\times(\vec{v}_1\times\vec{v}_2)\right\}. \end{aligned} \tag{76}$$

obviously every two electromagnetically interacting particles i, j in a given system will yield similar expressions, which must be summed through Equation (71) to obtain the center of mass acceleration which is not null. Now according to our previous discussion for the prevalent classical charged particle the most important force is electromagnetic while other interactions (gravitational & nuclear) can be neglected (but must be also retarded). We not that the force between any particles affecting the center of mass acceleration is partitioned into a force which reduces slowly as a function of inter particle distances (that is as $\frac{1}{R_{12}}$), and a force which depends on velocities and reduces in a Coulomb way (that is as $\frac{1}{R_{12}^2}$). As we assume slow moving particles this second contribution must be small. The contribution to the total momentum of those two interacting particles is:

$$\vec{P}_T^{[2]} = -\frac{kq_1q_2}{2c^2R_{12}}[(\vec{v}_1 + \vec{v}_2) + ((\vec{v}_1 + \vec{v}_2)\cdot\hat{R}_{12})\hat{R}_{12}], \qquad \Rightarrow \qquad \vec{v}_{cm}(t) = \frac{\vec{P}_T^{[2]}}{M}. \tag{77}$$

And again we stress that in order to obtain the correct center of mass velocity of a system contributions from all interacting particles must be considered.

Now although it obvious that every conceivable system is a "retarded engine" in the sense that its center of mass must move, in many systems this effect is rather small. In fact it takes great care to design a retarded engine with a measurable effect [24,25,27–29]. We will not give further examples of such devices, and refer the reader to the original literature.

7. Bohm's Interpretation of Quantum Mechanics and Quantum Engines

We shall now study the motion of center of mass, in quantum mechanics as interpreted by Bohm. We recall that in Bohm's quantum mechanics each point particle has a trajectory which can be calculated in principle by integrating Equation (46). Thus the center of mass defined in Equation (66) is well defined and moves with a velocity:

$$\vec{v}_{Bcm}(t) = \frac{d\vec{R}_{cm}}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d\vec{x}_i}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{v}_{qi}(t). \tag{78}$$

in which $\vec{v}_{qi}$ is defined in Equation (45). To prove the existence of an engine we need to show that provided that $\vec{R}_{cm}(0) = \vec{0}$ and $\vec{v}_{Bcm}(0) = \vec{0}$, one may have: $\frac{d\vec{v}_{Bcm}}{dt}|_{t=0} \neq \vec{0}$ without affecting the system externally. However:

$$\frac{d\vec{v}_{Bcm}}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d\vec{v}_{qi}}{dt}. \tag{79}$$

Hence we need to study the derivative of the quantum velocity of each particle in the system. This is done through the Bohm representation of Schrödinger's Equation (30) which relies on a phase amplitude representation of the wave function given in Equation (36). Assuming the Hamiltonian to be given in Equation (32), we can split the complex Schrödinger's Equation (30) into two real equations (see Holland [17] Equations (7.1.2) and (7.1.3) but there without a vector potential):

$$\partial_t a^2 + \sum_{i=1}^{N} \vec{\nabla}_i \cdot (a^2 \vec{v}_{qi}) = 0 \tag{80}$$

which according to the Born propensity rule is a probability conservation equation, and the phase equation:

$$\partial_t S + \sum_{i=1}^{N} \left\{ -\frac{\hbar^2}{2m_i}\frac{\nabla_i^2 a}{a} + \frac{1}{2} m_i v_{qi}^2 + q_i \Phi_i \right\} = 0. \tag{81}$$

This motivates the definition of the quantum potential:

$$Q = -\sum_{i=1}^{N} \frac{\hbar^2}{2m_i a} \nabla_i^2 a \tag{82}$$

Taking the $\vec{\nabla}_i$ of Equation (30) and making some tedious by straight forward manipulations one arrived at the quantum version of Newton's second law (compare to Equation (7.1.8) of [17]):

$$m_i \frac{d\vec{v}_{qi}}{dt} = \vec{F}_{qi} \equiv -\vec{\nabla}_i Q + q_i \Big(\vec{E}(\vec{x}_i, t) + \vec{v}_{qi} \times \vec{B}(\vec{x}_i, t) \Big). \tag{83}$$

The quantum force $\vec{F}_{qi}$ can be partitioned into a part which is of a purely quantum origin and an electromagnetic part:

$$\vec{F}_{qi} = \vec{F}_{qqi} + \vec{F}_{qemi}, \qquad \vec{F}_{qqi} \equiv -\vec{\nabla}_i Q \qquad \vec{F}_{qemi} \equiv q_i \Big(\vec{E}(\vec{x}_i, t) + \vec{v}_{qi} \times \vec{B}(\vec{x}_i, t) \Big). \tag{84}$$

Inserting Equation (83) into Equation (79) will lead to:

$$\frac{d\vec{v}_{Bcm}}{dt} = \frac{1}{M}\sum_{i=1}^{N} \vec{F}_{qi} = \frac{1}{M}\sum_{i=1}^{N} \Big[-\vec{\nabla}_i Q + q_i \Big(\vec{E}(\vec{x}_i, t) + \vec{v}_{qi} \times \vec{B}(\vec{x}_i, t) \Big) \Big]. \tag{85}$$

The above equation leads to the concept of the quantum engine. It is obvious that the first term of the quantum force which is due to the quantum potential $\vec{F}_{qqi}$ cannot be written generally as a sum of contributions in which each originates from a specific particle j:

$$\vec{F}_{qqi} \neq \sum_{j=1\ j\neq i}^{N} \vec{F}_{qqij} \tag{86}$$

rather the quantum potential Q will depend in a complicated way on the wave function of the entire system. In the case of a system of statistically independent (uncorrelated) particles:

$$a(\vec{X},t) = \prod_{i=1}^{N} a_i(\vec{x}_i,t), \qquad \Rightarrow \qquad Q = \sum_{i=1}^{N} Q_i, \qquad Q_i(\vec{x}_i,t) = -\frac{\hbar^2}{2m_i a_i}\frac{\nabla_i^2 a_i}{}, \tag{87}$$

in this case:

$$F_{qqi} = -\vec{\nabla}_i Q_i \tag{88}$$

thus it depends only on the location of the particle i and not on the location of any other particle so Equation (86) can be written as:

$$\vec{F}_{qqi} = \sum_{j=1}^{N} \vec{F}_{qqij}, \qquad \vec{F}_{qqij} = 0 \quad i \neq j, \qquad F_{qqii} = -\vec{\nabla}_i Q_i. \tag{89}$$

It follows that this force will exist even for a system containing a single particle. This force clearly does not confirm to Newton's third law, as even in the uncorrelated case it appears to be the action of the particle own wave function on its motion and does not take in most cases the classical interaction form. It follows that if one carefully prepares a quantum system (which means choosing initial conditions for the wave function and choosing the Hamiltonian) making sure that the quantum forces add up constructively one can obtain a quantum engine that is a self propelling system, even if retardation effects are negligible. Moreover, in the quantum case one may write the sources of the electromagnetic field in the forms given in Equations (41) and (51). As the expressions for the fields are linear in the sources (see Equations (15) and (16)). We may write:

$$\vec{E}(\vec{x}) = \sum_{i=1}^{N} \vec{E}_{qi}(\vec{x})$$
$$\vec{E}_{qi}(\vec{x}) = -k\int d^3x' \frac{1}{R^2}\left[\left(\rho_{qi}(\vec{x}',t_{ret}) + \left(\frac{R}{c}\right)\partial_t \rho_{qi}(\vec{x}',t_{ret})\right)\frac{\vec{R}}{R} + \left(\frac{R}{c}\right)^2 \frac{\partial_t \vec{J}_{qi}(\vec{x}',t_{ret})}{R}\right]. \tag{90}$$

$$\vec{B}(\vec{x}) = \sum_{i=1}^{N} \vec{B}_{qi}(\vec{x})$$
$$\vec{B}_{qi}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{R}}{R^3} \times \left(\vec{J}_q(\vec{x}',t_{ret}) + \left(\frac{R}{c}\right)\partial_t \vec{J}_q(\vec{x}',t_{ret})\right). \tag{91}$$

It follows that the following expression for the quantum force is permissible:

$$\vec{F}_{qi} = \vec{F}_{qqi} + \sum_{j=1}^{N} \vec{F}_{qemij}, \qquad \vec{F}_{qemij} = q_i\left(\vec{E}_{qj}(\vec{x}_i,t) + \vec{v}_{qi} \times \vec{B}_{qj}(\vec{x}_i,t)\right). \tag{92}$$

Thus we may rewrite Equation (85) in the form:

$$\begin{aligned} \frac{d\vec{v}_{Bcm}}{dt} &= \frac{1}{M}\sum_{i=1}^{N}\vec{F}_{qi} = \frac{1}{M}\left[\sum_{i=1}^{N}\vec{F}_{qqi} + \sum_{i=1}^{N}\sum_{j=1}^{N}\vec{F}_{qemij}\right] \\ &= \frac{1}{M}\left[\sum_{i=1}^{N}(\vec{F}_{qqi} + \vec{F}_{qemii}) + \sum_{\text{all pairs }(i,j)\ j\neq i}^{\frac{N}{2}(N-1)}\left(\vec{F}_{qemji} + \vec{F}_{qemij}\right)\right]. \end{aligned} \tag{93}$$

we notice two important facts. The first fact is that the (reduced) quantum wave function of the particle may have an electromagnetic effect on the same particle, this does not happen in the classical case. The second fact is that their is no reason to think the field force generated by the sources associated with particle j will affect particle i in an opposite direction to the field force generated by the sources associated with particle j when they affect particle i. That is generically speaking:

$$\vec{F}_{qemij} \neq -\vec{F}_{qemji}, \qquad i \neq j. \tag{94}$$

The identity of F_{qemij} and $-\vec{F}_{qemji}$ will reappear only in the classical limit and the neglect of retardation. Hence in the Bohm picture retardation is not needed to cause motion, one can generate motion by a purely quantum effect.

8. The Copenhagen Interpretation of Quantum Mechanics and the Ehrenfest Theorem

While the classical and Bohm pictures of reality ascribe a trajectory to a point particle the Copenhagen interpretation of quantum mechanics denies the existence of such a trajectory. And even if such a trajectory does exist it cannot be calculated. Thus it is not meaningful to discuss the trajectory of the center of mass either, because it is defined through the trajectories of the composing point particles. From a more conservative interpretations of quantum mechanics we can only discuss the attributes of the wave function. For examples we may ask where is the wave function centered? Or in the language of the theory of random variables, what is the expectation of the center of mass? This will be given in the form:

$$< \vec{R}_{cm} > (t) =< \Psi(t)|\vec{R}_{cm}|\Psi(t) >= \int d^{3N}X|\Psi(\vec{X},t)|^2\vec{R}_{cm} = \int d^{3N}Xa^2(\vec{X},t)\vec{R}_{cm} \tag{95}$$

The above expression can be easily written in terms of the expectation values of marginal probabilities as follows:

$$\begin{aligned} < \vec{R}_{cm} > (t) &= \frac{1}{M}\sum_{i=1}^{N} m_i \int d^3x_i a_i^2(\vec{x}_i,t)\vec{x}_i = \frac{1}{M}\sum_{i=1}^{N} m_i \int d^3x' a_i^2(\vec{x}',t)\vec{x}' \\ &= \frac{1}{M}\sum_{i=1}^{N} m_i < \vec{x}_i > (t). \end{aligned} \tag{96}$$

In the classical limit: $a_i^2(\vec{x}',t) \to \delta^3(\vec{x}' - \vec{x}_i(t))$ and thus $< \vec{R}_{cm} > (t) \to \vec{R}_{cm}(t)$.

The calculation of the temporal derivative of $< \vec{R}_{cm} > (t)$ is most easily done using the Ehrenfest theorem [15,22]. The theorem asserts that for any quantum operator A_o with an expected value:

$$< A_o >= \int d^{3N}X\Psi^\dagger A_o\Psi \tag{97}$$

This equality remains true:

$$\frac{d < A_o >}{dt} =< \frac{\partial A_o}{\partial t} > + \frac{1}{i\hbar} < [A_o, \hat{H}] >, \quad [A_o, \hat{H}] \equiv A_o\hat{H} - \hat{H}A_o. \tag{98}$$

The operators representing position and velocity of the particle i, as defined in Griffiths' [22] work, are:

$$\vec{x}_{io} \equiv \vec{x}_i, \qquad \vec{v}_{io} \equiv \frac{1}{m_i}\Big(\vec{p}_{io} - q_i\vec{A}(\vec{x}_i,t)\Big) = \frac{1}{m_i}\Big(-i\hbar\vec{\nabla}_i - q_i\vec{A}(\vec{x}_i,t)\Big). \tag{99}$$

in which the quantum momentum operator is associated with the canonical momentum given in Equation (28) and not with the classical linear momentum defined in Equation (29). Griffiths [22] derived a set of equations for a single particle without spin, his results can be trivially generalized using the multi particle Hamiltonian given in Equation (32). By substituting the aforementioned operators into the Ehrenfest theorem Equation (98):

$$\begin{aligned}
\frac{d<\vec{x}_{io}>}{dt} &= \frac{1}{i\hbar}<[\vec{x}_i,\hat{H}]>=\frac{1}{i\hbar}<[\vec{x}_i,\sum_{k=1}^{N}\hat{H}_k]>=\frac{1}{i\hbar}<[\vec{x}_i,\hat{H}_i]>=<\vec{v}_{io}> \\
\frac{d<\vec{v}_{io}>}{dt} &= <\frac{\partial\vec{v}_{io}}{\partial t}>+\frac{1}{i\hbar}<[\vec{v}_{io},\hat{H}]>=<\frac{\partial\vec{v}_{io}}{\partial t}>+\frac{1}{i\hbar}<[\vec{v}_{io},\sum_{k=1}^{N}\hat{H}_k]> \\
&= <\frac{\partial\vec{v}_{io}}{\partial t}>+\frac{1}{i\hbar}<[\vec{v}_{io},\hat{H}_i]> \\
&= \frac{q_i}{2m_i}<\vec{v}_{io}\times\vec{B}(\vec{x}_i,t)-\vec{B}(\vec{x}_i,t)\times\vec{v}_{io}>+\frac{q_i}{m_i}<\vec{E}(\vec{x}_i,t)>
\end{aligned} \tag{100}$$

Written in terms of expectation values the quantum forces (defined in Equation (84)) disappear, and we are left with electromagnetic terms which bare a superficial similarity to the classical Lorentz force. We recall that despite the superficial similarity of the above equations to Newton's second law, they are not the same. In particular the familiar Lorentz force form is only obtained when $\vec{B}$ is uniform over the reduced probability density support of the particle i in which case:

$$\frac{d^2<\vec{x}_i>}{dt^2} = \frac{q_i}{m_i}(<\vec{v}_{io}\times\vec{B}(\vec{x}_i,t)>+<\vec{E}(\vec{x}_i,t)>). \tag{101}$$

Moreover, even in the absence of a magnetic flux density, this is not identical to Newton's second law as generically:

$$<\vec{E}(\vec{x}_i,t)>\neq\vec{E}(<\vec{x}_i>,t), \tag{102}$$

unless the electric field is linear in the coordinates. A straightforward calculation shows that the expectation value of the velocity operator $\vec{v}_{io}$ is identical to the expectation value of the quantum velocity defined in Equation (45):

$$<\vec{v}_{io}>=\int d^{3N}Xa^2\vec{v}_{qi}=<\vec{v}_{qi}>. \tag{103}$$

This can also be written in terms of the reduced quantum velocity defined in Equation (48), such that:

$$<\vec{v}_{io}>=<\vec{v}_{qi}>=\int d^3x_i d^{3N-3}X_i a^2\vec{v}_{qi}=\int d^3x' a_i^2(\vec{x}')\vec{v}_{rqi}(\vec{x}',t)=<\vec{v}_{rqi}>. \tag{104}$$

The velocity of the quantum center of mass must be defined as:

$$\vec{v}_{qcm}(t)\equiv\frac{d<\vec{R}_{cm}>}{dt}. \tag{105}$$

And using Equations (96), (100), (104) and (105) we arrive at the result:

$$\begin{aligned} \vec{v}_{qcm}(t) &= \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d < \vec{x}_i >}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i < \vec{v}_{io} > \\ &= \frac{1}{M}\sum_{i=1}^{N} m_i < \vec{v}_{qi} >= \frac{1}{M}\sum_{i=1}^{N} m_i < \vec{v}_{rqi} > . \end{aligned} \tag{106}$$

The existence of a retarded quantum engine demands that an object at rest will move without an external influence. In the quantum case this means that an object located say the origin $< \vec{R}_{cm} > (0) = 0$ at $t = 0$ and is at rest that is $\vec{v}_{qcm}(0) = 0$ will move due to internal forces. Defining the acceleration of the quantum center of mass as:

$$\overrightarrow{ac}_{qcm}(t) \equiv \frac{d\vec{v}_{qcm}}{dt}. \tag{107}$$

we need to investigate if is possible to obtain $\overrightarrow{ac}_{qcm}(0) \neq 0$. Now according to Equation (106) this can be answered by calculating the temporal derivative of the expectation value of each velocity component and then calculating the weighted sum:

$$\begin{aligned} \overrightarrow{ac}_{qcm}(t) &= \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d < \vec{v}_{io} >}{dt} \\ &= \frac{1}{M}\sum_{i=1}^{N} q_i \left[\frac{1}{2} < \vec{v}_{io} \times \vec{B}(\vec{x}_i, t) - \vec{B}(\vec{x}_i, t) \times \vec{v}_{io} > + < \vec{E}(\vec{x}_i, t) >\right]. \end{aligned} \tag{108}$$

A somewhat lengthy but straight forward calculation leads to the following expression:

$$\overrightarrow{ac}_{qcm}(t) = \frac{1}{M}\sum_{i=1}^{N} \vec{F}_{Lqi}, \qquad \vec{F}_{Lqi} \equiv \int d^3x' \left[\rho_{qi}(\vec{x}')\vec{E}(\vec{x}') + \vec{J}_{qi}(\vec{x}') \times \vec{B}(\vec{x}')\right]. \tag{109}$$

in which ρ_{qi} is defined in Equation (41) and $\vec{J}_{qi}$ is defined in Equation (51). Moreover, using the definition of ρ_q defined in Equation (41) and $\vec{J}_q$ defined in Equation (51) we obtain the simple form:

$$\overrightarrow{ac}_{qcm}(t) = \frac{\vec{F}_{Lq}}{M}, \qquad \vec{F}_{Lq} \equiv \int d^3x' \left[\rho_q(\vec{x}')\vec{E}(\vec{x}') + \vec{J}_q(\vec{x}') \times \vec{B}(\vec{x}')\right]. \tag{110}$$

The expression $\vec{F}_{Lq}$ is identical to the Lorentz force expression [4,27], except that the classical charge and current densities are now replaced by their quantum analogues. In a closed system the electric and magnetic fields are produces by the charge and current densities of the same system, thus they satisfy Equations (90) and (91). Thus we write Equation (109) in the form:

$$\overrightarrow{ac}_{qcm}(t) = \frac{1}{M}\sum_{i=1}^{N}\sum_{k=1}^{N} \vec{F}_{Lq\ ik}, \qquad \vec{F}_{Lq\ ik} \equiv \int d^3x' \left[\rho_{qi}(\vec{x}')\vec{E}_{qk}(\vec{x}') + \vec{J}_{qi}(\vec{x}') \times \vec{B}_{qk}(\vec{x}')\right]. \tag{111}$$

The double sum may be written in terms of pairs as follows (compare to Equation (93)):

$$\overrightarrow{ac}_{qcm}(t) = \frac{1}{M}\left[\sum_{i=1}^{N} \vec{F}_{Lq\ ii} + \sum_{\text{all pairs }(i,j)\ j\neq i}^{\frac{N}{2}(N-1)} \left(\vec{F}_{Lq\ ij} + \vec{F}_{Lq\ ji}\right)\right]. \tag{112}$$

We notice that a quantum system can be propelled by the self interaction terms $\vec{F}_{Lq\ ii}$ which are absent in the classical picture, this has to do with the fact that charge and current densities are extended in quantum systems which is not the case for classical point like charges. As to the pair contribution, they must vanish as in the classical case unless retardation is taken into account, in this case the contribution is second order in $\frac{1}{c}$ and takes the form [27] (see also Equation (74)):

$$\begin{aligned}
& \vec{F}^{[2]}_{TLq\ ij} = \vec{F}^{[2]}_{Lq\ ij} + \vec{F}^{[2]}_{Lq\ ji} \\
= & \frac{\mu_0}{4\pi}\partial_t \int\int d^3x'_1 d^3x'_2 \left[\frac{1}{2}\left(\rho_{qj}(\vec{x}'_2)\partial_t\rho_{qi}(\vec{x}'_1) - \rho_{qi}(\vec{x}'_1)\partial_t\rho_{qj}(\vec{x}'_2)\right)\hat{R}\right. \\
- & \left.\left(\rho_{qi}(\vec{x}'_1)\vec{J}_{qj}(\vec{x}'_2) + \rho_{qj}(\vec{x}'_2)\vec{J}_{qi}(\vec{x}'_1)\right)R^{-1}\right]. \qquad (113)
\end{aligned}$$

9. Wheeler-Feynman Time-Symmetric Approach

The Wheeler–Feynman absorber theory [33,34], also known as the Wheeler–Feynman time-symmetric theory, is named after its developers, physicists Richard Feynman and John Archibald Wheeler. This theory of electrodynamics is a relativistic extension of action-at-a-distance interactions between electron particles (and thus neglecting their true quantum nature). It proposes that there is no independent electromagnetic field. Instead, the entire theory is described by the Lorentz-invariant action S of particle trajectories $a^\mu(\tau)$, $b^\mu(\tau)$, $\cdots$, defined as follows: The action S is given by:

$$S = -\sum_a m_a c \int \sqrt{-da_\mu da^\mu} + \sum_{a<b} \frac{e_a e_b}{c} \int\int da_\nu db^\nu\, \delta^4(ab_\mu ab^\mu), \qquad (114)$$

where $ab_\mu \equiv a_\mu - b_\mu$. In this equation, the first term represents the action of individual particles with mass m_a moving along their trajectories, while the second term accounts for the interaction between pairs of point particles a and b with charges e_a and e_b, incorporating a four-dimensional delta function to ensure relativistic invariance. The four electromagnetic potential is defined as:

$$A^{(b)}_\nu(x) = e_b \int \delta(xb_\mu xb^\mu) db_\nu(b) \qquad (115)$$

(vector potential of particle b at point x). The field tensor can be calculated as:

$$F^{(b)}_{\mu\nu} = \partial_\nu A^{(b)}_\mu - \partial_\mu A^{(b)}_\nu \qquad (116)$$

Instead of being the retarded solution Equation (115) in the Wheeler-Feynman time-symmetric approach is the time-symmetric half-advanced and half-retarded solution. The same applies to the fields generated by the particles. Suppressing the indices we may write Equation (116) as:

$$F^{(b)} = \frac{1}{2}\left[F^{(b)}_{ret} + F^{(b)}_{adv}\right] \qquad (117)$$

This field is present in the past as well as the future light cone of b.

Although the influence of a single charged particle, like particle b, is time-symmetric, the collective influence of all particles in the universe, which are affected by the movement of b, may not exhibit time symmetry. This was the main argument presented by Wheeler and Feynman in their 1945 work. Wheeler and Feynman argued that when charge b is moved, it creates a disturbance affecting all other charges in the universe. These charges then react. To calculate this reaction in a static Minkowski universe with a uniform distribution of electric charges, they developed a consistent method. They found that the reaction to the motion of charge b can be determined accurately, yielding the following result:

$$R^{(b)} = \frac{1}{2}\left[F^{(b)}_{ret} - F^{(b)}_{adv}\right] \qquad (118)$$

Therefore, a test particle near charge b experiences a net total field $F^{(b)}_{tot}$, which is the sum of the direct field $F^{(b)}$ and the reaction field $R^{(b)}$. This total field can be expressed as:

$$F^{(b)}_{tot} = F^{(b)} + R^{(b)} = F^{(b)}_{ret}. \tag{119}$$

Therefore, for all practical purposes he Wheeler-Feynman time-symmetric approach allows us to use retarded fields for any practical application in any system which does not contain the entire universe but only a small subset. Thus even in this approach we are allowed to use the retarded field expressions Equations (15) and (16) and the rest of our paper follows just as before.

In the words of Wheeler & Feynman (page 3 of [33]): "Advanced actions appear to conflict both with experience and with elementary notions of causality. Experience refers not to the simple case of two charges, however, but to a universe containing a very large number of particles. In the limiting case of a universe in which all electromagnetic disturbances are ultimately absorbed it may be shown that the advanced fields combine in such a way as to make it appear**—except for the phenomenon of radiative reaction—**that each particle generates only the usual and well-verified **retarded field**. It is only necessary to make the natural postulate that we live in such a completely absorbing universe to escape the apparent contradiction between advanced potentials and observation."

As we are not dealing with radiative reaction, we must accept the teachings of Wheeler & Feynman, that each particle generates only the usual and well-verified retarded field.

10. Advanced Potentials

Despite the conclusions of the previous section, we shall analyze a retarded engine in the case that there is no reaction force (given in Equation (118)), yet the Wheeler–Feynman time-symmetric theory is assumed to be correct. The purpose of this exercise is to see what changes are entailed in such a universe with respect to previous analysis as described in [27]. This will require to concentrate on the clearly separated piece of Equation (118). In the following we repeat the analysis of [27] with the required advanced potentials.

Consider the electric $\vec{E}$ and magnetic $\vec{B}$ fields which are generated by a charged body 1 and acts upon a charged body 2. Those bodies are composed of atoms, ions and free electrons, so we may write the Lorentz force:

$$\vec{F}_{21} = \int d^3x_2 \rho_{i2}(\vec{E} + \vec{v}_{i2} \times \vec{B}) + \int d^3x_2 \rho_{e2}(\vec{E} + \vec{v}_{e2} \times \vec{B}). \tag{120}$$

We integrate over the entire volume of 2. ρ_{i2} and ρ_{e2} denote the charge density of the ions and charge density of the electrons respectively, $\vec{v}_{i2}$ and $\vec{v}_{e2}$ are the ion and electron velocity fields. The total charge density is the sum of the ions and free electrons charge density, thus:

$$\rho_2 = \rho_{i2} + \rho_{e2}. \tag{121}$$

The electric terms in the above force formula are added and we obtain:

$$\vec{F}_{21} = \int d^3x_2 \rho_2 \vec{E} + \int d^3x_2 \rho_{i2} \vec{v}_{i2} \times \vec{B} + \int d^3x_2 \rho_{e2} \vec{v}_{e2} \times \vec{B}. \tag{122}$$

We assume that the ions are at immobile such that: $\vec{v}_{i2} = 0$. It follows that:

$$\vec{F}_{21} = \int d^3x_2 \rho_2 \vec{E} + \int d^3x_2 \rho_{e2} \vec{v}_{e2} \times \vec{B}. \tag{123}$$

Introducing the current density: $\vec{J}_2 = \rho_{e2}\vec{v}_{e2}$, we obtain:

$$\vec{F}_{21} = \int d^3x_2 \left(\rho_2 \vec{E} + \vec{J}_2 \times \vec{B}\right). \tag{124}$$

Electric and magnetic fields can be expresses in terms of vector and scalar potentials [4] as given in Equations (9) and (10). If the field is the result of a charge ρ_1 and current $\vec{J}_1$ densities in 1, we can calculate the retarded and advanced potentials [4]:

$$\Phi_{ret}(\vec{x}_2) = k \int d^3x_1 \frac{\rho_1(\vec{x}_1, t_{ret})}{R}. \tag{125}$$

$$\Phi_{adv}(\vec{x}_2) = k \int d^3x_1 \frac{\rho_1(\vec{x}_1, t_{adv})}{R}, \qquad t_{adv} \equiv t + \frac{R}{c}. \tag{126}$$

$$\vec{A}_{ret}(\vec{x}_2) = \frac{\mu_0}{4\pi} \int d^3x_1 \frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}. \tag{127}$$

$$\vec{A}_{adv}(\vec{x}_2) = \frac{\mu_0}{4\pi} \int d^3x_1 \frac{\vec{J}_1(\vec{x}_1, t_{adv})}{R}. \tag{128}$$

Thus in the time-symmetric Wheeler & Feynman theory [33], we have:

$$\Phi(\vec{x}_2) = \frac{1}{2}(\Phi_{ret}(\vec{x}_2) + \Phi_{adv}(\vec{x}_2)). \tag{129}$$

$$\vec{A}(\vec{x}_2) = \frac{1}{2}\left(\vec{A}_{ret}(\vec{x}_2) + \vec{A}_{adv}(\vec{x}_2)\right). \tag{130}$$

Combining Equation (127) and Aadv with Equation (10), we arrive at:

$$\vec{B}_{ret}(\vec{x}_2) = \vec{\nabla}_{\vec{x}_2} \times \vec{A}_{ret}(\vec{x}_2) = \frac{\mu_0}{4\pi} \int d^3x_1 \vec{\nabla}_{\vec{x}_2} \times \left(\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}\right). \tag{131}$$

$$\vec{B}_{adv}(\vec{x}_2) = \vec{\nabla}_{\vec{x}_2} \times \vec{A}_{adv}(\vec{x}_2) = \frac{\mu_0}{4\pi} \int d^3x_1 \vec{\nabla}_{\vec{x}_2} \times \left(\frac{\vec{J}_1(\vec{x}_1, t_{adv})}{R}\right). \tag{132}$$

Such that the total magnetic field in the time symmetric theory is:

$$\vec{B}(\vec{x}_2) = \frac{1}{2}\left(\vec{B}_{ret}(\vec{x}_2) + \vec{B}_{adv}(\vec{x}_2)\right). \tag{133}$$

However, notice that:

$$\vec{\nabla}_{\vec{x}_2} \times \left(\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}\right) = \vec{\nabla}_{\vec{x}_2} R \times \partial_R \left(\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}\right). \tag{134}$$

Since:

$$\vec{\nabla}_{\vec{x}_2} R = -\frac{\vec{R}}{R} \tag{135}$$

And:

$$\partial_R \left(\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}\right) = -\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R^2} - \frac{\partial_t \vec{J}_1(\vec{x}_1, t_{ret})}{Rc}. \tag{136}$$

Hence:

$$\vec{\nabla}_{\vec{x}_2} \times \left(\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}\right) = \frac{\vec{R}}{R^3} \times \left(\vec{J}_1(\vec{x}_1, t_{ret}) + \left(\frac{R}{c}\right)\partial_t \vec{J}_1(\vec{x}_1, t_{ret})\right). \tag{137}$$

Inserting Equation (137) into Equation (131), we arrive at Jefimenko's equations [4,23] for the retarded magnetic field:

$$\vec{B}_{ret}(\vec{x}_2) = \frac{\mu_0}{4\pi} \int d^3x_1 \frac{\vec{R}}{R^3} \times \left(\vec{J}_1(\vec{x}_1, t_{ret}) + \left(\frac{R}{c}\right)\partial_t \vec{J}_1(\vec{x}_1, t_{ret})\right). \tag{138}$$

For the same reason:

$$\vec{\nabla}_{\vec{x}_2} \times \left(\frac{\vec{J}_1(\vec{x}_1, t_{adv})}{R}\right) = \frac{\vec{R}}{R^3} \times \left(\vec{J}_1(\vec{x}_1, t_{adv}) - \left(\frac{R}{c}\right)\partial_t \vec{J}_1(\vec{x}_1, t_{adv})\right). \tag{139}$$

and thus:

$$\vec{B}_{adv}(\vec{x}_2) = \frac{\mu_0}{4\pi} \int d^3x_1 \frac{\vec{R}}{R^3} \times \left(\vec{J}_1(\vec{x}_1, t_{adv}) - \left(\frac{R}{c}\right)\partial_t \vec{J}_1(\vec{x}_1, t_{adv})\right). \tag{140}$$

The retarded electric field is the sum of two contributions given in Equation (9), one contribution from the scalar potential and another from the vector potential:

$$\vec{E}_{ret} = \vec{E}_{a\ ret} + \vec{E}_{b\ ret}, \qquad \vec{E}_{a\ ret} \equiv -\partial_t \vec{A}_{ret}, \qquad \vec{E}_{b\ ret} \equiv -\vec{\nabla}\Phi_{ret}. \tag{141}$$

Thus from Equation (127) we have:

$$\vec{E}_{a\ ret}(\vec{x}_2) = -\frac{\mu_0}{4\pi} \int d^3x_1 \frac{\partial_t \vec{J}_1(\vec{x}_1, t_{ret})}{R}. \tag{142}$$

And according to Equation (125):

$$\vec{E}_{b\ ret}(\vec{x}_2) = -k \int d^3x_1 \vec{\nabla}_{\vec{x}_2}\left(\frac{\rho_1(\vec{x}_1, t_{ret})}{R}\right). \tag{143}$$

The above equation can also be written as:

$$\vec{E}_{b\ ret}(\vec{x}_2) = -k \int d^3x_1 \left[\frac{1}{R}\vec{\nabla}_{\vec{x}_2}\rho_1(\vec{x}_1, t_{ret}) + \rho_1(\vec{x}_1, t_{ret})\vec{\nabla}_{\vec{x}_2}\frac{1}{R}\right], \tag{144}$$

however:

$$\vec{\nabla}_{\vec{x}_2}\frac{1}{R} = \frac{\hat{R}}{R^2}, \qquad \hat{R} \equiv \frac{\vec{R}}{R} \tag{145}$$

and also:

$$\vec{\nabla}_{\vec{x}_2}\rho_1(\vec{x}_1, t_{ret}) = \vec{\nabla}_{\vec{x}_2}R\ \partial_R \rho_1(\vec{x}_1, t_{ret})|_{\vec{x}_1} = \frac{\hat{R}}{c}\partial_t \rho_1(\vec{x}_1, t_{ret}). \tag{146}$$

It thus follows that:

$$\vec{E}_{b\ ret}(\vec{x}_2) = -k \int d^3x_1 \frac{\hat{R}}{R^2}\left[\rho_1(\vec{x}_1, t_{ret}) + \left(\frac{R}{c}\right)\partial_t \rho_1(\vec{x}_1, t_{ret})\right]. \tag{147}$$

Adding $\vec{E}_{b\ ret}$ and $\vec{E}_{a\ ret}$, we derive Jefimenko's [4,23] retarded electric field:

$$\begin{aligned} \vec{E}_{ret}(\vec{x}_2) &= -k \int d^3x_1 \frac{1}{R^2}\left[\left(\rho_1(\vec{x}_1, t_{ret}) + \left(\frac{R}{c}\right)\partial_t \rho_1(\vec{x}_1, t_{ret})\right)\hat{R} \right. \\ &+ \left. \left(\frac{R}{c}\right)^2 \frac{\partial_t \vec{J}_1(\vec{x}_1, t_{ret})}{R}\right]. \end{aligned} \tag{148}$$

A similar calculation will lead to the following advanced electric field:

$$\begin{aligned} \vec{E}_{adv}(\vec{x}_2) &= -k \int d^3x_1 \frac{1}{R^2}\left[\left(\rho_1(\vec{x}_1, t_{adv}) - \left(\frac{R}{c}\right)\partial_t \rho_1(\vec{x}_1, t_{adv})\right)\hat{R} \right. \\ &+ \left. \left(\frac{R}{c}\right)^2 \frac{\partial_t \vec{J}_1(\vec{x}_1, t_{adv})}{R}\right]. \end{aligned} \tag{149}$$

The total time symmetric electric field is:

$$\vec{E}(\vec{x}_2) = \frac{1}{2}\left(\vec{E}_{ret}(\vec{x}_2) + \vec{E}_{adv}(\vec{x}_2)\right). \tag{150}$$

Substituting Equations (133) and (150) into Equation (124), the Lorentz force of subsystem 1 on subsystem 2 is derived.

Consider the charge density evaluated at a retarded time: $\rho(\vec{x}', t_{ret}) = \rho(\vec{x}', t - \frac{R}{c})$, aasuming $\frac{R}{c}$ is minute one can expand using a Taylor series around t:

$$\rho(\vec{x}', t_{ret}) = \rho(\vec{x}', t - \frac{R}{c}) = \sum_{n=0}^{\infty} \frac{\partial_t^n \rho(\vec{x}', t)}{n!} (-\frac{R}{c})^n. \tag{151}$$

∂_t^n is the partial derivative with respect to time of order n. Similiarly:

$$\vec{J}(\vec{x}', t_{ret}) = \vec{J}(\vec{x}', t - \frac{R}{c}) = \sum_{n=0}^{\infty} \frac{\partial_t^n \vec{J}(\vec{x}', t)}{n!} (-\frac{R}{c})^n. \tag{152}$$

A Taylor expansion is of sufficient accuracy only for an environment of t which is unique to the function involved, We thus have a convergence radius T_{max} different for each function. Thus Equations (151) and (152) are accurate enough only in the domain $[t - T_{max}, t + T_{max}]$. As we expand in $\frac{R}{c}$ the expansion is accurate only for the range:

$$R < R_{max} \equiv c\, T_{max}. \tag{153}$$

Thus we are limited to a near field approximation, however, as c is a large, R_{max} is quite large for many systems. Inserting Equation (151) into Equation (125) we derive the expression:

$$\begin{aligned} \Phi_{1\,ret}(\vec{x}_2) &= k \int d^3x_1 \frac{\rho_1(t_{ret})}{R} = k \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3x_1 \partial_t^n \rho_1(\vec{x}', t) \frac{1}{R} (-\frac{R}{c})^n \\ &= k \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{c}\right)^n \int d^3x_1 \partial_t^n \rho_1(\vec{x}_1, t) R^{n-1} \end{aligned} \tag{154}$$

Similarly inserting Equation (152) into Equation (127) we obtain:

$$\begin{aligned} \vec{A}_{1\,ret}(\vec{x}_2) &= \frac{\mu_0}{4\pi} \int d^3x_1 \frac{\vec{J}_1(t_{ret})}{R} = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3x_1 \partial_t^n \vec{J}_1(\vec{x}', t) \frac{1}{R} (-\frac{R}{c})^n \\ &= \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{c}\right)^n \int d^3x_1 \partial_t^n \vec{J}_1(\vec{x}_1, t) R^{n-1} \end{aligned} \tag{155}$$

As $\frac{\mu_0}{4\pi} = \frac{\mu_0 \epsilon_0}{4\pi\epsilon_0} = \frac{k}{c^2}$ it follows that:

$$\begin{aligned} \vec{A}_{1\,ret}(\vec{x}_2) &= k \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{c}\right)^{n+2} \int d^3x_1 \partial_t^n \vec{J}_1(\vec{x}_1, t) R^{n-1} \\ &= k \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \left(-\frac{1}{c}\right)^n \int d^3x_1 \partial_t^{n-2} \vec{J}_1(\vec{x}_1, t) R^{n-3} \end{aligned} \tag{156}$$

Similar expressions can be obtained for the advanced potentials:

$$\Phi_{1\,adv}(\vec{x}_2) = k \sum_{n=0}^{\infty} \frac{1}{n!} \left(+\frac{1}{c}\right)^n \int d^3x_1 \partial_t^n \rho_1(\vec{x}_1, t) R^{n-1} \tag{157}$$

$$\vec{A}_{1\,adv}(\vec{x}_2) = k \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \left(+\frac{1}{c}\right)^n \int d^3x_1 \partial_t^{n-2} \vec{J}_1(\vec{x}_1, t) R^{n-3} \tag{158}$$

We are now at a position in which we can calculate the time symmetric potentials. Thus according to Equation (129) the scalar potential is:

$$\Phi_1(\vec{x}_2) = k \sum_{n=0,\ n \text{ even}}^{\infty} \frac{1}{n!}\left(+\frac{1}{c}\right)^n \int d^3x_1 \partial_t^n \rho_1(\vec{x}_1, t) R^{n-1}. \tag{159}$$

This means that even terms in the potential are identical to similar terms in the retarded theory while odd terms are null and thus are radically different from the predictions of the retarded theory. Similarly for the vector potential of the time symmetrical theory we obtain:

$$\vec{A}_1(\vec{x}_2) = k \sum_{n=2,\ n \text{ even}}^{\infty} \frac{1}{(n-2)!}\left(+\frac{1}{c}\right)^n \int d^3x_1 \partial_t^{n-2} \vec{J}_1(\vec{x}_1, t) R^{n-3}. \tag{160}$$

The same conclusion holds here, again even terms in the time symmetric theory are the same as in the standard retarded theory while odd terms are null and thus are radically different. Electric and magnetic fields can be calculated from Equations (159) and (160). However, before we continue we introduce $G^{[n]}$ be the contribution of order $\frac{1}{c^n}$ to some quantity G:

$$G = \sum_{n=0}^{\infty} G^{[n]} \tag{161}$$

Thus:

$$\Phi_1^{[n]}(\vec{x}_2) = \frac{k}{n!}\left(+\frac{1}{c}\right)^n \int d^3x_1 \partial_t^n \rho_1(\vec{x}_1, t) R^{n-1} \text{ for n even,} \quad \Phi_1^{[n]}(\vec{x}_2) = 0 \text{ for n odd.} \tag{162}$$

$$\begin{aligned} \vec{A}_1^{[n]}(\vec{x}_2) &= \frac{k}{(n-2)!}\left(+\frac{1}{c}\right)^n \int d^3x_1 \partial_t^{n-2} \vec{J}_1(\vec{x}_1, t) R^{n-3} \quad \text{for } n >= 2 \text{ and even} \\ \vec{A}_1^{[0]} &= \vec{A}_1^{[n]} = 0, \text{for } n \text{ odd.} \end{aligned} \tag{163}$$

It now follows that:

$$\begin{aligned} &\vec{E}_{1a}^{[n]} \equiv -\partial_t \vec{A}_1^{[n]} = \\ &-\frac{k}{(n-2)!}\left(+\frac{1}{c}\right)^n \int d^3x_1 \partial_t^{n-1} \vec{J}_1(\vec{x}_1, t) R^{n-3} \quad \text{for } n >= 2 \text{ and even} \\ &\vec{E}_{1a}^{[0]} = \vec{E}_{1a}^{[n]} = 0, \text{for } n \text{ odd.} \end{aligned} \tag{164}$$

and

$$\begin{aligned} \vec{E}_{1b}^{[n]} &= -\vec{\nabla}_{\vec{x}_2} \Phi^{[n]} = -\frac{k}{n!}\left(+\frac{1}{c}\right)^n \int d^3x_1 \partial_t^n \rho_1(\vec{x}_1, t) \vec{\nabla}_{\vec{x}_2} R^{n-1} \quad \text{for } n \text{ even,} \\ \vec{E}_{1b}^{[n]} &= 0 \quad \text{for } n \text{ odd.} \end{aligned} \tag{165}$$

Now:

$$\vec{\nabla}_{\vec{x}_2} R^{n-1} = \vec{\nabla}_{\vec{x}_2} R \, \partial_R R^{n-1} = -(n-1) R^{n-2} \hat{R}. \tag{166}$$

thus we have for even n:

$$\vec{E}_{1b}^{[n]} = \frac{k(n-1)}{n!}\left(+\frac{1}{c}\right)^n \int d^3x_1 \partial_t^n \rho_1(\vec{x}_1, t) R^{n-2} \hat{R}. \tag{167}$$

Contribution of the zeroth order comes only from the scalar potential term of the electric field, this is the Coulomb field:

$$\vec{E}_1^{[0]} = \vec{E}_{1b}^{[0]} = -k \int d^3x_1 \frac{\rho_1(\vec{x}_1, t)}{R^2} \hat{R} \tag{168}$$

similar result is thus obtained in both retarded [27] and time-symmetric theories. Odd terms and hence first order terms are absent in the time-symmetric theory, however, it is also absent in the retarded theory. Thus up to first order in $\frac{1}{c}$ both theories give the same predictions. The first term with both the scalar and vector potentials contributions to the electric field is second order:

$$\vec{E}_1^{[2]} = \vec{E}_{1a}^{[2]} + \vec{E}_{1b}^{[2]} = k\left(\frac{1}{c}\right)^2 \int d^3x_1 \left[\frac{1}{2}\partial_t^2 \rho_1(\vec{x}_1, t)\hat{R} - \partial_t \vec{J}_1(\vec{x}_1, t)R^{-1}\right] \tag{169}$$

As $\frac{1}{c}$ is small, it will in many cases suffice to consider up to second order terms and no more. The reader is reminded that for even terms (the second order term included) both time symmetric and retarded theories give identical results. Calculating the magnetic field using Equations (10) and (163):

$$\begin{aligned}
&\vec{B}_1^{[n]}(\vec{x}_2) = \vec{\nabla}_{\vec{x}_2} \times \vec{A}_1^{[n]}(\vec{x}_2) = \\
&\frac{k}{(n-2)!}\left(-\frac{1}{c}\right)^n \int d^3x_1 \vec{\nabla}_{\vec{x}_2} \times \left[\partial_t^{n-2} \vec{J}_1(\vec{x}_1, t)R^{n-3}\right] \quad \text{for } n >= 2 \text{ and even.} \\
&\vec{B}_1^{[0]} = \vec{B}_1^{[n]} = 0, \text{for } n \text{ odd.}
\end{aligned} \tag{170}$$

However:

$$\vec{\nabla}_{\vec{x}_2} \times \left[\partial_t^{n-2} \vec{J}_1(\vec{x}_1, t)R^{n-3}\right] = \vec{\nabla}_{\vec{x}_2} R^{n-3} \times \partial_t^{n-2} \vec{J}_1(\vec{x}_1, t) \tag{171}$$

and also:

$$\vec{\nabla}_{\vec{x}_2} R^{n-3} = \vec{\nabla}_{\vec{x}_2} R\, \partial_R R^{n-3} = (3-n)\, R^{n-4}\, \hat{R} \tag{172}$$

Thus we may write for even n:

$$\vec{B}_1^{[n]}(\vec{x}_2) = k\frac{(3-n)}{(n-2)!}\left(-\frac{1}{c}\right)^n \int d^3x_1 R^{n-4}\, \hat{R} \times \partial_t^{n-2} \vec{J}_1(\vec{x}_1, t) \quad \text{for } n >= 2. \tag{173}$$

or more specifically:

$$\vec{B}_1^{[2]}(\vec{x}_2) = k\left(\frac{1}{c}\right)^2 \int d^3x_1 R^{-2}\, \hat{R} \times \vec{J}_1(\vec{x}_1, t). \tag{174}$$

The electric and magnetic fields expressions are now used to derive the force nth order contribution taking advantage of Equation (124):

$$\vec{F}_{21}^{[n]} = \int d^3x_2 \left(\rho_2(\vec{x}_2)\vec{E}_1^{[n]}(\vec{x}_2) + \vec{J}_2(\vec{x}_2) \times \vec{B}_1^{[n]}(\vec{x}_2)\right). \tag{175}$$

For the zeroth order we deduce:

$$\vec{F}_{21}^{[0]} = \int d^3x_2 \rho_2(\vec{x}_2)\vec{E}_1^{[0]}(\vec{x}_2). \tag{176}$$

Inserting Equation (168), we derive Coulomb's force:

$$\vec{F}_{21}^{[0]} = -k \int\int d^3x_1 d^3x_2 \rho_1(\vec{x}_1)\rho_2(\vec{x}_2)\frac{1}{R^2}\hat{R} = -\vec{F}_{12}^{[0]}. \tag{177}$$

This (quasi) static force satisfies Newton's third law:

$$\vec{F}_T^{[0]} = \vec{F}_{21}^{[0]} + \vec{F}_{12}^{[0]} = 0. \tag{178}$$

All odd orders of the force are null in the time symmetric theory since all odd orders in the fields are null in this theory. This includes the force which is first order in $\frac{1}{c}$ being null, thus:

$$\vec{F}_T^{[1]} = \vec{F}_{21}^{[1]} = \vec{F}_{12}^{[1]} = 0, \tag{179}$$

this is not different from the result of the retarded theory. Calculating the second order force term, we first partition the force given in Equation (175) into electric and magnetic terms:

$$\begin{aligned} \vec{F}_{21}^{[2]} &= \vec{F}_{21e}^{[2]} + \vec{F}_{21m}^{[2]} \\ \vec{F}_{21e}^{[2]} &\equiv \int d^3x_2\, \rho_2(\vec{x}_2)\vec{E}_1^{[2]}(\vec{x}_2) \\ \vec{F}_{21m}^{[2]} &\equiv \int d^3x_2\, \vec{J}_2(\vec{x}_2) \times \vec{B}_1^{[2]}(\vec{x}_2) \end{aligned} \tag{180}$$

Inserting Equation (169) into Equation (180), we derive the electric force:

$$\vec{F}_{21e}^{[2]} = \left(\frac{k}{c^2}\right) \int\int d^3x_1 d^3x_2 \left[\frac{1}{2}\rho_2\partial_t^2\rho_1\hat{R} - \rho_2\partial_t\vec{J}_1 R^{-1}\right] \tag{181}$$

The magnetic force is obtained by substituting Equation (174) into Equation (180):

$$\begin{aligned} \vec{F}_{21m}^{[2]} &= \left(\frac{k}{c^2}\right) \int\int d^3x_1 d^3x_2\, \vec{J}_2 \times \left(R^{-2}\hat{R} \times \vec{J}_1\right) \\ &= \left(\frac{k}{c^2}\right) \int\int d^3x_1 d^3x_2 \left[\hat{R}\frac{\vec{J}_1 \cdot \vec{J}_2}{R^2} - \vec{J}_1 R^{-2}(\hat{R} \cdot \vec{J}_2)\right] \end{aligned} \tag{182}$$

Notice that,

$$R^{-2}\hat{R} = \vec{\nabla}_{\vec{x}_2} R^{-1} \tag{183}$$

which results in:

$$\vec{F}_{21m}^{[2]} = \left(\frac{k}{c^2}\right) \int\int d^3x_1 d^3x_2 \left[\hat{R}\frac{\vec{J}_1 \cdot \vec{J}_2}{R^2} - \vec{J}_1\left((\vec{\nabla}_{\vec{x}_2} R^{-1}) \cdot \vec{J}_2\right)\right] \tag{184}$$

Inspecting the integral:

$$\int d^3x_2 (\vec{\nabla}_{\vec{x}_2} R^{-1}) \cdot \vec{J}_2 = \int d^3x_2 \left[\vec{\nabla}_{\vec{x}_2} \cdot \left(\frac{\vec{J}_2}{R}\right) - \frac{1}{R}\vec{\nabla}_{\vec{x}_2} \cdot \vec{J}_2\right] \tag{185}$$

Taking into account Gauss theorem and Equation (14), we arrive at:

$$\int d^3x_2 (\vec{\nabla}_{\vec{x}_2} R^{-1}) \cdot \vec{J}_2 = \oint d\vec{S}_2 \cdot \left(\frac{\vec{J}_2}{R}\right) + \int d^3x_2 \frac{1}{R}\partial_t\rho_2. \tag{186}$$

The surface is encapsulating the volume, if the volume includes all space the surface will be at infinity. If there currents at infinity are null:

$$\int d^3x_2 (\vec{\nabla}_{\vec{x}_2} R^{-1}) \cdot \vec{J}_2 = \int d^3x_2 \frac{1}{R}\partial_t\rho_2. \tag{187}$$

Inserting Equation (187) into Equation (184), we derive:

$$\vec{F}_{21m}^{[2]} = \left(\frac{k}{c^2}\right) \int\int d^3x_1 d^3x_2 \left[\hat{R}\frac{\vec{J}_1 \cdot \vec{J}_2}{R^2} - \frac{\vec{J}_1}{R}\partial_t\rho_2\right]. \tag{188}$$

We notice that the second order is the lowest order for the magnetic field. The force is thus a sum of two parts, one that satisfies Newton's third law, and one that does not. Calculating the total electromagnetic force by adding Equations (181) and (188):

$$\begin{aligned}\vec{F}_{21}^{[2]} &= \vec{F}_{21e}^{[2]} + \vec{F}_{21m}^{[2]} \\ &= \left(\frac{k}{c^2}\right) \int\int d^3x_1 d^3x_2 \left[\frac{1}{2}\rho_2 \partial_t^2 \rho_1 \hat{x}_{12} - \partial_t(\rho_2 \vec{J}_1) R^{-1} + \hat{x}_{12} \frac{\vec{J}_1 \cdot \vec{J}_2}{R^2}\right]\end{aligned} \tag{189}$$

Introducing the notation $\hat{R} = \hat{x}_{12}$. It is rather easy to calculate $F_{12}^{[2]}$ by exchanging the indices 1 and 2:

$$\begin{aligned}\vec{F}_{12}^{[2]} &= \\ \left(\frac{k}{c^2}\right) \int\int d^3x_1 d^3x_2 &\left[\frac{1}{2}\rho_1 \partial_t^2 \rho_2 \hat{x}_{21} - \partial_t(\rho_1 \vec{J}_2) R^{-1} + \hat{x}_{21} \frac{\vec{J}_1 \cdot \vec{J}_2}{R^2}\right]\end{aligned} \tag{190}$$

Adding Equations (189) and (190) and considering that $\hat{x}_{12} = -\hat{x}_{21}$, it follows:

$$\begin{aligned}\vec{F}_{T}^{[2]} &= \vec{F}_{12}^{[2]} + \vec{F}_{21}^{[2]} = \\ \left(\frac{k}{c^2}\right) \int\int d^3x_1 d^3x_2 &\left[\frac{1}{2}\left(\rho_2 \partial_t^2 \rho_1 - \rho_1 \partial_t^2 \rho_2\right) \hat{R} - \partial_t(\rho_1 \vec{J}_2 + \rho_2 \vec{J}_1) R^{-1}\right]\end{aligned} \tag{191}$$

Now as $\frac{k}{c^2} = \frac{\mu_0}{4\pi}$ and since:

$$\rho_2 \partial_t^2 \rho_1 - \rho_1 \partial_t^2 \rho_2 = \partial_t(\rho_2 \partial_t \rho_1 - \rho_1 \partial_t \rho_2) \tag{192}$$

It also follows that:

$$\vec{F}_{T}^{[2]} = \frac{\mu_0}{4\pi} \partial_t \int\int d^3x_1 d^3x_2 \left[\frac{1}{2}(\rho_2 \partial_t \rho_1 - \rho_1 \partial_t \rho_2) \hat{R} - (\rho_1 \vec{J}_2 + \rho_2 \vec{J}_1) R^{-1}\right] \tag{193}$$

Which is the same formula derived in the retarded theory [27]. Of course if we calculate third order terms the time symmetric theory the calculations will yield null results, but this will not be the case for the retarded theory. Thus in principle one can decide if the time symmetric theory is correct by measuring this term. Notice, however, that taking into account the reaction term of Wheeler & Feynman [33], the retarded theory is recovered and again there are no means to distinguish between the two theories.

11. Conclusions

The current paper does not present an alternative theory. All theories discussed in the paper are well known. Those include classical electrodynamics, classical mechanics, Schrödinger's quantum mechanics and Bohm's quantum mechanics. The purpose of the paper is to study a retarded engine in the framework of those theories and elucidate its properties. In all parts of the paper the field is described by classical electrodynamics (not QED), however, we compare different theoretical representations of the material part (classical mechanics, Schrödinger's quantum mechanics and Bohm's quantum mechanics) and show the differences and similarities of the predictions those theories. This is done without forming any apriori opinion on the validity (or invalidity) of any of the theories. Our main findings are:

1. A Bohmian system does not need retardation to self propel its center of mass. Although the fact that the total momentum is not generically conserved in a Bohmian system was already noted by Holland [17] the practical and technological implications were not underlined.
2. The lack of linear momentum conservation in a classical mechanical system coupled to an electromagnetic field was already noted by Feynman [5] for a two particles system. Feynman correctly explained this lack of conservation by the transfer of linear momentum from matter to the field. This result was expanded here to a N

point particle system coupled to a retarded electromagnetic field. Thus supplying a theoretical underpinning to the macroscopic description of a classical retarded engine [27].

3. The same conclusion regarding the existence of a retarded engine is proved for a N particle quantum system of the Schrödinger type. Using the Ehrenfest theorem we were able to show that the expectation value of the center of mass will move due to the retarded electromagnetic field. This is a new result, although atomic retarded motors were considered before [28,29] the reasoning given in those previous papers for the motion of center of mass was classical (using the Lorentz force and Newton's classical equations of motion).
4. In a quantum system it seems that self propulsion is possible due to self electromagnetic interaction of the particle wave function with itself, this is not possible in classical mechanics as a point particle cannot interact with itself (this can be shown by explicitly solving the field equations for a point particle and showing that the field vanishes in the 4D event in which the particle is located). This is another type of a "quantum engine" that has arised from the current study.
5. The time-symmetric Wheeler–Feynman theory is studied here for the first time in the context of a retarded motor. It is shown that such a theory if correct will not affect the basic retarded engine effect which is second order but can have implications if third order corrections are to be considered.

We have limited our study to spinless particles which move at slow velocities (with respect to the speed of light in vacuum c), that is nonrelativistic particles. Of course the effects of spin on a retarded engine deserve a separate discussion and so do relativistic particles of both the classical and quantum types. This will require a deeper analysis of both the Pauli and Dirac formalism and is left for future studies.

Finally we mention some quantitative results regarding quantum retarded engines that were obtained in previous papers [28,29] and are repeated here for completeness (for quantitative results of classical retarded engines see [27]).

A Wave Packet Retarded Quantum Engine

Assume an electron of mass m_e, with an associate wave packet of the form:

$$\psi = Ae^{ik'x}, \qquad A = \begin{cases} \sqrt{\tilde{\rho}_c} & r < R_{max} \\ 0 & r \geq R_{max} \end{cases} \tag{194}$$

k' and $\tilde{\rho}_c$ are constant real numbers. Normalization dictates the following:

$$\tilde{\rho}_c = \frac{3}{4\pi} R_{max}^{-3} \tag{195}$$

The phase of the wave function is thus linear, and its amplitude is uniform inside a sphere of radius R_{max} and zero elsewhere.

The current density is calculated using Equation (51):

$$\vec{J} = -\frac{\hbar e}{m_e}\tilde{\rho}\vec{\nabla}\phi = -\frac{\hbar k e}{m_e}\tilde{\rho}_c\hat{x} = -ev\tilde{\rho}_c\hat{x} \qquad \text{for} \quad r < R_{max}, \qquad v \equiv \frac{\hbar k'}{m_e}, \tag{196}$$

v is a quantity with velocity units.

We now calculate the force using Equation (113), in which system 1 is the (static) proton and system 2 is an electron which is in a wave packet state as described. The much heavier proton is modelled as a classical point charge situated at the origin:

$$\rho_1 = e\delta^{(3)}(\vec{x}_1) \tag{197}$$

It thus follows that the center of mass momentum of such a system will be:

$$\vec{P}(t) = -\frac{\mu_0}{4\pi}\int\int d^3x_1 d^3x_2\, \rho_1 \vec{J}_2 R^{-1} == -\frac{\mu_0 e}{4\pi}\int d^3x_2\, \vec{J}_2 r_2^{-1}. \quad (198)$$

Plugging Equations (196) and (197) will result in:

$$\vec{P}(t) = \frac{1}{2}\mu_0 e^2 v \tilde{\rho}_c R_{max}^2 \hat{x}. \quad (199)$$

Taking into account the value of ρ_c given in Equation (195) we derive:

$$\vec{P}(t) = \frac{3}{8\pi}\mu_0 e^2 v R_{max}^{-1} \hat{x}. \quad (200)$$

Thus the momentum accumulated by a retarded motor is linear in the electron's "velocity" v and is proportional to the wave packet size R_{max} inversely. The maximal velocity generated by this motor occurs when the engine is not loaded by any cargo:

$$v_{max} = \frac{P}{m_p} = \frac{3}{8\pi m_p}\mu_0 e^2 v R_{max}^{-1}. \quad (201)$$

the electron mass is neglected relative to the proton mass m_p. Thus a wave packet of the radius:

$$R_{max} = \frac{3\mu_0 e^2 v}{8\pi m_p v_{max}}. \quad (202)$$

is needed to obtain a velocity v_{max} . We shall be considering a scenario where the velocity v is close to the speed of light c, in order to have a larger R_{max}. It's important to note that in real situations, a smaller R_{max} would likely be required to achieve the desired speed. Additionally, for a relativistic electron, the Schrödinger equation isn't appropriate; instead, the Dirac equation should be used. The previous discussion on the relativistic engine assumes that the engine's components are moving slowly, with the relativistic effects arising from the delay in the electromagnetic signal. However, if the components themselves are moving at relativistic speeds, a different mathematical approach would be required. With this in mind, we can make some preliminary observations: for a typical car aiming to reach a maximum speed of 50 m/s (which is equivalent to 180 km/h), we obtain the following results.

$$R_{50} \simeq 1.4 \times 10^{-11}\text{ m} = 0.26\, a_0 \quad (203)$$

thus the wave packet dimension is quarter of the bohr radius. Provided the hydrogen retarded motor is to reach the velocity of $v_{max} = 11.2$ km/s which is earth's escape velocity we obtain:

$$R_{escape\ velocity} \simeq 6.1 \times 10^{-14}\text{ m} \simeq 10^{-3}\, a_0 \simeq 73\, r_p \quad (204)$$

in which the proton charge radius is $r_p = 8.4\ 10^{-16}$ m. At such high velocities, the wave packet associated with the particle is comparable in size to a nucleus, rather than being on the scale of an atom. If we consider the relativistic engine accelerating to the maximum speed possible in a Lorentzian space-time for a particle that starts below the speed of light (subluminal), we can conclude that this maximum speed would approach the speed of light $v_{max} \simeq c$.

$$R_{light\ speed} \simeq 2.3 \times 10^{-18}\text{ m} \simeq 3 \times 10^{-3}\, r_p \quad (205)$$

thus the dimension of the wave packet is sub nuclear. The above result is highly inaccurate as our analysis so far is essentially valid only for slow moving particles (with respect to the speed of light).

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