BPS states and BPS amplitudes in string theory

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Contents

Publications

This manuscript is based on the two publications [BCHP1], [BCHP2] and the submitted preprint [BCHP3] below.

[BCHP1] G. Bossard, C. Cosnier-Horeau and B. Pioline, *Protected couplings and BPS dyons in half-maximal supersymmetric string vacua*, Phys. Lett. B765, 377 (2017), doi: 10.1016/j.physletb.2016.12.035, arXiv: hep-th/1608.01660.

[BCHP2] G. Bossard, C. Cosnier-Horeau and B. Pioline, *Four-derivative couplings and BPS dyons in heterotic CHL orbifolds*, SciPost Phys. 3(1), 008 (2017), doi: 10.21468/SciPostPhys.3.1.008, arXiv: hep-th/1702.01926.

[BCHP3] G. Bossard, C. Cosnier-Horeau and B. Pioline, *Exact effective interactions and 1/4-BPS dyons in heterotic CHL orbifolds*, preprint (June 2018), arXiv: hep-th/1806.03330.

Chapter 0

Résumé long en français

Ce résumé du manuscrit de thèse en français est une directive de l'École Doctorale de Physique en Île-de-France pour faciliter la propagation du savoir en langue française. Cette tâche est doublement délicate, principalement en raison de la difficulté à définir le lectorat d'un tel document : d'une part, bien qu'une introduction de base soit nécessaire, il ne s'agit pas de se substituer à un travail de vulgarisation, d'autre part, si le travail exposé dans ce manuscrit est effectivement original, il ne peut prétendre à une présentation francophone autonome, c'est-à-dire indépendante de la recherche non-francophone publiée durant le demi-siècle dernier. Pour ces raisons, et pour tenter de répondre convenablement à cette double attente, ce résumé propose une approche à deux têtes, c'est-à-dire une traduction partielle du chapitre 1 d'introduction, nécessaire à la contextualisation de ce travail de thèse, ainsi qu'une rapide présentation des résultats des chapitres 2, 3 et 4, permettant au lectorat plus curieux de se repérer dans la partie anglophone du manuscrit.

Dans le reste du document, nous utilisons un système standard de conventions où $\hbar = c = 1$, c'est-à-dire qu'une unité de temps est égale à une unité de longueur, et à l'inverse d'une unité d'énergie. La signature utilisée pour l'espace-temps est $(-, +, \dots, +)$.

0.1 Introduction

La gravité quantique est l'une des grandes problématiques de la physique théorique moderne. Sa version classique, telle qu'elle fut exprimée par Newton et Einstein, ne permet pas d'en établir une description microscopique malgré la longue liste d'outils théoriques développés lors du siècle dernier. Obtenir une description complète de la théorie de gravité fut considéré comme un problème majeur, notamment pour pouvoir expliquer des phénomènes physiques se déroulant dans des régions de l'espace-temps où la force de gravité est réputée extrêmement forte, comme les trous noirs ou "l'origine de l'espace-temps". Au-delà des applications pratiques, ce problème est souvent énoncé comme celui pouvant offrir la pièce de puzzle manquante entre les théories quantiques décrivant la physique des particules, les théories quantiques des champs, et la théorie classique décrivant la dynamique de l'espace-temps, à savoir la théorie de la relativité générale d'Einstein.

À ce propos, l'équation d'Einstein illustre ses deux faces élégamment

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \ . \tag{1}$$

Le membre gauche de (1) décrit une géométrie non-quantifiée (classique) de l'espace-temps, tandis que le côté droit décrit de la matière quantifiée. Ce paradoxe n'a pas été exploré davantage à l'époque des travaux d'Einstein, car la physique macroscopique (par opposition à la physique microscopique) était suffisante pour décrire tous les objets astrophysiques du milieu interstellaire nous entourant.

Tout laisse à croire que la nature quantique de l'espace-temps serait observable à l'échelle de Planck $M_{Pl}^2 = \frac{\hbar c}{G} \sim 10^{28}\,\mathrm{eV}$, où les effets quantique et gravitationnels sont comparables. Malheureusement, cette échelle d'énergie est hors d'atteinte de tout collisionneur artificiel – les techniques actuelles nous permettant de sonder des énergies de collision de l'ordre $10^{13}\,\mathrm{eV}$, malgré tout, de nombreuses études en théories effectives des champs et modèles phénoménologiques ont été inspirées par la recherche en gravité quantique [3, 4].

Comme présentés dans l'introduction anglophone de la suite de ce manuscrit, les deux obstacles majeurs à la quantification de la gravité sont les problèmes dits d'unitarité et de renormalisabilité. Pour y remédier, deux grands paradigmes ont émergé dans le dernier quart du XX^e siècle, deux approches distinctes qui diffèrent sur la question de la quantification. D'un côté, la gravité est considérée comme une théorie complètement non-perturbative, et les problèmes sus-mentionnés d'unitarité et de renormalisabilité sont interprétés comme des artefacts de l'approche perturbative. Cette approche s'appelle la Gravité Quantique à Boucles, où l'on utilise la notion de boucle dans l'espace pour mesurer la courbure de celui-ci, et l'on quantifie la dite variable de boucle. De l'autre côté, la gravité classique est considérée comme la limite à basse énergie d'une théorie plus fondamentale, et l'on décide de quantifier la métrique perturbativement autour d'un espace-temps plat, avec un contenu de matière plus riche et symétrique que celui connu, comme en supergravité ou en théorie des cordes. Cette approche est celle poursuivie dans le reste de ce manuscrit.

0.1.1 Trous noirs

Les théories couplées à la gravité permettent l'existence de solutions d'espace-temps de type trous noirs. Décrire leur entropie – que nous présentons maintenant – est un des points centraux de [BCHP1], [BCHP3].

Les trous noirs sont caractérisés par une surface hypothétique, l'horizon des évennements, ayant la propriété spéciale d'être une surface de type lumière : tout objet se trouvant à sa surface a deux alternatives, tomber vers l'intérieur du trou noir si sa vitesse est inférieure à celle de la lumière, ou bien se déplacer tangent à celle-ci s'il évolue à la vitesse de la lumière. Les objets voyageant à une vitesse supérieure à celle de la lumière pourraient, en principe, échapper à l'attraction du trou noir, mais ce comportement contredit le principe de causalité, ce qui est formellement interdit dans toute théorie raisonnable. Cependant, les effets quantiques, comme la polarisation du vide, permettent à certaines particules de s'en échapper (voir figure 1). Des paires particule-antiparticule se séparent constamment à l'horizon des événements, occasionant une radiation semblable à celle d'un corps noir de température finie. Cette température dépend des caractéristiques du trou noir, et est appelée la température de Hawking [10, 23]

$$T = \frac{\hbar\kappa}{2\pi}\,, \tag{2}$$

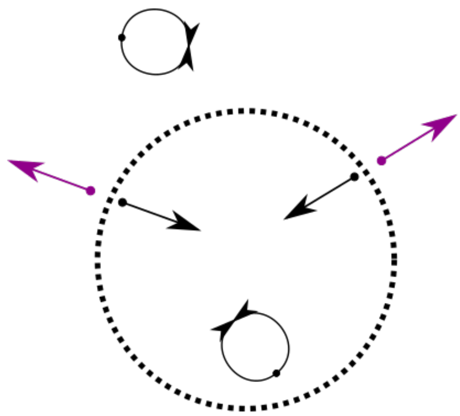


Fig. 1 *Illustration de l'émission de matière par un trou noir. Une paire particule-antiparticule se créé spontanément au voisinage de l'horizon des événements. L'une d'entre elles tombe à l'intérieur du trou noir tandis que l'autre s'en échappe. Particules et antiparticules sont représentées par une flèche.*

où κ est appelée gravité de surface, mesurant la force de gravité à l'horizon. De plus, les trous noirs se comportent comme des systèmes thermodynamiques caractérisés par leur température et d'autres *variables d'état*[1]: le théorème de 'calvitie' – ou d'absence de chevelure[2] – affirme qu'un trou noir est entièrement décrit par sa masse, ses charges et son moment angulaire, impliquant que son énergie interne peut être perçue comme une *fonction d'état*.[3] Cette analogie avec la fonction d'état d'un système thermodynamique a été utilisée pour identifier [24]

$$S_{BH} = \frac{A}{4\hbar G}\,, \tag{3}$$

la célèbre *entropie* de Bekenstein-Hawking[4]. Dans le paragraphe anglophone 4.1, nous présentons une méthode reconnue pour calculer l'entropie d'une certaine classe de solutions de trou noir, qui prend en compte les possibles corrections à l'action d'Einstein-Hilbert.

Ce résultat est pour le moins étrange, car on s'attend à voir l'entropie d'un objet étendu dans l'espace varier proportionnellement à son volume. Cependant, la loi (3) peut sembler raisonnable si l'on prend en compte la compression arbitrairement grande de l'espace dans la direction radiale au voisinage de l'horizon du trou noir, comme illustré figure 2. Si cette analogie est correcte, il est naturel de se poser la question suivante : est-ce qu'une théorie quantique de la gravité peut permettre une compréhension en termes statistiques de cette entropie ? En effet, l'entropie thermodynamique d'un système satisfait à la célèbre loi de Boltzmann

$$S = k_B \ln \Omega\,, \tag{4}$$

[1]La température, le volume, la pression et autres quantités macroscopiques qui sont définies à l'équilibre thermodynamique seulement.

[2]Son nom le plus courant étant "no-hair theorem".

[3]Par définition, une fonction d'état ne dépend que des variables d'état du système.

[4]Pour donner un ordre de grandeur, un trou noir de la taille de notre soleil aurait une très petite température $T \sim 10^{-7}K$, et une très grande entropie $S = 10^{77}$.

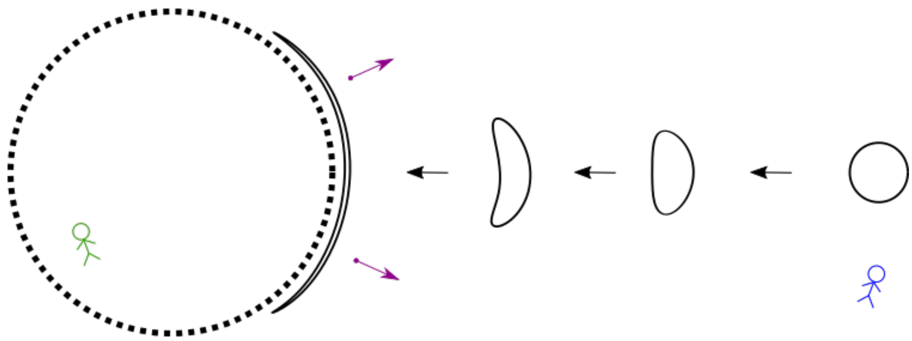


Fig. 2 *Illustration de la chute d'un objet vers l'intérieur d'un trou noir depuis un point de vue externe (en bleu) : la géométrie de l'espace-temps déforme infiniment la forme de l'objet vers une surface en deux dimensions, et de la même manière, ralentit infiniment le temps nécessaire pour que l'objet passe à travers l'horizon. Depuis le point de vue externe, les objets 'tombant' dans le trou noir ne font que s'accumuler tout autour sur une sphère entourant l'horizon.*

où Ω est le nombre de microétats accessibles à un même état d'équilibre thermodynamique. Dans [BCHP1], [BCHP3], nous concentrons notre attention sur une classe particulière d'objets où cette question a été répondue affirmativement, une sous-classe de solutions de trou noir extrémales [25]. Dans le paragraphe anglophone §4.2, nous présentons un cas particulier de théorie permettant une compréhension statistique des microétats de trous noirs extrémaux, où il a été montré avec grande précision que le dénombrement de ces microétats concorde avec l'entropie classique des trous noirs correspondants.

Les variables thermodynamiques d'un trou noir extrémal saturent l'inégalité définissant l'état à température nulle. Nous nous concentrerons sur le cas de solutions stationnaires à symétrie sphérique avec charges électromagnétiques – appelée la solution de Reissner-Nordström – où l'inégalité sur les charges est $M^2 \geq Q^2 + P^2$, avec Q la charge électrique et P la charge magnétique. L'entropie de Bekenstein-Hawking de ces trous noirs extrémaux (1.8) peut-être obtenue à partir d'une théorie dite de Einstein-Maxwell en quatre dimensions

$$S_{BH}(Q, P) = \pi(Q^2 + P^2)\,. \tag{5}$$

Il est important de souligner que, même en tant qu'approximation classique, l'expression (5) ne dépend d'aucun paramètre de la théorie. Dans une théorie invariante sous difféomorphismes, l'entropie d'un trou noir correspond à l'intégrale d'une 2-forme, charge de Noether définie par Wald, sur l'horizon [26]. L'entropie de trous noirs extrémaux, tels que (5), est aussi indépendante de la valeur asymptotique des champs scalaires, ce qui est une conséquence de la généralisation du mécanisme d'attracteur pour les solutions de trous noirs de supergravité [27, 28]. Ceci peut être facilement compris en écrivant la fonction entropie comme une fonctionnelle de la densité lagrangienne, tel que décrit dans le paragraphe anglophone §4.1.

Dans certaines théories où les couplages correspondent à la valeur asymptotique d'un champs dynamique, comme la théorie des cordes, cela implique que l'entropie d'un système ne change pas lorsque l'on varie le couplage. On peut ainsi décrire ses microétats à faible couplage, lorsque les techniques actuelles le permettent, et prolonger le résultat aux régimes de couplage suffisamment forts, lorsque le système devient un trou noir.

Dans les paragraphes anglophones §4.1 et §4.3, nous proposons une introduction des types de solutions de trous noirs pertinentes pour le reste de ce manuscrit. Ce sont des solutions sphériques de supergravité en quatre dimensions, et dans le paragraphe anglophone §4.2 nous présentons le comptage de leurs microétats à partir d'une leur description en théorie des cordes. Enfin, dans le paragraphe anglophone §4.4, nous montrons comment les calculs publiés dans [BCHP3] permettent de retrouver et de généraliser les résultats présentés en §4.2.

0.1.2 Supergravité

Les théories de supergravité constituent une tentative de modifier le comportement ultraviolet de la gravité d'Einstein, et peuvent également être la limite de basse énergie d'une théorie des cordes telle que celles que l'on considérera dans la suite de ce manuscrit. L'espace-temps habituel est plongé dans un espace de dimension supérieure appelé *superespace*, où les nouvelles coordonnées sont définies par des nombres de Grassmann anticommutants – ou *supernombres*, *i.e.* les degrés de liberté fermioniques.

Les symétries de cette nouvelle géométrie constituent un groupe de Poincaré (translations, rotations, boosts) étendu par des symétries locales anticommutantes appelées *supercharges*, et se transformant comme des spineurs sous la symétrie de Lorentz. Ces symétries locales contraignent également les champs de matière contenus dans la théorie quantique des champs, ce qui simplifie drastiquement la dynamique de ces théories (voir ci-après, tableau 1). Cette extension du groupe de Poincaré est toute particulière dans la mesure où elle correspond au seul exemple possible où les symétries de l'espace-temps se mélangent non-trivialement avec les symétries internes de la théorie quantique des champs sous-jacente, ce qui contredit l'esprit du théorème de Coleman-Mandula [29].

Les théories de supergravité sont généralement classifiées par leur nombre de supercharges : de 4 en quatre dimensions d'espace-temps à 32 pour l'extension maximale, cette dernière étant définie en toute dimensions jusqu'à $D = 11$ [30, 31]. Par la suite, nous référerons au nombre $\mathcal{N}$ de supercharges spinorielles en quatre dimensions seulement lorsque nous nous restreindrons à quatre dimensions, i.e. $\mathcal{N} = 8$ et $\mathcal{N} = 4$ correspondent respectivement aux supergravités maximales et demi-maximales en quatre dimensions d'espace-temps. De plus, les supergravités peuvent être séparées en deux types de construction, les constructions $(2, 2)$ et les constructions $(4, 0)$.[5] Ces dernières sont celles que nous étudierons dans le reste de ce manuscrit. En particulier, dans le paragraphe anglophone §3.1.1, nous présentons une réduction dimensionnelle de supergravité demi-maximale de dix à quatre dimensions, et nous exhibons la manière avec laquelle les champs de jauge et les modules peuvent s'arranger dans des représentations des groupes de symétrie globaux listés dans le tableau 1.

Les théories de supergravité contenant une grande extension de supersymétrie, dont certaines son listées tableau 1, ont un spectre bien plus riche que celui de la gravité d'Einstein (qui, par comparaison, se restreint à un seul champ de spin 2). Comme dit plus haut, cette complexité est réduite au niveau de l'action effective ou des amplitudes de diffusion, notamment à cause des contraintes de supersymétrie, mais aussi à cause des

[5] Cette notation fait référence à la chiralité des supercharges du modèle sigma de théorie des cordes – voir la section suivante – à ne pas confondre avec la supersymétrie d'espace-temps.

	$s=2$	$s=\frac{3}{2}$	$s=1$	$s=1/2$	$s=0$		
$\mathcal{N}=8$	1	8	28	56	70	E_7	
$\mathcal{N}=4$	1	4	6	4	2	$SO(6)\times SL(2,\mathbb{R})$	
	1	4	6+2	4+8	2+12	$SO(2,6)\times SL(2,\mathbb{R})$	
	1	4	6+4	4+16	2+24	$SO(4,6)\times SL(2,\mathbb{R})$	N=7
	1	4	6+6	4+24	2+36	$SO(6,6)\times SL(2,\mathbb{R})$	N=5
	1	4	6+10	4+40	2+60	$SO(10,6)\times SL(2,\mathbb{R})$	N=3
	1	4	6+14	4+56	2+84	$SO(14,6)\times SL(2,\mathbb{R})$	N=2
	1	4	6+22	4+88	2+132	$SO(22,6)\times SL(2,\mathbb{R})$	N=1

Table 1: Champs de matière classifiés par spin contenus dans des théories avec spin maximal 2 en quatre dimensions de certaines supergravités $\mathcal{N}=8$ et $\mathcal{N}=4$. La taille des représentations de spin est fixée depuis le plus haut spin par supersymétrie. Les deux premières lignes correspondent à des supergravités pures (champs de gravité de spin 2 et partenaires supersymétriques uniquement), et les suivantes sont couplées à un nombre spécifique de multiplets vectoriels préservant la symétrie globale non-perturbative $SL(2,\mathbb{R})$. Les deux dernières colonnes correspondent aux symétries globales attendues du spectre de masse nulle, et au paramètre d'orbifold de la théorie CHL avec orbifold $\mathbb{Z}_N$ correspondante.

symétries géométriques, ou symétries globales. Une partie de ce manuscrit a pour but de montrer les simplifications engendrées par les symétries géométriques, et de même que les contraintes imposées par supersymétrie sur les amplitudes de diffusion, comme rapporté dans les paragraphes anglophones §3.3 et §3.4.

Ces supergravités ont été étudiées vers la fin des années 70 et 80, après quoi s'est développé un consensus selon lequel les contraintes de supersymétrie ne seraient pas suffisantes pour éradiquer complètement le problème des divergences non-renormalisables. Cependant, ces théories ont concentré un regain d'intérêt lors des dernières années et il a été montré par le biais de calculs explicites que la supergravité $\mathcal{N}=8$ ne rencontre aucun divergence pathologique en dimension d'espace-temps $D<4+6/L$ avec $L=2,3,4$ boucles ($\mathcal{R}^4$, $D^2\mathcal{R}^4$) [32, 33]. Ce résultat redonna un élan d'optimisme, étant donné l'impressionante concordance avec le comportant ultraviolet d'une théorie de jauge pure, $\mathcal{N}=4$ super-Yang-Mills, qui est elle-même finie dans l'ultraviolet. Cependant, des études utilisant les symétries de dualité [34, 35, 36, 37, 38] ont prédit un changement abrupt dans le comportement critique de la supergravité $\mathcal{N}=8$ à $L=5$ boucles à cause d'un possible contre-terme $D^8\mathcal{R}^4$. Si ce contre-terme ne s'annule pas à 5 boucles, alors il y a fort à parier que la supergravité $\mathcal{N}=8$ est finie seulement pour $D<2+14/L$ pour $L>=5$, prédisant ainsi une divergence non-renormalisable en quatre dimensions à 7 boucles et au-delà. De fait, la présence de $D^8\mathcal{R}^4$ à 5 boucles a été récemment confirmée par un calcul explicite [39].

Dans ce manuscrit, nous ne nous intéresserons pas au cas de la supergravité $\mathcal{N}=8$ et nous concentrerons nos efforts sur les supergravités $\mathcal{N}=4$ présentées dans le tableau 1. À cause de leur supersymétrie $\mathcal{N}=4$, ces supergravités peuvent être couplées à des multiplets de matière [40] qui, par ailleurs, causent davantage de divergences. La richesse de leur structure leur permet également d'être réalisées en tant que limite à basse énergie de différents modèles de théorie des cordes, et, en particulier, la théorie des cordes *hétérotique* compactifiée sur T^6 avec un orbifold $\mathbb{Z}_N$ que nous décrivons plus en détails dans le chapitre anglophone 2. La présence de l'orbifold a notamment pour effet de réduire le spectre des états physiques, comme on peut le constater depuis le tableau 1.

0.1.3 Théorie des cordes

La théorie des cordes, disposant d'une longue et intéressante histoire, sera le centre du reste de ce manuscrit. Elle fut initialement développée autour de l'année 1968 avec l'amplitude de Veneziano, et plus tard, l'amplitude de Virasoro-Shapiro

$$M^{\text{VS}}(s,t,u) = \frac{\Gamma(-1-\alpha's/4)\Gamma(-1-\alpha't/4)\Gamma(-1-\alpha'u/4)}{\Gamma(-2-\alpha't/4-\alpha'u/4)\Gamma(-2-\alpha's/4-\alpha'u/4)\Gamma(-2-\alpha's/4-\alpha't/4)}\,. \tag{6}$$

Celles-ci étaient sensées donner une description du spectre de matière hadronique provoqué par les interactions fortes [41, 42]. Dans (6), s, t et u sont les invariants de Mandelstam définis par les quatre impulsions entrantes, respectivement, $-(k_1+k_2)^2$, $-(k_1+k_4)^2$ et $-(k_1+k_3)$, et α' fut appelée la pente de Regge. Il fut compris plus tard que ces amplitudes décrivent l'interaction de cordes ouvertes et fermées de taille $\ell_s = \sqrt{\alpha'}$ et de tension $T = 1/(2\pi\alpha')$,[6] tandis qu'une particule de masse nulle et de spin 2 était identifiée dans le spectre des cordes fermées comme une possible candidate pour le graviton [43]. Il fut ensuite rapidement compris que la quantification, l'invariance sous le groupe de Lorentz, et les contraintes d'unitarité imposent à ces cordes de se propager dans un espace à 26 dimensions d'espace-temps.[7]. De plus, leur spectre contient une infinité d'états, générés par les oscillations se déplaçant le long d'une corde et dont la masse et le spin s'expriment en terme du nombre de quantas d'oscillations n

$$m^2 \propto \frac{n}{\alpha'}, \qquad J \leq \alpha' m^2 + 1, \qquad n = -1,\, 0,\, 1, \ldots, +\infty\,. \tag{7}$$

Cette candidate pour une théorie de la gravité fut plus attirante qu'aucune autre théorie des champs, car elle a la particularité d'être manifestement finie dans l'ultraviolet. Ceci est dû notamment à la taille finie de la longueur de la corde ℓ_s, impliquant des interactions non-locales. En effet, à une distance très supérieure à la taille d'une corde, celles-ci se comportent comme des particules ponctuelles pouvant se rencontrer. De plus près, leur taille finie donne une épaisseur à leur trajectoire dans l'espace-temps, que l'on ne nomme plus ligne d'univers, mais *feuille d'univers*. Cette feuille d'univers à deux dimensions héberge une théorie des champs conforme (CFT), dont les symétries permettent par ailleurs une compréhension alternative de la dimension critique $D = 26$ [44], et impose également au graviton de satisfaire le graviton les équations d'Einstein (1.1) comme équations du mouvement, avec des corrections à tous les ordres en α'. Le gros problème de ces théories, aussi bien celle des cordes ouvertes que celle des cordes fermées, est qu'elles contiennent un tachyon (l'état correspondant à $n = -1$, dont la masse est un imaginaire pur) qui rend

[6] Notons que l'amplitude de Virasoro-Shapiro est invariante sous l'échange des trois variables de Mandelstam, comme attendu pour une amplitude de cordes fermées à l'ordre des arbres, tandis que l'amplitude de Veneziano, décrivant l'interaction de cordes ouvertes, n'est invariante que sous l'échange de s et t.

[7] Cette contrainte sur la dimension d'espace-temps protège l'unitarité de la théorie dans la mesure où elle élimine les états de norme négative du spectre.

incohérente la théorie en en brisant la causalité. Une théorie des supercordes fut alors élaborée plus tard, et Gliozzi, Scherk et Olive proposèrent un mécanisme pour projetter cet état tachyonic hors du spectre physique [45]. Les théories de supercordes vivent en 10 dimensions d'espace-temps, et disposent d'un secteur d'états de masse nulle contenant le spectre de la supergravité maximale [46] mentionné dans la section précédente.

La supergravité maximale a ainsi été étudiée longuement pour savoir si elle était cohérente par elle-même, ou si elle devait être complétée à une théorie des cordes afin d'obtenir un comportement unitaire et régulier dans l'ultraviolet. En effet, les amplitudes de théorie des cordes sont caractérisées par leur comportement régulier dans la limite des hautes énergies. En utilisant les invariants cinématiques de Mandelstam, une amplitude à 4 points à l'ordre des arbres, dans la limite de forte collision – $s, t \to +\infty$, angle fixe – se conduit telle que

$$M^{VS}(s,t) \sim \exp\Big(- \frac{\alpha'}{2}(s \ln s + t \ln t + u \ln u)\Big) \ , \tag{8}$$

où le comportement souple à large impulsion peut être attribué à la tour infinie d'états massifs présents dans le spectre.

Par ailleurs, le problème des divergences ultraviolettes est résolu par l'existence d'une longueur finie de cordes ℓ_s. Celle-ci implique la présence d'une taille minimale $\ell_s = \sqrt{\alpha'}$ pour les phénomènes se réalisant dans l'espace-temps. À l'ordre des boucles, cela peut se comprendre en remarquant que le lieu géométrique dans l'espace des impulsions correspondant aux divergences de l'amplitude est absente à tous les ordres. Ceci est une conséquence générale de l'invariance modulaire de l'intégrant : pour les tores et les surfaces de Riemann de plus haut genre, la partie divergente de la région ultraviolette est absente de l'espace des paramètres de l'amplitude.

La limite basse énergie de la théorie des cordes donne lieu à une théorie de la gravité couplée à des champs de matière. Ces théories peuvent donc décrire des solutions de trou noir, et constituent ainsi un cadre idéal pour étudier les propriétés quantiques des trous noirs.

Dualités non-perturbatives en théorie des cordes. Bien que la théorie des supercordes soit à l'époque l'unique candidate pour une théorie renormalisable de la gravité quantique, un certain manque d'intérêt fut présent à ses premiers instants, notamment en raison de l'existence de plusieurs réalisations différentes du principe des supercordes (théories appelées Type I, Type IIA et Type IIB). De plus, aucune d'entre elles ne semblait compatible avec les exigences de la Nature : les théories de type II n'avaient que des groupes de jauge abéliens, contrairement aux théories électrofaibles et d'interactions fortes, et la théorie de type I semblait pouvoir donner l'illusion de posséder un groupe de jauge arbitraire. Cependant, en 1985, il fut découvert que l'ensemble des groupes de jauges possibles est sévèrement contraint par l'absence d'anomalies, condition indispensable à tout théorie de jauge cohérente, et qu'une autre théorie des cordes, appelée théorie des cordes hétérotique,[8] avec groupe de jauge $SO(32)/\mathbb{Z}_2$ ou $E_8 \times E_8$ (où E_8 est l'un des

[8] Ce nom vient de la construction particulière de cette théorie. Le oscillations allant dans un sens le long des cordes fermées se donnent lieu à un spectre purement bosonique en 26 dimensions, tandis que les oscillations se déplaçant dans l'autre sens le long des cordes donnent lieu à un spectre supersymétrique en 10 dimensions.

groupes exceptionnels dans la classification de Cartan). Le dernier s'avéra être plus attrayant phénoménologiquement et devint très populaire pour son aptitude à produire des théories de grande unification du modèle standard à partir du sous-groupe exceptionnel E_6.

Bien que ces découvertes aient amélioré la réputation de cette théorie-candidate, il devint rapidement évident qu'une compréhension non-perturbative des effets de théorie des cordes serait nécessaire pour produire un modèle d'unification des forces de la nature. Plus précisément, la physique en quatre dimensions d'espace-temps dépend de manière critique du type de compactification utilisé pour réduire le nombre de dimensions d'espace-temps de dix à quatre. Certains de ces espaces de compactification possèdent de nombreuses symétries, facilitant ainsi les calculs analytiques, et d'autres sont parfois reliés entre eux par des dualités, les dualités T, permettant de comprendre un modèle de compactification à partir d'un autre. Cependant, beaucoup de compactifications ne sont nullement reliées, et restent insondables par nos techniques actuelles. De plus, le type de compactification est déterminé par la dynamique des champs à très haute énergie, et la sélection de cette compactification requiert également une information non-perturbative à propos du potentiel sur les vides possibles de théorie des cordes.

Il fut compris, en parallèle, que les effets non-perturbatifs de certaines théories de jauge étaient accessibles sans calcul explicite de toutes les contributions instantoniques. Ceci est rendu possible par la présence d'une symétrie dite non-perturbative, c'est-à-dire reliant de manière hautement non-triviale les effets à faible et fort couplage. Comme mentionné en §1.4, une telle symétrie fut conjecturée dans le modèle de Georgi-Glashow en 1977 [49], mais fut reçue avec scepticisme jusqu'à qu'un argument plus fort soit présenté dans une extension supersymétrique $\mathcal{N}=2$ de cette théorie [50]. Cette symétrie entre fort et faible couplage, appelée dualité S, relie deux phases d'une théorie super-Yang-Mills,[9] l'une à grande valeur du couplage avec l'autre à petite valeur du couplage. Lorsque la théorie est super-conforme, elle n'est jamais dans une de ces deux phases, mais toujours à un point critique, et la valeur du couplage ainsi que d'autres grandeurs plus complexe peuvent être obtenues en utilisant leur propriété d'invariance sous la dualité S.[10]

Ces dualités S furent au même moment conjecturées dans de nombreuses théories des cordes, telles que les théories hétérotiques et type IIB, à la fois en tant que symétries et en tant que dualités reliant *différentes* théories entre elles, comme par exemple les cordes hétérotiques à faible (fort) couplage avec les cordes de type I à fort (faible) couplage. Cette symétrie est notamment utilisée en [BCHP3] pour déduire les propriétés de l'action effective de basse énergie de l'interaction à quatre photons des cordes de type I.

Plus tard, une autre symétrie fut conjecturée, ce qui marqua un tournant important dans l'histoire de la théorie des cordes. Celle-ci reliait les cordes de type IIA à fort couplage à une théorie en onze dimensions d'espace-temps appelée théorie M. Cette même dualité envoie la théorie hétérotique avec groupe de jauge $E_8 \times E_8$ à fort couplage à une version de la théorie M avec des bords physiques sur la onzième dimension. L'existence de cette dernière théorie fut plus tard corroborée par la découverte d'une supergravité en onze dimensions d'espace-temps, mentionnée dans §0.1.2. Cette étape historique dans la recherche en théorie des cordes donna lieu à bien d'autres conjectures similaires qui ont

[9] Elle relie deux théorie différentes lorsque que le groupe de jauge n'est pas simplement lacé [51].

[10] C'est la cas des théories $\mathcal{N}=4$ et $\mathcal{N}=2$ où $N_s = 2N_c$, avec N_s et N_c les nombres de saveur et de couleur.

préparé la voie de ce travail de thèse.

Ainsi, en étudiant le régime perturbatif de certaines théories des cordes, à faible couplage, il est possible d'utiliser la dualité entre couplages fort et faible pour extraire certaines informations non-perturbatives du régime à fort couplage, dans la même théorie ou dans une autre, comme étudié dans les sections anglophones §3.3 et §3.4. Cette dualité S peut nous aider à comprendre les théories des cordes à grande et petite valeurs du couplage, mais aussi à nous donner le contrôle sur certaines informations contenues dans le secteur instantonique, comme proposé dans le chapitre anglophone §4.

0.1.4 Dualités non-perturbatives en théorie des champs

Les symétries non-perturbatives ont joué un rôle très important en physique, aussi bien dans la compréhension de la théorie des cordes que dans certaines théories quantiques des champs. Elles restent l'un de nos seuls outils théoriques permettant de décrire des effets physiques non-perturbatifs, c'est-à-dire indétectables par les techniques d'étude au voisinage des équations classiques du mouvement. Dans cette section, à défaut de résumer toute leur histoire, nous introduisons quelques détails et concepts clés des dualités entre couplage fort et faible dans les théories des champs à quatre dimensions. Ceux-ci seront utiles au profane pour comprendre la déclinaison de ces symétries en théorie des cordes et en supergravité $\mathcal{N}=4$, revue dans la prochaine section.

Théorie de jauge non-abélienne en quatre dimensions. Nous rappelons ici quelques détails sur la dualité entre faible et fort couplage dans la plus vieille théorie de l'électromagnétisme, la théorie de Maxwell, et nous commenterons ensuite sur ce type de dualité dans le cas des théories des champs (voir le paragraphe suivant), ainsi qu'en supergravité et en théorie des cordes (voir §3.1). Ces symétries sont au centre des propositions de couplage exacts faites dans [BCHP1], [BCHP2], [BCHP3], comme nous l'introduisons dans la section §0.3.

Dans le vide, la théorie de Maxwell de l'électromagnétisme a une symétrie de jauge $U(1)$ permettant une rotation de référentiel entre les champs électrique et magnétique

$$E + \mathrm{i}B \to e^{\mathrm{i}\alpha}(E + \mathrm{i}B)\,. \tag{9}$$

Celle-ci permet également d'échanger le champ électrique avec le champ magnétique $(E, B) \to (-B, E)$. Dans la formulation relativiste de cette théorie, où les champs électrique et magnétique sont exprimés respectivement par les entrées du tenseur d'intensité de champs F^{0i} et son dual de Hodge $\star F_{0i}$, la dualité $(E, B) \to (B, -E)$ peut s'exprimer simplement $F_{\mu\nu} \to \star F_{\mu\nu}$.

Lorsque l'on étend cette dualité au spectre chargé, celle-ci prédit l'existence de monopôles magnétiques,[11] c'est-à-dire des états de charge magnétique q_m non-nulle. Si présentes, ces charges doivent satisfaire à la condition de quantification de Dirac-Schwinger-Zwanziger [64, 65, 66]

$$q_e q_m' - q_e' q_m = 4\pi n\,, \qquad n \in \mathbb{Z}\,, \tag{10}$$

où (q_e, q_m) et (q_e', q_m') sont les charges électriques et magnétiques de deux particules présentes dans le spectre. Puisque des électrons de charges $(ge, 0)$, avec $e \in \mathbb{Z}$ et g la

[11] Ne pas confondre avec le monopole commercial, sans ô.

constante de couplage électromagnétique, existent dans la nature, la condition de quantification (1.19) pour un monopôle magnétique hypothétique requiert $geq_m = 2\pi n$. Ainsi, la charge magnétique possible d'un monopôle est donnée par

$$q_m = \frac{4\pi}{g} m \,, \qquad m \in \mathbb{Z} \,. \tag{11}$$

Ceci implique notamment que l'échange entre les champs électrique et magnétique, en présence de matière chargée, impose une correspondance entre les charges électriques et magnétiques, jointe à une dualité entre fort et faible couplage g

$$(e, m) \to (-m, e) \quad \Rightarrow \quad g \to \frac{4\pi}{g} \,. \tag{12}$$

Cette dualité du spectre de charge est également développée en supergravité $\mathcal{N} = 4$ dans la section anglophone §3.1.1.

Symétries non-perturbative de la théorie des champs $\mathcal{N} = 4$. In 1974, des solutions magnétiquement chargées furent trouvées dans des théories de jauge non-abéliennes avec brisure spontanée de symétrie vers des théories de jauge abéliennes [67, 68]. Montonen et Olive ont ainsi conjecturé l'existence qu'une dualité échangeant un triplet de jauge composé de ces états de monopôle et du photon avec les bosons W de la brisure spontanée de symétrie de la théorie non-abélienne [49]. Dans le cas général, cette conjecture fut falsifiée par les corrections quantiques à la masse, ou par la différence de spin entre ces deux triplets. Cependant, pour la théorie super-Yang-Mills $\mathcal{N} = 4$, l'action effective n'obtient aucune correction de couplages à plus de deux dérivées, et il fut montré que la quantification des modes zéro fermioniques autour d'une solution de monopôle en faisait un triplet vectoriel massif $\mathcal{N} = 4$ [69]. Cela laissa bon espoir quant à l'existence d'une dualité entre couplage fort et faible.

On peut ainsi donner une valeur moyenne dans le vide $\langle\phi\rangle$ à l'un des six scalaires adjoints de la théorie $\mathcal{N} = 4$ avec groupe de jauge G. Ensuite, le boson W, ou la fluctuation de la composante E_α du champ de jauge, avec α une racine de G, obtient une masse donnée par

$$\mathcal{M}_W(\alpha) = g\, |\alpha \cdot \langle\phi\rangle| \,. \tag{13}$$

D'un autre côté, chaque racine α donne une solution de monopôle similaire à la solution de Bogomol'nyi-Prasad-Sommerfield $G = SU(2)$ [70, 71]. Ces états, formant une représentation de dimension seize de l'algèbre de supersymétrie $\mathcal{N} = 4$, sont annihilés par la moitié des seize supercharges, et leur masses et charges satisfont une relation particulière, on dit qu'elles saturent l'inégalité de Bogomol'nyi. Dans le cas présent, cette relation est donnée par

$$\mathcal{M}_M(\alpha) = \frac{4\pi}{g} |\alpha^\vee \cdot \langle\phi\rangle| \,, \tag{14}$$

où $\alpha^\vee$ est une coracine de G.[12]

Le spectre de bosons W et de monopôles de la théorie peut être échangé si le système de racines est dual à lui-même par rapport à la projection orthogonale, *i.e.* si le système

[12] Le système des coracines est aussi un système de racines. Il est dual au système de racines par rapport à la projection orthogonale.

de racines et le système de coracines sont isomorphes.[13] Enfin, on peut également voir qu'échanger les monopôles de G avec les bosons W de G amène à la même dualité entre fort et faible couplage que (12).

Cette présentation superficielle de la dualité S en théorie de super-Yang-Mills $\mathcal{N} = 4$ peut être complétée par [72, 73, 74].

Dans la section anglophone §3.1.1, nous rappelons l'invariance de masse du spectre de charges du secteur BPS dans le cas de la supergravité $\mathcal{N} = 4$, de manière à motiver l'existence d'une dualité non-perturbative de la théorie des cordes complète. Dans les sections §0.3.1 et §0.3.2, nous utilisons cette dualité ainsi que les contraintes de supersymétrie pour conjecturer l'existence de couplages exacts F^4 et $\nabla^2 F^4$. Ce dernier nous permettra d'extraire l'information relative à la dégénérescence des trous noirs quart-BPS en supergravité $\mathcal{N} = 4$, dans le chapitre anglophone §4.

0.2 Amplitudes de supercordes et développement perturbatif

Dans le chapitre anglophone §3, nous étudions les amplitudes à une et deux boucles de théories des cordes hétérotiques $\mathcal{N} = 4$. Dans le cas le plus simple, la description effective du secteur de masse nulle de cette théorie correspond à la réduction toroïdale d'une supergravité $\mathcal{N} = 1$ couplée à une théorie de super-Yang-Mills $\mathcal{N} = 1$. Quelques modèles de ce type sont donnés dans le tableau 1, et certain d'entre eux étant réalisables par compactification toroïdale d'une théorie des cordes hétérotiques, la colonne N est l'ordre de l'action libre du groupe $\mathbb{Z}_N$ de la construction orbifold.

Nous commençons par rappeler certaines bases des amplitudes de cordes fermées en théorie des cordes, et nous présentons ensuite le calcul à une et deux boucles de l'interaction à quatre photons. Les résultats à deux boucles sont basés sur le célèbre calcul de D'Hoker et Phong [75, 76, 77, 78, 79, 80].

0.3 Contraintes non-perturbatives et de supersymétrie

Dans le chapitre anglophone 3, nous étudions les symétries entre fort et faible couplage dans un contexte de théorie des cordes, d'abord pour le modèle hétérotique entier, puis dans les modèles CHL de rang réduit. Nous voulons motiver l'existence de ces symétries afin de les utiliser dans la construction d'amplitudes exactes pour les interactions à quatre photons, dans le but *in fine* d'en extraire le comptage de dégénérescence de trous noirs en supergravité $\mathcal{N} = 4$ (chapitre 4). Ces symétries de théorie des cordes sont très similaires dans leur forme aux symétries de théorie des champs présentées dans l'introduction §0.1.4.

Dans la section anglophone 3.1, nous présentons le cas de la théorie hétérotique complète compactifiée sur le tore, et nous revoyons, à partir de [102, 72], les motivations pour la symétrie entre fort et faible couplage pour l'action effective de la théorie de supergravité en quatre dimensions, ainsi que le spectre de charges et d'états BPS. Nous réexaminons

[13] En général, la dualité S envoie une théorie avec un groupe de jauge de système de racines Φ vers une théorie avec un groupe de jauge de système de racines $\Phi^\vee$, où $\Phi^\vee$ est le système de coracines associé à Φ. La symétrie sous dualité S est ainsi possible uniquement si $\Phi^\vee \simeq \Phi$, ce qui est le cas des groupes simplement lacés seulement.

ensuite cette symétrie en trois dimensions, et présentons le groupe beaucoup plus grand de symétries non-perturbatives qui en découle, $G_3(\mathbb{Z})$ [103, 104].

Dans la section anglophone 3.2, nous revoyons quelques détails des modèles CHL $\mathbb{Z}_N$, avec N prime, dont le groupe de jauge est de rang réduit, depuis une perspective de cordes hétérotique [59, 60, 62], et nous argumentons ensuite pour la présence d'une symétrie entre couplage fort et faible pour ces théories en quatre et trois dimensions [105].

Finalement, dans les sections 0.3.1 et 0.3.2, nous exposons les conjectures de [BCHP1], [BCHP2], et [BCHP3] qui proposent que des couplages exacts à quatre scalaires dans la limite de basse énergie de l'action effective en trois dimensions – nommément $(\nabla\phi)^4$ et $\nabla^2(\nabla\phi)^4$ – sont donnés par des intégrales modulaires de certaines formes modulaires spécifiques, multipliées par les fonctions de partition pour le réseau non-perturbatif de Narain invariant sous le groupe complet des symétries non-perturbatives $G_3(\mathbb{Z})$. Ces couplages sont obtenus par covariantisation des coefficients des couplages perturbatifs respectifs $F^{(1)}_{abcd}$ et $G^{(2)}_{ab,cd}$ sous le groupe de symétries non-perturbatives $G_3(\mathbb{Z})$. Ceux-ci sont également motivés par les contraintes de supersymétrie que nous exposons dans les sections anglophones §3.3 et §3.4, et sont vérifiés en utilisant des résultats perturbatifs extraits de la littérature dans le régime de faible couplage en théorie des cordes hétérotique et de type II dans les sections §3.3.1 et §3.4.1 respectivement.

0.3.1 Conjecture pour le couplage exact F^4

Les arguments pour l'existence d'un groupe de dualités non-perturbatives $\widetilde{O}(r-4,8,\mathbb{Z})$, revus dans les sections anglophones 3.1 and 3.2, ainsi que les contraintes de supersymétrie revues section 3.3 ont motivé notre conjecture pour l'existence du couplage exact $(\nabla\phi)^4$ sous la forme d'une intégrale modulaire

$$F^{(r-4,8)}_{abcd}(\varphi) = \text{R.N.} \int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{d\tau_1 d\tau_2}{\tau_2^2} \frac{\Gamma_{\Lambda_{r-4,8}}[P_{abcd}]}{\Delta_k(\tau)}\,, \tag{15}$$

qui est construite comme la généralisation du couplage perturbative à une boucle – voir dans la partie anglophone (2.24) pour le rang maximal, ou (3.87) pour $N = 2,3,5,7$ – où nous avons remplacé le réseau de Narain $\Lambda_{r-5,7}$ par son extension non-perturbative $\Lambda_{r-4,8}$ (3.89).

La fonction (15) est manifestement invariante sous les dualités non-perturbatives mentionnées section 3.1.2, et a également la propriété de satisfaire aux contraintes de supersymétrie (3.95), et en particulier à l'équation différentielle (3.97), voir §3.2 de [BCHP2].

Afin de prouver qu'une solution de la contrainte différentielle de supersymétrie correspond au couplage exact escompté, nous devons vérifier qu'elle satisfait également aux bonnes conditions aux bords, par exemple, en vérifiant qu'elle redonne bien le résultat perturbatif pour F^4 dans la limite de faible couplage. Dans la sous-section suivante, nous calculons cette limite de faible couplage en étudiant la décomposition de Fourier de $F_{abcd}(\varphi)$ à l'approche du cusp $g_s \to 0$ dans le cas des cordes hétérotiques en trois dimensions, et dans le cas des cordes de type II en quatre dimensions. Nous montrons que le mode zéro dans la décomposition de Fourier de (15) correspond aux résultats perturbatifs obtenus dans la littérature.

Limite faible couplage en trois dimensions du couplage $(\nabla\phi)^4$**.** Dans [BCHP2], nous avons calculé la décomposition de Fourier de la fonction $F^{(2k,8)}_{abcd}$ au cusp $g_3 \to 0$ de

$$G_{2k,8} \simeq \mathbb{R}^+_{1/g_3^2} \times \left[\tfrac{O(r-5,7)}{O(r-5)\times O(7)}/O(r-5,7,\mathbb{Z})\right] , \tag{16}$$

ce qui correspond à la limite de faible couplage des cordes hétérotiques à $D = 3$. Le réseau non-perturbatif se décompose selon [BCHP2]

$$\Lambda_{2k,8} = \Lambda_{2k-1,7} \oplus I\!I_{1,1}[N] , \tag{17}$$

Pour interpréter les résultats dans un langage perturbatif, nous devons rappeler que la fonction covariante sous U-dualité $F^{(2k,8)}_{abcd}(\varphi)$ est le coefficient du couplage $(\nabla\phi)^4$ dans l'action de basse énergie écrite en référentiel d'Einstein, de sorte que la métrique γ_E est invariant sous U-dualité

$$S_3 = \int \mathrm{d}^3x \sqrt{-\gamma_E} \left[\mathcal{R}[\gamma_E] - (2\delta_{\hat a\hat b}\delta_{\hat c\hat d} - \delta_{\hat a\hat c}\delta_{\hat b\hat d})F^{(2k,8)}_{abcd}(\varphi)\, \gamma_E^{\mu\rho}\gamma_E^{\nu\sigma} P^{a\hat a}_\mu P^{b\hat b}_\nu P^{c\hat c}_\rho P^{d\hat d}_\sigma\right] + \dots . \tag{18}$$

Dans le référentiel de théorie des cordes, $\gamma = \gamma_E g_3^4$ et on trouve

$$S_3 = \int \mathrm{d}^3x \sqrt{-\gamma} \left[\frac{1}{g_3^2}\mathcal{R}[\gamma] - g_3^2\,(2\delta_{\hat a\hat b}\delta_{\hat c\hat d} - \delta_{\hat a\hat c}\delta_{\hat b\hat d})F^{(2k,8)}_{abcd}(\varphi)\, \gamma^{\mu\rho}\gamma^{\nu\sigma} P^{a\hat a}_\mu P^{b\hat b}_\nu P^{c\hat c}_\rho P^{d\hat d}_\sigma\right] + \dots . \tag{19}$$

En utilisant $c_k(0) = k$ pour les modèles CHL avec $N > 1$ ou $c(0) = 2k$ dans le cas de rang maximal, ainsi que $\xi(2) = \frac{\pi}{6}$, les résultats de [BCHP2] donnent

$$g_3^2\, F^{(2k,8)}_{abcd} = \frac{3}{2\pi g_3^2}\delta_{(ab}\delta_{cd)} + F^{(2k-1,7)}_{abcd} + \sum_{Q\in\Lambda_{2k-1,7}}{}^{\prime}\; \bar c_k(Q) e^{-\frac{2\pi\sqrt{2}\,|Q_R|}{g_3^2} + 2\pi i a\cdot Q} P^{(*)}_{abcd} , \tag{20}$$

où nous avons omis la forme détaillée des corrections exponentiellement supprimées, et où $\bar c_k(Q)$ est la mesure de sommation

$$\bar c_k(Q) = \sum_{\substack{d\geq 1,\\ Q/d\in\Lambda_{2k-1,7}}} d\, c_k\Big(-\frac{Q^2}{2d^2}\Big) + \sum_{\substack{d\geq 1,\\ Q/d\in N\Lambda^*_{2k-1,7}}} N\, d\, c_k\Big(-\frac{Q^2}{2Nd^2}\Big) . \tag{21}$$

Les deux premiers termes dans (20) devraient correspondre aux contributions à l'ordre des arbres et à une boucle respectivement. En effet, la réduction dimensionnelle du couplage hétérotique à l'ordre des arbres en dix dimensions $\mathcal{R}^2 + (\mathrm{Tr}F^2)^2$ [128, 129] implique l'existence d'un terme à l'ordre des arbres $(\nabla\phi)^4$ en $D = 3$, avec un coefficient indépendant de N. Le second terme dans 20 correspond au terme perturbatif à une boucle (2.24) par construction. Les termes non-perturbatifs restant peuvent être interprétés comme des contributions instantoniques de branes euclidiennes NS5, KK5 hétérotiques et de monopôles H enroulés autour de tout T^6 au sein du T^7 de compactification [130]. Plus précisément, les charges des branes NS5 et KK5 correspondent aux charges de moments et d'enroulements dans le réseau hyperbolique $I\!I_{1,1}[N] \oplus I\!I_{k-2,k-2}$ contenu dans $\Lambda_m \oplus I\!I_{1,1}$, tandis que les monopôles H correspondent aux charges dans le réseau de jauge $\Lambda_{k,8-k}$ (pour les cordes hétérotiques compactifiées sur T^7, ces sous-réseaux doivent être remplacés par $I\!I_{7,7}$ et $E_8 \oplus E_8$ ou D_{16}, respectivement).

Limite de faible couplage de la théorie des cordes de type II compactifiée sur $K3 \times T^2$. L'axiodilaton hétérotique S correspond respectivement au module Kähler du 2-tore $T_{\rm A}$ en type IIA, et à la structure complexe du 2-tore $U_{\rm B}$ en type IIB, tandis que l'axiodilaton de type II $S_{\rm A} = S_{\rm B}$ correspond au module de Kähler T du 2-tore du côté hétérotique

$$S = T_{\rm A} = U_{\rm B} \ , \qquad T = S_{\rm A} = S_{\rm B} \ , \qquad U = U_{\rm A} = T_{\rm B} \ . \tag{22}$$

Afin de développer à faible couplage de type II, *i.e.* à large T_2, nous décomposons le réseau selon

$$\Lambda_{2k-2,6} = \Lambda_{2k-4,4} \oplus I\!I_{1,1} \oplus I\!I_{1,1}[N] \, . \tag{23}$$

Pour simplifier, nous utiliserons les modules S_B, T_B, U_B de type IIB dans cette section, avec $S_{B2} = 1/{g_s}^2$. De plus, nous ne considérerons que les termes perturbatifs pour les champs de Maxwell dans le secteur RR correspondant aux indices $\alpha, \beta, \dots$ le long du sous-réseau $\Lambda_{2k-4,4}$. La limite de faible couplage de type IIB de l'interaction exacte F^4 donne ainsi [BCHP3]

$$\begin{aligned} \widehat{F}^{(2k-2,6)}_{\alpha\beta\gamma\delta\ \rm II} &= \frac{1}{{g_s}^2} F^{(2k-4,4)}_{\alpha\beta\gamma\delta\ \rm II} + \frac{3}{2\pi}\delta_{(\alpha\beta}\delta_{\gamma\delta)} \Big(\frac{\hat{\mathcal{E}}_1(NT_{\rm B}) + \hat{\mathcal{E}}_1(T_{\rm B}) + \hat{\mathcal{E}}_1(NU_{\rm B}) + \hat{\mathcal{E}}_1(U_{\rm B}) + \frac{12}{\pi}\log g_s}{N+1} \Big) \\ &= \frac{1}{{g_s}^2} F^{(2k-4,4)}_{\alpha\beta\gamma\delta}(t) - \frac{3}{8\pi^2}\delta_{(\alpha\beta}\delta_{\gamma\delta)} \log({g_s}^{-2k} T^k_{\rm B2} U^k_{\rm B2} |\Delta_k(T_{\rm B})\Delta_k(U_{\rm B})|^2) \ , \end{aligned} \tag{24}$$

où le premier terme correspond au couplage à l'ordre des arbres calculé dans [131], tandis que le second terme est lié par supersymétrie au couplage $\mathcal{R}^2$ calculé dans [132, 123].

0.3.2 Conjecture pour le couplage exact $\nabla^2(\nabla\phi)^4$

Comme pour le couplage $(\nabla\phi)^4$ de la section précédente, les arguments motivant l'existence d'un groupe de dualités non-perturbatives $\widetilde{O}(r-4, 8, \mathbb{Z})$, revus dans les sections anglophones §3.1 et §3.2, ainsi que les contraintes de supersymétrie, §3.3, ont motivé la conjecture [BCHP3] spécifiant le couplage exact $\nabla^2(\nabla\phi)^4$ comme l'intégrale modulaire

$$G_{ab,cd}(\varphi) = \text{R.N.} \int_{\Gamma_{2,0}(N)\backslash\mathcal{H}_2} \frac{\mathrm{d}\Omega_1 \mathrm{d}\Omega_2}{|\Omega_2|^2} \frac{\Gamma^{(2)}_{\Lambda_{r-4,8}}[P_{ab,cd}]}{\Phi_{k-2}(\Omega)} \, . \tag{25}$$

Celle-ci est construite comme une généralisation de l'amplitude perturbative à deux boucles (2.37) (avec $\Gamma_{2,0}(1) = Sp(4, \mathbb{Z})$), où l'on a remplacé le réseau de Narain $\Lambda_{r-5,7}$ avec son extension non-perturbative $\Lambda_{r-4,8}$ (3.89). Dans le cas des modèles CHL, la construction de (25) est nettement plus technique que dans le cas de genre un, mais celle-ci a été explicité en détails pour $N = 2$ dans l'appendice B.2.2 of [BCHP3], et généralisée à $N = 3, 5, 7$ avec une argumentation dans l'esprit de celle proposée dans la présentation de genre un de la section anglophone §3.2.1 de ce manuscrit.

La fonction (3.118) est manifestement invariante sous le groupe des dualités non-perturbatives mentionné dans la section anglophone 3.1.2, et satisfait aux contraintes de supersymétrie (3.114), et en particulier à l'équation (3.115) (voir la section §3.3 de [BCHP3]).

Dans la prochaine sous-section, nous nous intéressons à la limite de faible couplage du modèle hétérotique en trois dimensions, et de type II en quatre dimensions. Le mode zéro dans le développement de Fourier de (25) correspond ainsi à la réponse attendue d'un calcul perturbatif lorsqu'elle est connue, ou à une prédiction lorsqu'elle est inconnue.

Limite de faible couplage du couplage exact $\nabla^2(\nabla\phi)^4$. La décomposition de Fourier des fonctions $F^{(r-4,8)}_{abcd}$, et $G^{(r-4,8)}_{ab,cd}$ au cusp $g_3 \to 0$ (3.88) correspondant à la limite de faible couplage hétérotique en trois dimensions ont été calculé dans [BCHP2] et [BCHP3] respectivement. Dans cette limite, le réseau $\Lambda_{2k,8}$ se décompose en

$$\Lambda_{2k-1,7} \oplus II_{1,1}[N]\,, \tag{26}$$

où le rayon du second facteur est relié au couplage des cordes hétérotiques par $g_3 = 1/\sqrt{R}$, et le groupe d'U-dualité est brisé en $\widetilde{O}(2k-1,7,\mathbb{Z})$, le groupe des automorphismes restreints de $\Lambda_{2k-1,7}$. Afin d'interpréter les résultats en termes de contributions perturbatives à l'interaction $\nabla^2(\nabla\phi)^4$, il peut être pratique de multiplier le coefficient du couplage par g_3^6, qui prend sa source dans le redimensionnement de Weyl $\gamma_E = \gamma_s/g_3^4$ pour passer du référentiel d'Einstein au référentiel de cordes, voir la section §4.3 de [BCHP2]. Le développement à faible couplage peut être ensuite extrait de la section §4.1 de [BCHP3] en remplaçant $q = 8$, $v = 1$, et donne ainsi

$$\begin{aligned} {g_3}^6 G^{(2k,8)}_{\alpha\beta,\gamma\delta} = & -\frac{3}{4\pi {g_3}^2}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} - \frac{1}{4}\delta_{\langle\alpha\beta,}G^{(2k-1,7)}_{\gamma\delta\rangle}(\varphi) + {g_3}^2\, G^{(2k-1,7)}_{\alpha\beta,\gamma\delta}(\varphi) \\ & + \sum_{Q\in\Lambda^*_{2k-1,7}}{}' \frac{3e^{-\frac{2\pi}{g_3^2}\sqrt{2Q_R^2}+2\pi\mathrm{i}Q\cdot a}}{2Q_R^2}\,\bar{G}^{(2k-1,7)}_{\langle\alpha\beta,}(Q,\varphi)\Big(Q_{L\gamma}Q_{L\delta\rangle}\Big(\sqrt{2Q_R^2}+\tfrac{{g_3}^2}{2\pi}\Big) - \tfrac{{g_3}^2}{8\pi}\delta_{\gamma\delta\rangle}\Big) \\ & + \sum_{Q\in\Lambda^*_{2k-1,7}}{}' e^{-\frac{4\pi}{g_3^2}\sqrt{2Q_R^2}}G_{\alpha\beta,\gamma\delta}(g_3,Q_L,Q_R)\,. \end{aligned} \tag{27}$$

les trois premiers termes dans (27) correspondent respectivement à la contribution à deux boucles calculée en (2.37), la contribution à une boucle (2.29), et la contribution du point singulier où la surface de Riemann se factorise en deux surfaces de Riemann de genre un liées par un point. Cette dernière reproduit la contribution à l'ordre des arbres $\nabla^2(\nabla\phi)^4$, obtenue par réduction dimensionelle du couplage $\nabla^2 F^4$ en dix dimensions.

Les termes exponentiellement supprimés de la seconde ligne de (27) peuvent être interprétés comme des instantons de branes NS5 euclidiennes enroulées respectivement sur tous les T^6 possibles à l'intérieur du T^7 de compactification, des branes KK (6,1) enroulées avec toutes les fibres Taub-NUT S^1 possibles à l'intérieur du T^7 de compactification, et des monopôles H enroulés sur le T^7 tout entier. Leur expression précise peut être trouvée dans [BCHP3]. Bien que nous obtenions une expression précise de ces contributions, la méthode des orbites utilisée dans [BCHP3] manque plusieurs types de contributions exponentiellement supprimées ne dépendant pas des axions a. L'existence de ces termes est assurée par la contrainte différentielle de supersymétrie (3.117), car le coefficient du couplage $(\nabla\phi)^4$, F_{abcd}, apparaissant dans le membre droit contient des termes de type instantons anti-instantons indépendants des axions. Malheureusement, nos outils actuels ne nous permettent pas d'extraire ces contributions à partir de la méthode des orbites.

Pour finir, il est important de préciser que bien que les contributions perturbatives $G^{(2k-1,7)}_{ab}$ et $G^{(2k-1,7)}_{ab,cd}$ soient singulières sur des lieux géométriques de codimention 7 à l'intérieur de $\mathcal{M}_3$ aux points d'agrandissement du groupe de symétrie de jauge, le couplage exact composé des contributions perturbatives et non-perturbatives (3.118) est singulier sur des lieux géométriques de codimension 8 seulement.

Limite de faible couplage de cordes de type II compactifiées sur $K3 \times T^2$ Le développement des termes $\nabla^2 F^4$ et $\mathcal{R}^2 F^2$ exacts en quatre dimensions a été obtenu en section §5.3.1 de [BCHP3], et nous considérons maintenant la limite de faible couplage des cordes de type II. Rappelons que $S = T_{\rm A} = U_{\rm B}$, *i.e.* l'axiodilaton hétérotique correspond au module Kähler du 2-tore en type IIA, et à la structure complexe du 2-tore en type IIB, tandis que l'axiodilaton de type II $S_{\rm A} = S_{\rm B} = T$ correspond au module de Kähler hétérotique (3.106).

À large $T_{\rm B2}$, *i.e.* faible couplage de type II, le réseau se décompose similairement à (3.107), et le coefficient exact du couplage $\nabla^2 F^4$ a été obtenu section §5.3.1 de [BCHP3], après s'être séparé des termes en $\log R$

$$\begin{aligned}\widehat{G}^{(2k-2,6)}_{ab,cd\;{\rm NP}}(U_{\rm B},\varphi) = \;& \widehat{G}^{(2k-2,6)}_{ab,cd}(\varphi) - \frac{3}{4\pi}\delta_{\langle ab,}\delta_{cd\rangle}\Big(\frac{\hat{\mathcal{E}}_1(NU_{\rm B})+\hat{\mathcal{E}}_1(U_{\rm B})}{N+1}\Big)^2 \\ & -\frac{1}{4}\delta_{\langle ab,}\Big(\frac{N\hat{\mathcal{E}}_1(NU_{\rm B})-\hat{\mathcal{E}}_1(U_{\rm B})}{N^2-1}\widehat{G}^{(2k-2,6)}_{cd\rangle}(\varphi) + \frac{N\hat{\mathcal{E}}_1(U_{\rm B})-\hat{\mathcal{E}}_1(NU_{\rm B})}{N^2-1}\,{}^{\varsigma}\widehat{G}^{(2k-2,6)}_{cd\rangle}(\varphi)\Big)\;,\end{aligned} \tag{28}$$

où φ appartient à la grassmannienne paramétrant $\Lambda_{2k-2,6}$. Nous négligeons ici les corrections non-perturbatives et utilisons la décomposition de $\widehat{G}^{(2k-2,6)}_{ab,cd}(\varphi)$ pouvant être obtenue à partir de la section §5.3.1 de [BCHP3] en replaçant les modules par $R^2 = S_{\rm B2} = \frac{1}{g_s^2}$, et en dénotant par $\varphi = t$ les modules de K3 appartenant à la grassmannienne $G_{(2k-4,4)}$. Après avoir développé autour de $q = 6 + 2\epsilon$, on obtient

$$\begin{aligned}\widehat{G}^{(2k-2,6)}_{\alpha\beta,\gamma\delta}(\varphi) \sim \;& \frac{1}{g_s^{\,4}}\widehat{G}^{(2k-4,4)}_{\alpha\beta,\gamma\delta}(t) - \frac{3}{4\pi}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\Big(\frac{\hat{\mathcal{E}}_1(NT_{\rm B})+\hat{\mathcal{E}}_1(T_{\rm B})+\frac{12}{\pi}\log g_s}{N+1}\Big)^2 \\ -\frac{1}{4g_s^{\,2}}\delta_{\langle\alpha\beta,}\Big(& \frac{\frac{N\hat{\mathcal{E}}_1(NT_{\rm B})-\hat{\mathcal{E}}_1(T_{\rm B})}{N-1}+\frac{6}{\pi}\log g_s}{N+1}\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t) + \frac{\frac{N\hat{\mathcal{E}}_1(T_{\rm B})-\hat{\mathcal{E}}_1(NT_{\rm B})}{N-1}+\frac{6}{\pi}\log g_s}{N+1}\,{}^{\varsigma}\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t)\Big)\,.\end{aligned} \tag{29}$$

Pour calculer les termes en puissance de $\widehat{G}^{(2k-2,6)}_{ab}(\varphi)$, on peut développer autour de $q = 6 + 2\epsilon$ et négliger les contributions non-perturbatives. S'agissant de ${}^{\varsigma}\widehat{G}^{(2k-2,6)}_{ab}(\varphi)$, il peut être utile d'agir avec la dualité de Fricke sur le module de Kähler $T_{\rm B}$ pour obtenir $T_{\rm B} \to -\frac{1}{NT_{\rm B}}$, ainsi que sur le module de K3 t avec l'involution ς. En récupérant toutes ces contributions, nous obtenons le développement perturbatif complet du couplage $\nabla^2 F^4$ en quatre dimensions

$$\begin{aligned}\widehat{G}^{(2k-2,6)}_{\alpha\beta,\gamma\delta\;{\rm II}} = \;& \frac{1}{g_s^{\,4}}\widehat{G}^{(2k-4,4)}_{\alpha\beta,\gamma\delta}(t) \\ -\frac{1}{4(N+1)g_s^{\,2}}\delta_{\langle\alpha\beta,}\Big(& \Big(\frac{N\hat{\mathcal{E}}_1(NT_{\rm B})-\hat{\mathcal{E}}_1(T_{\rm B})+N\hat{\mathcal{E}}_1(NU_{\rm B})-\hat{\mathcal{E}}_1(U_{\rm B})}{N-1}+\frac{6}{\pi}\log g_s\Big)\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t) \\ & + \Big(\frac{N\hat{\mathcal{E}}_1(T_{\rm B})-\hat{\mathcal{E}}_1(NT_{\rm B})+N\hat{\mathcal{E}}_1(U_{\rm B})-\hat{\mathcal{E}}_1(NU_{\rm B})}{N-1}+\frac{6}{\pi}\log g_s\Big)\,{}^{\varsigma}\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t) \\ & -2N\delta_{\gamma\delta\rangle}\frac{(\hat{\mathcal{E}}_1(T_{\rm B})-\hat{\mathcal{E}}_1(NT_{\rm B}))(\hat{\mathcal{E}}_1(U_{\rm B})-\hat{\mathcal{E}}_1(NU_{\rm B}))}{N-1}\Big) \\ & -\frac{3}{4\pi}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\Big(\frac{\hat{\mathcal{E}}_1(NT_{\rm B})+\hat{\mathcal{E}}_1(T_{\rm B})+\hat{\mathcal{E}}_1(NU_{\rm B})+\hat{\mathcal{E}}_1(U_{\rm B})+\frac{12}{\pi}\log g_s}{N+1}\Big)^2\,.\end{aligned} \tag{30}$$

Le résultat (30) est manifestement invariant sous l'échange de $U_{\rm B}$ et $T_{\rm B}$, et est ainsi identique dans les cordes de type IIA et type IIB. Il est également invariant sous les dualités de Fricke combinées $T_{\rm B} \to -\frac{1}{NT_{\rm B}}$, $U_{\rm B} \to -\frac{1}{NU_{\rm B}}$, $t \to \varsigma t$ [105], ceci étant vrai par construction de la proposition (3.118).

La limite $N = 1$ de ce cas est légèrement subtile. Le résultat (30) doit être remplacé par

$$G^{(22,6)}_{\alpha\beta,\gamma\delta\ {\rm II}} = \frac{1}{{g_s}^4}\widehat{G}^{(20,4)}_{\alpha\beta,\gamma\delta}(t) + \frac{3}{4\pi {g_s}^2}\delta_{\langle\alpha\beta,}\Big(\log(T_{\rm B2}|\eta(T_{\rm B})|^4) + \log(U_{\rm B2}|\eta(U_{\rm B})|^4) - 2\log g_s\Big) G^{(20,4)}_{\gamma\delta\rangle}(t) - \frac{27}{4\pi^3}\delta_{\langle\alpha\beta}\delta_{\gamma\delta\rangle}\Big(\log(T_{\rm B2}|\eta(T_{\rm B})|^4) + \log(U_{\rm B2}|\eta(U_{\rm B})|^4) - 2\log g_s\Big)^2 . \tag{31}$$

Il serait intéressant de vérifier ces prédictions par un calcul perturbatif explicite en cordes de type II. Pour simplifier le résultat, on peut utiliser les relations

$$\frac{\hat{\mathcal{E}}_1(NT_{\rm B}) + \hat{\mathcal{E}}_1(T_{\rm B})}{N+1} = -\frac{1}{4\pi}\log(T^k_{\rm B2}|\Delta_k(T_{\rm B})|) , \qquad \hat{\mathcal{E}}_1(T_{\rm B}) = -\frac{1}{4\pi}\log(T^{12}_{\rm B2}|\Delta(T_{\rm B})|) , \tag{32}$$

afin de réécrire la contribution à deux boucles dans la dernière ligne de (30) as

$$-\frac{3}{(4\pi)^3}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\big(\log(g_s^{-2k}T^k_{\rm B2}U^k_{\rm B2}|\Delta_k(T_{\rm B})\Delta_k(U_{\rm B})|^2)\big)^2 . \tag{33}$$

0.4 Compter les microétats de trou noir avec des instantons

Dans le chapitre anglophone 4, nous faisons état des résultats de [BCHP1], [BCHP3] et illustrons leur application au comptage de microétats de trous noirs quart-BPS en supergravité $\mathcal{N} = 4$.

Ces trous noirs sont invariants sous certaines transformations de supersymétrie et leur masse sature l'inégalité de Bogomol'nyi (3.37). Ils sont donc extrémaux et n'émettent pas de radiation de Hawking. En vertu de ce fait, ce sont des objets stables et stationnaires, c'est-à-dire des solitons. Les microétats correspondants ont été étudiés à faible couplage, où les effets gravitationnels induits par le système peuvent être ignorés, et les résultats furent ensuite prolongés à fort couplage, où le système peut être décrit comme un trou noir. L'entropie de ces objets a notamment la particularité d'être inaffectée par les variations du couplage gravitationnel [25]. Par ailleurs, leur stabilité nous permet aisément de comprendre la dynamique des configurations microscopiques correspondantes, et celle-ci implique de nombreux objets de théorie des cordes, enroulés ou étendus dans des directions compactes de la variété de compactification, comme décrit dans le paragraph anglophone §4.2. Dans le régime de grands trous noirs, une découverte historique fut de découvrir que l'entropie de certains trous noirs en cinq dimensions d'espace-temps satisfait [25, 134, 135, 136]

$$S_{BH}(Q,P) = S_{stat}(Q,P) , \tag{34}$$

où $S_{HB}(Q,P)$ est l'entropie de Bekenstein-Hawking d'un trou noir extrémal de charge (Q,P), et $S_{stat}(Q,P)$ correspond à l'entropie statistique obtenue par comptage des microétats de charge (Q,P)

$$S_{stat} = \ln d(Q,P) . \tag{35}$$

Cette formule de Bekenstein-Hawking (1.8) reste valide si la taille de l'horizon est grande comparée à la courbure de l'espace-temps et d'autres intensité de champ à l'horizon, *i.e.* , pour de grandes charges. Dans ce régime, la taille de l'horizon est assez large pour que l'intensité de la courbure de l'espace-temps et des champs de jauge soit petite devant l'horizon. Dans un régime où ce n'est plus vrai, il faut alors se soucier des corrections de plus haute dérivée à l'action effective dans la limite de basse énergie [26, 137, 138, 139].

D'un autre côté, la limite des grandes charges simplifie également les calculs statistiques. Dans cette approche, un trou noir extrémal correspond à un état de la théorie conforme à large valeur propre de L_0 et valeur propre nulle de $\bar{L}_0$ (ou inversement). Pour $\bar{L}_0 = 0$, par exemple, on peut calculer la dégénérescence de cet état en utilisant la formule de Cardy en termes de la charge centrale du secteur gauche (left) c_L de la théorie conforme

$$S_{stat}(Q) \simeq 2\pi\sqrt{\frac{c_L L_0}{6}}\,, \tag{36}$$

où c_L est proportionelle à un produit de charges physiques du trou noir [25]. On trouve ainsi une concordance parfaite dans cette limite entre les deux calculs (4.1).

Dans le cas de supergravités $\mathcal{N} = 4$ pouvant être réalisées comme des modèles CHL avec orbifold $\mathbb{Z}_N$, ce résultat a été obtenu pour des trous noirs en quatre dimensions par [140, 141].[14] Le chapitre anglophone 4 est donc dévoué à la démonstration de la concordance des résultats de [BCHP3], et nous montrons en particulier comment obtenir la dégénérescence des trous noirs quart-BPS de supergravité $\mathcal{N} = 4$ à partir du calcul de l'interaction exacte $\nabla^2(\nabla\phi)^4$ en théorie des cordes en trois dimensions.

Dans le paragraphe anglophone §4.1, nous donnons une rapide description du formalisme d'entropie pour les trous noirs stationnaires en quatre dimensions[139, 142, 143, 124]. La fonction d'entropie est obtenue en tant que valeur extremum d'une fonctionnelle de la densité lagrangienne, ce qui par ailleurs nous assure qu'elle reste indépendante de la valeur asymptotique des modules à l'infinie [138, 142].

Dans §4.2, nous rappelons la célèbre formule de Dijkgraaf-Verlinde-Verlinde [140] dans le cas des modèles CHL [141, 144, 145].

Dans §4.3, nous rappelons le formalisme de base utilisé pour décrire les solutions de trou noir quart-BPS en supergravité $\mathcal{N} = 4$ [146].

Finalement, dans §4.4 nous rapportons les résultats de [BCHP1], [BCHP3], où les contributions instantoniques quart-BPS dans la limite de décompactification de $G^{(2k,8)}_{ab,cd}$ ont été utilisées pour prédire la dégénérescence des solutions de trou noir quart-BPS. Ces résultats concordent avec les prédictions présentées en amont §4.2 [140, 141, 144, 145] et les étendent à d'autres type de trous noirs quart-BPS, tout en déclinant correctement la prescription de contour proposée dans [147, 148].

0.5 Questions ouvertes

L'un des buts de ce manuscrit est de présenter de manière simplifiée et cohérente certains des résultats obtenus lors de ce travail de thèse de trois années. Beaucoup de questions restent cependant ouvertes. Elles sont présentées en anglais dans le chapitre §5.

[14] Cela fut originalement calculé dans la description de type II.

Chapter 1

Introduction

We use throughout this paper the standard system of conventions where $\hbar = c = 1$, which means that one unit of time equals one unit of length equals the inverse of a unit of energy. Our space-time signature is $(-, +, \ldots, +)$.

1.1 The ultraviolet catastrophe of quantum gravity

For long time, quantum gravity has been one of the main focus of modern theoretical physics. Gravity, as Newton and Einstein expressed it, is missing a microscopic description. Having a complete theory was considered to be a big concern to account for phenomena in regions of spacetime where the gravity force becomes extremely strong, like black holes or the 'origin' of spacetime. Beyond practical applications, this problem is often stated as being the missing puzzle piece between quantum theories describing particle physics, quantum fields theories, and the classical theory describing the dynamics of spacetime, Eistein's general relativity.

The first part of this picture, quantum field theory, was studied after the seminal computation by Hans Bethe [1] of the Lamb-Retherford shift [2], in 1947, explaining a color difference between two types of hydrogen atoms[1] which was unpredicted by the Dirac equation, the guiding theory at the time. This computation was then enhanced and developped in a more general framework called quantum theory of electrodynamism by Feynman, Schwinger, Stuekelberg, Tomonaga and Dyson. This computation used the fundamental idea – that we will use later on in the context of black holes – of 'vacuum polarisation', namely, that pairs of particles and anti-particles are constantly populating the vacuum at a microscopic level. This picture was completed in the late 60's, extending the construction to the weak and strong forces – all the forces known today except gravity – by many important physicists such as Glashow, Salam, Weinberg and Gell-Mann.

On the second side of this picture stands general relativity, a non-quantum theory of gravitational interactions elaborated by Einstein in 1915 that describes the dynamics of spacetime itself. Einstein's equation illustrates the two sides of the problem elegantly

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \ . \tag{1.1}$$

[1]The first being excited to the $2S_{1/2}$ orbital, the second to the $2P_{1/2}$.

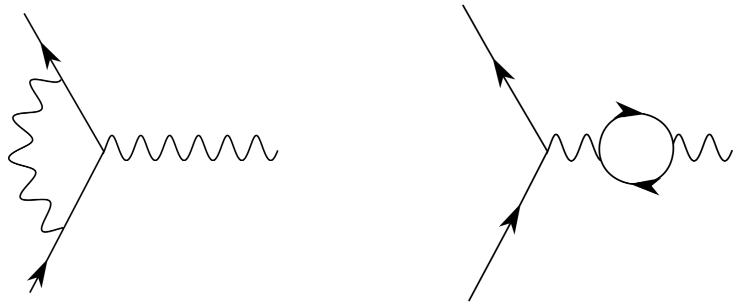


Fig. 1.1 *Diagrams of the most prominent corrections to quantum mechanics when describing an electron (straight line) interacting with the hydrogen atom through the emission of a photon (wiggly line). In 1.1a, the electron emits a photon before interacting and reabsorbs it after the interaction, while in 1.1b, the exchanged photon experiences the creation of a particle-antiparticle pair.*

The left hand side of (1.1) describes a non-quantised geometry of spacetime, while its right hand side describes quantised matter and energy contained in spacetime. This paradox did not get pursued extensively at the time since macroscopic physics (as opposed to quantum physics) was sufficient to describe all the observed objects in the interstellar medium around us.

The quantum nature of spacetime is expected to be observable at the Planck scale, $M_{Pl}^2 = \frac{\hbar c}{G} \sim 10^{28}\,\text{eV}$, where gravitational and quantum effects are comparable. This energy range is far away from the reach of any human-made collider – which is currently at 10^{13} eV, but still many phenomenological and effective theory studies have been inspired by quantum gravity research [3, 4].

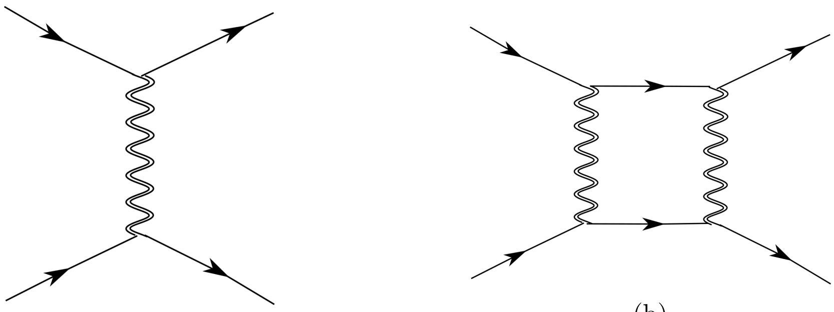
Renormalisability and unitarity. Naive quantisation of gravity is known to fail because of non-renormalisability and unitarity violation, or, in other words, its inability to be tracked down at arbitrary small scales and have a consistent quantum interpretation. The study of gravity divergences and its non-renormalisability dates back to 't Hooft and Veltman [5] in 1974. The same year, Llewellyn-Smith proposed that non-renormalisibility of a quantum field theory was equivalent to unitarity violation at the classical level [6], and that the growth of scattering processes in energy could be used as a criterium. In the case of gravity, the linearised Einstein-Hilbert action in D dimensions with a single dimensionless scalar field writes, symbolically,

$$S_{EH} = \int d^D x \left(\frac{1}{2} \partial h \partial h (1 + 2\kappa h) + \frac{1}{2G} \partial \phi \partial \phi (1 + \kappa h) \right) + \ldots , \tag{1.2}$$

where we linearised the curved metric around the Minkowski metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \ , \tag{1.3}$$

with $\kappa^2 = 32\pi G$ and $G = M_{Pl}^{2-D}$ the gravity coupling constant, and with $h_{\mu\nu}$ the graviton field. The second and fourth term in this expansion indicate that a graviton interacts with another graviton and any other matter field with a double derivative term. Thus, when considering the event of two massless fields exchanging a single graviton, as shown figure 1.2a, one obtains an amplitude proportional to E^2/κ, where E is the energy of the process, with E^4/κ^2 coming from the vertices and κ/E^2 from the external legs. This violates unitarity for processes beyond the Planck scale, i.e. when $E \gg \kappa$.[2] The rule proposed by Llewellyn-Smith follows from the fact that any process of this type will arise with a UV divergence when two gravitons get exchanged, as in the figure 1.2b. In pure gravity, these divergences occur at two loops [8, 9]. These issues will persist for all loops



(a) (b)

Fig. 1.2 *Classical and one-loop interaction of two massless fields through gravity (doubled wiggly lines)*

using the same reasoning, as long as $D > 2$. This can be seen as a consequence of the positive mass dimension of the gravity coupling $1/G$.[3] This infinite number of divergences must be compensated by introducing as many counter-terms or arbitrariness in the theory, and therefore makes it ill-defined.

Scattering amplitudes could also induce a breaking of unitarity for another very subtle reason: evaporating black holes. Although this is not the direction this manuscript will take, let us mention this point of on-going debate and introduce some basic concepts about black holes. In 1975, Hawking argued that vacuum polarisation, in the region near a black hole's horizon, would cause particle emission [10]. Since this vacuum polarisation behaves as a purely thermal fluctuation, the emitted radiation cannot contain any information, and in particuler not the one that "felt" in the black hole, which leads to another unitarity infringement.

Large black holes are not perturbative objects. However, small black holes have a non-zero probability to be created in an energetic collision process, also known as trans-Planckian process [11, 12, 13, 14, 15, 16]. Black hole physics is thus very relevant at high energy, and this would implies another type of perturbative inconsistency.

In the 80's, an argument called complementarity developped by Susskind et al. [17], suggested that information infalling the black hole could be located both inside and outside the black hole. Namely, from the point of view of asymptotic observers, time

[2]More precisely, for a theory of gravity coupled to N_s scalars, N_f fermions and N_V vectors, one expects unitarity to be violated at energy $E_{CM}^2 = \frac{60}{\kappa^2(2N_s+3N_f+12N_V)}$ [7]

[3]In naive dimensional analysis, we say that the critical dimension of the gravitational coupling constant is $D = 2$, and thus, it is non-renormalisable for $D > 2$

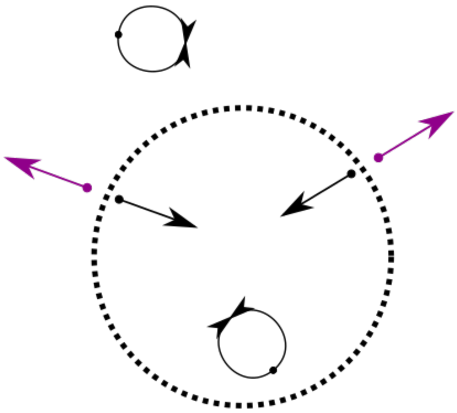


Fig. 1.3 *Schematical representation of radiation emission by a black hole. A pair of particle-antiparticle spontaneously emerges near the horizon (dotted circle), one fall inside the black hole while the other goes outward. Both particles and antiparticles are symbolised indistinguishly by an arrow.*

intervals near the horizon become arbitrary dilated while space intervals in the falling direction become arbitrarily thin, which prevents these observers from seeing the matter passing through the black hole horizon in a finite time. Infalling objects spend an arbitrary long time around the black hole's horizon, and information can be radiated away through Hawking's radiation. On the other hand, an observer within the black hole can see the matter entering in a finite time, but this one can never communicate with the exterior, which seems to prevent a paradox.

A very subtle point that was learnt from this debate is that, in the classical approximation, information must be stored on the black hole's surface, i.e. in a two-dimensional space, with gravity playing no dynamical role. This is in contrast with the three-dimensional interior of the black hole, where gravity is of course central. This led to the popularisation of the notion of holography, that was first initiated by 't Hooft and developped in the early 90's [18, 17], when, in 1997, Maldacena made a precise statement out of the idea above, by conjecturing that string theory under certain circumstances – when understood as a quantum theory of gravity – is equivalent to a quantum field theory without gravity in a spacetime with one space dimension less [19, 20, 21]. This latter argument has been considered to be almost a proof that decay of small black holes[4] was consistent with unitarity: if the thermalisation process is described by a quantum field theory without gravity, it must be a unitary process by definition.

Although locally, the evaporation process is unitary, the decay of larger black holes is still puzzling at present.[5] In 2010, Almheiri, Marolf, Polchinski, and Sully found a self-contradiction in the complementarity argument while studying this evaporation process under certain circumstances [22]. The thermal radiation becomes problematic after a certain time, because it cannot be maximally correlated with both the radiation inside the black hole – which is assumed to maintain spacetime regularity at the horizon – and

[4] Smaller in size than the length of the *AdS* spacetime, such that it is similar to asymptotically flat black holes.

[5] Large black holes do not evaporate entirely in the holographic picture.

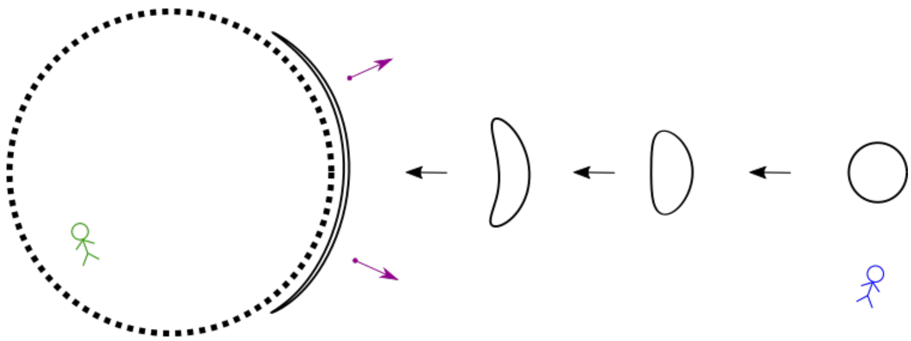


Fig. 1.4 *Schematic representation of an object falling into a black hole from an outsider's perpective (in blue) : the spacetime geometry deforms infinitely the shape of the object to a 2D surface, as well as the time needed to go through the horizon. From the outsider's perspective, the infalling objects accumulate in a sphere around the surface horizon.*

the past radiation outside – which is assumed to preserve unitarity. This paradox does not seem to be explicable by arguments from fundamental theory of gravity, like string theory, and goes under the name of "firewall paradox".[6]

All the modern issues about renormalisibility and unitarity have generated different programs in the realm of research in quantum gravity. The two most developped paradigms differ on how they approach quantisation. On the one hand, gravity is believed to be fully non-perturbative theory, and above-mentioned issues are artefacts of the perturbative treatment of quantisation as (1.3). This is the philosophy of Loop Quantum Gravity, where one uses the notion of loops in spaces to measure its curvature, and quantises these so-called loop variables. On the other hand, classical gravity is believed to be the low-energy limit of a more fundamental theory, and one quantises the metric perturbatively around flat spacetime theories with richer and more symmetric matter content, like a supergravity or a string theory. This is the direction followed in this manuscript.

UV divergences. In quantum field theories, ultraviolet divergences are of central importance when considering effective theories because they point out our lack of understanding of the ultraviolet behavior: identification of the wrong degrees of freedom. They will be used in section 1.2, as well as in chapter 3 to analyse the results of [BCHP1], [BCHP2].

As we will be interested in theories coupled to matter, and in particular, gauge vectors, we will focus on interactions in the low-energy effective action of the type of (1.2) from higher order scattering events of n_1-gravitons and n_2-photons. These interactions must respect symmetries of the theory, in particular diffeomorphism and gauge invariance. These symmetries force interactions to be formulated in terms of the Riemann tensor $\mathcal{R}_{\mu\nu\alpha\beta}$ and the electromagnetic field strength tensor $F_{\mu\nu}$, and derivative of these. For n-point amplitudes, with $n = n_1 + n_2$, the possible effective couplings are

$$\nabla^{m_1}\mathcal{R}^{n_1}, \quad \nabla^{m_2}F^{n_2}, \quad \nabla^{m_1}\mathcal{R}^{n_1}\nabla^{m_2}F^{n_2} \tag{1.4}$$

[6] Local irregularity of spacetime at the horizon was thought to produce a dramatic 'firewall'.

where ∇ is the covariant derivative and $m_i \geq 0$. These operators have mass dimension

$$[\nabla^{m_1}\mathcal{R}^{n_1}\nabla^{m_2}F^{n_2}] = M^{m_1+m_2+2n_1+n_2} \,. \tag{1.5}$$

The superficial degree of divergence provides an upper bound on the possible divergences that one might have to cancel. It is given by counting the degree in momentum of the most divergent graphs. In the case of graviton and photon interactions, the most divergent graph at leading order in the gravity coupling constant is obtained by concatenating 3-valent vertices with two powers of the momentum each (associated to the double derivatives). The naive superficial behavior of a $L-$loop n_1-graviton and n_2-photon amplitude is[7]

$$M_n^{L-\text{loop}} = \nabla^{m_1}\mathcal{R}^{n_1}\nabla^{m_2}F^{m_2} \times \Lambda^{L(D-2)+2-m_1+m_2-2n_1-n_2} \,. \tag{1.6}$$

This simple calculation will allow us to compare the superficial degree of divergence with explicit computation presented in the section 1.2.

Black holes and entropy. Theories coupled to gravity typically have black hole solutions, and describing their entropy – that we now introduce – will be one of the central points of [BCHP1], [BCHP3].

They are caracterised by an hypothetical surface, the event horizon, that has the special property of being lightlike: any object at the surface would either fall inward if traveling slower than light, or remain tangent to the surface if traveling at the speed of light. Objects traveling faster than light could in principle escape from the black hole's pull, but this behavior would break causality and is thus forbidden in any sensible theory. However, quantum effects, as the vacuum polarisation mentioned above, allow some particles to escape their fate (see figure 1.3). Pairs of particles-antiparticles separate at the horizon, causing black holes to emit black body radiation of finite temperature, called the Hawking temperature [10, 23]

$$T = \frac{\hbar\kappa}{2\pi} \,, \tag{1.7}$$

where κ is the so-called surface gravity, which measures the strength of gravity at the horizon. Furthermore, they behave as thermodynamic systems characterised by their temperature and other *state quantities*:[8] the no-hair theorem states that black holes can solely be described by their mass, charge and spin, implying that their internal energy can be seen as a state function.[9] The analogy with the state function of thermodynamical system has been used to identify [24]

$$S_{BH} = \frac{A}{4\hbar G} \,, \tag{1.8}$$

the so-called Bekenstein-Hawking *entropy*[10]. In §4.1, we present how to compute the entropy of some specific class of black hole solutions, taking into account higher derivative correction in the effective action.

[7] One can make use of the Euler formula for connected graphs with V vertices, I internal legs and L loops $V - I + L = 1$. Each vertex (internal line) will add a factor k^2 in the numerator (denominator).

[8] Temperature, volume, pressure and other non-vanishing macroscopic quantities that are only defined at thermodynamic equilibrium of the system.

[9] By definition, a state function only depends on the state quantities of the system.

[10] Stellar-size black holes, for instance, have very small temperature $T \sim 10^{-7}K$, and very large entropy $S = 10^{77}$.

This result is quite peculiar, since entropy of extended objects is expected to scale as their volume, but it seems reasonable given the arbitrary large spacetime stretching happening near the event horizon, as advocated figure 1.4. However, if this analogy is correct, can a quantum theory of gravity provide an understanding of this entropy from a statistical viewpoint ? Indeed, the entropy of thermodynamical systems is known to satisfy

$$S = k_B \ln \Omega \,, \tag{1.9}$$

where Ω is the number of possible microscopic states underlying a given thermodynamical state. In [BCHP1], [BCHP3], we focused our attention on a specific class of objects where this question has been answered in the affirmative, a particular subclass of extremal black holes [25]. In §4.2, we present a specific setup providing a statistical understanding of black hole microstates, where it has been shown to match with high accuracy black hole's classical entropy.

Thermodynamic variables of extremal black holes saturate a bound corresponding to states with zero temperature. We will focus on the case of stationary spherical black holes with electromagnetic charge – called the Reissner-Nordström solution – where the charge bound is $M^2 \geq Q^2 + P^2$ with electric and magnetic charges Q and P. These extremal black holes thus have $M = \sqrt{Q^2 + P^2}$, and their Bekenstein-Hawking entropy (1.8) can be obtained from Einstein-Maxwell theory in four dimensions

$$S_{BH}(Q, P) = \pi(Q^2 + P^2) \,. \tag{1.10}$$

It is important to note that (1.10), even as a classical approximation, does not depend on any parameter of the theory. In any diffeomorphism invariant theory, the black hole entropy is the integral of the Noether charge over the event horizon [26], and it is thus always invariant under any non-singular field redefinition. For extremal black holes, such as (1.10), it is also independent of the asymptotic values of the fields parametrising the metric, which is a generalisation of the attractor mechanism for black holes in supergravity theories [27, 28]. This latter fact can be easily understood by writing the entropy function as the extremum value of a functional of the Lagrangian density, as we come back to in §4.1.

In some specific theories, where the coupling constants corresponds to the asymptotic value of a dynamical field, like a string theory, it implies that the entropy of the system does not change as we vary coupling constants from a sufficiently large value where the black hole description is valid, to a sufficiently low value where the microscopic description can be handled with the current technical tools.

In §4.1 and §4.3, we introduce the black holes of interest in more details, which are spherical solutions of a four-dimensional supergravity, and we present in §4.2 how their microstates have been counted in a string theory description. In §4.4, we present how the calculations in [BCHP3] can recover and extend the results presented in §4.2.

1.2 Supergravities

They constitute an attempt at modifying the UV behavior of Einstein's gravity, and are the low-energy limits of superstring theories considered in the rest of this manuscript. The usual spacetime is embedded in a higher-dimensional space called *superspace*, where the coordinates of the new dimensions are labelled by anticommuting Grassmann numbers, the so-called "fermionic" degrees of freedom.

The new symmetry of this geometry is the usual Poincaré group (translations, rotations, boosts) extended by local anti-commuting generators called *supercharges*. These anti-commuting symmetries also constrain the matter content of the field theorie, leading to drastic dynamical simplifications (see table 1.1 hereafter). This extension is the only instance where spacetime symmetries mix non-trivially with internal symmetries of the quantum field theory, which goes against in spirit to the Coleman-Mandula theorem [29].

Theories are usually classified by the number of supercharges : from 4 in four dimensions to 32 for the maximal extension, which is defined in any dimension up to $D = 11$ [30, 31]. In the following, we shall refer to the number $\mathcal{N}$ of four-dimensional supercharges only when we restrict ourselves to four dimensions, i.e. $\mathcal{N} = 8$ and $\mathcal{N} = 4$ correspond respectively to supergravity and half-maximal supergravity in four dimensions. Half-maximal supergravities can be separated into two types of constructions, the $(2, 2)$ and $(4, 0)$.[11] The latter will be the one of interest in this manuscript. In particular, in §3.1.1 we present the dimensional reduction of half-maximal supergravity from ten to four dimensions, and exhibit how the gauge and moduli fields can be arranged in representations of the global symmetry groups listed table 1.1.

	$s=2$	$s=\frac{3}{2}$	$s=1$	$s=1/2$	$s=0$		
$\mathcal{N}=8$	1	8	28	56	70	E_7	
$\mathcal{N}=4$	1	4	6	4	2	$SO(6)\times SL(2,\mathbb{R})$	
	1	4	6+2	4+8	2+12	$SO(2,6)\times SL(2,\mathbb{R})$	
	1	4	6+4	4+16	2+24	$SO(4,6)\times SL(2,\mathbb{R})$	N=7
	1	4	6+6	4+24	2+36	$SO(6,6)\times SL(2,\mathbb{R})$	N=5
	1	4	6+10	4+40	2+60	$SO(10,6)\times SL(2,\mathbb{R})$	N=3
	1	4	6+14	4+56	2+84	$SO(14,6)\times SL(2,\mathbb{R})$	N=2
	1	4	6+22	4+88	2+132	$SO(22,6)\times SL(2,\mathbb{R})$	N=1

Table 1.1: Spin content of the massless supersymmetry representation with maximal spin 2 in four dimensions of some $\mathcal{N} = 8$ and $\mathcal{N} = 4$ supergravities. The size of representations with decreasing spin are fixed from the highest by supersymmetry. The first two rows correspond to the pure supergravities and the ones below are coupled to a given number of vector multiplets preserving the non-perturbative $SL(2, \mathbb{R})$ global symmetry. The two last columns correspond to the expected global symmetry of the massles spectrum, and the orbifold parameter of the corresponding $\mathbb{Z}_N$ CHL string theory.

The supergravity theories with large supersymmetry extension, some of which being listed in table 1.1, have much richer spectrum than Einstein's supergravity (which, for comparison, can be restricted to a single spin $s = 2$ field). However, this complexity is reduced at the level of the effective action and scattering amplitudes, mainly because of the geometric symmetries and supersymmetries of the theory. Part of this manuscript will be aimed at understanding simplifications induced by the geometric symmetries, as well as the constraints imposed by supersymmetries on the scattering amplitudes, as reviewed §3.3 and §3.4.

[11] This notation refer to the sigma model, not to confuse with the spacetime supersymmetry.

These supergravities were studied in depth in the late 70's and 80's, giving the consensus opinion that supersymmetry alone wouldn't be sufficient to eradicate non-renormalisable divergences completely – although they would start to appear at a larger number of quantum loops than in Einstein gravity. However, these theories have focused a resurgence of interest in the recent years, and direct calculations by the leading experts have shown that $\mathcal{N} = 8$ supergravity is finite in spacetime dimensions $D < 4 + 6/L$ for $L = 2, 3, 4$ loops ($\mathcal{R}^4$, $D^2\mathcal{R}^4$) [32, 33], showing an impressive concordance with the UV behavior of a purely gauge theory, $\mathcal{N} = 4$ super-Yang-Mills, which is UV-finite in four-dimension. However, duality symmetry analysis [34, 35, 36, 37, 38] have predicted an abrupt change in the critical behavior of $\mathcal{N} = 8$ supergravity at $L = 5$ loops, due to an allowed $D^8\mathcal{R}^4$ counterterm. Non-vanishing of this counterterm at five loops would indicate that supergravity is finite only for $D < 2+14/L$, predicting a non-renormalisable divergence in four dimensions at seven loops and beyond. Unfortunately, the presence of $D^8\mathcal{R}^4$ at five loops was recently confirmed by a long-awaited computation [39].

In this manuscript, we will disregard the case of $\mathcal{N} = 8$ and focus on the $\mathcal{N} = 4$ supergravity theories presented in the table 1.1. Because of their reduced supersymmetry, they allow coupling to matter multiplets [40] which trigger more divergences. Their richer structure also allows them to be realised as the low energy limit of various string models, and in particular, the heterotic string on T^6 with $\mathbb{Z}_N$ orbifold that we describe in chapter 2.

1.3 String theory

String theories have a very long and interesting history, and will be the main focus of the rest of the manuscript. It was initially developped around 1968, through the Veneziano amplitude and later the Virasoro-Shapiro amplitude

$$M^{\text{VS}}(s,t,u) = \frac{\Gamma(-1-\alpha's/4)\Gamma(-1-\alpha't/4)\Gamma(-1-\alpha'u/4)}{\Gamma(-2-\alpha't/4-\alpha'u/4)\Gamma(-2-\alpha's/4-\alpha'u/4)\Gamma(-2-\alpha's/4-\alpha't/4)}\,, \tag{1.11}$$

to give an account of the observed spectrum in hadronic matter caused by strong interactions [41, 42]. In (1.11), s, t and u are the usual kinematic Mandelstam invariants defined by the four incoming momenta, respectively, $-(k_1+k_2)^2$, $-(k_1+k_4)^2$ and $-(k_1+k_3)$, and α' was called the Regge slope. These amplitudes were later understood to describe interactions of open and closed strings of size $\ell_s = \sqrt{\alpha'}$ and tension $T = 1/(2\pi\alpha')$,[12] while a massless particle of spin two was identified as a possible graviton candidate in the closed string sector [43]. Quantisation, Lorentz invariance and unitarity constraints impose that these strings must propagate in a 26-dimensional spacetime[13]. Moreover, their spectrum contains an infinite tower of states caused by the oscillations running onto the string, with quantised mass and spin given by

$$m^2 \propto \frac{n}{\alpha'}, \qquad J \leq \alpha' m^2 + 1, \qquad n = -1,\, 0,\, 1, \ldots, +\infty\;. \tag{1.12}$$

[12] Note that the Virasoro-Shapiro amplitude is invariant under exchanges of s ,t and u, as expected for a tree-level closed string interaction, while the Veneziano amplitude, describing open string interactions, is only invariant under exchange of s and t

[13] This latter constraint protects unitarity in the sense that it eliminates negative normed states created by X^μ.

This candidate as a quantum theory of gravity was more attractive than usual field theories because of the finiteness of all amplitudes. Such fortunate phenomenon is due to the string length being finite, and hence the interaction points being non-local as we will discuss later. From a far away perspective, strings behave like pointlike particles that can join and split like in the Feynman diagrams in figure 1.1 and 1.2, but their finite length gives their trajectory in spacetime, their worldline, a one-dimensional 'thickness' named *worldsheet*. This embedded two-dimensional worldsheet hosts a conformal field theory (CFT), whose symmetries give an alternative understanding of the critical dimension $D = 26$ [44], but are responsible for obtaining Einstein's equations of motion for the graviton (1.1), with corrections at all orders in α'. Both theories of open and closed strings contain a problematic tachyon (the state with $n = -1$, of imaginary mass), which would spoil the theory by breaking causality. A theory of superstrings was later elaborated, and it was proposed by Gliozzi, Scherk and Olive to project out this tachyonic state of the spectrum [45]. Superstrings live in a 10-dimensional spacetime and were shown to possess in its massless sector the content of maximal supergravity spectrum [46] that was mentioned in the previous section.

It has been since long a topic of research to inquire whether maximal supergravity necessitates to be completed to a string theory to exhibit a safe UV behavior and unitarity. Indeed, string theory amplitudes show in the high energy limit a soft behavior compatible with unitary. Using the kinematic Madelstam invariant s, t and u in a 4-point scattering event, the hard scattering limit – $s, t \to +\infty$, fixed angle – at tree level behaves as

$$M^{VS}(s,t) \sim \exp\Big(-\frac{\alpha'}{2}(s\ln s + t\ln t + u\ln u)\Big) , \tag{1.13}$$

where the soft behavior at large momenta, the same responsible for the mismatch between string and strong interactions, can be attributed to the infinite tower of massive states.

On the other hand, the problem of UV divergences is rather solved by the finiteness of the string length. In string theory, the spacetime phenomena have a minimum size : the string length $\sqrt{\alpha'}$. At loop amplitude, this can be understood by noticing that the usual UV divergent part (large momenta, or small distances) is absent at all genera. This is a general consequence of modular invariance of integrand : for tori and Riemann surfaces of higher genera, the UV divergent region is absent of the parametrisation of the amplitude.

Low energy limit of string theory gives rise to gravity coupled to matter fields. These theory must then have black hole solutions, and thus constitute a framework for studying classical and quantum properties of black holes.

Heterotic strings Within the landscape of possible string theoretic constructions, one has been particularly studied for phenomenological purposes, despite its peculiarity. The heterotic strings [47, 48] have an asymmetry between the left-moving sector, which is purely bosonic, and the right-moving sector, which is supersymmetric, where left- and right- designate the direction of the oscillation along the strings. Their critical dimension is ten, as for superstring theories, but sixteen 'extra' left-movers are required by the critical dimension 26 of a bosonic string mentioned in §1.3. Since no string boundary condition can be consistent with this peculiar asymmetry between the right- and left-moving sector, heterotic strings can only be closed strings. The sixteen 'extra' direction of one sector must be compactified, and windings and momenta along the 16 compact directions are counted by a sixteen-dimensional even self-dual lattice, *i.e.* the lattices of either $E_8 \times E_8$ or $SO(32)$ (where E_8 is one of the exceptional groups). The two possible gauge groups resulting from the massless sector of these 'extra' dimension turn out to be different models are sometimes being refered to as *HE* and *HO* respectively. It is through this construction that we will continue most of the discussion, although some of our results in [BCHP2] and [BCHP3] can be interpreted in other string constructions.

Non-perturbative dualities in string theories. Although superstring theory is the only candidate for a renormalisable quantum theory of gravity, a lack of interest was noticeable in its early days, notably because of the existence of many possible realisation of string theories (called Type I, Type IIA and Type IIB). None of them seemed to be in compatible with Nature : type II theories only had abelian gauge group, unlike the electroweak and strong forces, and in type I the gauge group for super-Yang-Mills was thought to be arbitrary. However, in 1985, it was learned that the possible gauge groups were restricted by the absence of anomalies, and the heterotic string was discovered. The latter was phenomenologically attractive and became popular for its aptitude to produce grand unified theories starting from the exceptional subgroup E_6.

Although this discoveries attracted many researchers into the field, it became soon clear that understanding non-perturbative effects of string theories was crucial to produce a grand unified model of nature. More precisely, four-dimensional physics depends crucially on the type of compactification which is used to reduce from ten to four dimensions. Numerous symmetries relating different compactifications are known, they are named T-duality, but there remains a large class of compactifications which are not related any any way. In principle, the type of compactification by the dynamics at very high energy, however, the selection of the correct compactification scheme requires non-perturbative information on the potential over the landscape of string vacua.

In parallel, it has been understood that many non-perturbative features of some specific four-dimensional gauge theories could be understood without performing an explicit instanton calculation. This is made possible by the presence of a so-called non-perturbative symmetry, *i.e.* highly non-trivial symmetry between the weak and strong coupling effects. As we mention in §1.4, such symmetry was conjectured in the Georgi-Glashow model in 1977 [49], but was received with skepticism until convincing evidence was presented for the case of a $\mathcal{N} = 2$ extension [50]. This strong-weak coupling symmetry, dubbed S-duality, relates two phases of a super-Yang-Mills theory,[14] one at large value of the coupling with the other at small value of the coupling. If the theory is superconformal, it cannot be in either of two aforementioned phases and the coupling is fixed by self-duality under S-duality.[15]

These S-dualities symmetries were at the same time conjectured to be present as *self*-symmetries of various string theories such as the heterotic or type IIB string theory, but as connectors between *different* string theories such as heterotic at weak (strong) coupling to type I at strong (weak) coupling. This duality is in fact used in [BCHP3] in order to

[14] It relates two differents theories when the gauge groups are non-simply-laced [51].

[15] This is the case of both $\mathcal{N} = 4$, or $\mathcal{N} = 2$ with $N_f = 2N_c$ with N_f and N_c being the flavor and color numbers.

deduce some properties of the low energy effective action of four-photon interactions of type I strings.

Later, another symmetry was conjectured to map the type IIA theory at strong coupling into an eleven-dimensional theory called M-theory, and which maps the heterotic superstring with gauge group $E_8 \times E_8$ at strong coupling into a version of M-theory with boundaries. The existence of the latter theory was corroborated by the discovery of the eleven-dimensional supergravity mentioned previously in §1.2. This historical step in string theory research induced many other non-perturbative symmetry conjectures and important works that paved the path for this present work.

Thus, by studying the perturbative regime of string theories where the coupling constant is small, one can use the strong-weak duality non-perturbative information where the coupling constant is large in the same, or in another theory. S-duality symmetries may help in understanding superstring theory at very small and very large values of the coupling constants, but also to gain control over some relevant informations contained in the instantonic effects. This is the direction pursued in this manuscript.

1.4 Non-perturbative dualities

Non-perturbative dualites have played an important role in understanding string theory as well as certain quantum theories. They remain one of the only theoretical tool to access effects that are invisible in the neighborhood of the solutions to the equations of motion. We do not intend to recapitulate their history here, but only to introduce some details and concepts that will be relevant in the following chapters, and in particular to motivate the presence of a strong-weak duality in $\mathcal{N} = 4$ supergravities.

Kramers-Wannier duality The first instance of a non-perturbative duality was found by Kramers and Wannier [52] in the Ising model. It is presented here as an example to introduce the concept of dualities in toroidal string compactifications in the next paragraph.

In a statistical physics, a model is said to be self-dual if its partition function is left invariant under a transformation interchanging to physical variables. The partition function of physical system is of the utmost importance, as it sums the probability weights of all the possible states of the system. The partition function of a system on a lattice, like a field theory, is relevant for its similarities with lattice partition function in string theory that counts the states generated by winding and momenta in the compact flat directions.

Consider a square lattice, on which each site hosts a particle of spin with up and down state $\sigma_i = \pm 1$. The partition function at temperature $T = 1/\beta$ is given by summing the probability weight of a system configuration $\{\sigma_i\}$ over all possible spin states

$$Z = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} \exp(\beta J\, \sigma_i \sigma_j) = \cosh(\beta J)^{2N} \sum_{\{\sigma\}} \prod_{<ij>} (1 + \tanh(\beta J) \sigma_i \sigma_j)\,, \tag{1.14}$$

where $\langle ij \rangle$ designate nearest neighbouring sites, and where the second equality is obtained by noticing that $e^{\beta J \sigma_i \sigma_j} = \cosh(\beta J) + \sigma_i \sigma_j \sinh(\beta J)$. Using the formula (1.14), one can study the high temperature expansion $\beta \ll 1$, *i.e.* to study configurations close to the configurations where all spins are unaligned with their nearest neighbors. The expansion shows configurations will only contribute when they produce a polynomial of even degree in the spin variables σ_i, $\sigma_j, \ldots$, which can be represented as closed loops on the spin lattice, see figure 1.5. The high temperature expansion of the partition function thus counts closed loops of length $2n$, with $n \in \mathbb{Z}$. In the expansion of the partition function

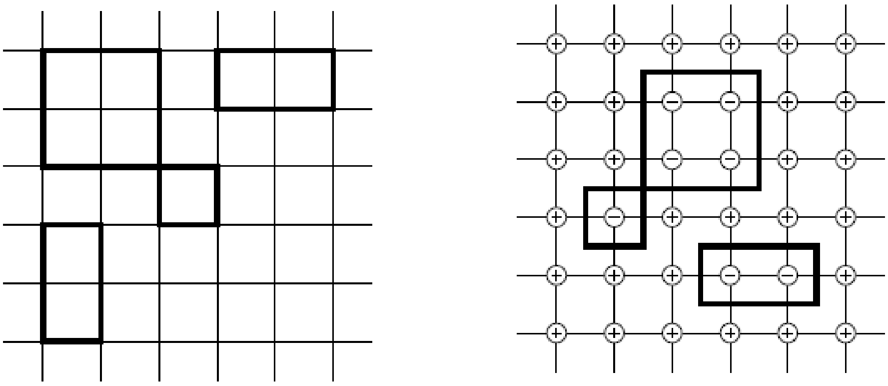


Fig. 1.5 *Schematic representation of the contributions to the partition function at large temperature (left, $\beta \ll 1$) and et low temperature (right, $\beta \gg 1$).*

at low temperature, $\beta \gg 1$, one expects to obtain the ordered phase where all spins are pointing in the same direction (either up our down), up to some patches of one or several spins pointing in the other direction. These are caused by small fluctuations due to the non zero temperature, and can be circled as in the right picture of figure 1.5. As one can judge in figure 1.5, the links and interfaces drawn in these two pictures form identical patterns, and this is so because 1)the polygons drawn to circle the blobs in the low temperature phase lie on the lattice dual to the spin lattice with respect to the nearest neighbor pairing $\langle ij \rangle$, and 2)the dual lattice to a square lattice is also a square lattice. These arguments prove that the partition function of the Ising model is invariant under low-/high-temperature duality

$$Z(\beta) = Z(-1/\beta)\,. \tag{1.15}$$

These results can extended to the case where the spin lattice is not a self-dual lattice by coupling several Ising models [53]. They naturally generalise to abelian gauge theories on four-dimensional lattice, where a larger group $SL(2,\mathbb{Z})$ acts non-perturbatively on both the gauge coupling and a topological "theta term" added in the action, which can be used to recover critical points of phase transition [54, 55]. Although we will not review this work further, we will exhibit how the charged spectrum of Maxwell's theory transforms under this duality 1.4, as well as massive BPS states in $\mathcal{N}=4$ super-Yang-Mills 1.4. We will also review a generalisation of these results to $\mathcal{N}=4$ supergravities descending from heterotic string theory in §3.1 to motivate the conjecture of exact F^4 and $\nabla^2 F^4$ couplings in sections 3.3 and 3.4.

We now exhibit briefly how the non-perturbative symmetry shown above is often encountered as a perturbative symmetry in a compactified string theory model 1.4.

Perturbative symmetry in toroidal string compactification. The string theory partition function counting string states with momenta and winding around the internal flat directions, exhibits self duality in a very similar fashion. In particular, toroidal compactification of heterotic string theory can be viewed as the compactification of independent left- and right-movers $(L, \tilde{L})$ on tori on which momenta en windings are represented by an even Lorentzian self-dual lattices in $\mathbb{R}^{16+d,d}$[56], with $D = 10 - d$ being the spacetime dimension after compactification. In the case of compactification on one circle, the duality symmetry, which acts on the radius of the circle as $R \to 1/(2R)$, maps different Lorentzian even lattices and sends the lattice vectors as [57, 58]

$$(L, \tilde{L}) \to (L, -\tilde{L})\,, \tag{1.16}$$

thus preserving the norm $L^2 - \tilde{L}^2$, and thus the self-duality of the lattice. This symmetry is here perturbative, as it leaves the string coupling constant unchanged. Compactifying down to D spacetime dimension, d-tori is parametrised by dynamical massless scalar fields called moduli that span the non-compact Riemannian symmetric space

$$G_{16+d,d} = \frac{O(16+d,d)}{O(16+d) \times O(d)}\,, \tag{1.17}$$

and whose transformation under the duality symmetries can be understood from the supergravity effective description, as we discuss in §3.1. String states being labelled by their discrete winding and momenta along the compactified directions, the group of global symmetries is $SO(16+d,d,\mathbb{Z})$. Other effective $\mathcal{N} = 4$ effective supergravities can be obtained by quotienting the internal lattice by a discrete $\mathbb{Z}_N$ rotation [59, 60, 61, 62]. Note that these types of $\mathcal{N} = 4$ model with reduced gauge group were first discovered and studied through a type I string construction by Bianchi, Pradisi and Sagnotti [63]. We will focus on some of these models, the ones listed in table 1.1 §1.2.

Four-dimensional abelian gauge theory. We now study the strong-weak coupling duality in the oldest theory of electromagnetism. We will further comment on this type of duality in the case of field theories (see next paragraph), as well as in supergravity and string theory (see §3.1). It is moreover at the center of the exact coupling proposal of [BCHP1], [BCHP2], [BCHP3], as we review in chapter 3.

In the vacuum, Maxwell's theory of electromagnetism has a $U(1)$ symmetry allowing rotations between the electric and the magnetic field as

$$E + \mathrm{i}B \to e^{\mathrm{i}\alpha}(E + \mathrm{i}B)\,, \tag{1.18}$$

which thus contain the exchange between the electric and the magnetic field $(E, B) \to (-B, E)$. In the relativistic formalism where the electric and magnetic fields are given respectively by F^{0i} and its Hodge dual $\star F_{0i}$, the duality $(E, B) \to (B, -E)$ can be simply expressed as $F_{\mu\nu} \to \star F_{\mu\nu}$.

Extending this duality to the charged spectrum predicts the existence of magnetic monopoles, *i.e.* states with non vanishing charge q_m under the magnetic field. The presence of such charge would have to satisfy the Dirac-Schwinger-Zwanziger quantisation condition [64, 65, 66]

$$q_e q'_m - q'_e q_m = 4\pi n\,, \qquad n \in \mathbb{Z}\,, \tag{1.19}$$

where (q_e, q_m) and (q'_e, q'_m) are the electric and magnetic charges of two particles. Since in nature there are electrons of charge $(ge, 0)$, with $e \in \mathbb{Z}$ and g the gauge coupling constant, the charge quantisation condition (1.19) for a hypothetical magnetic monopole of charge (q_e, q_m) requires $geq_m = 2\pi n$. Thus the allowed magnetic monopole charge reads

$$q_m = \frac{4\pi}{g} m \,, \qquad m \in \mathbb{Z} \,. \tag{1.20}$$

This implies that the exchange of magnetic and electric field, in the presence of charged matter, imply the following identification of the charges, joined with a strong-weak duality in g

$$(e, m) \to (-m, e) \quad \Rightarrow \quad g \to \frac{4\pi}{g} \,. \tag{1.21}$$

This picture will be exhibited for the electro-magnetic spectrum of $\mathcal{N} = 4$ supergravity in §3.1.1.

Self-duality of $\mathcal{N} = 4$ four-dimensional gauge theory. In 1974, magnetically charged solution were found in non-abelian gauge theories with spontaneous symmetry breaking to abelian gauge groups [67, 68]. Montonen and Olive conjectured a duality exchanging a gauge triplet made of this monopole states with the photon, together with the W bosons of the spontaneously broken non-abelian gauge theory [49]. In the general case, this conjecture was either falsified by the mass quantum corrections or the matching of the two triplets' spin. However, for $\mathcal{N} = 4$ super-Yang-Mills, the effective action does not obtain corrections beyond two derivative couplings and it was shown that the quantisation of fermionic zero modes around the monopole solution makes it into a $\mathcal{N} = 4$ massive vector multiplet [69].

One can give a vacuum expectation value $\langle \phi \rangle$ to one of the six adjoint scalars of the $\mathcal{N} = 4$ theory with gauge group G. Then, the W-boson, or fluctuation of the component E_α of the gauge field, with α is a root of G, obtains a mass given by

$$\mathcal{M}_W(\alpha) = g \, |\alpha \cdot \langle \phi \rangle| \,. \tag{1.22}$$

On the other hand, each root α gives a monopole solution similar to Bogomol'nyi-Prasad-Sommerfield solution for $G = SU(2)$ [70, 71]. These states belong to the 16 dimensional representation of the $\mathcal{N} = 4$ supersymmetry algebra, are annihilated by half of the sixteen supercharges, and their mass and charge satisfying a definite relation. In the present case, the mass of the solution is given by

$$\mathcal{M}_M(\alpha) = \frac{4\pi}{g} |\alpha^\vee \cdot \langle \phi \rangle| \,, \tag{1.23}$$

where $\alpha^\vee$ is a coroot of G.

Then, the spectrum of W bosons and the monopoles of the theory can be exchanged if the root lattice is self-dual with respect to the orthogonal projection, *i.e.* , if the root system is isomorphic to the coroot system.[16] One can see that exchanging monopoles of

[16] In general, the S-duality maps a theory with gauge group of root system Φ to a theory with gauge group of root system $\Phi^\vee$, where $\Phi^\vee$ is the system of coroots of the former. Self-duality is hence possible only if $\Phi^\vee \simeq \Phi$, which is the case for simply-laced groups only.

G with W-boson of G leads to the same strong-weak duality as (1.21). This superficial recount of the S-duality in $\mathcal{N} = 4$ theories can completed by [72, 73, 74].

In §3.1.1, we will recall invariance of the mass spectrum of the BPS sector in the case of $\mathcal{N} = 4$ supergravity, as a motivation of a non-perturbative duality of the full theory. In sections 3.3 and 3.4, we use this duality together with supersymmetry considerations to conjecture the existence of exact F^4 and $\nabla^2 F^4$ couplings.

Structure of the manuscript

Below is a quick summary of the organisation of this manuscript.

In chapter 2, we introduce some basics of perturbative string theory and recall the low-energy computation of perturbative F^4 interaction at one loop, and $\nabla^2 F^4$ interaction at one and two loops.

In chapter 3, we introduce the conjectures of [BCHP1], [BCHP2] and [BCHP3] for the exact F^4 and $\nabla^2 F^4$ couplings in three dimensions, and discuss their perturbative limit in weak heterotic coupling in three dimensions and weak type II coupling in four dimensions. To motivate these conjectures, we first review the dimensional reduction of half-maximal supergravity to four and three dimensions, and give argument for S-duality in four dimensions. We then recall some of the specificities of CHL models with prime N, and give a brief presentation of the construction of their one-loop partition function.

In chapter 4, we discuss their decompactification limit from three to four dimensions presented in [BCHP1], [BCHP3], and show how they allow to compute the degeneracy of quarter-BPS black hole solutions. These results reproduce and extend the Dijgkraaf-Verlinde-Verlinde (DVV) formula for prime CHL models, as well as the exact contour prescription. To introduce these results, we first review the entropy formalism for stationary four-dimensional black holes, and then the famous DVV formula in the case of CHL models. Finally, we recall some techniques to describe quarter-BPS black holes in $\mathcal{N} = 4$ supergravities.

Finally, in chapter 5, we give some directions for the outlook.

Chapter 2

Superstring amplitudes and perturbative expansion

In this chapter, we study the one- and two-loop amplitudes of half-maximal heterotic string theories. In the simplest case, this theory is the toroidal reduction of $\mathcal{N} = 1$ supergravity coupled to $\mathcal{N} = 1$ super-Yang-Mills. Such models are given table 1.1, some of which being realisable as a toroidal compactifiction of heterotic string theory, with N in the rightmost column being the order of the free action of the Z_N orbifold.

We first review some general facts about closed string theory amplitudes, and then present the one-loop and two-loop computations of four-photon interactions, where the latter are based on a calculation by D'Hoker and Phong [75, 76, 77, 78, 79, 80].

2.1 Closed string theory amplitudes

2.1.1 Bosonic string

Scattering processes in string theory, or S-matrix elements, are computed in a first quantised formalism. The string trajectory X describes the wordlsheet. It can be thought as a map from the string-surface Σ (indices $m, n = 1, 2$) to the D-dimensional spacetime manifold M (indices $\mu, \nu = 0, \ldots, D-1$). The fields X are scalars from the view point of Σ which are governed by the Polyakov action[1] [44]

$$S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{g}\, g^{mn} \partial_m X^\mu \partial_n X^\nu\, G_{\mu\nu}(X) \ , \tag{2.1}$$

where σ, τ and g^{mn} are the coordinates and metric on Σ, and where $G_{\mu\nu}(X)$ is the metric on M. It has the special feature of being renormalisable as a QFT, as well as local in X, g, and G, and invariant under orientation preserving diffeomorphisms of Σ and diffeomorphisms of M. Strings can only interact through 'joinings' and 'splittings' because of Lorentz invariance and mathematical consistency. In particular, no specific point on the worldsheet can be singled out as interaction point, since it depends upon the Lorentz frame chosen to observe the process. One should note that an *open string theory* always contain closed strings, since endpoints of an open string can always join. However a *closed string theory* refer to a theory containing only closed strings.

[1] We refrain from presenting the historic Nambu-Goto action because of its difficult quantisation.

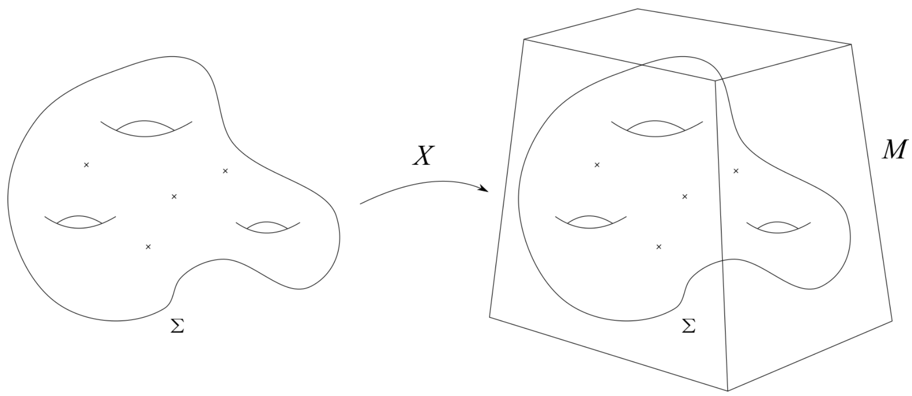


Fig. 2.1 *Map from the string-surface* Σ *to the spacetime manifold* M*. The little crosses symbolise the insertion points, i.e. the ingoing or outgoing states*

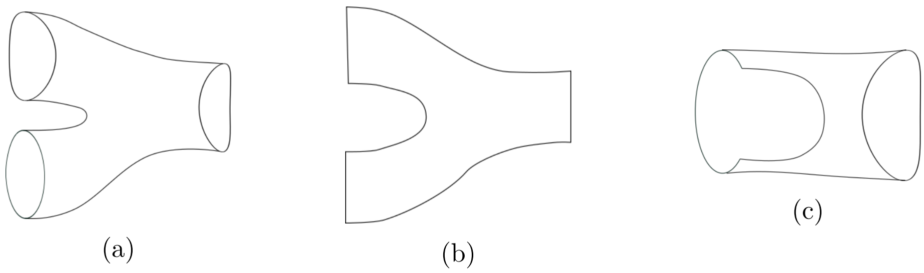


Fig. 2.2 *String interactions. Closed and open strings can interact as in 2.2a and 2.2b, respectively. Open strings can also join their ends to form a closed string 2.2c.*

As in a first quantised path integral formalism, the transition amplitudes between two specified external string states is obtained by the sum over all possible worldsheets, i.e. all possible surfaces Σ and trajectories in spacetime X,

$$A = \sum_{\Sigma} \sum_{X} e^{-S[X,\Sigma,M]} \ , \tag{2.2}$$

and must be normalised by the overall volume given by the diffeomorphism invariance to be determined later. Another important ingredient is to notice that the action S is also invariant under *Weyl* rescalings, which can be spoiled by anomalies, but is nonetheless crucial for the consistency of the theory. Assuming this, the sum over all geometries on Σ collapses to the sum over all topologies of genus h, and all metrics of surfaces Σ_h for each h. Finally, a second simplification concerns the boundary data specifying the ingoing and outgoing states of the scattering process, which can be geometrically reduced to a simple point on the compact surface Σ_h, where states' data is mapped by inserting so-called vertex operators $V_1, \ldots, V_N$ to be constructed later. The amplitudes thus writes

$$A = \sum_{h=0}^{\infty} \int_{\text{Met}(\Sigma_h)} Dg \, \frac{1}{\mathfrak{N}(g)} \int_{\text{Maps}(\Sigma_h,M)} DX \, V_1 \ldots V_N \, e^{-S[X,g]} \tag{2.3}$$

where $\text{Met}(\Sigma_h)$ is the space of metrics on Σ_h, $\mathfrak{N}(g)$ is a normalisation factor compensating the diffeomorphism and Weyl invariance of the action S such that $Dg/\mathfrak{N}(g)$ reduces

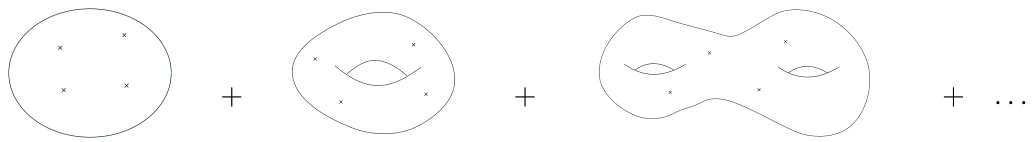


Fig. 2.3 *The sum over all string worldsheets decompose into a sum over all topologies because of Weyl invariance.*

naturally to the canonical measure on $\mathcal{M}_h = \mathrm{Met}(\Sigma_h)/\mathrm{Diff}(\Sigma_h) \times Weyl(\Sigma_h)$, the moduli space of Riemann surfaces of genus h. The action S that we considered is not the most general one, and one can consider having a manifold M with extra structure. The manifold can carry an anti-symmetric tensor field $B_{\mu\nu} \in \Omega^{(2)}(M)$, a dilaton field $\Phi \in \Omega^{(0)}(M)$ coupled to the Gauss curvature of Σ, and a tachyon field $T \in \Omega^{(0)}(M)$. The tachyon field will not be consider as sensible field in general, but the other will lead to an effective action at low energy in terms of kinetic terms for the graviton, the dilaton, and the anti-symmetric B-field.

2.1.2 Superstrings

The sector of string theory that we considered previously is usually named the bosonic string theory. It is necessary to include bosonic excitation on strings, but these alone don't allow for a physically sensible theory, as Nature clearly displays fermionic states (electrons, quarks, etc.) or states transforming under a spinor representation of the Lorentz group. The purely bosonic string in flat spacetime also contains a tachyonic state, as stated before, which leads to a violation of the physical principal of causality.

These problems can be solved by adding extra degrees of freedom on the string worldsheet, which will result in introducing fermionic string states in the physical Hilbert space, but also in changing the critical dimension D of the embedding manifold M.

- This can be done in the *Green-Schwarz (GS) formulation* [81, 82], where one considers strings to move in a "super spacetime". The coordinates X^μ are supplemented with the fermionic ones θ^α, respectively transforming under a vectorial and fermionic representation of $SO(1, D-1)$. Thus ground state is thus degenerate and bosonic and fermionic states are obtained by applying to it the latter two fields. The drawback of this formulation is the difficulty to quantise it in a manifestly Lorentz invariant way.

- A way to bypass this difficulty was to use a twistor-like constraint to gauge-fix differently the purely bosonic action, leading to *Berkovits' pure spinor formalim* [83, 84] It has lead to several important results in the past year, among which the first computation of a three-loop four-graviton amplitude [85][2].

- The *Ramond-Neveu-Schwarz (RNS) formulation* two fundamental spacetime vector fields in the theory, X^μ and ψ^μ, where the latter is a Grassmannian variable. There

[2]Above genus five amplitudes, the prescription to compute the pure spinor ghost path integral has to be changed [86].

are thus two sectors in the Hilbert space, built by applying both fields on either the Neveu-Schwartz bosonic ground state, or the Ramond fermionic ground state. However, this field produces further negative norm states independent of those of X^μ, which are eliminated by imposing a local Grassmann symmetry, or *local supersymmetry*. This symmetry has to be local to remove the entire field component ψ_0 of ψ_μ. And, it also implies the existence of the spin $3/2$ field χ_α, a spinor-vector field sometimes called the superpartner to the metric field, gravitino, or Rarita-Schwinger field.

The supersymmetric Deser-Zumino-Brink-Di Vecchia-Howe-Polyakov action[3] thus involves X^μ, ψ^μ, g_{mn} and χ_m – also refered to as the $\mathcal{N}=1$ supergravity action – writes, for flat target spacetime M

$$S[X,\psi;g,\chi] = \frac{1}{4\pi}\int_\Sigma d\sigma d\tau\sqrt{g}\Big[\frac{1}{2\alpha'}g^{mn}\partial_m X^\mu\partial_n X_\mu + \psi^\mu\gamma^a e_a^m\partial_m\psi_\mu - \frac{1}{\sqrt{\alpha'}}\psi^\mu\gamma^a\gamma^b\chi_a e_b^m\partial_m X_\mu - \frac{1}{4}(\psi^\mu\gamma^m\gamma^n\chi_m)(\chi_n\psi_\mu)\Big]\,, \tag{2.4}$$

where γ^m satisfy the 2-dimension algebra $\{\gamma^m,\gamma^n\} = -g^{mn}$, and e_a^m is the local frame field satisfying $e_a^m e_b^n\delta^{ab} = g^{mn}$. The integrand of (2.4) is a single-valued function when ψ^μ and χ_m have the same spin structure, and the total action under three additional symmetries with respect to (2.1): Weyl-invariance over Σ, super-Weyl invariance $\delta e_a = 0$, $\delta\chi_m = \gamma_m\delta\Lambda$, and local supersymmetry

$$\begin{cases}\delta X^\mu = \zeta\psi^\mu\\ \delta\psi^\mu = \gamma^m(\partial_m X^\mu - \frac{1}{2}\chi_m\psi^\mu)\zeta\\ \delta e_m^a = \zeta\gamma^a\chi_m\\ \delta\chi_m^\sigma = -2\nabla_m\zeta^\sigma\,.\end{cases} \tag{2.5}$$

The supersymmetrised Polyakov action coupled to a conformal field theory thus defines a superconformal field theory. The critical dimension for the superstring theory, which can be computed by requiring the Weyl-anomaly to vanish, is $D = 10$. Furthermore, gauging the supergravity fields on a genus-h super-Riemann surface with n-insertion point induces an integration over $3h-3+n$ bosonic and $2h-2+n$ fermionic moduli. They parametrise the space of all frame and gravitino fields $\{e_m^a,\,\chi_m^\sigma\}$, denoted $\mathrm{sMet}(\Sigma_\mathrm{h})$, quotiented by the group of superdiffeomorphisms $\{\mathrm{Diff}(\Sigma_h), \text{local SUSY}\}$, super-Weyl transformations $\{\mathrm{Weyl}(\Sigma_h), \text{super }\delta\lambda\}$, and local $SO(2)$ frame rotations $\mathrm{Lorentz}(\Sigma_h)$ [89, 90, 91]

$$\mathrm{s}\mathcal{M}_\mathrm{h} = \mathrm{sMet}(\Sigma_\mathrm{h})/(\mathrm{sDiff}(\Sigma_h)\times\mathrm{sWeyl}(\Sigma_h)\times\mathrm{Lorentz}(\Sigma_h)) \tag{2.6}$$

The scattering amplitudes or N string states are thus expressed in terms of the fields X, ψ, g and χ

$$A = \sum_{h=0}^{\infty}\sum_{\nu,\bar\nu} w(\nu,\bar\nu)\int_{\mathrm{sMet}(\Sigma_h)} D\chi Dg\,\frac{1}{\mathfrak{N}(g,\chi)}\int_{\mathrm{Maps}(\Sigma_h,M)} DX\int D\psi\,V_1\ldots V_N\,e^{-S[X,\psi;g,\chi]}\,, \tag{2.7}$$

[3]Although it corresponds to the supersymmetrized version of the Polyakov action, it was discovered beforehand by Deser, Zumino, and independently by Brink, Di Vecchia, Howe [87, 88].

where ν, $\bar{\nu}$ are the spin structure, $w(\nu,\bar{\nu})$ is a weight factor, and $V_1 \ldots V_N$ a collection of vertex operators for the RNS string.

The procedure to compute integrals of the form (2.7) was believed to rely in the existence of a global holomorphic section of $s\mathcal{M}_h$ [91, 92, 93]. The existence of such section makes space of super-Riemann surfaces *split*, implying that the odd moduli can be integrated alone and the amplitude reduces to an integral over its Riemann base. However, for $h \geq 5$, it is known that $s\mathcal{M}_h$ is not holomorphically projected [94], while the question remains for $h = 3, 4$.

2.2 One-loop four-photon couplings

We start this section by recalling the form of one-loop photon amplitude in string theory. At four-point in heterotic string, they write as a correlation function of a product of vertex operators

$$\mathcal{A}^{(1)}_{abcd} = \frac{\alpha'^2}{(2\pi \mathrm{i})^4} \int_{\mathcal{F}_1} \frac{\mathrm{d}^2\tau}{\tau_2^2} \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{\mathrm{d}^2 z_i}{\tau_2} \langle V_{1a}(z_1) V_{2b}(z_2) V_{3c}(z_3) V_{4d}(z_4) \rangle \,, \tag{2.8}$$

where the domain $\mathcal{F}_1$ has been defined in the previous chapter, and the z_i belong to $\mathcal{T} = \{z \in \mathbb{C},\, -1/2 < \mathrm{Re}\, z \leq 1/2,\, 0 < \mathrm{Im}\, z < \tau_2\}$. One vertex operator is fixed to $z_4 = \mathrm{i}\tau_2$ by conformal invariance. The heterotic gauge bosons propagate on the left-moving sector, and the vertex operators for the gauge boson read:[4]

$$V_a(z) = \mathrm{i}\rho_{a\mu} \tilde{\mathcal{P}}^\mu \mathcal{P}^a e^{ik\cdot X(z,\bar{z})} \tag{2.9}$$

with $\mu = 1, \ldots, d$ labelling the transverse spacetime directions, $a = 1, \ldots, 16 + d$ labelling the internal lattice dimensions, and where $\tilde{\mathcal{P}}^\mu$ is the supersymmetric right-moving momentum operator

$$\tilde{\mathcal{P}}^\mu = \partial_\tau X^\mu + \tfrac{1}{16} k^\nu \bar{\psi} \gamma^{\mu\nu} \psi \,, \tag{2.10}$$

with ψ the RNS spacetime Grassmann variable, and where $\partial_\tau X^\mu$ and $\mathcal{P}^a$ are the left-moving spacetime momentum and gauge lattice operators

$$\partial_\tau X^\mu = \tfrac{1}{2} p^\mu + \sum_{n \neq 0} \alpha_n^\mu e^{-2\mathrm{i}n(\tau-\sigma)} \qquad , \qquad \mathcal{P}^a = p^a + \sum_{n \neq 0} \tilde{\alpha}_n^a \, e^{-2\mathrm{i}n(\tau+\sigma)} \,. \tag{2.11}$$

The periodicity conditions for the fermionic fields ψ^μ, $\bar{\psi}^\nu$ upon transport along the torus of complex structure τ define *spin structures*, denoted by $\alpha, \beta \in \{0, 1\}$, such that

$$\psi^\mu(z+1) = e^{\mathrm{i}\pi\alpha} \psi^\mu(z) \,, \qquad \psi^\mu(z+\bar{\tau}) = e^{\mathrm{i}\pi\beta} \psi^\mu(z) \,. \tag{2.12}$$

One must sum all these sectors to ensure modular invariance, with a relative sign dictated by the GSO projection [45]. The partition function of a supersymmetric sector of spin structure α, β writes[5]

$$Z^{\alpha\beta}(\bar{\tau}) \equiv \frac{\theta\left[{\alpha \atop \beta}\right](\bar{\tau}, 0)}{\eta(\bar{\tau})^{12}} \,, \tag{2.13}$$

[4] The vertex operators V_i can all be chosen in the (0) superghost picture since the superghost background charge is zero on the torus.

[5] Note that for orbifold models, such as the one presented in the next section, GSO boundary conditions can be mixed with target-space shifts, implying non-trivial boundary conditions for the fields X^μ, ψ^μ [95, 96].

with the Riemann theta $\theta\left[{\alpha \atop \beta}\right](\tau,z)$ and eta $\eta(\tau)$ functions

$$\theta\left[{\alpha \atop \beta}\right](\tau,z) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-\frac{\alpha}{2})^2} e^{2\pi\mathrm{i}(z-\frac{\beta}{2})(n-\frac{\alpha}{2})}\,, \qquad \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{2.14}$$

The GSO projection gives rise the so-called supersymmetric cancellation identities on the worldsheet, namely

$$\begin{aligned} &\sum_{\substack{\alpha,\beta=0,1\\ \alpha\beta=0}} (-1)^{\alpha+\beta+\alpha\beta} Z^{\alpha\beta} = 0 \\ &\sum_{\substack{\alpha,\beta=0,1\\ \alpha\beta=0}} (-1)^{\alpha+\beta+\alpha\beta} Z^{\alpha\beta} \prod_{i=1}^{4} S_{\alpha,\beta}(\bar\tau, z_i - z_{i+1}) = -(2\pi)^4\,, \end{aligned} \tag{2.15}$$

with the fermionic correlators $S_{\alpha,\beta} = \langle\psi^\mu(z_i)\psi^\nu(z_j)\rangle_{\alpha,\beta}$ of spin structure α, β. The first of these identities ensures that the string self-energy vanishes, while the second will produce the t_8F^4 tensor when there are exactly four bilinears $:\psi\psi:$. These are a consequence of supersymmetric simplifications on the worldsheet in the RNS formalism, and details of these computation can be found in standard textbooks [82, 97].

Contraction of the spacetime and internal momenta also leads the so-called kinematical Koba-Nielsen factor, coming from the plane-wave part of the vertex operator

$$\langle: e^{ik\cdot X(z_1,\bar z_1)} : \ldots : e^{ik\cdot X(z_1,\bar z_1)} :\rangle_h = \exp\Big(\sum_{i<j} k_i\cdot k_j \langle X(z_i,\bar z_i)X(z_j,\bar z_j)\rangle_h\Big)\,. \tag{2.16}$$

Since its expression does not change for all genera, we will denote using the genus-h holomorphic two-point function

$$G_h(\tau, z_i - z_j) = \langle X(z_i)X(z_j)\rangle_h \tag{2.17}$$

and

$$\chi_{ij}^{(h)} = e^{G_h(\tau,z_i-z_j)}\,. \tag{2.18}$$

The one loop amplitude becomes, denoting $J_a(z)$ the internal current bilinear in the fermions, and s, t, u the Mandelstam variables

$$\begin{aligned} \mathcal{A}^{(1)}_{abcd} = \frac{\alpha'^2}{(2\pi\mathrm{i})^4}\int_{\mathcal{F}_1}\frac{\mathrm{d}^2\tau}{\tau_2^2}\frac{1}{\Delta(\tau)}\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{\mathrm{d}^2 z_i}{\tau_2}(\chi_{12}\chi_{34})^{\alpha' s}(\chi_{13}\chi_{42})^{\alpha' t}(\chi_{14}\chi_{23})^{\alpha' u} \\ \times\langle J_a(z_1)J_b(z_2)J_c(z_3)J_d(z_4)\rangle\,, \end{aligned} \tag{2.19}$$

where the torus Green function reads

$$g(z) = -\log|\theta_1(\tau,z)/\eta(\tau)|^2 + \tfrac{2\pi}{\tau_2}(\mathrm{Im}z)^2\,, \tag{2.20}$$

and the discriminant function

$$\Delta(\tau) = \eta(\tau)^{24}\,. \tag{2.21}$$

The partition function with four current insertion evaluates to

$$\langle J_a(z_1)J_b(z_2)J_c(z_3)J_d(z_4)\rangle = \Gamma_{p,q}[P_{abcd}] - \frac{1}{4\pi^2}\Big(\delta_{ab}\Gamma_{p,q}[P_{ab}]\partial^2 g(z_1-z_2) + 5\,\text{perms}\Big) + \frac{1}{16\pi^4}\Big(\delta_{ab}\delta_{cd}\Gamma_{p,q}\partial^2 g(z_1-z_2)\partial^2 g(z_3-z_4) + 2\,\text{perms}\Big)\,, \tag{2.22}$$

where P_{ab} and P_{abcd} are modular polynomial homogeneous in (Q_L^2, ρ_2^{-1}) explicited in [BCHP2], such that for any modular polynomial P of degree n, and integer lattice $\Lambda_{p,q}$ of signature (p,q), the partition function

$$\Gamma_{\Lambda_{p,q}}[P] = \tau_2^{q/2} \sum_{Q\in\Lambda_{p,q}} P(Q_{La},\tau) e^{i\pi\tau Q_L^2} e^{-i\pi\bar\tau Q_R^2} \tag{2.23}$$

is a modular form of weight $(2n+\frac{p-q}{2},0)$. Upon expanding the one-loop amplitude (2.8) in α', the so-called low-energy expansion, the leading term reproduces the one-loop contribution to the t_8F^4 coupling in $D=10-d$ dimensions

$$F^{(1)}_{abcd} = \text{R.N.}\int_{\mathcal{F}_1} \frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2} \frac{\Gamma_{\Lambda_{16+d,d}}[P_{abcd}]}{\Delta(\tau)}\,, \tag{2.24}$$

where P_{abcd} denotes a polynomial in Q_{La} defined in [BCHP1], [BCHP2], and where R.N. denotes a regularisation procedure introduced in [98, 99], which is in particular need to make sense of the integral when $d\geq 6$.

At next to leading order in α', the term linear in the Mandelstam variables s,t,u reduces to

$$G^{(1)}_{ab,cd} = \int_{\mathcal{F}_1} \frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\frac{1}{\Delta}\int_{\mathcal{E}^4}\prod_{i=1}^4 \frac{\mathrm{d}z_i\mathrm{d}\bar z_i}{2\mathrm{i}\tau_2}\Big[g(z_1-z_2)\,\partial^2 g(z_1-z_2)\,\delta_{ab}\,\Gamma_{\Lambda_{d+16,d}}[P_{cd}] + 5\,\text{perms}\Big]\,, \tag{2.25}$$

since all other terms at this order are total derivatives with respect to z_i. The integral over z can be computed by using the Poincaré series representation of the Green function,

$$g(\tau,z) = \frac{1}{\pi}\sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{\tau_2}{|m\tau+n|^2} e^{\frac{\pi}{\tau_2}[\bar z(m\tau+n)-z(m\bar\tau+n)]}\,, \tag{2.26}$$

leading to

$$\int_{\mathcal{E}} \frac{\mathrm{d}z\mathrm{d}\bar z}{2\mathrm{i}\tau_2} g(z-w)\partial^2 g(z-w) = \lim_{s\to 0}\sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m\tau+n)^2|m\tau+n|^{2s}} = \frac{\pi^2}{6}\hat E_2\,, \tag{2.27}$$

where the sum over (m,n) was regularized à la Kronecker, with $\hat E_2$ the non-holomorphic modular form of weight two

$$\hat E_2 = \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m\tau+n)^2} - \frac{3}{\pi\tau_2}\,. \tag{2.28}$$

Up to an overall numerical factor, we therefore find that the one-loop contribution to the coefficient of ∇^2F^4 coupling for the maximal rank model is given by

$$G^{(1)}_{ab,cd} \propto \delta_{\langle ab}\, G^{(d+16,d)}_{cd\rangle}\,, \qquad G^{(p,q)}_{ab} = \text{R.N.}\int_{\mathcal{F}_1}\frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\rho_2^2}\frac{\hat E_2}{\Delta(\tau)}\Gamma_{\Lambda_{p,q}}[P_{ab}]\,. \tag{2.29}$$

2.3 Two-loop $\nabla^2 F^4$ coupling

At two-loop, the scattering amplitude of four gauge bosons in ten-dimensional heterotic string theory was computed in [80, 100]. Upon compactifying on a torus T^d, one obtains

$$\mathcal{A}^{(2)}_{abcd} = \int_{\mathcal{F}_2} \frac{\mathrm{d}^3\Omega_1\,\mathrm{d}^3\Omega_2}{|\Omega_2|^3} \frac{1}{\Phi_{10}} \times \int_{\Sigma^4} \overline{\mathcal{Y}}_S \prod_{i=1}^{4} \mathrm{d}z_i\, (\chi_{12}\chi_{34})^{\alpha' s} (\chi_{13}\chi_{24})^{\alpha' t} (\chi_{14}\chi_{23})^{\alpha' u} \langle J_a(z_1)\, J_b(z_2)\, J_c(z_3)\, J_d(z_4)\rangle \tag{2.30}$$

where Σ is a genus-two Riemann surface with period matrix Ω, $\overline{\mathcal{Y}}_S$ is a specific $(1,1)$ form in each of the coordinates z_i on Σ [80, (11.32)],

$$\mathcal{Y}_S = t\,\Delta(z_1,z_2)\,\Delta(z_3,z_4) - s\,\Delta(z_1,z_4)\,\Delta(z_2,z_3)\ , \tag{2.31}$$

where $\Delta(z,w) = \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z)$, $\chi_{ij} = e^{G(\Omega, z_i - z_j)}$ and $G(\Omega,z)$ is the scalar Green function on Σ. At leading order in α', χ_{ij} can be set to one, and the integrated current correlator $\int_\Sigma J^a(z)\mathrm{d}z\,\overline{\omega_I(z)}$ can be expressed as a multiple derivative [101]

$$\langle \int_{\Sigma^4} J^a(z_1)J^b(z_2)J^c(z_3)J^d(z_4) \prod_{i=1}^{4} \mathrm{d}z_i\,\overline{\omega_I(z_i)}\,\rangle = \frac{\frac{1}{3}(\varepsilon_{rr'}\varepsilon_{ss'} + \varepsilon_{rs'}\varepsilon_{sr'})\partial^4}{(2\pi\mathrm{i})^4 \partial y^r_a\,\partial y^s_b\,\partial y^{r'}_c\,\partial y^{s'}_d} \Gamma^{(2)}_{\Lambda_{d+16,d}}(y)|_{y=0} \tag{2.32}$$

where $\Gamma^{(2)}_{\Lambda_{d+16,d}}(y)$ is the partition function of the compact bosons deformed by the currents $y^r_a J^a$ integrated along the r-th A-cycle of Σ,

$$\Gamma^{(2)}_{\Lambda_{p,q}}(y) = |\Omega_2|^{q/2} \sum_{Q\in\Lambda^{\otimes 2}_{p,q}} e^{\mathrm{i}\pi Q^r_{La}\,\Omega_{rs}\,Q^{s\ a}_L - \mathrm{i}\pi Q^r_{R\hat a}\,\bar\Omega_{rs}\,Q^{s\ \hat a}_R + 2\pi\mathrm{i} Q^r_{La} y^a_r + \frac{\pi}{2} y^a_r \Omega_2^{rs} y^{as}}\ . \tag{2.33}$$

The Siegel modular form $\Phi_{10}(\Omega)$ is given by the square product of all genus two even theta series

$$\Phi_{10}(\Omega) = 2^{-12}\Big[\theta^{(2)}\big[{}^{00}_{00}\big]\theta^{(2)}\big[{}^{00}_{01}\big]\theta^{(2)}\big[{}^{00}_{10}\big]\theta^{(2)}\big[{}^{00}_{11}\big]\theta^{(2)}\big[{}^{10}_{00}\big]\theta^{(2)}\big[{}^{01}_{00}\big]\theta^{(2)}\big[{}^{11}_{00}\big]\theta^{(2)}\big[{}^{10}_{10}\big]\theta^{(2)}\big[{}^{01}_{01}\big]\theta^{(2)}\big[{}^{11}_{11}\big]\Big]^2, \tag{2.34}$$

where $\theta^{(2)}\big[{}^{a_1a_2}_{b_1b_2}\big] = \theta^{(2)}\big[{}^{a_1a_2}_{b_1b_2}\big](\Omega|0)$, and

$$\theta^{(2)}\big[{}^{a_1a_2}_{b_1b_2}\big](\Omega|\zeta) = \sum_{n_1,n_2\in\mathbb{Z}} e^{\mathrm{i}\pi(n_1+\frac{a_1}{2}, n_2+\frac{a_2}{2})\cdot\Omega\cdot\begin{pmatrix} n_1+\frac{a_1}{2} \\ n_2+\frac{a_2}{2}\end{pmatrix} + 2\pi\mathrm{i}(n_1+\frac{a_1}{2}, n_2+\frac{a_2}{2})\cdot\begin{pmatrix}\zeta_1+\frac{b_1}{2} \\ \zeta_2+\frac{b_2}{2}\end{pmatrix}}\ . \tag{2.35}$$

The Siegel modular form $1/\Phi_{10}(\Omega)$ has an order one singularity at $v=0$,[6] corresponding the separating degeneration where the genus two surface degenerates to the connected sum of two genus one surfaces. In this limit

$$\Phi_{10}(\rho,\sigma,v) \sim (2\pi\mathrm{i}v)^2\Delta(\rho)\Delta(\sigma)\,, \tag{2.36}$$

[6]The singularity also exists at all its $Sp(4,\mathbb{Z})$ images, but only $v=0$ intersects with the fundamental domain of $Sp(4,\mathbb{Z})$.

where the discriminant function $\Delta(\tau)$ is defined (2.21).

Evaluating the derivatives explicitly, we obtain the result announced in [BCHP3] for the two-loop $\nabla^2 F^4$ coupling in the maximal rank case,

$$G_{ab,cd}^{(d,d+16)} = \text{R.N.} \int_{\mathcal{F}_2} \frac{\mathrm{d}^3\Omega_1\, \mathrm{d}^3\Omega_2}{|\Omega_2|^3} \frac{\Gamma^{(2)}_{\Lambda_{d,\,d\,+\,16}}[P_{ab,cd}]}{\Phi_{10}} \tag{2.37}$$

where $P_{ab,cd}$ is the quartic polynomial defined in [BCHP1], and more details about the regularization procedure can be found in [BCHP3].

Chapter 3

Non-perturbative and supersymmetry constraints

In the introduction we motivated the study of strong-weak duality symmetries in the context of string theory as a useful feature to count the microstates of black holes solutions in $\mathcal{N} = 4$ supergravity theories. Such symmetries are similar to the non-perturbative symmetries presented in §1.4 in the context of fields theories.

In section 3.1, we first consider the case of full heterotic string theory compactified on the torus, and we review the arguments motivating S-duality symmetries at the level of the four-dimensional effective action, the charge and the BPS spectrum, following [102, 72]. We review how this strong-weak symmetry generalises to a much bigger group $G_3(\mathbb{Z})$ of non-perturbative symmetries for the descending three-dimensional theory upon dimensional reduction on circle [103, 104].

In section 3.2, we review some details of the $\mathbb{Z}_N$ CHL models for prime N from an heterotic string perspective [59, 60, 62], and arguments for the presence of strong-weak dualities in both four and three dimensions [105].

Finally, in section 3.3 and 3.4 we expose the conjectures of [BCHP1], [BCHP2], [BCHP3] stating that exact four-scalar interactions in the low-energy three-dimensional effective action – namely $(\nabla\phi)^4$ and $\nabla^2(\nabla\phi)^4$ couplings – are given by modular integrals of specific modular forms times partition function for the non-perturbative Narain lattice invariant under the full group of non-perturbative symmetries $G_3(\mathbb{Z})$. These exact interactions are obtained by covariantisation of the respective perturbative $F^{(1)}_{abcd}$ and $G^{(2)}_{ab,cd}$ coupling coefficients under the group of non-perturbative symmetries $G_3(\mathbb{Z})$. They are motivated in addition by supersymmetry constraints that we expose in §3.3 and §3.4, and are checked against known pertubative results extracted from the literature in the weak coupling regime for both the heterotic and type II string in §3.3.1 and §3.4.1 respectively.

3.1 Dualities and applications

Our intention here is not to provide an exhaustive recapitulation of the knowledge about four-dimensional half-maximal theories, but rather to introduce the main arguments motivating the presence of strong-weak dualities in these compactified string theories.

In general, analysis of compactified string theories benefits from isometries of the target space introduced in chapter 2, the so-called T-dualities. In perticular, for toroidal compactification of the heterotic string, they are composed of reparametrisations of the 16+d-dimensional left-moving sector and d-dimensional right-moving sector, as introduced in §1.4. These act trivially on the coupling constant g_s and are thus valid order by order in string perturbation theory. Four-dimensional compactification of the heterotic string exhibits such symmetry, which we come back to in this section, but it possesses another type of symmetries that act non-trivially on the coupling constant g_s, the so-called strong-weak duality or S-duality. Such symmetry is very reminiscent of the strong-weak symmetry of $\mathcal{N} = 4$ super-Yang-Mills evocated in §1.4. As such, this property cannot be realised order by order in the g_s-expansion, and since our modern computational method are mostly perturbative, we still lack tools to prove the existence of S-duality on the full theory. However, it is possible to verify this property on some quantities that can be known exactly. This will be the focus of this section, and will serve as a motivation for anticipating the exact answers to questions that are not fully understood perturbatively.

In section 3.1.1, we review the dimensional reduction of the two-derivative low energy effective action to four dimensions, the realisation of the the strong-weak duality on the charge spectrum, as well as on the spectrum of massive BPS states. In section 3.1.2, we review the arguments for the presence of a $O(24, 8, \mathbb{Z})$ duality symmetry in three dimensions.

3.1.1 Strong-weak duality in four-dimensional string theory

To introduce heterotic string theory on a six dimensional torus, we start by writing the low energy effective action as $\mathcal{N} = 1$ supergravity theory coupled with $\mathcal{N} = 1$ super Yang-Mills theory in ten dimensions, and then reduce this action from ten to four dimensions [106]. The moduli space in $D = 4$ is a quotient

$$\mathcal{M}_4 = \frac{SL(2,\mathbb{R})}{SO(2)} \times \frac{O(22,6)}{O(22)\times O(6)}\,, \tag{3.1}$$

where $SL(2,\mathbb{R})/SO(2)$ is parametrised by the heterotic axiodilaton S while the Grassmaniann factor $Gr_{22,6}$ is parametrised by the scalars in the vector multiplets – respectively the 2 and the 132 in 1.1 – that we will come back to later. Since we will mostly be interested in the theory at a generic point of the moduli, we restrict the gauge group to its abelian subgroup $U(1)^{16}$. The bosonic part of the ten-dimensional action is given by, with $0 \leq M,\, N \leq 9$ and $I, J, \ldots$ indexing the 16 gauge directions,

$$\int \mathrm{d}^{10}x\sqrt{-G}e^{-\Phi}(\mathcal{R}_G + G^{MN}\partial_M\Phi\partial_N\Phi - \frac{1}{12}\mathcal{H}_{MNP}\mathcal{H}^{MNP} - \frac{1}{4}\eta_{IJ}\mathcal{F}^I_{MN}\mathcal{F}^{JMN})\ , \tag{3.2}$$

where G_{MN}, $\mathcal{B}_{MN}$, $\mathcal{A}^I_M$, Φ are the ten-dimensional metric, anti-symmetric tensor field, $U(1)$ gauge fields and the scalar dilaton field respectively, where η_{IJ} is the positive-definite metric on the internal gauge lattice, and

$$\mathcal{F}^I_{MN} = \partial_M\mathcal{A}^I_N - \partial_N\mathcal{A}^I_M \quad , \quad \mathcal{H}_{MNP} = \partial_M\mathcal{B}_{NP} - \frac{1}{2}\eta_{IJ}\mathcal{A}^I_M\mathcal{F}^J_{NP} + \text{cyclic perm}\,. \tag{3.3}$$

We will then denote by $m, n, \ldots$ the directions along the six-dimensional torus, and by $\mu, \nu, \ldots$ the non-compact ones. The compactification Ansatz for toroidal compactification

$$E^A_M = \begin{pmatrix} \tilde{e}^\alpha_\mu & A^{(G)n}_\mu E^a_n \\ 0 & E^a_m \end{pmatrix} \tag{3.4}$$

leads the the following decomposition of the Riemann tensor, when all fields are assumed to be $y-$independant

$$\sqrt{-G}\, e^{-\Phi}\, \mathcal{R}_G = \sqrt{-\tilde{g}} e^{-\phi}\Big(\mathcal{R}_{\tilde{g}} + \partial_\mu\phi\partial_\nu\phi + \frac{1}{4}\mathrm{Tr}(\partial_\mu G \partial^\mu G) - \frac{1}{4}\widehat{G}_{mn}F^{(G)m}_{\mu\nu}F^{(G)n\mu\nu}\Big), \tag{3.5}$$

where we introduced the shifted dilaton field $\phi = \Phi - \frac{1}{2}\log\det(\widehat{G})$. Using bold letters for tensors along the non-compact four-dimensional space

$$\begin{aligned} \mathcal{A}^I &= a^I_m(dy^m + A^{(G)m}) + A^{(F)I} \\ \mathcal{B} &= b_{mn}(dy^m + A^{(G)m})(dy^n + A^{(G)n}) + B_m(dy^m + A^{(G)m}) + B\,. \end{aligned} \tag{3.6}$$

The field strengths thus become

$$\begin{aligned} \mathcal{F}^I_m &= da^I_m \\ \mathcal{F}^I &= d\big[\mathcal{A}^{(F)I} - a^I_m A^{(G)m}\big] + a^I_m dA^{(G)m} \\ &\equiv dA^{(F)I} + a^I_m dA^{(G)m} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \mathcal{H}_{mn} &= db_{mn} + \frac{1}{2}\eta_{IJ}(a^I_m da^J_n - a^I_n da^J_m) \\ \mathcal{H}_m &= d\big[B_m + b_{mn}A^{(G)n} + \frac{1}{2}\eta_{IJ}a^I_m(\mathcal{A}^{(F)J} - a^J_n A^{(G)n})\big] - (b_{mn} + \frac{1}{2}\eta_{IJ}a^I_m a^J_n)dA^{(G)n} \\ &\quad - \eta_{IJ}a^I_m d(\mathcal{A}^{(F)J} - a^J_n A^{(G)n}) \\ &\equiv dA^{(B)}_m - (\widehat{B}_{mp} + \widehat{C}_{mp})dA^{(G)p} - \eta_{IJ}a^I_m dA^{(F)J} \\ \mathcal{H} &= d\big[B + \frac{1}{2}A^{(G)m}\wedge A^{(B)}_m - b_{mn}A^{(G)m}A^{(G)n}\big] - \frac{1}{2}(A^{(G)m}dA^{(B)}_m + A^{(B)m}dA^{(G)}_m) \\ &\quad + \frac{1}{2}\eta_{IJ}A^{(F)I}dA^{(F)J} - \frac{1}{2}\eta_{IJ}a^I_m A^{(F)J}dA^{(G)m}\,. \end{aligned} \tag{3.8}$$

To make the full $O(22,6)$ symmetry explicit, one can define, using the notations introduced in (3.7),(3.8), a new gauge vector A^I where the indices $I, J, \ldots$ now denote the full 22 gauge directions

$$\begin{aligned} &A^I = \begin{pmatrix} A^{(B)}_m \\ -A^{(F)I} \\ A^{(G)m} \end{pmatrix}, \qquad B^{(B)} = B + \frac{1}{2}A^{(G)m}\wedge A^{(B)}_m - b_{mn}A^{(G)m}A^{(G)n} \\ &\tilde{g}_{\mu\nu} = G_{\mu\nu} - G_{m\mu}G_{n\nu}(\widehat{G}^{-1})^{mn}\,, \end{aligned} \tag{3.9}$$

so that the kinetic term of this 28-dimensional gauge field A^I can be deduced from 28×28 dimensional matrix

$$M = \begin{pmatrix} \widehat{G}^{-1} & \widehat{G}^{-1}a & \widehat{G}^{-1}(b+c) \\ a^\intercal\widehat{G}^{-1} & \mathbb{1}_{16} + a^\intercal\widehat{G}^{-1}a & a\widehat{G}^{-1}(\widehat{G}+b+c) \\ (-b+c)\widehat{G}^{-1} & (\widehat{G}-b+c)\widehat{G}^{-1}a & (\widehat{G}-b+c)\widehat{G}^{-1}(\widehat{G}+b+c) \end{pmatrix} \tag{3.10}$$

where $c_{mn} = \frac{1}{2} a^I_m a^I_n$ and $\mathbb{1}_n$ is the $n \times n$ identity matrix. Note that this matrix consists of the 132 scalars paramatrising $G_{22,6}$ (3.1). This matrix is an element of $O(22,6)$, satisfying

$$MLM^\intercal = L \qquad , \qquad L = \begin{pmatrix} 0 & 0 & -\mathbb{1}_6 \\ 0 & \eta & 0 \\ -\mathbb{1}_6 & 0 & 0 \end{pmatrix} . \tag{3.11}$$

The four-dimensional action can thus be re-written as, using a shifted dilaton field $\phi \to \phi - \frac{1}{2} \log \det G$

$$\begin{aligned} \mathcal{S} = \int \mathrm{d}^4x \sqrt{-\tilde{g}} e^{-\phi} (\mathcal{R}_{\tilde{g}} + \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - F^I_{\mu\nu} (M^{-\intercal})_{IJ} F^{J\mu\nu} \\ + \frac{1}{8} \mathrm{Tr}(\partial_\mu ML \partial^\mu ML)) \ , \end{aligned} \tag{3.12}$$

where all the indices are raised by the metric $\tilde{g}_{\mu\nu}$, where $\mathcal{R}_{\tilde{g}}$ is now the curvatuge associated to the latter, while other dynamical fields can be found in [107].

This effective action is indeed invariant under $O(22,6)$ tranformations [107]

$$M \to \Omega M \Omega^\intercal \,, \qquad A^I_\mu \to \Omega^I{}_J A^J_\mu \,, \qquad g_{\mu\nu} \to g_{\mu\nu} \,, \qquad B_{\mu\nu} \to B_{\mu\nu} \,, \qquad \phi \to \phi \,, \tag{3.13}$$

where Ω is an $O(22,6)$ matrix. However, charge quantisation will break this symmetry to its largest discrete subgroup, $O(22,6,\mathbb{Z})$, which is also known as the T-duality group of this theory. Part of this symmetry exchanges the Kaluza-Klein modes of the theory, *i.e.* the states carrying momenta along the internal directions with the string wrapped around the internal directions.

To exhibit the string-weak symmetry, let us go to the Einstein frame metric $g_{\mu\nu} = e^{-\phi} \tilde{g}_{\mu\nu}$, and define a scalar field b dual to the antisymmetric tensor field

$$H^{\mu\nu\rho} = -(\sqrt{-g})^{-1} e^{2\phi} \varepsilon^{\mu\nu\rho\sigma} \partial_\sigma \mathrm{b} \,. \tag{3.14}$$

Let us introduce the axiodilaton

$$S = \mathrm{b} + \mathrm{i} e^{-\phi} \equiv S_1 + \mathrm{i} S_2 \,, \tag{3.15}$$

which is the complex scalar field parametrising the coset $SL(2,\mathbb{R})/SO(2)$ in (3.1), and rewrite the effective action with this field redefinition

$$\begin{aligned} \mathcal{S} = \int \mathrm{d}^4x \sqrt{-g} \Big[\mathcal{R}_g - \frac{1}{2{S_2}^2} \partial_\mu S \partial^\mu \bar{S} - S_2 F^I_{\mu\nu} (M^{-\intercal})_{IJ} F^{J\mu\nu} + S_1 F^I_{\mu\nu} \star F^{\mu\nu}_I \\ + \frac{1}{8} \mathrm{Tr}(\partial_\mu ML \partial^\mu ML) \Big] \ . \end{aligned} \tag{3.16}$$

where we denote the Hodge-dual field strength by

$$\star\, F^{I\mu\nu} = \frac{1}{2\sqrt{-g}} \varepsilon^{\mu\nu\rho\sigma} F^I_{\rho\sigma} \,, \qquad \star F^{\mu\nu}_I = L_{IJ} \star F^{J\mu\nu} \,, \tag{3.17}$$

where we will use L_{IJ} to indices of other tensors in the following.

The equation of motion of (3.16) for the field strength and the complex scalar field are

$$\begin{aligned} & D_\mu (S_2 (M^{-\intercal})_{IJ} F^{J\mu\nu} - S_1 \star F^{\mu\nu}_I) = 0 \\ & \frac{1}{{S_2}^2} D^\mu D_\mu S + \frac{2}{{S_2}^3} D_\mu S_2 D^\mu S - \mathrm{i} F^I_{\mu\nu} (M^{-\intercal})_{IJ} F^{J\mu\nu} + F^I_{\mu\nu} \star F^{\mu\nu}_I = 0 \,, \end{aligned} \tag{3.18}$$

where D_μ is the standard covariant derivative constructed from the metric $g_{\mu\nu}$. The other equation of motion can be found in [102, 72]. In order to exhibit the electromagnetic duality in a canonical way, we introduce the dual field $\tilde{F}_I^{\mu\nu}$ as suggested the equation of motion (3.18)

$$\tilde{F}_I^{\mu\nu} = S_2(M^{-\intercal})_{IJ} \star F^{J\mu\nu} - S_1 F_I^{\mu\nu} , \tag{3.19}$$

such that the equations of motion for the gauge field strength and its Bianchi identities, respectively

$$\mathrm{d}\tilde{F}^I = 0 , \qquad \mathrm{d}F = 0 , \tag{3.20}$$

are dual one another, as suggests their expression above, and are related by the projection

$$((ML)^I_J - \mathrm{i} \star) \frac{1}{\sqrt{S_2}} (\tilde{F}^J + S F^J) = 0 . \tag{3.21}$$

Strong-weak duality of the effective theory As stated earlier, the strong-weak symmetry acts non-trivially on $e^{-\phi}$. It is not an explicit symmetry of the four-dimensional action (3.16), but it can be exhibited in the equations of motion like (3.18). Using the dual field notation (3.19), one can check that the $SL(2,\mathbb{R})$ transformations [40, 108, 109]

$$S \to S' = \frac{aS+b}{cS+d} , \qquad \begin{pmatrix} \tilde{F}^I \\ F^I \end{pmatrix} \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} \tilde{F}^I \\ F^I \end{pmatrix} , \qquad g_{\mu\nu} \to g_{\mu\nu} . \tag{3.22}$$

Considering that $S_2^{-1} = e^\phi$ can be identified to the string coupling constant, this set of transformation contains the strong-weak coupling duality given by $a = 0$, $b = 1$, $c = -1$, $d = 0$,

$$S \to -\frac{1}{S} , \qquad F_I^{\mu\nu} \to \tilde{F}_I^{\mu\nu} , \qquad \tilde{F}_I^{\mu\nu} \to -F_I^{\mu\nu} \tag{3.23}$$

which, together with the shift of S_1

$$S_1 \to S_1 + b , \qquad \tilde{F}_I^{\mu\nu} \to \tilde{F}_I^{\mu\nu} - bF_I^{\mu\nu} , \tag{3.24}$$

generates the full $SL(2,\mathbb{R})$ group, with all other fields remaining invariant.

Note that the $SL(2,\mathbb{R})$ symmetry is not explicitely realised at the level of the effective action (3.16), but only at the level of the equations of motion. It is possible to introduce auxiliary variable to make both $SL(2,\mathbb{R})$ and $O(22,6)$ symmetries explicit, but this is at the cost of losing explicit general covariance [110]. Since the $SL(2,\mathbb{Z})$ symmetry exchanges the electric fields E_i^I with the magnetic fields $(ML)^I{}_J B^{iJ}$ (3.23), general covariance cannot be kept explicit. It is only possible at the cost of breaking the T-duality symmetry at the level of the effective action, although it is recovered in the equations of motions [110].

Note that $SL(2,\mathbb{R})$ cannot be a symmetry of the full theory, in the same way that $O(22,6)$ is broken to $O(22,6,\mathbb{Z})$ when requiring quantisation of the charges. However, $SL(2,\mathbb{Z})$ can be. In (3.16), we rewrote the four-dimensional action in a way that made explicit the coupling between S_1 and the topological density $F^I_{\mu\nu} L_{IJ} \star F^{J\mu\nu}$. The latter obtains contributions from gauge instantons which exist only for a discrete set of value – usually refered to as *instanton number*, and thus the translational symmetry (3.24) must be broken to a discrete subgroup of translations. Note that even if Abelian gauge field do not lead to gauge instantons, the theory considered here does produces gauge instantons since the fields A^I descend from non-Abelian gauge fields at a generic point of the moduli space. One can compute the contribution of such instanton [108], and show that the action can be normalised such that e^{iS} remains invariant under $S_1 \to S_1+1$. The translational symmetry is thus broken to $S \to S+1$ in the path integral formalism, and one can show that, together with $S \to -1/S$, it generates $SL(2,\mathbb{Z})$ the group of integer matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ab - cd = 1$.

Invariance of the charge spectrum. The full four-dimensional string theory contains charged fields. Although massive states decouple from the low energy effective action, they are a good playground to investigate $SL(2,Z)$ invariance of the spectrum. The gauge fields couple to their currents J_μ through the action

$$-\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}A^I_\mu J^\mu_I\,, \tag{3.25}$$

and one can identify the variables related the the electromagnetic field strength at large distance

$$\mathcal{Q}^I_e = \lim_{r\to\infty} r^2 F^I_{0r}\,, \qquad \mathcal{Q}_{Im} = \lim_{r\to\infty} r^2 \star F_{I0r}\,, \tag{3.26}$$

and use the equations of motion from the effective action [110] together with (3.25) to identify the electric charge as

$$\mathcal{Q}^I_e = \frac{1}{S_2^\infty} M^\infty_{IJ} Q_I\,, \tag{3.27}$$

where the superscript $^\infty$ stands for the asymptotic values of the fields and will be kept implicit in the following, and where Q_I is the integrated charge density

$$Q_I = \int \mathrm{d}^3x\sqrt{-g}\,J^0_I\,. \tag{3.28}$$

The electric charge vectors Q_I must belong to an even self-dual Lorentzian lattice Λ_e with metric L defined in (3.11) [111]. One can find the magnetic charges by imposing the Dirac-Schwinger-Zwanziger quantisation rule [64, 65, 66]. Considering an elementary string that can only carry an electric charge

$$(\mathcal{Q}^I_e, \mathcal{Q}^I_m) = \Big(\frac{1}{S_2} M^{IJ} Q_J, 0\Big)\,, \tag{3.29}$$

and generic solitonic state, the quantisation rule constrain the magnetic charge to satisfy

$$S_2\,\mathcal{Q}^I_m (LML)_{IJ} \frac{1}{S_2} M^{JK} Q_K = \mathcal{Q}^I_m Q_I \in \mathbb{Z}\,, \tag{3.30}$$

which corresponds to the definition of the dual lattice. Thus we have $\Lambda_m = \Lambda_e^*$, and the magnetic charges can be defined as

$$\mathcal{Q}^I_m = \frac{1}{S_2} L^{IJ} P_I\,, \qquad P_I \in \Lambda_e^*\,, \tag{3.31}$$

where Q_e^I and Q_m^I are canonically contracted with the canonical lattice metric L_{IJ}. Note that these lattice charge are directly obtained a the canonical definition using the Gauss law

$$Q_I = \frac{1}{4\pi}\int_{S^2}\tilde{F}_I\,, \quad P_I = \frac{1}{4\pi}\int_{S^2}F_I\,, \tag{3.32}$$

where here S_2 designates a 2-sphere enclosing the charge, and where the definition of the dual field is given in (3.19).

We now want to know the general charge for dyonic state with both Q_I and P_I non-zero. For $S_1 = 0$, or vanishing field strength $H_{\mu\nu\rho}$, the topological term in (3.16) is turned off from the action and we expect the electric charge of dyonic states to be quantised in units of integral electric charges [112], but it not the case in general. By using the canonical description (3.32), and relating the definition of $(\mathcal{Q}_e, \mathcal{Q}_m)$ (3.26), one obtains

$$(\mathcal{Q}_e^I, \mathcal{Q}_m^I) = \frac{1}{S_2}\Big(M^{IJ}(Q_J + S_1 P_J), L^{IJ}P_J\Big)\,. \tag{3.33}$$

The class of states described by (3.33) consists of all possible states charged under the gauge field, including the purely electric states. One can notice that when $S_1 \neq 0$, there does not exist electrically neutral magnetic monopoles [112].

To verify invariance of the charged spectrum under the strong-weak duality one can perform a generic $SL(2,\mathbb{Z})$ transformation on the field strength (3.22) and deduce the effect on the lattice charges $Q_I \in \Lambda_e$, $P_I \in \Lambda_m$

$$\begin{aligned} \mathcal{Q}_e^I &\to (cS_1 + d)\mathcal{Q}_e^I + cS_2(ML)^I{}_J Q_m^I = \frac{1}{S_2'}M^{IJ}(Q_I' + S_1'P_I') \\ \mathcal{Q}_m^I &\to (cS_1 + d)\mathcal{Q}_m^I - cS_2(ML)^I{}_J Q_e^I = \frac{1}{S_2'}L^{IJ}P_J'\,. \end{aligned} \tag{3.34}$$

where $S' = \frac{aS+b}{cS+d}$ and

$$\begin{pmatrix} Q' \\ P' \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix}. \tag{3.35}$$

This show that $SL(2,\mathbb{Z})$ transformations (3.22) preserve the expression of the charge spectrum, up to a linear transformation of the bases (3.35), $\Lambda_e \times \Lambda_m \to \Lambda_e$ and $\Lambda_e \times \Lambda_m \to \Lambda_m$. These transformations are well defined if the lattice is self-dual

$$\Lambda_e = \Lambda_m\,, \tag{3.36}$$

but might lead to further restrictions on the S-duality group otherwise. It is important to notice that for generic $\mathcal{N} = 4$ string theory models, the lattice is not necessarily self-dual and only subgroup of $SL(2,\mathbb{Z})$ can be preserved. In the case where both lattices are included into each other up to a integer coefficient, for instance if $\Lambda^* \subset \Lambda$ and $N\Lambda \subset \Lambda^*$, a subgroup of $SL(2,\mathbb{Z})$ with $c = 0 \bmod N$ – the congruent subgroup $\Gamma_0(N)$ – may be a symmetry of the charged spectrum. We come back to this in section 3.2.

Invariance of the mass spectrum. $SL(2,\mathbb{Z})$ invariance of the full string theory includes invariance of the allowed charge spectrum, but also of the full mass spectrum. This statement is more challenging to verify because of all the perturbative and non-perturbative quantum corrections to be considered. Here, we focus on a special class of states that are protected from any quantum corrections, as in the case of $\mathcal{N}=4$ super-Yang-Mills 1.4. Namely, states saturating the Bogomol'nyi bound [70] have their mass fixed as a function of their charge, and the latter can be obtained from the asymptotic value of the 28-dimensional field strength (3.26), as we shall now see.

The Bogomol'nyi lower bound on the mass squared of a state is given by $\mathcal{M}^2 \geq \mathcal{M}(Q,P)^2$ with [102]

$$\begin{aligned}\mathcal{M}(Q,P)^2 &= S_2\big(\mathcal{Q}_e^I(M^{-\intercal}-L)_{IJ}\mathcal{Q}_e^J + \mathcal{Q}_m^I(LML-L)_{IJ}\mathcal{Q}_m^J\big)\\ &= \frac{(M-L)^{IJ}}{S_2}\begin{pmatrix}Q_I & P_I\end{pmatrix}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}\begin{pmatrix}Q_J\\ P_J\end{pmatrix}.\end{aligned} \tag{3.37}$$

Note that for vanishing charge along the compactified directions, $(Q_I,P_I)=(0,0)$ for $1\leq I\leq 6$ and $22<I\leq 28$,

$$\mathcal{M}(Q,P)^2 = S_2\eta_{IJ}\big((\mathcal{Q}_e^I a_n^J)(\widehat{G}^{-1})^{np}(Q_e^I a_p^J) + (Q_m^I a_n^J)(\widehat{G}^{-1})^{np}(Q_m^I a_p^J)\big)\,, \tag{3.38}$$

with η_{IJ} the metric on the internal sixteen-dimensional lattice, is precisely Osborn's formula [69] presented in section 1.4 for the case of $\mathcal{N}=4$ super-Yang-Mills, where the fields a_m^I should be interpreted as the vacuum expectation value of the Higgs fields.

Using the definition of the Grassmaniann projectors defined in [BCHP2], one finds that $(M-L)^IJ = 2p^I_{R\hat{a}}p_R^{J\,\hat{a}}$ and can rewrite $\mathcal{M}(Q,P)^2$ as

$$\mathcal{M}(Q,P)^2 = \frac{2}{S_2}|Q_R+SP_R|^2\,. \tag{3.39}$$

The two expressions (3.37) and (3.39) are explicitely invariant under $O(22,6,\mathbb{Z})$ and $SL(2,\mathbb{Z})$ transformations given respectively by (3.13) and $S\to\frac{aS+b}{cS+d}$, up to self-duality of the electro-magnetic charge lattice discussed under (3.35) [102, 113, 114]. In other words, two states saturating the Bogomol'nyi bound have the same mass if their electro-magnetic charge numbers (Q,P), and the asymptotic values of M and S are related by an $SL(2,\mathbb{Z})$ transformation.

To establish the invariance of the complete mass spectrum for such states, it thus remains to show that the degeneracy $N(Q,P)$ of states carrying electromagnetic numbers Q_I, P_I, is an $SL(2,\mathbb{Z})$ invariant. Such task has not been perfomed completely in the literature, but we will see how far it has been pushed. We first consider states of vanishing magnetic charge, *i.e.* string excitations, and then identify the specific magnetic monopoles and dyons that close their $SL(2,\mathbb{Z})$ orbit.[1] It will be convenient to specify a duality frame where the matrix M equals the identity

$$M\to\Omega M\Omega^\intercal = \mathbb{1}_{28}\,,\qquad \Lambda_e\to\Omega^{-\intercal}\Lambda_e = \widetilde{\Lambda}_e\,. \tag{3.40}$$

In the chosen normalisation, the mass formurla for string excitations in the Neveu-Schwarz sector[2] is [102]

$$\mathcal{M}_{str}(Q)^2 = \frac{2}{S_2}(Q_R^2+2N_R-1)\,, \tag{3.41}$$

[1]For a monopole soliton to be a plausible dual state, one needs to ensure that the carries the same quantum and classical properties than the given string excitations. We will thus not consider singular or non-asymptotically-flat monopole solutions.

[2]The Ramond and Neveu-Schwarz states being degenerate with each other due to space-time supersymmetry, it is enough to study the Neveu-Schwarz sector only.

where Q_R, N_R and -1 are respectively the internal momenta, oscillator and ghost contributions to $\bar{L}_0$ in the world-sheet theory. All elementary string states saturating the Bogomol'nyi bound have $N_R = 1/2$, so that $\mathcal{M}_{str}(Q) = \mathcal{M}(Q, 0)$.

On the other hand, the monopole solitons can be of various origin [115, 116, 117, 118], but realistic known ones can only be gauge monopole solutions or H-monopoles. The former can be obtained in a gauge where vacuum expectation value of the gauge field is directed along a fixed direction [116, 118], and then rotate back the solution to the frame (3.40)

$$(Q_I, P_I) = (p, 1)e_I\,, \qquad e_I e^I = 2\,, \qquad e_I \in \Lambda_e\,, \tag{3.42}$$

where $p \in \mathbb{Z}$. The H-monopole is a solution associated with the ten dimensional field $\mathcal{H}_{MNP}$ [117, 118], and was constructed in [118] a finite sized gauge five-brane solution around the torus. They are pure magnetic monopoles in terms of quantum numbers (Q, P), and correspond to charge vectors

$$(Q_I, P_I) = (0, m_I)\,, \qquad m_I m^I = 0 \qquad m_I \in \Lambda_m\,. \tag{3.43}$$

These monopole solutions contains an $SU(2)$ gauge field, and thus only exist in a codimension one locus over the moduli space. In other words, they can only be constructed for specific class of M where the gauge group has a non-abelian enhancement [72].

One can now review what type of states are known in the $SL(2, \mathbb{Z})$-orbit of string excitation saturating the Bogomol'nyi bound

- In the case $Q^2 = e_I e^I = 2$, one can infer from $Q^2 = Q_L^2 - Q_R^2 = 2(1 - N_L)$ that there are no left-moving oscillators. Such states are mapped by strong-weak duality onto purely magnetic gauge solitons (3.42) with $p = 0$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_I \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ e_I \end{pmatrix}, \tag{3.44}$$

  which exists at any point of the moduli space. One can also show, similarly to $\mathcal{N} = 4$ super-Yang-Mills [69], that both of them fall in the vector representation of the $\mathcal{N} = 4$ super-Poincaré algebra. For a generic $SL(2, \mathbb{Z})$ transformation, one obtains

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_I \\ 0 \end{pmatrix} = \begin{pmatrix} a e_I \\ c e_I \end{pmatrix}, \tag{3.45}$$

  where the unit determinant condition of $SL(2, \mathbb{Z})$ imposes a and c to be coprimes. The gauge soliton states (3.42) are a special case of (3.45) for $c = 1$, but the others have not been constructed yet, and can been seen as a prediction of the strong-weak duality.

- In the case $Q^2 = 0$, the states satisfy $N_L = 1$. The oscillators associated with the 22 internal directions transform as scalars under four-dimensional Lorentz transformations and thus are vectors of super-Poincaré algebra. The Inversion of the type (3.44) maps elementary string states to H-monopole solutions – which only have been constructed in a background where the gauge group hasn't been totally broken. Ths other states obtained from (3.45) have not been constructed either.

- In the case $Q^2 < 0$, we get $N_L \geq 2$ and thus states of the form

$$(Q_I, P_I) = (n_I, 0) \tag{3.46}$$

are mapped to states that haven't been constructed yet. However, they are not expected to be constructible from the massless fields of the low energy effective action, except at special point of the moduli space where their mass vanishes.

3.1.2 Strong-weak duality in three dimensions

In this section, we study the duality group of heterotic string theory compactified on a seven-dimensional torus. In the low energy limit, this results in a three-dimensional supergravity theory with eight local supersymmetries [119]. The only massless bosonic fields are the non-propagating spin-2 graviton, and a set of scalar fields: all the gauge fields from dimensional reduction can be dualised to scalars in three spacetime dimensions. The scalar fields parametrise a single coset space

$$\mathcal{M}_3 = \frac{O(24,8)}{O(24) \times O(8)} \tag{3.47}$$

containing the four-dimensional moduli space, the holonomies of the four-dimensional gauge fields, the Kaluza-Klein vector and the circle radius [119, 120]. This symmetry enhancement can also be noticed from the perspective of vector multiplets: the 23 vector multiplets have manifest R-symmetry $Spin(7)/SO(8)$, while the gravity multiplet consists of 8 fermions and 7 vectors which can be dualised and completed with the dilaton to give 8 scalars and 8 fermions with $SO(8)$ symmetry.

The full string theory possess $O(23, 7, \mathbb{Z})$ target space duality, but the theory can also be seen as the four-dimensional theory of section 3.1.1 compactified on a circle, whose $SL(2, \mathbb{Z})$ invariance should remain unbroken since it does not act on space-time. When seen from the three-dimensional perspective, $O(23, 7, \mathbb{Z})$ target space duality implies that there are seven ways of decompactify back to a four-dimensional theory, and since $SL(2, \mathbb{Z})$ acts on the moduli space of the decompactifying circle, these seven different compactification lead to seven different $SL(2, \mathbb{Z})$ strong-weak duality groups. Since these transformations do not commute with each other, they generate a larger non-abelian discrete subgroup of $O(24, 8)$, which happens to be $O(24, 8, \mathbb{Z})$.

In the following, we first use the picture elaborated in §3.1.1 to reduce from ten to three dimensions directly and show, at the level of the bosonic effective action, how all the vector fields can be arranged in a 30-dimensional multiplet and parametrize the Grassmaniann $G_{24,8}$ (3.47), together with moduli of $G_{23,7}$ and the dilaton. In a second and complementary paragraph, we work out for latter use the dimensional reduction from four to three dimensions in a more general context with $2k + 4$ vector fields.

Starting from the ten-dimensional action of $\mathcal{N} = 1$ supergravity coupled with $\mathcal{N} = 1$ super-Yang-Mills (3.2), the reduction on T^7 leads to a matrix of scalar similar to (3.10), where the upperblock 7×7 instead of 6×6, and satisfies

$$MLM^{\intercal} = L\,, \qquad M^T = M\,, \qquad L = \begin{pmatrix} 0 & 0 & -\mathbb{1}_7 \\ 0 & \eta & 0 \\ -\mathbb{1}_7 & 0 & 0 \end{pmatrix}, \tag{3.48}$$

where η is the metric on the internal gauge lattice. The action can be written in the Einstein frame as

$$\mathcal{S} = \int \mathrm{d}^3x\sqrt{-\tilde{g}}(\mathcal{R}_{\tilde{g}} + \partial_\mu\phi\partial^\mu\phi - \frac{e^{-4\phi}}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - e^{-2\phi}F^I_{\mu\nu}(LML)_{IJ}F^{J\mu\nu} + \frac{1}{8}\mathrm{Tr}(\partial_\mu ML\partial^\mu ML)) \ , \tag{3.49}$$

and is invariant under $O(23,7)$ transformations $\hat{\Omega}$

$$M \to \hat{\Omega}M\hat{\Omega}^\intercal \, , \qquad A^I \to \hat{\Omega}_{IJ}A^J \, , \tag{3.50}$$

with $g_{\mu\nu}$, $B_{\mu\nu}$, ϕ remaining invariant.

In three dimensions, the B-field has no physical degree of freedom and its field strength can be fixed to 0, which implies that the equations of motion of the gauge fields A^I, $1 \leq I \leq 27$,

$$\partial_\mu(e^{-2\phi}\sqrt{-g}(ML)_{IJ}F^I_{\mu\nu}) = 0 \, , \tag{3.51}$$

can be used to introduce the 30 scalars ψ^I

$$e^{-2\phi}\sqrt{-g}(ML)_{IJ}F^J_{\mu\nu} = \frac{1}{2}\eta_{IJ}\varepsilon^{\mu\nu\rho}\partial_\rho\psi^J \, . \tag{3.52}$$

One can thus introduce the 32×32 matrix

$$\widetilde{M} = \begin{pmatrix} M + e^{2\phi}\psi\psi^\intercal & ML\psi + \frac{1}{2}e^{2\phi}\psi(\psi^\intercal L\psi) & -e^{2\phi}\psi \\ \psi^\intercal LM + \frac{1}{2}e^{2\phi}\psi^\intercal(\psi^\intercal L\psi) & e^{-2\phi} + \psi^\intercal LML\psi + \frac{1}{4}e^{2\phi}(\psi^\intercal L\psi) & -\frac{1}{2}e^{2\phi}\psi^\intercal L\psi \\ -e^{2\phi}\psi^\intercal & -\frac{1}{2}e^{2\phi}\psi^\intercal L\psi & e^{2\phi} \end{pmatrix} , \tag{3.53}$$

which belongs to $O(24,8)$. For $\mathcal{H}_{\mu\nu\rho} = 0$, the action can be rewritten as

$$S = \int \mathrm{d}^3x\sqrt{-g}[\mathcal{R}_g + \frac{1}{8}\,\mathrm{tr}\,(\partial_\mu\widetilde{M}\partial^\mu\widetilde{M})] \, , \tag{3.54}$$

which is manifestly invariant under the $O(24,8)$ transformation

$$\widetilde{M} \to \Omega\widetilde{M}\Omega^\intercal \, , \qquad g_{\mu\nu} \to g_{\mu\nu} \, , \tag{3.55}$$

with

$$\Omega\widetilde{L}\Omega^\intercal = \widetilde{L} \, , \qquad \widetilde{L} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & L & 0 \\ -1 & 0 & 0 \end{pmatrix} . \tag{3.56}$$

One can then show that this $O(24,8)$ symmetry of the action can be understood in terms of the $O(23,7)$ symmetry (3.50) and the $SL(2,\mathbb{R})$ symmetry of the four-dimensional action, by exhibiting the element acting on the S module (3.15) [103]. It can shown that the $O(24,8)$ group is generated as a combination of the $O(23,7)$ transformations (3.50), and the $SL(2,\mathbb{Z})$ tranformation. To show that $O(24,8,\mathbb{Z})$ is the invariance group of the full theory, one needs to show that the $SL(2,\mathbb{Z})$ invariance of the four-dimensional theory is not destroyed when we compactify on one of the three space-like dimension of this theory. Some of the monopole solitons necessary for the $O(24,8,\mathbb{Z})$ invariance of the theory are identified in [103].

Dimensional reduction from 4 to 3 dimensions. It is instructive to perform this dimensional reduction from a four-dimensional perspective. Let us thus consider again the effective action in four dimensions (3.16) and reduce the metric along a vector $K_\mu = (Rk_m, R)$, with $m = 0, 1, 2$

$$g_{\mu\nu} = \begin{pmatrix} -\frac{h_{mn}}{R} + Rk_m k_n & Rk_n \\ Rk_n & R \end{pmatrix}, \tag{3.57}$$

where h_{mn} is the scaled metric on the three dimensional space. Note that this reduction can be performed irrespective of the signature of the Killing vector field, and we have used notations for time-translation for the study of instantons in §4.4, *i.e.* h_{mn} is positive definite and $R > 0$. For axial rotations, the metric has signature $(-++)$ and $R < 0$.

Supposing we are interested in configuration allowing a Killing vector K, all the fields A^I_μ will depend only on the remaining three coordinates x^m, and decompose as $A^I_\mu = (\tilde{A}^I_m + k_m B^I, B^I)$, with $\tilde{A}^I_m$ and A^I perpendicular and parallel to K. Thus, the effective action (3.16) rewrites as – apart from surface terms –

$$\begin{aligned}\tilde{\mathcal{S}}^{(3)} = \int \mathrm{d}^3x\sqrt{-h}\Big[\mathcal{R}_h - \frac{1}{2{S_2}^2}\partial_m S\partial^m \bar{S} + S_1(F^I_{mn} + k_{mn}A^I)L_{IJ}\frac{2}{\sqrt{-h}}\varepsilon^{mnp}\partial_p A^J \\ - S_2(F^I_{mn} + k_{mn}A^I)(M^{-\intercal})_{IJ}(F^{Jmn} + k^{mn}A^J) + \frac{R^2}{4}k_{mn}k^{mn} \\ + \frac{2}{R}\partial_m A^I (M^{-\intercal})_{IJ}\partial^m A^J + \frac{1}{2R^2}\partial_m R\partial^m R - \frac{1}{8}\mathrm{Tr}(\partial_\mu ML\partial^\mu ML)\Big]\,,\end{aligned} \tag{3.58}$$

where $\mathcal{R}_h$ is the scalar curvature for h, $k_{mn} = \partial_m k_n - \partial_n k_m$, $F^I_{mn} = \partial_m \tilde{A}^I_n - \partial_n \tilde{A}^I_m$. The equations of motion for $\tilde{A}^I_m$ and k_m can then be considered as Bianchi identities for the dual fields $\tilde{B}^I$ and the *twist* potential ψ. One can treat the fields $\tilde{A}^I$ and their dual $\tilde{B}^I$ in a self-dual way by introducing

$$\tilde{\mathcal{A}}^I = \begin{pmatrix} \tilde{A}^I \\ \tilde{B}^I \end{pmatrix}, \qquad \tilde{F}^I = \mathrm{d}\tilde{\mathcal{A}}^I\,, \tag{3.59}$$

and the matrices

$$Y = \begin{pmatrix} 0 & L^{-\intercal}/4 \\ -L^{-1}/4 & 0 \end{pmatrix}, \qquad \bar{M} = \frac{4}{S_2}\begin{pmatrix} S_2^2 M^{-\intercal} + S_1^2 M^{-1} & S_1 M^{-1} \\ S_1 M^{-1} & M^{-1} \end{pmatrix}. \tag{3.60}$$

Treating them as independent variables, one ca rewrite the full action as

$$\begin{aligned}\mathcal{S}^{(3)} = \int \mathrm{d}^3x\sqrt{-h}\Big[\mathcal{R}_h - \frac{1}{2{S_2}^2}\partial_m S\partial^m \bar{S} + \frac{S_2}{4R}(\partial_m \mathcal{A}^\intercal)\bar{M}(\partial^m \mathcal{A}) \\ - \frac{1}{R^2}\partial_m R\partial^m R - \frac{1}{2R^2}\Omega_m\Omega^m - \frac{1}{8}\mathrm{Tr}(\partial_\mu ML\partial^\mu ML)\Big]\,.\end{aligned} \tag{3.61}$$

where

$$\Omega_m = \partial_m\psi - \mathcal{A}^\intercal Y^{-\intercal}\partial_m\mathcal{A}\,. \tag{3.62}$$

(3.61) is in agreement with the graded decomposition of the Lie algebra $\mathfrak{so}_{2k,8}$

$$\mathfrak{so}_{2k,8} \simeq \ldots \oplus (\mathfrak{gl}_1 \oplus \mathfrak{sl}_2 \oplus \mathfrak{so}_{2k-2,6})^{(0)} \oplus (\mathbf{2} \otimes (\mathbf{2k+4}))^{(1)} \oplus \mathbf{1}^{(2)}\,, \tag{3.63}$$

where the radius R, the axiodilaton S and the scalar matrix M are the grade-0 fields, $(\tilde{A}^I, \tilde{B}^I)$ is the grade-1 doublet, and ψ is the grade-2 singlet. This indicates that the action can be rewritten as a non-linear sigma-model over the coset space $\frac{O(2k,8)}{O(2k)\times O(8)}$, as in (3.54).

3.2 CHL models in heterotic string

Chaudhuri-Hockney-Lykken models [59, 60, 62, 61] are asymmetric orbifolds of the heterotic string compatified on $T^d \times S^1$ that preserve all of the half-maximal supersymmetry.[3] They exist in type I string constructions, where they were originally discovered [63], as well as in type II string descriptions [60, 61].

In the following sections we review from an heterotic string perspective some details of three- and four-dimensional $\mathbb{Z}_N$ CHL models with prime N, and argue for the presence of strong-weak dualities in these constructions.

3.2.1 CHL moduli space in four dimensions

We consider theories that are freely acting orbifold of the maximal rank model, where a $\mathbb{Z}_N$ rotation acts on the heterotic lattice $\Lambda_{22,6}$ together with an order N shift along one circle inside T^6. This projection removes $28 - r$ of the gauge fields in four dimensions, along with their fermionic and scalar partners. For simplicity we shall restrict ourselves to CHL orbifolds with N prime with $k = 24/(N+1)$. In this case, one can decompose

$$\Lambda_{22,6} = \Lambda_{Nk,8-k} \oplus I\!I_{1,1} \oplus I\!I_{k-3,k-3}\,, \tag{3.64}$$

such that the $\mathbb{Z}_N$ action acts on the first term by a $\mathbb{Z}_N$ rotation, on the second term by an order N shift, leaving $I\!I_{k-3,k-3}$ invariant.[4] We denote by $\Lambda_{k,8-k}$ the quotient of $\Lambda_{Nk,8-k}$ under the $\mathbb{Z}_N$ rotation (see Table 3.1). One thus obtains

$$\Lambda_{r-6,6} = \Lambda_{k,8-k} \oplus I\!I_{1,1}[N] \oplus I\!I_{k-3,k-3}\ , \tag{3.65}$$

i.e. the subgroup of the automorphism group of $\Lambda_{r-6,6}$ which acts trivially on the discriminant group $\Lambda^*_{r-6,6}/\Lambda_{r-6,6}$. The 6 from $\Lambda_{r-6,6}$ always corresponds to the gravity multiplet, while

$$r - 6 = \frac{24}{N+1} + \frac{2(11-N)}{N+1}\,, \tag{3.66}$$

corresponds the number of vector multiplets: the first term can be interpreted as the number of vector multiplets left after orbifolding of the 16-dimensional gauge vectors together with (when $N > 2$) some of the fields descending from dimensional reduction, while the second term can be interpreted as the number of compactified dimensions, or equivalently, the number of vector multiplets that unaffected by the orbifolding. Note also that the second term in (3.66) must always be greater or equal to one for the model to exist since the compactification involves an additional orbifolded circle $S^1/\mathbb{Z}_N$.

Here and below, for any lattice Λ, we denote by $\Lambda[\alpha]$ the same lattice with a quadratic form rescaled by a factor α.[5] Note that the lattice (3.65) is still even, but it is no longer

[3] A given CHL model is defined $d \geq d^*$, where d^* increases monotoneously with the order of the orbifold for the cases considered in this manuscript. See 3.1.

[4] See § A2 of [BCHP2] for details on this construction.

[5] This is equivalent to rescaling the lattice vectors by $\sqrt{\alpha}$.

N	Cycle Shape	k	r	$\Lambda_{k,8-k}$	$\Lambda_m \cong \Lambda_e^*$	$\lvert\Lambda_m^*/\Lambda_m\rvert$
1	1^{24}	12	28		$E_8 \oplus E_8 \oplus I\!I_{6,6}$	1
2	$1^8 2^8$	8	20	$E_8[2]$	$E_8[2] \oplus I\!I_{1,1}[2] \oplus I\!I_{5,5}$	2^{10}
3	$1^6 3^6$	6	16	$D_6[3] \oplus D_2[-1]$	$A_2 \oplus A_2 \oplus I\!I_{3,3}[3] \oplus I\!I_{3,3}$	3^8
5	$1^4 5^4$	4	12	$D_4[5] \oplus D_4[-1]$	$I\!I_{3,3}[5] \oplus I\!I_{3,3}$	5^6
7	$1^3 7^3$	3	10	$D_3[7] \oplus D_5[-1]$	$\left[\begin{smallmatrix}-4 & -1\\ -1 & -2\end{smallmatrix}\right] \oplus I\!I_{2,2}[7] \oplus I\!I_{2,2}$	7^5

Table 3.1: The class of $\mathbb{Z}_N$ CHL orbifolds studied in this manuscript. Here $k = 24/(N+1)$ is the weight of the cusp form whose inverse counts half-BPS states, $r = 2k+4$ is the rank of the gauge group and Λ_m is the lattice of magnetic charges in four dimensions. The discriminant group Λ_m^*/Λ_m is isomorphic to $\mathbb{Z}_N^{k+2}$. $D_0[-1]$ is the null vector and $D_2[-1] = A_1[-1] \oplus A_1[-1]$. Agreement between the lattice Λ_m listed here and $\Lambda_{r-6,6}$ defined in (3.65) follows from the lattice isomorphisms listed in [105].

unimodular, rather it is a lattice of level N, *i.e.*

$$Q \in \Lambda_{r-6,6} \Rightarrow Q^2 \in 2\mathbb{Z}\,, \qquad Q \in \Lambda_{r-6,6}^* \Rightarrow Q^2 \in 2\mathbb{Z}/N\,. \tag{3.67}$$

One can see from (3.67) and arguments similar to § 3.1.1, that the U-duality group $G_4(\mathbb{Z})$ includes $\Gamma_1(N)\times\widetilde{O}(r-6,6,\mathbb{Z})$, where $\Gamma_1(N)$ is the congruence subgroup of $SL(2,\mathbb{Z})$ corresponding to matrices $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ with $c = 0 \bmod N, a = d = 1 \bmod N$, and $\widetilde{O}(r-6,6,\mathbb{Z})$ is the restricted automorphism group of the lattice. A brief, but technical, review of the construction of the CHL partition function [BCHP2] is given a the end of this subsection.

Motivation for strong-weak duality. In [105] it was observed that the Gauss-Bonnet coupling

$$-\frac{1}{(8\pi)^2}\int \mathrm{d}^4x\,\sqrt{-g}\,\log(S_2^k|\Delta_k(S)|^2)(\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2) \tag{3.68}$$

is in fact invariant under the larger group $\widehat{\Gamma}_0(N)$, obtained by adjoining to $\Gamma_0(N)$ the Fricke involution, which acts on modular forms of weight k under $\Gamma_0(N)$ via $f_k(\tau) \mapsto \hat{f}_k(\tau) = (-\mathrm{i}\tau\sqrt{N})^{-k} f_k(-1/(N\tau))$. Based on a detailed study of geometric dualities in the type II dual description, it was conjectured[6] that the full U-duality group in $D=4$ also includes the so-called Fricke S-duality, which acts on the axiodilaton modulus S by the Fricke involution $S \mapsto -1/(NS)$, accompanied by a suitable action $\varsigma \in O(r-6,6)$ on the second factor. Additional evidence for the existence of Fricke S-duality comes from the spectrum of BPS states, to which we now turn.

Moreover, it was observed in [105] that the lattice Λ_m is in fact N-modular, *i.e.* it satisfies

$$\Lambda_m^* \simeq \Lambda_m[1/N]\,. \tag{3.69}$$

In other words, there exists an $O(r-6,6)$ matrix ς such that $\sqrt{N}\varsigma$ maps the lattice Λ_m into itself and such that

$$\Lambda_m^* = \frac{\varsigma}{\sqrt{N}}\Lambda_m \quad (\supset \Lambda_m)\,. \tag{3.70}$$

[6] More generally, Fricke S-duality is conjectured to hold whenever the cycle shape satisfies the balancing condition $m(a) = m(N/a)$ for all $a|N$. [105]

A simple example of N-modular lattice is $\Lambda_{d,d}[N] \oplus \Lambda_{d,d}$, which is relevant for $N = 5$ above. In this case one can parametrise an element in the lattice in $(\mathbb{Z}^d, N\mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d)$ and an element of the dual lattice in $(\mathbb{Z}^d/N, \mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d)$ and define $\varsigma \in O(2d, 2d)$ such that

$$\frac{\varsigma}{\sqrt{N}} = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{N}}\mathbb{1}_{d,d} & 0 \\ 0 & 0 & 0 & \sqrt{N}\mathbb{1}_{d,d} \\ \sqrt{N}\mathbb{1}_{d,d} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{N}}\mathbb{1}_{d,d} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{N}\mathbb{1}_{d,d} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{d,d} \\ \mathbb{1}_{d,d} & 0 & 0 & 0 \\ 0 & \frac{1}{N}\mathbb{1}_{d,d} & 0 & 0 \end{pmatrix} . \tag{3.71}$$

This latter fact is essential to obtain a strong-weak duality acting non-trivially on the charge spectrum.

Half-BPS Charge spectrum. Point-like particles in $D = 4$ carry electric and magnetic charges $(Q, P) \in \Lambda_{em}$ under the r Maxwell fields, where

$$\Lambda_{em} = \Lambda_e \oplus \Lambda_m \ , \quad \Lambda_m = \Lambda_{r-6,6} = \Lambda_e^* \ . \tag{3.72}$$

The lattice Λ_m is tabulated in the sixth column of Table 1.1, taken from [105]. In view of the remarks below (3.65), one has, for any $(Q, P) \in \Lambda_{em}$,

$$Q^2 \in \frac{2}{N}\mathbb{Z} \ , \quad P^2 \in 2\mathbb{Z} \ , \quad P \cdot Q \in \mathbb{Z} \ . \tag{3.73}$$

The last property in particular ensures that the Dirac-Schwinger-Zwanziger pairing $Q \cdot P' - Q' \cdot P$ is integer.

The map (3.70), provided by the N-modularity of the electromagnetic charge lattice, defines the action

$$(Q, P) \mapsto (-\varsigma \cdot P/\sqrt{N}, \varsigma^{-1} \cdot Q\sqrt{N}) \, , \tag{3.74}$$

of the Fricke S-duality on Λ_{em}, which maps $(Q^2, P^2, P \cdot Q) \mapsto (P^2/N, NQ^2, -P \cdot Q)$ and therefore preserves the quantisation conditions (3.73).

Covariance of both the spectrum and the level$-N$ modular form Δ_k hints at a possible strong-weak duality for the F^4 interaction acting as $S \to -1/(NS)$. As we see in the next paragraph, the coefficient of this coupling can be written as a modular integral over the fundamental domain $\Gamma_0(N)\backslash\mathcal{H}_1$, which is itself invariant under the Fricke duality. Indeed, $\Gamma_0(N)\backslash\mathcal{H}_1$ possesses two cusps, $i\infty$ and 0, of width 1 and N respectively, which are exchanged under this duality. Unfortunately, deeper understanding of this strong-weak duality is not available at present.[7]

Construction of the F^4 coupling for CHL models. We give in spirit the construction detailed in [BCHP2] for the freely acting $\mathbb{Z}_N$ orbifolds. To implement the quotient, it is simpler to work at the point of the moduli space $G_{d+16,d}$ where the lattice partition function factorises as (3.64), and $\mathbb{Z}_N$ acts on the lattice $\Lambda_{d+16,d}$ by a permutation with cycle shape $1^k N^k$ (recall $k(N+1) = 24$). One can then construct the CHL lattice $\Lambda_{k,8-k}$

[7] In particular, it does not descend naively from the unorbifolded theory since the Fricke duality cannot be written as a $SL(2, \mathbb{Z})$ element.

in (3.65) for $N = 2, 3, 5, 7$ by applying a Wick rotation on the Niemeier lattices D_k^{N+1}, replacing one D_k by $D_{8-k}[-1]$[8]

$$\begin{aligned} N=2: &\quad D_8^3 &\Rightarrow&\quad D_8^2 \oplus D_0[-1] \\ N=3: &\quad D_6^4 &\Rightarrow&\quad D_6^3 \oplus D_2[-1] \\ N=5: &\quad D_4^6 &\Rightarrow&\quad D_4^5 \oplus D_4[-1] \\ N=7: &\quad D_3^8 &\Rightarrow&\quad D_3^7 \oplus D_5[-1] \end{aligned} \tag{3.75}$$

so that the new lattice is an even self-dual Lorentzian lattice with signature $(Nk, 8-k)$ [121, § A.4]. In particular, it enjoys $\mathbb{Z}_N$ symmetry σ acting by cyclic permutations of the N D_k factors. Insertion in the partition function over (3.75) of elements σ^g is equivalent to changing the lattices as

$$\begin{aligned} N=2: &\quad D_8^2 \oplus D_0[-1] &\Rightarrow&\quad D_8[2] \oplus D_0[-1] \\ N=3: &\quad D_6^3 \oplus D_2[-1] &\Rightarrow&\quad D_6[3] \oplus D_2[-1] \\ N=5: &\quad D_4^5 \oplus D_4[-1] &\Rightarrow&\quad D_4[5] \oplus D_4[-1] \\ N=7: &\quad D_3^7 \oplus D_5[-1] &\Rightarrow&\quad D_3[7] \oplus D_5[-1]\,, \end{aligned} \tag{3.76}$$

so that one is left with the signature $(k, 8-k)$ lattice $\Lambda_{k,8-k}$ advertised in (3.65). Denoting the blocs over this lattice $Z_{k,k-8}\big[{h \atop g}\big]$, we have the natural set of transformation rules, for $h \neq 0 \bmod N$,

$$Z_{k,k-8}\big[{h \atop g}\big](\tau) = Z_{k,k-8}\big[{h \atop 0}\big](\tau + gh^{-1}) \tag{3.77}$$

where h^{-1} is the inverse of h in the multiplicative group $\mathbb{Z}_N$. Then, the untwisted unprojected block decomposes as

$$\begin{aligned} Z_{k,k-8}\big[{0 \atop 0}\big] &= \frac{\Gamma_{\Lambda_{k,\,k-8}}}{\Delta_k\big[{0 \atop 1}\big]} + \sum_{g=0}^{N-1} \frac{\Gamma_{\Lambda^*_{k,\,k-8}}\big[(-1)^{gQ^2}\big]}{\Delta_k\big[{1 \atop g}\big]}\,, \\ &= \sum_{\gamma \in \Gamma_0(N)\backslash SL(2,\mathbb{Z})} \frac{\Gamma_{\Lambda_{k,\,k-8}}}{\Delta_k}\Big|_\gamma\,, \end{aligned} \tag{3.78}$$

i.e. a sum over the coset $\Gamma_0(N)\backslash SL(2,\mathbb{Z}) = \{1, S, TS, \ldots, T^{N-1}S\}$, where

$$\Delta_k = \Delta_k\big[{0 \atop 1}\big] = \eta^k(\tau)\eta^k(N\tau)\,, \qquad \Delta_k\big[{1 \atop g}\big] = e^{\frac{i\pi gk}{12}}\,\eta(\tau)^k\,\eta\left(\tfrac{\tau+g}{N}\right)^k\,, \tag{3.79}$$

and where $\Gamma_{\Lambda_{p,q}}[P(Q)]$ is the standard partition function over $\Lambda_{p,q}$ with insertion of $P(Q)$.

The full partition function can be obtained by multiplying the blocks $Z_{k,k-8}\big[{h \atop g}\big]$, with the orbifold blocks $\Gamma_{\Lambda_{k,\,8-k}}\big[{h \atop g}\big](\tau)$ for the lattice with the shifted partition function for the remaining $d-8+k$ compact directions, with $d > 8-k$,

$$\Gamma_{\Lambda_{d-8+k,\,d-8+k}}\big[{h \atop g}\big] = \tau_2^{\frac{d-8+k}{2}} \sum_{Q \in \Lambda_{d-8+k,d-8+k} + \frac{h}{N}\delta} (-1)^{\frac{2}{N} g\,\delta\cdot Q}\, q^{\frac12 Q_L^2}\, \bar{q}^{\frac12 Q_R^2}\,, \tag{3.80}$$

[8]The lattices D_k^{N+1} consist of the adjoint lattice plus gluing vectors associated to a certain representations of a product group of type $V^{N+1}C^{N+1}S^{N+1}$. See details in § A2 [BCHP2].

where $\delta/N = (0^d; 0^{d-1}, 1/N)$ represents the $1/N$ translation along a S^1 inside T^d.[9] One can then notice that the orbifold blocks written as sums over

$$\tilde{\Lambda} = \Lambda_{k,8-k} \oplus I\!I_{d-8+k,d-8+k} \ , \tag{3.81}$$

rearrange as

$$\begin{aligned}
\widetilde{Z}_{d+2k-8,d}\big[{}^0_1\big] &= \frac{\tau_2^{d/2}}{\Delta_k\big[{}^0_1\big]} \sum_{Q\in\tilde{\Lambda}} \left[1 + (-1)^{\frac{2\delta}{N}\cdot Q} + \ldots + (-1)^{\frac{2(N-1)\delta}{N}\cdot Q}\right] q^{\frac{1}{2}Q_L^2} \bar{q}^{\frac{1}{2}Q_R^2} \\
\widetilde{Z}_{d+2k-8,d}\big[{}^1_0\big] &= \frac{\tau_2^{d/2}}{\Delta_k\big[{}^1_0\big]} \Big[\sum_{Q\in\tilde{\Lambda}^*} + \sum_{Q\in\tilde{\Lambda}^*+\frac{1}{N}\delta} + \ldots + \sum_{Q\in\tilde{\Lambda}^*+\frac{N-1}{N}\delta} \Big] q^{\frac{1}{2}Q_L^2} \bar{q}^{\frac{1}{2}Q_R^2} \\
&\vdots \\
\widetilde{Z}_{d+2k-8,d}\big[{}^1_g\big] &= \frac{\tau_2^{d/2}}{\Delta_k\big[{}^1_g\big]} \Big[\sum_{Q\in\tilde{\Lambda}^*} + \sum_{Q\in\tilde{\Lambda}^*+\frac{1}{N}\delta} + \ldots + \sum_{Q\in\tilde{\Lambda}^*+\frac{N-1}{N}\delta} \Big] (-1)^{gQ^2} q^{\frac{1}{2}Q_L^2} \bar{q}^{\frac{1}{2}Q_R^2} \ ,
\end{aligned} \tag{3.82}$$

which, altogether, rewrites similarly as (3.78)

$$\widetilde{Z}_{d+2k-8,d}\big[{}^0_1\big] + \sum_{g=0}^{N-1} \widetilde{Z}_{d+2k-8,d}\big[{}^1_g\big] = \sum_{\gamma\in\Gamma_0(N)\backslash SL(2,\mathbb{Z})} \widetilde{Z}_{d+2k-8,d}\big[{}^0_1\big]\Big|_\gamma \ . \tag{3.83}$$

Finally, the insertion in $\widetilde{Z}_{d+2k-8,d}\big[{}^0_1\big]$ act as a projection (up to a factor N) onto charges $Q \in \tilde{\Lambda}_{d+2k-8,d}$ which have vanishing component modulo N along the S^1 subjected to the orbifolding. Thus, this projection amounts to reduce the lattice $\tilde{\Lambda}_{d+2k-8,d}$ to the one adviertised in (3.65)

$$\Lambda_{r-6,6} = \Lambda_{k,8-k} \oplus I\!I_{1,1}[N] \oplus I\!I_{k-3,k-3} \ , \tag{3.84}$$

and the full partition function rewrites naturally as

$$\begin{aligned}
\frac{1}{N}\widetilde{Z}_{d+2k-8,d}\big[{}^0_1\big] + \frac{1}{N}\sum_{g=0}^{N-1} \widetilde{Z}_{d+2k-8,d}\big[{}^1_g\big] &= \frac{\Gamma_{\Lambda_{d+2k-8,d}}}{\Delta_k\big[{}^0_1\big]} + \frac{1}{N}\sum_{g=0}^{N-1} \frac{\Gamma_{\Lambda^*_{d+2k-8,d}}\big[(-1)^{gQ^2}\big]}{\Delta_k\big[{}^1_g\big]} \\
&= \sum_{\gamma\in\Gamma_0(N)\backslash SL(2,\mathbb{Z})} \frac{\Gamma_{\Lambda_{d+2k-8,d}}}{\Delta_k\big[{}^0_1\big]}\Big|_\gamma \ .
\end{aligned} \tag{3.85}$$

This description is in agreement with the results stated in Table 1.1, thanks to the isomorphisms

$$\begin{aligned}
D_6[3] \oplus D_2[-1] &\simeq A_2 \oplus A_2 \oplus I\!I_{2,2}[3] \\
D_4[5] \oplus D_4[-1] &\simeq I\!I_{2,2}[5] \oplus I\!I_{2,2} \\
D_3[7] \oplus D_5[-1] &\simeq \begin{pmatrix} -4 & -1 \\ -1 & -2 \end{pmatrix} \oplus I\!I_{1,1}[7] \oplus I\!I_{2,2}
\end{aligned} \tag{3.86}$$

Indeed, both lattices on each line have the same discriminant group $L^*/L = \mathbb{Z}_N^k$.

[9] In more generality, it is a null modulo N vector

Finally, we can obtain the one-loop F^4 amplitude by an insertion of the polynomial P_{abcd} explicited in [BCHP2], and integrating over the fundamental domain $\mathcal{H}/SL(2,\mathbb{Z})$. The integral can thus be unfolded onto a fundamental domain $\Gamma_0(N)\backslash\mathcal{H}$ for the action of $\Gamma_0(N)$ on $\mathcal{H}$, at the expense of keeping only the block $\left[{0 \atop 1}\right]$,

$$F^{(\text{1-loop})}_{abcd} = \text{R.N.} \int_{\Gamma_0(N)\backslash\mathcal{H}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2} \frac{\Gamma_{\Lambda_{d+2k-8,d}}[P_{abcd}]}{\Delta_k} \,, \tag{3.87}$$

where $\Delta_k \equiv \Delta_k\left[{0 \atop 1}\right]$, thus establishing the F^4 coupling coefficient for this class of models.

While the U-duality group in four dimension $G_4(\mathbb{Z})$ must certainly include $\Gamma_1(N) \times \widetilde{O}(r-6,6,\mathbb{Z})$, it may actually be larger. Moreover, special BPS observables may well be invariant under an even larger group. Indeed, the partition function of the coupling coefficient (3.87) turns out to be invariant under the action of the larger duality group $\Gamma_0(N) \times O(r-6,6,\mathbb{Z})$, where $\Gamma_0(N)$ is the subgroup of matrices $\left({a \atop c}\ {b \atop d}\right)$ with $c = 0 \bmod N$ and $O(r-6,6,\mathbb{Z})$ is the full automorphism group of the lattice $\Lambda_{r-6,6}$.

3.2.2 CHL moduli space in three dimensions

Upon further compactification on a circle, additional moduli arise from the radius R of the circle, from the holonomies a^{1I} of the r gauge fields, and from the scalars a^{2I}, ψ dual to the r Maxwell fields and to the Kaluza–Klein gauge field in three dimensions. The U-duality group $G_3(\mathbb{Z})$ includes $G_4(\mathbb{Z})$, the Heisenberg group of large gauge transformations acting on $a^{I,i}, \psi$, and a subgroup of $O(r-5,7,\mathbb{Z})$ containing the restricted automorphism group $\widetilde{O}(r-5,7,\mathbb{Z})$ of the Narain lattice $\Lambda_{r-5,7} = \Lambda_{r-6,6} \oplus I\!I_{1,1}$. The action of these subgroups is most easily seen in the vicinity of the cusps $R \to \infty$ and $g_3 \to 0$, corresponding to the decompactification limit to $D = 4$ and the weak heterotic coupling limit in $D = 3$, where the moduli space reduces to

$$\mathcal{M}_3 \to \begin{cases} \mathbb{R}^+_R \times \mathcal{M}_4 \times \tilde{T}^{2r+1} \\ \mathbb{R}^+_{1/g_3^2} \times \left[\frac{O(r-5,7)}{O(r-5)\times O(7)} / O(r-5,7,\mathbb{Z})\right] \times T^{r+2} \,, \end{cases} \tag{3.88}$$

and where $\mathcal{M}_4$ is parametrised as in (3.1) with the value of the asymptotic scalar fields listed in table 1.1, the T^{2r+1} is parametrised by the grade-1 r Wilson lines $\tilde{A}^I$, their duals $\tilde{B}^I$, and the grade-2 twit potential ψ (3.59). In the second line of (3.88), the T^{r+2} is parametrised be the $r+2$ grade-1 scalars dual to the gauge fields.

For $r = 28$, it is well-known that these subgroups generate the automorphism group $O(24,8,\mathbb{Z})$ of the 'non-perturbative Narain lattice' $\Lambda_{24,8} = \Lambda_{22,6} \oplus I\!I_{2,2}$, as we discussed in section 3.1.2. The U-duality group for CHL models does not seem to have been discussed in the literature, but it is natural to expect that it includes the restricted automorphism group $\widetilde{O}(r-4,8,\mathbb{Z})$ of an extended Narain lattice of the form

$$\Lambda_{r-4,8} = \Lambda_m \oplus I\!I_{1,1} \oplus I\!I_{1,1}[N] \,, \tag{3.89}$$

where $I\!I_{1,1}[N]$ is the standard hyperbolic lattice with quadratic form rescaled by a factor of N, such that $\Lambda^*_{r-4,8}/\Lambda_{r-4,8} \simeq \mathbb{Z}_N^{k+4}$. In terms of the usual construction of $I\!I_{2,2}$ by windings $(n_1,n_2) \in \mathbb{Z}^2$, momenta $(m_1,m_2) \in \mathbb{Z}^2$ and quadratic form $2m_1n_1 + 2m_2n_2$, we define $I\!I_{1,1} \oplus I\!I_{1,1}[N]$ as the sublattice of $I\!I_{2,2}$ where n_2 is restricted to be a multiple of

N. The restricted automorphism group of $I\!I_{1,1} \oplus I\!I_{1,1}[N]$ was determined in [105, 122], and includes $\sigma_{T\leftrightarrow S} \ltimes [\Gamma_1(N) \times \Gamma_1(N)]$, acting by fractional linear transformations on the moduli (T, S) parametrising $G_{2,2}$, such that $|m_1 + Sm_2 + Tn_1 + STn_2|^2/(S_2T_2)$ is invariant (see [123, § C], case V for $N = 2$, or [124, § 3.1.3] for arbitrary N). In the present context, T is interpreted as $\psi + \mathrm{i}R^2$, while S is the heterotic axiodilaton. Thus, $\widetilde{O}(r-4, 8, \mathbb{Z})$ contains the S-duality group $\Gamma_1(N)$ and T-duality group $\widetilde{O}(r-6, 6, \mathbb{Z})$ in four dimensions. In addition, Fricke S-duality in four dimensions follows from the fact that the non-perturbative lattice (3.89) is itself N-modular,

$$\Lambda^*_{r-4,8} \simeq \Lambda_{r-4,8}[1/N] \ , \tag{3.90}$$

which can be checked easily using the table 3.1, and the relation $I\!I_{1,1}[N]^* = I\!I_{1,1}[1/N]$.

3.3 Exact F^4 coupling from supersymmetry constraints

In three dimensional supergravity with half-maximal supersymmetry, linearised supersymmetry invariants can be obtained from the action of supercharge derivatives D^i_α on any homogeneous function of the linearised superfield $W_{\hat{a}a}$ [119, 125, 126]

$$D^i_\alpha W_{\hat{a}a} = (\Gamma_{\hat{a}})^{i\hat{j}} \chi_{\alpha\hat{j}a} \, , \qquad D^i_\alpha \chi_{\beta\hat{j}a} = -\mathrm{i}(\sigma^\mu)_{\alpha\beta}(\Gamma^{\hat{a}})_{\hat{j}}{}^i \partial_\mu W_{\hat{a}a} \, , \tag{3.91}$$

where $\hat{a} = 1 \dots 8$ is a vectorial index for $O(8)$, $i = 1 \dots 8$ for the positive chirality Weyl spinor of $Spin(8)$ and $\hat{\imath} = 1 \dots 8$ the negative chirality Weyl spinor.

In particular, couplings with k derivatives are obtained by acting with $2k$ supercharge derivatives D^i_α, and are thus said to be protected by supersymmetry for $k < 8$. The coupling F^4, of the type $D^8 f(W)$, is thus said to be half-BPS.

At the non-linear level, derivatives of the scalar fields only appear through the pull-back of the right-invariant form $P_{a\hat{b}}$ defining the metric on $G_{r-8,8}$ as $G_{\mu\nu} = 2P_{\mu a\hat{b}} P^{a\hat{b}}_\nu$, and the covariant derivative in tangent frame acting on a symmetric tensor with unhatted indices as

$$\mathcal{D}_{a\hat{b}} A_{a_1\dots a_m} \equiv P_{\mu a\hat{b}} G^{\mu\nu}(\partial_\nu A_{a_1\dots a_m} + m\omega_{\nu(a_1}{}^c A_{a_2\dots a_m)c}) \ . \tag{3.92}$$

The supersymmetry invariant associated to a tensor F_{abcd} on the Grassmanian defines a Lagrangian density $\mathcal{L}$ that decomposes naturally as

$$\begin{aligned}\mathcal{L} = F_{a_1a_2a_3a_4}\mathcal{L}^{a_1a_2a_3a_4} + \mathcal{D}_{(a_1}{}^{\hat{a}} F_{a_2a_3a_4a_5)} \mathcal{L}^{a_1\dots a_5}{}_{\hat{a}} + \mathcal{D}_{(a_1}{}^{\hat{a}_1} \mathcal{D}_{a_2}{}^{\hat{a}_2} F_{a_3a_4a_5a_6)} \mathcal{L}^{a_1\dots a_6}{}_{\hat{a}_1\hat{a}_2} \\ + \mathcal{D}_{(a_1}{}^{\hat{a}_1} \mathcal{D}_{a_2}{}^{\hat{a}_2} \mathcal{D}_{a_3}{}^{\hat{a}_3} F_{a_4a_5a_6a_7)} \mathcal{L}^{a_1\dots a_7}{}_{\hat{a}_1\hat{a}_2\hat{a}_3} \\ + \mathcal{D}_{(a_1}{}^{\hat{a}_1} \mathcal{D}_{a_2}{}^{\hat{a}_2} \mathcal{D}_{a_3}{}^{\hat{a}_3} \mathcal{D}_{a_4}{}^{\hat{a}_4} F_{a_5\dots a_8)} \mathcal{L}^{a_1\dots a_8}{}_{\hat{a}_1\dots\hat{a}_4} \ ,\end{aligned} \tag{3.93}$$

where the polynomials $\mathcal{L}^{n+4}{}_n$ are $O(r-8, 8)$ invariant functions of the covariant scalar field strength, the dreibeins and the gravitini fields. Since non-linear invariants define linear invariants by truncation to lowest order, the covariant polynomials $\mathcal{L}^{4+n}_n$ reduce at lowest order to homogeneous polynomials of degree $n + 4$ in the covariant fields,

$$\mathcal{L}^{abcd} = \sqrt{-g}\big(2P^{(a}_{\mu}{}_{\hat{a}} P^{\mu\, b}{}_{\hat{b}} P^{c|\hat{a}}_{\nu} P^{\nu\, d)\hat{b}} - P^{(a}_{\mu}{}_{\hat{a}} P^{\mu\, b|\hat{a}} P^{c}_{\nu\hat{b}} P^{\nu\, d)\hat{b}} + \dots\big) \ . \tag{3.94}$$

The important conclusion to draw from the linearised analysis in [BCHP2] is that the $O(r-8, 8)$ right-invariant polynomials $\mathcal{L}^{n+4}_n$ appearing in the ansatz (3.93) are symmetric in both sets of indices and traceless in the $O(8)$ indices. Checking the supersymmetry invariance (modulo a total derivative) of $\mathcal{L}$ (3.93) in this basis, one finds that the tensor F_{abcd} must satisfy the constraints [BCHP2]

$$\mathcal{D}_{[a}{}^{[\hat{a}}\mathcal{D}_{b]}{}^{\hat{b}]}F_{cdef} = 0 \ , \qquad \mathcal{D}_{[e}{}^{\hat{a}}F_{a]bcd} = 0 \ . \tag{3.95}$$

Similarly, because the polynomials $\mathcal{L}^{n+4}_n$ are traceless in the $O(8)$ indices, the $O(8)$ singlet component of $\delta(\mathcal{D}F)\mathcal{L}^5_1$ can only be cancelled by terms coming from $F\delta\mathcal{L}^4$, and thus the tensor F_{abcd} satisfies an equation of the form

$$\mathcal{D}_e{}^{\hat{a}}\mathcal{D}_{f\hat{a}}F_{abcd} = 5b_1\delta_{e(f}F_{abcd)} + 5b_2\,\delta_{(fa}F_{bcd)e} \ , \tag{3.96}$$

for the numerical constants b_1, b_2 can be fixed by consistency. In particular, the integrability condition on the component antisymmetric in e and f implies $b_2 = 2b_1 + 4$.

One can then generalise F_{abcd} to a completely symmetric tensor $F^{(p,q)}_{abcd}$ on a general Grassmanian $G_{p,q}$, which would arise by considering a superfield in $D = 10-q$ dimensions with $3 \leq q \leq 6$. The tensor F_{abcd} is thus subject to the constraints (3.95) and

$$\mathcal{D}_e{}^{\hat{a}}\mathcal{D}_{f\hat{a}}F^{(p,q)}_{abcd} = b_1\,\delta_{ef}F^{(p,q)}_{abcd} + 2b_2\delta_{f(a}F^{(p,q)}_{bcd)e} + (2b_2-q)\delta_{e(a}F^{(p,q)}_{bcd)f} + 3b_3\,\delta_{(ab}F^{(p,q)}_{cd)ef} \ . \tag{3.97}$$

with coefficients b_1, b_2, b_3 a priori depending on p and q.

A first integrability condition implies $b_1 = \frac{2b_2-q}{4}$ and $b_3 = b_2$, and considering the antisymmetrised action of three covariant derivatives, one finds that $b_2 = 1$

$$\mathcal{D}_e{}^{\hat{a}}\mathcal{D}_{f\hat{a}}F^{(p,q)}_{abcd} = 5\frac{2-q}{4}\delta_{e(f}F^{(p,q)}_{abcd)} + 5\delta_{(fa}F^{(p,q)}_{bcd)e} \ . \tag{3.98}$$

Finally, let us note that the discussion only applies so far to the local Wilsonian effective action. The Ward identity satisfied by the renormalised coupling $\hat{F}_{abcd}$ is corrected in four dimensions (for $q = 6$) because of the 1-loop divergence of the supergravity amplitude [127], leading a source term in given in [BCHP2].

3.3.1 Conjecture for exact F^4 coupling

The arguments for the existence of a non-perturbative duality group $\widetilde{O}(r-4, 8, \mathbb{Z})$, reviewed section 3.1 and 3.2, as well as the supersymmetry constraints section 3.3 motivated the conjecture of the exact $(\nabla\phi)^4$ coupling as the one-loop integral

$$F^{(r-4,8)}_{abcd}(\Phi) = \text{R.N.}\int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\frac{\Gamma_{\Lambda_{r-4,8}}[P_{abcd}]}{\Delta_k(\tau)} \ , \tag{3.99}$$

which is constructed as the generalisation of one-loop perturbative function – see (2.24) for the full rank case or (3.87) for $N = 2, 3, 5, 7$ – where we replaced the Narain lattice $\Lambda_{r-5,7}$ with its non-perturbative extension $\Lambda_{r-4,8}$ (3.89).

The function (3.99) is manifestly invariant under the non-perturbative dualities mentioned in section 3.1.2, and has the property of satisfying the supersymmetry constraints (3.95), and in particular the equation (3.97), see § 3.2 of [BCHP2].

In order to prove that a solution of the supersymmetry differential constraints corresponds to the expected exact coupling, we must verify that it statisfies the right the boundary condition, by, for instance, matching the perturbative F^4 coupling in the weak string coupling limit. In the next subsection, we compute its weak coupling limit by studying the Fourier decomposition of $F_{abcd}(\Phi)$ near the cusp $g_s \to 0$ in the cases of heterotic string in three dimensions and type II string in four dimensions. We show that zero mode in the Fourier decomposition of (3.99) matches with perturbative answers from the literature.

Weak coupling limit of three-dimensional exact $(\nabla\phi)^4$ couplings. In [BCHP2], we computed the Fourier decomposition of the function $F^{(2k,8)}_{abcd}$ at the cusp $g_3 \to 0$ of

$$G_{2k,8} \simeq \mathbb{R}^+_{1/g_3^2} \times \left[\tfrac{O(r-5,7)}{O(r-5)\times O(7)}/O(r-5,7,\mathbb{Z})\right] , \tag{3.100}$$

corresponding to the weak heterotic coupling limit in $D = 3$. Decomposing as

$$\Lambda_{2k,8} = \Lambda_{2k-1,7} \oplus I\!I_{1,1}[N] , \tag{3.101}$$

the limit studied in this section corresponds to the expansion of the exact $(\nabla\Phi)^4$ couplings in $D = 3$ [BCHP2]. To interpret the resulting contributions in the language of heterotic perturbation theory, one should remember that the U-duality function $F^{(2k,8)}_{abcd}(\Phi)$ is the coefficient of the $(\nabla\Phi)^4$ coupling in the low-energy action written in Einstein frame, such that the metric γ_E is inert under U-duality,

$$S_3 = \int \mathrm{d}^3x \sqrt{-\gamma_E}\, \left[\mathcal{R}[\gamma_E] - (2\delta_{\hat a\hat b}\delta_{\hat c\hat d} - \delta_{\hat a\hat c}\delta_{\hat b\hat d}) F^{(2k,8)}_{abcd}(\Phi)\, \gamma_E^{\mu\rho}\gamma_E^{\nu\sigma} P_\mu^{a\hat a} P_\nu^{b\hat b} P_\rho^{c\hat c} P_\sigma^{d\hat d}\right] + \dots \, . \tag{3.102}$$

In terms of the string frame metric $\gamma = \gamma_E g_3^4$, one finds

$$S_3 = \int \mathrm{d}^3x \sqrt{-\gamma}\, \left[\frac{1}{g_3^2}\mathcal{R}[\gamma] - g_3^2\, (2\delta_{\hat a\hat b}\delta_{\hat c\hat d} - \delta_{\hat a\hat c}\delta_{\hat b\hat d}) F^{(2k,8)}_{abcd}(\Phi)\, \gamma^{\mu\rho}\gamma^{\nu\sigma} P_\mu^{a\hat a} P_\nu^{b\hat b} P_\rho^{c\hat c} P_\sigma^{d\hat d}\right] + \dots \, . \tag{3.103}$$

Using $c_k(0) = k$ for CHL orbifolds with $N > 1$ or $c(0) = 2k$ in the maximal rank case, and $\xi(2) = \frac{\pi}{6}$, the results from [BCHP2] read

$$g_3^2\, F^{(2k,8)}_{abcd} = \frac{3}{2\pi g_3^2}\delta_{(ab}\delta_{cd)} + F^{(2k-1,7)}_{abcd} + \sum_{Q\in\Lambda_{2k-1,7}}{}^{'}\, \bar c_k(Q) e^{-\frac{2\pi\sqrt{2}\,|Q_R|}{g_3^2} + 2\pi \mathrm{i} a\cdot Q} P^{(*)}_{abcd} \, , \tag{3.104}$$

where we omit the detailed form of exponentially suppressed corrections, and where $\bar c_k(Q)$ is as summation measure

$$\bar c_k(Q) = \sum_{\substack{d\geq 1,\\ Q/d\in\Lambda_{2k-1,7}}} d\, c_k\Big(-\frac{Q^2}{2d^2}\Big) + \sum_{\substack{d\geq 1,\\ Q/d\in N\Lambda^*_{2k-1,7}}} N\, d\, c_k\Big(-\frac{Q^2}{2Nd^2}\Big) \, . \tag{3.105}$$

The first two terms in (3.104) should match the tree-level and one-loop contributions, respectively. Indeed, the dimensional reduction of the tree-level $\mathcal{R}^2 + (\mathrm{Tr}F^2)^2$ coupling in ten-dimensional heterotic string theory [128, 129] leads to a tree-level $(\nabla\Phi)^4$ coupling in $D = 3$, with a coefficient independent of N by construction. The second term in (3.104) matches the one-loop contribution (2.24) by construction. The remaining non-perturbative terms can be interpreted as heterotic NS5-brane, KK5-brane and H-monopoles wrapped on any possible T^6 inside T^7 [130]. More precisely, NS5-brane and KK5-brane charges correspond to momentum and winding charges in the hyperbolic part $I\!I_{1,1}[N] \oplus I\!I_{k-2,k-2}$ of $\Lambda_m \oplus I\!I_{1,1}$, while H-monopoles correspond to charges in the gauge lattice $\Lambda_{k,8-k}$ (for the heterotic string compactification on T^7, these sublattices must be replaced by $I\!I_{7,7}$ and $E_8 \oplus E_8$ or D_{16}, respectively).

Weak coupling limit in type II string theory compactified on $K3 \times T^2$. The heterotic axiodilaton S corresponds respectively to the 2-torus Kähler modulus $T_{\rm A}$ in type IIA, and the 2-torus complex structure modulus $U_{\rm B}$ in type IIB, while the type II axiodilaton $S_{\rm A} = S_{\rm B}$ corresponds to the Kähler modulus T of the 2-torus on the heterotic side

$$S = T_{\rm A} = U_{\rm B} \ , \qquad T = S_{\rm A} = S_{\rm B} \ , \qquad U = U_{\rm A} = T_{\rm B} \ . \tag{3.106}$$

In order to expand at small type II string coupling, *i.e.* at large T_2, we decompose the lattice as

$$\Lambda_{2k-2,6} = \Lambda_{2k-4,4} \oplus I\!I_{1,1} \oplus I\!I_{1,1}[N] \, . \tag{3.107}$$

For simplicity we shall use the type IIB moduli in this section, with $S_{B2} = 1/{g_s}^2$. Moreover, we shall only consider the perturbative terms for the Maxwell fields in the RR sector corresponding to indices $\alpha, \beta, \ldots$ along the sublattice $\Lambda_{2k-4,4}$. The type IIB weak coupling limit of the exact F^4 interaction gives [BCHP3]

$$\begin{aligned} \widehat{F}^{(2k-2,6)}_{\alpha\beta\gamma\delta\,{\rm II}} &= \frac{1}{{g_s}^2} F^{(2k-4,4)}_{\alpha\beta\gamma\delta\,{\rm II}} + \frac{3}{2\pi} \delta_{(\alpha\beta}\delta_{\gamma\delta)} \Big(\frac{\hat{\mathcal{E}}_1(NT_{\rm B}) + \hat{\mathcal{E}}_1(T_{\rm B}) + \hat{\mathcal{E}}_1(NU_{\rm B}) + \hat{\mathcal{E}}_1(U_{\rm B}) + \frac{12}{\pi}\log g_s}{N+1} \Big) \\ &= \frac{1}{{g_s}^2} F^{(2k-4,4)}_{\alpha\beta\gamma\delta}(t) - \frac{3}{8\pi^2} \delta_{(\alpha\beta}\delta_{\gamma\delta)} \log({g_s}^{-2k} T_{\rm B2}^k U_{\rm B2}^k |\Delta_k(T_{\rm B}) \Delta_k(U_{\rm B})|^2) \ , \end{aligned} \tag{3.108}$$

where the first term matches the tree-level coupling computed in [131], while the second term is related by supersymmetry to the $\mathcal{R}^2$ coupling computed in [132, 123].

3.4 Exact $\nabla^2 F^4$ coupling and differential Ward identities

This analysis is a generalisation of section 3.3. A six-derivative coupling is obtained by acting with twelve supercharge derivatives on an homogeneous function of the linearized superfield $W_{\hat{a}a}$ for a specific measure [BCHP3], and is thus said to be quarter-BPS. The integral vanishes unless the integrand includes at least the factor $W^{[\hat{a}}_{[a} W^{\hat{b}]}_{b]} W^{[\hat{c}}_{[c} W^{\hat{d}]}_{d]}$, such that the non-trivial integrands are defined as the homogeneous polynomials of degree $4 + 2n + m$ in $W_{\hat{a}a}$.

It follows from the analysis in [BCHP3] that the non-linear invariant only depends on the scalar fields through the tensor $F_{ab,cd}$, its covariant derivatives $\mathcal{D}^n F_{ab,cd}$ and covariant polynomials $\mathcal{L}_{[n,m]}$ in the corresponding irreducible representation of highest weight $m\Lambda_1 + n\Lambda_2$ of $SO(8)$, where the latter only depends on the scalar fields through the covariant scalar field strength $P_{\mu\, a\hat{b}}$, fermions and their covariant derivative, the dreibeins and the gravitini fields. Using the known structure of the $t_8 {\rm tr} \nabla_\mu F \nabla^\mu F {\rm tr} FF$ invariant in ten dimensions [128],[10] one can compute the first covariant density $\mathcal{L}_{[0,0]}$ bosonic component

$$\begin{aligned}\mathcal{L}^{ab,cd} &= \frac{\sqrt{-g}}{8\pi}\Big(2P_{(\mu}{}^{[a}{}_{\hat a}\nabla_\sigma P_{\nu)}{}^{b]\hat a}P^{\mu[c}{}_{\hat b}\nabla^\sigma P^{\nu|d]\hat b} + 2P_\mu{}^{[a}{}_{(\hat a}\nabla_\sigma P^{\mu|b]}{}_{\hat b)}P^{\nu[c|\hat a}\nabla^\sigma P_\nu{}^{d]\hat b} \\ &\quad - P_\mu{}^{[a}{}_{\hat a}\nabla_\sigma P^{\mu\,|b]\hat a}P_\nu{}^{[c}{}_{\hat b}\nabla^\sigma P^{\nu\,|d]\hat b} - 4P_{[\mu}{}^{[a|\hat a}\nabla_\sigma P_{\nu]}{}^{b]\hat b}P^{\mu[c}{}_{[\hat a}\nabla^\sigma P^{\nu\,|d]}{}_{\hat b]} + \dots\Big)\ .\end{aligned} \tag{3.109}$$

The factor of π is introduced by convenience for the definition (2.37) to hold. Moreover, the only non-vanishing tensors in this mass dimension are the polynomials $\mathcal{L}_{[n,m]}$ with $0 \le n \le 2$ and $0 \le m \le 4$, such that the invariant $\mathcal{L}$ admits the decomposition

$$\begin{aligned}\mathcal{L} &= F_{ab,cd}\mathcal{L}^{ab,cd} + \mathcal{D}_e{}^{\hat a}F_{ab,cd}\mathcal{L}^{ab,cd,e}_{\hat a} + \mathcal{D}_{(e}{}^{(\hat a}\mathcal{D}_{f)}{}^{\hat b)}F_{ab,cd}\mathcal{L}^{ab,cd,e,f}_{\hat a,\hat b} + \mathcal{D}_{[e}{}^{[\hat a}\mathcal{D}_{f]}{}^{\hat b]}F_{ab,cd}\mathcal{L}^{ab,cd,ef}_{\hat a\hat b} \\ &\quad + \dots + \mathcal{D}_{(b_1}{}^{(\hat b_1}\cdots\mathcal{D}_{b_4)}{}^{\hat b_4)}\mathcal{D}_{a_1}{}^{\hat a_1}\cdots\mathcal{D}_{a_4}{}^{\hat a_4}F_{a_5a_6,a_7a_8}\mathcal{L}^{a_1a_2,a_3a_4,a_5a_6,a_7a_8,b_1,b_2,b_3,b_4}_{\hat a_1\hat a_2,\hat a_3\hat a_4,\hat b_1,\hat b_2,\hat b_3,\hat b_4}\ ,\end{aligned} \tag{3.110}$$

where the $\mathcal{L}^{a_1a_2,\dots,a_{2n+3}a_{2n+4},b_1,\dots,b_m}_{\hat a_1\hat a_2,\dots,\hat a_{2n-1}\hat a_{2n},\hat b_1,\dots,\hat b_m}$ are in the irreducible representation of highest weight $m\check\alpha_1 + n\check\alpha_2$ of $SO(8)$ and admit the symmetry of the Young tableau $[n+2,m]$ with respect to the permutation of the $SO(p)$ indices. In particular, $F_{ab,cd}$ transforms according to ⊞, realised by first symmetrising along the columns and then antisymmetrising along the rows $[ab],[cd]$.

Checking the supersymmetry invariance (modulo a total derivative) of $\mathcal{L}$ in this basis, one finds that the tensor $F_{ab,cd}$ must satisfy the constraints [BCHP3]

$$\mathcal{D}_{[a_1}{}^{\hat a}F_{a_2a_3],bc} = 0\ , \quad \mathcal{D}_{[a_1}{}^{[\hat a_1}\mathcal{D}_{a_2}{}^{\hat a_2]}F_{a_3]b,cd} = 0\ , \quad \mathcal{D}_{[a_1}{}^{[\hat a_1}\mathcal{D}_{a_2}{}^{\hat a_2}\mathcal{D}_{a_3]}{}^{\hat a_3]}F_{cd,ef} = 0\ . \tag{3.111}$$

Similarly, because the $\mathcal{L}_{[n,m]}$ are traceless in the $SO(8)$ indices, the $SO(8)$ singlet component of $\delta(\mathcal{D}F)\mathcal{L}_{[0,1]}$ can only be cancelled by terms coming from $F\delta\mathcal{L}_{[0,0]}$, and thus the tensor $F_{ab,cd}$ must obey an equation of the form [BCHP3]

$$\begin{aligned}\mathcal{D}_e{}^{\hat a}\mathcal{D}_{f\hat a}F_{ab,cd} &= b_1\Big(-\delta_{ef}F_{ab,cd} + \delta_{e[a}F_{b]f,cd} + \delta_{e[c}F_{d]f,ab}\Big) \\ &\quad - 3b_2\big(\delta_{f[a}F_{b]e,cd} + \delta_{f[c}F_{d]e,ab}\big) - 4b_2\delta_{c][a}F_{b](e,f)[d}\ ,\end{aligned} \tag{3.112}$$

where the numerical constants b_1, b_2 can be fixed by consistency. In particular the integrability condition on the component antisymmetric in e and f implies $b_1 = 4 - 3b_2$.

One can then generalise $F_{ab,cd}$ to a tensor $F^{(p,q)}_{ab,cd}$ on a general Grassmanian $G_{p,q}$, which would arise by considering a superfield in $D = 10 - q$ dimensions with $4 \le q \le 6$. The same argument leads to the conclusion that $F^{(p,q)}_{ab,cd}$ satisfies to (3.112) with $b_1 = \frac{q}{2} - 3b_2$, and another integrability condition gives $b_2 = \frac{1}{2}$ [BCHP3]. One can then represent a tensor with the symmetry ⊞ with two pairs of indices that are manifestly symmetric, *i.e.* $G_{ab,cd} = G_{ba,cd} = G_{ab,dc} = G_{cd,ab}$ such that $G_{(ab,c)d} = 0$, such that

$$F_{ab,cd} = G_{c][a,b][d}\ , \qquad G_{ab,cd} = -\tfrac{4}{3}F_{a)(c,d)(b}\ . \tag{3.113}$$

The tensor $G_{ab,cd}$ thus satisfies the constraints

$$\mathcal{D}_{[a_1}{}^{\hat a}G^{(p,q)}_{a_2|b|,a_3]c} = 0\ , \quad \mathcal{D}_{[a_1}{}^{[\hat a_1}\mathcal{D}_{a_2}{}^{\hat a_2]}G^{(p,q)}_{a_3]b,cd} = 0\ , \quad \mathcal{D}_{[a_1}{}^{[\hat a_1}\mathcal{D}_{a_2}{}^{\hat a_2}\mathcal{D}_{a_3]}{}^{\hat a_3]}G^{(p,q)}_{cd,ef} = 0\ . \tag{3.114}$$

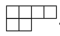
[10] with $t_8F^4 = F_{\mu\nu}F^{\nu\sigma}F_{\sigma\rho}F^{\rho\mu} - 1/4(F_{\mu\nu}F^{\mu\nu})^2$.

and

$$\mathcal{D}_{(e}{}^{\hat{a}}\mathcal{D}_{f)\hat{a}}G^{(p,q)}_{ab,cd} \;=\; \tfrac{3-q}{2}\delta_{ef}G^{(p,q)}_{ab,cd} + \tfrac{6-q}{2}\big(\delta_{e)(a}G^{(p,q)}_{b)(f,cd} + \delta_{e)(c}G^{(p,q)}_{d)(f,ab}\big) + \tfrac{3}{2}\delta_{\langle ab,}G^{(p,q)}_{cd\rangle,ef}\;. \tag{3.115}$$

The discussion only applies so far to a supersymmetry invariant modulo the classical equations of motion, whereas one must take into account the first correction in $(\nabla\Phi)^4$. This implies that corrections to the differential equations must be quadratic source terms in the coefficient $F^{(p,q)}_{abcd}$ defining the $(\nabla\Phi)^4$ coupling (2.24). Constraints on $F^{(p,q)}_{abcd}$ (3.111) indicates that there is no possible correction to (3.114), and contributions to (3.115) can be restrained by looking at their possible representations. Because  is trivially satisfied

$$\frac{1}{2}\mathcal{D}_{[a_1|}{}^{\hat{a}}\mathcal{D}_{b\hat{a}}F^{(p,q)}_{|a_2a_3],cd} = -\frac{q}{4}\delta_{b[a_1}F^{(p,q)}_{a_2a_3],cd} - \frac{q}{4}\delta_{c][a_1}F^{(p,q)}_{a_2a_3],b[d}\;, \tag{3.116}$$

the source term quadratic in $F^{(p,q)}_{abcd}$ must belong to the representation . One finally finds that the only possible source term that also satisfies to the constraint (3.114) is $F^{(p,q)}_{|e)\langle ab,}{}^{g}F^{(p,q)}_{cd\rangle(f|g}$.

We conclude that the correct supersymmetry constraint for $G^{(p,q)}_{ab,cd}$ reads

$$\begin{aligned}\mathcal{D}_{(e}{}^{\hat{a}}\mathcal{D}_{f)\hat{a}}G^{(p,q)}_{ab,cd} = \tfrac{3-q}{2}\delta_{ef}G^{(p,q)}_{ab,cd} + \tfrac{6-q}{2}\big(\delta_{e)(a}G^{(p,q)}_{b)(f,cd} + \delta_{e)(c}G^{(p,q)}_{d)(f,ab}\big) + \tfrac{3}{2}\delta_{\langle ab,}G^{(p,q)}_{cd\rangle,ef}\\ - \frac{3\varpi}{2}F^{(p,q)}_{|e)\langle ab,}{}^{g}F^{(p,q)}_{cd\rangle(f|g}\;,\end{aligned} \tag{3.117}$$

where ϖ is an undetermined numerical coefficient at this stage. In [BCHP3], we show by an explicit calculation that the genus-two modular integral $G^{(p,q)}_{ab,cd}$ satisfies (3.117) with $\varpi = \pi$.

Let us note that this discussion only applies to the Wilsonian effective action. The differential Ward identity satisfied by the renormalised coupling $\hat{G}_{ab,cd}$ from the 1PI effective action is corrected in four dimensions ($q = 6$) by constant terms and by terms linear in $\hat{F}_{abcd}$ [BCHP3].

3.4.1 Conjecture for $\nabla^2(\nabla\phi)^4$

As for the $(\nabla\phi)^4$ coupling of the previous section, the arguments for the existence of a non-perturbative duality group $\widetilde{O}(r-4,8,\mathbb{Z})$, reviewed in §3.1 and §3.2, as well as the supersymmetry constraints, §3.3, motivated to conjecture the exact $\nabla^2(\nabla\phi)^4$ coupling as the two-loop integral

$$G_{ab,cd}(\Phi) = \text{R.N.}\int_{\Gamma_{2,0}(N)\backslash\mathcal{H}_2} \frac{\mathrm{d}\Omega_1\mathrm{d}\Omega_2}{|\Omega_2|^2}\,\frac{\Gamma^{(2)}_{\Lambda_{r-4,8}}[P_{ab,cd}]}{\Phi_{k-2}(\Omega)}\,, \tag{3.118}$$

which is constructed as the generalisation of two-loop perturbative function (2.37) (with $\Gamma_{2,0}(1) = Sp(4,\mathbb{Z})$), where we replaced the Narain lattice $\Lambda_{r-5,7}$ with its non-perturbative extension $\Lambda_{r-4,8}$ (3.89). In the case of CHL models, the construction of (3.118) is more involved than the genus one case, but has been worked out in details for $N = 2$ in § B.2.2 of [BCHP3], and generalised to $N = 3, 5, 7$ with a line of argument similar in spirit to the genus one presentation in § 3.2.1 of this manuscript.

The function (3.118) is manifestly invariant under the non-perturbative dualities mentioned in section 3.1.2, and satisfies the supersymmetry constraints (3.114), and in particular the equation (3.115), see § 3.3 of [BCHP3].

In the next subsection, we look at the weak coupling limit of heterotic string in three dimensions and type II string in four dimensions. We show that the zero mode in the Fourier decomposition of (3.118) matches with expected perturbative computations when known, and consider them as predictions otherwise.

Weak coupling limit of three-dimensional exact $\nabla^2(\nabla\phi)^4$ couplings. The Fourier decomposition of the function $F^{(r-4,8)}_{abcd}$ at the cusp $g_3 \to 0$ (3.88) corresponding to the weak heterotic coupling limit in $D = 3$ was computed in [BCHP2]. In this limit, the lattice $\Lambda_{2k,8}$ decomposes into

$$\Lambda_{2k-1,7} \oplus I\!I_{1,1}[N]\,, \tag{3.119}$$

where the 'radius' of the second factor is related to the heterotic string coupling by $g_3 = 1/\sqrt{R}$, and the U-duality group is broken to $\widetilde{O}(2k-1, 7, \mathbb{Z})$, the restricted automorphic group of $\Lambda_{2k-1,7}$. In order to interpret the results as perturbative contributions to the $\nabla^2(\nabla\phi)^4$ interaction, it is convenient to multiply the coupling by a factor of g_3^6, which arises due to the Weyl rescaling $\gamma_E = \gamma_s/g_3^4$ from the Einstein frame to the string frame, see §4.3 of [BCHP2]. The weak coupling expansion can be extracted from §4.1 of [BCHP3] upon setting $q = 8$, $v = 1$, and reads

$$\begin{aligned} g_3{}^6 G^{(2k,8)}_{\alpha\beta,\gamma\delta} = & -\frac{3}{4\pi g_3{}^2}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} - \frac{1}{4}\delta_{\langle\alpha\beta,}G^{(2k-1,7)}_{\gamma\delta\rangle}(\varphi) + g_3{}^2\, G^{(2k-1,7)}_{\alpha\beta,\gamma\delta}(\varphi) \\ & + \sum_{Q\in\Lambda^*_{2k-1,7}}{}' \frac{3e^{-\frac{2\pi}{g_3^2}\sqrt{2Q_R^{\,2}}+2\pi i Q\cdot a}}{2Q_R^{\,2}}\, \bar{G}^{(2k-1,7)}_{\langle\alpha\beta,}(Q,\varphi)\Big(Q_{L\gamma}Q_{L\delta\rangle}\Big(\sqrt{2Q_R^2}+\tfrac{g_3{}^2}{2\pi}\Big) - \tfrac{g_3{}^2}{8\pi}\delta_{\gamma\delta\rangle}\Big) \\ & + \sum_{Q\in\Lambda^*_{2k-1,7}}{}' e^{-\frac{4\pi}{g_3^2}\sqrt{2Q_R^{\,2}}}\, G_{\alpha\beta,\gamma\delta}(g_3, Q_L, Q_R)\,. \end{aligned} \tag{3.120}$$

The three first terms in (3.120) correspond to the two-loop perturbative contribution computed in (2.37), the one-loop contribution (2.29), and the splitting degeneration contribution. The latter reproduces the tree-level $\nabla^2(\nabla\phi)^4$, obtained by dimensional reduction of the $\nabla^2 F^4$ coupling in 10 dimensions.

The exponentially suppressed terms in the second line of (3.120) can be interpreted as instantons from Euclidean NS five-branes wrapped respectively on any possible T^6 inside T^7, KK (6,1)-branes wrapped with any S^1 Taub-NUT fiber in T^7, and H-monopoles wrapped on T^7. Their precise expression can be found in [BCHP3]. Although we obtain a definite answer for such contributtion, the orbit method misses exponentially suppressed terms which do not depend on the axions a in the last line of (3.120). The existence of these terms is clear from the differential constraint (3.117), since the $(\nabla\phi)^4$ coupling F_{abcd} appearing on the right-hand side contains both instanton and anti-instanton contributions. Unfortunately, our current tools do not allow us to extract these contributions from the unfolding method at present.

Finally, it is worth stressing that while the perturbative contributions $G^{(2k-1,7)}_{ab}$ and $G^{(2k-1,7)}_{ab,cd}$ have singularities in codimension 7 inside $\mathcal{M}_3$ at points of enhanced gauge symmetry, the full instanton-corrected coupling (3.118) has only singularities in codimension 8.

Weak coupling limit in type II string theory compactified on $K3\times T^2$ The expansion of the exact $\nabla^2 F^4$ and $\mathcal{R}^2F^2$ terms in $D=4$ is obtained in §5.3.1 of [BCHP3], and we now consider the weak coupling limit on the type II side. Recall that $S=T_{\rm A}=U_{\rm B}$, *i.e.* the heterotic axiodilaton corresponds to the 2-torus Kähler modulus in type IIA, and the 2-torus complex structure modulus in type IIB, while the type II axiodilaton $S_{\rm A}=S_{\rm B}=T$ corresponds to the heterotic Kähler modulus (3.106).

At large $T_{\rm B2}$, *i.e.* small type II coupling, the lattice decomposes as (3.107), and the exact ∇^2F^4 interaction is obtained from §5.3.1 of [BCHP3] after dropping the logarithmic terms in R,

$$\begin{aligned}\widehat{G}^{(2k-2,6)}_{ab,cd\,\rm NP}(U_{\rm B},\varphi) = &\ \widehat{G}^{(2k-2,6)}_{ab,cd}(\varphi)-\frac{3}{4\pi}\delta_{\langle ab,}\delta_{cd\rangle}\Big(\frac{\hat{\mathcal{E}}_1(NU_{\rm B})+\hat{\mathcal{E}}_1(U_{\rm B})}{N+1}\Big)^2 \\ &-\frac{1}{4}\delta_{\langle ab,}\Big(\frac{N\hat{\mathcal{E}}_1(NU_{\rm B})-\hat{\mathcal{E}}_1(U_{\rm B})}{N^2-1}\widehat{G}^{(2k-2,6)}_{cd\rangle}(\varphi)+\frac{N\hat{\mathcal{E}}_1(U_{\rm B})-\hat{\mathcal{E}}_1(NU_{\rm B})}{N^2-1}\,{}^{\varsigma}\widehat{G}^{(2k-2,6)}_{cd\rangle}(\varphi)\Big)\ ,\end{aligned} \tag{3.121}$$

where φ belongs to the Grassmannian on $\Lambda_{2k-2,6}$. We neglect the non-perturbative contributions and use the decomposition of $\widehat{G}^{(2k-2,6)}_{ab,cd}(\varphi)$, which can be obtained from §5.3.1 of [BCHP3] by replacing the moduli as $R^2=S_{\rm B2}=\frac{1}{g_s^2}$ and $\varphi=t$ the K3 moduli of the Grassmanian $G_{(2k-4,4)}$. After expanding around $q=6+2\epsilon$,[11] we find

$$\begin{aligned}\widehat{G}^{(2k-2,6)}_{\alpha\beta,\gamma\delta}(\varphi)\sim &\ \frac{1}{{g_s}^4}\widehat{G}^{(2k-4,4)}_{\alpha\beta,\gamma\delta}(t)-\frac{3}{4\pi}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\Big(\frac{\hat{\mathcal{E}}_1(NT_{\rm B})+\hat{\mathcal{E}}_1(T_{\rm B})+\frac{12}{\pi}\log g_s}{N+1}\Big)^2 \\ &-\frac{1}{4{g_s}^2}\delta_{\langle\alpha\beta,}\Big(\frac{\frac{N\hat{\mathcal{E}}_1(NT_{\rm B})-\hat{\mathcal{E}}_1(T_{\rm B})}{N-1}+\frac{6}{\pi}\log g_s}{N+1}\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t)+\frac{\frac{N\hat{\mathcal{E}}_1(T_{\rm B})-\hat{\mathcal{E}}_1(NT_{\rm B})}{N-1}+\frac{6}{\pi}\log g_s}{N+1}\,{}^{\varsigma}\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t)\Big)\ .\end{aligned} \tag{3.122}$$

To compute the power-like terms of $\widehat{G}^{(2k-2,6)}_{ab}(\varphi)$, one proceeds as in [BCHP2] and finds after expanding around $q=6+2\epsilon$ and neglecting the non-perturbative contributions

$$\begin{aligned}\widehat{G}^{(2k-2,6)}_{\alpha\beta}(\varphi)\sim\frac{1}{{g_s}^2}\Big(&\widehat{G}^{(2k-4,4)}_{\alpha\beta}(t)+\frac{2N}{N+1}\delta_{\alpha\beta}(\hat{\mathcal{E}}_1(T_{\rm B})-\hat{\mathcal{E}}_1(NT_{\rm B}))\Big) \\ &+\frac{12}{N+1}\frac{1}{2\pi}\delta_{\alpha\beta}\Big(\frac{12}{\pi}\log(g_s)+\hat{\mathcal{E}}_1(T_{\rm B})+\hat{\mathcal{E}}_1(NT_{\rm B})\Big)\ ,\end{aligned} \tag{3.123}$$

while the ones of ${}^{\varsigma}\widehat{G}^{(2k-2,6)}_{ab}(\varphi)$ are obtained by acting on the Kähler moduli $T_{\rm B}$ by Fricke duality $T_{\rm B}\to-\frac{1}{NT_{\rm B}}$, and on the K3 moduli t with the involution ς, so that

$$\begin{aligned}{}^{\varsigma}\widehat{G}^{(2k-2,6)}_{\alpha\beta}(\varphi)\sim\frac{1}{{g_s}^2}\Big(&{}^{\varsigma}\widehat{G}^{(2k-4,4)}_{\alpha\beta}(t)+\frac{2N}{N+1}\delta_{\alpha\beta}(\hat{\mathcal{E}}_1(NT_{\rm B})-\hat{\mathcal{E}}_1(T_{\rm B}))\Big) \\ &+\frac{12}{N+1}\frac{1}{2\pi}\delta_{\alpha\beta}\Big(\frac{12}{\pi}\log(g_s)+\hat{\mathcal{E}}_1(T_{\rm B})+\hat{\mathcal{E}}_1(NT_{\rm B})\Big)\ .\end{aligned} \tag{3.124}$$

[11] Note that the lattice $\Lambda_{2k-2,6}$ is kept fixed, and the expansion in $q=6+2\epsilon$ only applies to the numerical value of the various exponents, just as if one introduced a regularising factor of $|\Omega_2|^\epsilon$ in the genus 2 integral.

Collecting all terms, we obtain the complete perturbative $\nabla^2 F^4$ coupling in $D = 4$,

$$\begin{aligned}
\widehat{G}^{(2k-2,6)}_{\alpha\beta,\gamma\delta\ \mathrm{II}} &= \frac{1}{g_s^{\,4}}\widehat{G}^{(2k-4,4)}_{\alpha\beta,\gamma\delta}(t) \\
&- \frac{1}{4(N+1)g_s^{\,2}}\delta_{\langle\alpha\beta,}\Big(\Big(\frac{N\hat{\mathcal{E}}_1(NT_{\rm B})-\hat{\mathcal{E}}_1(T_{\rm B})+N\hat{\mathcal{E}}_1(NU_{\rm B})-\hat{\mathcal{E}}_1(U_{\rm B})}{N-1}+\frac{6}{\pi}\log g_s\Big)\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t) \\
&\quad + \Big(\frac{N\hat{\mathcal{E}}_1(T_{\rm B})-\hat{\mathcal{E}}_1(NT_{\rm B})+N\hat{\mathcal{E}}_1(U_{\rm B})-\hat{\mathcal{E}}_1(NU_{\rm B})}{N-1}+\frac{6}{\pi}\log g_s\Big)\,{}^{\varsigma}\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t) \\
&\qquad -2N\delta_{\gamma\delta\rangle}\frac{(\hat{\mathcal{E}}_1(T_{\rm B})-\hat{\mathcal{E}}_1(NT_{\rm B}))(\hat{\mathcal{E}}_1(U_{\rm B})-\hat{\mathcal{E}}_1(NU_{\rm B}))}{N-1}\Big) \\
&- \frac{3}{4\pi}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\Big(\frac{\hat{\mathcal{E}}_1(NT_{\rm B})+\hat{\mathcal{E}}_1(T_{\rm B})+\hat{\mathcal{E}}_1(NU_{\rm B})+\hat{\mathcal{E}}_1(U_{\rm B})+\frac{12}{\pi}\log g_s}{N+1}\Big)^2 . \qquad (3.125)
\end{aligned}$$

The terms involving $\log g_s$ originate from the mixing between the local and non-local terms in the effective action [34]. The result (3.125) is manifestly invariant under the exchange of $U_{\rm B}$ and $T_{\rm B}$, and is thus identical in type IIA and type IIB strings. It is also invariant under the combined Fricke duality $T_{\rm B} \to -\frac{1}{NT_{\rm B}}$, $U_{\rm B} \to -\frac{1}{NU_{\rm B}}$, $t \to \varsigma t$ [105], which is built in the conjecture (3.118).

The limit $N = 1$ in this case is subtle, and for the full rank case (3.125) must be replaced by [12]

$$\begin{aligned}
G^{(22,6)}_{\alpha\beta,\gamma\delta\ \mathrm{II}} &= \frac{1}{g_s^{\,4}}\widehat{G}^{(20,4)}_{\alpha\beta,\gamma\delta}(t)+\frac{3}{4\pi g_s^{\,2}}\delta_{\langle\alpha\beta,}\Big(\log(T_{\rm B2}|\eta(T_{\rm B})|^4)+\log(U_{\rm B2}|\eta(U_{\rm B})|^4)-2\log g_s\Big)G^{(20,4)}_{\gamma\delta\rangle}(t) \\
&- \frac{27}{4\pi^3}\delta_{\langle\alpha\beta}\delta_{\gamma\delta\rangle}\Big(\log(T_{\rm B2}|\eta(T_{\rm B})|^4)+\log(U_{\rm B2}|\eta(U_{\rm B})|^4)-2\log g_s\Big)^2 . \qquad (3.126)
\end{aligned}$$

It would be interesting to check these predictions by explicit perturbative computations in type II string theory. To simplify the results, one can use the formulae

$$\frac{\hat{\mathcal{E}}_1(NT_{\rm B})+\hat{\mathcal{E}}_1(T_{\rm B})}{N+1} = -\frac{1}{4\pi}\log(T_{\rm B2}^{k}|\Delta_k(T_{\rm B})|) , \qquad \hat{\mathcal{E}}_1(T_{\rm B}) = -\frac{1}{4\pi}\log(T_{\rm B2}^{12}|\Delta(T_{\rm B})|) , \qquad (3.127)$$

to rewrite the two-loop contribution on the last line of (3.125) as

$$-\frac{3}{(4\pi)^3}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\big(\log(g_s^{-2k}T_{\rm B2}^{k}U_{\rm B2}^{k}|\Delta_k(T_{\rm B})\Delta_k(U_{\rm B})|^2)\big)^2 . \qquad (3.128)$$

The $(\log g_s)^2$ term is consistent with the two-loop logarithmic divergence of the four-photon amplitude [133] (recall that the $\log g_s$ can be traced back to the logarithm of the Mandelstam variables in the full amplitude, and therefore to the logarithm supergravity divergences [34][BCHP2]). The term linear in $\log g_s$ in (3.128), corresponding to the $t_8 F^4$

[12] Note that $G^{(20,4)}_{\alpha\beta}$ is finite for the full rank case, whereas $\widehat{G}^{(2k-4,4)}_{\alpha\beta}$ requires in general a regularisation due to the 1-loop supergravity divergence in six dimensions.

form factor divergence, can be rewritten as

$$\begin{aligned}
&-\frac{3k}{4\pi}\log g_s \delta_{\langle\alpha\beta,}\Big(\frac{1}{12{g_s}^2}(\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t)+{}^{\varsigma}\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t))-\delta_{\gamma\delta\rangle}\frac{1}{8\pi^2}\log(T_{\mathrm{B}2}^kU_{\mathrm{B}2}^k|\Delta_k(T_{\mathrm{B}})\Delta_k(U_{\mathrm{B}})|^2)\Big)\\
&\qquad=-\frac{3}{4\pi}\log g_s\,\delta_{\langle\alpha\beta,}\Big(\frac{1}{{g_s}^2}F^{(2k-4,4)\eta}_{\gamma\delta\rangle\eta}(t)-\delta_{\gamma\delta\rangle}\frac{2k}{(4\pi)^2}\log(T_{\mathrm{B}2}^kU_{\mathrm{B}2}^k|\Delta_k(T_{\mathrm{B}})\Delta_k(U_{\mathrm{B}})|^2)\Big)\\
&\qquad=-\frac{3}{4\pi}\log g_s\,\delta_{\langle\alpha\beta,}\widehat{F}^{(2k-2,6)c}_{\gamma\delta\rangle c\ \mathrm{II}}\ ,
\end{aligned}\tag{3.129}$$

where one uses integration by part on the definition of $F^{(2k-2,6)}_{\alpha\beta\gamma\delta}$ with $-\frac{1}{i\pi}\frac{\partial}{\partial\tau}\frac{1}{\Delta_k(\tau)}=\frac{k}{12}(E_2(\tau)+NE_2(N\tau))/\Delta_k(\rho)$, and $\delta_{(ab}\delta_{cd)}\delta^{cd}=\frac{2k}{3}\delta_{ab}$. Ignoring these logarithmic contributions, the two-loop coupling (3.128) does not depend on the K3 moduli, as required by supersymmetry, and might be computable in topological string theory.

The amplitudes with two photons in the Ramond sector and two gravitons can be obtained in the same way. It is non vanishing only when both photons have the same polarisation and the gravitons' differ to one another. In type IIB string, the complex amplitude is obtained through the Kähler derivative of the same function (3.125) with respect to U_{B}, *e.g.* in the full rank case

$$\begin{aligned}
R^{(22,6)}_{\alpha\beta\ \mathrm{II}}=&-\frac{9}{2\pi^3}\delta_{\alpha\beta}\hat{E}_2(U_{\mathrm{B}})\Big(\log(T_{\mathrm{B}2}|\eta(T_{\mathrm{B}})|^4)+\log(U_{\mathrm{B}2}|\eta(U_{\mathrm{B}})|^4)-2\log g_s\Big)\\
&+\frac{1}{4\pi{g_s}^2}\hat{E}_2(U_{\mathrm{B}})\,G^{(20,4)}_{\alpha\beta}(t)\ ,
\end{aligned}\tag{3.130}$$

or with respect to T_{A} in type IIA. The $\log g_s$ term can be interpreted as the divergence of the form factor of the operator $\mathcal{R}\,F_R^2$ (where $F_R^{\hat{\alpha}}$ are the graviphoton field strengths) belonging to the $\mathcal{R}^2$-type supersymmetric invariant.

Chapter 4

Black hole counting from instantonic corrections

In this chapter we review the application of [BCHP1], [BCHP3] to the counting of quarter-BPS black holes in $\mathcal{N} = 4$ supergravities.

As we reviewed in the introducing paragraph 1.1, these black holes do not emit Hawking radiation, and are thus stable stationary objects, or solitons. There are moreover invariant under a certain number of supersymmetry transformations, and their mass saturate the Bogomol'nyi bound (3.37). The stability linked to these properties allows some control over the dynamics of the microscopic configurations corresponding to these black holes, which involve various object depending on the string theoretic description as we describe in §4.2. Furthermore, the entropy of these objects has the particularity of being unaffected by variations of the gravitational coupling [25]. The corresponding microscopic states have thus been studied in the weak coupling regime where the gravitational back-reaction of the system can be ignored, and the results were continued to strong coupling, where the system can be described as a black hole. In the regime where the size of the black hole is large, it was found for certain five-dimensional black holes that [25, 134, 135, 136]

$$S_{BH}(Q, P) = S_{stat}(Q, P)\,, \tag{4.1}$$

where $S_{HB}(Q, P)$ denotes the Bekenstein-Hawing entropy of an extremal black hole with charge (Q, P), and $S_{stat}(Q, P)$ denotes the entropy of the corresponding microstates, obtained as the logarithm of their degeneracy given by the statistical calculation

$$S_{stat} = \ln d(Q)\,. \tag{4.2}$$

This Bekenstein-Hawking formula (1.8) remains valid as long as the size of the horizon is large compared to the space-time curvature and other field strengths at the horizon, *i.e.* for large values of the charges. Typically, in this regime the size of the horizon is large enough so that the strength of the curvature of space-time and gauge fields is small at the horizon. Otherwise, one must worry about higher derivative corrections to the effective action in the low energy limit [26, 137, 138, 139].

On the other hand, the large charge limit also simplifies the statistical computation, where an extremal black hole corresponds to a state of the conformal field theory with large L_0 eigenvalue and zero $\bar{L}_0$ eigenvalue (or inversely). For $\bar{L}_0 = 0$, for instance, one can compute the degeneracy of such state using the Cardy formula in terms of the left-moving central charge c_L of the conformal field theory

$$S_{stat}(Q) \simeq 2\pi\sqrt{\frac{c_L L_0}{6}}\,, \tag{4.3}$$

where c_L is proportional to (a product of) the physical charges of the black holes [25]. One finds that the two computations give the same answer (4.1).

In the case of $\mathcal{N}=4$ supergravities that can be realised as $\mathbb{Z}_N$ CHL orbifolds, this result was first obtained for four-dimensional black holes by [140, 141].[1] This chapter is devoted to demonstrating how the degeneracy of quarter-BPS black hole can be obtained from the exact $\nabla^2(\nabla\phi)^4$ interaction in the three-dimensional heterotic string.

In §4.1, we give a rapid description of the entropy formalism for stationary four-dimensional black holes [139, 142, 143, 124]. the entropy function is obtained as the extremum value of a functional of the Lagrangian density, which ensures it cannot depend on the value of the asymptotic moduli at infinity [138, 142].

In §4.2, we review the famous DVV formula [140] in the case of CHL models [141, 144, 145].

In §4.3, we recall the formalism used to describe quarter-BPS black holes in $\mathcal{N}=4$ supergravities [146].

Finally, in §4.4 we review the results of [BCHP1], [BCHP3], where quarter-BPS instantonic contributions in the decompactification limit of $G^{(2k,8)}_{ab,cd}$ were used to predict the degeneracy of quarter-BPS black hole solutions. These results recover and extend the predictions presented in §4.2 [140, 141, 144, 145] as well as the exact contour prescription [147, 148].

4.1 The black hole entropy function

Let us consider a four-dimensional theory of gravity coupled to abelian gauge fields A^I_μ and scalars S_1, S_2 and M_{ij}. They are described by a Lagrangian density expressed solely in terms of the metric $g_{\mu\nu}$, the Riemann tensor $\mathcal{R}_{\mu\nu\rho\sigma}$, the gauge field strengths $F^I_{\mu\nu}$ and covariant derivative of the fields. It is invariant under reparametrisations and gauge transformations. In such a theory, the 'near horizon' limit of the extremal Reissner-Nordstrom solution is a spherically symmetric extremal solution of the equation of motion, the so-called Bertotti-Robinson geometry

$$\begin{aligned}
&\mathrm{d}s^2 = v_1\Big(-r^2\mathrm{d}t^2+\frac{\mathrm{d}r^2}{r^2}\Big)+v_2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)\,,\\
&S_1 = u_{S_1}\,, \qquad S_2 = u_{S_2}\,, \qquad M_{ij} = u_{M_{ij}}\,,\\
&F^I_{rt} = e_I\,, \qquad F^I_{\theta\phi} = p_I\sin\theta\,,
\end{aligned} \tag{4.4}$$

which has the particularity of being the product of two spaces, namely $AdS_2\times S_2$, and thus enjoys $SO(2,1)\times SO(3)$ as an isometry group, generated by the three-dimensional rotation and

$$L_1 = \partial_t\,, \qquad L_0 = t\partial_t - r\partial_r\,, \qquad L_{-1} = \frac{1}{2}\Big(\frac{1}{r^2}+t^2\Big)\partial_t - tr\partial_r\,, \tag{4.5}$$

[1] It was originally computed in the type II string theory description.

which is a near-horizon symmetry of all extremal spherically symmetric black holes in four dimensions. Note that the gauge fields strength in (4.4) are also invariant under these transformations.

Let us denote $f(u_s, v_i, e_I, p_I)$ the Lagrangian density $\sqrt{-g}\mathcal{L}$ evaluated for the geometry (4.4) and averaged over the two-sphere

$$f(u_s, v_i, e_I, p_I) = \frac{1}{4\pi}\int_{S^2} \mathrm{d}\theta\mathrm{d}\phi\sqrt{-g}\mathcal{L}\,, \tag{4.6}$$

For spherical solutions, we expect all the field equations of motions to be encapsulated in f. For instance, the scalar and metric field equations rewrite simply as

$$\frac{\partial f}{\partial u_s} = 0\,, \qquad \frac{\partial f}{\partial v_i} = 0\,, \tag{4.7}$$

while for the gauge field equations of motions, the Gauss law can be applied to recover

$$\frac{\partial f}{\partial e_I} = Q_I\,, \qquad p_I = P_I\,. \tag{4.8}$$

It is convenient to take the Legendre transform of f

$$\mathcal{E}(u_s, v_i, e^I, Q_I, P^I) = e_I Q_I - f(u_s, v_i, e_I, P_I)\,, \tag{4.9}$$

which is equivalent to a reparametrisation changing equations (4.7), (4.8) into extremisation of the function $\mathcal{E}$. It has been proven [142] that the general formula for black hole entropy in the presence of higher derivative terms [26, 149, 150] for a metric (4.4) was correctly reproduced by the value of $\mathcal{E}$ at its extremum. Subsequently, since $\mathcal{E}$ only depends on Lagrangian density and the charges, it cannot depend on any asymptotic moduli. Note that this procedure has been extended to rotating black holes in [124].

Let us choose a specific action for the fields (4.4) reminiscent of the $\mathcal{N}=4$ supergravity effective action (3.16), in the string frame metric $S_2 G_{\mu\nu} = g_{\mu\nu}$

$$\begin{aligned}\mathcal{S} = \int \mathrm{d}^4x\sqrt{-G}S2\Big[\mathcal{R}_G - \frac{1}{2{S_2}^2}\partial_\mu S\partial^\mu \bar{S} - F^I_{\mu\nu}(M^{-\intercal})_{IJ}F^{J\mu\nu} + \frac{S_1}{S_2}F^I_{\mu\nu}\star F_I^{\mu\nu} \\ + \frac{1}{8}\mathrm{Tr}(\partial_\mu ML\partial^\mu ML)\Big]\,.\end{aligned} \tag{4.10}$$

The expression of $\mathcal{E}$ with the explicit Lagrangian density (4.10) simplifies to

$$\mathcal{E} = \frac{\pi}{2u_{S_2}}\left[{u_{S_2}}^2(v_2 - v_1) + \frac{v_1}{v_2}(Q^\intercal u_M Q + (u^2_{S_1} + u^2_{S_2})P^\intercal u_M P - 2u_{S_1}Q^\intercal u_M P)\right]. \tag{4.11}$$

This expression is explicitely $O(2k-2,6)$ invariant, but one can check that it also satisfies $SL(2,\mathbb{R})$ invariance on the moduli fields (3.22), which also acts in the metric fields v_1, v_2

$$u_S \to \frac{pu_S + q}{ru_S + s}\,, \qquad \begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} p & -q \\ -r & s \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix}, \qquad v_i \to |ru_S + s|^2 v_i\,. \tag{4.12}$$

One can take advantage of these symmetries, and go to a duality frame where

$$(\mathbb{1}_{2k+4} - L)_{IJ}Q^I = 0\,, \qquad (\mathbb{1}_{2k+4} - L)_{IJ}Q^I = 0\,, \tag{4.13}$$

such that $u_M = \mathbb{1}_{2k+4}$ extremizes (4.11), and extremize $\mathcal{E}$ with respect to v_1, v_2, u_{S_1} and u_{S_2}. One obtains [142]

$$v_1 = v_2 = 2P^2\,, \qquad u_{S_2} = \frac{\sqrt{Q^2\,P^2 - (Q\cdot P)^2}}{P^2}\,, \qquad u_{S_1} = \frac{Q\cdot P}{P^2}\,. \tag{4.14}$$

Hence, the black hole entropy, which the value of $\mathcal{E}$ at this point, writes

$$S_{BH} = \pi\sqrt{Q^2\,P^2 - (Q\cdot P)^2}. \tag{4.15}$$

One-loop correction to the effective action. As mentioned in the introduction and recalled in (3.68), the one-loop effective action contains a Gauss-Bonnet $\mathcal{R}^2$ term of the form [151, 132]

$$\Delta\mathcal{L} = -\frac{1}{(8\pi)^2}\log(S_2^k|\Delta_k(S)|^2)(\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2)\,, \tag{4.16}$$

where $\mathcal{R}_{\mu\nu\rho\sigma}$ is the Riemann tensor constructed from the Einstein frame metric $g_{\mu\nu}$. This function is manifestly invariant under S-duality, using the $\Gamma_0(N)$-modular properties of $S_2^k|\Delta_k(S)|^2$. Furthermore, it does not depend on the $O(2k-2,6)$ moduli, and is thus T-duality invariant as well.

This effect gives a correction the the black hole entropy [152], and leads to a difference in $\mathcal{E}$

$$\Delta\mathcal{E} = -\int \mathrm{d}\theta\mathrm{d}\phi\sqrt{-g}\Delta\mathcal{L} = \frac{1}{2\pi}\log(u_{S_2}^k|\Delta_k(u_S)|^2)\,. \tag{4.17}$$

The minimization with respect to u_{S_1} and u_{S_2}, implies the equations

$$\begin{aligned} &u_{S_1}P^2 + Q\cdot P + u_{S_2}\frac{\partial}{\partial u_{S_1}}\log(u_{S_2}^k|\Delta_k(u_S)|^2) = 0 \\ &(Q + u_{S_1}P)^2 + u_{S_2}^2P^2 + 2u_{S_2}^2\frac{\partial}{\partial u_{S_2}}\log(u_{S_2}^k|\Delta_k(u_S)|^2) = 0\,. \end{aligned} \tag{4.18}$$

These equations can be solved iteratively so that in the large charge limit one obtains [142]

$$\Delta S_{BH}(Q,P) = -\log(u_{S_2}^k|\Delta_k(u_S)|^2) + \ldots\,, \qquad u_{S_1} = \frac{Q\cdot P}{P^2}\,, \qquad u_{S_2} = \frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{P^2}\,, \tag{4.19}$$

where ... denotes correction terms that are further suppressed by inverse powers of the charges.

4.2 Dyon counting and marginal stability

Statistical entropy computation have been performed for some the dyonic quarter-BPS black holes of interest in this manuscript. The original DVV formula for the degeneracy was proposed in [140, 153, 154, 155, 156] in the case of type II string theory compactified on $K3\times T^2$, or heterotic string compactified on T^6, and extended to specific dyonic charges of more general models such as CHL models and asymmetric type II orbifolds in [141, 157, 158].

The computation was performed on type IIB strings compactified on $K3 \times \widetilde{S}^1 \times S^1$, which is dual to the four-dimensional toroidal compactification of heterotic string after performing a strong-weak duality, a T-duality on $\widetilde{S}^1$ and then a string-string duality. This chain of dualities holds for the CHL models [61, 62], where the $\mathbb{Z}_N$ action on the type IIB compactification manifold acts by a $1/N$ translation along S^1 and an order N automorphism on $K3$. On this side of the duality, [144, 145, 159] considered dyonic states consisting of Q_5 D5-branes wrapped on $K3 \times S^1$, Q_1 D1-branes wrapped on S^1, a single Kaluza-Klein monopole associated with the $\widetilde{S}^1$ with negative magnetic charge, momentum $-n/N$ along S^1 and momentum J along $\widetilde{S}^1$. This also known as a BPMV black hole at the center of Taub-NUT space, and corresponds to charge vectors

$$Q = \begin{pmatrix} 0 \\ -n/N \\ 0 \\ -1 \\ \vec{0} \end{pmatrix}, \qquad P = \begin{pmatrix} Q_1 - Q_5 \\ -J \\ Q_5 \\ 0 \\ \vec{0} \end{pmatrix}, \tag{4.20}$$

which gives

$$Q^2 = 2n/N\,, \qquad P^2 = 2(Q_1 - Q_5)Q_5\,, \qquad Q\cdot P = J\,. \tag{4.21}$$

The degeneracy $d(Q,P)$ for these states, which counts the number of bosonic minus fermionic quarter BPS supermultiplets carrying this particular charge, is given by, for $n \neq 0 \bmod N$ and P primitive,

$$\begin{aligned} d(Q,P) &= \frac{(-1)^{Q\cdot P+1}}{N} \int_{\mathcal{C}} \mathrm{d}\rho \mathrm{d}\sigma \mathrm{d}v \frac{e^{\mathrm{i}\pi\left[\rho N Q^2 + \sigma P^2/N + 2vQ\cdot P\right]}}{\tilde{\Phi}_{k-2}(\rho,\sigma,v)} \\ &= \frac{(-1)^{Q\cdot P+1}}{N} \int_{\mathcal{C}'} \mathrm{d}\rho \mathrm{d}\sigma \mathrm{d}v \frac{e^{\mathrm{i}\pi\left[\rho Q^2 + \sigma P^2 + 2vQ\cdot P\right]}}{\tilde{\Phi}_{k-2}(\rho,\sigma,v)} \end{aligned} \tag{4.22}$$

where $\mathcal{C}$ is a three-dimensional cube of width $(1,N,1)$ in (ρ_1,σ_1,v_1) at position $(\rho_2,\sigma_2,v_2) = (M_1,M_2,-M_3)$, with M_1,M_2,M_3 being large positive numbers and $M_1, M_2 \gg M_3$. The function $\tilde{\Phi}_{k-2}(\rho,\sigma,v)$

$$\begin{aligned} \tilde{\Phi}(\rho,\sigma,v) &= e^{2\pi\mathrm{i}(\alpha\rho+\gamma\sigma+v)} \\ &\times \prod_{b=0}^{1} \prod_{r=0}^{N-1} \prod_{\substack{k\in\mathbb{Z}+\frac{r}{N}, l\in\mathbb{Z}, j\in 2\mathbb{Z}+b \\ k,l\geq 0, j\leq 0 \text{ for } k=l=0}} \left(1 - e^{2\pi\mathrm{i}(k\sigma+l\rho+jv)}\right)^{\sum_{s=0}^{N-1} e^{-2\pi\mathrm{i}sl/N} c_b^{(r,s)}(4kl-j^2)}, \end{aligned} \tag{4.23}$$

was first constructed in [141] from CFT counting, and proven to be a modular form of the congruent subgroup $\Gamma_{2,0}$ of $Sp(2,\mathbb{Z})$. Let us mention briefly that the historical case was described in type IIB for the same type of vector (4.20) with $N = 1$. It can notably be computed by collecting the degeneracies of a system of one D5 and n D1-branes from the Fourier coefficients of the elliptic genus $\chi_n(\rho,v)$ of a symmetric orbifold of n K3's [135], where it was further shown that the weighted sum of these elliptic genera $\sum_{n\geq 0} \chi_n(\rho,v)\, e^{2\pi\mathrm{i}\,n\rho}$, together with contributions from a single fivebrane, gives back the inverse of Φ_{10}.

In the case of $\tilde{\Phi}_{k-2}$, this modular form can also be described as the image of the level-N Siegel modular form $\Phi_{k-2}(\rho,\sigma,v)$

$$\tilde{\Phi}_{k-2}(\rho,\sigma,v) = (\sqrt{N})^k(-\mathrm{i}\rho)^{-(k-2)}\Phi_{k-2}\Big(-\frac{1}{\rho},\sigma-\frac{v^2}{\rho},\frac{v}{\rho}\Big)\,, \tag{4.24}$$

which transforms as a weight-$(k-2)$ form under $\Gamma_{2,0}(N)$, the group of $Sp(4,\mathbb{Z})$ matrices with lower left block congruent to $0_{2\times 2}$ modulo N.

Because of ambiguities in the expansion of $1/\tilde{\Phi}_{k-2}$, the result of an integral of the type of (4.22) is very sensitive to the contour $\mathcal{C}$ [160, 161]. Mathematically, this is due to terms proportional to

$$\frac{e^{2\pi\mathrm{i}v}}{(1-e^{2\pi\mathrm{i}v})^2}\,, \tag{4.25}$$

in the expansion of $1/\Phi_{k-2}$, that have different Taylor expansions depending on whether $e^{-2\pi v_2}$ is smaller or larger than one.

These singular terms describe walls of marginal stability where quarter-BPS states marginaly decay into a pair of half-BPS states. They are double poles given by all images of the locus $v=0$ under $\Gamma_{2,0}(N)$, which are mapped by the integration (4.22) to one-codimensional subspaces of the asymptotic moduli space, on which the mass of quarter-BPS states becomes equal to the sum of masses of two half-BPS states carrying the same total charge. Thus, crossing a wall in the moduli space amounts to going from a region where a quarter-BPS bound state is energetically favoured, to a region where it becomes disfavoured for a pair of unbound half-BPS states with same total charge. In other words, the spectrum of quarter-BPS states of a given charge changes discontinously as asymptotic moduli pass through any of these walls, which is quite a generic phenomenon for quarter-BPS states in $\mathcal{N}=4$ supersymmetric string theories [162].

The quarter-BPS mass can be expressed as

$$\mathcal{M}(Q,P)^2 = \frac{2|Q_R+SP_R|^2}{S_2} + 4\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}\,, \tag{4.26}$$

which is manifestly invariant under S-duality[2] and T-duality (3.67). General decay can occur as

$$\begin{pmatrix}Q\\P\end{pmatrix} = \begin{pmatrix}p\\r\end{pmatrix}\frac{sQ-qP}{ps-qr} + \begin{pmatrix}q\\s\end{pmatrix}\frac{pP-rQ}{ps-qr}\,, \tag{4.27}$$

with $\begin{pmatrix}p & q\\ r & s\end{pmatrix}\in M_2(\mathbb{Z})$ and such that $\begin{pmatrix}p & q\\ r & s\end{pmatrix}^{-1}\begin{pmatrix}Q\\P\end{pmatrix}\in\Lambda_m^*\oplus\Lambda_m^*$. Thus, walls of marginal stability are defined as loci in the moduli space where the masses of the states in (4.27) satisfy

$$\mathcal{M}(Q,P) = \mathcal{M}\Big(p\frac{sQ-qP}{ps-qr},r\frac{sQ-qP}{ps-qr}\Big) + \mathcal{M}\Big(q\frac{pP-rQ}{ps-qr},s\frac{pP-rQ}{ps-qr}\Big)\,, \tag{4.28}$$

which can be reformulated as [124]

$$\Big(S_1-\frac{ps+qr}{2rs}\Big)^2 + \Big(S_2+\frac{E}{2rs}\Big)^2 = \frac{1}{(2rs)^2}(1+E^2)\,, \tag{4.29}$$

[2]Either $SL(2,\mathbb{Z})$ (3.22) for the full rank model, or $\Gamma_0(N)$ plus the Fricke transformation (3.74) for CHL models.

where

$$E = \frac{rsQ_R^2 + pqP_R^2 - (ps+qr)Q_R \cdot P_R}{\left(Q_R^2 P_R^2 - (Q_R \cdot P_R)^2\right)^{1/2}}\,, \tag{4.30}$$

where all the moduli fields are implicitely evaluated at infinity. For fixed Q, P, and moduli $\phi \in G_{2k-2,6}$, the walls of marginal stability describe lines and circles in the S complex plane [124]. One can see that, in the relevant upper-half plane $S_2 > 0$, the circles only intersect with other walls at rational points of the real axis $S_2 = 0$

$$p/r \quad \text{and} \quad q/s\,, \tag{4.31}$$

while the straight lines intersect other walls in integer points b of the real axis as well as at i∞. These intersection points have the special property of depending on the decay $(Q,P) = (Q_1,P_1) + Q_2,P_2)$ only, and not on the moduli $\phi \in G_{2k-2,6}$ – not even on the charges themselves. They are thus invariant under continuous change of the moduli, which is not the case of the walls (4.29) whose precise shape depends on ϕ.

It is then natural to question the validity of the formula (4.22) for a generic domain defined in the S-plane by its vertices, as well as the generality of the formula itself for other type of quarter-BPS dyonic charges (Q,P). These formulae were worked out at weak coupling limit of the type IIB string and other moduli finite, which translates in the heterotic description to finite S_1 and S_2, with P_R^2, $Q_R \cdot P_R \ll Q_R^2$. In this regime, the walls (4.29) with circle shape lie close to the real axis, while straight ones are almost vertical lines passing through the integers $S = b$. Thus, the coupling region in type IIB string with $-1 < S_1 < 1$ is mapped in the heterotic description to two neighbouring domains bound by the lines $b = -1, 0, 1$ together with a set of circle segments at the bottom. The domain with $S_1 > 0$ is described by the formula (4.22), while the other one is describe only by a similar formula where the contour cC has been changed by $M_3 \to -M_3$. In general, using S-duality with the formula (4.22) allows to express the degeneracy of other dyonic charges within other domains. Invariance of the theory can thus be used to obtain the degeneracy formula for other types of vectors, and it will be at the cost of changing the imaginary part of the contour $\mathcal{C}$. T-duality can also be used to express the degeneracy of another dyonic charge (Q', P') with the same T-duality invariants $(Q'^2,\, P'^2,\, Q'\cdot P') = (Q^2,\, P^2,\, Q\cdot P)$.

The quarter-BPS charges considered in [124] are a subset of ones that strictly belong to electromagnetic lattice $(Q,P) \underline{\in} \Lambda_e \oplus \Lambda_m$, where *strictly* is understood as 'does not belong to a smaller lattice in the graph of inclusion'[3]

$$\begin{matrix} N\Lambda_e \oplus N\Lambda_e \\ \Lambda_m \oplus N\Lambda_m \end{matrix} \begin{matrix} \subset \\ \subset \end{matrix} \Lambda_m \oplus N\Lambda_e \begin{matrix} \subset \\ \subset \end{matrix} \begin{matrix} \Lambda_m \oplus \Lambda_m \\ \Lambda_e \oplus N\Lambda_e \end{matrix} \begin{matrix} \subset \\ \subset \end{matrix} \Lambda_e \oplus \Lambda_m\,. \tag{4.32}$$

When primitive vectors belong to one of the lattice above, it may experience splits into pairs of half-BPS charges that are neither twisted nor primitive. One may consider using S-duality to reach another type of vector contained in a lattice smaller than $\Lambda_e \oplus \Lambda_m$, but the latter is in fact self-dual under Fricke duality (3.74)

$$(Q,P) \in \Lambda_e \oplus \Lambda_m \mapsto (-\varsigma \cdot P/\sqrt{N}, \varsigma^{-1} \cdot Q\sqrt{N}) \in \Lambda_e \oplus \Lambda_m\,, \tag{4.33}$$

[3] A lattice is considered smaller than another one if it is included in it.

besides, lattices in the graph (4.32) are drawn such that Fricke duality acts as a reflection with respect to the horizontal axis. As for using T-duality invariance of the theory, one should keep in mind that two dyonic charges with the same T-duality invariants (Q^2, P^2, $Q \cdot P$) need not be in the same T-duality orbit.[4] Another point is that T-duality acts non-trivially on the asymptotic moduli, which makes identifying the degeneracy of a dyonic charge in a given region difficult.

All these options were considered in [124], and the degeneracy of primitive vector in $\Lambda_e \oplus \Lambda_m$ even more generic than (4.20) were shown to be correctly given by (4.22) [144, 145, 159]. Note that it was further noticed in [155, 163, 158] that a class of quarter-BPS dyons arises from string networks which lift to M5-branes wrapped on $K3$ times a genus-two curve. Part of the motivations of [BCHP1]and [BCHP3] were to make explicit and quantitative the nature of the connection between quarter-BPS black holes and genus-two surfaces.

4.3 Quarter-BPS solutions in $\mathcal{N} = 4$ supergravity

In the case of generic quarter-BPS black holes, it is natural to look for more general solutions that go beyond the static case (4.4). Indeed, to include quarter-BPS solutions realised as two-center bound states of half-BPS states, one has to allow for more general spacetimes. In [164], it was argued that BPS time-independent configurations require a metric that can be expressed in the form

$$\mathrm{d}s^2 = -e^{2U}(\mathrm{d}t + \omega_i \mathrm{d}x^i)^2 + e^{-2U}\mathrm{d}x^i\mathrm{d}x^i\,, \tag{4.34}$$

where U is an arbitrary function of space coordinates, ω is time-independent, and both vanish at space infinity, since we consider asymptotically flat spacetimes. Fields configuration allowing a timelike Killing vector give rise to a dimensionally reduced three-dimensional theory [104]. The three dimensional effective action was well-defined for small ω when rewritten in terms of the metric field U, the scalar moduli, and vector fields only [146, § 7.1]. It is equivalent to consider (3.49) and dualise the scalar component of the gauge fields A^I. The BPS constraints onto the equation of motion imply that

$$\begin{aligned} e^{-2U} &= \Big[(\mathcal{H}^{I1}\mathcal{H}^1_I)(\mathcal{H}^{I2}\mathcal{H}^2_I) - (\mathcal{H}^{I1}\mathcal{H}^2_I)^2\Big]^{1/2} \\ \star\mathrm{d}\omega_i &= \varepsilon_{ijk}\mathcal{H}^{Ij}\mathrm{d}\mathcal{H}^k_I\,, \end{aligned} \tag{4.35}$$

with $\mathcal{H}^{Ii}$ the harmonic function

$$\mathcal{H}^{Ii} = \sum_A \frac{\Gamma^{Ii}_A}{|x - x_A|} - p^I_{R\hat{a}} v^{-1i}_\mu \frac{Z^{\hat{a}\mu}_+(\Gamma)}{\mathcal{M}(Q,P)}\,, \tag{4.36}$$

with $\Gamma = (Q,P) = \sum_A \Gamma_A$, and the central charge $Z = \frac{2}{\sqrt{S_2}}(Q_R + SP_R) = Z_+ + Z_-$ decomposing into two components

$$Z_\pm = \frac{1}{\sqrt{S_2}}\Big[(1,S)\cdot\begin{pmatrix} Q_R \\ P_R \end{pmatrix} \pm \frac{\mathrm{i}}{|Q_R \wedge P_R|}(-S,1)\cdot\begin{pmatrix} P_R^2 Q_R - (Q_R\cdot P_R)P_R \\ Q_R^2 P_R - (Q_R \cdot P_R)Q_R \end{pmatrix}\Big]\,, \tag{4.37}$$

[4]For instance, a dyonic charge with non-integer components and even duality invariants is certainly in the same duality orbit than dyonic charges with integer components only. However, the T-duality group being a discrete integer group, it cannot map the former to the latter.

with $Z^{\hat{a}}_{+1} = \mathrm{Re} Z^{\hat{a}}_{+}$, $Z^{\hat{a}}_{+2} = \mathrm{Im} Z^{\hat{a}}_{+}$, and $\mathcal{M}(Q,P) = |Z_+| = \sqrt{Z_{\hat{\alpha}} \bar{Z}^{\hat{\alpha}}}$ the mass of states with charge (Q,P) saturating the Bogomol'nyi bound (3.37). It is convenient to write $Z_{+\hat{\alpha}} = (z_1 + \mathrm{i} z_2)_{\hat{\alpha}} \mathcal{M}(Q,P)$ with z_1 and z_2 vectors of $SO(6)$ satisfying $z_1^2 + z_2^2 = 1$. All the other fields are determined in terms of the harmonic function (4.36) [146].

Using $\langle \Delta \mathcal{H}, \mathcal{H} \rangle = 0$, with $\Delta\, |x - x_A|^{-1} = -4\pi \delta^3(x - x_A)$, one obtains from (4.36)

$$\sum_B \frac{\langle \Gamma_A, \Gamma_B \rangle}{|x_A - x_B|} = \frac{\langle \Gamma_{AR}, v_\mu^{-1} Z_+^\mu(\Gamma) \rangle}{\mathcal{M}(Q,P)} \,. \tag{4.38}$$

Thus, for two centers with charge Γ_1, Γ_2, the distance $|x_1 - x_2|$

$$|x_1 - x_2| = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{\langle \Gamma_{1R}, v_\mu^{-1} Z_+^\mu(\Gamma) \rangle} \mathcal{M}(Q,P) \tag{4.39}$$

might fail to be positive depending on the sign difference between $\langle \Gamma_1, \Gamma_2 \rangle$ and $\langle \Gamma_{1R}, v_\mu^{-1} Z_+^\mu(\Gamma) \rangle$. In particular, two charges will be driven to infinite distance from each other when one of the walls of marginal stability defined by $\langle \Gamma_{1R}, v_\mu^{-1} Z_+^\mu(\Gamma) \rangle = 0$ is approached. We come back to this condition the next section.

These multicenter solutions can have intrisic angular momentum. Defining the angular momentum vector J from the asymptotic metric as [165]

$$\omega_i = 2\varepsilon_{ijk} J^j \frac{x^k}{r^3} + O(\frac{1}{r^3}) \qquad \text{as} \qquad r \to \infty, \tag{4.40}$$

one can use (4.35) and (4.38), one obtains for a two-center solution [146]

$$J = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \frac{x_1 - x_2}{|x_1 - x_2|} \,. \tag{4.41}$$

This quantity is independent of the details of the solution and is quantised such that $2|J| \in \mathbb{Z}$ (3.73).

4.4 Black holes degeneracy from exact $\nabla^2 F^4$ coupling

Since stationary solutions in four dimension beneficiate from timelike Killing vector, they can be described by the Euclidean field theory resulting from the dimensional reduction of the same theory along this vector [104]. As described section 3.1.2, the four-dimensional effective theory could be reformulated in terms of the three-dimensional fields, and this dimensional reduction can be done irrespective of the signature of the Killing vector field. The moduli spaces G_4/K_4 and G_3/K_3 of for four and three spacetime dimensions were explicited in (3.1) and (3.47) respectively, while for the Euclidean three-dimensional theory the moduli space G_3/K_{3E} is given by

$$G_3/K_{3E} = \frac{O(2k, 8)}{O(2k-2, 2) \times O(2, 6)} \,, \tag{4.42}$$

where K_{3E} is a non-compact version of K_3 such that G_3/K_{3E} is a symmetric space with indefinite metric where the signature of the scalars which arose from vectors fields has been changed. Stationary four-dimensional black hole solutions allow both a time-like and space-like Killing vector fields, and will thus be preserved by both reductions. This feature is generic for dimensional reduction of four-dimensional gravity models coupled to scalars in non-compact Riemannian symmetric space [104]. Recalling the effective action obtained after reduction (3.61)

$$\begin{aligned}\mathcal{S}^{(3)} = \int \mathrm{d}^3x\sqrt{-h}\Big[\mathcal{R}_h - \frac{1}{2{S_2}^2}\partial_m S\partial^m \bar{S} + \frac{S_2}{4R}(\partial_m \mathcal{A}^\intercal)\bar{M}(\partial^m\mathcal{A}) \\ - \frac{1}{R^2}\partial_m R\partial^m R - \frac{1}{2R^2}\Omega_m\Omega^m - \frac{1}{8}\mathrm{Tr}(\partial_\mu ML\partial^\mu ML)\Big]\,,\end{aligned} \tag{4.43}$$

where we used notations for the reduction along a time-like Killing vector field, one can notice that all the kinetic terms of the scalar fields ensure definiteness of the action in the path integral formalism, and thus of the non-trivial instanton saddle points. The kinetic terms of the gauge fields, however, is problematic. In [146], Denef showed that a well-defined form of the action was obtained by dualizing all the scalar field obtained after dimensional reduction to gauge fields §7.1 [146], and we use this as a postulate to study non-trivial saddle points of the Euclidean action.

Size of the automorphic representation. Although the contribution of black holes as instantons is not understood in general, in the present case one can notice a relation between the electromagnetic charge (Q, P) characterising the black hole solution and the size of the automorphic representation of $G^{(2k,8)}_{ab,cd}$.

In general, the tensor $G^{(p,q)}_{ab,cd}$ does not belong strickly speaking to an automorphic representation of $SO(p,q)$, because of the quadratic source term in (3.117), but one can nonetheless define a generalisation of this notion. From the linearised analysis §3.4, the homogeneous differential equation is attached to the $SO(p,q)$ representation of the nilpotent orbit. A representative of the nilpotent orbit in the unipotent associated to the maximal parabolic $GL(2)\times SO(p-2,q-2)\ltimes\mathbb{R}^{2(p+q-4)+1}$ must satisfy the constraint [5]

$$Q_{[i}{}^{[m}Q_j{}^nQ_{k]}{}^{p]} = 0\ , \qquad Q_{[i}{}^mQ_j{}^nK_{kl]} = 0\,, \tag{4.44}$$

which admits a subspace of solutions of dimension $2(p+q-4)$ for $Q_i{}^m \in \mathbb{R}^{2(p+q-4)}$ and a subspace of dimension 1 for $K_{ij}\in\mathbb{R}$. Therefore, the total subspace of solutions is of dimension $2(p+q-4)+1$.

On the other hand, in the Euclidean three-dimensional theory, *i.e.* $(p,q) = (2k, 8)$, the quarter-BPS black hole solutions are associated to a real nilpotent orbit of $SO(2k,8)$. Their electromagnetic charge (Q,P) lies in the grade-1 component of $SO(2k,8)$

$$\mathfrak{so}_{2k,8} \simeq \ldots \oplus (\mathfrak{gl}_1 \oplus \mathfrak{sl}_2 \oplus \mathfrak{so}_{2k-2,6})^{(0)} \oplus \big(\mathbf{2}\otimes(\mathbf{2k+4})\big)^{(1)} \oplus \mathbf{1}^{(2)}\,. \tag{4.45}$$

They thus coincide with the dimension $2(p+q-4)$ of the representation attached to the linear differential equations (4.44) for vanishing K_{ij}. The Fourier coefficients associated to these black hole solutions thus saturate the dimension associated to the automorphic representation, and one can expect those Fourier coefficients to be proportional to the helicity supertrace of these states.

[5]The unipotent being non-Abelian, one cannot generally define the Fourier coefficients for (Q_i^m, K_{ij}), but one must consider separately the Abelian Fourier coefficient with $K_{ij} = 0$, from the non-Abelian ones.

4.4.1 Decompactification limit of exact $\nabla^2(\nabla\phi)^4$ couplings

The Fourier decomposition of the function $G^{(2k,8)}_{ab,cd}$ at the cusp $R \to \infty$ (3.88) corresponding to the decompactification limit of the three-dimensional theory was computed in [BCHP3].[6] In this limit, the lattice $\Lambda_{2k,8}$ decomposes into

$$\Lambda_{2k-2,6} \oplus I\!I_{1,1}[N] \oplus I\!I_{1,1} \,. \tag{4.46}$$

We find that the conjectured exact $\nabla^2(\nabla\phi)^4$ coupling (3.118) has the large radius expansion

$$G^{(2k,8)}_{\alpha\beta,\gamma\delta} \;=\; G^{(0)}_{\alpha\beta,\gamma\delta} + G^{(1)}_{\alpha\beta,\gamma\delta} + G^{(2)}_{\alpha\beta,\gamma\delta} + G^{(\mathrm{TN})}_{\alpha\beta,\gamma\delta} \tag{4.47}$$

corresponding to the constant terms, half-BPS and quarter-BPS Abelian Fourier modes and finally, the non-Abelian Fourier modes with non-zero Taub-NUT charge. The constant part $G^{(0)}_{\alpha\beta,\gamma\delta}$ contains a terms proportional to R^4 which reproduce the exact $\nabla^2 F^4$ couplings in the four-dimensional effective action. It is given in detail together with the half-BPS and non-Abelian modes in [BCHP3].

In this section, we focus on the contributions from the the Abelian – with vanishing Taub-NUT charge – quarter-BPS contributions, that we associate to quarter-BPS black holes solution in four dimensions. Indeed, these Fourier coefficients correspond to non-perturbative corrections associated to spacetime instantons, or equivalently, solutions of the Euclidean three-dimensional action which can be interpreted as stationary solution in the four-dimensional theory, as argue in the beginning of this section 4.4.

Black hole solutions and quarter-BPS instantons. Decomposing

$$G^{(2)}_{ab,cd} = \sum_{\substack{\Gamma\in\Lambda_m^*\oplus\Lambda_m \\ Q\wedge P\neq 0}} G^{(2,\Gamma)}_{ab,cd}\, e^{2\pi \mathrm{i}(a_1 Q + a_2 P)} \tag{4.48}$$

with $\Gamma = (Q,P)$, $G^{(2,\Gamma)}_{\alpha\beta,\gamma\delta}$ can be expressed as

$$G^{(2,\Gamma)}_{ab,cd} = 2R^4 \int_{\mathcal{P}} \mathrm{d}^3\Omega_2\, \bar{C}_{k-2}(Q,P;\Omega_2)\, P_{ab,cd}(Q_L,P_L,\Omega_2)\, e^{-\pi \mathrm{Tr}\left[\frac{R^2}{S_2}\Omega_2^{-1}\left(\begin{smallmatrix}1 & S_1\\ S_1 & |S|^2\end{smallmatrix}\right)+2\Omega_2\left(\begin{smallmatrix}Q_R^2 & Q_R P_R\\ Q_R P_R & P_R^2\end{smallmatrix}\right)\right]}\,, \tag{4.49}$$

where $\mathcal{P}$ is the set of positive-definite matrices, and $P_{\alpha\beta,\gamma\delta}(Q_L,P_L,\Omega_2)$ a polynomial explicited in [BCHP3] § H. Notice that the function $\bar{C}_{k-2}(Q,P,\Omega_2)$, obtained from the Fourier coefficients of $1/\Phi_{k-2}$ and $1/\tilde{\Phi}_{k-2}$, depends on both the charge $\Gamma = (Q,P)$ and the integration variable $\Omega_2 \in \mathcal{P}$. In the full rank model, it is given by

$$\bar{C}(Q,P;\Omega_2) = \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\Gamma\in\Lambda_{22,6}\oplus\Lambda_{22,6}}} |A|\, C\big[A^{-1}\left(\begin{smallmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{smallmatrix}\right)A^{-\intercal};\, A^\intercal \Omega_2 A\big]\,, \tag{4.50}$$

where $\Lambda_{22,6} = \Lambda_m$ is the magnetic lattice of the full rank model, and $C\big[\left(\begin{smallmatrix}2m & l\\ l & 2n\end{smallmatrix}\right);\Omega_2\big]$ are the Fourier coefficients of $1/\Phi_{10}$ [140].[7]. For CHL models with $N = 2,3,5,7$, it is instead

[6] The large circle is of course orthogonal to the one involved in the orbifold action.

[7] A detailed study of the properties of $1/\Phi_{10}$ can be found in [166].

given by

$$\begin{aligned}\bar{C}_{k-2}(Q,P;\Omega_2) = & \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda_m\oplus\Lambda_m}} |A| C_{k-2}\Big[A^{-1}\begin{pmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{pmatrix}A^{-\intercal}; A^\intercal\Omega_2 A\Big] \\ & + \sum_{\substack{A\in M_{2,0}(N)/[\mathbb{Z}_2\ltimes\Gamma_0(N)]\\ A^{-1}\binom{Q}{P}\in\Lambda^*_m\oplus\Lambda_m}} |A|\widetilde{C}_{k-2}\Big[A^{-1}\begin{pmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{pmatrix}A^{-\intercal}; A^\intercal\Omega_2 A\Big] \\ & + \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P/N}\in\Lambda^*_m\oplus\Lambda^*_m}} |A| C_{k-2}\Big[A^{-1}\begin{pmatrix}-NQ^2 & -Q\cdot P\\ -Q\cdot P & -P^2/N\end{pmatrix}A^{-\intercal}; A^\intercal\Omega_2 A\Big] \quad (4.51)\end{aligned}$$

where $C_{k-2}\big[\begin{pmatrix}2m & l\\ l & 2n\end{pmatrix};\Omega_2\big]$ and $\widetilde{C}_{k-2}\big[\begin{pmatrix}2m & l\\ l & 2n\end{pmatrix};\Omega_2\big]$ denote the Fourier coefficients of $1/\Phi_{k-2}(\Omega)$ and $1/\widetilde{\Phi}_{k-2}(\Omega)$ defined in (4.23), (4.24).

The functions $1/\Phi_{k-2}(\Omega)$ and $1/\widetilde{\Phi}_{k-2}(\Omega)$ are meromorphic with poles, so that their Fourier coefficients are piecewise constant functions of Ω_2, with discontinuities as well as delta-function singularities at the boundary between distinct chambers.[8] However, the integral (4.49) is dominated by a saddle point $\Omega_2 = \Omega_2^*$,

$$\Omega_2^\star = \frac{R}{\mathcal{M}(\Gamma)} A^\intercal \Big(\tfrac{1}{\sqrt{S_2}}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix} + \tfrac{1}{|Q_R\wedge P_R|}\begin{pmatrix}P_R^2 & -Q_R\cdot P_R\\ -Q_R\cdot P_R & Q_R^2\end{pmatrix}\Big) A\,, \tag{4.52}$$

in the neighborhood of which their Fourier coefficients are constant for generic moduli S and φ. Due to this non-trivial Ω_2-dependence, one cannot compute (4.49) analytically, but the leading contribution can be computed at large radius by a saddle point approximation with $\bar{C}_{k-2}(Q,P;\Omega_2) \sim \bar{C}_{k-2}(Q,P;\Omega_2^\star)$ kept constant in the integrand, see eq. (5.77) of [BCHP3].

These leading contributions are exponentially suppressed in $e^{-2\pi R\mathcal{M}(\Gamma)}$, with $\Gamma = (Q,P)$ and $\mathcal{M}(\Gamma)$ the BPS mass of a black hole of charge Γ (4.26). Given a charge Γ for which there is no $d\neq 1$ such that $d^{-1}\Gamma\in\Lambda^*_m\oplus\Lambda_m$, only $A=1$ contributes to the measures (4.50), (4.51), and one can interpret the measure factor (up to an overall sign) as the helicity supertrace counting string theory states of charge Γ, as given by the formula (4.22). Indeed, the value of Ω_2 at the saddle point (4.52) reproduces the contour prescription of [147, 148] when both electric and magnetic charges are separately primitive in Λ^*_m and Λ_m, and $d^{-1}Q\wedge P\in\Lambda^*_m\wedge\Lambda_m$ for $d=1$ only. More generally, the contour prescription depends on the set of matrices A such that $A^{-1}\binom{Q}{P}$ belongs to the electromagnetic lattice. In the full rank case, for instance, all primitive charges (Q,P) are in the U-duality orbit of a charge of the form [167]

$$Q = e_1 + q\,e_2\,,\qquad P = p\,e_2\,,\qquad Q\wedge P = p\,e_1\wedge e_2\,, \tag{4.53}$$

with e_1 and e_2 primitive in $\Lambda_{22,6}$, such that (4.50) simplifies to

$$\bar{C}(Q,P;\Omega_2^\star) = \sum_{\substack{d\geq 1\\ d|p}} d\, C\Big[\begin{pmatrix}Q^2 & QP/d\\ QP/d & P^2/d^2\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & d\end{pmatrix}\Omega_2^\star\begin{pmatrix}1 & 0\\ 0 & d\end{pmatrix}\Big]\,, \tag{4.54}$$

[8] Moreover, they are generically well-defined only for $|\Omega_2| > \frac{1}{4}$, otherwise the integration domain $\mathcal{C}$ (4.22) generically crosses the poles at small values of $|\Omega_2|$.

in agreement with [168], with added precision on the contour prescription. If we consider (4.53) in CHL orbifolds for e_1 primitive in Λ_m^* and not in Λ_m, e_2 primitive in Λ_m and not in $N\Lambda_m^*$, and with p not divisible by N, only the second line in (4.51) contributes and the result reduces similarly to

$$\bar{C}_{k-2}(Q,P;\Omega_2^\star) = \sum_{\substack{d\geq 1\\ d|p}} d\, \widetilde{C}_{k-2}\Big[\big(\begin{smallmatrix} Q^2 & QP/d \\ QP/d & P^2/d^2\end{smallmatrix}\big), \big(\begin{smallmatrix} 1 & 0 \\ 0 & d\end{smallmatrix}\big)\Omega_2^\star\big(\begin{smallmatrix} 1 & 0 \\ 0 & d\end{smallmatrix}\big)\Big] , \tag{4.55}$$

in agreement with [141] for $p=1$. For general 'untwisted' charges such that Q can be in Λ_m or P in $N\Lambda_m^*$, all three terms in (4.51) contribute the degeneracy, generalising (4.22). Note that the result is manifestly invariant under U-duality, including Fricke duality.

Bound states and pairs of half-BPS instantons. As mentioned in the previous paragraph, the saddle point approximation to (4.49) has subleading corrections reflecting the non-trivial behavior of quarter-BPS black hole solutions at walls of marginal stability (4.28).[9]

For fixed total charge $\Gamma=(Q,P)$, we expect contributions from all pairs of half-BPS states with charges Γ_1 and Γ_2 such that $\Gamma=\Gamma_1+\Gamma_2$. Such splitting is parametrised by a non-degenerate matrix $B=\big(\begin{smallmatrix} p & q \\ r & s\end{smallmatrix}\big)\in M_2(\mathbb{Z})$, such that

$$\begin{pmatrix} Q_1 \\ P_1\end{pmatrix} = \begin{pmatrix} p \\ r\end{pmatrix}\frac{sQ-qP}{ps-qr} = B\pi_1 B^{-1}\begin{pmatrix} Q \\ P\end{pmatrix} , \quad \begin{pmatrix} Q_2 \\ P_2\end{pmatrix} = \begin{pmatrix} q \\ s\end{pmatrix}\frac{pP-rQ}{ps-qr} = B\pi_2 B^{-1}\begin{pmatrix} Q \\ P\end{pmatrix} , \tag{4.56}$$

where $\pi_1=\big(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\big)$ and $\pi_2=\big(\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\big)$. It follows from the calculation of these Fourier modes that all splittings of a given charge Γ are in one-to-one correspondence with the matrices $B\in M_2(\mathbb{Z})/\mathrm{Stab}(\pi_i)$ with

$$M_2(\mathbb{Z})/\mathrm{Stab}(\pi_i) = \Big\{\gamma\cdot\begin{pmatrix} 1 & j' \\ 0 & k'\end{pmatrix} , \quad \gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4 , \quad 0\leq j'<k' , \quad (j',k')=1\Big\} , \tag{4.57}$$

such that $B\pi_1 B^{-1}\Gamma\in\Lambda_m^*\oplus\Lambda_m$. This can be proven with some effort to generalise to CHL models, see § C.2 of [BCHP3].

Focusing on the maximal rank case for simplicity, function (4.50) on the domain $|\Omega_2|>\frac{1}{4}$ reads (for $N>1$, see (5.92) in [BCHP3])

$$\begin{aligned}\bar{C}(Q,P;\Omega_2) =& \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\Gamma\in\Lambda_m\oplus\Lambda_m}} |A|\, C^F\big[A^{-1}\big(\begin{smallmatrix} -Q^2 & -Q\cdot P \\ -Q\cdot P & -P^2\end{smallmatrix}\big)A^{-\intercal}\big]\\ &+\sum_{\substack{\Gamma_i\in\Lambda_m\oplus\Lambda_m\\ Q_i\wedge P_i=0,\,\Gamma_1+\Gamma_2=\Gamma}} \bar{c}(\Gamma_1)\bar{c}(\Gamma_2)\Big(-\frac{\delta([\hat{B}^\intercal\Omega_2\hat{B}]_{12})}{4\pi}+\frac{\langle\Gamma_1,\Gamma_2\rangle}{2}(\mathrm{sign}(\langle\Gamma_1,\Gamma_2\rangle)-\mathrm{sign}([\hat{B}^\intercal\Omega_2\hat{B}]_{12}))\Big) ,\end{aligned} \tag{4.58}$$

[9] The contributions to $G^{(2,\Gamma)}_{\alpha\beta,\gamma\delta}$ due to the deviation of $\bar{C}_{k-2}(Q,P,\Omega_2)$ from its saddle point value are investigated in § F [BCHP3]. In particular, at large $|\Omega_2|$, it is shown that these contributions are exponentially suppressed in $e^{-2\pi R(\mathcal{M}(\Gamma_1)+\mathcal{M}(\Gamma_2))}$, and can therefore be ascribed to two-instanton effects associated to two unbound half-BPS states of charges Γ_1 and Γ_2.

where the so-called finite coefficients $C^F\left[\left(\begin{smallmatrix} m & l \\ l & n\end{smallmatrix}\right)\right]$ are globally constant functions of Ω_2 [166], and where $\hat{B} \in SL(2,\mathbb{Q})/\mathrm{Stab}(\pi_i,\mathbb{Q})$ are defined as

$$\hat{B} = B\begin{pmatrix}1 & 0 \\ 0 & |B|^{-1}\end{pmatrix} = \gamma \cdot \begin{pmatrix}1 & \frac{j'}{k'} \\ 0 & 1\end{pmatrix} \,, \tag{4.59}$$

determined such that $\Gamma_i = \hat{B}\pi_i\hat{B}^{-1}\Gamma$ and where $[\hat{B}^\intercal \Omega_2 \hat{B}]_{ij}$ denotes the components ij of the matrix.

In §4.3, we introduced the adequate formalism to express the distance within a quarter-BPS two-center black hole solution with charge Γ (4.39)

$$|x_1 - x_2| = \frac{\langle \Gamma_1, \Gamma_2\rangle}{\langle \Gamma_{1R}, v_\mu^{-1} Z_+^\mu(\Gamma)\rangle}\mathcal{M}(Q,P)\,. \tag{4.60}$$

The Z_+ component of the central charge (4.37) is related to the saddle point value $\Omega_2 = \Omega_2^\star$ (4.52) through

$$v_\mu^{-1} Z_+^\mu(\Gamma) = \frac{1}{R}\Omega_2^\star \begin{pmatrix} Q_R \\ P_R\end{pmatrix} \,, \tag{4.61}$$

implying that the distance $|x_1 - x_2|$ (4.60) satisfies

$$\frac{\langle \Gamma_1, \Gamma_2\rangle}{|x_1 - x_2|} = -\frac{|Q_R \wedge P_R|}{R}[\hat{B}^\intercal \Omega_2^\star \hat{B}]_{12} \,. \tag{4.62}$$

In other words, the bound state of charge Γ is only defined when $\langle \Gamma_1, \Gamma_2\rangle$ and $[\hat{B}^\intercal \Omega_2^\star \hat{B}]_{12}$ have opposite signs, or equivalently (4.28)

$$\mathcal{M}(Q,P) < \mathcal{M}(Q_1,P_1) + \mathcal{M}(Q_2,P_2)\,. \tag{4.63}$$

If so, it contributes to $G^{(2,\Gamma)}_{ab,cd}$ at leading order with measure factor $\bar{c}(\Gamma_1)\bar{c}(\Gamma_2)|\langle \Gamma_1, \Gamma_2\rangle|$ (4.58).

In contrast, when $[\hat{B}^\intercal \Omega_2^\star \hat{B}]_{12}$ and $\langle \Gamma_1, \Gamma_2\rangle$ have the same sign, the bound state is not allowed and the last term in (4.58) vanishes at the saddle point $\Omega_2 = \Omega_2^\star$ (4.52). It still contributes to the integral (4.49) with an exponential suppression $e^{-2\pi R(\mathcal{M}(\Gamma_1)+\mathcal{M}(\Gamma_2))}$, but is then subdominant since the inequality (4.63) is reversed.

We conclude that (4.49) receives contributions of each possible splitting $\Gamma = \Gamma_1 + \Gamma_2$, weighted by the product of the half-BPS measures $\bar{c}(\Gamma_1)\,\bar{c}(\Gamma_2)$ and further exponentially suppressed by $e^{-2\pi R(\mathcal{M}(\Gamma_1)+\mathcal{M}(\Gamma_2))}$. It is important to distinguish these two-instanton contributions from one-instanton contributions due to bound states of half-BPS states, and to notice that the full function $G^{(2,\Gamma)}_{ab,cd}$ (4.49) is made continuous at the walls of marginal stability by the defining equality $\mathcal{M}(\Gamma) = \mathcal{M}(\Gamma_1) + \mathcal{M}(\Gamma_2)$. This discussion easily generalises to the CHL case [BCHP3].

Properties of the differential equation. It is interesting to understand these properties from the perspective of the inhomogeneous differential equation imposed by the supersymmetry Ward identities (3.117). The leading contribution to the Fourier coefficient (4.49) – associated to quarter-BPS states – solves the homogeneous equation associated to (3.117), whereas contributions due to discontinuities of $\bar{C}_{k-2}(Q,P;\Omega_2)$ give a particular solution to the full inhomogeneous equation (3.117). For a given quarter-BPS charge Γ, the Fourier coefficients of F_{abcd} contribute with a source term proportional to $\bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2)$ for all possible splittings $\Gamma = \Gamma_1 + \Gamma_2$, which matches the structure of the measure in (4.58). Furthermore, the Fourier coefficients of F_{abcd} being associated to instantonic half-BPS states [BCHP2], it is consistent for the leading contributions to $G^{(2,\Gamma)}_{ab,cd}$ sourced by $(F_{abcd})^2$ terms to be associated to unbound pairs of half-BPS instantons.

The explicit check of the differential equation, in § E.3 [BCHP3], demonstrates that the unfolding procedure used for the computation of the Fourier modes $G^{(2,\Gamma)}_{ab,cd}$ reproduces the correct Abelian Fourier coefficients, at least up to terms exponentially suppressed in $e^{-2\pi R^2}$ associated to Taub-NUT anti-Taub-NUT instantons. This is an important consistency check since the same unfolding procedure fails to reproduce the non-perturbative contributions to the constant terms, which are also required to solve the differential equation.

Chapter 5

Outlook

One of the aims of this manuscript was to collect and present in a simple and coherent way the publications produced with the work of the author during their three years of PhD. Many open questions remains after this work, in addition to those that were encountered through this journey.

Non-perturbative contributions to the zero modes. There exists missing contributions to the zero modes of $G^{(p,q)}_{ab,cd}$, which induce non-perturbative terms in the exact low-energy effective action at finite coupling in both of the heterotic or type II descriptions. In the case of decompactification from three to four dimensions, they are particular solutions to the full inhomogeneous equations obtained from the supersymmetry constraints, *e.g.* § E.1 of [BCHP3], for instance

$$\begin{aligned}&\Big(2\mathcal{D}_{(\mu}{}^{\tau}\mathcal{D}_{\nu)\tau} - (\partial_\phi + 6)\mathcal{D}_{\mu\nu} + \tfrac{1}{8}(\partial_\phi+8)(\partial_\phi+6)\delta_{\mu\nu}\Big)G^{\Gamma}_{\sigma\rho,\kappa\lambda} \\ &= -\frac{3\pi}{4}\delta_{\langle\sigma\rho,}\delta_{\kappa\lambda\rangle}\sum_{\Gamma_1\in\Lambda_m^*\oplus\Lambda_m}\big(F^{\Gamma_1}_{\varsigma d(\mu}{}^{\varsigma}F^{\Gamma-\Gamma_1\lambda d}_{\nu)\lambda} - F^{\Gamma_1}_{\varsigma d(\mu}{}^{\lambda}F^{\Gamma-\Gamma_1\varsigma d}_{\nu)\lambda}\big) - 3\pi F^{\Gamma}_{\mu\nu,\sigma\rho,\kappa\lambda}\ ,\end{aligned} \tag{5.1}$$

where all the indices were chosen in the $SL(2,\mathbb{R})/SO(2)$ direction, while the index d runs in all eight directions, and with $R = e^{-\phi}$ the radius of the decompactified dimension. The first source terms in the r.h.s. behave as $e^{-2\pi R}$ and result from half-BPS instanton-anti-instanton contributions from Abelian modes in $F^{(2k,8)}$, while the last term in (5.1) behaves as $e^{-2\pi R^2}$, induced by instanton-anti-instantons with non-zero Taub-NUT charge. It would be interesting to recover these contributions by solving these sets of differential equations.

Unfolding method and meromorphic functions. Although connected to the previous question this issue is slightly more general, and is related to handling of the unfolding method in the case of meromorphic functions. In particular, the missing contributions mentioned above seem to be related to the presence of overlapping singularities on the modular domain of genus-two Riemann surfaces. It would be useful to understand these complications in a more rigourous manner, and/or to find other tools to extract these contribution directly, in particular for cases where one would not be able to use differential equations to recover the correct results.

Black holes with non-prime charges. The measure of instantons with non-prime vectors (Q, P) gets contributions from all the possible ways of obtaining the given charge (Q, P) by multiplying a lattice charge (Q', P') with the instantonic winding k – winding along the U-duality and/or decompactification direction, *i.e.* such that $(Q, P) = k(Q', P')$. It would be interesting to understand if and how this type of multiplicity counting is related to multiplicities of BPS solitons. This would allow to extend the results presented in this manuscript to all possible quarter-BPS dyonic solitons.

Beyond prime $\mathbb{Z}_N$ CHL models. An obvious generalisation of the results presented in this manuscript would be to extend these calculation to the case of non-prime $\mathbb{Z}_N$ CHL models, with $N = 4, 6$ in particular, whose frame shapes are $1^4 2^2 4^4$ and $1^2 2^3 3^2 6^2$ respectively. The number of cusps in the modular plane is no longer two – which is the case of prime orbifolds – but one can expect general lines of the calculation to hold. It would be interesting to confirm whether the degeneracy of half-BPS states follows

$$\bar{c}_k(Q, P) = \sum_{a|N} \sum_{\substack{d \geq 1 \\ (Q,P)/d \in \Lambda_{em}[a]}} c_k\Big(- \tfrac{\gcd(NQ^2, P^2, Q\cdot P)}{2a\, d^2} \Big) \,, \tag{5.2}$$

as suggestively written in [BCHP2], and to understand the counting of quarter-BPS black holes in these cases. Some initial steps can be found in [169, 105, 170]. Note that there exists other prime CHL models that are not freely acting orbifold, but their frame shape is not balanced and in general one does not expect strong-weak Fricke dualities to be a symmetries of these theories, but rather to map one CHL model to another. A list of such models with their properties can be found in [105, 170].

Matching the perturbative effective actions. The perturbative limits of our conjecture was presented in §3.3 and §3.4, in the case of weak heterotic coupling in three dimensions and weak type II couplings in four dimensions. Other perturbative limit were given in [BCHP2] and [BCHP3]. It would be interesting to match these results with explicit calculations.

Matching classical entropy for all prime vectors (Q, P). In §4.1, we review the black hole entropy calculation for extremal solutions in $\mathcal{N} = 4$ supergravities. The matching between black hole and statistical entropy was performed with high accuracy in [142], and it would be interesting to pursue this calculation for the other types of black hole solution with degeneracy predicted in [BCHP3].

Counting bound states of three half-BPS black holes. It would be also interesting to comprehend the results of [171], conjecturing the degeneracies for three- and two-center bound states of half-BPS states to be Fourier coefficients of a degree three Siegel modular form, and to determine whether it can be understood using the langage of string amplitudes. It is however clear that these states belong to long quarter-BPS multiplets, and thus do not contribute to the helicity supertrace related to the degeneracies studied in this manuscript.

Generalisation to $\mathcal{N} = 2$ theories. Given the success in finding the dyon spectrum in $\mathcal{N} = 4$ supersymmetric string compactifications, one could hope that the dyon degeneracy in $\mathcal{N} = 2$ supersymmetric string theories will also be given by a similar formula:

$$d(Q, P) = \int_{\mathcal{C}} \mathrm{d}M \, f(Q, P, M) \,, \tag{5.3}$$

where M denotes a set of complex variables, $\mathcal{C}$ is a contour in the complex manifold labelled by the variables M. A suggestion has been made for the STU model in [172, 173], although the expected relevant BPS coupling should be the metric over the moduli space of three-dimensional vector multiplets.

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Appendix A [BCHP1]

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Protected couplings and BPS dyons in half-maximal supersymmetric string vacua

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We analyze four- and six-derivative couplings in the low energy effective action of $D = 3$ string vacua with half-maximal supersymmetry. In analogy with an earlier proposal for the $(\nabla\Phi)^4$ coupling, we propose that the $\nabla^2(\nabla\Phi)^4$ coupling is given exactly by a manifestly U-duality invariant genus-two modular integral. In the limit where a circle in the internal torus decompactifies, the $\nabla^2(\nabla\Phi)^4$ coupling reduces to the $\nabla^2 F^4$ and $\mathcal{R}^2 F^2$ couplings in $D = 4$, along with an infinite series of corrections of order e^{-R}, from four-dimensional 1/4-BPS dyons whose wordline winds around the circle. Each of these contributions is weighted by a Fourier coefficient of a meromorphic Siegel modular form, explaining and extending standard results for the BPS index of 1/4-BPS dyons.

String vacua with half-maximal supersymmetry offer an interesting window into the non-perturbative regime of string theory and the quantum physics of black holes, unobstructed by intricacies present in vacua with less supersymmetry. In particular, the low-energy effective action at two-derivative order does not receive any quantum corrections, and all higher-derivative interactions are expected to be invariant under the action of an arithmetic group $G(\mathbb{Z})$, known as the U-duality group, on the moduli space G/K of massless scalars [1–4]. This infinite discrete symmetry also constrains the spectrum of BPS states, and allows to determine, for any values of the electromagnetic charges, the number of BPS black hole micro-states (counted with signs) in terms of Fourier coefficients of certain modular forms [5–7]. This property has been used to confirm the validity of the microscopic stringy description of BPS black holes at an exquisite level of precision, both for small black holes (preserving half of the supersymmetries of the background) [8, 9] and for large black holes (preserving a quarter of the same) [10–18].

In this letter, we shall exploit U-duality invariance and supersymmetry Ward identities to determine certain higher-derivative couplings in the low-energy effective action of three-dimensional string vacua with 16 supercharges, for all values of the moduli. These protected couplings are analogues of the $\mathcal{R}^4$ and $\nabla^6\mathcal{R}^4$ couplings in toroidal compactifications of type II strings, which have been determined exactly in [19, 20] and in many subsequent works. Our motivation for studying these protected couplings in $D = 3$ is that they are expected to encode the infinite spectrum of BPS black holes in $D = 4$, in a way consistent with the U-duality group $G_3(\mathbb{Z})$. The latter contains the four-dimensional U-duality group $G_4(\mathbb{Z})$, but is potentially far more constraining. Thus, these protected couplings provide analogues of 'black hole partition functions', which do not suffer from the usual difficulties in defining thermodynamical partition functions in theories of quantum gravity, and are manifestly automorphic [21].

The fact that solitons in $D = 4$ may induce instanton corrections to the quantum effective potential in dimension $D = 3$ is well known in the context of gauge theories with compact $U(1)$ [22]. In the context of quantum field theories with 8 rigid supersymmetries, BPS dyons in four dimensions similarly correct the moduli space metric after reduction on a circle [23, 24]. In string vacua with 16 local supersymmetries, one similarly expects that 1/2-BPS dyons in $D = 4$ will contribute to four-derivative scalar couplings of the form $F_{abcd}(\Phi)\,\nabla\Phi^a\nabla\Phi^b\nabla\Phi^c\nabla\Phi^d$ in $D = 3$, while both 1/2-BPS and 1/4-BPS dyons in $D = 4$ will contribute to six-derivative scalar couplings of the form $G_{ab,cd}(\Phi)\,\nabla(\nabla\Phi^a\nabla\Phi^b)\nabla(\nabla\Phi^c\nabla\Phi^d)$ (here, ∇ denote space-time derivatives, contracted so as to make a Lorentz scalar). In either case, the contribution of a four-dimensional BPS state with electric and magnetic charges (Q, P) is expected to be suppressed by $e^{-2\pi R\mathcal{M}(Q,P)}$, where $\mathcal{M}(Q, P)$ is the BPS mass and R the radius of the circle on which the four-dimensional theory is compactified, and weighted by a suitable BPS index $\Omega(Q, P)$ counting the number of BPS states with given charges. In addition, coupling to gravity implies additional $\mathcal{O}(e^{-R^2/\ell_P^2})$ corrections from gravitational Taub-NUT instantons, which are essential for invariance under $G_3(\mathbb{Z})$ (here, ℓ_P is the Planck length in four dimensions).

For simplicity we shall restrict attention to the simplest three-dimensional string vacuum with 16 supercharges, obtained by compactifying the ten-dimensional heterotic string on T^7. Our construction can however be generalized to other half-maximal supersymmetric models with reduced rank [25] with some effort [26]. The moduli space in three dimensions is the symmetric space $\mathcal{M}_3 = G_{24,8}$ [28], where $G_{p,q} = O(p, q)/O(p) \times O(q)$ denotes the orthogonal Grassmannian of q-dimensional positive planes in $\mathbb{R}^{p,q}$. In the limit where the heterotic string coupling g_3 becomes small, $\mathcal{M}_3$ decomposes as

$$G_{24,8} \to \mathbb{R}^+ \times G_{23,7} \ltimes \mathbb{R}^{30} , \qquad (1)$$

where the first factor corresponds to g_3, the second factor to the Narain moduli space (parametrizing the metric, B-field and gauge field on T^7), and $\mathbb{R}^{30}$ to the scalars a^I dual to the gauge fields in three dimensions. At each order in g_3^2, the low-energy effective action is known to be invariant under the T-duality group $O(23,7,\mathbb{Z})$, namely the automorphism group of the even self-dual Narain lattice $\Lambda_{23,7}$ [27]. The latter leaves g_3 invariant, acts on $G_{23,7}$ by left multiplication and on the last factor in (1) by the defining representation. U-duality postulates that this symmetry is extended to $G_3(\mathbb{Z}) = O(24,8,\mathbb{Z})$, the automorphism group of the 'non-perturbative Narain lattice' $\Lambda_{24,8} = \Lambda_{23,7} \oplus \Lambda_{1,1}$, where $\Lambda_{1,1}$ is the standard even-self dual lattice of signature $(1,1)$ [29].

In the limit where the radius R of one circle of the internal torus becomes large, $\mathcal{M}_3$ instead decomposes as

$$G_{24,8} \to \mathbb{R}^+ \times [G_{2,1} \times G_{22,6}] \ltimes \mathbb{R}^{56+1} , \qquad (2)$$

where the first factor now corresponds to $R^2/(g_4^2 \ell_H^2) = R/(g_3^2 \ell_H) = R^2/\ell_P^2$ (with ℓ_H being the heterotic string scale and g_4 the string coupling in $D=4$), the second correspond to the moduli space $\mathcal{M}_4$ in 4 dimensions, the third factor to the holonomies a^{1I}, a^{2I} of the 28 electric gauge fields fields and their magnetic duals along the circle, along with the NUT potential ψ, dual to the Kaluza–Klein gauge field. The factor $G_{2,1} \cong SL(2)/U(1)$ is parametrized by the axio-dilaton $S = S_1 + \mathrm{i} S_2 = B + \mathrm{i}/g_4^2$, while $G_{22,6}$ is the Narain moduli space of T^6, with coordinates ϕ. In the limit $R \to \infty$, the U-duality group is broken to $SL(2,\mathbb{Z}) \times O(22,6,\mathbb{Z})$, where the first factor $SL(2,\mathbb{Z})$ is the famous S-duality in four dimensions [1, 2].

Besides being automorphic under $G_3(\mathbb{Z})$, the couplings F_{abcd} and $G_{ab,cd}$ must satisfy supersymmetric Ward identities. To state them, we introduce the covariant derivative $\mathcal{D}_{a\hat{b}}$ on the Grassmannian $G_{p,q}$, defined by its action on the projectors $p^I_{L,a}$ and $p^I_{R,\hat{a}}$ on the time-like p-plane and its orthogonal complement (here and below, $a, b, \ldots$, $\hat{a}, \hat{b} \ldots$, $I, J \ldots$ take values 1 to p, q, and $p+q$, respectively):

$$\mathcal{D}_{a\hat{b}}\, p^I_{L,c} = \tfrac12 \delta_{ac}\, p^I_{R,\hat{b}} , \quad \mathcal{D}_{a\hat{b}}\, p^I_{R,\hat{c}} = \tfrac12 \delta_{\hat{b}\hat{c}}\, p^I_{L,a} . \qquad (3)$$

The trace of the operator $\mathcal{D}^2_{ef} = \mathcal{D}_{(e}{}^{\hat{g}} \mathcal{D}_{f)\hat{g}}$ is equal to (1/2 times) the Laplacian on $G_{p,q}$. On-shell linearized superspace methods indicate that F_{abcd} and $G_{ab,cd}$ have to satisfy [26]

$$\mathcal{D}^2_{ef}\, F_{abcd} = c_1\, \delta_{ef}\, F_{abcd} + c_2\, \delta_{e)(a}\, F_{bcd)(f} + c_3\, \delta_{(ab}\, F_{cd)ef} , \qquad (4)$$

$$\begin{aligned} \mathcal{D}^2_{ef} G_{ab,cd} =& c_4 \delta_{ef} G_{ab,cd} + c_5 \left[\delta_{e)(a} G_{b)(f,cd} + \delta_{e)(c} G_{d)(f,ab}\right] \\ &+ c_6 \left[\delta_{ab}\, G_{ef,cd} + \delta_{cd}\, G_{ef,ab} - 2\delta_{a)(c}\, G_{ef,d)(b}\right] \\ &+ c_7 \left[F_{abk(e}\, F_{f)cd}{}^k - F_{c)ka(e}\, F_{f)b(d}{}^k\right] , \end{aligned} \qquad (5)$$

$$\mathcal{D}_{[e}{}^{[\hat{e}} \mathcal{D}_{f]}{}^{\hat{f}]} F_{abcd} = 0 , \quad \mathcal{D}_{[e}{}^{[\hat{e}} \mathcal{D}_{f}{}^{\hat{f}} \mathcal{D}_{g]}{}^{\hat{g}]} G_{ab,cd} = 0 . \qquad (6)$$

The first two constraints are analogous to those derived in [30] for H^4 and $\nabla^2 H^4$ couplings in Type IIB string theory on K3. The numerical coefficients c_1, ... c_7 will be fixed below from the knowledge of perturbative contributions.

EXACT $(\nabla\Phi)^4$ COUPLINGS IN $D=3$

Based on the known one-loop contribution [31–33], it was proposed in [34] (a proposal revisited in [35]) that the four-derivative scalar coupling F_{abcd} is given exactly by the genus-one modular integral

$$F^{(24,8)}_{abcd} = \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\partial^4}{(2\pi \mathrm{i})^4 \partial y^a \partial y^b \partial y^c \partial y^d}\bigg|_{y=0} \frac{\varGamma_{24,8}}{\Delta} \qquad (7)$$

where $\mathcal{F}_1$ is the standard fundamental domain for the action of $SL(2,\mathbb{Z})$ on the Poincaré upper half-plane, $\Delta = \eta^{24}$ is the unique cusp form of weight 12, and $\varGamma_{24,8}$ is the partition function of the non-perturbative Narain lattice,

$$\varGamma_{24,8} = \rho_2^4 \sum_{Q \in \Lambda_{24,8}} e^{\mathrm{i}\pi Q_L^2 \rho - \mathrm{i}\pi Q_R^2 \bar{\rho} + 2\pi \mathrm{i} Q_L \cdot y + \frac{\pi (y\cdot y)}{2\rho_2}} \qquad (8)$$

where $Q_L \equiv p^I_L Q_I$, $Q_R \equiv p^I_R Q_I$, and $|Q|^2 = Q_L^2 - Q_R^2$ takes even values on $\Lambda_{24,8}$. It will be important that the Fourier coefficients of $1/\Delta = \sum_{m \geq -1} c(m)\, q^m$ count the number of 1/2-BPS states in the $D=4$ vacuum obtained by decompactifying a circle inside T^7. This is obvious from the fact that these states are dual to perturbative string states carrying only left-moving excitations [5, 8]. It is also worth noting that the ansatz (7) is a special case of a more general class of modular integrals, which we shall denote by $F^{(q+16,q)}_{abcd}$, where the lattice $\Lambda_{24,8}$ is replaced by an even self-dual lattice $\Lambda_{q+16,q}$ and the factor ρ_2^4 by $\rho_2^{q/2}$. The integral $F^{(q+16,q)}_{abcd}$ converges for $q < 6$, and is defined for $q \geq 6$ by a suitable regularization prescription. For $q \leq 7$, the modular integral $F^{(q+16,q)}_{abcd}$ controls the one-loop contribution to the F^4 coupling in heterotic string compactified on T^q [31–33]. For any value of q, it can be checked that $F^{(q+16,q)}_{abcd}$ satisfies (4) and (6) with $c_1 = \frac{2-q}{4}$, $c_2 = 4 - q$, $c_3 = 3$.

By construction, the ansatz (7) is a solution of the supersymmetric Ward identity, which is manifestly invariant under $G_3(\mathbb{Z})$. Its expansion at weak coupling (corresponding to the parabolic decomposition (1), such that the non-perturbative Narain lattice $\Lambda_{24,8}$ degenerates to $\Lambda_{23,7} \oplus \Lambda_{1,1}$) can be computed using the standard unfolding trick. For simplicity, we shall assume that none of the indices $abcd$ lies along $\Lambda_{1,1}$:

$$\begin{aligned} F^{(24,8)}_{\alpha\beta\gamma\delta} =& \frac{c_0}{16\pi g_3^4}\, \delta_{(\alpha\beta} \delta_{\gamma\delta)} + \frac{F^{(23,7)}_{\alpha\beta\gamma\delta}}{g_3^2} + 4 \sum_{k=1}^{3} \sum_{Q \in \Lambda^*_{23,7}} P^{(k)}_{\alpha\beta\gamma\delta} \\ &\times \bar{c}(Q)\, g_3^{2k-9}\, |\sqrt{2} Q_R|^{k-\frac72}\, K_{k-\frac72}\left(\tfrac{2\pi}{g_3^2} |\sqrt{2} Q_R|\right) e^{-2\pi \mathrm{i} a^I Q_I} \end{aligned} \qquad (9)$$

where $c_0 = 24$ is the constant term in $1/\Delta$, $\Lambda^\star = \Lambda\backslash\{0\}$, $P^{(1)}_{\alpha\beta\gamma\delta}(Q) = Q_{L\alpha}Q_{L\beta}Q_{L\gamma}Q_{L\delta}$, $P^{(2)}_{\alpha\beta\gamma\delta} = -\frac{3}{2\pi}\delta_{(\alpha\beta}Q_{L\gamma}Q_{L\delta)}$, $P^{(3)}_{\alpha\beta\gamma\delta} = \frac{3}{16\pi^2}\delta_{(\alpha\beta}\delta_{\gamma\delta)}$, $K_\nu(z)$ is the modified Bessel function of the second kind, behaving as $\sqrt{\frac{\pi}{2z}}e^{-z}(1+\mathcal{O}(1/z))$ for large positive values of z, and

$$\bar{c}(Q) = \sum_{d|Q} d\, c\big(-\tfrac{|Q|^2}{2d}\big)\ . \tag{10}$$

After rescaling from Einstein frame to string frame, the first and second terms in (9) are recognized as the tree-level and one-loop $(\nabla\Phi)^4$ coupling in perturbative heterotic string theory, while the remaining terms correspond to NS5-brane and KK5-branes wrapped on any possible T^6 inside T^7 [34].

In the large radius limit (corresponding to the parabolic decomposition (2), such that the non-perturbative Narain lattice $\Lambda_{24,8}$ degenerates to $\Lambda_{22,6}\oplus\Lambda_{2,2}$), we get instead (in units where $\ell_P = 1$)

$$\begin{aligned} F^{(24,8)}_{\alpha\beta\gamma\delta} &= R^2\left(\frac{c_0}{16\pi}\,\widehat{E}_1(S)\,\delta_{(\alpha\beta}\delta_{\gamma\delta)} + F^{(22,6)}_{\alpha\beta\gamma\delta}\right) \\ &+4\sum_{k=1}^{3}\sum_{Q'\in\Lambda^\star_{22,6}}\sum_{m,n}{}' c\big(-\tfrac{|Q'|^2}{2}\big)\,R^{5-k}\,P^{(k)}_{\alpha\beta\gamma\delta} \\ &K_{k-\frac{7}{2}}\left(\tfrac{2\pi R|mS+n|}{\sqrt{S_2}}|\sqrt{2}Q'_R|\right)\,e^{-2\pi\mathrm{i}(ma^1+na^2)\cdot Q'}+\dots \end{aligned} \tag{11}$$

where $\widehat{E}_1(S) = -\frac{3}{\pi}\log S_2|\eta(S)|^4$. The first term in (11) originates from the dimensional reduction of the $\mathcal{R}^2$ and F^4 couplings in $D=4$ [33, 36], after dualizing the gauge fields into scalars. The term $F^{(22,6)}_{\alpha\beta\gamma\delta}$ can also be traced to the four-derivative scalar couplings studied in [32]. The second term in (11) is of order $e^{-2\pi R\mathcal{M}(Q,P)}$, where $\mathcal{M}$ is the mass of a four-dimensional 1/2-BPS state with electromagnetic charges $(Q,P) = (mQ', nQ')$. The phase factor is the expected minimal coupling of a dyonic state to the holonomies of the electric and magnetic gauge fields along the circle. Fixing charges (Q,P) such that Q and P are collinear, the sum over (m,n) induces a measure factor

$$\mu(Q,P) = \sum_{d|(Q,P)} c\big(-\tfrac{\gcd(Q^2,P^2,Q\cdot P)}{2d^2}\big)\ , \tag{12}$$

which is recognized as the degeneracy of 1/2-BPS states with charges (Q,P). In particular for a purely electric state ($P=0$) with primitive charge, it reduces to the well-known result $c(-|Q|^2/2)$ [5]. The dots in (11) stand for terms of order $e^{-2\pi R^2|k|+2\pi\mathrm{i}k\psi}$, characteristic of a Kaluza–Klein monopole of the form $\mathrm{TN}_k\times T^6$, where TN_k is Euclidean Taub–NUT space with charge k. These contributions will be discussed in [26].

EXACT $\nabla^2(\nabla\Phi)^4$ COUPLINGS IN $D=3$

We now turn to the six-derivative coupling $G_{ab,cd}$, which is expected to receive both 1/2-BPS and 1/4-BPS instanton contributions. Based on U-duality invariance, supersymmetric Ward identities and the known two-loop contribution [37, 38], it is natural to conjecture that $G_{ab,cd}$ is given by the genus-two modular integral

$$G^{(24,8)}_{ab,cd} = \int_{\mathcal{F}_2}\frac{\mathrm{d}^3\Omega_1\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\frac{\frac{1}{2}(\varepsilon_{ik}\varepsilon_{jl}+\varepsilon_{il}\varepsilon_{jk})\partial^4}{(2\pi\mathrm{i})^4\partial y^a_i\partial y^b_j\partial y^c_k\partial y^d_l}\bigg|_{y=0}\frac{\varGamma_{24,8,2}}{\Phi_{10}}\ , \tag{13}$$

where $\mathcal{F}_2$ is the standard fundamental domain for the action of $Sp(4,\mathbb{Z})$ on the Siegel upper half-plane of degree two [39], $|\Omega_2|$ is the determinant of the imaginary part of $\Omega = \Omega_1 + \mathrm{i}\Omega_2$, Φ_{10} is the unique cusp form of weight 10 under the Siegel modular group $Sp(4,\mathbb{Z})$ (whose inverse counts micro-states of 1/4-BPS black holes [6]), and $\varGamma_{24,8,2}$ is the genus-two partition function of the non-perturbative Narain lattice,

$$\varGamma_{24,8,2} = |\Omega_2|^4\sum_{Q^i\in\Lambda^{\otimes 2}_{24,8}} e^{\mathrm{i}\pi(Q^i_L\Omega_{ij}Q^j_L - Q^i_R\bar{\Omega}_{ij}Q^j_R + 2Q^i_L y_i)+\frac{\pi}{2}y^a_i\Omega_2^{-1ij}y_{ja}} \tag{14}$$

Acting with the y^a_i-derivatives results in the insertion of a polynomial $P_{\alpha\beta,\gamma\delta}(Q^i_L,\Omega_2^{-1})$ of degree 4 and 2 in its first and second arguments. We shall denote by $G^{(q+16,q)}_{ab,cd}$ the analogue of (14) where the lattice $\Lambda_{24,8}$ is replaced by $\Lambda_{q+16,q}$ and the power of $|\Omega_2|$ by $q/2$. The integral $G^{(q+16,q)}_{ab,cd}$ is convergent for $q<6$, and defined for $q\geq 6$ by a suitable regularization prescription [40]. For $q\leq 7$, the modular integral $G^{(q+16,q)}_{ab,cd}$ controls the two-loop contribution to the ∇^2F^4 coupling in heterotic string compactified on T^q [37, 38].

For any value of q, one can show that $G^{(q+16,q)}_{ab,cd}$ satisfies (5) and (6) with $c_4 = \frac{3-q}{2}, c_5 = \frac{6-q}{2},\ c_6 = \frac{1}{2},\ c_7 = -\pi$. In particular, the quadratic source term on the r.h.s. of (5) originates from the pole of $1/\Phi_{10}$ on the separating degeneration divisor, similar to the analysis in [30, 40]. Thus, $G^{(24,8)}$ is a solution of the supersymmetric Ward identity, which is manifestly invariant under $G_3(\mathbb{Z})$. It remains to check that it produces the expected terms at weak coupling, when $\Lambda_{24,8}$ degenerates to $\Lambda_{23,7}\oplus\Lambda_{1,1}$. This limit can be studied using a higher-genus version of the unfolding trick [41, 42]. Using results about the Fourier–Jacobi expansion of $1/\Phi_{10}$ from [16], we find

$$\begin{aligned} G^{(24,8)}_{\alpha\beta,\gamma\delta} &= \frac{G^{(23,7)}_{\alpha\beta,\gamma\delta}}{g_3^4} - \frac{\delta_{\alpha\beta}G^{(23,7)}_{\gamma\delta}+\delta_{\gamma\delta}G^{(23,7)}_{\alpha\beta}-2\delta_{\gamma)(\alpha}G^{(23,7)}_{\beta)(\delta}}{12g_3^6} \\ &-\frac{1}{2\pi g_3^8}\left[\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha(\gamma}\delta_{\delta)\beta}\right]+\dots \end{aligned} \tag{15}$$

where

$$G^{(q+16,q)}_{ab} = \int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\partial^2}{(2\pi\mathrm{i})^2\partial y^a\partial y^b}\bigg|_{y=0}\frac{\widehat{E}_2\,\varGamma_{q+16,q}}{\Delta}\ , \tag{16}$$

with $\widehat{E}_2 = \frac{12}{\mathrm{i}\pi}\partial_\rho\log\eta - \frac{3}{\pi\rho_2}$ the almost holomorphic Eisenstein series of weight 2. The first and second terms in (15) corresponds to the zero and rank 1 orbits, respectively. The third term is necessary for consistency with the quadratic term on the r.h.s. of the supersymmetric Ward identity (5), although a naive unfolding procedure fails to produce it, presumably due to the singularity of the integrand in the separating degeneration limit. After rescaling to string frame, the first three terms in (15) correspond to the expected two-loop [37, 38], one-loop [43] and tree-level contributions [44, 45] to the $\nabla^2(\nabla\Phi)^4$ coupling in heterotic string on T^7, while the dots stand for terms of order e^{-1/g_3^2} ascribable to NS5-brane and KK5-brane instantons, which will be discussed in [26]. Note that the tree-level single trace $\nabla^2 F^4$ term in [44] proportional to $\zeta(3)$ vanishes on the Cartan subalgebra [46], and does not contribute to this coupling.

Having shown that our ansatz (13) passes all consistency conditions in $D=3$, let us now analyze its large radius limit, where $\Lambda_{24,8}$ degenerates to $\Lambda_{22,6}\oplus\Lambda_{2,2}$. Again, the unfolding trick gives

$$\begin{aligned} &G^{(24,8)}_{\alpha\beta,\gamma\delta} \qquad (17)\\ &= R^4\Big[G^{(22,6)}_{\alpha\beta,\gamma\delta} - \frac{\widehat{E}_1(S)}{12}\left(\delta_{\alpha\beta}G^{(22,6)}_{\gamma\delta}+\delta_{\gamma\delta}G^{(22,6)}_{\alpha\beta}-2\delta_{\gamma)(\alpha}G^{(22,6)}_{\beta)(\delta}\right)\\ &\quad +g(S)(\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha(\gamma}\delta_{\delta)\beta})\Big]+G^{(1)}_{\alpha\beta,\gamma\delta}+G^{(2)}_{\alpha\beta,\gamma\delta}+G^{(\mathrm{KKM})}_{\alpha\beta,\gamma\delta}\end{aligned}$$

The two terms on the first line (which correspond to the constant term with respect to the parabolic decomposition (2)) originate from the reduction of the $\nabla^2 F^4$ and $\mathcal{R}^2F^2$ couplings in four dimensions. The term proportional to $g(S)$ originates presumably from the separating degeneration divisor, and is determined by the differential equation (6). The terms $G^{(1)}$ and $G^{(2)}$ are independent of the NUT potential ψ, and correspond to the Abelian Fourier coefficients. They are both suppressed as $e^{-2\pi R\mathcal{M}(Q,P)}$, but $G^{(1)}$ has support on electromagnetic charges (Q,P) which Q and P collinear, hence corresponds to contributions of 1/2-BPS states winding the circle, while $G^{(2)}$ has support on generic charges, corresponding to 1/4-BPS states. The last term $G^{(\mathrm{KKM})}$ includes all terms with non-zero charge with respect to the NUT potential, corresponding to Kaluza–Klein monopole contributions.

In this letter, we focus on the contribution $G^{(2)}$ from 1/4-BPS black holes. This contribution originates from the 'Abelian rank 2 orbit', whose stabilizer is the parabolic subgroup $GL(2,\mathbb{Z})\ltimes\mathbb{Z}^3$ inside $Sp(4,\mathbb{Z})$. Thus, the integral can be unfolded onto $\mathcal{P}_2/PGL(2,\mathbb{Z})\times[0,1]^3$, where $\mathcal{P}_2$ denotes the space of positive definite 2×2 matrices Ω_2:

$$\begin{aligned} G^{(2)}_{\alpha\beta,\gamma\delta} =& R^4\int_{\mathcal{P}_2}\frac{\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\int_{[0,1]^3}\mathrm{d}^3\Omega_1\,\frac{(\varepsilon_{ik}\varepsilon_{jl}+\varepsilon_{il}\varepsilon_{jk})\partial^4}{(2\pi\mathrm{i})^4\partial y_i^a\partial y_j^b\partial y_k^c\partial y_l^d}\Big|_{y=0}\\ &\times\frac{\left\langle e^{-2\pi\mathrm{i}a^{iI}A_{ij}Q_I^j}\right\rangle_{22,6,2}}{\Phi_{10}}\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ |A|\neq 0}} e^{-\frac{\pi R^2}{S_2}\mathrm{Tr}\left[\Omega_2^{-1}\cdot A^\intercal\cdot\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}\cdot A\right]}\end{aligned} \qquad (18)$$

where $\langle f(Q)\rangle_{22,6,2}$ denotes the partition function $\Gamma_{22,6,2}$ with an insertion of $f(Q)$ in the sum. The integral over Ω_1 at fixed Ω_2 extracts the Fourier coefficient $C\left[\begin{pmatrix} -\frac12|Q|^2 & -Q\cdot P\\ -Q\cdot P & -\frac12|P|^2\end{pmatrix};\Omega_2\right]$ of $1/\Phi_{10}$. Due to the zeros of Φ_{10}, the latter is a locally constant function of Ω_2, discontinuous across certain real codimension 1 walls in $\mathcal{P}_2$ [47, 48]. For large R however, the remaining integral over Ω_2 is dominated by a saddle point $\Omega_2^\star$ (see (24) below), so to all orders in $1/R$ around the saddle point, we can replace the above Fourier coefficient by its value at $\Omega_2^\star$. The remaining integral over Ω_2 can be computed using

$$\int_{\mathcal{P}_2}\mathrm{d}^3S\,|S|^{\delta-\frac32}\,e^{-\pi\mathrm{Tr}\,(SA+S^{-1}B)} = 2\left(\frac{|B|}{|A|}\right)^{\delta/2}\widetilde{B}_\delta(AB)\,, \qquad (19)$$

where $\widetilde{B}_\delta(Z)$ is a matrix-variate generalization of the modified Bessel function [49][60],

$$\widetilde{B}_\delta(Z)=\int_0^\infty\frac{\mathrm{d}t}{t^{3/2}}\,e^{-\pi t-\frac{\pi\mathrm{Tr}Z}{t}}\,K_\delta\left(\frac{2\pi\sqrt{|Z|}}{t}\right)\,. \qquad (20)$$

In the limit where all entries in Z are large, one has

$$\widetilde{B}_\delta(Z)\sim\frac12\left[|Z|\big(\mathrm{Tr}Z+2\sqrt{|Z|}\big)\right]^{-\frac14}\,e^{-2\pi\sqrt{\mathrm{Tr}Z+2\sqrt{|Z|}}}\,. \qquad (21)$$

Further relabelling $\binom{Q}{P}=A\binom{Q_1}{Q_2}$, we find

$$\begin{aligned} G^{(2)}_{\alpha\beta,\gamma\delta} &= 2R^7\sum_{Q,P\in\Lambda^\star_{22,6}} e^{-2\pi\mathrm{i}(a^1Q+a^2P)}\frac{\mu(Q,P)}{|2P_R\wedge Q_R|^{\frac32}}\\ &\times P_{\alpha\beta,\gamma\delta}\left(\tfrac{1}{\sqrt{S_2}}\begin{pmatrix}1 & S_1\\ 0 & S_2\end{pmatrix}\begin{pmatrix}Q_L\\ P_L\end{pmatrix},-\tfrac{1}{\pi R^2}\tfrac{\partial}{\partial Y}\right)\cdot \qquad (22)\\ &\left(|Y|^{\frac34}\widetilde{B}_{\frac32}\left[Y\,\tfrac{2R^2}{S_2}\begin{pmatrix}1 & S_1\\ 0 & S_2\end{pmatrix}\begin{pmatrix}|Q_R|^2 & P_R\cdot Q_R\\ P_R\cdot Q_R & |P_R|^2\end{pmatrix}\begin{pmatrix}1 & 0\\ S_1 & S_2\end{pmatrix}\right]\right)\Big|_{Y=\mathbb{1}}\end{aligned}$$

where $|P_R\wedge Q_R|=\sqrt{(P_R^2)(Q_R^2)-(P_R\cdot Q_R)^2}$, $P_{\alpha\beta,\gamma\delta}$ is the polynomial defined below (14),

$$\mu(Q,P)=\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda_{22,6}^{\otimes 2}}}|A|\,C\left[A^{-1}\begin{pmatrix}-\frac12|Q|^2 & -Q\cdot P\\ -Q\cdot P & -\frac12|P|^2\end{pmatrix}A^{-\intercal};\Omega_2^\star\right] \qquad (23)$$

and $\Omega_2^\star$ is the location of the afore-mentioned saddle point,

$$\Omega_2^\star=\frac{R}{\mathcal{M}(Q,P)}A^\intercal\left[\tfrac{1}{S_2}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}+\tfrac{1}{|P_R\wedge Q_R|}\begin{pmatrix}|P_R|^2 & -P_R\cdot Q_R\\ -P_R\cdot Q_R & |Q_R|^2\end{pmatrix}\right]A\,. \qquad (24)$$

Using (21), we see that these contributions behave as $e^{-2\pi R\mathcal{M}(Q,P)}$ in the limit $R\to\infty$, where

$$\mathcal{M}(Q,P)=\sqrt{2\frac{|Q_R+SP_R|^2}{S_2}+4\sqrt{\begin{vmatrix}|Q_R|^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & |P_R|^2\end{vmatrix}}} \qquad (25)$$

is recognized as the mass of a 1/4-BPS dyon with electromagnetic charges (Q,P) [50, 51]. Moreover, in cases where only $A=\mathbb{1}$ contributes to (23), the instanton measure $\mu(Q,P)$ agrees with the BPS index $\Omega(Q,P;S,\phi)$ in the corresponding chamber of the moduli space $\mathcal{M}_4$ in

$D = 4$, computed with the contour prescription in [52]. Our result (23) generalizes this prescription to arbitrary electromagnetic charges (Q, P) and recovers the results of [53–55] for dyons with torsion, fixing a subtlety in the choice of chamber. Additional (exponentially suppressed) contributions to $G^{(2)}$ arise from the difference between $C\left[\begin{pmatrix} -\frac{1}{2}|Q|^2 & -Q\cdot P \\ -Q\cdot P & -\frac{1}{2}|P|^2 \end{pmatrix}; \Omega_2\right]$ and its value at the saddle point. The relation between the jumps of these Fourier coefficients and the possible splittings of a 1/4-BPS bound state into two 1/2-BPS constituents [47] is crucial for consistency with the quadratic source term in the supersymmetric Ward identity (5). These contributions, along with the terms $G^{(2)}$ and $G^{(\mathrm{KKM})}$ which we have ignored here, will be discussed in [26].

DISCUSSION

In this work, we have conjectured the exact form of the $(\nabla\Phi)^4$ and $\nabla^2(\nabla\Phi)^4$ couplings in the low energy effective action of $D = 3$ string vacua with half-maximal supersymmetry, focussing on the simplest model, heterotic string compactified on T^7. Our ansätze (7) and (13) are manifestly U-duality invariant, satisfy the requisite supersymmetric Ward identities, reproduce the known perturbative contributions at weak heterotic coupling and the known $F^4, \mathcal{R}^2, D^2F^4$ and $\mathcal{R}^2F^2$ couplings in $D = 4$ in the limit where the radius of one circle inside T^7 becomes large. While we do not yet have a rigorous proof that these constraints uniquely determine the functions F_{abcd} and $G_{ab,cd}$, we expect that additional contributions from cusp forms are ruled out by the supersymmetric Ward identities (4) and (5), by the same type of arguments which apply for the $\mathcal{R}^4$ and $\nabla^6\mathcal{R}^4$ couplings.

In the limit where the radius of one circle inside T^7 becomes large, we find, in addition to the aforementioned power-like terms, an infinite series of corrections of order $e^{-2\pi R\mathcal{M}(Q,P)}$ which are interpreted as Euclidean counterparts of four-dimensional BPS states with mass $\mathcal{M}(Q, P)$, whose worldline winds around the circle. Rather remarkably, the contribution from a 1/4-BPS dyon is weighted by the BPS index $\Omega(Q, P; S, \phi)$, extracted from the Siegel modular form $1/\Phi_{10}$ using the very same contour prescription as in [52]. Indeed, it was suggested in [56] (see also [57, 58]) to represent 1/4-BPS dyons as heterotic strings wrapped on a genus-two curve holomorphically embedded in a T^4 inside T^7. This picture was further used in [59] to justify the contour prescription of [52]. Our analysis of the $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ gives a concrete basis to these heuristic ideas, and explains why 1/4-BPS dyons in $D = 4$ are counted by a Siegel modular form of genus two. We emphasize that the introduction of the Siegel modular form $1/\Phi_{10}$ in the conjectured formula (13) is necessary to match the perturbative 2-loop amplitude, where it appears explicitly [37, 38]. A more detailed analysis of the weak coupling and large radius expansions of the $\nabla^2(\nabla\Phi)^4$ coupling will appear in [26], with particular emphasis on the consequences of wall-crossing for three-dimensional couplings.

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Appendix B [BCHP2]

Four-derivative couplings and BPS dyons in heterotic CHL orbifolds

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Abstract

Three-dimensional string models with half-maximal supersymmetry are believed to be invariant under a large U-duality group which unifies the S and T dualities in four dimensions. We propose an exact, U-duality invariant formula for four-derivative scalar couplings of the form $F(\Phi)(\nabla\Phi)^4$ in a class of string vacua known as CHL $\mathbb{Z}_N$ heterotic orbifolds with N prime, generalizing our previous work which dealt with the case of heterotic string on T^6. We derive the Ward identities that $F(\Phi)$ must satisfy, and check that our formula obeys them. We analyze the weak coupling expansion of $F(\Phi)$, and show that it reproduces the correct tree-level and one-loop contributions, plus an infinite series of non-perturbative contributions. Similarly, the large radius expansion reproduces the exact F^4 coupling in four dimensions, including both supersymmetric invariants, plus infinite series of instanton corrections from half-BPS dyons winding around the large circle, and from Taub-NUT instantons. The summation measure for dyonic instantons agrees with the helicity supertrace for half-BPS dyons in 4 dimensions in all charge sectors. In the process we clarify several subtleties about CHL models in $D = 4$ and $D = 3$, in particular we obtain the exact helicity supertraces for 1/2-BPS dyonic states in all duality orbits.

Contents

1 Introduction

In the absence of a first principle non-pertubative formulation of superstring theory, the study of string vacua with extended supersymmetry continues to be one of the few sources of insight into the strong coupling regime. By exploiting invariance under U-dualities, which the full quantum theory is believed to enjoy [1, 2, 3, 4], as well as supersymmetric Ward identities, it is often possible to determine certain couplings in the low energy effective action exactly, for all values of the moduli (as demonstrated by [5] and numerous subsequent works). The expansion of these couplings near boundaries of the moduli space, corresponding to cusps of the U-duality group, then reveals, beyond power-like terms computable in perturbation theory, infinite series of exponentially suppressed corrections interpreted as semi-classical contributions in the putative string field theory. A particularly interesting class of examples is that of BPS saturated couplings in three-dimensional string vacua: in the limit where a circle in the internal space decompactifies, these couplings receive exponentially suppressed contributions from BPS states in four dimensions, along with further suppressed contributions from Taub-NUT instantons. These couplings can therefore be viewed as BPS black hole partitions, which encode the exact degeneracies (or more precisely, helicity supertraces) of BPS black hole micro-states [6, 7, 8].

In the recent letter [7], we investigated the $F(\Phi)(\nabla\Phi)^4$ and $G(\Phi)\nabla^2(\nabla\Phi)^4$ couplings in the low energy effective action of three-dimensional string vacua with 16 supercharges, focussing on the simplest example of such vacua, namely heterotic strings compactified on a torus T^7, or equivalently, type II strings compactified on $K3\times T^3$. Based on the known perturbative contributions to these couplings, we conjectured exact formulae for the coefficients $F(\Phi)$ and $G(\Phi)$ for all values of the moduli Φ, which satisfy the requisite supersymmetric Ward identities and are manifestly invariant under U-duality. In the limit where one circle inside T^7 decompactifies, we claimed that these formulae reproduce the correct helicity supertraces for 1/2-BPS and 1/4-BPS states with primitive charges, for all values of the moduli ϕ in four dimensions.

The goal of the present work is to demonstrate these claims in the case of the $(\nabla\Phi)^4$ coupling,[1] revisiting the analysis in [9], and extend our conjecture to a class of string vacua with 16 supercharges known as CHL orbifolds [10], restricting to $\mathbb{Z}_N$ orbifolds with N prime for simplicity.

In Section 2, after reviewing relevant aspects of heterotic CHL vacua with 16 supercharges in four and three dimensions, we state the helicity supertraces of 1/2-BPS dyons with arbitrary charge in four dimensions (referring to Appendix A for the derivation of the perturbative BPS spectrum), and determine the precise form of the U-duality group $G_3(\mathbb{Z})$ in three dimensions, consistent with S-duality and T-duality in four dimensions. We then propose a manifestly U-duality invariant formula (2.27) for the coefficient $F_{abcd}(\Phi)$ of the $(\nabla\Phi)^4$ couplings, obtained by covariantizing the known one-loop contribution under $G_3(\mathbb{Z})$, extending the proposal in [7] for the maximal rank case ($N=1$).

In Section 3, using superspace arguments we establish the supersymmetric Ward identities (2.23) which constrain the coupling $F_{abcd}(\Phi)$, and show that the proposal (2.27) satisfies these relations.

In Section 4, we analyze (2.27) in the limit where $g_3\to 0$, and show that it reproduces the known tree-level and one-loop contributions in heterotic perturbation theory, plus an infinite series of NS5-brane, Kaluza–Klein monopole and H-monopole instanton corrections.

In Section 5, we similarly analyze (2.27) in the large radius limit $R\to\infty$, and show that it reproduces the known F^4 and $\mathcal{R}^2$ couplings in $D=4$, along with an infinite series of exponentially suppressed corrections of order $e^{-R\mathcal{M}(Q,P)}$ with Q and P collinear, weighted by the helicity supertrace $\Omega_4(Q,P)$, and further exponentially suppressed corrections from Taub-NUT monopoles.

In most computations, we allow for lattices of arbitrary signature (p,q), before specifying to the most relevant case $(p,q)=(2k,8)$ at the end. Details of some computations are relegated to Appendices. The one-loop vacuum amplitude for heterotic CHL models, from which the perturbative BPS spectrum, F^4 and $(\nabla\Phi)^4$ couplings are easily read off, is constructed in

[1] An analysis of the $\nabla^2(\nabla\Phi)^4$ couplings will appear in a separate publication.

Appendix §A. In §B we decompose the Ward identity on all Fourier modes in the degeneration limit $O(p,q) \to O(p-1,q-1)$, and show that all Fourier coefficients are uniquely determined up to a moduli-independent summation measure. In §C and §D we collect some notations which arise in the analysis of §4 and §5. In Appendix §E we obtain a Poincaré series representation of the relevant genus-one modular integrals, and use the same method to construct Eisenstein series for $O(p,q,\mathbb{Z})$.

2 Dualities, BPS spectrum and $(\nabla\Phi)^4$ couplings in CHL vacua

In this section, we recall relevant aspects of heterotic CHL vacua with 16 supercharges in four and three dimensions, restricting to the case of $\mathbb{Z}_N$ orbifolds with N prime for simplicity. While most of the results are well known, we pay special attention to the quantization conditions for the electromagnetic charges of 4D dyons, and to the precise form of the U-duality groups in $D=4$ and $D=3$. Finally, we state our proposal for the non-perturbative $(\nabla\Phi)^4$ coupling, which is the focus of the remainder of this work.

2.1 Moduli space and 1/2-BPS dyons in $D=4$

Recall that in four-dimensional string vacua with 16 supercharges, the moduli space is locally a product

$$\mathcal{M}_4 = \left[\frac{SL(2,\mathbb{R})}{SO(2)} \times G_{r-6,6}\right]/G_4(\mathbb{Z}) \ , \tag{2.1}$$

where $G_{p,q} \equiv O(p,q)/[O(p)\times O(q)]$ denotes the orthogonal Grassmannian of positive q-planes in a fiducial vector space $\mathbb{R}^{p,q}$ of signature (p,q) (a real symmetric space of dimension pq), r is the rank of the Abelian gauge group, and $G_4(\mathbb{Z})$ is an arithmetic subgroup of $SL(2,\mathbb{R})\times O(r-6,6,\mathbb{R})$. In heterotic string theory compactified on a torus T^6, the first factor is parametrized by the axiodilaton $S = b + 2\pi\mathrm{i}/g_4^2$, where b is the scalar dual to the Kalb-Ramond two-form, while the second factor, with $r=28$, is the Narain moduli space [11]. The U-duality group $G_4(\mathbb{Z})$ is then the product of the S-duality group $SL(2,\mathbb{Z})$, acting on S by fractional linear transformations $S \mapsto \frac{aS+b}{cS+d}$ [1, 2], and of the T-duality group $O(22,6,\mathbb{Z})$, which is the automorphism group of the even self-dual Narain lattice $\Lambda_{22,6} = E_8 \oplus E_8 \oplus I\!I_{6,6}$, where E_8 denotes the root lattice of E_8 and $I\!I_{d,d}$ denotes d copies of the standard hyperbolic lattice $I\!I_{1,1}$. The effective action is singular on real codimension-6 loci where the projection Q_R of a vector $Q\in\Lambda_{22,6}$ with norm $Q^2=2$ on the negative 6-plane parametrized by $G_{r-6,6}$ vanishes, corresponding to points of gauge symmetry enhancement. The same moduli space (2.1) arises in type IIA string compactified on $K3\times T^2$, where the first factor parametrizes the Kähler modulus of T^2, while the second factor parametrizes the axiodilaton, the complex modulus of T^2, the $K3$ moduli and the holonomies of the RR gauge fields on $T^2\times K3$. These two string vacua are in fact related by heterotic/type II duality [12], which in particular turns S-duality into a geometrical symmetry.

Vacua with lower values of r can be constructed as freely acting orbifolds of the maximal rank model with $r=28$ [10, 13, 14, 15]. On the heterotic side, one mods out by a $\mathbb{Z}_N$ rotation of the heterotic lattice $\Lambda_{22,6}$ at values of the Narain moduli where such a symmetry exists, combined with an order N shift along one circle inside T^6. This projection removes $28-r$ of the gauge fields in 4 dimensions, along with their scalar partners. On the type II side, one can similarly mod out by a symplectic automorphism of order N on K3, combined

N	Cycle Shape	k	r	$\Lambda_{k,8-k}$	$\Lambda_m \cong \Lambda_e^*$	$\lvert\Lambda_m^*/\Lambda_m\rvert$
1	1^{24}	12	28		$E_8 \oplus E_8 \oplus I\!I_{6,6}$	1
2	$1^8 2^8$	8	20	$E_8[2]$	$E_8[2] \oplus I\!I_{1,1}[2] \oplus I\!I_{5,5}$	2^{10}
3	$1^6 3^6$	6	16	$D_6[3] \oplus D_2[-1]$	$A_2 \oplus A_2 \oplus I\!I_{3,3}[3] \oplus I\!I_{3,3}$	3^8
5	$1^4 5^4$	4	12	$D_4[5] \oplus D_4[-1]$	$I\!I_{3,3}[5] \oplus I\!I_{3,3}$	5^6
7	$1^3 7^3$	3	10	$D_3[7] \oplus D_5[-1]$	$\left[\begin{smallmatrix}-4 & -1\\ -1 & -2\end{smallmatrix}\right] \oplus I\!I_{2,2}[7] \oplus I\!I_{2,2}$	7^5

Table 1: A class of $\mathbb{Z}_N$ CHL orbifolds. Here $k = 24/(N+1)$ is the weight of the cusp form whose inverse counts 1/2 BPS states, $r = 2k+4$ is the rank of the gauge group and Λ_m is the lattice of magnetic charges in four dimensions. The discriminant group Λ_m^*/Λ_m is isomorphic to $\mathbb{Z}_N^{k+2}$. Agreement between the lattice Λ_m listed here and $\Lambda_{r-6,6}$ defined in (2.2) follows from the lattice isomorphisms (A.33).

with an order N shift on T^2. It is convenient to label this action by the data $\{m(a), a|N\}$ and the associated cycle shape $\prod_{a|N} a^{m(a)}$ such that $\sum_{a|N} am(a) = 24$, corresponding to the cycle decomposition of the $\mathbb{Z}_N$ action on the even homology lattice $H_{\text{even}}(K3) \sim \mathbb{Z}^{24}$. For simplicity we shall restrict ourselves to CHL orbifolds with N prime and cycle shape $1^k N^k$ with $k = 24/(N+1)$. In this case, one can decompose $\Lambda_{22,6} = \Lambda_{Nk,8-k} \oplus I\!I_{1,1} \oplus I\!I_{k-3,k-3}$, such that the $\mathbb{Z}_N$ action acts on the first term by a $\mathbb{Z}_N$ rotation, on the second term by an order N shift, leaving $I\!I_{k-3,k-3}$ invariant (see §A.2 for details on this construction). We denote by $\Lambda_{k,8-k}$ the quotient of $\Lambda_{Nk,8-k}$ under the $\mathbb{Z}_N$ rotation (see Table 1). The U-duality group $G_4(\mathbb{Z})$ includes $\Gamma_1(N) \times \widetilde{O}(r-6,6,\mathbb{Z})$, where $\Gamma_1(N)$ is the congruence subgroup of $SL(2,\mathbb{Z})$ corresponding to matrices $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ with $c = 0 \bmod N, a = d = 1 \bmod N$, and $\widetilde{O}(r-6,6,\mathbb{Z})$ is the restricted automorphism group of the lattice

$$\Lambda_{r-6,6} = \Lambda_{k,8-k} \oplus I\!I_{1,1}[N] \oplus I\!I_{k-3,k-3} \ , \tag{2.2}$$

i.e. the subgroup of the automorphism group of $\Lambda_{r-6,6}$ which acts trivially on the discriminant group $\Lambda_{r-6,6}^*/\Lambda_{r-6,6}$. Here and below, for any lattice Λ, we denote by $\Lambda[\alpha]$ the same lattice with a quadratic form rescaled by a factor α (which is equivalent to rescaling the lattice vectors by $\sqrt{\alpha}$). Note that the lattice (2.2) is still even, *i.e.* $Q^2 \in 2\mathbb{Z}$ for $Q \in \Lambda_{r-6,6}$, but it is no longer unimodular, rather it is a lattice of level N, in the sense that $Q^2 \in 2\mathbb{Z}/N$ for any $Q \in \Lambda_{r-6,6}^*$. Singularities now occur on codimension-q loci where $Q_R^2 = 0$ for a norm 2 vector $Q \in \Lambda_{r-6,6}$, or for a norm $2/N$ vector $Q \in \Lambda_{r-6,6}^*$.

While the U-duality group $G_4(\mathbb{Z})$ must certainly include $\Gamma_1(N) \times \widetilde{O}(r-6,6,\mathbb{Z})$, it may actually be larger. Moreover, special BPS observables may well be invariant under an even larger group. In particular the four-derivative couplings in $D = 4$ turn out to be invariant under the action of the larger duality group $\Gamma_0(N) \times O(r-6,6,\mathbb{Z})$, where $\Gamma_0(N)$ is the subgroup of matrices $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ with $c = 0 \bmod N$ and $O(r-6,6,\mathbb{Z})$ is the full automorphism group of the lattice $\Lambda_{r-6,6}$. For example, the exact $\mathcal{R}^2$ coupling in the low-energy effective action is given by [19, 20, 21]

$$-\frac{1}{(8\pi)^2} \int \mathrm{d}^4x \sqrt{-g} \log(S_2^k |\Delta_k(S)|^2)(\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2) \tag{2.3}$$

where Δ_k is the unique cusp form of weight k under $\Gamma_0(N)$, nowhere vanishing except at the cusps $\mathrm{i}\infty$ and 0,

$$\Delta_k(\tau) = \eta^k(\tau)\,\eta^k(N\tau) \ . \tag{2.4}$$

In the weak coupling limit $S_2 \to \infty$, the expansion

$$-\log(S_2^k|\Delta_k(S)|^2) = 4\pi S_2 - k\log S_2 + k\sum_{m=1}^{\infty}\left(\sum_{d|m} d + \sum_{Nd|m} Nd\right)\frac{q_S^m+\bar{q}_S^m}{m} \tag{2.5}$$

with $q_S = e^{2\pi \mathrm{i} S}$ reveals, beyond the expected tree-level contribution and logarithmic mixing with the non-local part of the effective action, an infinite series of exponentially suppressed corrections ascribed to NS5-branes wrapped on T^6 [19]. While not all $\Gamma_0(N)\times O(r-6,6,\mathbb{Z})$ transformations are expected to be U-dualities of the full theory but only of the BPS sector, for brevity we shall refer to them respectively as S- and T-dualities.

In [18] it was observed that the coupling (2.3) is in fact invariant under the larger group $\widehat{\Gamma}_0(N)$, obtained by adjoining to $\Gamma_0(N)$ the Fricke involution, which acts on modular forms of weight k under $\Gamma_0(N)$ via $f_k(\tau)\mapsto \hat{f}_k(\tau) = (-\mathrm{i}\tau\sqrt{N})^{-k} f_k(-1/(N\tau))$. Based on a detailed study of geometric dualities in the type II dual description, it was conjectured[2] that the full U-duality group in $D=4$ also includes the so-called Fricke S-duality, which acts on the first factor in (2.1) by the Fricke involution $S\mapsto -1/(NS)$, accompanied by a suitable action of $O(r-6,6,\mathbb{R})$ on the second factor. Additional evidence for the existence of Fricke S-duality comes from the spectrum of BPS states, to which we now turn.

Point-like particles in $D=4$ carry electric and magnetic charges $(Q,P)\in\Lambda_{em}$ under the r Maxwell fields, where

$$\Lambda_{em} = \Lambda_e \oplus \Lambda_m\ , \quad \Lambda_m = \Lambda_{r-6,6} = \Lambda_e^*\ . \tag{2.6}$$

The lattice Λ_m is tabulated in the sixth column of Table 1, taken from [18]. It agrees with the result (2.2) upon making use of the lattice isomorphisms (A.33). In view of the remarks below (2.2), one has, for any $(Q,P)\in\Lambda_{em}$,

$$Q^2 \in \frac{2}{N}\mathbb{Z}\ , \quad P^2\in 2\mathbb{Z}\ , \quad P\cdot Q\in\mathbb{Z}\ . \tag{2.7}$$

The last property in particular ensures that the Dirac-Schwinger-Zwanziger pairing $Q\cdot P' - Q'\cdot P$ is integer. Moreover, it was observed in [18] that the lattice Λ_m is in fact N-modular, *i.e.* it satisfies

$$\Lambda_m^* \simeq \Lambda_m[1/N]\ . \tag{2.8}$$

In other words, there exists an $O(r-6,6,\mathbb{R})$ matrix σ such that $\sqrt{N}\sigma$ maps the lattice Λ_m into itself and such that

$$\Lambda_m^* = \frac{\sigma}{\sqrt{N}}\Lambda_m \quad (\supset \Lambda_m)\ . \tag{2.9}$$

A simple example of N-modular lattice is $\Lambda_{d,d}[N]\oplus\Lambda_{d,d}$, which is relevant for $N=5$ above. In this case one can parametrize an element in the lattice in $(\mathbb{Z}^d, N\mathbb{Z}^d,\mathbb{Z}^d,\mathbb{Z}^d)$ and an element of the dual lattice in $(\mathbb{Z}^d/N,\mathbb{Z}^d,\mathbb{Z}^d,\mathbb{Z}^d)$ and define $\sigma\in O(2d,2d,\mathbb{R})$ such that

$$\frac{\sigma}{\sqrt{N}} = \frac{1}{\sqrt{N}}\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{N}}\mathbb{1}_{d,d} & 0\\ 0 & 0 & 0 & \sqrt{N}\mathbb{1}_{d,d}\\ \sqrt{N}\mathbb{1}_{d,d} & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{N}}\mathbb{1}_{d,d} & 0 & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{N}\mathbb{1}_{d,d} & 0\\ 0 & 0 & 0 & \mathbb{1}_{d,d}\\ \mathbb{1}_{d,d} & 0 & 0 & 0\\ 0 & \frac{1}{N}\mathbb{1}_{d,d} & 0 & 0\end{pmatrix}\ . \tag{2.10}$$

[2] More generally, Fricke S-duality is conjectured to hold whenever the cycle shape satisfies the balancing condition $m(a)=m(N/a)$ for all $a|N$. [18]

The map (2.9) defines the action $(Q,P) \mapsto (-\sigma \cdot P/\sqrt{N}, \sigma^{-1} \cdot Q\sqrt{N})$ of the Fricke S-duality on Λ_{em}, which maps $(Q^2, P^2, P \cdot Q) \mapsto (P^2/N, NQ^2, -P \cdot Q)$ and therefore preserves the quantization conditions (2.7). It also allows to identify $N\Lambda_m^*$ as a sublattice of Λ_m

$$N\Lambda_m^* = \sqrt{N}\sigma\Lambda_m \subset \Lambda_m \ . \tag{2.11}$$

Electric charge vectors $Q \in \Lambda_m \subset \Lambda_e$ are called untwisted, while vectors $Q \in \Lambda_e \smallsetminus \Lambda_m$ are called twisted. More generally, we shall call dyonic charge vectors (Q,P) lying in $\Lambda_m \oplus N\Lambda_e \subset \Lambda_e \oplus \Lambda_m$ untwisted, and twisted otherwise.[3] Untwisted dyons are in particular such that

$$Q^2 \in 2\mathbb{Z} \ , \quad P^2 \in 2N\mathbb{Z} \ , \quad P \cdot Q \in N\mathbb{Z} \ . \tag{2.12}$$

Half-BPS states exist only when Q, P are collinear. Their mass is then determined in terms of the charges via

$$\mathcal{M}^2(Q,P) = \frac{2}{S_2}(Q_R - SP_R) \cdot (Q_R - \bar{S}P_R) \tag{2.13}$$

where, for a vector $Q^I \in \mathbb{R}^{p,q}$ $(I = 1 \dots p+q)$, we denote by Q_L^a $(a = 1 \dots p)$ and $Q_R^{\hat{a}}$ $(\hat{a} = 1 \dots q)$ its projections on the positive p-plane and its orthogonal complement parametrized by the orthogonal Grassmannian $G_{p,q}$, such that $Q^2 = Q_L^2 - Q_R^2$.

For primitive purely electric states (such that $Q \in \Lambda_e$ but $Q/d \notin \Lambda_e$ for all $d > 1$), corresponding to left-moving excitations in the twisted sectors of the perturbative heterotic string, it is known that the helicity supertrace $\Omega_4(Q,0)$ is given by [22, 17, 24, 25, 23]

$$\Omega_4(Q,0) = c_k\left(-\tfrac{NQ^2}{2}\right) \ , \quad \frac{1}{\Delta_k(\tau)} = \sum_{\substack{m\in\mathbb{Z}\\ m\geq -1}} c_k(m)\, q^m = \frac{1}{q} + k + \dots , \tag{2.14}$$

where $q = 2^{2\pi i \tau}$ and $\Delta_k(\tau)$ is the same cusp form (2.4) which enters in the exact $\mathcal{R}^2$ coupling. In Appendix A, we rederive this result by constructing the one-loop vacuum amplitude for the CHL models under consideration, and show that primitive purely electric states corresponding to left-moving excitations in the untwisted sector have an additional contribution (first observed for $N = 2$ in [26])

$$\Omega_4(Q,0) = c_k\left(-\tfrac{Q^2}{2}\right) + c_k\left(-\tfrac{NQ^2}{2}\right) \ . \tag{2.15}$$

Invariance under both $\Gamma_0(N)$ and Fricke S-duality implies that the same formulae apply to generic primitive dyons with Q^2 being replaced by $\frac{1}{N}\gcd(NQ^2, P^2, Q \cdot P)$. It follows that the helicity supertrace for general 1/2 BPS primitive dyons is given by

$$\Omega_4(Q,P) = c_k\left(-\tfrac{\gcd(NQ^2,P^2,Q\cdot P)}{2}\right) \ . \tag{2.16}$$

for twisted electromagnetic charge $(Q,P) \in (\Lambda_e \oplus \Lambda_m) \smallsetminus (\Lambda_m \oplus N\Lambda_e)$, and by

$$\Omega_4(Q,P) = c_k\left(-\tfrac{\gcd(NQ^2,P^2,Q\cdot P)}{2}\right) + c_k\left(-\tfrac{\gcd(NQ^2,P^2,Q\cdot P)}{2N}\right) \ . \tag{2.17}$$

for untwisted charge $(Q,P) \in \Lambda_m \oplus N\Lambda_e$. In contrast, primitive 1/2-BPS states of the maximal rank theory have a single contribution

$$\Omega_4(Q,P) = c\left(-\tfrac{\gcd(Q^2,P^2,Q\cdot P)}{2}\right) \ , \quad \frac{1}{\Delta(\tau)} = \sum_{\substack{m\in\mathbb{Z}\\ m\geq -1}} c(m)\, q^m = \frac{1}{q} + 24 + \dots \tag{2.18}$$

[3] Note that this terminology is defined to be consistent with Fricke and $\Gamma_0(N)$ S-duality, but twisted magnetic charges do not correspond to any twisted sector in the conventional sense.

2.2 Moduli space and 1/2-BPS couplings in $D = 3$

Upon further compactification on a circle, additional moduli arise from the radius R of the circle, from the holonomies a^{1I} of the r gauge fields, and from the scalars a^{2I}, ψ dual to the r Maxwell fields and to the Kaluza–Klein gauge field in three dimensions, extending (2.1) to [27]

$$\mathcal{M}_3 = G_{r-4,8}/G_3(\mathbb{Z}) \ . \tag{2.19}$$

The U-duality group $G_3(\mathbb{Z})$ includes $G_4(\mathbb{Z})$, the Heisenberg group of large gauge transformations acting on $a^{I,i}, \psi$, and the automorphism group $O(r-5,7,\mathbb{Z})$ (or rather a subgroup containing $\widetilde{O}(r-5,7,\mathbb{Z})$) of the Narain lattice $\Lambda_{r-5,7} = \Lambda_{r-6,6} \oplus I\!I_{1,1}$ corresponding to T-duality in heterotic string compactified on T^7. The action of these subgroups is most easily seen in the vicinity of the cusps $R \to \infty$ and $g_3 \to 0$, corresponding to the decompactification limit to $D = 4$ and the weak heterotic coupling limit in $D = 3$, where (2.19) reduces to

$$\mathcal{M}_3 \to \begin{cases} \mathbb{R}^+_R \times \mathcal{M}_4 \times \tilde{T}^{2r+1} \\ \mathbb{R}^+_{1/g_3^2} \times \left[\frac{O(r-5,7)}{O(r-5)\times O(7)} / O(r-5,7,\mathbb{Z}) \right] \times T^{r+2} \end{cases} \tag{2.20}$$

Here, $\tilde{T}^{2r+1}$ is a circle bundle over the torus T^{2r} parametrized by the holonomies $a^{i,I}$, with fiber parametrized by the NUT potential ψ, while T^{r+2} corresponds to the scalars dual to the Maxwell gauge fields after compactifying the heterotic string on T^7. In heterotic perturbation theory, the effective action in $D = 3$ is singular on codimension-7 loci where $Q_R^2 = 0$ for a norm 2 vector $Q \in \Lambda_{r-5,7}$, or for a norm $2/N$ vector $Q \in \Lambda^*_{r-5,7}$.

For $r = 28$, it is well-known that these subgroups generate the automorphism group $O(24,8,\mathbb{Z})$ of the 'non-perturbative Narain lattice' $\Lambda_{24,8} = \Lambda_{22,6} \oplus I\!I_{2,2}$ [28]. To the extent of our knowledge, the U-duality group for CHL models has not been discussed in the literature, but it is natural to expect that it includes the restricted automorphism group $\widetilde{O}(r-4,8,\mathbb{Z})$ of an extended Narain lattice of the form $\Lambda_{r-4,8} = \Lambda_m \oplus \Lambda_{2,2}$. We find that the following choice reproduces the correct S and T-dualities in $D = 4$:

$$\Lambda_{r-4,8} = \Lambda_m \oplus I\!I_{1,1} \oplus I\!I_{1,1}[N] \ , \tag{2.21}$$

where $I\!I_{1,1}[N]$ is the standard hyperbolic lattice with quadratic form rescaled by a factor of N, such that $\Lambda^*_{r-4,8}/\Lambda_{r-4,8} \simeq \mathbb{Z}_N^{k+4}$. In terms of the usual construction of $I\!I_{2,2}$ by windings $(n_1, n_2) \in \mathbb{Z}^2$, momenta $(m_1, m_2) \in \mathbb{Z}^2$ and quadratic form $2m_1n_1 + 2m_2n_2$, we define $I\!I_{1,1} \oplus I\!I_{1,1}[N]$ as the sublattice of $I\!I_{2,2}$ where n_2 is restricted to be a multiple of N. The restricted automorphism group of $I\!I_{1,1} \oplus I\!I_{1,1}[N]$ was determined in [18, 29], and includes $\sigma_{T\leftrightarrow S} \ltimes [\Gamma_1(N) \times \Gamma_1(N)]$, acting by fractional linear transformations on the moduli (T, S) parametrizing $G_{2,2}$, such that $|m_1 + Sm_2 + Tn_1 + STn_2|^2/(S_2T_2)$ is invariant (see [20, §C], case V for $N = 2$, or [30, §3.1.3] for arbitrary N). In the present context, T is interpreted as $\psi + \mathrm{i}R^2$, while S is the heterotic axiodilaton. Thus, $\widetilde{O}(r-4,8,\mathbb{Z})$ contains the S-duality group $\Gamma_1(N)$ and T-duality group $\widetilde{O}(r-6,6,\mathbb{Z})$ in four dimensions. In addition, Fricke S-duality in four dimensions follows from the fact that the non-perturbative lattice (2.21) is itself N-modular,

$$\Lambda^*_{r-4,8} \simeq \Lambda_{r-4,8}[1/N] \ . \tag{2.22}$$

More evidence for the claim (2.21) will come from the analysis of BPS couplings in $D = 3$, to which we now turn.

In this work, we focus on the coupling of the form $F(\Phi)(\nabla\Phi)^4$ in the low energy effective action in $D=3$, where $F(\Phi)$ is a symmetric rank four tensor $F_{abcd}(\Phi)$, and $(\nabla\Phi)^4$ is a short-hand notation for a particular contraction of the pull-back of the right-invariant one-forms $P_{a\hat{a}}$ on $G_{r-4,8}$ to $\mathbb{R}^{2,1}$ (see (3.15)). As stated in [7], and further explained below, supersymmetry requires that the coefficient $F_{abcd}(\Phi)$ satisfies the tensorial differential equations

$$\mathcal{D}_{(e}{}^{\hat{g}}\mathcal{D}_{f)\hat{g}}\,F_{abcd} \;=\; \tfrac{2-q}{4}\,\delta_{ef}\,F_{abcd}+(4-q)\,\delta_{e)(a}\,F_{bcd)(f}+3\,\delta_{(ab}\,F_{cd)ef}+\tfrac{15k}{(4\pi)^2}\delta_{(ab}\delta_{cd}\delta_{ef)}\delta_{q,6}\ , \tag{2.23a}$$

$$\mathcal{D}_{[e}{}^{[\hat{e}}\mathcal{D}_{f]}{}^{\hat{f}]}F_{abcd} \;=\; 0\ , \qquad \mathcal{D}_{[e}{}^{\hat{a}}F_{a]bcd}=0\ , \tag{2.23b}$$

where the constant term in the first line occurs from the regularisation in $q=6$ (see 3.57), and where $\mathcal{D}_{a\hat{b}}$ are the covariant derivatives in tangent frame on $G_{p,q}$. In fact, we shall show that all components of the tensor F_{abcd} can be recovered from its trace $F_{\rm tr}(\Phi)\equiv F_{ab}^{ab}(\Phi)$ by acting with the differential operators $\mathcal{D}_{a\hat{b}}$ (see (3.26)). Supersymmetry requires that $F_{\rm tr}(\Phi)$ be an eigenmode of the Laplacian on $G_{r-4,8}$ with a specified eigenvalue, while U-duality requires that it should be invariant under $\widetilde{O}(r-4,8,\mathbb{Z})$. (Note however that the second order differential equations satisfied by $F_{\rm tr}(\Phi)$ does not imply (2.23), so it should not be thought of as a prepotential for F_{abcd}.)

In CHL $\mathbb{Z}_N$ orbifold of heterotic string on T^7, F_{abcd} gets tree-level and one-loop contributions, both of which are solutions of (2.23), invariant under the full T-duality group $O(r-5,7,\mathbb{Z})$. As we show in Appendix A, the one-loop contribution is given by a modular integral[4]

$$F^{(\text{1-loop})}_{abcd}=\text{R.N.}\int_{\Gamma_0(N)\backslash\mathcal{H}}\frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\frac{\Gamma_{\Lambda_{r-5,7}}[P_{abcd}]}{\Delta_k(\tau)}\ , \tag{2.24}$$

where $\Delta_k(\tau)$ is the same cusp form (2.4) which appeared in the $\mathcal{R}^2$ couplings in $D=4$, and $\Gamma_{\Lambda_{p,q}}[P_{abcd}]$ denotes the Siegel–Narain theta series for the lattice $\Lambda_{p,q}$,

$$\Gamma_{\Lambda_{p,q}}[P_{abcd}]=\tau_2^{q/2}\sum_{Q\in\Lambda_{p,q}}P_{abcd}(Q)\,e^{\mathrm{i}\pi Q_L^2\tau-\mathrm{i}\pi Q_R^2\bar{\tau}}\ , \tag{2.25}$$

with an insertion of the polynomial

$$P_{abcd}(Q)=Q_{L,a}Q_{L,b}Q_{L,c}Q_{L,d}-\frac{3}{2\pi\tau_2}\delta_{(ab}Q_{L,c}Q_{L,d)}+\frac{3}{16\pi^2\tau_2^2}\delta_{(ab}\delta_{cd)}, \tag{2.26}$$

$\Gamma_0(N)\backslash\mathcal{H}$ is any fundamental domain for the action of $\Gamma_0(N)$ on the Poincaré upper half-plane $\mathcal{H}$, and R.N. denotes a suitable regularization prescription (see (3.30)). In view of the form of the one-loop contribution, it is therefore natural to conjecture [9, 7] that the exact $(\nabla\Phi)^4$ coupling is the obvious generalization of (2.24), where the Narain lattice $\Gamma_{\Lambda_{r-5,7}}$ is replaced by its non-perturbative extension (2.21),

$$F_{abcd}(\Phi)=\text{R.N.}\int_{\Gamma_0(N)\backslash\mathcal{H}}\frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\frac{\Gamma_{\Lambda_{r-4,8}}[P_{abcd}]}{\Delta_k(\tau)}\ . \tag{2.27}$$

A similar formula holds for the trace part $F_{\rm tr}(\Phi)\equiv\delta^{ab}\delta^{cd}F_{abcd}(\Phi)$,

$$F_{\rm tr}(\Phi)=\text{R.N.}\int_{\Gamma_0(N)\backslash\mathcal{H}}\frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\,\Gamma_{\Lambda_{r-4,8}}\cdot D_{-k+2}D_{-k}\frac{1}{\Delta_k(\tau)}\ , \tag{2.28}$$

[4] A similar computation for four-graviton couplings in CHL models was performed in [31].

where $D_w = \frac{\mathrm{i}}{\pi}(\partial_\tau - \frac{\mathrm{i}w}{2\tau_2})$ is the Maass raising operator, mapping modular forms of weight w to weight $w+2$. The proposals (2.27) and (2.28) are manifestly invariant (or covariant) under the full automorphism group $O(r-4,8,\mathbb{Z})$ of the non-perturbative lattice (2.21), which contains the true U-duality group in $D=3$. Moreover, since the latter is N-modular, $\Gamma_{\Lambda_{r-4,8}}$ is invariant under the combined action of the Fricke involution on $\mathcal{H}$ and the rotation $\sigma \in O(r-4,8,\mathbb{R})$ realizing the isomorphism (2.22),

$$\Gamma_{\Lambda_{r-4,8}}(\Phi,\tau)[P_{abcd}] = \left(-\mathrm{i}\tau\sqrt{N}\right)^{-k} \Gamma_{\Lambda_{r-4,8}}[P_{abcd}]\left(\sigma\cdot\Phi, -\frac{1}{N\tau}\right) . \tag{2.29}$$

Since Δ_k is also an eigenmode of the Fricke involution on $\mathcal{H}$, and since the fundamental domain $\Gamma_0(N)\backslash\mathcal{H}$ can be chosen to be invariant under this involution, it follows that $F_{abcd}(\Phi)$ (and therefore $F_{\mathrm{tr}}(\Phi)$) is covariant (invariant) under the action of σ on $G_{r-4,8}$. As already anticipated, this action descends to Fricke S-duality in $D=4$.

It is also important to note that the couplings (2.27) and (2.28) are singular on codimension-8 loci where $Q_R^2 = 0$ for some norm 2 vector $Q \in \Lambda_{r-4,8}$, or norm $2/N$ vector $Q \in \Lambda^*_{r-4,8}$. When the vector Q is of the form $Q = (0,\widetilde{Q},0) \in \Lambda_{r-4,8}$ with $\widetilde{Q} \in \Lambda_{r-5,7}$, this singularity is visible at the level of the one-loop correction to the $(\nabla\Phi)^4$ coupling, and is due to additional states becoming massless. However, the one-loop correction is singular in real codimension 7, while the full non-perturbative coupling (assuming that (2.27) is correct) is singular in real codimension 8. Indeed, the invariant norm $Q_R^2 = \widetilde{Q}_R^2 + \frac{1}{2}g_3^2(\widetilde{Q}\cdot a)^2$ vanishes only when both $\widetilde{Q}_R^2 = 0$ and $\widetilde{Q}\cdot a = 0$. This partial resolution may be seen as an analogue of the resolution of the conifold singularity on the vector multiplet branch in type II strings compactified on a CY threefold times a circle, or equivalently on the hypermultiplet branch in the mirror description [32]. Singularities associated to generic vectors $Q \in \Lambda_{r-4,8}$ are not visible at any order in perturbation theory, and are associated to 'exotic' particles in $D=3$ becoming massless [33, 34].

3 Establishing and solving supersymmetric Ward identities

In this section, we establish the supersymmetric Ward identities (2.23), from linearized superspace considerations, relate the components of the tensor F_{abcd} to its trace $F_{\mathrm{tr}} \equiv F_{ab}^{\ \ ab}$, and show that the genus-one modular integral (2.27) obeys this identity. For completeness, we solve the first equation of (2.23) in appendix B, and show that it is satisfied by each Fourier mode of F_{abcd}.

3.1 $(\nabla\Phi)^4$ type invariants in three dimensions

In three dimensional supergravity with half-maximal supersymmetry, the linearised superfield $W_{\hat{a}a}$ satisfies the constraints [27, 35, 36]

$$D^i_\alpha W_{\hat{a}a} = (\Gamma_{\hat{a}})^{i\hat{\jmath}}\chi_{\alpha\hat{\jmath}a} , \qquad D^i_\alpha \chi_{\beta\hat{\jmath}a} = -\mathrm{i}(\sigma^\mu)_{\alpha\beta}(\Gamma^{\hat{a}})_{\hat{\jmath}}{}^{i}\partial_\mu W_{\hat{a}a} , \tag{3.1}$$

with $\hat{a} = 1$ to 8 for the vector of $O(8)$, $i = 1$ to 8 for the positive chirality Weyl spinor of $Spin(8)$ and $\hat{\imath} = 1$ to 8 for the negative chirality Weyl spinor. The 1/2 BPS linearised invariants are defined using harmonics of $Spin(8)/U(4)$ parametrizing a $Spin(8)$ group element

$u^r{}_i, u_{ri}$ in the Weyl spinor representation of positive chirality [37],

$$2u_{r(i}u^r{}_{j)} = \delta_{ij}\ , \qquad \delta^{ij}u_{ri}u^s{}_j = \delta^s_r\ , \qquad \delta^{ij}u_{ri}u_{sj} = 0\ , \qquad \delta^{ij}u^r{}_iu^s{}_j = 0\ , \tag{3.2}$$

$u_{r\hat\imath}, u^r{}_{\hat\imath}$ in the Weyl spinor representation of negative chirality,

$$2u_{r(\hat\imath}u^r{}_{\hat\jmath)} = \delta_{\hat\imath\hat\jmath}\ , \qquad \delta^{\hat\imath\hat\jmath}u_{r\hat\imath}u^s{}_{\hat\jmath} = \delta^s_r\ , \qquad \delta^{\hat\imath\hat\jmath}u_{r\hat\imath}u_{s\hat\jmath} = 0\ , \qquad \delta^{\hat\imath\hat\jmath}u^r{}_{\hat\imath}u^s{}_{\hat\jmath} = 0\ , \tag{3.3}$$

and $u^+{}_{\hat a}, u^{rs}{}_{\hat a}, u^-{}_{\hat a}$ in the vector representation,

$$\begin{aligned}&2u^+{}_{(\hat a}u^-{}_{\hat b)} + \frac{1}{2}\varepsilon_{rstu}u^{rs}{}_{\hat a}u^{tu}{}_{\hat b} = \delta_{\hat a\hat b}\ , \quad \delta^{\hat a\hat b}u^+{}_{\hat a}u^-{}_{\hat b} = 1\ , \quad \delta^{\hat a\hat b}u^{rs}{}_{\hat a}u^{tu}{}_{\hat b} = \frac{1}{2}\varepsilon^{rstu}\ ,\\ &\delta^{\hat a\hat b}u^+{}_{\hat a}u^+{}_{\hat b} = 0\ , \quad \delta^{\hat a\hat b}u^+{}_{\hat a}u^{rs}{}_{\hat b} = 0\ , \quad \delta^{\hat a\hat b}u^-{}_{\hat a}u^{rs}{}_{\hat b} = 0\ , \quad \delta^{\hat a\hat b}u^-{}_{\hat a}u^-{}_{\hat b} = 0\ ,\end{aligned} \tag{3.4}$$

with $r = 1$ to 4 of $U(4)$. They are related through the relations

$$\begin{aligned}u^+{}_{\hat a}u_{ri}(\Gamma^{\hat a})^{i\hat\jmath} &= \sqrt{2}u_{r\hat\imath}\delta^{\hat\imath\hat\jmath}\ , \quad u_{ri}u_{s\hat\jmath}(\Gamma_{\hat a})^{i\hat\jmath} = \varepsilon_{rstu}u^{tu}{}_{\hat a}\ , \quad u^{rs}{}_{\hat a}u_{ti}(\Gamma^{\hat a})^{i\hat\jmath} = 2\delta_t^{[r}u^{s]}{}_{\hat\imath}\delta^{\hat\imath\hat\jmath}\ ,\\ u_{ri}u^s{}_{\hat\jmath}(\Gamma_{\hat a})^{i\hat\jmath} &= \sqrt{2}u^-{}_{\hat a}\delta^s_r\ , \quad u^r{}_iu_{s\hat\jmath}(\Gamma_{\hat a})^{i\hat\jmath} = \sqrt{2}u^+{}_{\hat a}\delta^r_s\ , \quad u^r{}_iu^s{}_{\hat\jmath}(\Gamma_{\hat a})^{i\hat\jmath} = 2u^{rs}{}_{\hat a}\ .\end{aligned} \tag{3.5}$$

The superfield $W^+_a \equiv u^{+\hat a}W_{\hat a a}$ then satisfies the G-analyticity property

$$u^r{}_iD^i_\alpha u^{+\hat a}W_{\hat a a} \equiv D^r_\alpha W^+_a = 0\ . \tag{3.6}$$

One can obtain a linearised invariant from the action of the eight derivatives $D_{\alpha r} \equiv u_{ri}D^i_\alpha$'s on any homogeneous function of the W^+_a's. After integrating over the harmonic variables with the normalisation $\int du = 1$ and using

$$\int du\, u^-{}_{\hat a_1}\dots u^-{}_{\hat a_n}W^{+a_1}\cdots W^{+a_n} = \frac{6!\,n!}{(6+2n)(5+n)!}W^{a_1}{}_{(a_1}\cdots W^{a_n}{}_{a_n)'}\ , \tag{3.7}$$

with the projection $(\hat a_1\dots\hat a_n)'$ on the traceless symmetric component (recall that $u^-{}_{\hat a}u^{-\hat a} = 0$), one gets [5]

$$\begin{aligned}&\frac{(6+2n)(5+n)!}{6!\,n!}\int du\, u^-{}_{\hat a_1}\dots u^-{}_{\hat a_n}[D^8]\tfrac{1}{(n+4)!}c_{a_1\dots a_{n+4}}W^{+a_1}\dots W^{+a_{n+4}}\\ =\ &\frac{1}{n!}c_{a_1\dots a_nabcd}W^{a_1}{}_{(\hat a_1}W^{a_2}{}_{\hat a_2}\dots W^{a_n}{}_{\hat a_n)'}\mathcal{L}^{(0)abcd}\\ &+\frac{1}{(n-1)!}c_{a_1\dots a_nabcd}W^{a_2}{}_{(\hat a_2}W^{a_3}{}_{\hat a_3}\dots W^{a_n}{}_{\hat a_n}\mathcal{L}^{(0)a_1abcd}_{\hat a_1)'} + \dots\\ &\quad+\frac{1}{(n-4)!}c_{a_1\dots a_nabcd}W^{a_5}{}_{(\hat a_5}W^{a_6}{}_{\hat a_6}\dots W^{a_n}{}_{\hat a_n}\mathcal{L}^{(0)a_1a_2a_3a_4abcd}_{\hat a_1\hat a_2\hat a_3\hat a_4)'} + \partial(\dots)\ ,\end{aligned} \tag{3.9}$$

[5] In particular for a single vector multiplet

$$\begin{aligned}[D^8]\frac{1}{(n+4)!}(W^+)^{n+4} &= \frac{1}{n!}(W^+)^n\big(2\partial_\mu W_{rs}\partial_\nu W^{rs}\partial^\mu W_{tu}\partial^\nu W^{tu} - \partial_\mu W_{rs}\partial^\mu W^{rs}\partial_\nu W_{tu}\partial^\nu W^{tu}\big)\\ &\quad-\frac{8}{(n+1)!}(W^+)^{n+1}\partial_\mu W_{rs}\partial_\nu W^{rs}\partial^\mu\partial^\nu W^- + \frac{8}{(n+2)!}(W^+)^{n+2}\partial_\mu\partial_\nu W^-\partial^\mu\partial^\nu W^- + \dots\ .\end{aligned} \tag{3.8}$$

where the $\mathcal{L}^{n+4}_n$ are symmetric tensors consisting of a homogeneous polynomial of order $4+n$ in $\partial_\mu W^{a\hat{a}}$, $\chi_{\alpha\hat{\imath}a}$ and $\partial_\mu\chi_{\alpha\hat{\imath}a}$, *i.e.*

$$\begin{aligned}
\mathcal{L}^{(0)abcd} &= 2\partial_\mu W^{(a}{}_{\hat{a}}\partial^\mu W^b{}_{\hat{b}}\partial_\nu W^{c|\hat{a}}\partial^\nu W^{d)\hat{b}} - \partial_\mu W^{(a}{}_{\hat{a}}\partial^\mu W^{b|\hat{a}}\partial_\nu W^c{}_{\hat{b}}\partial^\nu W^{d)\hat{b}} + \dots \\
\mathcal{L}^{(0)abcde}_{\hat{a}} &\sim 4\times\chi^2(\partial W)^3 + 5\times\chi^3\partial\chi\partial W \\
\mathcal{L}^{(0)a_1a_2abcd}_{\hat{a}_1\hat{a}_2} &\sim 6\times\chi^4(\partial W)^2 + 2\times\chi^5\partial\chi \\
\mathcal{L}^{(0)a_1a_2a_3abcd}_{\hat{a}_1\hat{a}_2\hat{a}_3} &\sim \chi^6\partial W \\
\mathcal{L}^{(0)a_1a_2a_3a_4abcd}_{\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4} &\sim \chi^8
\end{aligned} \tag{3.10}$$

where we only wrote the bosonic part of the first polynomial, and only indicated the number of independent structures for the others, such that χ^8 is for example the unique Lorentz singlet in the irreducible representation of $O(8)$ with four symmetrised indices without trace and eight symmetrised $O(r-8)$ indices. A total derivative has been extracted in (3.9) in order to remove all second derivative terms $\partial_\mu\partial_\nu W^{a\hat{a}}$.

At the non-linear level, derivatives of the scalar fields only appear through the pull-back of the right-invariant form $P_{a\hat{b}}$ defined from the Maurer–Cartan form

$$\mathrm{d}g\, g^{-1} = \begin{pmatrix} \mathrm{d}p_{La}{}^I\eta_{IJ}p_{Lb}{}^J & -\mathrm{d}p_{La}{}^I\eta_{IJ}p_{R\hat{b}}{}^J \\ \mathrm{d}p_{R\hat{a}}{}^I\eta_{IJ}p_{Lb}{}^J & -\mathrm{d}p_{R\hat{a}}{}^I\eta_{IJ}p_{R\hat{b}}{}^J \end{pmatrix} \equiv \begin{pmatrix} -\omega_{ab} & P_{a\hat{b}} \\ P_{b\hat{a}} & -\omega_{\hat{a}\hat{b}} \end{pmatrix}\,, \tag{3.11}$$

where η_{IJ} is the $O(r-8,8)$ metric and $p_{L,a}{}^I$, $p_{R,\hat{b}}{}^I$ are the left and right projections parametrised by the Grassmaniann $G_{r-8,8}$. The right-invariant metric on $G_{r-8,8}$ is defined as $G_{\mu\nu} = 2P_{\mu a\hat{b}}P^{a\hat{b}}_\nu$ and the covariant derivative in tangent frame acts on a symmetric tensor as

$$\mathcal{D}_{a\hat{b}}A_{a_1\dots a_m,\hat{b}_1\dots\hat{b}_n} \equiv P_{\mu a\hat{b}}G^{\mu\nu}\big(\partial_\nu A_{a_1\dots a_m,\hat{b}_1\dots\hat{b}_n} + m\omega_{\nu(a_1}{}^cA_{a_2\dots a_m)c,\hat{b}_1\dots\hat{b}_n} + n\omega_{\nu(\hat{b}_1}{}^{\hat{c}}A_{a_1\dots a_m,|\hat{b}_2\dots\hat{b}_n)\hat{c}}\big)\,. \tag{3.12}$$

The supersymmetry invariant associated to a tensor F_{abcd} on the Grassmanian defines a Lagrange density $\mathcal{L}$ that decomposes naturally as

$$\begin{aligned}
\mathcal{L} = F_{a_1a_2a_3a_4}\mathcal{L}^{a_1a_2a_3a_4} &+ \mathcal{D}_{(a_1}{}^{\hat{a}}F_{a_2a_3a_4a_5)}\mathcal{L}^{a_1\dots a_5}{}_{\hat{a}} + \mathcal{D}_{(a_1}{}^{\hat{a}_1}\mathcal{D}_{a_2}{}^{\hat{a}_2}F_{a_3a_4a_5a_6)}\mathcal{L}^{a_1\dots a_6}{}_{\hat{a}_1\hat{a}_2} \\
&+ \mathcal{D}_{(a_1}{}^{\hat{a}_1}\mathcal{D}_{a_2}{}^{\hat{a}_2}\mathcal{D}_{a_3}{}^{\hat{a}_3}F_{a_4a_5a_6a_7)}\mathcal{L}^{a_1\dots a_7}{}_{\hat{a}_1\hat{a}_2\hat{a}_3} \\
&+ \mathcal{D}_{(a_1}{}^{\hat{a}_1}\mathcal{D}_{a_2}{}^{\hat{a}_2}\mathcal{D}_{a_3}{}^{\hat{a}_3}\mathcal{D}_{a_4}{}^{\hat{a}_4}F_{a_5\dots a_8)}\mathcal{L}^{a_1\dots a_8}{}_{\hat{a}_1\dots\hat{a}_4}\,,
\end{aligned} \tag{3.13}$$

where the $\mathcal{L}^{n+4}{}_n$ are $O(r-8,8)$ invariant polynomial functions of the following covariant fields:

$$P_{\mu\, a\hat{b}} = \partial_\mu\phi^\mu P_{\mu\, a\hat{b}}\,,\quad \chi_{\alpha\hat{\imath}a}\,,\quad \mathcal{D}_\mu\chi_{\alpha\hat{\imath}a} = \nabla_\mu\chi_{\alpha\hat{\imath}a} + \partial_\mu\phi^\mu\Big(\omega_{\mu\, a}{}^b\chi_{\alpha\hat{\imath}a} + \frac{1}{4}\omega_{\mu\,\hat{a}\hat{b}}(\Gamma^{\hat{a}\hat{b}})_{\hat{\imath}}{}^{\hat{\jmath}}\chi_{\alpha\hat{\jmath}a}\Big)\,, \tag{3.14}$$

and the dreibeins and the gravitini fields. Because non-linear invariants define a linear invariant by truncation to lowest order in the fields (3.14), the covariant densities $\mathcal{L}^{4+n}_n$ reduce at lowest order to homogeneous polynomials of degree $n+4$ in the covariant fields (3.14) that coincide with the linearised polynomials $\mathcal{L}^{(0)n+4}_n$, in particular

$$\mathcal{L}^{abcd} = \sqrt{-g}\big(2P^{(a}_\mu{}_{\hat{a}}P^{\mu\, b}{}_{\hat{b}}P^{c|\hat{a}}_\nu P^{\nu\, d)\hat{b}} - P^{(a}_\mu{}_{\hat{a}}P^{\mu\, b|\hat{a}}P^c_{\nu\hat{b}}P^{\nu\, d)\hat{b}} + \dots\big)\,. \tag{3.15}$$

The important conclusion to draw from the linearised analysis is that the $O(r-8,8)$ right-invariants tensors $\mathcal{L}^{n+4}_n$ appearing in the ansatz (3.13) are symmetric in both sets of indices and traceless in the $O(8)$ indices. Checking the supersymmetry invariance (modulo a total derivative) of $\mathcal{L}$ in this basis, one finds that there is no term to cancel the supersymmetry variation

$$\delta F_{abcd} = \big(\bar\epsilon_i(\Gamma^{\hat f})^{i\hat\jmath}\chi^e_{\hat\jmath}\big)\mathcal{D}^{e\hat f}F_{abcd} \tag{3.16}$$

of the tensor F_{abcd} and of its derivative when open $O(r-8)$ indices are antisymmetrized, hence the tensor F_{abcd} must satisfy the constraints

$$\mathcal{D}_{[a}{}^{[\hat a}\mathcal{D}_{b]}{}^{\hat b]}F_{cdef} = 0\ , \qquad \mathcal{D}_{[e}{}^{\hat a}F_{a]bcd} = 0\ . \tag{3.17}$$

Similarly, because the $\mathcal{L}^{n+4}_n$ are traceless in the $O(8)$ indices, the $O(8)$ singlet component of $\delta(\mathcal{D}F)\mathcal{L}^5_1$ can only be cancelled by terms coming from $F\delta\mathcal{L}^4$, *i.e.*

$$F_{abcd}\delta\mathcal{L}^{abcd} + \frac{1}{8}\mathcal{D}_e{}^{\hat a}\mathcal{D}_{f\hat a}F_{abcd}(\bar\epsilon\,\Gamma^{\hat c}\chi^e)\mathcal{L}^{abcdf}_{\hat c} \sim 0 \tag{3.18}$$

modulo terms arising from the supercovariantisation,[6] so that the covariant components must satisfy

$$\delta\mathcal{L}^{abcd} + \frac{5b_1}{8}(\bar\epsilon\,\Gamma^{\hat c}\chi_e)\mathcal{L}^{abcde}_{\hat c} + \frac{5b_2}{8}(\bar\epsilon\,\Gamma^{\hat c}\chi^{(a})\mathcal{L}^{bcd)e}_{\hat c}{}_e = \nabla_\mu(\dots) \tag{3.19}$$

and the tensor F_{abcd} an equation of the form

$$\mathcal{D}_e{}^{\hat a}\mathcal{D}_{f\hat a}F_{abcd} = 5b_1\delta_{e(f}F_{abcd)} + 5b_2\,\delta_{(fa}F_{bcd)e}\ , \tag{3.20}$$

for some numerical constants b_1, b_2 which are fixed by consistency. In particular the integrability condition on the component antisymmetric in e and f implies $b_2 = 2b_1 + 4$.

Before determining the constants b_i, it is convenient to generalize F_{abcd} to a completely symmetric tensor $F^{(p,q)}_{abcd}$ on a general Grassmanian $G_{p,q}$, which would arise by considering a superfield in $D = 10-q$ dimensions with $3 \le q \le 6$, with harmonics parametrizing similarly the Grassmannian $G_{q-2,2}$ [40]. The corresponding invariant takes the form $\mathcal{L} = F^{(p,q)}_{abcd}\,\mathcal{L}^{abcd} + \dots$ with

$$\begin{aligned}\mathcal{L}^{abcd} = \sqrt{-g}\Big(&F^{(a}_{\mu\nu}F^{b|\nu\sigma}F^{b}_{\sigma\rho}F^{d)\rho\mu} - \frac{1}{4}F^{(a}_{\mu\nu}F^{b|\mu\nu}F^{b}_{\sigma\rho}F^{d)\sigma\rho}\\ &+ (4F^{(a}_{\mu\sigma}F^{b|\sigma}_{\nu} - \eta_{\mu\nu}F^{(a}_{\sigma\rho}F^{b|\sigma\rho})P^{\mu|c}{}_{\hat a}P^{\nu|d)\hat a}\\ &+ 2P^{(a}_{\mu}{}_{\hat a}P^{\mu\,b}{}_{\hat b}P^{c|\hat a}_{\nu}P^{\nu\,d)\hat b} - P^{(a}_{\mu}{}_{\hat a}P^{\mu\,b|\hat a}P^{c}_{\nu\hat b}P^{\nu\,d)\hat b} + \dots\Big)\end{aligned} \tag{3.21}$$

where F_{abcd} is subject to the constraints (3.17) and

$$\mathcal{D}_e{}^{\hat a}\mathcal{D}_{f\hat a}F^{(p,q)}_{abcd} = b_1\,\delta_{ef}F^{(p,q)}_{abcd} + 2b_2\delta_{f(a}F^{(p,q)}_{bcd)e} + (2b_2-q)\delta_{e(a}F^{(p,q)}_{bcd)f} + 3b_3\,\delta_{(ab}F^{(p,q)}_{cd)ef}\ . \tag{3.22}$$

with coefficients b_1, b_2, b_3 a priori depending on p, q.

A first integrability condition for (3.22) is obtained through

$$\begin{aligned}0 = \mathcal{D}_e{}^{\hat a}(\mathcal{D}_{f\hat a}F^{(p,q)}_{abcd} - \mathcal{D}_{(a|\hat a}F^{(p,q)}_{bcd)f}) \;&=\; \Big(b_1 - \frac{2b_2-q}{4}\Big)(\delta_{ef}F^{(p,q)}_{abcd} - \delta_{e(a}F^{(p,q)}_{bcd)f})\\ &+\frac{3}{2}(b_2-b_3)(\delta_{f(a}F^{(p,q)}_{bcd)e} - \delta_{(ab}F^{(p,q)}_{cd)ef})\ ,\end{aligned} \tag{3.23}$$

[6] The same construction in superspace implies that the lift of $\mathcal{L}$ in superspace is d-closed [38], such that $\mathrm{d}_\omega\mathcal{L}^{abcd} = \frac{15}{16}P^{\hat c}{}_e\wedge\mathcal{L}^{abcde}_{\hat c} - \frac{5}{8}P^{\hat c(a}\wedge\mathcal{L}^{bcd)e}_{\hat c}{}_e$, in agreement with equation (3.19). Therefore, the terms associated to the variation of the gravitini that we disregard here do not spoil the argument [39].

which implies $b_1 = \frac{2b_2-q}{4}$ and $b_3 = b_2$, consistently with (3.20). Similarly, considering

$$\begin{aligned} & \mathcal{D}_g{}^{\hat a}\big(\mathcal{D}_e{}^{\hat b}\mathcal{D}_{f\hat b}F^{(p,q)}_{abcd}\big) - \mathcal{D}_f{}^{\hat a}\big(\mathcal{D}_e{}^{\hat b}\mathcal{D}_{g\hat b}F^{(p,q)}_{abcd}\big) = 2b_1\delta_{e[f}\mathcal{D}_{g]}{}^{\hat a}F^{(p,q)}_{abcd} + 2b_2\delta_{a)[f}\mathcal{D}_{g]}{}^{\hat a}F^{(p,q)}_{e(bcd} \\ = \; & [\mathcal{D}_g{}^{\hat a},\mathcal{D}_e{}^{\hat b}]\mathcal{D}_{f\hat b}F^{(p,q)}_{abcd} - [\mathcal{D}_f{}^{\hat a},\mathcal{D}_e{}^{\hat b}]\mathcal{D}_{g\hat b}F^{(p,q)}_{abcd} + \mathcal{D}_e{}^{\hat b}[\mathcal{D}_{[g}{}^{\hat a},\mathcal{D}_{f]\hat b}]F^{(p,q)}_{abcd} \\ = \; & \frac{2-q}{2}\delta_{e[f}\mathcal{D}_{g]}{}^{\hat a}F^{(p,q)}_{abcd} + 2\delta_{a)[f}\mathcal{D}_{g]}{}^{\hat a}F^{(p,q)}_{e(bcd}\,, \end{aligned} \tag{3.24}$$

and therefore $b_1 = \frac{2-q}{4}$ and $b_2 = 1$ and so $b_3 = 1$ so that

$$\mathcal{D}_e{}^{\hat a}\mathcal{D}_{f\hat a}F^{(p,q)}_{abcd} = 5\frac{2-q}{4}\delta_{e(f}F^{(p,q)}_{abcd)} + 5\delta_{(fa}F^{(p,q)}_{bcd)e}\,. \tag{3.25}$$

Taking traces of this equation one can show that the entire tensor is determined by its trace component $F^{(p,q)}_{\rm tr} \equiv F^{(p,q)ab}_{ab}$ through

$$F^{(p,q)}_{abcd} = \frac{1}{(8+p-q)(6+p-q)}\Big(2\mathcal{D}_{(a}{}^{\hat e}\mathcal{D}_{b|\hat e}\mathcal{D}_c{}^{\hat f}\mathcal{D}_{d)\hat f} + (2q-7)\delta_{(ab}\mathcal{D}_c{}^{\hat e}\mathcal{D}_{d)\hat e} + \frac{3(q-2)(q-4)}{8}\delta_{(ab}\delta_{cd)}\Big)F^{(p,q)}_{\rm tr}\,. \tag{3.26}$$

The function $F^{(p,q)}_{\rm tr}$ is an eigenmode of the Laplacian $\Delta_{G_{p,q}} \equiv 2\mathcal{D}_{a\hat b}\mathcal{D}^{a\hat b}$ on $G_{p,q}$, and satisfies

$$\Delta_{G_{p,q}}F^{(p,q)}_{\rm tr} = -\frac{1}{2}(p+4)(q-6)F^{(p,q)}_{\rm tr}\,, \qquad \mathcal{D}_{[a}{}^{[\hat a}\mathcal{D}_{b]}{}^{\hat b]}F^{(p,q)}_{\rm tr} = 0\,. \tag{3.27}$$

It is worth noting, however, that Eq. (3.25) for the tensor defined by (3.26) is an additional constraint on the function $F_{\rm tr}$, which does not follow by integrability from the two equations (3.27).

Finally, let us note that the discussion so far only applies to the local Wilsonian effective action. As we shall see in the next subsection, the Ward identity satisfied by the renormalized coupling $\hat F_{abcd}$ is corrected in four dimensions (for $q = 6$) because of the 1-loop divergence of the supergravity amplitude [41], leading to the source term in (2.23).

3.2 The modular integral solves the Ward identities

In this subsection we shall prove that the modular integral (2.27) is a solution of the supersymmetric Ward identities (2.23). More generally, we shall show that the modular integral

$$F^{(p,q)}_{abcd}(\Phi) = \text{R.N.}\int_{\Gamma_0(N)\backslash\mathcal{H}}\frac{d\tau_1 d\tau_2}{\tau_2^2}\frac{\Gamma_{\Lambda_{p,q}}[P_{abcd}]}{\Delta_k(\tau)}\,, \tag{3.28}$$

where $\Delta_k(\tau)$ is the cusp form (2.4) of weight k under $\Gamma_0(N)$, $\Lambda_{p,q}$ is a level N even lattice of signature (p,q) with $\frac{p-q}{2}+4 = k$, and P is the quartic polynomial (2.26), satisfies the constraints (3.17) and (3.22). Moreover, its trace $\delta^{ab}\delta^{cd}F^{(p,q)}_{abcd}(\Phi)$ is given by

$$F^{(p,q)}_{\rm tr}(\Phi) = \text{R.N.}\int_{\Gamma_0(N)\backslash\mathcal{H}}\frac{d\tau_1 d\tau_2}{\tau_2^2}\Gamma_{\Lambda_{p,q}}\cdot D_{-k+2}D_{-k}\frac{1}{\Delta_k(\tau)}\,. \tag{3.29}$$

Before going into the proof however, it will be useful to spell out the regularization prescription which we use to define these otherwise divergent modular integrals. We follow the procedure developed in [42, 43, 44], whereby the integral is first carried out on the truncated fundamental domain $\mathcal{F}_{N,\Lambda} = \mathcal{F}_N \cap \{\tau_2 < \Lambda\} \cap \{\frac{\tau_2}{N|\tau|^2} > \Lambda\}$, where $\mathcal{F}_N$ is the standard fundamental domain for $\Gamma_0(N)\backslash\mathcal{H}$, invariant under the Fricke involution $\tau \mapsto -1/(N\tau)$, and then the limit $\Lambda \to \infty$ is taken after subtracting any divergent term in Λ. In the case of the integral (3.28), the divergent term originates from the contribution of the vector $Q = 0$ in $\Gamma_{\Lambda_{p,q}}[P_{abcd}]$, so the regularized integral is defined for $q \neq 6$ by

$$F^{(p,q)}_{abcd}(\Phi) = \lim_{\Lambda\to\infty}\left[\int_{\mathcal{F}_{N,\Lambda}} \frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\frac{\Gamma_{\Lambda_{p,q}}[P_{abcd}]}{\Delta_k(\tau)} - \frac{3\alpha_k}{16\pi^2}\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\,\delta_{(ab}\delta_{cd)}\right] , \tag{3.30}$$

where $\alpha_{12} = (1+\upsilon)k = (1+\upsilon)\frac{c(0)}{2}$, and $\alpha_k = (1+\upsilon)c_k(0)$ for prime CHL models. In the case of interest $\upsilon = 1$, but it depends on the lattice volume in general and $\upsilon = 1/N$ for the non-perturbative Narain lattice (2.21). For $q < 6$, no subtraction is necessary, as long as the integral is carried out first along $\tau_1 \in [-\frac{1}{2}, \frac{1}{2}]$ in the region $\tau \to \infty$. For $q = 6$, the integral is logarithmically divergent, and the regularized integral is defined instead by

$$\widehat{F}^{(p,6)}_{abcd}(\Phi) = \lim_{\Lambda\to\infty}\left[\int_{\mathcal{F}_{N,\Lambda}} \frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\frac{\Gamma_{\Lambda_{p,6}}[P_{abcd}]}{\Delta_k(\tau)} - \frac{3(2k)}{16\pi^2}\log\Lambda\,\delta_{(ab}\delta_{cd)}\right] . \tag{3.31}$$

The logarithmic divergence at $q = 6$ is consistent with the expected divergence in the one-loop scattering amplitude of four gauge bosons in $D = 4$ supergravity [41]. Equivalently, following [45] one may consider the modular integral

$$F^{(p,q)}_{abcd}(\Phi,\epsilon) = \int_{SL(2,\mathbb{Z})\backslash\mathcal{H}} \frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^{2-\epsilon}} \sum_{\gamma\in\Gamma_0(N)\backslash SL(2,\mathbb{Z})} \left.\frac{\Gamma_{\Lambda_{p,q}}[P_{abcd}]}{\Delta_k(\tau)}\right|_\gamma , \tag{3.32}$$

which converges for $\mathrm{Re}(\epsilon) < \frac{6-q}{2}$, and defines the renormalized integral as the constant term in the Laurent expansion at $\epsilon = 0$ of the analytical continuation of $F^{(p,q)}_{abcd}(\Phi,\epsilon)$. The result will then differ from (3.31) by an irrelevant additive constant. In what follows, we shall often abuse notation and omit the hat in $\hat{F}^{(p,q)}_{abcd}$ when stating properties valid for arbitrary q. It is also important to note that while the regularized integral (3.30) or (3.31) is finite at generic points on $G_{p,q}$, it diverges on a real codimension-q loci of $G_{p,q}$, where $Q_{R,\hat{a}} = 0$ for a vector $Q \in \Lambda_{p,q}$ with $Q^2 = 2$, or for a vector $Q \in \Lambda^*_{p,q}$ with $Q^2 = 2/N$ (see (E.12)).

In order to establish that $F^{(p,q)}_{abcd}$ satisfies the constraints (3.22), we shall first establish differential equations for a general class of lattice partition functions

$$\Gamma_{\Lambda_{p,q}}[P] = \tau_2^{\frac{q}{2}} \sum_{Q\in\Lambda_{p,q}} P(Q)\, e^{\mathrm{i}\pi Q_L^2\tau - \mathrm{i}\pi Q_R^2\bar\tau} , \tag{3.33}$$

where the polynomial $P(Q)$ is obtained by acting with the operator $\tau_2^n e^{-\frac{\Delta}{8\pi\tau_2}}$, with

$$\Delta \equiv \sum_a \left(\frac{\partial}{\partial Q_L^a}\right)^2 + \sum_{\hat a}\left(\frac{\partial}{\partial Q_R^{\hat a}}\right)^2 , \tag{3.34}$$

on a homogeneous polynomial of bidegree (m,n) in (Q_L, Q_R), respectively. As shown in [45], $\Gamma_{\Lambda_{p,q}}[P]$ satisfies

$$\Gamma_{\Lambda_{p,q}}[P](-1/\tau) = \frac{(-\mathrm{i})^{\frac{p-q}{2}}\,\tau^{\frac{p-q}{2}+m-n}}{\sqrt{|\Lambda^*_{p,q}/\Lambda_{p,q}|}}\Gamma_{\Lambda^*_{p,q}}[P](\tau) , \tag{3.35}$$

which implies that it transforms as a modular form of weight $\frac{p-q}{2}+m-n$ under $\Gamma_0(N)$. More specifically, we shall consider $\Gamma_{\Lambda_{p,q}}\big[P_{a_1\dots a_m,\hat b_1\dots\hat b_n}\big]$ with

$$P_{a_1\dots a_m,\hat b_1\dots\hat b_n}=\tau_2^n e^{-\frac{\Delta}{8\pi\tau_2}}\left(Q_{L,a_1}\dots Q_{L,a_m}\,Q_{R,\hat b_1}\dots Q_{R,\hat b_n}\right)\,. \tag{3.36}$$

The quartic polynomial P_{abcd} defined in (2.26) arises in the case $(m,n)=(4,0)$, so that $\Gamma_{\Lambda_{p,q}}[P_{abcd}]$ is a modular form of weight $\frac{p-q}{2}+4=k$, ensuring the modular invariance of the integrands in (3.28) and (3.29). Upon contracting the indices, it is easy to check that $\delta^{ab}\delta^{cd}\Gamma_{\Lambda_{p,q}}[P_{abcd}]=D_{k-2}D_{k-4}\Gamma_{\Lambda_{p,q}}[1]$, so the claim that (3.29) gives the trace of (3.28) follows by integration by parts.

To obtain the differential equations satisfied by (3.28), we shall act with the covariant derivative $\mathcal{D}_{a\hat b}$, defined in (3.11) and (3.12). As mentioned below (2.13), ${p_{L,a}}^I$, ${p_{R,\hat b}}^I$ are the left and right orthogonal projectors on the Grassmaniann $G_{p,q}=O(p,q)/\left[O(p)\times O(q)\right]$. Using the derivative rules

$$\mathcal{D}_{a\hat b}\,{p_{L,c}}^I=\frac{1}{2}\delta_{ac}\,{p_{R,\hat b}}^I\,,\qquad\qquad \mathcal{D}_{a\hat b}\,{p_{R,\hat c}}^I=\frac{1}{2}\delta_{\hat b\hat c}\,{p_{L,a}}^I\,, \tag{3.37}$$

one can effectively define the action of the covariant derivative on a function that only depends on Q_L and Q_R as

$$\mathcal{D}_{a\hat b}=\frac{1}{2}\left(Q_{L,a}\partial_{\hat b}+Q_{R,\hat b}\partial_a\right)\,, \tag{3.38}$$

where $\partial_a=\frac{\partial}{\partial Q_L^a}$, $\partial_{\hat b}=\frac{\partial}{\partial Q_R^b}$. Acting with $\mathcal{D}_{e\hat g}$ on (3.33) we get

$$\mathcal{D}_{e\hat g}\Gamma_{\Lambda_{p,q}}\big[P_{a_1\dots a_m,\hat b_1\dots\hat b_n}\big]=\Gamma_{\Lambda_{p,q}}\Big[\left(\mathcal{D}_{e\hat g}-2\pi\tau_2\,Q_{L,e}Q_{R,\hat g}\right)\,P_{a_1\dots a_m\hat b_1\dots\hat b_n}\Big]\,. \tag{3.39}$$

Using (3.38), one computes the commutation relations

$$[\Delta,\mathcal{D}_{e\hat g}]=2\partial_e\partial_{\hat g}\,,\qquad\qquad [\Delta,Q_{L,e}Q_{R,\hat g}]=4\mathcal{D}_{e\hat g}\,, \tag{3.40}$$

$$[\Delta,Q_{L,e}Q_{L,f}]=2\delta_{ef}+4Q_{L,(e}\partial_{f)}\,,\qquad\qquad [\Delta,Q_{L,(e}\partial_{f)}]=2\partial_e\partial_f\,. \tag{3.41}$$

Using them along with the Baker-Campbell-Hausdorff formula

$$e^{\frac{\Delta}{8\pi\tau_2}}\mathcal{O}\,e^{-\frac{\Delta}{8\pi\tau_2}}=\mathcal{O}+\frac{1}{8\pi\tau_2}[\Delta,\,\mathcal{O}]+\frac{1}{2!}\frac{1}{(8\pi\tau_2)^2}[\Delta,\,[\Delta,\,\mathcal{O}]]+\dots\,, \tag{3.42}$$

one easily obtains

$$\mathcal{D}_{e\hat g}\Gamma_{\Lambda_{p,q}}\big[P_{a_1\dots a_m,\hat b_1\dots\hat b_n}\big]=-2\pi\tau_2\,\Gamma_{\Lambda_{p,q}}\Big[e^{-\frac{\Delta}{8\pi\tau_2}}\Big(Q_{L,e}Q_{R,\hat g}-\frac{1}{(4\pi\tau_2)^2}\partial_e\partial_{\hat g}\Big)e^{\frac{\Delta}{8\pi\tau_2}}P_{a_1\dots a_m,\hat b_1\dots\hat b_n}\Big]\,. \tag{3.43}$$

Note that the similarity transformation is such that the operator acts on the simple monomial in $Q_{a_1}\dots Q_{a_m}Q_{\hat b_1}\dots Q_{\hat b_n}$ according to (3.36), such that it directly follows from (3.43) that

$$\mathcal{D}_{e\hat g}\Gamma_{\Lambda_{p,q}}\big[P_{a_1\dots a_m,\hat b_1\dots\hat b_n}\big]=\Gamma_{\Lambda_{p,q}}\Big[-2\pi\,P_{ea_1\dots a_m,\hat g\hat b_1\dots\hat b_n}+\tfrac{mn}{8\pi}\delta_{e(a_1}P_{a_2\dots a_m),(\hat b_2\dots\hat b_n}\delta_{\hat b_1)\hat g}\Big]\,. \tag{3.44}$$

Upon antisymmetrizing in (e,a_1), we get

$$\mathcal{D}_{[e}{}^{\hat g}\Gamma_{\Lambda_{p,q}}\big[P_{a_1]\dots a_m,\hat b_1\dots\hat b_n}\big]=\frac{1}{8\pi^2\tau_2^2}\,\Gamma_{\Lambda_{p,q}}\Big[e^{-\frac{\Delta}{8\pi\tau_2}}\partial_{[e}\partial^{\hat g}\,e^{\frac{\Delta}{8\pi\tau_2}}P_{a_1]\dots a_m,\hat b_1\dots\hat b_n}\Big]. \tag{3.45}$$

which vanishes when $n = 0$ since $e^{\frac{\Delta}{8\pi\tau_2}} P_{a_1\dots a_m}$ does not depend on Q_R. Acting a second time with $\mathcal{D}_{a\hat{b}}$ and antisymmetrizing, we get

$$\mathcal{D}_{[e}{}^{[\hat{e}}\mathcal{D}_{f]}{}^{\hat{f}]}\Gamma_{\Lambda_{p,q}}\left[P_{a_1\dots a_m,\hat{b}_1\dots\hat{b}_n}\right] = -2\Gamma_{\Lambda_{p,q}}\left[e^{-\frac{\Delta}{8\pi\tau_2}}\, Q_{L,[e}Q_R{}^{[\hat{e}}\partial_{f]}\partial^{\hat{f}]}\, e^{\frac{\Delta}{8\pi\tau_2}} P_{a_1\dots a_m,\hat{b}_1\dots\hat{b}_n}\right], \quad (3.46)$$

which similarly vanishes when $n = 0$. Setting $m = 4$, we conclude that the modular integral (3.28) satisfies

$$\mathcal{D}_{[e}{}^{\hat{a}}F_{a]bcd} = 0\ , \qquad \mathcal{D}_{[e}{}^{[\hat{e}}\mathcal{D}_{f]}{}^{\hat{f}]}F_{abcd} = 0\ , \quad (3.47)$$

which therefore establishes the last two equations in (2.23). Note that these two equations do not rely on any particular property of the function $1/\Delta_k$.

Now, the first equation of (2.23) arises from applying the quadratic operator $\mathcal{D}^2_{ef} \equiv \mathcal{D}_{(e}{}^{\hat{g}}\mathcal{D}_{f)\hat{g}}$ on the partition function with polynomial insertion,

$$\begin{aligned}4\mathcal{D}^2_{ef}\Gamma_{\Lambda_{p,q}}\left[P_{a_1\dots a_m,\hat{b}_1\dots\hat{b}_n}\right] = \Gamma_{\Lambda_{p,q}}\Big[\Big(&4\mathcal{D}^2_{ef} - 8\pi\tau_2 Q_{L,(e}Q_R{}^{\hat{g}}\mathcal{D}_{f)\hat{g}} \\ &+16\pi^2\tau_2^2\left(Q_{L,e}Q_{L,f} - \tfrac{\delta_{ef}}{4\pi\tau_2}\right)\left(Q_R^2 - \tfrac{q}{4\pi\tau_2}\right) - q\delta_{ef}\Big)P_{a_1\dots a_m\hat{b}_1\dots\hat{b}_n}\Big]\ ,\end{aligned} \quad (3.48)$$

which gives, using (3.40) and (3.42)

$$\begin{aligned}4\mathcal{D}^2_{ef}\Gamma_{\Lambda_{p,q}}\left[P_{a_1\dots a_m,\hat{b}_1\dots\hat{b}_n}\right] = \Gamma_{\Lambda_{p,q}}\Big[e^{-\frac{\Delta}{8\pi\tau_2}}\Big(&16\pi^2\tau_2^2\, Q_R^2\, Q_{L,e}Q_{L,f} + \frac{\partial_e\partial_f\partial_R^2}{16\pi^2\tau_2^2} \\ &- Q_{L,(e}\partial_{f)}(2Q_R{}^{\hat{g}}\partial_{\hat{g}} + q) - \delta_{ef}(Q_R{}^{\hat{g}}\partial_{\hat{g}} + q)\Big)e^{\frac{\Delta}{8\pi\tau_2}} P_{a_1\dots a_m\hat{b}_1\dots\hat{b}_n}\Big]\end{aligned} \quad (3.49)$$

The first term on the r.h.s. can be rewritten as the action of the Maass lowering operator $\bar{D}_w = -\mathrm{i}\pi\tau_2^2\partial_{\bar{\tau}}$ mapping modular forms of weight w to weight $w-2$. Indeed,

$$\begin{aligned}\bar{D}_w\Gamma_{\Lambda_{p,q}}\left[P_{efa_1\dots a_m,\hat{b}_1\dots\hat{b}_n}\right] =& -\pi^2\tau_2^2\,\Gamma_{\Lambda_{p,q}}\left[\left(Q_R^2 - \tfrac{q+2n}{4\pi\tau_2}\right)P_{efa_1\dots a_m,\hat{b}_1\dots\hat{b}_n}\right] \\ &+ \frac{1}{16}\Gamma_{\Lambda_{p,q}}\left[\Delta\, P_{efa_1\dots a_m,\hat{b}_1\dots\hat{b}_n}\right] \\ =&\Gamma_{\Lambda_{p,q}}\left[e^{-\frac{\Delta}{8\pi\tau_2}}\left(\tfrac{1}{16}\partial_L^2 - (\pi\tau_2 Q_R)^2\right)e^{\frac{\Delta}{8\pi\tau_2}} P_{efa_1\dots a_m,\hat{b}_1\dots\hat{b}_n}\right].\end{aligned} \quad (3.50)$$

where in the second line, we used the fact that Δ commutes with $e^{-\frac{\Delta}{8\pi\tau_2}}$. The r.h.s. of (3.49) can thus be written as

$$\begin{aligned}&4\mathcal{D}^2_{ef}\Gamma_{\Lambda_{p,q}}\left[P_{a_1\dots a_m,\hat{b}_1\dots\hat{b}_n}\right] = (2-(q+n))\delta_{ef}\Gamma_{\Lambda_{p,q}}\left[P_{a_1\dots a_m,\hat{b}_1\dots\hat{b}_n}\right] \\ &+ m(4-(q+2n))\delta_{|e)(a_1}\Gamma_{\Lambda_{p,q}}\left[P_{a_2\dots a_m)(f|,\hat{b}_1\dots\hat{b}_n}\right] + m(m-1)\delta_{(a_1a_2}\Gamma_{\Lambda_{p,q}}\left[P_{a_3\dots a_m)ef,\hat{b}_1\dots\hat{b}_n}\right] \\ &+ \tfrac{m(m-1)n(n-1)}{16\pi^2}\delta_{e(a_1}\delta_{|f|a_2}\Gamma_{\Lambda_{p,q}}\left[P_{a_3\dots a_m),(\hat{b}_1\dots\hat{b}_{n-2}}\right]\delta_{\hat{b}_{n-1}\hat{b}_n)} - 16\bar{D}_w\Gamma_{\Lambda_{p,q}}\left[P_{efa_1\dots a_m,\hat{b}_1\dots\hat{b}_n}\right],\end{aligned} \quad (3.51)$$

where only the last term remains to be computed explicitely. Specializing to the case of main interest, we obtain

$$\square_{ef}\cdot\Gamma_{\Lambda_{p,q}}[P_{abcd}] = -4\bar{D}_w\,\Gamma_{\Lambda_{p,q}}[P_{abcdef}] \quad (3.52)$$

where, for any tensor F_{abcd}, we denote

$$\square_{ef} \cdot F_{abcd} \equiv \mathcal{D}^2_{ef} F_{abcd} + \frac{(q-2)}{4}\delta_{ef}\, F_{abcd} + (q-4)\delta_{(e|(a}F_{bcd)|f)} - 3\delta_{(ab}F_{cd)ef} \tag{3.53}$$

We can now integrate both sides of (3.52) times $1/\Delta_k$ on the truncated fundamental domain $\mathcal{F}_{N,\Lambda}$, leading to

$$\square_{ef} \int_{\mathcal{F}_{N,\Lambda}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2} \frac{\Gamma_{\Lambda_{p,q}}[P_{abcd}]}{\Delta_k} = -4 \int_{\mathcal{F}_{N,\Lambda}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2} \frac{1}{\Delta_k} \bar{D}_{k+2} \Gamma_{\Lambda_{p,q}}[P_{abcdef}] \tag{3.54}$$

The r.h.s. is a boundary term, because $\bar{D}_{-k}(1/\Delta_k) = 0$ by holomorphicity. To compute the boundary term we use Stokes' theorem in the form

$$\int_{\partial \mathcal{F}_{N,\Lambda}} f\, g\, \mathrm{d}\tau = \int_{\mathcal{F}_\Lambda} \mathrm{d}(f\, g\, \mathrm{d}\tau) = \frac{2}{\pi} \int_{\mathcal{F}_{N,\Lambda}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2} (\bar{D}_w f\, g + f\, \bar{D}_{w'}\, g), \tag{3.55}$$

where f and g are any modular forms of weight w and $w' = -w + 2$ and $2\mathrm{d}\tau_1\mathrm{d}\tau_2 = \mathrm{i}\mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}$. By modular invariance, the boundary term reduces to an integral along the segment $\{1/2 \leq \tau_1 < 1/2,\, \tau_2 = \Lambda\}$ and its image under the Fricke involution (for $N > 1$). The latter can be mapped to the former upon using (3.35). At generic points on the Grassmannian $G_{p,q}$, the contributions of non-zero vectors in $\Lambda_{p,q}$ and $\Lambda^*_{p,q}$ are exponentially suppressed, leaving only the contribution of $Q = 0$:

$$\square_{ef} \int_{\mathcal{F}_{N,\Lambda}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2} \frac{\Gamma_{\Lambda_{p,q}}[P_{abcd}]}{\Delta_k} = \Lambda^{\frac{q-6}{2}} \frac{15\,\alpha_k}{2(4\pi)^2} \delta_{(ab}\delta_{cd}\delta_{ef)}, \tag{3.56}$$

where we recall that $\alpha_k = (1+\upsilon)k = (1+\upsilon)\frac{c(0)}{2}$ for the heterotic string compactifications and $\alpha_k = (1+\upsilon)k = (1+\upsilon)c_k(0) = \frac{48}{N+1}$ for CHL models, see table 1. $\upsilon = 1$ for the case of interest but $\upsilon = 1/N$ for the non-perturbative lattice (2.21). Physically, $2k-2$ is the number of vector multiplets. Acting with the same operator $\mathcal{D}^2_{ef}$ on the subtraction in (3.30), we see that the term proportional to $\Lambda^{(q-6)/2}$ cancels, except for $q = 6$ where the substraction in (3.31) leaves a finite remainder. Thus, we find, as claimed earlier, that the modular integral (3.28) is annihilated by the second-order differential operator $\square_{ef}$ defined in (3.53), up to a constant source term present when $q = 6$,

$$\square_{ef}\, F^{(p,q)}_{abcd} = \frac{15(2k)}{2(4\pi)^2} \delta_{(ab}\delta_{cd}\delta_{ef)}\, \delta_{q,6}\ . \tag{3.57}$$

In B, as a consistency check we show that this equation is verified by each Fourier mode in the degeneration limit $O(p,q) \to O(p-1, q-1)$.

4 Weak coupling expansion of exact $(\nabla\Phi)^4$ couplings

In this section, we study the expansion of the proposal (2.27) in the limit where the heterotic string coupling g_3 goes to zero, and show that it reproduces the known tree-level and one-loop amplitudes, along with an infinite series of NS5-brane, Kaluza–Klein monopole and H-monopole instanton corrections. We start by analyzing the expansion of the tensorial modular integral defining the coupling and its trace

$$F^{(p,q)}_{abcd}(\Phi) = \text{R.N.}\int_{\Gamma_0(N)\backslash\mathcal{H}} \frac{d\tau_1 d\tau_2}{\tau_2^2} \frac{\Gamma_{\Lambda_{p,q}}[P_{abcd}]}{\Delta_k(\tau)}\ , \tag{4.1a}$$

$$F^{(p,q)}_{\rm tr}(\Phi) = \text{R.N.}\int_{\Gamma_0(N)\backslash\mathcal{H}} \frac{d\tau_1 d\tau_2}{\tau_2^2} \Gamma_{\Lambda_{p,q}} D_{-k+2}D_{-k}\frac{1}{\Delta_k(\tau)}\ , \tag{4.1b}$$

for a level N even lattice $\Lambda_{p,q}$ of arbitrary signature (p,q), in the limit near the cusp where $O(p,q)$ is broken to $O(1,1)\times O(p-1,q-1)$, so that the moduli space decomposes into

$$G_{p,q} \to \mathbb{R}^+ \times G_{p-1,q-1} \ltimes \mathbb{R}^{p+q-2}\ . \tag{4.2}$$

For simplicity, we first discuss the maximal rank case $N=1$, $p-q=16$, where the integrand is invariant under the full modular group, before dealing with the case of N prime, where the integrand is invariant under the Hecke congruence subgroup $\Gamma_0(N)$. The reader uninterested by the details of the derivation may skip to §4.3, where we specialize to the values $(p,q)=(r-4,8)$ relevant for the $(\nabla\Phi)^4$ couplings in $D=3$ and interpret the various contributions as perturbative and non-perturbative effects in heterotic string theory compactified on T^7. In §4.5 we discuss the case $(p,q)=(r-7,5)$ relevant for H^4 couplings in type IIB string theory compactified on $K3$.

4.1 $O(p,q)\to O(p-1,q-1)$ for even self-dual lattices

We first consider the case where the lattice $\Lambda_{p,q}$ is even self-dual and factorizes in the limit (4.2) as

$$\Lambda_{p,q} \to \Lambda_{p-1,q-1}\oplus I\!I_{1,1}\ . \tag{4.3}$$

We shall denote by R the coordinate on $\mathbb{R}^+$ and by a^I, $I=2\dots p+q-1$ the coordinates on $\mathbb{R}^{p+q-2}$. R parametrizes a one-parameter subgroup e^{RH_0} in $O(p,q)$, such that the action of the non-compact Cartan generator H_0 on the Lie algebra $\mathfrak{so}_{p,q}$ decomposes into

$$\mathfrak{so}_{p,q} \simeq (\mathbf{p}+\mathbf{q}-\mathbf{2})^{(-2)} \oplus (\mathfrak{gl}_1\oplus\mathfrak{so}_{p-1,q-1})^{(0)} \oplus (\mathbf{p}+\mathbf{q}-\mathbf{2})^{(2)}\ . \tag{4.4}$$

while the coordinates a^I parametrize the unipotent subgroup obtained by exponentiating the grade 2 component in this decomposition. A generic charge vector $Q_{\mathcal{I}}\in\Lambda_{p,q}\simeq \mathbf{1}^{(-2)}\oplus(\mathbf{p}+\mathbf{q}-\mathbf{2})^{(0)}\oplus\mathbf{1}^{(2)}$ decomposes into $Q_{\mathcal{I}}=(m,\widetilde{Q}_I,n)$ where $(m,n)\in I\!I_{1,1}=\mathbb{Z}^2$ and $\widetilde{Q}_I\in\Lambda_{p-1,q-1}$, such that $Q^2=-2mn+\widetilde{Q}^2$. The orthogonal projectors defined by $Q_L\equiv p_L^{\mathcal{I}}Q_{\mathcal{I}}$ and $Q_R\equiv p_R^{\mathcal{I}}Q_{\mathcal{I}}$ decompose according to

$$\begin{aligned} p^{\mathcal{I}}_{L,1}Q_{\mathcal{I}} =& \frac{1}{R\sqrt{2}}\left(m+a\cdot\widetilde{Q}+\frac12 a\cdot a\,n\right)-\frac{R}{\sqrt2}n,\\ p^{\mathcal{I}}_{L,\alpha}Q_{\mathcal{I}} =& \widetilde{p}^{I}_{L,\alpha}(\widetilde{Q}_I+na_I),\\ p^{\mathcal{I}}_{R,1}Q_{\mathcal{I}} =& \frac{1}{R\sqrt{2}}\left(m+a\cdot\widetilde{Q}+\frac12 a\cdot a\,n\right)+\frac{R}{\sqrt2}n,\\ p^{\mathcal{I}}_{R,\hat\alpha}Q_{\mathcal{I}} =& \widetilde{p}^{I}_{R,\hat\alpha}(\widetilde{Q}_I+na_I), \end{aligned} \tag{4.5}$$

where $\widetilde{p}^I_{L,\alpha},\widetilde{p}^I_{R,\hat\alpha}$ ($\alpha=2\dots d+16$, $\hat\alpha=2\dots d$) are orthogonal projectors in $G_{p-1,q-1}$ satisfying $\widetilde{Q}^2=\widetilde{Q}_L^2-\widetilde{Q}_R^2$. In the following we shall denote $|Q_R|\equiv\sqrt{\widetilde{Q}_R^2}$.

To study the behavior of (4.1) in the limit $R \gg 1$,[7] it is useful to perform a Poisson resummation on m. For a lattice partition function $\Gamma_{\Lambda_{p,q}}$ with no insertion, as in the scalar integral (4.1b), this gives

$$\Gamma_{\Lambda_{p,q}} = R \sum_{(m,n)\in\mathbb{Z}^2} e^{-\frac{\pi R^2 |n\tau+m|^2}{\tau_2}} \tau_2^{\frac{q-1}{2}} \sum_{\widetilde{Q}\in\Lambda_{p-1,q-1}} e^{2\pi i m(a\cdot\widetilde{Q}+\frac{1}{2}a\cdot a\, n)}\, q^{\frac{1}{2}\widetilde{Q}_L^2}\, q^{\frac{1}{2}\widetilde{Q}_R^2} \tag{4.6}$$

In the case of a lattice sum with momentum insertion, as in the tensor integral $F^{(p,q)}_{abcd}$ (4.1a), we must distinguish whether the indices $abcd$ lie along the direction 1 or along the directions α. Denoting by h the number of indices along direction 1, the previous result generalizes to

$$\begin{aligned}\Gamma_{\Lambda_{p,q}}\left[e^{-\frac{\Delta}{8\pi\tau_2}}\left[(Q_{L,1})^h Q_{L,\alpha_1}\dots Q_{L,\alpha_{4-h}}\right]\right] &= R \sum_{(m,n)\in\mathbb{Z}^2}\left(\frac{R(n\bar\tau+m)}{\mathrm{i}\tau_2\sqrt{2}}\right)^h e^{-\frac{\pi R^2|n\tau+m|^2}{\tau_2}} \\ &\times \Gamma_{\Lambda_{p-1,p-1}+na}\left[e^{-\frac{\Delta}{8\pi\tau_2}}\left[\widetilde{Q}_{L,\alpha_1}\dots\widetilde{Q}_{L,\alpha_{4-h}}\right] e^{2\pi \mathrm{i} m(\widetilde{Q}-\frac{1}{2}a\,n)\cdot a}\right]. \end{aligned}\tag{4.7}$$

In this representation, modular invariance is manifest, since a transformation $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ can be compensated by a linear transformation $(n,m)\mapsto(n,m)\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, under which the second line of (4.7) transforms with weight $12-h$. As a relevant example for what follows, consider the case $(n,m)=k(c,d)$, $k=\gcd(m,n)$, then using an transformation $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$

$$\begin{aligned}&\sum_{\widetilde{Q}\in\Lambda_{p-1,q-1}+kc\,a} e^{-\frac{\Delta}{8\pi\tau_2}}\left[\widetilde{Q}_{L,\alpha_1}\dots\widetilde{Q}_{L,\alpha_{4-h}}\right] e^{2\pi\mathrm{i}\,kd(\widetilde{Q}-\frac{1}{2}a\,kc)\cdot a}\, q^{\frac{1}{2}\widetilde{Q}_L^2}\,\bar q^{\frac{1}{2}\widetilde{Q}_R^2} = \\ &(c\tau+d)^{12-h}\sum_{\widetilde{Q}\in\Lambda_{p-1,q-1}} e^{-\frac{\Delta}{8\pi\tau_2}}\left[\widetilde{Q}_{L,\alpha_1}\dots\widetilde{Q}_{L,\alpha_{4-h}}\right] e^{2\pi\mathrm{i}k\,\widetilde{Q}\cdot a}\, q^{\frac{1}{2}\widetilde{Q}_L^2}\,\bar q^{\frac{1}{2}\widetilde{Q}_R^2}\,. \end{aligned}\tag{4.8}$$

We can therefore compute the integral using the orbit method [46, 47, 48], namely decompose the sum over (m,n) into various orbits under $SL(2,\mathbb{Z})$, and for each orbit $\mathcal{O}$, retain the contribution of a particular element $\varsigma\in\mathcal{O}$ at the expense of extending the integration domain $\mathcal{F}_1 = SL(2,\mathbb{Z})\backslash\mathcal{H}$ to $\Gamma_\varsigma\backslash\mathcal{H}$, where Γ_ς is the stabilizer of ς in $SL(2,\mathbb{Z})$,[8] by using the identity

$$\bigcup_{\gamma\in\Gamma_\varsigma\backslash SL(2,\mathbb{Z})} \gamma\cdot\mathcal{F}_1 = \Gamma_\varsigma\backslash\mathcal{H}. \tag{4.9}$$

The coset representative $\varsigma\in\mathcal{O}$, albeit arbitrary, is usually chosen so as to make the unfolded domain $\Gamma_\varsigma\backslash\mathcal{H}$ as simple as possible. In the present case, there are two types of orbits:

The trivial orbit $(n,m)=(0,0)$ produces, up to a factor of R, the integrals (4.1) for the lattice $\Lambda_{p-1,q-1}$, provided none of the indices $abcd$ lie along the direction 1,

$$F^{(p,q),0}_{\alpha\beta\gamma\delta} = R\,F^{(p-1,q-1)}_{\alpha\beta\gamma\delta}\ ,\quad F^{(p,q),0}_{\rm tr} = R\,F^{(p-1,q-1)}_{\rm tr}\ , \tag{4.10}$$

while it vanishes otherwise (*i.e.* when $h>0$).

[7]Since $1/\Delta$ grows as $e^{\frac{2\pi}{\tau_2}}$ at $\tau_2\to 0$, the following treatment which relies on exchanging the sum and the integral for unfolding is justified for $R^2>2$.

[8]This unfolding procedure requires particular care since the integrand is not of rapid decay near the cusp. We suppress these details here, and refer to [42, 45, 49, 43, 50, 44] for rigorous treatments.

The rank-one orbit corresponds to terms with $(n,m)\neq(0,0)$. Setting $(n,m)=k(c,d)$, with $\gcd(c,d)=1$ and $k\neq 0$, the doublet (c,d) can always be rotated by an element of $SL(2,\mathbb{Z})$ into $(0,1)$, whose stabilizer inside $SL(2,\mathbb{Z})$ is $\Gamma_\infty=\{\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right),n\in\mathbb{Z}\}$. Thus, doublets (c,d) with $\gcd(c,d)=1$ are in one-to-one correspondence with elements of $\Gamma_\infty\backslash SL(2,\mathbb{Z})$. For each k, one can therefore unfold the integration domain $SL(2,\mathbb{Z})\backslash\mathcal{H}$ to $\mathcal{S}=\Gamma_\infty\backslash\mathcal{H}=\mathbb{R}^+_{\tau_2}\times(\mathbb{R}/\mathbb{Z})_{\tau_1}$, the unit width strip, provided one keeps only the term $(c,d)=(0,1)$ in the sum. The resulting contribution to the tensor integral (4.1a) are

$$\begin{aligned}
F^{(p,q),1}_{\alpha\beta\gamma\delta}&=R\int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^2}\int_{\mathbb{R}/\mathbb{Z}}\mathrm{d}\tau_1\sum_{k\neq0}e^{-\pi R^2k^2/\tau_2}\frac{\Gamma_{\Lambda_{p-1,q-1}}\left[\tilde{P}_{\alpha\beta\gamma\delta}\,e^{2\pi\mathrm{i}ka^I\widetilde{Q}_I}\right]}{\Delta}\ ,\\
F^{(p,q),1}_{11\gamma\delta}&=R\int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^2}\int_{\mathbb{R}/\mathbb{Z}}\mathrm{d}\tau_1\sum_{k\neq0}\left(\frac{Rk}{\mathrm{i}\tau_2\sqrt{2}}\right)^2e^{-\pi R^2k^2/\tau_2}\frac{\Gamma_{\Lambda_{p-1,q-1}}\left[\tilde{P}_{\alpha\beta}\,e^{2\pi\mathrm{i}ka^I\widetilde{Q}_I}\right]}{\Delta}\ ,\\
F^{(p,q),1}_{1111}&=R\int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^2}\int_{\mathbb{R}/\mathbb{Z}}\mathrm{d}\tau_1\sum_{k\neq0}\left(\frac{Rk}{\mathrm{i}\tau_2\sqrt{2}}\right)^4e^{-\pi R^2k^2/\tau_2}\frac{\Gamma_{\Lambda_{p-1,q-1}}\left[e^{2\pi\mathrm{i}ka^I\widetilde{Q}_I}\right]}{\Delta}\ ,
\end{aligned}\tag{4.11}$$

where

$$\tilde{P}_{\alpha_1\dots\alpha_{4-h}}=e^{-\frac{\Delta}{8\pi\tau_2}}\left[\widetilde{Q}_{L,\alpha_1}\dots\widetilde{Q}_{L,\alpha_{4-h}}\right]\ ,\tag{4.12}$$

while the contribution to its trace is

$$F^{(p,q),1}_{\mathrm{tr}}=R\int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^2}\int_{\mathbb{R}/\mathbb{Z}}\mathrm{d}\tau_1\sum_{k\neq0}e^{-\pi R^2k^2/\tau_2}\,\Gamma_{\Lambda_{p-1,q-1}}\left[e^{2\pi\mathrm{i}ka^I\widetilde{Q}_I}\right]D^2\left(\frac{1}{\Delta}\right)\ .\tag{4.13}$$

The integral over $\mathcal{S}$ can be computed by inserting the Fourier expansion

$$\frac{1}{\Delta}=\sum_{\substack{m\in\mathbb{Z}\\m\geq-1}}c(m)\,q^m\ ,\quad D^2\frac{1}{\Delta}=a_2\,c(0)+\sum_{\substack{m\in\mathbb{Z}-\{0\}\\m\geq-1}}\sum_{\ell=0}^{2}a_\ell\,m^{2-\ell}c(m)\,q^m\tau_2^{-\ell}\tag{4.14}$$

where

$$a_0=4\ ,\quad a_1=\frac{p-q+6}{\pi}\ ,\quad a_2=\frac{(p-q+6)(p-q+8)}{16\pi^2}\ .\tag{4.15}$$

The integral over τ_1 picks up the Fourier coefficient $c(m)$ with $m=-\frac12\widetilde{Q}^2$. The remaining integral over τ_2 can be computed after expanding $\tilde{P}_{\alpha_1\dots\alpha_{4-h}}=\sum_{\ell=0}^{\lfloor\frac{4-h}{2}\rfloor}\tilde{P}^{(\ell)}_{\alpha_1\dots\alpha_{4-h}}\tau_2^{-\ell}$, where $\tilde{P}^{(\ell)}_{\alpha_1\dots\alpha_{4-h}}$ is a polynomial in $\widetilde{Q}$ of degree $4-h-2\ell$, or zero when $2\ell>4-h$. Contributions with $\widetilde{Q}=0$ lead to power-like terms,

$$\begin{aligned}
F^{(p,q),1,0}_{\alpha\beta\gamma\delta}&=R^{q-6}\,\xi(q-6)\,\frac{3c(0)}{8\pi^2}\delta_{(\alpha\beta}\delta_{\gamma\delta)},\\
F^{(p,q),1,0}_{11\alpha\beta}&=R^{q-6}\,\xi(q-6)\,(7-q)\frac{c(0)}{8\pi^2}\delta_{\alpha\beta}\ ,\\
F^{(p,q),1,0}_{1111}&=R^{q-6}\,\xi(q-6)\,(7-q)(9-q)\frac{c(0)}{8\pi^2}\ ,
\end{aligned}\tag{4.16}$$

while the result vanishes for an odd number of indices along the direction 1, and for its trace

$$F^{(p,q),1,0}_{\mathrm{tr}}=R^{q-6}\,\xi(q-6)\,(p-q+6)(p-q+8)\,\frac{c(0)}{8\pi^2}\ .\tag{4.17}$$

Here we used $\tilde{P}^{(2)}_{abcd}(0) = \frac{3}{16\pi^2}\delta_{(ab}\delta_{cd)}$, $\tilde{P}^{(1)}_{ab}(0) = -\frac{1}{4\pi}\delta_{ab}$, and $\tilde{P}^{(0)} = 1$. Note that (4.17) and (4.16) have a simple pole at $q = 6$, which is subtracted by the regularization prescription mentioned below (3.32). For $q = 7$, the pole in (4.17), (4.16) cancels against the pole from the trivial orbit contribution (4.10).

In contrast, non-zero vectors $\widetilde{Q}$ lead to exponentially suppressed contributions, which depend on the axions through a phase factor $e^{2\pi i k a\cdot\widetilde{Q}}$. After rescaling $\widetilde{Q} \mapsto Q/k$, we find that the Fourier coefficient with charge $Q \in \Lambda_{p-1,q-1} \smallsetminus \{0\}$ is given by

$$\begin{aligned}
F^{(p,q),1,Q}_{\alpha\beta\gamma\delta} &= 4\,\bar{c}(Q)\,R^{\frac{q-1}{2}} \sum_{\ell=0}^{2} \frac{\tilde{P}^{(\ell)}_{\alpha\beta\gamma\delta}(Q)}{R^\ell} \frac{K_{\frac{q-3}{2}-\ell}\left(2\pi R\sqrt{2|Q_R|^2}\right)}{\sqrt{2|Q_R|^2}^{\frac{q-3}{2}-\ell}} \\
F^{(p,q),1,Q}_{1\alpha\beta\gamma} &= 4\,\bar{c}(Q)\,R^{\frac{q-1}{2}} \sum_{\ell=0}^{1} \frac{\tilde{P}^{(\ell)}_{\alpha\beta\gamma}(Q)}{\mathrm{i}\sqrt{2}R^\ell} \frac{K_{\frac{q-5}{2}-\ell}\left(2\pi R\sqrt{2|Q_R|^2}\right)}{\sqrt{2|Q_R|^2}^{\frac{q-5}{2}-\ell}} \\
&\vdots \\
F^{(p,q),1,Q}_{1111} &= 4\bar{c}(Q)\,R^{\frac{q-1}{2}} \frac{\tilde{P}^{(0)}}{4} \frac{K_{\frac{q-11}{2}}\left(2\pi R\sqrt{2|Q_R|^2}\right)}{\sqrt{2|Q_R|^2}^{\frac{q-11}{2}}}
\end{aligned} \tag{4.18}$$

for the tensor integral, and

$$F^{(p,q),1,Q}_{\mathrm{tr}} = 4\,\bar{c}(Q)\,R^{\frac{q-1}{2}} \sum_{\ell=0}^{2} \frac{a_\ell}{R^\ell}\left(-\frac{Q^2}{2}\right)^{2-\ell} \frac{K_{\frac{q-3}{2}-\ell}\left(2\pi R\sqrt{2|Q_R|^2}\right)}{\sqrt{2|Q_R|^2}^{\frac{q-3}{2}-\ell}} \tag{4.19}$$

for its trace. In either case,

$$\bar{c}(Q) = \sum_{d|Q} c\left(-\frac{Q^2}{2d^2}\right) d^{q-7}\ . \tag{4.20}$$

The physical interpretation of these results will be discussed in §4.3, after generalizing them to $\mathbb{Z}_N$ orbifolds.

4.2 Extension to $\mathbb{Z}_N$ CHL orbifolds

The degeneration limit (4.2) of the modular integrals (4.1) for $\mathbb{Z}_N$ CHL models with $N = 2, 3, 5, 7$ can be treated similarly by adapting the orbit method to the case where the integrand is invariant under the Hecke congruence subgroup $\Gamma_0(N)$ [51, 52, 44]. In (4.1), Δ_k is the cusp form of weight $k = \frac{24}{N+1}$ defined in (2.4), and $\Gamma_{\Lambda_{p,q}}$ is the partition function for a lattice

$$\Lambda_{p,q} = \tilde{\Lambda}_{p-1,q-1} \oplus I\!I_{1,1}[N]\ , \tag{4.21}$$

where $\tilde{\Lambda}_{p-1,q-1}$ is a level N even lattice of signature $(p-1, q-1)$. The lattice $I\!I_{1,1}[N]$ is obtained from the usual unimodular lattice $I\!I_{1,1}$ by restricting the winding and momentum to $(n, m) \in N\mathbb{Z} \oplus \mathbb{Z}$. After Poisson resummation on m, Eq. (4.6) and (4.7) continue to hold, except for the fact that n is restricted to run over $N\mathbb{Z}$. The sum over (n, m) can then be decomposed into orbits of $\Gamma_0(N)$:[9]

[9]Since $1/\Delta_k$ grows as $e^{\frac{2\pi}{N\tau_2}}$ at $\tau_2 \to 0$, the following treatment which relies on exchanging the sum and the integral for unfolding is justified for $NR^2 > 2$.

Trivial orbit The term $(n,m)=(0,0)$ produces the same modular integral, up to a factor of R,

$$F^{(p,q),0}_{\alpha\beta\gamma\delta}=R\,\tilde F^{(p-1,q-1)}_{\alpha\beta\gamma\delta}\ ,\qquad F^{(p,q),0}_{\rm tr}=R\,\tilde F^{(p-1,q-1)}_{\rm tr}\ ,\tag{4.22}$$

where $\tilde F^{(p-1,q-1)}_{\alpha\beta\gamma\delta}$, $\tilde F^{(p-1,q-1)}_{\rm tr}$ are the integrals (4.1) for the lattice $\tilde\Lambda_{p-1,q-1}$ defined by (4.21).

Rank-one orbits Terms with $(n,m)=k(c,d)$ with $k\neq 0$ and $\gcd(c,d)=1$ fall into two different classes of orbits under $\Gamma_0(N)$:

- Doublets $k(c,d)$ such that $c=0\bmod N$ and $k\in\mathbb{Z}$ can be rotated by an element of $\Gamma_0(N)$ into $(0,1)$, whose stabilizer in $\Gamma_0(N)$ is $\Gamma_\infty=\{\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right),n\in\mathbb{Z}\}$. For these elements, one can unfold the integration domain $\Gamma_0(N)\backslash\mathcal{H}$ into the unit width strip $\mathcal{S}=\Gamma_\infty\backslash\mathcal{H}=\mathbb{R}^+_{\tau_2}\times(\mathbb{R}/\mathbb{Z})_{\tau_1}$;
- Doublets $k(c,d)$ such that $c\neq 0\bmod N$ and $k=0\bmod N$ can be rotated by an element of $\Gamma_0(N)$ into $(1,0)$, whose stabilizer in $\Gamma_0(N)$ is $S\,\Gamma_{\infty,N}\,S^{-1}$, where $\Gamma_{\infty,N}=\{\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right),n\in N\mathbb{Z}\}$ and $S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. One can unfold the integration domain $\Gamma_0(N)\backslash\mathcal{H}$ into $S\,\Gamma_{\infty,N}\,S^{-1}\backslash\mathcal{H}$, and change variable $\tau\to-1/\tau$ so as to reach $\mathcal{S}_N=\Gamma_{\infty,N}\backslash\mathcal{H}=\mathbb{R}^+_{\tau_2}\times(\mathbb{R}/N\mathbb{Z})_{\tau_1}$, the width-$N$ strip. Under this change of variable, the level-N weight-k cusp form transforms as $\Delta_k(-1/\tau)=(i\sqrt{N})^{-k}\tau^k\Delta_k(\tau/N)$, while the partition function for the sublattice $\tilde\Lambda_{p-1,q-1}$ transforms as

$$\Gamma_{\tilde\Lambda_{p-1,q-1}}[P_{\alpha\beta\gamma\delta}](-1/\tau)=\tilde\upsilon N^{-\frac{k}{2}-1}(-\mathrm{i})^{\frac{p-q}{2}}\tau^k\,\Gamma_{\tilde\Lambda^*_{p-1,q-1}}[P_{\alpha\beta\gamma\delta}](\tau)\ ,\tag{4.23}$$

where $\Gamma_{\tilde\Lambda^*_{p-1,q-1}}(\tau)$ denotes the sum over the dual lattice $\tilde\Lambda^*_{p-1,q-1}$, and $\tilde\upsilon N^{-\frac{k}{2}-1}=\left|\tilde\Lambda^*_{p-1,q-1}/\tilde\Lambda_{p-1,q-1}\right|^{-1/2}$ (Note that $\tilde\upsilon=N^{1-\delta_{q,8}}$ for $q\le 8$ in the cases of interest).

For the simplest component $F^{(p,q),1}_{\alpha\beta\gamma\delta}$, the sum of the two classes of orbits then reads

$$\begin{aligned}F^{(p,q),1}_{\alpha\beta\gamma\delta}&=R\int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^2}\int_{\mathbb{R}/\mathbb{Z}}\mathrm{d}\tau_1\,\frac{1}{\Delta_k(\tau)}\sum_{k\neq0}e^{-\pi R^2k^2/\tau_2}\Gamma_{\tilde\Lambda_{p-1,q-1}}\left[e^{2\pi\mathrm{i}ka^I\widetilde{Q}_I}\,P_{\alpha\beta\gamma\delta}\right]\\&+R\int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^2}\int_{\mathbb{R}/(N\mathbb{Z})}\mathrm{d}\tau_1\,\frac{\tilde\upsilon}{N}\,\frac{1}{\Delta_k(\tau/N)}\sum_{\substack{k\neq0\\k=0\bmod N}}e^{-\pi R^2k^2/\tau_2}\Gamma_{\tilde\Lambda^*_{p-1,q-1}}\left[e^{2\pi\mathrm{i}ka^I\widetilde{Q}_I}\,P_{\alpha\beta\gamma\delta}\right]\ .\end{aligned}\tag{4.24}$$

The contributions from $\widetilde{Q}=0$ lead to power-like terms,

$$\begin{aligned}F^{(p,q)(1,0)}_{\alpha\beta\gamma\delta}&=R^{q-6}\xi(q-6)\left(1+\tilde\upsilon N^{q-7}\right)\frac{3c_k(0)}{8\pi^2}\delta_{(\alpha\beta}\delta_{\gamma\delta)}\ ,\\F^{(1,0)}_{11\alpha\beta}&=R^{q-6}\,\xi(q-6)\,(7-q)\left(1+\tilde\upsilon N^{q-7}\right)\frac{c_k(0)}{8\pi^2}\delta_{\alpha\beta}\ ,\\F^{(1,0)}_{1111}&=R^{q-6}\,\xi(q-6)\,(7-q)(9-q)\left(1+\tilde\upsilon N^{q-7}\right)\frac{c_k(0)}{8\pi^2}\ ,\end{aligned}\tag{4.25}$$

for the tensor integral and

$$F^{(p,q)(1,0)}_{\rm tr}=R^{q-6}\xi(q-6)(p-q+6)(p-q+8)\left(1+\tilde\upsilon N^{q-7}\right)\frac{c_k(0)}{8\pi^2}\tag{4.26}$$

for its trace, where $c_k(0)=k$ is the constant term in $1/\Delta_k$. As in (4.17) and (4.16), the pole at $q=6$ is subtracted by the regularization prescription (3.30), while the pole at $q=7$ cancels against the pole from the zero orbit contribution (4.22).

The terms with non-zero vector $\widetilde{Q}$ produce exponentially suppressed corrections of the same form as in the maximal rank case (4.18), but with a different summation measure, namely

$$\bar{c}_k(Q) = \sum_{\substack{d\geq 1,\\ Q/d\in\tilde{\Lambda}_{p-1,q-1}}} c_k\Big(-\frac{Q^2}{2d^2}\Big)\, d^{q-7} + \ \tilde{\upsilon} \sum_{\substack{d\geq 1,\\ Q/d\in N\tilde{\Lambda}^*_{p-1,q-1}}} c_k\Big(-\frac{Q^2}{2Nd^2}\Big)\,(Nd)^{q-7}\,, \tag{4.27}$$

where the first term, arising from the first class of orbits, has support on $\tilde{\Lambda}_{p-1,q-1}$, and the second term, arising from the second class of orbits, has support on the sublattice $N\tilde{\Lambda}^*_{p-1,q-1}\subset\tilde{\Lambda}_{p-1,q-1}$. In the latter contribution, notice that one factor of N in the numerator of the Fourier coefficient comes from the matching condition with $1/\Delta_k(\tau/N)$, and two factors of N in its denominator come from all the divisors being originally multiples of N.

It will also be useful to consider a different degeneration limit of the type (4.2) where the lattice decomposes as

$$\Lambda_{p,q}=\Lambda_{p-1,q-1}\oplus I\!I_{1,1}\ , \tag{4.28}$$

where $I\!I_{1,1}$ is the usual unimodular even lattice, with no restriction on the windings and momenta (n,m), and $\Lambda_{p-1,q-1}$ is a level N even lattice of signature $(p-1,q-1)$, not to be confused with the lattice $\tilde{\Lambda}_{p-1,q-1}$ above. The sum over $(n,m)\in\mathbb{Z}\oplus\mathbb{Z}$ can then be decomposed into orbits of $\Gamma_0(N)$. The trivial orbit is similar to (4.22), but now $F^{(p-1,q-1)}_{\alpha\beta\gamma\delta}$ and $F^{(p-1,q-1)}_{\rm tr}$ are the modular integrals for the lattice $\Lambda_{p-1,q-1}$. For the rank-one orbit, the discussion goes as before, except that the second class of orbits $(m,n)=k(c,d)$ with $k=\gcd(m,n)$ and $c\neq 0 \bmod N$ has no restriction on k. For the simplest component $F^{(p,q),1}_{\alpha\beta\gamma\delta}$, the sum of the two classes of orbits then reads

$$\begin{aligned}F^{(p,q),1}_{\alpha\beta\gamma\delta} &= R\int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^2}\int_{\mathbb{R}/\mathbb{Z}}\mathrm{d}\tau_1\,\frac{1}{\Delta_k(\tau)}\sum_{k\neq 0} e^{-\pi R^2k^2/\tau_2}\Gamma_{\Lambda_{p-1,q-1}}\Big[e^{2\pi\mathrm{i}ka^I\widetilde{Q}_I}\,P_{\alpha\beta\gamma\delta}\Big]\\ &+R\int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^2}\int_{\mathbb{R}/(N\mathbb{Z})}\mathrm{d}\tau_1\,\frac{1}{\Delta_k(\tau/N)}\,\frac{\upsilon}{N}\sum_{k\neq 0} e^{-\pi R^2k^2/\tau_2}\Gamma_{\Lambda^*_{p-1,q-1}}\Big[e^{2\pi\mathrm{i}ka^I\widetilde{Q}_I}\,P_{\alpha\beta\gamma\delta}\Big]\ ,\end{aligned} \tag{4.29}$$

where $\upsilon N^{-\frac{k}{2}-1}=\big|\Lambda^*_{p-1,q-1}/\Lambda_{p-1,q-1}\big|^{-1/2}$ (which now simplifies to $\upsilon=N^{-\delta_{q,8}}$ for $q\leq 8$ in the cases of interest). The contributions from $\widetilde{Q}=0$ lead to power-like terms,

$$\begin{aligned}F^{(p,q)(1,0)}_{\alpha\beta\gamma\delta} &= R^{q-6}\xi(q-6)\big(1+\upsilon\big)\frac{3c_k(0)}{8\pi^2}\delta_{(\alpha\beta}\delta_{\gamma\delta)},\\ F^{(1,0)}_{11\alpha\beta} &= R^{q-6}\,\xi(q-6)\,(7-q)\big(1+\upsilon\big)\frac{c_k(0)}{8\pi^2}\delta_{\alpha\beta},\\ F^{(1,0)}_{1111} &= R^{q-6}\,\xi(q-6)\,(7-q)(9-q)\big(1+\upsilon\big)\frac{c_k(0)}{8\pi^2}\end{aligned} \tag{4.30}$$

for the tensor integral and

$$F^{(p,q)(1,0)}_{\rm tr} = R^{q-6}\xi(q-6)(p-q+6)(p-q+8)\big(1+\upsilon\big)\frac{c_k(0)}{8\pi^2} \tag{4.31}$$

for its trace, where $c_k(0) = k$ is the constant term in $1/\Delta_k$.

The terms with non-zero vector $\widetilde{Q}$ produce exponentially suppressed corrections of the same form as in the maximal rank case (4.18), but with a different summation measure, namely

$$\bar{c}_k(Q) = \sum_{\substack{d\geq 1,\\ Q/d\in\Lambda_{p-1,q-1}}} c_k\Big(-\frac{Q^2}{2d^2}\Big)\, d^{q-7} + \ \upsilon \sum_{\substack{d\geq 1,\\ Q/d\in\Lambda^*_{p-1,q-1}}} c_k\Big(-\frac{NQ^2}{2d^2}\Big)\, d^{q-7}, \tag{4.32}$$

where the first term, arising from the first class of orbits, has support on $\Lambda_{p-1,q-1}$, and the second term, arising from the second class of orbits, has support on the dual lattice $\Lambda^*_{p-1,q-1}$. In the latter contribution, notice that one factor of N in the numerator of the Fourier coefficient comes from the matching condition with $1/\Delta_k(\tau/N)$.

4.3 Perturbative limit of exact $(\nabla\Phi)^4$ couplings in $D = 3$

Specializing to $(p, q) = (2k, 8) = (r-4, 8)$, and decomposing as $\Lambda_{2k,8} = \Lambda_{2k-1,7} \oplus I\!I_{1,1}[N]$, the limit (4.2) studied in this section corresponds to the expansion of the exact $(\nabla\Phi)^4$ couplings in $D = 3$ in the limit where the heterotic string coupling $g_3 = 1/\sqrt{R}$ becomes weak. To interpret the resulting contributions in the language of heterotic perturbation theory, one should remember that the U-duality function $F^{(2k,8)}_{abcd}(\Phi)$ is the coefficient of the $(\nabla\Phi)^4$ coupling in the low-energy action written in Einstein frame, such that the metric γ_E is inert under U-duality,

$$S_3 = \int \mathrm{d}^3x\,\sqrt{-\gamma_E}\,\Big[\mathcal{R}[\gamma_E] - (2\delta_{\hat{a}\hat{b}}\delta_{\hat{c}\hat{d}} - \delta_{\hat{a}\hat{c}}\delta_{\hat{b}\hat{d}})F^{(2k,8)}_{abcd}(\Phi)\,\gamma_E^{\mu\rho}\gamma_E^{\nu\sigma}P^{a\hat{a}}_{\mu}P^{b\hat{b}}_{\nu}P^{c\hat{c}}_{\rho}P^{d\hat{d}}_{\sigma}\Big] + \dots \tag{4.33}$$

In terms of the string frame metric $\gamma = \gamma_E g_3^4$, one finds

$$S_3 = \int \mathrm{d}^3x\,\sqrt{-\gamma}\,\Big[\frac{1}{g_3^2}\mathcal{R}[\gamma] - g_3^2\,(2\delta_{\hat{a}\hat{b}}\delta_{\hat{c}\hat{d}} - \delta_{\hat{a}\hat{c}}\delta_{\hat{b}\hat{d}})F^{(2k,8)}_{abcd}(\Phi)\,\gamma^{\mu\rho}\gamma^{\nu\sigma}P^{a\hat{a}}_{\mu}P^{b\hat{b}}_{\nu}P^{c\hat{c}}_{\rho}P^{d\hat{d}}_{\sigma}\Big] + \dots\,. \tag{4.34}$$

Using $c_k(0) = k$ for CHL orbifolds with $N > 1$ or $c(0) = 2k$ in the maximal rank case, and $\xi(2) = \frac{\pi}{6}$, the results from §4.1 and §4.2 read

$$g_3^2\,F^{(2k,8)}_{abcd} = \frac{3}{2\pi g_3^2}\delta_{(ab}\delta_{cd)} + F^{(2k-1,7)}_{abcd} + \sideset{}{'}\sum_{Q\in\Lambda_{2k-1,7}} \bar{c}_k(Q)e^{-\frac{2\pi\sqrt{2|Q_R|^2}}{g_3^2}+2\pi\mathrm{i}a\cdot Q}\,P^{(*)}_{abcd}\,, \tag{4.35}$$

where we omit the detailed form of exponentially suppressed corrections, and the summation measure is read off from (4.27)

$$\bar{c}_k(Q) = \sum_{\substack{d\geq 1,\\ Q/d\in\Lambda_{2k-1,7}}} d\,c_k\Big(-\frac{Q^2}{2d^2}\Big) + \sum_{\substack{d\geq 1,\\ Q/d\in N\Lambda^*_{2k-1,7}}} N\,d\,c_k\Big(-\frac{Q^2}{2Nd^2}\Big)\,, \tag{4.36}$$

The first two terms in (4.35), originating from the zero orbit and rank-one orbit, respectively, should match the tree-level and one-loop contributions, respectively. Indeed, the dimensional reduction of the tree-level $\mathcal{R}^2+(\mathrm{Tr}F^2)^2$ coupling in ten-dimensional heterotic string theory [53, 54] leads to a tree-level $(\nabla\Phi)^4$ coupling in $D=3$, with a coefficient which is by construction independent of N. A more detailed analysis of the ten-dimensional origin of this term will be given in §5.3.1. The second term in (4.35) of course matches the one-loop contribution (2.24) by construction. The remaining non-perturbative terms can be interpreted as heterotic NS5-brane, KK5-brane and H-monopoles wrapped on any possible T^6 inside T^7 [9]. More precisely, NS5-brane and KK5-brane charges correspond to momentum and winding charges in the hyperbolic part $I\!I_{1,1}[N]\oplus I\!I_{k-2,k-2}$ of $\Lambda_m\oplus I\!I_{1,1}$, while H-monopoles correspond to charges in the gauge lattice $\Lambda_{k,8-k}$ (for the heterotic string compactification on T^7, these sublattices must be replaced by $I\!I_{7,7}$ and $E_8\oplus E_8$ or D_{16}, respectively). Note that [9] studied these corrections on a special locus in moduli space, corresponding to $T^4/\mathbb{Z}_2$ realization of K3 surfaces on the type II side, and did not keep track of all gauge charges, which resulted in a different summation measure.

4.4 Decompactification limit of one-loop F^4 couplings

For general $(p,q)=(d+2k-8,d)=(d+r-12,d)$ with $q\leq 7$, the modular integral (4.1a) is interpreted as the one-loop F^4 amplitude in a heterotic CHL orbifold compactified down to dimension $D=10-d$. The decomposition (4.21) corresponds to the case (a) where the radius R of a circle in T^d orthogonal to the $\mathbb{Z}_N$ orbifold action becomes large, while the limit (4.28) corresponds to the case (b) where the radius R of the circle in T^d singled out by the $\mathbb{Z}_N$ orbifold action becomes large in string units.

The power-like terms contributions in R come in part from the trivial orbit, and from the zero-charge contribution to the rank-one orbit:

$$\begin{aligned} a):\quad F^{(p,q)}_{\alpha\beta\gamma\delta} &= RF^{(p-1,q-1)}_{\alpha\beta\gamma\delta}+R^{q-6}\xi(q-6)\frac{3(2k)}{8\pi^2}\delta_{(\alpha\beta}\delta_{\gamma\delta)}+\dots \\ b):\quad F^{(p,q)}_{\alpha\beta\gamma\delta} &= R\tilde{F}^{(p-1,q-1)}_{\alpha\beta\gamma\delta}+R^{q-6}\xi(q-6)\frac{3k(1+N^{q-6})}{8\pi^2}\delta_{(\alpha\beta}\delta_{\gamma\delta)}+\dots \end{aligned} \tag{4.37}$$

The first term reproduces, up to a volume factor of R, the one-loop F^4 amplitude in $D+1$ dimensions (4.10), either in the same CHL model (case a), or in the full heterotic string compactification (case b). Indeed, in the latter case, the partition function $\Gamma_{\Lambda_{p-1,q-1}}$ factorizes into $\Gamma_{I\!I_{d+k-9,d+k-9}}\times\Gamma_{\Lambda_{k,8-k}}$. The fundamental domain $\Gamma_0(N)\backslash\mathcal{H}$ can be extended to $SL(2,\mathbb{Z})\backslash\mathcal{H}$, at the expense of replacing $\Gamma_{\Lambda_{k,8-k}}/\Delta_k$ by the sum over its images under $\Gamma_0(N)\backslash SL(2,\mathbb{Z})=\{1,S,TS,\dots,T^{N-1}S\}$. As explained in §A, this sum reproduces $\Gamma_{\Lambda_{d+15,d-1}}/\Delta$, the partition function for the maximal rank theory in dimension $D+1$.

The second term, originating from the zero-charge contribution to the rank-one orbit, can instead be understood as the limit $s\to 0$ of an infinite tower of terms of the schematic form $\sum_{m\neq 0}(\frac{m^2}{R^2}-s)^{3-\frac{d}{2}}F^4$ in the low-energy effective action, where s is a Mandelstam variable, arising from threshold contributions of Kaluza–Klein excitations of the massless supergravity states in dimension $D+1$. In the limit $R\to\infty$, this infinite series along with the term $m=0$ from the non-local part of the action in dimension D sums up to the contribution of massless supergravity states to the non-local part of the action in dimension $D+1$. The pole at $q=6$ in the second term of (4.37) originates from the logarithmic infrared divergence in the local part of the string effective action in dimension $D=4$, and matches the expected ultraviolet divergence in 4-dimensional supergravity. The apparent pole at $q=7$ cancels against a pole in the first term, due to the same logarithmic divergence. Indeed, the $1/\epsilon$ pole of the full amplitude $F^{(p,6)}_{abcd}(\Phi,\epsilon)$ can be extracted from its Laurent expansion at $\epsilon=0$, namely

$$F^{(p,6)}_{abcd}(\Phi,\epsilon) = -\frac{3(2k)}{16\pi^2\epsilon}\,\delta_{(ab}\delta_{cd)} + \mathcal{O}(1) \tag{4.38}$$

In addition, massive perturbative BPS states with non-vanishing charge $Q\in\Lambda_{d+2k-9,d-1}$ in dimension $D+1$ and mass $\mathcal{M}(Q)$ lead to exponentially suppressed terms of order $e^{-2\pi R\mathcal{M}(Q)}$, weighted by the helicity supertrace $\Omega_4(Q)$, as expected on general grounds.

4.5 Perturbative limit of exact H^4 couplings in type IIB on $K3$

Here we briefly consider the case $q=5$, $N=1$, corresponding to type IIB string theory compactified on $K3$. In Einstein frame, the low energy effective action takes the form

$$S_6 = \int \mathrm{d}^6x\,\sqrt{-\gamma_E}\,\Big[\mathcal{R}[\gamma_E] - F^{(21,5)}_{abcd}(\Phi)\,H^a_{\mu\nu\kappa}H^b_{\rho\sigma}{}^{\kappa}H^{c\,\mu\nu\lambda}H^{d\rho\sigma}{}_{\lambda}\Big] + \dots \tag{4.39}$$

where the three-form H^α with $\alpha\neq 1$ are the self-dual field-strengths of the reduction of the RR two-form, four-form and six-form on the self-dual part of the homology lattice $H^{\rm even}(K3) = E_8\oplus E_8\oplus I\!I_{4,4}$, while H^1 is the self-dual component of the NS-NS two-form field-strength. We shall restrict for simplicity to the components $\alpha,\beta,\gamma,\delta\neq 1$. In terms of the string frame metric $\gamma=g_s\gamma_E$ and setting $\mathcal{H}^a = g_s H^a$ (since Ramond-Ramond field are normalized as $H\sim 1/g_s$ in type II perturbation theory), we get

$$S_6 = \int d^6x\,\sqrt{-\gamma}\,\left[\frac{1}{g_s^2}\mathcal{R}[\gamma] - \frac{1}{g_s}\,F^{(21,5)}_{\alpha\beta\gamma\delta}(\Phi)\,\mathcal{H}^\alpha_{\mu\nu\kappa}\mathcal{H}^\beta_{\rho\sigma}{}^{\kappa}\mathcal{H}^{\gamma\,\mu\nu\lambda}\mathcal{H}^{\delta\rho\sigma}{}_{\lambda}\right] + \dots \tag{4.40}$$

Identifying $R=1/g_s$, the large radius expansion of $F^{21,5}_{\alpha\beta\gamma\delta}$ becomes, schematically,

$$\frac{1}{g_s}F^{(21,5)}_{\alpha\beta\gamma\delta} = \frac{1}{g_s^2}F^{(20,4)}_{\alpha\beta\gamma\delta}(\Phi) + \frac{3}{2\pi}\delta_{(\alpha\beta}\delta_{\gamma\delta)} + \sum_{Q\in\Lambda_{20,4}}{}^{'}\,\bar{c}(Q)e^{-\frac{2\pi\sqrt{2|Q_R|^2}}{g_s}-2\pi \mathrm{i}a\cdot Q}P^{(*)}_{\alpha\beta\gamma\delta}\,. \tag{4.41}$$

The first term proportional to $F^{(20,4)}_{abcd}$ is now recognized as a tree-level correction in type II on $K3$, the second term is a one-loop correction which to our knowledge has not been computed independently yet, and the remaining terms originate from D3, D1, D(-1) branes wrapped on $K3$ [55]. It is worth noting that decompactification limits of the form $O(2k,8)\to O(2k-3,5)$ exist in principle for all CHL models listed in Table 1, however, they cannot be interpreted in terms of six-dimensional chiral string vacua, due to anomaly cancellation constraints.

5 Large radius expansion of exact $(\nabla\Phi)^4$ couplings

In this section, we study the expansion of the proposal (2.27) in the limit where the radius R of one circle in the internal space goes to infinity. We show that it reproduces the known F^4 and $\mathcal{R}^2$ couplings in $D=4$, along with an infinite series of $\mathcal{O}(e^{-R})$ corrections from 1/2-BPS dyons whose wordline winds around the circle, as well as an infinite series of $\mathcal{O}(e^{-R^2})$ corrections from Taub-NUT instantons. We start by analyzing the expansion of genus-one modular integrals (4.1b) and (4.1a) for arbitrary values of (p,q), in the limit near the cusp where $O(p,q)$ is broken to $O(2,1)\times O(p-2,q-2)$, so that the moduli space decomposes into

$$G_{p,q} \to \mathbb{R}^+ \times \left[\frac{SL(2)}{SO(2)} \times G_{p-2,q-2}\right] \ltimes \mathbb{R}^{2(p+q-4)} \times \mathbb{R} \tag{5.1}$$

As in the previous section, we first discuss the maximal rank case $N=1$, $p-q=16$, where the integrand is invariant under the full modular group, before dealing with the case of N prime. The reader uninterested by the details of the derivation may skip to §5.3, where we specialize to the values $(p,q)=(r-4,8)$ relevant for the $(\nabla\Phi)^4$ couplings in $D=3$, and interpret the various contributions arising in the decompactification limit to $D=4$.

5.1 $O(p,q)\to O(p-2,q-2)$ for even self-dual lattices

We first consider the case where the lattice $\Lambda_{p,q}$ is even self-dual and factorizes in the limit (5.1) as

$$\Lambda_{p,q} \to \Lambda_{p-2,q-2} \oplus I\!I_{2,2} \ . \tag{5.2}$$

In order to study the behavior of the modular integral (4.1a) in the limit (5.1), we denote by $R, S, \phi, a^{I,i}, \psi$ the coordinates for each factors in (5.1), where $i=1,2$ and $I=3,\dots,p+q-2$. The coordinate R (not to be confused with the one used in §4) parametrizes a one-parameter subgroup e^{RH_1} in $O(p,q)$, such that the action of the non-compact Cartan generator H_1 on the Lie algebra $\mathfrak{so}_{p,q}$ decomposes into

$$\mathfrak{so}_{p,q} \simeq \dots \oplus (\mathfrak{gl}_1 \oplus \mathfrak{so}_{p-2,q-2})^{(0)} \oplus (\mathbf{2}\otimes(\mathbf{p}+\mathbf{q}-\mathbf{4}))^{(1)} \oplus \mathbf{1}^{(2)}, \tag{5.3}$$

while (a^{iI},ψ) parametrize the unipotent subgroup obtained by exponentiating the grade 1 and 2 components in this decomposition. We parametrize the $SO(2)\backslash SL(2,\mathbb{R})$ coset representative ${v_\mu}^i$ and the symmetric $SL(2,\mathbb{R})$ element $M\equiv v^T v$ by the complex upper half-plane coordinate $S=S_1+\mathrm{i}S_2$

$${v_\mu}^i = \frac{1}{\sqrt{S_2}}\begin{pmatrix}1 & S_1\\ 0 & S_2\end{pmatrix} \ , \quad M^{ij} = \delta^{\mu\nu}{v_\mu}^i{v_\nu}^j = \frac{1}{S_2}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix} \ . \tag{5.4}$$

A generic charge vector $Q_{\mathcal{I}} \in \Lambda_{p,q} \simeq \mathbf{p}+\mathbf{q} \simeq \mathbf{2}^{(-1)} \oplus (\mathbf{p}+\mathbf{q}-\mathbf{4})^{(0)} \oplus \mathbf{2}^{(1)}$ decomposes into $Q=(m^i,\widetilde{Q}_I,n_j)$, where $(m^i,n_i)\in I\!I_{2,2}$ and $\widetilde{Q}_I\in\Lambda_{p-2,q-2}$ such that $Q^2=-2m^in_i+\widetilde{Q}^2$. The projectors defined by $Q_L\equiv p_L^{\mathcal{I}}Q_{\mathcal{I}}$ and $Q_R\equiv p_R^{\mathcal{I}}Q_{\mathcal{I}}$ decompose according to

$$\begin{aligned}
p^{\mathcal{I}}_{L,\mu}Q_{\mathcal{I}} =& \frac{v^{-1}_{i\mu}}{R\sqrt{2}}\left(m^i + a^i\cdot\widetilde{Q} + (\psi\epsilon^{ij}+\frac{1}{2}a^i\cdot a^j)n_j\right) - \frac{R}{\sqrt{2}}{v_\mu}^i n_i\\
p^{\mathcal{I}}_{L,\alpha}Q_{\mathcal{I}} =& \tilde{p}^I_{L,\alpha}(\widetilde{Q}_I + n_i a^i_I)\\
p^{\mathcal{I}}_{R,\mu}Q_{\mathcal{I}} =& \frac{v^{-1}_{i\mu}}{R\sqrt{2}}\left(m^i + a^i\cdot\widetilde{Q} + (\psi\epsilon^{ij}+\frac{1}{2}a^i\cdot a^j)n_j\right) + \frac{R}{\sqrt{2}}{v_\mu}^i n_i\\
p^{\mathcal{I}}_{R,\hat\alpha}Q_{\mathcal{I}} =& \tilde{p}^I_{R,\hat\alpha}(\widetilde{Q}_I + n_i a^i_I)
\end{aligned} \tag{5.5}$$

where $\tilde{p}^I_{L,\alpha}, \tilde{p}^I_{R,\hat\alpha}$ ($\alpha=3\dots p$, $\hat\alpha=3\dots q$) are orthogonal projectors in $G_{p-2,q-2}$ satisfying $\widetilde{Q}^2=\widetilde{Q}_L^2-\widetilde{Q}_R^2$.

In order to study the region $R \gg 1$ it is useful to perform a Poisson resummation on the momenta m^i along $I\!I_{2,2}$. Note that this analysis is in principle valid for a region containing $R > \sqrt{2}$. In the case of the scalar integral (4.1b), one obtains

$$\Gamma_{\Lambda_{p,q}} = R^2 \sum_{A\in\mathbb{Z}^{2\times 2}} e^{-\frac{\pi}{\tau_2}\frac{R^2}{S_2}\left|(1,S)A\binom{\tau}{1}\right|^2 - 2\pi\mathrm{i}\,(\psi+\mathrm{i}R^2)\det A}\,\Gamma_{\Lambda_{p-2,q-2}}\left[e^{2\pi\mathrm{i}\, m_i(\widetilde{Q}\cdot a^i + \frac{a^i\cdot a^j}{2}n_j)}\right] , \tag{5.6}$$

where $A = \binom{n_1\, m_1}{n_2\, m_2}$. In the case of (4.1a), we must distinguish whether the indices $abcd$ lie along the direction $1,2$ or along the directions α. Denoting by h the number of indices of the first kind, we get

$$\begin{aligned}\Gamma_{\Lambda_{p,q}}\left[e^{-\frac{\Delta}{8\pi\tau_2}}\left(\prod_{i=1}^{h}(Q_{L,\mu_i})Q_{L,\alpha_1}\dots Q_{L,\alpha_{4-h}}\right)\right] &= R^2 \sum_{A\in\mathbb{Z}^{2\times 2}}\left(\frac{R}{\mathrm{i}\sqrt{2}}\right)^h \\ &\times \exp\left(-\frac{\pi}{\tau_2}\frac{R^2}{S_2}\left|(1,S)A\tbinom{\tau}{1}\right|^2 - 2\pi\mathrm{i}\,T\det A\right)\prod_{k=1}^{h}\left[\tfrac{1}{\sqrt{S_2}}\begin{pmatrix}1 & S_1\\ 0 & S_2\end{pmatrix}A\tbinom{\bar\tau}{1}\right]_{\mu_k} \\ &\times \Gamma_{\Lambda_{p-2,q-2+n_i a^i}}\left[e^{-\frac{\Delta}{8\pi\tau_2}}\left[\widetilde{Q}_{L,\alpha_1}\dots\widetilde{Q}_{L,\alpha_{4-h}}\right]e^{2\pi\mathrm{i}\, m_i(\widetilde{Q}\cdot a^i - \frac{a^i\cdot a^j}{2}n_j)}\right]\end{aligned} \tag{5.7}$$

In this representation, modular invariance is manifest, since a transformation $\tau \to \frac{a\tau+b}{c\tau+d}$ can be compensated by a linear action $A \to A\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$, under which the last line of (5.7) transforms with weight $12-h$. We can therefore decompose the sum over A into various orbits under $SL(2,\mathbb{Z})$ and apply the unfolding trick to each orbit:

The trivial orbit $A = 0$ produces, up to a factor of R^2, the integrals (4.1) or for the lattice $\Lambda_{p-2,q-2}$, provided none of the indices $abcd$ lie along the direction 1 or 2,

$$F^{(p,q),0}_{\alpha\beta\gamma\delta} = R^2\, F^{(p-2,q-2)}_{\alpha\beta\gamma\delta} \ , \quad F^{(p,q),0}_{\mathrm{tr}} = R^2\, F^{(p-2,q-2)}_{\mathrm{tr}} \ , \tag{5.8}$$

while it vanishes otherwise (*i.e.* when $h > 0$).

Rank-one orbit: Matrices with $\det A = 0$ but $A \neq 0$ can be decomposed into $A = \begin{pmatrix}0 & j\\ 0 & p\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, where $(j,p) \in \mathbb{Z}^2 \smallsetminus (0,0)$ and $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_\infty\backslash SL(2,\mathbb{Z})$. As before the fundamental domain $SL(2,\mathbb{Z})\backslash\mathcal{H}$ can be unfolded to the strip $\mathcal{S} = \Gamma_\infty\backslash\mathcal{H} = \mathbb{R}^+_{\tau_2}\times(\mathbb{R}/\mathbb{Z})_{\tau_1}$ using (4.9), leading to

$$\begin{aligned}F^{(p,q),1}_{\mu_1\dots\mu_h\alpha_1\dots\alpha_{4-h}} &= R^2 \sum_{(j,p)}{}' \prod_{i=1}^{h}\left(\frac{R}{\mathrm{i}\sqrt{2}}\right)^h\left[\tfrac{1}{\sqrt{S_2}}\begin{pmatrix}1 & S_1\\ 0 & S_2\end{pmatrix}\begin{pmatrix}j\\ p\end{pmatrix}\right]_{\mu_i} \\ &\times \int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^{2+h}}\int_{\mathbb{R}/\mathbb{Z}}\mathrm{d}\tau_1\,\frac{e^{-\frac{\pi}{\tau_2}\frac{R^2}{S_2}|j+pS|^2}}{\Delta}\,\Gamma_{\Lambda_{p-2,q-2}}\left[\tilde{P}_{\alpha_1\dots\alpha_{4-h}}e^{2\pi\mathrm{i}\,(j\widetilde{Q}\cdot a^1 + p\widetilde{Q}\cdot a^2)}\right] , \\ F^{(p,q),1}_{\mathrm{tr}} &= R^2 \sum_{(j,p)}{}' \int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^{2+h}}\int_{\mathbb{R}/\mathbb{Z}}\mathrm{d}\tau_1 e^{-\frac{\pi}{\tau_2}\frac{R^2}{S_2}|j+pS|^2}\,\Gamma_{\Lambda_{p-2,q-2}}\left[e^{2\pi\mathrm{i}\,(j\widetilde{Q}\cdot a^1 + p\widetilde{Q}\cdot a^2)}\right] D^2\left(\frac{1}{\Delta}\right) ,\end{aligned} \tag{5.9}$$

for the tensor integral with $0 \le h \le 4$ indices along the large torus and its trace respectively. Inserting the Fourier expansion (4.14), the integral over τ_1 picks up the Fourier coefficient $c(m)$ with $m = -\frac{1}{2}\widetilde{Q}^2$. The remaining integral over τ_2 can be computed after expanding $\tilde{P}_{\alpha_1\dots\alpha_{4-h}} = \sum_{\ell=0}^{\lfloor\frac{4-h}{2}\rfloor} \tilde{P}^{(\ell)}_{\alpha_1\dots\alpha_{4-h}}\tau_2^{-\ell}$, where $\tilde{P}^{(\ell)}_{\alpha_1\dots\alpha_{4-h}}$ is a polynomial in $\widetilde{Q}$ of degree $4-h-2\ell \ge 0$, or vanishing otherwise. The contribution of $\widetilde{Q} = 0$ produces power-like terms in R^2,

$$\begin{aligned} F^{(p,q),1,0}_{\alpha\beta\gamma\delta} &= R^{q-6}\frac{3c(0)}{8\pi^2}\,\mathcal{E}^\star(\tfrac{8-q}{2},S)\,\delta_{(\alpha\beta}\delta_{\gamma\delta)},\\ F^{(p,q),1,0}_{\mu\nu\gamma\delta} &= R^{q-6}\frac{c(0)}{4\pi^2}\left[\tfrac{8-q}{4}\delta_{\alpha\beta}\delta_{\mu\nu} - \delta_{\alpha\beta}\mathcal{D}_{\mu\nu}\right]\mathcal{E}^\star(\tfrac{8-q}{2},S),\\ F^{(p,q),1,0}_{\mu\nu\rho\sigma} &= R^{q-6}\frac{c(0)}{2\pi^2}\left[\mathcal{D}^2_{\mu\nu\rho\sigma} - \tfrac{10-q}{2}\delta_{(\mu\nu}\mathcal{D}_{\rho\sigma)} + \left(\tfrac{8-q}{2}\right)\left(\tfrac{10-q}{2}\right)\tfrac{3}{8}\delta_{(\mu\nu}\delta_{\rho\sigma)}\right]\mathcal{E}^\star(\tfrac{8-q}{2},S) \end{aligned} \tag{5.10}$$

for the tensor integral, and

$$F^{(p,q),1,0}_{\rm tr} = R^{q-6}\,\frac{c(0)}{8\pi^2}\,(p-q+6)(p-q+8)\,\mathcal{E}^\star(\tfrac{8-q}{2},S)\;, \tag{5.11}$$

for its trace. Here, $\mathcal{E}^\star(s,S)$ is the completed weight 0 non-holomorphic Eisenstein series,

$$\mathcal{E}^\star(s,S) = \frac{1}{2}\pi^{-s}\,\Gamma(s)\sum_{(m,n)\in\mathbb{Z}^2}{}^{'}\frac{S_2^s}{|nS+m|^{2s}} \equiv \xi(2s)\,\mathcal{E}(s,S)\;, \tag{5.12}$$

$\mathcal{D}_{\mu\nu}$ is the traceless differential operator on $\frac{SL(2,\mathbb{R})}{SO(2)}$ defined in appendix D, and $\mathcal{D}^2_{\mu\nu\rho\sigma} = \mathcal{D}_{(\mu\nu}\mathcal{D}_{\rho\sigma)} - \frac{1}{4}\delta_{(\mu\nu}\delta_{\rho\sigma)}\mathcal{D}_{\tau\kappa}\mathcal{D}^{\tau\kappa}$ is the traceless operator of degree 2 in the symmetric representation. The equalities used to write (5.10) are detailed in (D.8), and similar expressions using non-holomorphic series of non-zero weight are given in (D.7). Recall that $\mathcal{E}^\star(s,S)$ is invariant under $s \mapsto 1-s$, and has simple poles at $s=0$ and $s=1$. As in the previous section, the pole at $q=6$ is subtracted by the regularization prescription mentioned below (3.32), while the pole at $q=8$ cancels against the pole from the trivial orbit contribution (5.8).

Contributions of non-zero vectors $\widetilde{Q} \in \Lambda_{p-2,q-2}$, on the other hand, lead to exponentially suppressed contributions, *e.g.* for the trace of the tensor integral

$$\begin{aligned} 2R^{\frac{q}{2}}\sum_{\widetilde{Q}\in\Lambda_{p-2,q-2}}{}^{'}\;\sum_{(j,p)}{}^{'} e^{2\pi i(j\widetilde{Q}\cdot a_1 + p\widetilde{Q}\cdot a_2)}\sum_{\ell=0}^{2}\frac{a_\ell}{R^{2\ell}}\left(-\tfrac{\widetilde{Q}^2}{2}\right)^{2-\ell} c\left(-\tfrac{\widetilde{Q}^2}{2}\right)\left(\frac{2\widetilde{Q}_R^2 S_2}{|j+pS|^2}\right)^{\frac{q-4-2\ell}{4}}&\\ \times K_{\frac{q-4}{2}-\ell}\left(2\pi\sqrt{\tfrac{2R^2}{S_2}}|j+pS||\widetilde{Q}_R|\right)& \end{aligned} \tag{5.13}$$

Defining $(Q,P) = (j,p)\widetilde{Q}$, we see that the Fourier expansion with respect to (a_1,a_2) has support on collinear vectors (Q,P) with $Q,P \in \Lambda_{p-2,q-2}$. Extracting the greatest common divisor of (j,p), we find that the Fourier coefficients with charge $Q'^i = (Q,P)$ and mass

$\mathcal{M}(Q,P)=\sqrt{2Q'^i_R Q'^j_R M_{ij}}$ defined in (2.13) are given by

$$\begin{aligned}
F^{(p,q),1,Q'}_{\alpha\beta\gamma\delta} &= 4R^{\frac{q}{2}}\,\bar{c}(Q'^i)\sum_{\ell=0}^{2}\frac{\mathcal{P}^{(\ell)}_{\alpha\beta\delta\gamma}(Q'^i,S)}{R^{2\ell}}\left[\frac{|Q+SP|^2}{S_2}\right]^{\frac{q-8}{2}}\frac{K_{\frac{q-4}{2}-\ell}(2\pi R\mathcal{M}(Q,P))}{\mathcal{M}(Q,P)^{\frac{q-4}{2}-\ell}}\\
F^{(p,q),1,Q'}_{\mu\alpha\beta\gamma} &= 4R^{\frac{q}{2}}\,\bar{c}(Q'^i)\sum_{\ell=0}^{1}\frac{\mathcal{P}^{(\ell)}_{\mu\alpha\beta\delta}(Q'^i,S)}{\mathrm{i}\sqrt{2}R^{2\ell}}\left[\frac{|Q+SP|^2}{S_2}\right]^{\frac{q-8}{2}}\frac{K_{\frac{q-6}{2}-\ell}(2\pi R\mathcal{M}(Q,P))}{\mathcal{M}(Q,P)^{\frac{q-6}{2}-\ell}}\\
&\vdots\\
F^{(p,q),1,Q'}_{\mu\nu\rho\sigma} &= 4R^{\frac{q}{2}}\,\bar{c}(Q'^i)\frac{\mathcal{P}^{(0)}_{\mu\nu\sigma\rho}(Q'^i,S)}{4}\left[\frac{|Q+SP|^2}{S_2}\right]^{\frac{q-8}{2}}\frac{K_{\frac{q-12}{2}}(2\pi R\mathcal{M}(Q,P))}{\mathcal{M}(Q,P)^{\frac{q-12}{2}}}
\end{aligned}\tag{5.14}$$

for the tensor integral, and

$$F^{(p,q),1,Q'}_{\mathrm{tr}}=4\,R^{\frac{q}{2}}\,\bar{c}(Q'^i)\sum_{\ell=0}^{2}\frac{a_\ell}{R^{2\ell}}\left[-\frac{\gcd(Q'^i\cdot Q'^j)}{2}\right]^{2-\ell}\frac{K_{\frac{q-4}{2}-\ell}(2\pi R\mathcal{M}(Q,P))}{\mathcal{M}(Q,P)^{\frac{q-4}{2}-\ell}}\tag{5.15}$$

for its trace. The covariantized versions of $P_{abcd}(Q)$ with respect to the torus' metric, $\mathcal{P}^{(\ell)}_{\alpha\beta\gamma\delta},\ldots,\mathcal{P}^{(\ell)}_{\mu\nu\sigma\rho}$ are given in appendix C. Finally the degeneracy is given by

$$\bar{c}(Q,P)=\sum_{(Q,P)/d\in\Lambda^{\oplus 2}_{p-2,q-2}}\left(\frac{d^2}{\gcd(Q^2,Q\cdot P,P^2)}\right)^{\frac{q-8}{2}}c\left(-\tfrac{\gcd(Q^2,Q\cdot P,P^2)}{2d^2}\right),\tag{5.16}$$

with support $(Q,P)\in\Lambda_{p-2,q-2}\oplus\Lambda_{p-2,q-2}$.

Rank-two orbit Finally, rank-two matrices can be uniquely decomposed as $A=\left(\begin{smallmatrix}k & j\\ 0 & p\end{smallmatrix}\right)\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ where $k>j\geq 0$ and $p\neq 0$ and $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in SL(2,\mathbb{Z})$. The matrices A can therefore be restricted to $A=\left(\begin{smallmatrix}k & j\\ 0 & p\end{smallmatrix}\right)$, provided the integral is extended to the double cover of the upper half-plane $\mathcal{H}$. This leads to

$$\begin{aligned}
F^{(p,q),1}_{\mu_1\ldots\mu_h\alpha_1\ldots\alpha_{4-h}} &= 2R^2\sum_{\substack{k>j\geq 0\\ p\neq 0}}\left(\frac{R}{\mathrm{i}\sqrt{2}}\right)^h e^{-2\pi\mathrm{i}kp(\psi+\mathrm{i}R^2)}\int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^{2+h}}\int_{\mathbb{R}}\mathrm{d}\tau_1\,\frac{e^{-\frac{\pi}{\tau_2}\frac{R^2}{S_2}|k\tau+j+pS|^2}}{\Delta}\\
&\times\prod_{l=1}^{h}\left[\tfrac{1}{\sqrt{S_2}}\begin{pmatrix}1 & S_1\\ 0 & S_2\end{pmatrix}\begin{pmatrix}k\bar{\tau}+j\\ p\end{pmatrix}\right]_{\mu_l}\Gamma_{\Lambda_{p-2,q-2}+n_ia^i}\left[P_{\alpha_1\ldots\alpha_{4-h}}\,e^{2\pi\mathrm{i}\left(j(\widetilde{Q}-\frac12ka_1)\cdot a_1+p(\widetilde{Q}-\frac12ka_1)\cdot a_2\right)}\right]
\end{aligned}\tag{5.17}$$

for the tensor integral, and to

$$\begin{aligned}
F^{(p,q),1}_{\mathrm{tr}} &= 2R^2\sum_{\substack{k>j\geq 0\\ p\neq 0}}e^{-2\pi\mathrm{i}kp(\psi+\mathrm{i}R^2)}\int_{\mathbb{R}^+}\frac{\mathrm{d}\tau_2}{\tau_2^2}\int_{\mathbb{R}}\mathrm{d}\tau_1\,e^{-\frac{\pi}{\tau_2}\frac{R^2}{S_2}|k\tau+j+pS|^2}\\
&\times\Gamma_{\Lambda_{p-2,q-2}+n_ia^i}\left[e^{2\pi\mathrm{i}\left(j(\widetilde{Q}-\frac12ka_1)\cdot a_1+p(\widetilde{Q}-\frac12ka_1)\cdot a_2\right)}\right]D^2\left(\frac{1}{\Delta}\right)
\end{aligned}\tag{5.18}$$

for its trace.

Inserting the Fourier expansion (4.14), the integral over τ_1 is Gaussian while the integral over τ_2 is of Bessel type. The sum over $0 \leq j < k$ enforces a Kronecker delta function modulo k,

$$\sum_{j=0}^{k-1} \exp\left[2\pi\mathrm{i}\,\frac{j}{k}\left(\frac{\widetilde{Q}^2}{2}+m\right)\right] = \begin{cases} k & \text{if } \frac{\widetilde{Q}^2}{2}+m = lk, \quad l\in\mathbb{Z}, \\ 0 & \text{otherwise} \end{cases} \tag{5.19}$$

Relabelling the charges as $p\widetilde{Q} \to P$, $kp \to -M_1$ and $lp \to -M_2$, and defining $D = -\frac{P^2}{2} + M_1 M_2$ one obtains, for the trace of the tensor integral,

$$F_{\mathrm{tr}}^{(p,q),2} = \sum_{\substack{M_1\neq 0, M_2 \\ P\in\Lambda_{p-2,q-2}}} F_{\mathrm{tr}}^{(p,q),2,M_1}\left(P - M_1 a_1, M_2 - a_1\cdot P + \tfrac{1}{2}(a_1\cdot a_1)M_1\right) \\ \times e^{2\pi\mathrm{i}(P\cdot a_2 + M_1(\psi - \frac{1}{2}a_1\cdot a_2) + (M_2 - a_1\cdot P + \frac{1}{2}(a_1\cdot a_1)M_1)S_1)} \tag{5.20}$$

where $F_{\mathrm{tr}}^{(p,q),2,M_1}$ is the non-Abelian Fourier coefficient,

$$F_{\mathrm{tr}}^{(p,q),2,M_1}(P,M_2) = 4(R^2S_2)^{\frac{q-1}{2}}\,\bar{c}(M_1,M_2,P)\sum_{\ell=0}^{2}\frac{a_\ell\, D^{2-\ell}}{(R^2S_2)^\ell}\left(\frac{2\pi}{S_{\mathrm{cl}}}\right)^{\frac{q-3}{2}-\ell} K_{\frac{q-3}{2}-\ell}(S_{\mathrm{cl}})\ , \tag{5.21}$$

S_{cl} is the classical action

$$S_{\mathrm{cl}}(M_1,M_2,P) = 2\pi\sqrt{(R^2M_1+S_2M_2)^2 + 2R^2S_2P_R^2}\ , \tag{5.22}$$

and $\bar{c}(M_1,M_2,P)$ the summation measure

$$\bar{c}(M_1,M_2,P) = \sum_{\substack{d|(M_1,M_2) \\ P/d\in\Lambda_{p-2,q-2}}} c\left(\tfrac{D}{d^2}\right)\, d^{q-7}\ . \tag{5.23}$$

It is worth noting that (5.20) is the general expansion of a function of (S_1, a_1, a_2, ψ) invariant under discrete shifts $T_{b,\epsilon_1,\epsilon_2,\kappa}$ acting as

$$(S_1,a_1,a_2,\psi) \mapsto \left(S_1 + b, a_1+\epsilon_1, a_2+\epsilon_2+ba_1, \psi+\kappa+\tfrac{1}{2}[\epsilon_2(a_1+\epsilon_1) - \epsilon_1(a_2+ba_1)]\right) \tag{5.24}$$

with $b,\kappa\in\mathbb{Z}$ and $\epsilon_1,\epsilon_2\in\mathbb{Z}^{p-2,q-2}$. Invariance under $T_{b,0,\epsilon_2,\kappa}$ is manifest, while invariance under $T_{0,\epsilon_1,0,0}$ is realized by shifting $P\mapsto P+M_1\epsilon_1$, $M_2\mapsto M_2+\epsilon_1 P + \frac{1}{2}M_1\epsilon_1^2$, which leaves D and $\tilde{M}_2 = M_2 - a_1\cdot P + \frac{1}{2}(a_1\cdot a_1)M_1$ invariant. It is worth noting that in the special case $p=2$, P_R^2 vanishes identically so (5.22) simplifies to $S_{\mathrm{cl}} = 2\pi|R^2M_1+S_2M_2|$.

Similarly, for the tensor integral, we get

$$\begin{aligned}
F_{\alpha\beta\gamma\delta}^{(p,q),2,M_1}(P,M_2) &= 4\,(R^2S_2)^{\frac{q-2}{2}}\,\bar{c}(M_1,M_2,P)\sum_{\ell=0}^{2}\frac{\tilde{P}^{(\ell)}_{\alpha\beta\gamma\delta}(P)}{(R^2S_2)^\ell}\left(\frac{2\pi}{S_{\mathrm{cl}}}\right)^{\frac{q-3}{2}-\ell} K_{\frac{q-3}{2}-\ell}(S_{\mathrm{cl}}) \\
F_{2\alpha\beta\gamma}^{(p,q),2,M_1}(P,M_2) &= 4\,(R^2S_2)^{\frac{q-2}{2}}\,\bar{c}(M_1,M_2,P)\sum_{\ell=0}^{1}\frac{\tilde{P}^{(\ell)}_{\alpha\beta\gamma}(P)}{\mathrm{i}\sqrt{2}(R^2S_2)^{\ell+\frac{1}{2}}}\left(\frac{2\pi}{S_{\mathrm{cl}}}\right)^{\frac{q-5}{2}-\ell} K_{\frac{q-5}{2}-\ell}(S_{\mathrm{cl}}) \\
&\ \ \vdots \\
F_{2222}^{(p,q),2,M_1}(P,M_2) &= 4\,(R^2S_2)^{\frac{q-2}{2}}\,\bar{c}(M_1,M_2,P)\frac{\tilde{P}^{(0)}}{4(R^2S_2)^2}\left(\frac{2\pi}{S_{\mathrm{cl}}}\right)^{\frac{q-11}{2}} K_{\frac{q-11}{2}}(S_{\mathrm{cl}}),
\end{aligned} \tag{5.25}$$

where we restricted to the cases $\mu, \nu, \ldots = 2$ for simplicity.

5.2 Extension to $\mathbb{Z}_N$ CHL orbifolds

The degeneration limit (5.1) of the modular integrals (4.1) for $\mathbb{Z}_N$ CHL models with $N = 2, 3, 5, 7$ can be treated similarly by applying the orbit method. In (4.1), Δ_k is the cusp form of weight $k = \frac{24}{N+1}$ defined in (2.4), and $\Gamma_{\Lambda_{p,q}}[P_{abcd}]$ is the partition function with insertion of P_{abcd} for a lattice

$$\Lambda_{p,q} = \Lambda_{p-2,q-2} \oplus II_{1,1} \oplus II_{1,1}[N] \ , \tag{5.26}$$

where $\Lambda_{p-2,q-2}$ is a lattice of level N. The lattice $II_{1,1} \oplus II_{1,1}[N]$ is obtained from the usual unimodular lattice $II_{2,2}$ by restricting the windings and momenta to $(n_1, n_2, m_1, m_2) \in \mathbb{Z} \oplus N\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, hence breaking the automorphism group $O(2, 2, \mathbb{Z})$ to $\sigma_{S\leftrightarrow T} \ltimes [\Gamma_0(N) \times \Gamma_0(N)]$. After Poisson resummation on m_2, Eq. (5.6) and (5.7) continue to hold, except for the fact that n_2 is restricted to run over $N\mathbb{Z}$. The sum over $A = \left(\begin{smallmatrix} n_1 & m_1 \\ n_2 & m_2 \end{smallmatrix}\right)$ can then be decomposed into orbits of $\Gamma_0(N)$: [10]

Trivial orbit The contribution of $A = 0$ reduces, up to a factor of R^2, to the integrals (4.1) for the lattice $\Lambda_{p-2,q-2}$,

$$F^{(p,q),0}_{\alpha\beta\gamma\delta} = R^2 \, F^{(p-2,q-2)}_{\alpha\beta\gamma\delta} \ , \qquad F^{(p,q),0}_{\mathrm{tr}} = R^2 \, F^{(p-2,q-2)}_{\mathrm{tr}} \ , \tag{5.27}$$

Rank-one orbits Matrices A of rank-one fall into two different classes of orbits under $\Gamma_0(N)$. For simplicity, let us first consider the case where $(n_2, m_2) \neq (0, 0)$, and denote $(m_2, n_2) = p(n_2', m_2')$, with $p = \gcd(n_2, m_2)$:

- Matrices with $n_2' = 0 \bmod N$, as they are required to be rank-one, can be decomposed as $\left(\begin{smallmatrix} n_1 & m_1 \\ n_2 & m_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & j \\ 0 & p \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $(j, p) \in \mathbb{Z}^2 \smallsetminus \{(0, 0)\}$, $p \neq 0$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(N)$. For this class of orbit, one can thus unfold directly the domain $\Gamma_0(N)\backslash\mathcal{H}$ into the unit strip $\mathcal{S} = \Gamma_\infty\backslash\mathcal{H} = \mathbb{R}^+_{\tau_2} \times (\mathbb{R}/\mathbb{Z})_{\tau_1}$.
- Matrices with $n_2' \neq 0 \bmod N$ can be decomposed as $\left(\begin{smallmatrix} n_1 & m_1 \\ n_2 & m_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} j & 0 \\ p & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $(j, p) \in \mathbb{Z} \oplus N\mathbb{Z} \smallsetminus \{(0, 0)\}$, $p \neq 0$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S\,\Gamma_{\infty,N}\,S^{-1}\backslash\Gamma_0(N)$, where $\Gamma_{\infty,N} = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right), n \in N\mathbb{Z}\}$. One can then unfold the fundamental domain $\Gamma_0(N)\backslash\mathcal{H}$ into $S\,\Gamma_{\infty,N}\,S^{-1}\backslash\mathcal{H}$, and change variable $\tau \to -1/\tau$ as in the weak coupling case (4.24) to recover the integration domain $\mathcal{S}_N = \Gamma_{\infty,N}\backslash\mathcal{H} = \mathbb{R}^+_{\tau_2} \times (\mathbb{R}/N\mathbb{Z})_{\tau_1}$, the width-$N$ strip.

The remaining contributions A with $(n_2, m_2) = (0, 0)$ belong to the two classes of orbits above. Let $(n_1, m_1) = j(n_1', m_1')$, where $j = \gcd(n_1, m_1)$ and $j \in \mathbb{Z}$, then contributions with $n_1' = 0 \bmod N$ correspond to the cases $(j, p) = (j, 0)$ in the first class above; contributions with $n_1' \neq 0 \bmod N$ correspond to $(j, p) = (j, 0)$ in the second class above.

After unfolding and changing variable, the result for the simplest component $F^{(p,q),1}_{\alpha\beta\gamma\delta}$ reads

[10] Note that the subsequent analysis is valid in the region of the moduli space where $NR^2 > 2S_2$

(similarly to (4.24))

$$F^{(p,q),1}_{\alpha\beta\gamma\delta} = R^2 \int_{\mathbb{R}^+} \frac{\mathrm{d}\tau_2}{\tau_2^2} \int_{\mathbb{R}/\mathbb{Z}} \mathrm{d}\tau_1 \, \frac{1}{\Delta_k(\tau)} \sum_{(j,p)\in\mathbb{Z}^2}{}' e^{-\frac{\pi R^2}{\tau_2 S_2}|j+pS|^2} \Gamma_{\Lambda_{p-2,q-2}} \left[e^{2\pi \mathrm{i}(j\widetilde{Q}\cdot a_1 + p\widetilde{Q})\cdot a_2} P_{\alpha\beta\gamma\delta} \right]$$
$$+ R^2 \int_{\mathbb{R}^+} \frac{\mathrm{d}\tau_2}{\tau_2^2} \int_{\mathbb{R}/N\mathbb{Z}} \mathrm{d}\tau_1 \, \frac{1}{\Delta_k(\tau/N)} \frac{\upsilon}{N} \sum_{\substack{(j,p)\in\mathbb{Z}^2 \\ p=0 \bmod N}}{}' e^{-\frac{\pi R^2}{\tau_2 S_2}|j+pS|^2} \Gamma_{\Lambda^*_{p-2,q-2}} \left[e^{2\pi \mathrm{i}(j\widetilde{Q}\cdot a_1 + p\widetilde{Q})\cdot a_2} P_{\alpha\beta\gamma\delta} \right] , \tag{5.28}$$

where $\Gamma_{\Lambda^*_{p-2,q-2}}$ is the partition function of the dual lattice $\Lambda^*_{p-2,q-2}$ and where $\upsilon = N^{k/2+1}|\Lambda^*_{p-2,q-2}/\Lambda_{p-2,q-2}|^{-1/2}$ (which reduces to $\upsilon = N^{1-\delta_{q,8}}$ for $q \le 8$ in the cases of interest). The contributions from $\widetilde{Q} = 0$ thus give

$$\begin{aligned}
F^{(p,q),1,0}_{\alpha\beta\gamma\delta} &= R^{q-6} \frac{3(2c_k(0))}{8\pi^2} \frac{1}{2} \left(\mathcal{E}^{\star}_{\frac{8-q}{2}}(S) + \upsilon N^{\frac{q-8}{2}} \mathcal{E}^{\star}_{\frac{8-q}{2}}(NS) \right) \delta_{(\alpha\beta}\delta_{\gamma\delta)}, \\
F^{(p,q),1,0}_{\mu\nu\gamma\delta} &= R^{q-6} \frac{2c_k(0)}{4\pi^2} \left[\tfrac{8-q}{4} \delta_{\alpha\beta}\delta_{\mu\nu} - \delta_{\alpha\beta}\mathcal{D}_{\mu\nu} \right] \frac{1}{2} \left(\mathcal{E}^{\star}_{\frac{8-q}{2}}(S) + \upsilon N^{\frac{q-8}{2}} \mathcal{E}^{\star}_{\frac{8-q}{2}}(NS) \right), \\
F^{(p,q),1,0}_{\mu\nu\rho\sigma} &= R^{q-6} \frac{2c_k(0)}{2\pi^2} \\
&\quad \times \left[\mathcal{D}^2_{\mu\nu\rho\sigma} - \tfrac{10-q}{2} \delta_{(\mu\nu}\mathcal{D}_{\rho\sigma)} + \left(\tfrac{8-q}{2}\right)\left(\tfrac{10-q}{2}\right) \tfrac{3}{8} \delta_{(\mu\nu}\delta_{\rho\sigma)} \right] \frac{1}{2} \left(\mathcal{E}^{\star}_{\frac{8-q}{2}}(S) + \upsilon N^{\frac{q-8}{2}} \mathcal{E}^{\star}_{\frac{8-q}{2}}(NS) \right),
\end{aligned} \tag{5.29}$$

for the tensor integral, and

$$F^{(p,q),1,0}_{\mathrm{tr}} = R^{q-6}(p-q+6)(p-q+8) \frac{2c_k(0)}{8\pi^2} \frac{1}{2} \left(\mathcal{E}^{\star}_{\frac{8-q}{2}}(S) + \upsilon N^{\frac{q-8}{2}} \mathcal{E}^{\star}_{\frac{8-q}{2}}(NS) \right), \tag{5.30}$$

for its trace. Recall $c_k(0) = \frac{24}{N+1} = k$ is the zero mode of $1/\Delta_k = \sum_m c_k(m) q^m$. As in (5.11) and (5.10), the pole at $q = 6$ is minimally subtracted by the regularization prescription mentioned below (3.32), while the pole at $q = 8$ cancels against the pole from the zero orbit contribution (5.27).

The contributions with $\widetilde{Q} \neq 0$ are exponentially suppressed at large R, and have similar Fourier coefficients as in the full rank case (5.14), except for a different summation measure. Let us label the electromagnetic charges by $(Q,P) = (j,p)\widetilde{Q} = (j',p')\hat{Q}$ where (j',p') are coprime integers. It will be useful to classify all possible rank-one charges (Q,P) in orbits of the S-duality group $\Gamma_0(N)$ acting as $\binom{Q}{P} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{Q}{P}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

- Charges (Q,P) such that $p' = 0 \bmod N$ are in the same orbit as purely electric charges $(\hat{Q},0)$. Their Fourier coefficient gets contributions from both terms in (5.28) with $d = \gcd(j,p)$ and $\frac{\hat{Q}}{d} = \widetilde{Q} \in \Lambda_{p-2,q-2}$ in the first case and $\frac{\hat{Q}}{d} = \widetilde{Q} \in \Lambda^*_{p-2,q-2}$ in the second, such that they are weighted by the measure

$$\bar{c}_k(Q,P) = \sum_{\substack{d\ge1 \\ \hat{Q}/d\in\Lambda_{p-2,q-2}}} c_k\Big(-\frac{\hat{Q}^2}{2d^2}\Big) \left(\frac{d^2}{\hat{Q}^2}\right)^{\frac{q-8}{2}} + \upsilon \sum_{\substack{d\ge1 \\ \hat{Q}/d\in\Lambda^*_{p-2,q-2}}} c_k\Big(-\frac{N\hat{Q}^2}{2d^2}\Big) \left(\frac{d^2}{N\hat{Q}^2}\right)^{\frac{q-8}{2}} , \tag{5.31}$$

where the first contribution has support $Q \in \Lambda_{p-2,q-2} \subset \Lambda^*_{p-2,q-2}$, while the second has support on $Q \in \Lambda^*_{p-2,q-2}$. Notice that the latter is matched against $1/\Delta_k(\tau/N)$, which explains the N factor in the argument of c_k.

- Charges (Q,P) such that $p' \neq 0 \bmod N$ are in the same orbit as purely magnetic charges $(0,\hat{P})$, where we relabelled $\hat{Q}$ as $\hat{P}$ for convenience. Their Fourier coefficient gets contributions from both terms in (5.28) with $d = \gcd(j,p)$ and $\frac{\hat{P}}{d} = \tilde{Q} \in \Lambda_{p-2,q-2}$ in the first case and $Nd = \gcd(j,p)$ (because $j = 0 \bmod N$) and $\frac{\hat{P}}{Nd} = \tilde{Q} \in \Lambda^*_{p-2,q-2}$ in the second, such that they are weighted by the measure

$$\bar{c}_k(Q,P) = \sum_{\substack{d\geq 1\\ \hat{P}/d\,\in\Lambda_{p-2,q-2}}} c_k\Big(-\frac{\hat{P}^2}{2d^2}\Big)\left(\frac{d^2}{\hat{P}^2}\right)^{\frac{q-8}{2}} + \upsilon \sum_{\substack{d\geq 1\\ \hat{P}/d\,\in N\Lambda^*_{p-2,q-2}}} c_k\Big(-\frac{\hat{P}^2}{2Nd^2}\Big)\left(\frac{Nd^2}{\hat{P}^2}\right)^{\frac{q-8}{2}}, \quad (5.32)$$

where the first contribution has support $P \in \Lambda_{p-2,q-2}$, while the second has $P \in N\Lambda^*_{p-2,q-2} \subset \Lambda_{p-2,q-2}$. In the latter contribution, one N factor in the argument of c_k comes from the matching condition, and two N factors in its denominator come from all divisors d being originally multiples of N.

Rank-two orbit For the rank-two matrices A, the two classes of orbits are similarly given by studying $(n_2,m_2) = p(n'_2,m'_2)$, where $p = \gcd(n_2,m_2)$.

- Contributions for which $(n'_2,m'_2) = (0,1) \bmod N$ can be decomposed as $A = \left(\begin{smallmatrix} k & j\\ 0 & p\end{smallmatrix}\right)\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, $0 \leq j < k$, $p \in \mathbb{Z}\smallsetminus\{0\}$ and $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$, where its representative has trivial stabilizer. For this first class of orbits, the fundamental domain can be unfolded to the full upper half-plane $\mathcal{H} = \mathbb{R}^+_{\tau_2} \times \mathbb{R}_{\tau_1}$.

- Contributions for which $(n'_2,m'_2) = (1,0) \bmod N$ can have $A = \left(\begin{smallmatrix} j & k\\ p & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, $0 \leq j < Nk$, $p \in N\mathbb{Z} \smallsetminus \{0\}$ and $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$, where the representative has trivial stabilizer. For this second class of orbits, the fundamental domain can be unfolded to $\mathcal{H} = \mathbb{R}^+_{\tau_2} \times \mathbb{R}_{\tau_1}$ as well and the integrand can be brought back to the standard lattice sum representation using a transformation $\tau \to -1/\tau$, in the spirit of (5.28).

Both classes of contributions lead to the same type of non-Abelian Fourier coefficient as in the unorbifolded case (5.21) and (5.25), except for a different summation measure $\bar{c}(M_1,M_2,P)$. The first class have support $(M_1,M_2,P) \in \mathbb{Z}\oplus\mathbb{Z}\oplus\Lambda_{p-2,q-2}$, whereas the second class have support $(M_1,M_2,P) \in N\mathbb{Z}\oplus N\mathbb{Z}\oplus N\Lambda^*_{p-2,q-2}$. In fine the summation measure reads

$$\bar{c}_k(M_1,M_2,P) = \sum_{\substack{d|(M_1,M_2)\\ P/d\in\Lambda_{p-2,q-2}}} c_k\Big(\frac{D}{d^2}\Big)\, d^{q-7} + \upsilon \sum_{\substack{Nd|(M_1,M_2)\\ P/d\in N\Lambda^*_{p-2,q-2}}} c_k\Big(\frac{D}{Nd^2}\Big)\,(Nd)^{q-7}, \quad (5.33)$$

where we recall that $D = -\frac{1}{2}P^2 + M_1M_2$. For the second class of orbits, one factor of N in the argument of c_k comes from the matching condition, and two factors of $1/N$ come from the fact that all divisors were originally multiples of N.

5.3 Large radius limit and BPS dyons

Specializing to $(p,q) = (2k,8) = (r-4,8)$, and choosing $\Lambda_{p-2,q-2} = \Lambda_m$, the degeneration studied in this section corresponds to the limit of the exact $(\nabla\Phi)^4$ amplitude in heterotic string on T^7 in the limit where a circle inside T^7, orthogonal to the $\mathbb{Z}_N$ action, decompactifies. The coordinate R is identified as the radius of the large circle in units of the four-dimensional Planck length $l_P = g_4 l_H$. The contributions from the various orbits discussed in §5.1 and §5.2 are then interpreted as follows:

5.3.1 Effective action in $D = 4$

In the large R limit, $F^{(2k,8)}_{\alpha\beta\gamma\delta}$ should reproduce the exact four-dimensional F^4 coupling, up to exponentially suppressed corrections. As already mentioned below (5.10) and (5.29), the contribution of the vector $\widetilde{Q} = 0$ to the rank-one orbit has a pole at $q = 8$. Using the regularisation (3.32), that formally sets $q = 8 + 2\epsilon$, one obtains

$$\begin{aligned} F^{(2k,8),1,0}_{\alpha\beta\gamma\delta}(\epsilon) &= R^{2+2\epsilon}\frac{3(2k)}{(4\pi)^2}\left(\mathcal{E}^\star_{-\epsilon}(S) + N^\epsilon \mathcal{E}^\star_{-\epsilon}(NS)\right)\delta_{(\alpha\beta}\delta_{\gamma\delta)} \\ &= R^2\frac{3}{2(2\pi)^2}\left(\frac{k}{\epsilon} - \log(S_2^k|\Delta_k(S)|^2) + k\left(\log\left(\frac{R^2}{4\pi}\right) - \gamma\right)\right)\delta_{(\alpha\beta}\delta_{\gamma\delta)} + \mathcal{O}(\epsilon)\ , \end{aligned} \tag{5.34}$$

However, this pole cancels against the pole (4.38) in the trivial zero-orbit contribution (5.8), (5.27), leaving the finite result

$$F^{(2k,8)}_{\alpha\beta\gamma\delta} = R^2\left(-\frac{3}{2(2\pi)^2}\left(\log(S_2^k|\Delta_k(S)|^4) - 2k\log R\right)\delta_{(\alpha\beta}\delta_{\gamma\delta)} + \hat{F}^{(2k-2,6)}_{\alpha\beta\gamma\delta}(\Phi)\right) + \dots \tag{5.35}$$

where $\hat{F}^{(2k-2,6)}_{\alpha\beta\gamma\delta}$ is the renormalized 1-loop coupling, up to an irrelevant additive constant, and the dots denote exponentially suppressed terms.

Thus, the conjectural formula (2.27) for the exact $(\nabla\Phi)^4$ coupling in $D = 4$ predicts that the exact F^4 coupling in four dimensions should be given by

$$-\frac{3}{8\pi^2}\log(S_2^k|\Delta_k(S)|^2)\delta_{(ab}\delta_{cd)} + F^{(2k-2,6)}_{abcd}(\Phi)\ , \tag{5.36}$$

where for convenience we renamed the indices $\alpha, \beta, \dots$ into $a, b, \dots$ running from 1 to $2k-2$. Indeed, it is known that half-maximal supersymmetry in $D = 4$ allows for two types of supersymmetry invariants with four derivatives: the first one is determined in terms of a holomorphic function of S, the second depends on the $G_{2k-2,6}$ moduli only, as described in (3.21), and both contribute to F^4 couplings [56]. The first term in (5.36) corresponds the first invariant, which also includes the $\mathcal{R}^2$ coupling (2.3), while the second was considered in [55], it is by construction exact at 1-loop and includes a four-derivative scalar couplings studied in [57].

The relative coefficient of the two invariants in (5.36) is in fact fixed by unitarity. Indeed, the logarithmic dependence of the one-loop amplitude with respect to the Mandelstam variables ($s_1 = s$, $s_2 = t$, $s_3 = u$) is determined by the 1-loop divergence of the four-photon supergravity amplitude [41]. Because the genus-one string theory amplitude $F^{(2k-2,6)}_{abcd}(\Phi, s_i)$ is finite in the ultra-violet, the corresponding supergravity amplitude pole in dimensional regularisation $D=4-2\epsilon$ cancels by construction the pole of the coupling $F^{(2k-2,6)}_{abcd}(\Phi,\epsilon)$ regularised according to (3.32) (corresponding formally to $q=6+2\epsilon$). Thus, in the low energy limit $-\ell_s^{\,2}s_i\ll 1$ [11]

$$\begin{aligned} F^{(2k-2,6)}_{abcd}(\Phi,s_i) &\sim F^{(2k-2,6)}_{abcd}(\Phi,\epsilon)+\frac{3(2k)}{(4\pi)^2}\Big(\frac{1}{\epsilon}-\frac{1}{3}\sum_{i=1}^{3}\log(-\ell_s^{\,2}s_i)\Big)\delta_{(ab}\delta_{cd)} \\ &\sim \hat F^{(2k-2,6)}_{abcd}(\Phi)-\frac{3}{8\pi^2}\log(S_2^{\,k})\delta_{(ab}\delta_{cd)}-\frac{2k}{(4\pi)^2}\sum_{i=1}^{3}\log(-\ell_P^{\,2}s_i)\delta_{(ab}\delta_{cd)}\ , \end{aligned} \tag{5.37}$$

up to a fixed constant, where we used the relation $\frac{S_2}{2\pi}\ell_P^{\,2}=\ell_s^{\,2}$ between Planck length and string length. Therefore, the relative coefficient of the two invariants in (5.36) is indeed such that the logarithm of S_2 in the coupling disappears in string frame, consistently with the fact that string amplitudes depend analytically on the string coupling constant when formulated in string frame [58].

The overal normalisation of the 4-photon amplitude can be determined from the 1-loop divergence as [41, 59] (with $t_8f^4=f_{\mu\nu}f^{\nu\sigma}f_{\sigma\rho}f^{\rho\mu}-\frac{1}{4}(f_{\mu\nu}f^{\mu\nu})^2$)

$$A_4(S,\Phi,s_i)=\frac{\kappa^4}{8}\bigg(\frac{3}{8\pi^2}\log(S_2^{\,k}|\Delta_k(S)|^2)\delta_{(ab}\delta_{cd)}-F^{(2k-2,6)}_{abcd}(\Phi,s_i)\bigg)t_8F^aF^bF^cF^d\ . \tag{5.38}$$

More precisely, the 1PI effective action includes the local terms

$$\begin{aligned} S_4 \;=\; \int d^4x\sqrt{-g}\bigg(&\frac{1}{2\kappa^2}\mathcal{R}-\frac{S_2}{32\pi}(F^a_{\mu\nu}F^{\mu\nu}_a+F^{\hat a}_{\mu\nu}F^{\mu\nu}_{\hat a})+\frac{S_1}{64\pi\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma}(F^a_{\mu\nu}F_{\rho\sigma\,a}-F^{\hat a}_{\mu\nu}F_{\rho\sigma\,\hat a}) \\ &+\frac{\kappa^4}{8}\Big(\frac{3}{8\pi^2}\log(S_2^{\,k}|\Delta_k(S)|^2)\delta_{(ab}\delta_{cd)}-\hat F^{(2k-2,6)}_{abcd}(\Phi)\Big)t^{\mu\nu\rho\sigma\kappa\lambda\vartheta\tau}\Big(\frac{S_2}{8\pi}\Big)^2F^a_{\mu\nu}F^b_{\rho\sigma}F^c_{\kappa\lambda}F^d_{\vartheta\tau} \\ &-\frac{1}{(8\pi)^2}\log(S_2^{\,k}|\Delta_k(S)|^2)(\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}-4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}+\mathcal{R}^2) \\ &-\frac{\kappa^2}{(8\pi)^2}\mathcal{R}^{\mu\nu\rho\sigma}\Big(\mathcal{D}\log(S_2^{\,k}|\Delta_k(S)|^2)\frac{S_2}{8\pi}F^{\hat a-}_{\mu\nu}F^-_{\rho\sigma\hat a}+\overline{\mathcal{D}}\log(S_2^{\,k}|\Delta_k(S)|^2)\frac{S_2}{8\pi}F^{\hat a+}_{\mu\nu}F^+_{\rho\sigma\hat a}\Big) \\ &-\frac{\kappa^4}{(8\pi)^2}\mathcal{D}^2\log(S_2^{\,k}|\Delta_k(S)|^2)\Big(\frac{S_2}{8\pi}\Big)^2\Big(2F^{\hat a-}_{\mu\nu}F^-_{\rho\sigma\hat a}F^{\mu\nu}_{\hat b-}F^{\rho\sigma\hat b}_-+F^{\hat a-}_{\mu\nu}F^{\mu\nu}_{\hat a-}F^{\rho\sigma}_{\hat b-}F^{\hat b-}_{\rho\sigma}\Big) \\ &-\frac{\kappa^4}{(8\pi)^2}\overline{\mathcal{D}}^2\log(S_2^{\,k}|\Delta_k(S)|^2)\Big(\frac{S_2}{8\pi}\Big)^2\Big(2F^{\hat a+}_{\mu\nu}F^+_{\rho\sigma\hat a}F^{\mu\nu}_{\hat b+}F^{\rho\sigma\hat b}_++F^{\hat a+}_{\mu\nu}F^{\mu\nu}_{\hat a+}F^{\rho\sigma}_{\hat b+}F^{\hat b+}_{\rho\sigma}\Big)+\dots\bigg)\ , \end{aligned} \tag{5.39}$$

which includes in particular the exact $\mathcal{R}^2$ coupling (2.3). The components of (5.10), (5.29) with μ,ν indices correspond to scalar field parametrizing the circle radius R, the scalar field ψ dual to the Kaluza–Klein vector, and the axiodilaton scalar field S in four dimensions. The components involving the derivative of the function of S depend on the complex (anti)selfdual field $F^{\hat a\pm}_{\mu\nu}\equiv\frac{1}{2}F^{\hat a}_{\mu\nu}\pm\frac{i}{4\sqrt{-g}}\varepsilon_{\mu\nu}{}^{\rho\sigma}F^{\hat a}_{\rho\sigma}$, with the covariant derivative $\mathcal{D}$ defined as in Appendix D with $\mathcal{D}\equiv\mathcal{D}_0$ and $\mathcal{D}^2\equiv\mathcal{D}_2\mathcal{D}_0$.

Let us now discuss the decompactification limit of the 1PI effective action to ten dimensions, focussing for simplicity on the maximal rank case where the lattice decomposes as

$$\Lambda_{22,6}=D_{16}\oplus I\!I_{6,6}, \tag{5.40}$$

[11] Recall that $2k-2$ is the number of vector multiplets in $D=4$.

where D_{16} is the weight lattice of $Spin(32)/\mathbb{Z}_2$. Identifying $S_2 = \frac{2\pi(2\pi R)^6}{g_s^2}$, with g_s the heterotic string coupling constant in 10 dimensions, one obtains for a, b, c, d along D_{16},

$$-\frac{3}{8\pi^2}\log(S_2^k|\Delta_k(S)|^2)\delta_{(ab}\delta_{cd)} + \hat{F}^{(2k-2,6)}_{abcd}(\Phi) = (2\pi R)^6\Big(\frac{3}{g_s^2}\delta_{(ab}\delta_{cd)} + \frac{1}{2\pi^5}\delta_{abcd}\Big) + \dots \quad (5.41)$$

up to a threshold contribution and exponentially suppressed terms. Here $\delta_{abcd} = 1$ if all indices are identical, and zero otherwise, and we used

$$\begin{aligned}\int_{SL(2,\mathbb{Z})\backslash\mathcal{H}} \frac{d^2\tau}{\tau_2^2}\frac{\Gamma_{D_{16}}[P_{abcd}]}{\Delta} &= \int_{SL(2,\mathbb{Z})\backslash\mathcal{H}} \frac{d^2\tau}{\tau_2^2}\Big[\Big(\tfrac{E_4^3 - 2\hat{E}_2E_4E_6 + \hat{E}_2^2E_4^2}{48\Delta} - 24\Big)\delta_{(ab}\delta_{cd)} + 48\delta_{abcd}\Big] \\ &= 32\pi\delta_{abcd}\,. \end{aligned} \quad (5.42)$$

This equation follows from known results about the elliptic genus of the heterotic string [60]. Using an orthogonal basis for a Cartan subalgebra of $SO(32)$, one easily computes that this coupling gives the following trace combination in the vector representation of $SO(32)$

$$\Big(\frac{3}{g_s^2}\delta_{(ab}\delta_{cd)} + \frac{1}{2\pi^5}\delta_{abcd}\Big)t_8F^aF^bF^cF^d = \frac{(2\pi R)^6}{4}t_8\Big(\frac{3}{g_s^2}(\mathrm{Tr}F^2)^2 + \frac{1}{\pi^5}\mathrm{Tr}F^4\Big)\,. \quad (5.43)$$

Using $\kappa^2 = 4\alpha'$ and reabsorbing the $(2\pi R)^6\alpha'^3$ into the 6-torus volume one obtains in Einstein frame

$$\begin{aligned} S_{10} &= \int d^{10}x\sqrt{-g}\Big(\frac{1}{8\alpha'^4}\mathcal{R} + \frac{1}{8\alpha'^3}e^{-\frac{1}{2}\phi}\Big(\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2\Big) \\ &\qquad - \frac{1}{2\alpha'}t_8\Big(3e^{-\frac{3}{2}\phi}\mathrm{Tr}F^2\mathrm{Tr}F^2 + \frac{1}{\pi^5}e^{\frac{1}{2}\phi}\mathrm{Tr}F^4\Big) + \dots\Big)\,, \end{aligned} \quad (5.44)$$

which reproduces the tree level $\mathcal{R}^2$ and $(\mathrm{Tr}F^2)^2$ coupling computed in [53] upon identifying $\phi = \sqrt{2}\kappa D - 6\log 2$, and the 1-loop $\mathrm{Tr}F^4$ coupling computed in [61, 62].

5.3.2 BPS dyons

The contributions of non-zero vectors to the rank-one orbit yield exponentially suppressed corrections of order $e^{-2\pi R\mathcal{M}(Q,P)}$ (5.14), where $\mathcal{M}$ is the mass of a 1/2-BPS state of electromagnetic charge (Q, P) in four dimensions. The phase $e^{2\pi i(a^1Q+a^2P)}$ multiplying (5.14) is the expected minimal coupling of a dyonic state with charge (Q, P) to the holonomies of the electric and magnetic gauge fields along the circle. The corresponding instanton is a saddle point of the three-dimensional Euclidean supergravity theory obtained by formal reduction along a time-like Killing vector, in the duality frame where the axionic scalars a_1, a_2 are dualized into vector fields. Following the same steps as [63], one finds that the classical action is then $S_{\rm cl} = 2\pi R\mathcal{M}(Q, P)$.

In the maximal rank case, the summation measure (5.16) is given by

$$\bar{c}(Q,P) = \sum_{\substack{d\geq 1\\ (Q,P)/d\in\Lambda_{em}}} c\left(-\tfrac{\gcd(Q^2,P^2,Q\cdot P)}{2d^2}\right)\,, \quad (5.45)$$

where $c(m)$ are the Fourier coefficients of $1/\Delta$. For (Q, P) primitive, this agrees with the helicity supertrace (2.18) of 1/2-BPS states with charges (Q, P). In the case of CHL models, the summation measure is instead given by (5.31) or (5.32) with $q = 8$, $\tilde{v} = 1$, depending whether the dyon is related by $\Gamma_0(N)$, acting as $\binom{Q}{P} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{Q}{P}$, to a purely electric or a purely magnetic state. It is interesting to note that these two formulas can be combined as follows. We first notice using the decomposition $(Q,P) = (j',p')\hat{Q}$ and $(Q,P) = (j',p')\hat{P}$ when (Q,P) belong the electric and magnetic orbit respectively, with $(j',p') = 1$, one obtains

$$\begin{aligned} \frac{\hat{Q}}{d} \in \Lambda_m \quad &\Rightarrow \quad \frac{(Q,P)}{d} \in \Lambda_m \oplus N\Lambda_m \ , \\ \frac{\hat{P}}{d} \in N\Lambda_e \quad &\Rightarrow \quad \frac{(Q,P)}{d} \in N\Lambda_e \oplus N\Lambda_e \ , \end{aligned} \tag{5.46}$$

such that in both cases $(Q,P)/d \in \Lambda_m \oplus N\Lambda_e$. Moreover, if $(Q,P)/d \in \Lambda_m \oplus N\Lambda_e$, then $\hat{Q}/d \in \Lambda_m$ or $\hat{P}/d \in N\Lambda_e$, depending of the orbit to which (Q,P) belongs to, therefore one has the equivalence

$$\frac{(Q,P)}{d} \in \Lambda_m \oplus N\Lambda_e \quad \Leftrightarrow \quad \frac{\hat{Q}}{d} \in \Lambda_m \ \text{ or } \ \frac{\hat{P}}{d} \in N\Lambda_e \ , \tag{5.47}$$

for (Q,P) conjugate to either an electric charge $\hat{Q}$ or a magnetic charge $\hat{P}$. Similarly,

$$\begin{aligned} \frac{\hat{Q}}{d} \in \Lambda_e \quad &\Rightarrow \quad \frac{(Q,P)}{d} \in \Lambda_e \oplus N\Lambda_e \ , \\ \frac{\hat{P}}{d} \in \Lambda_m \quad &\Rightarrow \quad \frac{(Q,P)}{d} \in \Lambda_m \oplus \Lambda_m \ , \end{aligned} \tag{5.48}$$

such that

$$\frac{(Q,P)}{d} \in \Lambda_e \oplus \Lambda_m \quad \Leftrightarrow \quad \frac{\hat{Q}}{d} \in \Lambda_e \ \text{ or } \ \frac{\hat{P}}{d} \in \Lambda_m \ , \tag{5.49}$$

for (Q,P) conjugate to either an purely electric charge $(\hat{Q},0)$ or a purely magnetic charge $(0,\hat{P})$. Moreover, we have that $\gcd(NQ^2,P^2,Q\cdot P) = N\hat{Q}^2$ for a dyon in the $\Gamma_0(N)$ orbit of a purely electric charge, because then $\gcd(Nj'^2,p'^2,j'p') = N$ since $p' = 0 \mod N$, and $\gcd(NQ^2,P^2,Q\cdot P) = \hat{P}^2$ for a dyon in the $\Gamma_0(N)$ orbit of a purely magnetic charge, because then $\gcd(Nj'^2,p'^2,j'p') = 1$ since $p' \neq 0 \mod N$. Putting these observations together we conclude that the summation measure for a general 1/2 BPS dyon is given by

$$\bar{c}_k(Q,P) = \sum_{\substack{d\geq 1 \\ (Q,P)/d\in\Lambda_e\oplus\Lambda_m}} c_k\Big(-\tfrac{\gcd(NQ^2,P^2,Q\cdot P)}{d^2}\Big) + \sum_{\substack{d\geq 1 \\ (Q,P)/d\in\Lambda_m\oplus N\Lambda_e}} c_k\Big(-\tfrac{\gcd(NQ^2,P^2,Q\cdot P)}{2Nd^2}\Big) \ . \tag{5.50}$$

It is worth noting that $\gcd(NQ^2,P^2,Q\cdot P)$ is invariant under $\Gamma_0(N)$ and Fricke S-duality, so that each term in (5.50) is separately invariant under Fricke duality. Further noticing that $\Lambda_m \oplus N\Lambda_e \simeq \Lambda_e[N] \oplus \Lambda_m[N]$, (5.50) can be rewritten in a more suggestive way as

$$\bar{c}_k(Q,P) = \sum_{a|N} \sum_{\substack{d\geq 1 \\ (Q,P)/d\in\Lambda_{em}[a]}} c_k\Big(-\tfrac{\gcd(NQ^2,P^2,Q\cdot P)}{2a\,d^2}\Big) \ . \tag{5.51}$$

Most importantly, (5.51) agrees with the helicity supertrace $\Omega_4(Q,P)$ of a half-BPS dyon with primitive charge (Q,P) which was determined in (2.16) and (2.17).

5.3.3 Taub-NUT instantons

Finally, the rank-two orbit (5.25) yields contributions schematically of the form

$$\sum_{M_1\neq 0, M_2, P} \bar{c}(M_1, M_2, P)\, e^{-2\pi\sqrt{\left(R^2 M_1 + S_2\tilde{M}_2\right)^2 + 2R^2 S_2 \tilde{P}_R^2} + 2\pi\mathrm{i}(P\cdot a_2 + M_1(\psi - \frac{1}{2}a_1\cdot a_2) + \tilde{M}_2 S_1)} \tag{5.52}$$

where the summation measure (5.33) is given by

$$\bar{c}_k(M_1, M_2, P) = \sum_{\substack{d|(M_1,M_2)\\ P/d\in\Lambda_m}} d\, c_k\Big(\frac{D}{d^2}\Big) + \sum_{\substack{Nd|(M_1,M_2)\\ P/d\in N\Lambda_e}} N d\, c_k\Big(\frac{D}{Nd^2}\Big)\ , \tag{5.53}$$

and we denoted $\tilde{M}_2 = M_2 - a_1\cdot P + \frac{1}{2}(a_1\cdot a_1)M_1$, $\tilde{P} = P - M_1 a_1$, and $D = -\frac{1}{2}P^2 + M_1 M_2$. These $\mathcal{O}(e^{-2\pi R^2|M_1|})$ contributions are characteristic of an Euclidean Taub-NUT solution of the form $\mathrm{TN}_{M_1}\times T^6$, where the Taub-NUT space asymptotes to $\mathbb{R}^3\times S_1(R)$ at spatial infinity [64].

The detailed semi-classical interpretation of these effects is complicated by the fact that in a Taub-NUT background, similarly to the case of NS5-branes, large gauge transformations of the electric and magnetic holonomies a_1 and a_2 do not commute, thus cannot be diagonalized simultaneously. The representation (5.20) corresponds to the case where translations in a_2 and ψ are diagonalized. Accordingly, the argument of the exponential in (5.52) should be interpreted as the classical action in the duality frame in which the fields ψ, S_1, a_2 associated to the conserved charges M_1, M_2 and P are dualized into vector fields ω, B, A in three dimensions. In order to reach a positive definite action after dualization, one should first analytically continue the non-linear sigma model on $\frac{O(2k,8)}{O(2k)\times O(8)}$ into $\frac{O(2k,8)}{O(2k-1,1)\times O(7,1)}$ by taking ψ, S_1, a_2 to be purely imaginary. Equivalently, this is the non-linear sigma model obtained by reduction of a Euclidean four-dimensional theory. Denoting by U, ϕ, ζ the scalar fields whose asymptotic values are given by $\log R, -\frac{1}{2}\log S_2$ and a_1, the Lagrange density in three dimensions is

$$\begin{aligned}\mathcal{L} =& |\mathrm{d}U|^2 + \frac{1}{4}e^{4U}|\mathrm{d}\omega|^2 + |\mathrm{d}\phi|^2 + \frac{1}{4}e^{-4\phi}|\mathrm{d}B - (\zeta, \mathrm{d}A) + \tfrac{1}{2}(\zeta,\zeta)\mathrm{d}\omega|^2\\ &+ \frac{1}{4}e^{2U-2\phi}g(\mathrm{d}A - \zeta\mathrm{d}\omega, \mathrm{d}A - \zeta\mathrm{d}\omega) + \frac{1}{4}e^{-2U-2\phi}g(\mathrm{d}\zeta, \mathrm{d}\zeta) + P_{a\hat{b}}\star P^{a\hat{b}}\ ,\end{aligned} \tag{5.54}$$

where we denote $|f|^2 = f\wedge\star f$, $g(F,F)\equiv F_L^a\star F_{La} + F_R^{\hat{a}}\star F_{R\hat{a}}$. For simplicity we shall consider only instantons for which the electromagnetic fields vanish, $\mathrm{d}A = \zeta = 0$. One can then write the Lagrangian as a sum of squares

$$\mathcal{L} = \frac{1}{4}e^{4U}\Big|\star\mathrm{d}e^{-2U}\pm\mathrm{d}\omega\Big|^2 \pm\frac{1}{2}\mathrm{d}(e^{2U}\mathrm{d}\omega) + \frac{1}{4}e^{-4\phi}\Big|\star\mathrm{d}e^{2\phi}\pm\mathrm{d}B\Big|^2 \pm\frac{1}{2}\mathrm{d}(e^{-2\phi}\mathrm{d}B) + P_{a\hat{b}}\star P^{a\hat{b}}\ . \tag{5.55}$$

The corresponding 1/2-BPS solutions describe M_2 Euclidean NS5-branes on a self-dual Taub-NUT space of charge M_1, with $M_1 M_2 \geq 0$.[12] For simplicity we consider the NS5-branes at the tip of the Taub-NUT space, with

$$e^{-2U} = \frac{1}{R^2} + \frac{|M_1|}{r}\ ,\quad e^{2\phi} = \frac{1}{S_2} + \frac{|M_2|}{r}\ ,\quad \omega = -M_1\cos\theta\,\mathrm{d}\varphi\ ,\quad B = -M_2\cos\theta\,\mathrm{d}\varphi\ , \tag{5.56}$$

[12] Solutions with $M_1 M_2 \leq 0$ exist but do not preserve eight supercharges.

and the fields Φ on the Grassmannian $G_{r-6,6}$ are uniform. The action then reduces to the boundary term $S_{\rm cl} = 2\pi(R^2|M_1|+S_2|M_2|) = 2\pi|R^2M_1+S_2M_2|$. Note that the measure factor (5.53) vanishes for $P=0$ unless $M_1M_2 \geq -1$. We shall refrain from constructing 1/2-BPS instantons with generic magnetic charge P such that $D\geq 0$, although we expect that their action will reproduce $S_{\rm cl}$ in (5.22).

6 Discussion

In this work, we have proposed a formula (2.24) for the exact $(\nabla\Phi)^4$ coupling in a class of three-dimensional string vacua obtained as freely acting orbifolds of the heterotic string on T^7 under a $\mathbb{Z}_N$ action with N prime. Our formula is manifestly invariant under the U-duality group $G_3(\mathbb{Z})$, which unifies the S and T-duality in $D=4$ along with Fricke duality. We derived the supersymmetric Ward identities that the exact coupling function $F_{abcd}(\Phi)$ must satisfy, and showed that the formula (2.24) satisfies this constraint. Furthermore, we analyzed its behavior in the weak coupling regime $g_3\to 0$ and large radius regime $R\to\infty$, and found that it correctly reproduces the known tree-level and one-loop contributions in $D=3$, and the correct non-perturbative F^4 couplings in $D=4$. In addition, we extracted the exponential corrections to these power-like terms in both regimes, corresponding to non-zero Fourier coefficients with respect to parabolic subgroups $\mathbb{R}^+\times G_{2k-1,7}\ltimes\mathbb{R}^{2k+6}$ and $\mathbb{R}^+\times [SL(2)/SO(2)\times G_{2k-2,6}]\ltimes\mathbb{R}^{2\times(2k+4)}\times\mathbb{R}$, and found agreement with the expected form of the contributions of NS5-brane, Kaluza–Klein monopoles and H-monopole instantons as $g_3\to 0$, and the contributions of half-BPS dyons and Taub-NUT instantons as $R\to\infty$. In the case of half-BPS dyons, we found a precise match between the summation measure $\bar c_k(Q,P)$ and the helicity supertrace $\Omega_4(Q,P)$, at least when the charge vector (Q,P) is primitive. This vindicates the general expectation that BPS saturated couplings in dimension D encode BPS indices in dimension $D+1$. It would be interesting to determine the helicity supertrace $\Omega_4(Q,P)$ when (Q,P) is not primitive (which requires a careful treatment of threshold bound states), and compare with the summation measure $\bar c_k(Q,P)$.

It is natural to ask whether our formula (2.27) is the unique solution to the Ward identities (2.23) which is invariant under $G_3(\mathbb{Z})$, and reproduces the correct power-like terms in the weak coupling and large radius expansions $g_3\to 0$ and $R\to\infty$. Typically, theorems in the mathematical literature guarantee that smooth automorphic forms on $K\backslash G/G(\mathbb{Z})$ which vanish at all cusps and have sufficiently sparse Fourier coefficients (in mathematical terms, are attached to a sufficiently small nilpotent orbit) necessarily vanish; so that the only smooth automorphic functions satisfying to (3.27) are necessary Eisenstein series. However, these theorems are typically concerned with Chevalley subgroups of reductive groups in the split or quasi-split real form, which is not the case here ($G_3(\mathbb{Z})$ is a proper subgroup of the Chevalley group of $O(2k,8)$ for $N>1$), and smoothness away from the cusps is essential.

As far as the support of Fourier coefficients is concerned, the Ward identities (3.27), imply that the trace of the modular integral (3.29) is attached to the vectorial character of $O(p,q)$, corresponding to the next-to-minimal orbit. However, the constraints imposed by the differential equations (3.17), (3.20) are stronger than (3.27), *e.g.* we show in Appendix B that the tensor F_{abcd} derived from the scalar Eisenstein series defined in Appendix E.2 is not a solution to (3.20). The general form of the Fourier coefficients is in fact very reminiscent of the one for automorphic forms attached to the minimal orbit of $O(p,q)$: it allows for only two power-like terms at the cusp, rather than three for the next-to-minimal orbit; they involve ordinary Bessel function of one single variable, similarly to A_1 Whittaker vectors, rather than more complicated functions of two variables or the typical $2A_1$ Whittaker vectors which appear in the Fourier coefficients of generic vectorial Eisenstein series [65].

However, as we emphasized repeatedly, (3.28) has singularities in the bulk of $G_{p,q}$ on codimension q loci where the projection $P_R^{\hat{a}}$ of a vector P in $\Lambda_{p,q}$ with norm 2 (or the projection $Q_R^{\hat{a}}$ of a vector Q in $\Lambda^*_{p,q}$ with norm $2/N$) vanishes. In order to argue for uniqueness, it is crucial to ensure that the modular integral (2.24) correctly captures the behavior of the $(\nabla\Phi)^4$ coupling at all singular loci. Since (2.24) reproduces correctly the one-loop contribution to $(\nabla\Phi)^4$, it is clear that it correctly captures the singular behavior on the loci associated to vectors P, Q in the 'perturbative Narain lattice' $\Lambda_{r-5,7} \subset \Lambda_{r-4,8}$, at least in the weak coupling limit. Presumably, this suffices to guarantee agreement on all singular loci, but we do not know how to prove this rigorously.

Let us note finally that, independently of our proposed identification of the U-duality group in three dimensions, the general solution to the Ward identities (3.17), (3.20) derived in Appendix B implies that the exact coupling must be of the form (4.35), up to the determination of the measure factor $\bar{c}_k(Q)$. The property that we recover the exact coupling in four dimensions implies that the mesure factor is correct for null vectors by $O(r-5,7,\mathbb{Z})$ T-duality. Indeed, for $Q^2 = 0$, the summation measure in (4.36) reproduces the summation measure for NS5-brane instantons in (2.5). The computation of the BPS index associated to an arbitrary NS5-brane, Kaluza–Klein monopole, H-monopole instanton, would therefore give a direct proof of our result.

Clearly, it would be interesting to generalize our construction to the complete class of heterotic CHL models, whose duality properties and BPS spectrum in 4-dimensions are by now well understood. It is natural to conjecture that the duality group in $D = 3$ will still be given by the automorphism group of the non-perturbative Narain lattice (2.21), which naturally incorporates the S and T-duality symmetries in $D = 4$. More pressingly however, the present study was a warm-up towards the more challenging problem of understanding the 1/4-BPS saturated coupling $\nabla^2(\nabla\Phi)^4$ in four dimensions, which we shall address in forthcoming work.

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A Perturbative spectrum and one-loop F^4 couplings in heterotic CHL orbifolds

In this section, we construct the one-loop vacuum amplitude in CHL models obtained as a freely acting Z_N-orbifold of the standard heterotic string on T^d with N prime. From this, we deduce the helicity supertrace for perturbative BPS states, and the one-loop contribution to the F^4 and $(\nabla\Phi)^4$ couplings. We start with the simplest model with $N = 2$, and then generalize the construction to $N = 3, 5, 7$.

A.1 $\mathbb{Z}_2$ orbifold

The simplest CHL model is obtained by orbifolding the $E_8 \times E_8$ heterotic string compactified on T^d, by an involution σ which exchanges the two E_8 gauge groups and performs a translation by half a period along one circle in T^d [14]. This perturbative BPS spectrum in this model was further studied in [66, 26]. The symmetry σ exists only on a codimension $8d$ space inside the Narain moduli space $G_{d+16,d}$ and preserves only a $U(1)^{2d+8}$ subgroup of the original $U(1)^{2d+16}$ gauge symmetry, corresponding to the usual $2d$ Kaluza–Klein and Kalb-Ramond gauge fields, and the Cartan torus of the diagonal combination of the two E_8 gauge groups. To implement the quotient by σ, it is simplest to work at the point in $G_{d+16,d}$ where the lattice factorizes as

$$\Lambda_{d+16,d} = E_8 \oplus E_8 \oplus I\!I_{d,d} \ . \tag{A.1}$$

The integrand of the one-loop vacuum amplitude of the original heterotic string is then

$$\mathcal{A} = Z_{E_8\times E_8} \times \Gamma_{I\!I_{d,d}} \times \frac{1}{2} \sum_{\alpha,\beta\in\{0,1\}} (-1)^{\alpha\beta+\alpha+\beta} \frac{\bar\vartheta^4\left[{\alpha \atop \beta}\right]}{\tau_2^4 \eta^8 \bar\eta^{12}} \tag{A.2}$$

where

$$Z_{E_8\times E_8} = \left[\frac{\sum_{Q_1\in E_8} q^{\frac12 Q_1^2}}{\eta^8}\right] \left[\frac{\sum_{Q_2\in E_8} q^{\frac12 Q_2^2}}{\eta^8}\right] = \frac{[E_4(\tau)]^2}{\eta^{16}} \tag{A.3}$$

is the partition function of the 16 chiral bosons on the $E_8 \times E_8$ root lattice, and the last factor in (A.2) represents the contribution of the transverse bosonic and fermionic oscillators, while the sum over α, β implements the GSO projection. As a consequence of space-time supersymmetry, the integral (A.2) vanishes pointwise, but it will no longer be so in the presence of vertex operators. Note that the right-moving part in (A.2) will not play any role in our case, and will be later replaced by an insertion of the polynomial P_{abcd} (2.26).

Following standard rules, the one-loop partition function of the orbifold by σ is obtained by replacing $\mathcal{A}$ by a sum $\frac12 \sum_{h,g\in\{0,1\}} \mathcal{A}\left[{h \atop g}\right]$, where $\mathcal{A}\left[{h \atop g}\right]$ is obtained by twisting the boundary conditions of the fields by σ^g along the spatial direction of the string, and σ^h along the Euclidean time direction, so that $\frac12(\mathcal{A}\left[{0 \atop 0}\right] + \mathcal{A}\left[{0 \atop 1}\right])$ counts σ-invariant states in the untwisted sector, while $\frac12(\mathcal{A}\left[{1 \atop 0}\right] + \mathcal{A}\left[{1 \atop 1}\right])$ counts σ-invariant states in the twisted sector. Modular invariance permutes the three blocks $\left[{0 \atop 1}\right], \left[{1 \atop 0}\right], \left[{1 \atop 1}\right]$ according to

$$\mathcal{A}\left[{h \atop g}\right] \left(\tfrac{a\tau+b}{c\tau+d}\right) = \mathcal{A}\left[{ah+cg \atop bh+gd}\right](\tau) \tag{A.4}$$

where h, g are treated modulo 2. In particular, the block $\left[{0 \atop 1}\right]$ is invariant under the Hecke congruence subgroup $\Gamma_0(2)$, and all other blocks can be obtained by acting on it with elements of $SL(2,\mathbb{Z})/\Gamma_0(2) = \{1, S, ST\}$.

In the case at hand, the involution σ exchanges $Q_1 \leftrightarrow Q_2$ and the corresponding oscillators, so σ-invariant states must have $Q_1 = Q_2$ and the same oscillator state on both factors, thus

$$Z_{E_8\times E_8}\left[{0 \atop 1}\right](\tau) = \frac{\sum_{Q\in E_8} q^{Q^2}}{\eta^8(2\tau)} \ . \tag{A.5}$$

The two remaining orbifold blocks are then fixed by modular covariance,

$$\begin{aligned} Z_{E_8\times E_8}\left[{}^0_0\right] &= \frac{E_4^2(\tau)}{\eta^{16}(\tau)}\ , \quad Z_{E_8\times E_8}\left[{}^0_1\right] = \frac{E_4(2\tau)}{\eta^8(2\tau)}\ , \\ Z_{E_8\times E_8}\left[{}^1_0\right] &= \frac{E_4(\frac{\tau}{2})}{\eta^8(\frac{\tau}{2})}\ , \quad Z_{E_8\times E_8}\left[{}^1_1\right] = \frac{E_4(\frac{\tau+1}{2})}{e^{2i\pi/3}\eta^8(\frac{\tau+1}{2})}\ , \end{aligned} \tag{A.6}$$

As for the action of σ on the torus T^d, it can be taken into account by replacing the partition function $\Gamma_{I\!I_{d,d}}$ by

$$\Gamma_{I\!I_{d,d}}\left[{}^h_g\right] = \tau_2^{d/2} \sum_{Q\in I\!I_{d,d}+\frac{h}{2}\delta} (-1)^{g\,\delta\cdot Q}\, q^{\frac12 Q_L^2}\, \bar q^{\frac12 Q_R^2}\ . \tag{A.7}$$

where δ must be null modulo 2, and depends on the choice of circle S_1 inside T^d. The resulting one-loop vacuum amplitude is then the modular integral of

$$\mathcal{A}_{\rm orb} = \frac12 \sum_{h,g\in\{0,1\}} Z_{E_8\times E_8}\left[{}^h_g\right] \Gamma_{I\!I_{d,d}}\left[{}^h_g\right] \times \frac12 \sum_{\alpha,\beta\in\{0,1\}} (-1)^{\alpha\beta+\alpha+\beta} \frac{\bar\vartheta^4\left[{}^\alpha_\beta\right]}{\tau_2^4\eta^8\bar\eta^{12}}, \tag{A.8}$$

where the one-half factor is explained above (A.4). Now, a key observation is that the numerator in the blocks $Z_{E_8\times E_8}\left[{}^h_g\right]$ for $(h,g)\neq(0,0)$ can be written as a partition functions for the lattice $\Lambda = E_8[2]$ and for its dual $\Lambda^* = E_8[1/2]$,

$$\begin{aligned} Z_{E_8\times E_8}\left[{}^0_1\right] &= \frac{1}{\eta^8(2\tau)} \sum_{Q\in E_8[2]} q^{\frac12 Q^2} \\ Z_{E_8\times E_8}\left[{}^1_0\right] &= \frac{1}{\eta^8(\frac{\tau}{2})} \sum_{Q\in E_8[1/2]} q^{\frac12 Q^2} \\ Z_{E_8\times E_8}\left[{}^1_1\right] &= \frac{1}{e^{2i\pi/3}\eta^8(\frac{\tau+1}{2})} \sum_{Q\in E_8[1/2]} (-1)^{Q^2} q^{\frac12 Q^2}\ . \end{aligned} \tag{A.9}$$

Moreover, the untwisted, unprojected partition function satisfies

$$\begin{aligned} Z_{E_8\times E_8}\left[{}^0_0\right] &= \frac{E_4(2\tau)}{\eta^8(2\tau)} + \frac{E_4(\frac{\tau}{2})}{\eta^8(\frac{\tau}{2})} + \frac{E_4(\frac{\tau+1}{2})}{e^{2i\pi/3}\eta^8(\frac{\tau+1}{2})} \\ &= Z_{E_8\times E_8}\left[{}^0_1\right] + Z_{E_8\times E_8}\left[{}^1_0\right] + Z_{E_8\times E_8}\left[{}^1_1\right]\ . \end{aligned} \tag{A.10}$$

This relation can be checked using the explicit form of the blocks $Z_{E_8\times E_8}\left[{}^0_1\right]$, but more conceptually, it follows by decomposing $Z_{E_8\times E_8}\left[{}^0_0\right]$, the character of the level 1 representation of $\hat E_8\oplus\hat E_8$, into characters of level 2 representations of the diagonal $\hat E_8$ [67]. It follows from (A.9), (A.10) that the one-loop amplitude (A.8) can be written as

$$\mathcal{A}_{\rm orb} = \frac12 \sum_{h,g\in\{0,1\}}{}' \widetilde Z_{d+8,d}\left[{}^h_g\right] \times \frac12 \sum_{\alpha,\beta\in\{0,1\}} (-1)^{\alpha\beta+\alpha+\beta} \frac{\bar\vartheta^4\left[{}^\alpha_\beta\right]}{\tau_2^4\bar\eta^{12}} \tag{A.11}$$

where the sum over (h,g) no longer includes $(0,0)$. Here, we defined the eta products

$$\begin{aligned} \Delta_8\left[{}^0_1\right] &= \eta^8(\tau)\eta^8(2\tau) = 2^{-4}\eta^{12}\vartheta_2^4 \equiv \Delta_8(\tau) \\ \Delta_8\left[{}^1_0\right] &= \eta^8(\tau)\eta^8(\tfrac{\tau}{2}) = \eta^{12}\vartheta_4^4 = \Delta_8(\tfrac{\tau}{2}), \\ \Delta_8\left[{}^1_1\right] &= e^{2i\pi/3}\,\eta^8(\tau)\eta^8(\tfrac{\tau+1}{2}) = -\eta^{12}\vartheta_3^4 = \Delta_8(\tfrac{\tau+1}{2}),\ , \end{aligned} \tag{A.12}$$

satisfying

$$\Delta_8\left[{0 \atop 1}\right](-1/\tau) = 2^{-4}\tau^8\,\Delta_8\left[{1 \atop 0}\right](\tau)\ ,\quad \Delta_8\left[{1 \atop 0}\right](\tau+1) = \Delta_8\left[{1 \atop 1}\right](\tau)\ , \tag{A.13}$$

and the partition functions $\widetilde{Z}_{d+8,d}\left[{h \atop g}\right]$ are defined over $\tilde{\Lambda}_{d+8,d} = E_8[2]\oplus I\!I_{d,d}$ and its dual $\tilde{\Lambda}^*_{d+8,d} = E_8[1/2]\oplus I\!I_{d,d}$:

$$\begin{aligned}
\widetilde{Z}_{d+8,d}\left[{0 \atop 1}\right] &= \frac{\tau_2^{d/2}}{\Delta_8\left[{0 \atop 1}\right]}\sum_{Q\in\tilde{\Lambda}_{d+8,d}}\left[1+(-1)^{\delta\cdot Q}\right]q^{\frac12 Q_L^2}\bar{q}^{\frac12 Q_R^2}\\
\widetilde{Z}_{d+8,d}\left[{1 \atop 0}\right] &= \frac{\tau_2^{d/2}}{\Delta_8\left[{1 \atop 0}\right]}\Bigg[\sum_{Q\in\tilde{\Lambda}^*_{d+8,d}}+\sum_{Q\in\tilde{\Lambda}^*_{d+8,d}+\frac12\delta}\Bigg]q^{\frac12 Q_L^2}\bar{q}^{\frac12 Q_R^2}\\
\widetilde{Z}_{d+8,d}\left[{1 \atop 1}\right] &= \frac{\tau_2^{d/2}}{\Delta_8\left[{1 \atop 1}\right]}\Bigg[\sum_{Q\in\tilde{\Lambda}^*_{d+8,d}}+\sum_{Q\in\tilde{\Lambda}^*_{d+8,d}+\frac12\delta}\Bigg](-1)^{Q^2}q^{\frac12 Q_L^2}\bar{q}^{\frac12 Q_R^2}
\end{aligned} \tag{A.14}$$

These relations were derived at the special point where the lattice $\tilde{\Lambda}_{d+8,d}$ is factorized, but it is now clear that they hold at arbitrary points on the moduli space $G_{d+8,d}\subset G_{d+16,d}$ where the $\mathbb{Z}_2$ symmetry exists.

Choosing $\delta = (0^d;0^{d-1},1)$, so that the involution σ acts by a translation along the d-th circle by a half period, this can be further written as

$$\begin{aligned}
\Gamma_{\Lambda_{d+8,d}} &\equiv \tau_2^{d/2}\sum_{Q\in\Lambda_{d+8,d}} q^{\frac12 Q_L^2}\bar{q}^{\frac12 Q_R^2} = \frac12\Delta_8\left[{0 \atop 1}\right]\widetilde{Z}_{d+8,d}\left[{0 \atop 1}\right]\\
(2^5/\tau^4)\Gamma_{\Lambda_{d+8,d}}(-1/\tau) = \Gamma_{\Lambda^*_{d+8,d}} &\equiv \tau_2^{d/2}\sum_{Q\in\Lambda^*_{d+8,d}} q^{\frac12 Q_L^2}\bar{q}^{\frac12 Q_R^2} = \Delta_8\left[{1 \atop 0}\right]\widetilde{Z}_{d+8,d}\left[{1 \atop 0}\right]\\
\Gamma_{\Lambda^*_{d+8,d}}\left[(-1)^{Q^2}\right] &\equiv \tau_2^{d/2}\sum_{Q\in\Lambda^*_{d+8,d}} (-1)^{Q^2}q^{\frac12 Q_L^2}\bar{q}^{\frac12 Q_R^2} = \Delta_8\left[{1 \atop 1}\right]\widetilde{Z}_{d+8,d}\left[{1 \atop 1}\right]
\end{aligned} \tag{A.15}$$

where $\Lambda_{d+8,d}$ is related to $\tilde{\Lambda}_{d+8,d}$ by rescaling a $I\!I_{1,1}$ summand[13],

$$\Lambda_{d+8,d} = E_8[2]\oplus I\!I_{1,1}[2]\oplus I\!I_{d-1,d-1}\ . \tag{A.16}$$

Here $I\!I_{1,1}[2]$ is the usual sum over momentum m_d and winding n_d, with m_d running only over even integers. The dual lattice is

$$\Lambda^*_{d+8,d} = E_8[1/2]\oplus I\!I_{1,1}[1/2]\oplus I\!I_{d-1,d-1}\ , \tag{A.17}$$

where $I\!I_{1,1}[1/2]$ is the usual sum over momentum m_d and winding n_d, with n_d running over $\mathbb{Z}/2$. For $d=6$, since $\Lambda_{14,6}\subset\Lambda^*_{14,6}$, we see that the electric charges carried by excitations of the heterotic string lie in the lattice $\Lambda_e = \Lambda^*_{14,6}$, in agreement with the result stated in Table 1. Moreover, it is apparent that the degeneracy of perturbative BPS states with charge $Q\in\Lambda^*_{d+8,d}$, $Q\notin\Lambda_{d+8,d}$ in the twisted sector is given by the coefficient of $q^{-Q^2/2}$ in $1/\Delta_8\left[{1 \atop 0}\right] = 1/\Delta_8(\tau/2)$, or equivalently the coefficient of q^{-Q^2} in $1/\Delta_8$, while the degeneracy

[13] Note that this rescaling implies an extra volume factor upon Poisson resummation, namely $\Gamma_{\Lambda^*_{d+8,d}}(\tau) = (2^5/\tau^4)\Gamma_{\Lambda_{d+8,d}}(-1/\tau)$.

of perturbative BPS states with charge $Q \in \Lambda_{d+8,d} \subset \Lambda^*_{d+8,d}$ has an additional contribution from the coefficient of $q^{-Q^2/2}$ in $1/\Delta_8$, in agreement with (2.14) and (2.15), and the analysis in [66, 26].

At last, we can turn to the one-loop F^4 amplitude in this model. As is the case in the usual heterotic string, the insertion of four vertex operators replaces the right-moving contribution in the vacuum amplitude (A.11) by an insertion of the polynomial P_{abcd} in (2.26). Thus, we get

$$F^{\text{(1-loop)}}_{abcd} = \text{R.N.} \int_{SL(2,\mathbb{Z})\backslash\mathcal{H}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2} \sum_{\gamma\in\Gamma_0(2)\backslash SL(2,\mathbb{Z})} \left.\frac{\Gamma_{\Lambda_{d+8,d}}[P_{abcd}]}{\Delta_8}\right|_\gamma , \tag{A.18}$$

where $\Gamma_{\Lambda_{d+8,d}}[P_{abcd}]$ denotes the lattice partition function $\Gamma_{\Lambda_{d+8,d}}\left[\begin{smallmatrix}h\\g\end{smallmatrix}\right]$ with an insertion of the polynomial P as in (2.25). Equivalently, we can unfold the integral over a fundamental domain $\Gamma_0(2)\backslash\mathcal{H}$ for the action of $\Gamma_0(2)$ on $\mathcal{H}$, at the expense of keeping only the identity in the sum over cosets,

$$F^{\text{(1-loop)}}_{abcd} = \text{R.N.} \int_{\Gamma_0(2)\backslash\mathcal{H}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2} \frac{\Gamma_{\Lambda_{d+8,d}}[P_{abcd}]}{\Delta_8} , \tag{A.19}$$

which demonstrates (2.24) in this case.

A.2 $\mathbb{Z}_N$ orbifold with $N = 3, 5, 7$

The construction detailed in the previous section can be easily generalized to $\mathbb{Z}_N$ orbifolds, provided one can find a point in the moduli space $G_{d+16,d}$ where $\mathbb{Z}_N$ acts on the lattice $\Lambda_{d+16,d}$ by a permutation with cycle shape $1^k N^k$. It turns out that for $N = 3, 5, 7$, such a lattice can be obtained by applying a Wick rotation on the Niemeier lattices D_6^4, D_4^6 and D_3^8, respectively. Indeed, recall that given an even self-dual Euclidean lattice

$$\Lambda = \cup_{(\lambda,\lambda')\in\mathcal{G}} (D_k + \lambda) \oplus (\Lambda' + \lambda') \tag{A.20}$$

of dimension n, where the glue code $\mathcal{G}$ is a given sublattice of $D_k^*/D_k \oplus \Lambda'^*/\Lambda'$, one can obtain an even self-dual lattice of dimension $n-8$, by replacing D_k by D_{k-8}, while keeping the same glue code $\mathcal{G}$, using the fact that $\mathcal{G}_k = D_k^*/D_k$ is invariant under $k \mapsto k-8$[14],

$$\hat{\Lambda} = \cup_{(\lambda,\lambda')\in\mathcal{G}} (D_{k-8} + \lambda) \oplus (\Lambda' + \lambda') . \tag{A.21}$$

If $1 \le k < 8$, then D_{k-8} should be understood as $D_{8-k}[-1]$, so that the new lattice is a Lorentzian lattice with signature $(n-k, 8-k)$ [68, §A.4]. In this way, starting from the Niemeier lattice $\Lambda = D_k^{N+1}$ for $N = 3, 5, 7$, which is symmetric under cyclic permutations of the $N+1$ D_k factors, we obtain an even self-dual lattice $\hat{\Lambda} = D_k^N \oplus D_{8-k}[-1]$ of signature $(Nk, 8-k)$ with a $\mathbb{Z}_N$ symmetry σ acting by cyclic permutations of the N D_k factors. Using the explicit description of the glue code for Niemeier lattices given in [69, Table 16.1], it is possible to check that the only elements $(\lambda_1, \dots \lambda_{N+1})$ in the glue code $\mathcal{G} \subset \mathcal{G}_k^{N+1}$ which are invariant under $\mathbb{Z}_N$ are those of the form $(\lambda, \dots, \lambda)$ with λ running over $\mathcal{G}_k$. The partition function of the lattice $\hat{\Lambda}$ with an insertion of the element σ^g with $g \neq 0 \bmod N$ is thus

$$\begin{aligned} Z_{k,8-k}\left[\begin{smallmatrix}0\\g\end{smallmatrix}\right] = & \frac{\vartheta_3^k+\vartheta_4^k}{2\eta^k}(N\tau)\overline{\frac{\vartheta_3^{8-k}+\vartheta_4^{8-k}}{2\eta^{8-k}}} + \frac{\vartheta_3^k-\vartheta_4^k}{2\eta^k}(N\tau)\overline{\frac{\vartheta_3^{8-k}-\vartheta_4^{8-k}}{2\eta^{8-k}}} \\ & + \frac{\vartheta_2^k+\vartheta_1^k}{2\eta^k}(N\tau)\overline{\frac{\vartheta_2^{8-k}+\vartheta_1^{8-k}}{2\eta^{8-k}}} + \frac{\vartheta_2^k-\vartheta_1^k}{2\eta^k}(N\tau)\overline{\frac{\vartheta_2^{8-k}-\vartheta_1^{8-k}}{2\eta^{8-k}}} . \end{aligned} \tag{A.22}$$

[14] Indeed, $\mathcal{G}_k = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is k is even, or $\mathbb{Z}_4$ is k is odd, with the 4 elements in one-to-one correspondence with the highest weights $0, s, v, c$ of the adjoint, spinor, vector and conjugate spinor representations.

The other blocks are obtained by modular covariance, leading for $h \neq 0 \bmod N$ to

$$\begin{aligned} Z_{k,8-k}\left[\begin{smallmatrix} h \\ 0 \end{smallmatrix}\right] =& \tfrac{\vartheta_3^k+\vartheta_2^k}{2\eta^k}\left(\tfrac{\tau}{N}\right)\overline{\tfrac{\vartheta_3^{8-k}+\vartheta_2^{8-k}}{2\eta^{8-k}}} + \tfrac{\vartheta_3^k-\vartheta_2^k}{2\eta^k}\left(\tfrac{\tau}{N}\right)\overline{\tfrac{\vartheta_3^{8-k}-\vartheta_2^{8-k}}{2\eta^{8-k}}} \\ &+ \tfrac{\vartheta_4^k+\vartheta_1^k}{2\eta^k}\left(\tfrac{\tau}{N}\right)\overline{\tfrac{\vartheta_4^{8-k}+\vartheta_1^{8-k}}{2\eta^{8-k}}} + \tfrac{\vartheta_4^k-\vartheta_1^k}{2\eta^k}\left(\tfrac{\tau}{N}\right)\overline{\tfrac{\vartheta_4^{8-k}-\vartheta_1^{8-k}}{2\eta^{8-k}}} \ , \end{aligned} \tag{A.23}$$

while the remaining blocks with $g \neq 0 \bmod N$ follow by acting with $\tau \to \tau + 1$,

$$Z_{k,8-k}\left[\begin{smallmatrix} h \\ g \end{smallmatrix}\right](\tau) = Z_{k,8-k}\left[\begin{smallmatrix} h \\ 0 \end{smallmatrix}\right](\tau + gh^{-1}) \tag{A.24}$$

where h^{-1} is the inverse of h in the multiplicative group $\mathbb{Z}_N$. The untwisted, unprojected block is then

$$Z_{k,8-k}\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = Z_{k,8-k}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] + \sum_{g=0}^{N-1} Z_{k,8-k}\left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right] \ , \tag{A.25}$$

i.e. a sum over images of $Z_{\hat\Lambda}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ under $\Gamma_0(N)\backslash SL(2,\mathbb{Z}) = \{1, S, TS, \dots, T^{N-1}S\}$. As a consistency check, one can verify that the analogous sum for the Euclidean lattice Λ reproduces the partition function of the Niemeier lattice,

$$\frac{\Theta_{D_k^{N+1}}}{\eta^{24}} = Z_{2k-8}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] + \sum_{g=0}^{N-1} Z_{2k-8}\left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right] = \frac{E_4^3}{\eta^{24}} + 48k - 768 \ , \tag{A.26}$$

where $Z_{2k-8}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ is obtained by replacing $\overline{\vartheta_i^{8-k}/\eta^{8-k}}$ by $(\vartheta_i/\eta)^k$ in (A.22).

The integrand of the one-loop vacuum amplitude follows in the same way as in the previous subsection, by combining the orbifold blocks $Z_{k,8-k}\left[\begin{smallmatrix} h \\ g \end{smallmatrix}\right](\tau)$ for the lattice $\hat\Lambda$ with the shifted partition function for the remaining $d-8+k$ compact directions (where d is assumed to be greater that $8-k$)

$$\Gamma_{\Lambda_{d-8+k,d-8+k}}\left[\begin{smallmatrix} h \\ g \end{smallmatrix}\right] = \tau_2^{\frac{d-8+k}{2}} \sum_{Q\in\Lambda_{d-8+k,d-8+k}+\frac{h}{N}\delta} (-1)^{\frac{2}{N}g\,\delta\cdot Q}\, q^{\frac12 Q_L^2}\, \bar q^{\frac12 Q_R^2} \ . \tag{A.27}$$

After eliminating $Z_{k,8-k}\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ using (A.25), grouping terms into an orbit of $\Gamma_0(N)\backslash SL(2,\mathbb{Z})$, and rescaling a $I\!I_{1,1}$ factor in $\Lambda_{d+2k-8,d}$ as[15]

$$\Lambda_{d+2k-8,d} = D_k[N] \oplus D_{8-k}[-1] \oplus I\!I_{1,1}[N] \oplus I\!I_{d+k-9,d+k-9} \ , \tag{A.28}$$

with a glue code $\{(0,0),(s,s),(v,v),(c,c)\}$ for the first two factors, we find

$$\mathcal{A}_{\rm orb} = \left[\frac{\Gamma_{\Lambda_{d+2k-8,d}}}{\Delta_k\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} + \frac{1}{N}\sum_{g=0}^{N-1}\frac{\Gamma_{\Lambda^*_{d+2k-8,d}}[(-1)^{gQ^2}]}{\Delta_k\left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right]}\right] \times \frac12 \sum_{\alpha,\beta\in\{0,1\}} (-1)^{\alpha\beta+\alpha+\beta}\frac{\overline{\vartheta}^4\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]}{\tau_2^4\bar\eta^{12}} \tag{A.29}$$

where we defined the eta products

$$\Delta_k\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \eta(\tau)^k\,\eta(N\tau)^k \ , \quad \Delta_k\left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right] = e^{\frac{i\pi g k}{12}}\,\eta(\tau)^k\,\eta\left(\tfrac{\tau+g}{N}\right)^k \tag{A.30}$$

[15] Note that this rescaling implies $\Gamma_{\Lambda^*_{d+2k-8,d}}(\tau) = (N^{\frac{k}{2}+1}/\tau^{k-4})\Gamma_{\Lambda_{d+2k-8,d}}(-1/\tau)$.

and

$$\begin{aligned}\Gamma_{\Lambda_{d+2k-8,d}} &= \tau_2^{\frac{d}{2}} \sum_{Q\in\Lambda_{d+2k-8,d}} q^{\frac{1}{2}Q_L^2}\bar{q}^{\frac{1}{2}Q_R^2} \\ \Gamma_{\Lambda^*_{d+2k-8,d}}\left[(-1)^{gQ^2}\right] &= \tau_2^{\frac{d}{2}} \sum_{Q\in\Lambda^*_{d+2k-8,d}} (-1)^{gQ^2}\, q^{\frac{1}{2}Q_L^2}\bar{q}^{\frac{1}{2}Q_R^2} \,.\end{aligned} \tag{A.31}$$

From this description, it is apparent that the degeneracy of twisted perturbative BPS states with charge $Q\in\Lambda^*_{d+2k-8,d}$, $Q\notin\Lambda_{d+2k-8,d}$ is given by the coefficient of $q^{-Q^2/2}$ in $1/\Delta_k\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]=1/\Delta_k(\tau/N)$, or equivalently the coefficient of $q^{-NQ^2/2}$ in $1/\Delta_k$, while the degeneracy of perturbative BPS states with charge $Q\in\Lambda_{d+2k-8,d}\subset\Lambda^*_{d+2k-8,d}$ has an additional contribution from the coefficient of $q^{-Q^2/2}$ in $1/\Delta_k$, in agreement with (2.14) and (2.15). For four-dimensional vacua ($d=6$), we see that the electric charges carried by perturbative BPS states lie in the lattice $\Lambda_e=\Lambda_m^*$ where

$$\begin{aligned} N=3: &\quad \Lambda_m = D_6[3]\oplus D_2[-1]\oplus I\!I_{1,1}[3]\oplus I\!I_{3,3} \\ N=5: &\quad \Lambda_m = D_4[5]\oplus D_4[-1]\oplus I\!I_{1,1}[5]\oplus I\!I_{1,1} \\ N=7: &\quad \Lambda_m = D_3[7]\oplus D_5[-1]\oplus I\!I_{1,1}[7] \end{aligned} \tag{A.32}$$

This is in fact in agreement with the results stated in Table 1, thanks to the isomorphisms

$$\begin{aligned} D_6[3]\oplus D_2[-1] &\simeq A_2\oplus A_2\oplus I\!I_{2,2}[3] \\ D_4[5]\oplus D_4[-1] &\simeq I\!I_{2,2}[5]\oplus I\!I_{2,2} \\ D_3[7]\oplus D_5[-1] &\simeq \begin{pmatrix}-4 & -1\\ -1 & -2\end{pmatrix}\oplus I\!I_{1,1}[7]\oplus I\!I_{2,2} \end{aligned} \tag{A.33}$$

Indeed, both lattices on each line have the same genus, in particular the same discriminant group $L^*/L=\mathbb{Z}_N^k$. For $N=2$ (hence $k=8$), Eq. (A.28) continues to hold with the understanding that $D_8[2]\oplus D_0[-1]\equiv E_8[2]$.

Finally, we can obtain the one-loop F^4 amplitude by replacing the last factor in (A.29) by an insertion of the polynomial P_{abcd} in (2.26), and integrating over the fundamental domain $\mathcal{H}/SL(2,\mathbb{Z})$. As before, the integral can be unfolded onto a fundamental domain $\Gamma_0(N)\backslash\mathcal{H}$ for the action of $\Gamma_0(N)$ on $\mathcal{H}$, at the expense of keeping only the block $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$,

$$F^{(\text{1-loop})}_{abcd} = \text{R.N.}\int_{\Gamma_0(N)\backslash\mathcal{H}} \frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\,\frac{\Gamma_{\Lambda_{d+2k-8,d}}[P_{abcd}]}{\Delta_k}\,, \tag{A.34}$$

where $\Delta_k\equiv\Delta_k\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, thus establishing (2.24) for this class of models.

B Ward identity in the degeneration $O(p,q)\to O(p-1,q-1)$

In section 3.2, we proved that the differential equations (3.17) and (3.22) are satisfied by the one-loop modular integral F_{abcd} defined in (3.28). Here, we verify explicitly that the differential equation in (3.22) is verified by each Fourier mode in the degeneration limit $O(p,q)\to O(p-1,q-1)$, and that the solution is uniquely determined up to a moduli-independent summation measure.

Using the decomposition (4.4) and changing variable $R = e^{-\phi}$ for the non-compact Cartan generator of $O(p,q)$, the metric on moduli space reads

$$2P_{a\hat{b}}P^{a\hat{b}} = 2\mathrm{d}\phi^2 + 2P_{\alpha\hat{\beta}}P^{\alpha\hat{\beta}} + e^{2\phi}\big(p_{L\,\alpha\,I}p_L{}^{\alpha}{}_J + p_{R\,\hat{\alpha}\,I}p_R{}^{\hat{\alpha}}{}_J\big)\,\mathrm{d}a^I\,\mathrm{d}a^J \tag{B.1}$$

with

$$P_{0\hat{0}} = -\mathrm{d}\phi\ , \qquad P_{0\hat{\alpha}} = \frac{1}{\sqrt{2}}e^{\phi}p_{R\,\hat{\alpha}\,I}\,\mathrm{d}a^I, \qquad P_{\alpha\hat{0}} = \frac{1}{\sqrt{2}}e^{\phi}p_{L\,\alpha\,I}\,\mathrm{d}a^I\ . \tag{B.2}$$

Beware that in this section we use the same notations p_L and p_R for both $O(p,q)$ and $O(p-1,q-1)$, so $p_{L\,\alpha\,I}Q^I$ is not $p_{L\,a\,\mathcal{I}}Q^{\mathcal{I}}$ for $a=\alpha$.

One can compute the covariant derivative in tangent frame such that

$$\mathrm{d}Z_a = 2P^{b\hat{c}}\partial_{b\hat{c}}Z_a = 2P^{b\hat{c}}(\mathcal{D}_{b\hat{c}}Z_a - B_{b\hat{c}\,a}{}^dZ_d), \tag{B.3}$$

and similarly for hatted indices. This way one computes that, for any tensor $F_a = (F_0,\, F_\alpha,\, F_{\hat{\alpha}},\, F_{\hat{0}})$, $F_b = (F_0,\, F_\beta,\, F_{\hat{\beta}},\, F_{\hat{0}})$, ...

$$\begin{aligned}
\mathcal{D}_{0\hat{0}}F_a &= -\frac{1}{2}\frac{\partial}{\partial\phi}F_a\ ,\\
\mathcal{D}_{\alpha\hat{0}}F_a &= \frac{1}{\sqrt{2}}e^{-\phi}v^{\text{-}1I}{}_{\alpha}\frac{\partial}{\partial a^I}F_a + \frac{1}{2}\left(F_\alpha, -\delta_{\alpha\beta}F_0, 0, 0\right)\\
\mathcal{D}_{0\hat{\alpha}}F_a &= \frac{1}{\sqrt{2}}e^{-\phi}v^{\text{-}1I}{}_{\hat{\alpha}}\frac{\partial}{\partial a^I}F_b + \frac{1}{2}\left(0, 0, -\delta_{\alpha\beta}F_{\hat{0}}, F_{\hat{\alpha}}\right)\ ,
\end{aligned} \tag{B.4}$$

and finally the operator $\mathcal{D}_{\alpha\beta}$ will only be acting on the moduli fields through the projectors $p^I_{L\,\gamma}$, $p^I_{R\,\hat{\gamma}}$:

$$\mathcal{D}_{\alpha\hat{\beta}}p^I_{L\,\gamma} = \tfrac{1}{2}\delta_{\alpha\gamma}p^I_{R\,\hat{\beta}}, \qquad \mathcal{D}_{\alpha\hat{\beta}}p^I_{L\,\hat{\gamma}} = \tfrac{1}{2}\delta_{\hat{\beta}\hat{\gamma}}p^I_{R\,\alpha}\ . \tag{B.5}$$

Recall the differential equation (2.23)

$$\mathcal{D}^{\hat{c}}_{(e}\mathcal{D}_{f)\,\hat{c}}F_{abcd} = \frac{2-q}{4}\delta_{ef}F_{abcd} + (4-q)\delta_{a)(e}F_{f)(bcd} + 3\delta_{(ab}F_{cd)ef}\ . \tag{B.6}$$

For brevity we define the vector $\vec{F}$

$$\vec{F} = (F_{1111}, F_{111\alpha}, F_{11\alpha\beta}, F_{1\alpha\beta\gamma}, F_{\alpha\beta\gamma\delta})^{\intercal} \tag{B.7}$$

and $\vec{F}_Q$ such that $\vec{F} = \sum_Q \vec{F}_Q e^{2\pi\mathrm{i}Q\cdot a}$. The first component $(e,f)=(0,0)$gives

$$4\mathcal{D}_0{}^{\hat{c}}\mathcal{D}_{0\hat{c}}\vec{F}_Q = (\partial_\phi(\partial_\phi+q-1)-8\pi^2e^{-2\phi}Q_R^2)\vec{F}_Q = -\begin{pmatrix} 5(q-6)F_{1111}\\ 4(q-5)F_{111\alpha}\\ 3(q-4)F_{11\alpha\beta} - 2\delta_{\alpha\beta}F_{1111}\\ 2(q-3)F_{1\alpha\beta\gamma} - 6\delta_{(\alpha\beta}F_{111\gamma)}\\ (q-2)F_{\alpha\beta\gamma\delta} - 12\delta_{(\alpha\beta}F_{11\gamma\delta)}\end{pmatrix}\ . \tag{B.8}$$

Then the action of the differential operator

$$\begin{aligned}
2\mathcal{D}_0{}^{\hat{c}}\mathcal{D}_{\eta\hat{c}}\vec{F}_Q + 2\mathcal{D}_\eta{}^{\hat{c}}\mathcal{D}_{0\hat{c}}\vec{F}_Q &= -2\pi\mathrm{i}\sqrt{2}e^{-\phi}(Q_{L\,\eta}(\partial_\phi+q-2) + 2Q_{R\,\hat{\alpha}}\mathcal{D}_\eta{}^{\hat{\alpha}})\vec{F}_Q\\
&\quad -(\partial_\phi + \frac{q-2}{2})\begin{pmatrix} 4F_{111\eta}\\ 3F_{11\alpha\eta} - \delta_{\eta\alpha}F_{1111}\\ 2F_{1\alpha\beta\eta} - 2\delta_{\eta(\alpha}F_{111\beta)}\\ F_{\alpha\beta\gamma\eta} - 3\delta_{\eta(\alpha}F_{11\beta\gamma)}\\ -4\delta_{\eta(\alpha}F_{1\beta\gamma\delta)}\end{pmatrix}\ ,
\end{aligned} \tag{B.9}$$

allows to obtain the second component $(e,f)=(0,\alpha)$ of the differential equation

$$-2\pi\mathrm{i}\sqrt{2}e^{-\phi}(Q_{L\,\eta}(\partial_\phi+q-2)+2Q_{R\,\hat{\alpha}}\mathcal{D}_\eta{}^{\hat{\alpha}})\vec{F}_Q$$
$$=\begin{pmatrix} 4(\partial_\phi+4)F_{111\eta} \\ 3(\partial_\phi+3)F_{11\alpha\eta}-\delta_{\eta\alpha}(\partial_\phi+q-3)F_{1111} \\ 2(\partial_\phi+2)F_{1\alpha\beta\eta}-2\delta_{\eta(\alpha}(\partial_\phi+q-3)F_{111\beta)}+2\delta_{\alpha\beta}F_{111\eta} \\ (\partial_\phi+1)F_{\alpha\beta\gamma\eta}-3\delta_{\eta(\alpha}(\partial_\phi+q-3)F_{11\beta\gamma)}+6\delta_{(\alpha\beta}F_{11\gamma)\eta} \\ -4\delta_{\eta(\alpha}(\partial_\phi+q-3)F_{1\beta\gamma\delta)}+12\delta_{(\alpha\beta}F_{1\gamma\delta)\eta} \end{pmatrix}. \tag{B.10}$$

The final differential operator for $(e,f)=(\eta,\vartheta)$

$$\begin{aligned} 4\mathcal{D}_{(\eta}{}^{\hat{c}}\mathcal{D}_{\vartheta)\hat{c}}\vec{F}_Q \;=\;& (4\mathcal{D}_{(\eta}{}^{\hat{\gamma}}\mathcal{D}_{\vartheta)\hat{\gamma}}+\delta_{\eta\vartheta}\partial_\phi-8\pi^2e^{-2\phi}Q_{L\,\eta}Q_{L\,\vartheta})\vec{F}_Q \\ & +4\pi\mathrm{i}\sqrt{2}e^{-\phi}Q_{L\,(\eta}\begin{pmatrix} 4F_{111\vartheta)} \\ 3F_{11\alpha|\vartheta)}-\delta_{\vartheta)\alpha}F_{1111} \\ 2F_{1\alpha\beta|\vartheta)}-2\delta_{\vartheta)(\alpha}F_{111\beta)} \\ F_{\alpha\beta\gamma|\vartheta)}-3\delta_{\vartheta)(\alpha}F_{11\beta\gamma)} \\ -4\delta_{\vartheta)(\alpha}F_{1\beta\gamma\delta)} \end{pmatrix} \\ & +\begin{pmatrix} 12F_{11\eta\vartheta}-4\delta_{\eta\vartheta}F_{1111} \\ 6F_{1\alpha\eta\vartheta}-3\delta_{\eta\vartheta}F_{111\alpha}-7\delta_{\alpha(\eta}F_{111\vartheta)} \\ 2F_{\alpha\beta\eta\vartheta}-2\delta_{\eta\vartheta}F_{11\alpha\beta}-10\delta_{\alpha)(\eta}F_{11\vartheta)(\beta}+2\delta_{\alpha)(\eta}\delta_{\vartheta)(\beta}F_{1111} \\ -\delta_{\eta\vartheta}F_{1\alpha\beta\gamma}-9\delta_{\alpha)(\eta}F_{1\vartheta)(\beta\gamma}+6\delta_{\alpha)(\eta}\delta_{\vartheta)(\beta}F_{111\gamma} \\ -4\delta_{\alpha)(\eta}F_{\vartheta)(\beta\gamma\delta}+12\delta_{\alpha)(\eta}\delta_{\vartheta)(\beta}F_{11\gamma\delta} \end{pmatrix}, \end{aligned} \tag{B.11}$$

gives a third differential equation

$$(4\mathcal{D}_{(\eta}{}^{\hat{\gamma}}\mathcal{D}_{\vartheta)\hat{\gamma}}+\delta_{\eta\vartheta}\partial_\phi-8\pi^2e^{-2\phi}Q_{L\,\eta}Q_{L\,\vartheta})\vec{F}_Q+4\pi\mathrm{i}\sqrt{2}e^{-\phi}Q_{L\,(\eta}\begin{pmatrix} 4F_{111\vartheta)} \\ 3F_{11\alpha|\vartheta)}-\delta_{\vartheta)\alpha}F_{1111} \\ 2F_{1\alpha\beta|\vartheta)}-2\delta_{\vartheta)(\alpha}F_{111\beta)} \\ F_{\alpha\beta\gamma|\vartheta)}-3\delta_{\vartheta)(\alpha}F_{11\beta\gamma)} \\ -4\delta_{\vartheta)(\alpha}F_{1\beta\gamma\delta)} \end{pmatrix}$$
$$=-\begin{pmatrix} (q-6)\delta_{\eta\vartheta}F_{1111} \\ (q-5)\delta_{\eta\vartheta}F_{111\alpha}+(q-11)\delta_{\alpha(\eta}F_{111\vartheta)} \\ (q-4)\delta_{\eta\vartheta}F_{11\alpha\beta}-2\delta_{\alpha\beta}F_{11\eta\vartheta}+2(q-9)\delta_{\alpha)(\eta}F_{11\vartheta)(\beta}+2\delta_{\alpha)(\eta}\delta_{\vartheta)(\beta}F_{1111} \\ (q-3)\delta_{\eta\vartheta}F_{1\alpha\beta\gamma}-6\delta_{(\alpha\beta}F_{1\gamma)\eta\vartheta}+3(q-7)\delta_{\alpha)(\eta}F_{1\vartheta)(\beta\gamma}+6\delta_{\alpha)(\eta}\delta_{\vartheta)(\beta}F_{111\gamma} \\ (q-2)\delta_{\eta\vartheta}F_{\alpha\beta\gamma\delta}-12\delta_{(\alpha\beta}F_{\gamma\delta)\eta\vartheta}+4(q-5)\delta_{\alpha)(\eta}F_{\vartheta)(\beta\gamma\delta}+12\delta_{\alpha)(\eta}\delta_{\vartheta)(\beta}F_{11\gamma\delta} \end{pmatrix} \tag{B.12}$$

One can then check that the only exponentially suppressed solution to the three equations (B.8), (B.10) and (B.12) is given, up to a moduli-independent prefactor, by

$$\vec{F}_Q=\begin{pmatrix} F_1^{(4)} \\ Q_{L\,\alpha}F_1^{(3)} \\ Q_{L\,\alpha}Q_{L\,\beta}F_1^{(2)}+\delta_{\alpha\beta}F_2^{(2)} \\ Q_{L\,\alpha}Q_{L\,\beta}Q_{L\,\gamma}F_1^{(1)}+\delta_{(\alpha\beta}Q_{L\,\gamma)}(Q)F_2^{(1)} \\ Q_{L\,\alpha}Q_{L\,\beta}Q_{L\,\gamma}Q_{L\,\delta}F_1^{(0)}+\delta_{(\alpha\beta}Q_{L\,\gamma}Q_{L\,\delta)}F_2^{(0)}+\delta_{(\alpha\beta}\delta_{\gamma\delta)}F_3^{(0)} \end{pmatrix}, \tag{B.13}$$

$$
\begin{aligned}
F_1^{(k)} &= \Big(\frac{\mathrm{i}}{\sqrt{2}}\Big)^k 2^{\frac{q-2}{2}}(2\pi)^{\frac{q-3-2k}{2}}R^{\frac{q-1}{2}}\sqrt{2|Q_R|^2}^{\frac{2k+3-q}{2}}K_{\frac{2k+3-q}{2}}(2\pi R\sqrt{2|Q_R|^2})\\
F_2^{(k)} &= -\Big(\frac{\mathrm{i}}{\sqrt{2}}\Big)^k 2^{\frac{q-4}{2}}\frac{(4-k)(3-k)}{2}(2\pi)^{\frac{q-5-2k}{2}}R^{\frac{q-3}{2}}\sqrt{2|Q_R|^2}^{\frac{2k+5-q}{2}}K_{\frac{2k+5-q}{2}}(2\pi R\sqrt{2|Q_R|^2})\\
F_3^{(0)} &= 3\times 2^{\frac{q-6}{2}}(2\pi)^{\frac{q-7}{2}}R^{\frac{q-5}{2}}\sqrt{2|Q_R|^2}^{\frac{7-q}{2}}K_{\frac{7-q}{2}}(2\pi R\sqrt{2|Q_R|^2})\ ,
\end{aligned}
\tag{B.14}
$$

In particular, the tensorial part of the function $\vec{F}_Q$ is polynomial in $Q_{L\alpha}$, ..., and the rest only depends on the moduli through Q_R^2 and $R = e^{-\phi}$. We conclude that the Fourier coefficient $\vec{F}_Q$ for a fixed Q is uniquely determined by the differential equations (3.17) and (3.22) up to an overal constant corresponding to the measure factor.

The power-low terms satisfy to the same equations for $Q = 0$. One easily computes that the only two solutions are such that

$$
\vec{F} = \begin{pmatrix} (7-q)(9-q)c_0 R^{q-6} \\ 0 \\ (7-q)c_0 R^{q-6}\delta_{\alpha\beta} \\ 0 \\ 3c_0 R^{q-6}\delta_{(\alpha\beta}\delta_{\gamma\delta)} + RF^{p-1,q-1}_{\alpha\beta\gamma\delta} \end{pmatrix}\ ,
\tag{B.15}
$$

for an arbitrary constant c_0 and a solution $F^{p-1,q-1}_{\alpha\beta\gamma\delta}$ to (3.17) and (3.22) on $G_{p-1,q-1}$.

C Polynomials appearing in Fourier modes

In the degeneration limit $O(p,q) \to O(p-1,q-1)$ studied in §4, the monomials $\tilde{P}^{(\ell)}_{\alpha_{h+1}\dots\alpha_4}(Q)$ with $\ell \geq 0$ are of degree $4-2\ell-h$ in Q, and defined by

$$
\begin{aligned}
&\sum_{\ell\geq 0}\tilde{P}^{(\ell)}_{\alpha\beta\gamma\delta}(Q) = Q_{L,\alpha}Q_{L,\beta}Q_{L,\gamma}Q_{L,\delta} - \frac{3}{2\pi}\delta_{(\alpha\beta}Q_{L,\gamma}Q_{L,\delta)} + \frac{3}{16\pi^2}\delta_{(\alpha\beta}\delta_{\gamma\delta)},\\
&\sum_{\ell\geq 0}\tilde{P}^{(\ell)}_{\alpha\beta\gamma}(Q) = Q_{L,\alpha}Q_{L,\beta}Q_{L,\gamma} - \frac{3}{4\pi}Q_{L,(\alpha}\delta_{\beta\gamma)},\\
&\sum_{\ell\geq 0}\tilde{P}^{(\ell)}_{\alpha\beta}(Q) = Q_{L,\alpha}Q_{L,\beta} - \frac{1}{4\pi}\delta_{\alpha\beta},\\
&\sum_{\ell\geq 0}\tilde{P}^{(\ell)}_{\alpha}(Q) = Q_{L,\alpha},\\
&\sum_{\ell\geq 0}\tilde{P}^{(\ell)}(Q) = 1.
\end{aligned}
\tag{C.1}
$$

In the degeneration limit $O(p,q) \to O(p-2,q-2)$ studied in §5, the monomials

$\mathcal{P}^{(\ell)}_{\mu_1\dots\mu_h\,\alpha_{h+1}\dots\alpha_4}(Q'^i,S)$ with $\ell\geq 0$ are of degree $4-2\ell-h$ in Q'^i, and defined by

$$
\begin{aligned}
\sum_{\ell\geq 0}\mathcal{P}^{(\ell)}_{\alpha\beta\gamma\delta}(Q'^i,S) &= Q'^i_{L,(\alpha}Q'^j_{L,\beta}Q'^k_{L,\gamma}Q'^l_{L,\delta)}M_{ij}M_{kl}-\frac{3}{2\pi}\delta_{(\alpha\beta}Q'^i_{L,\gamma}Q'^j_{L,\delta)}M_{ij}+\frac{3}{16\pi^2}\delta_{(\alpha\beta}\delta_{\gamma\delta)},\\
\sum_{\ell\geq 0}\mathcal{P}^{(\ell)}_{\mu\alpha\beta\gamma}(Q'^i,S) &= Q'_{L,\mu(\alpha}Q'^i_{L,\beta}Q'^j_{L,\gamma)}M_{ij}-\frac{3}{4\pi}Q'_{L,\mu(\alpha}\delta_{\beta\gamma)},\\
\sum_{\ell\geq 0}\mathcal{P}^{(\ell)}_{\mu\nu\alpha\beta}(Q'^i,S) &= Q'_{L,\mu\alpha}Q'_{L,\nu\beta}-\frac{1}{4\pi}\delta_{\alpha\beta}\frac{Q'_\mu\cdot Q'_\nu}{Q'_\tau\cdot Q'^\tau},\\
\sum_{\ell\geq 0}\mathcal{P}^{(\ell)}_{\mu\nu\rho\alpha}(Q'^i,S) &= Q'_{L,\mu\alpha}\frac{Q'_\nu\cdot Q'_\rho}{Q'_\tau\cdot Q'^\tau},\\
\sum_{\ell\geq 0}\mathcal{P}^{(\ell)}_{\mu\nu\rho\sigma}(Q'^i,S) &= \frac{Q'_{(\mu}\cdot Q'_\nu\,Q'_\rho\cdot Q'_{\sigma)}}{(Q'_\tau\cdot Q'^\tau)^2},
\end{aligned}
\tag{C.2}
$$

where $M_{ij}=v_{i\mu}v^\mu{}_j$ is the torus metric (5.4), and $Q'_\mu\cdot Q'_\nu=\frac{1}{S_2}\begin{pmatrix}(Q+S_1P)^2 & (Q+S_1P)S_2P\\ (Q+S_1P)S_2P & S_2^2P^2\end{pmatrix}$.

D Tensorial Eisenstein series

In the degeneration limit $O(p,q)\to O(p-2,q-2)$ studied in §5, the power-like terms in (5.29) involve tensorial Eisenstein series that we rewrote as tensorial derivatives of real analytic Eisenstein series, using $\mathcal{D}_{\mu\nu}$ the traceless differential operator on $SL(2,\mathbb{R})/O(2)$. Here we exhibit these relations, and show how this operator can be rewritten in terms of lowering and raising operators $\mathcal{D}_w$ and $\overline{\mathcal{D}}_w$.

The non-holomorphic Eisenstein series

$$
\mathcal{E}_{s,w}(S)=\frac{1}{2\zeta(2s)}\sum_{(c,d)\in\mathbb{Z}^2\smallsetminus\{0,0\}}\frac{S_2^s}{(c+dS)^{s+\frac{w}{2}}(c+d\bar S)^{s-\frac{w}{2}}}
\tag{D.1}
$$

has modular weight $(\frac{w}{2},-\frac{w}{2})$ under $SL(2,\mathbb{Z})$. The raising and lowering operators, $\mathcal{D}_w=2\mathrm{i}S_2\partial_S+\frac{w}{2}$ and $\overline{\mathcal{D}}_w=-2\mathrm{i}S_2\partial_{\bar S}-\frac{w}{2}$ act on $\mathcal{E}_{s,w}(S)$ according to

$$
\mathcal{D}_w\,\mathcal{E}_{s,w}=\Big(s+\frac{w}{2}\Big)\mathcal{E}_{s,w+2},\qquad \overline{\mathcal{D}}_w\,\mathcal{E}_{s,w}=\Big(s-\frac{w}{2}\Big)\mathcal{E}_{s,w-2}.
\tag{D.2}
$$

Non-holomorphic Eisenstein series are thus eigenmodes of the laplacian $\Delta_w=\bar{\mathcal{D}}_{w+2}\mathcal{D}_w$ with eigenvalue $\big(s+\frac{w}{2}\big)\big(s-\frac{w}{2}-1\big)$.

Alternatively, one can denote the momenta and winding along a torus as $z_\mu=m_iv^i_\mu$ with $(m_1,m_2)=(c,d)$, $v_\mu{}^i$ is the vielbein defined in (5.4), such that $z_\mu z^\mu=\frac{1}{S_2}|c+dS|^2$ is invariant under $SL(2,\mathbb{Z})$. The traceless differential operator $\mathcal{D}_{\mu\nu}$ acts as

$$
\mathcal{D}_{\mu\nu}z_\rho=\frac{1}{2}\delta_{\rho(\mu}z_{\nu)}-\frac{1}{4}\delta_{\mu\nu}z_\rho.
\tag{D.3}
$$

One can show that they are related to the lowering and raising operator through

$$
\mathcal{D}_{\mu\nu}=-\frac{1}{2}\sigma^+_{\mu\nu}\,\mathcal{D}_w-\frac{1}{2}\sigma^-_{\mu\nu}\,\bar{\mathcal{D}}_w
\tag{D.4}
$$

where $\sigma^\pm = \frac{1}{2}(\sigma_3 \pm \mathrm{i}\sigma_1)$ and σ_i are the Pauli matrices. By acting on non-holomorphic Eisenstein series of weight 0 with $\mathcal{D}_{\mu\nu}$ and $\mathcal{D}_{(\mu\nu}\mathcal{D}_{\rho\sigma)}$, one obtains the relations

$$\begin{aligned}
&\frac{s}{2}\,\sigma^+_{\mu\nu}\mathcal{E}_{s,2} + \frac{s}{2}\,\sigma^-_{\mu\nu}\mathcal{E}_{s,-2} = \frac{s}{2\zeta(2s)}\sum_{(j,p)}{}' \frac{1}{(z_\tau z^\tau)^s}\left(\frac{z_\mu z_\nu}{z_\tau z^\tau} - \frac{1}{2}\delta_{\mu\nu}\right) \\
&\frac{s(s+1)}{4}\sigma^+_{(\mu\nu}\sigma^+_{\rho\sigma)}\mathcal{E}_{s,4} + \frac{s(s-1)}{4}\sigma^-_{(\mu\nu}\sigma^-_{\rho\sigma)}\mathcal{E}_{s,-4} + s(s-1)\left(\sigma^+_{(\mu\nu}\sigma^-_{\rho\sigma)} - \frac{1}{8}\delta_{(\mu\nu}\delta_{\rho\sigma)}\right)\mathcal{E}_{s,0} = \\
&\qquad\qquad \frac{s(s+1)}{2\zeta(2s)}\sum_{(j,p)}{}' \frac{1}{(z_\tau z^\tau)^s}\left(\frac{z_\mu z_\nu z_\rho z_\sigma}{(z_\tau z^\tau)^2} - \frac{\delta_{(\mu\nu}z_\rho z_{\sigma)}}{z_\tau z^\tau} + \frac{1}{8}\delta_{(\mu\nu}\delta_{\rho\sigma)}\right)
\end{aligned} \tag{D.5}$$

where the second line is traceless.

Now, the components $F^{(p,q),1,0}_{\alpha\beta\mu\nu}$ and $F^{(p,q),1,0}_{\mu\nu\rho\sigma}$ in (5.10) were obtained originally as

$$\begin{aligned}
F^{(p,q),1,0}_{\alpha\beta\mu\nu} &= R^{q-6}\frac{c(0)}{4\pi^2}\left(\frac{8-q}{2}\right)\frac{1}{2\zeta(8-q)}\sum_{(j,p)}{}' \frac{1}{(z_\tau z^\tau)^{\frac{8-q}{2}}}\frac{z_\mu z_\nu}{z_\tau z^\tau}, \\
F^{(p,q),1,0}_{\mu\nu\rho\sigma} &= R^{q-6}\frac{c(0)}{2\pi^2}\left(\frac{8-q}{2}\right)\left(\frac{10-q}{2}\right)\frac{1}{2\zeta(8-q)}\sum_{(j,p)}{}' \frac{1}{(z_\tau z^\tau)^{\frac{8-q}{2}}}\frac{z_\mu z_\nu z_\rho z_\sigma}{(z_\tau z^\tau)^2}
\end{aligned} \tag{D.6}$$

They can be written as in (5.10) by rewritting the relations above, for $s \neq -1$

$$\begin{aligned}
\frac{s}{2\zeta(2s)}\sum_{(j,p)}{}' \frac{1}{(z_\tau z^\tau)^s}\frac{z_\mu z_\nu}{z_\tau z^\tau} &= \frac{s}{2}\left(\delta_{\mu\nu}\mathcal{E}_{s,0} + \sigma^+_{\mu\nu}\mathcal{E}_{s,2} + \sigma^-_{\mu\nu}\mathcal{E}_{s,-2}\right), \\
\frac{s(s+1)}{2\zeta(2s)}\sum_{(j,p)}{}' \frac{1}{(z_\tau z^\tau)^s}\frac{z_\mu z_\nu z_\rho z_\sigma}{(z_\tau z^\tau)^2} &= \frac{s(s+1)}{4}\sigma^+_{(\mu\nu}\sigma^+_{\rho\sigma)}\mathcal{E}_{s,4} + \frac{s(s-1)}{4}\sigma^-_{(\mu\nu}\sigma^-_{\rho\sigma)}\mathcal{E}_{s,-4} \\
&\quad + \frac{s(s+1)}{2}\left(\delta_{(\mu\nu}\sigma^+_{\rho\sigma)}\mathcal{E}_{s,2} + \delta_{(\mu\nu}\sigma^-_{\rho\sigma)}\mathcal{E}_{s,-2}\right) \\
&\quad + \frac{s^2}{2}\left(\sigma^+_{(\mu\nu}\sigma^-_{\rho\sigma)} - \frac{1}{4}\delta_{(\mu\nu}\delta_{\rho\sigma)}\right)\mathcal{E}_{s,0} + \frac{3s(s+1)}{8}\delta_{(\mu\nu}\delta_{\rho\sigma)}\mathcal{E}_{s,0}
\end{aligned} \tag{D.7}$$

In other words, all the tensorial series in (5.29) appearing as low-energy propagators on the torus can be rewritten a combination of $\mathcal{E}_{s,0}$, $\mathcal{D}\mathcal{E}_{s,0}$, $\overline{\mathcal{D}}\mathcal{E}_{s,0}$, $\mathcal{D}^2\mathcal{E}_{s,0}$ and $\overline{\mathcal{D}}^2\mathcal{E}_{s,0}$. This is used extensively to rewrite the 1-PI effective action in four dimensions (5.39).

Similarly, they can also be rewritten using traceless differential operators $\mathcal{D}_{\mu\nu}$ and $\mathcal{D}^2_{\mu\nu\rho\sigma} = \mathcal{D}_{(\mu\nu}\mathcal{D}_{\rho\sigma)} - \frac{1}{4}\delta_{(\mu\nu}\delta_{\rho\sigma)}\mathcal{D}_{\tau\kappa}\mathcal{D}^{\tau\kappa}$

$$\begin{aligned}
\frac{s}{2\zeta(2s)}\sum_{(j,p)}{}' \frac{1}{(z_\tau z^\tau)^s}\frac{z_\mu z_\nu}{z_\tau z^\tau} &= \left(\frac{s}{2}\delta_{\mu\nu} - \mathcal{D}_{\mu\nu}\right)\mathcal{E}_{s,0} \\
\frac{s(s+1)}{2\zeta(2s)}\sum_{(j,p)}{}' \frac{1}{(z_\tau z^\tau)^s}\frac{z_\mu z_\nu z_\rho z_\sigma}{(z_\tau z^\tau)^2} &= \left(\mathcal{D}^2_{\mu\nu\rho\sigma} - (s+1)\delta_{(\mu\nu}\mathcal{D}_{\rho\sigma)} + \frac{3}{8}s(s+1)\delta_{(\mu\nu}\delta_{\rho\sigma)}\right)\mathcal{E}_{s,0}
\end{aligned} \tag{D.8}$$

E Poincaré series and Eisenstein series for $O(p,q,\mathbb{Z})$

In this section, we evaluate the modular integrals (3.28) and (3.29) using the method developed in [50, 44], which keeps invariance under the automorphism group $O(p,q,\mathbb{Z})$ of the lattice $\Lambda_{p,q}$ manifest. The result is expressed as a sum over lattice vectors with fixed norm, which is a special type of Poincaré series for $O(p,q,\mathbb{Z})$. In §E.2, we use a similar method to construct Eisenstein series for $O(p,q,\mathbb{Z})$.

E.1 Poincaré series representation of $F^{p,q}$

The method developed in [50, 44] relies on expressing the factor multiplying the lattice sum in the integrand in terms of a special type of Poincaré series for $\Gamma_0(N)$, known as the Niebur-Poincaré series of weight $w \in 2\mathbb{Z}$,

$$\mathcal{F}_N(s,\kappa,w;\tau) = \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}|_w\gamma \;, \tag{E.1}$$

where $\mathcal{M}_{s,w}(y)$ is the Whittaker function defined in [50, Eq. (2.7)], and $|_w\gamma$ is the Petersson slash operator, $[f|_w\gamma](\tau) = (c\tau+d)^{-k} f(\frac{a\tau+b}{c\tau+d})$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. The series converges absolutely for $\mathrm{Re}(s) > 1$, grows as $\frac{\Gamma(2s)}{\Gamma(s+\frac{w}{2})} q^{-\kappa}$ near the cusp $\tau \to i\infty$ and is regular at the cusp $\tau = 0$. It transforms under the Maass raising and lower operators according to

$$\begin{aligned} D\mathcal{F}_N(s,\kappa,w) =& 2\kappa\left(s+\tfrac{w}{2}\right) \mathcal{F}_N(s,\kappa,w+2) \;, \\ \bar{D}\mathcal{F}_N(s,\kappa,w) =& \frac{1}{8\kappa}\left(s-\tfrac{w}{2}\right) \mathcal{F}_N(s,\kappa,w-2) \;, \end{aligned} \tag{E.2}$$

which implies that it is an eigenmode of the weight w Laplacian on $\mathcal{H}$ with eigenvalue $(s-\frac{w}{2})(s-1+\frac{w}{2})$. In particular, for $w<0$ and $s = 1-\frac{w}{2}$, $\mathcal{F}_N(s,\kappa,w)$ is a harmonic Maass form of weight w. In cases where there exists no cusp form of weight $2-w$, it is actually a weakly holomorphic modular form of weight w [49]. The Fourier expansion of $\mathcal{F}_N(s,\kappa,w) \equiv \mathcal{F}_\infty(s,\kappa,w;\tau)$ around the cusps at ∞ and at 0 is given in [44, Eq. (5.8-10)], in terms of the Kloosterman sums $\mathcal{Z}_{\infty\infty}(m,n;s)$ and $\mathcal{Z}_{0\infty}(m,n;s)$ defined in Eq. A.3 and A.4 of loc. cit.

For $N=1$, one has, by matching the residue of the pole at $\tau = i\infty$,

$$\frac{1}{\Delta(\tau)} = \lim_{s\to 7} \frac{\mathcal{F}_1(s,1,-12;\tau)}{\Gamma(2s)} \;. \tag{E.3}$$

For $N = 2,3,5,7$, using the fact that Δ_k is invariant under the Fricke involution, one has instead

$$\frac{1}{\Delta_k(\tau)} = \lim_{s\to 1+\frac{k}{2}} \frac{\left[\mathcal{F}_N(s,1,-k;\tau) + \hat{\mathcal{F}}_N(s,1,-k;\tau)\right]}{\Gamma(2s)} \;, \tag{E.4}$$

where $\hat{\mathcal{F}}_N(s,\kappa,w;\tau)$ is the image of $\mathcal{F}_N(s,\kappa,w;\tau)$ under the Fricke involution.[16]

[16] For $N=7$, $1/\Delta_3$ is a modular form of odd weight with character $\chi = (\frac{\cdot}{7})$, so the Petersson slash operator $|_w\gamma$ in (E.1) involves an additional factor of $\chi(d)^{-1}$. This results in additional factors of $\chi(d)^{-1}$ and $\chi(c)^{-1}$ in the Kloosterman sums $\mathcal{Z}_{\infty\infty}(m,n;s)$ and $\mathcal{Z}_{0\infty}(m,n;s)$, respectively.

We shall compute the family of integrals

$$\begin{aligned} F^{(p,q)}(\Phi,s,\kappa) =& \frac{1}{\Gamma(2s)} \int_{\Gamma_0(N)\backslash\mathcal{H}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2}\, \Gamma_{\Lambda_{p,q}}\, \mathcal{F}_N(s,\kappa,-\tfrac{p-q}{2};\tau)\ , \\ F^{(p,q)}_{abcd}(\Phi,s,\kappa) =& \frac{1}{\Gamma(2s)} \int_{\Gamma_0(N)\backslash\mathcal{H}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2}\, \Gamma_{\Lambda_{p,q}}[P_{abcd}]\, \mathcal{F}_N(s,\kappa,-\tfrac{p-q}{2}-4;\tau)\ , \end{aligned} \tag{E.5}$$

which converges absolutely for $\mathrm{Re}(s) > \frac{p+q}{4}$. Here, $\Gamma_{\Lambda_{p,q}}[P_{abcd}]$ is the partition function of a N-modular lattice $\Lambda_{p,q}$ of signature (p,q). It follows from the N-modularity property that $|\Lambda^*_{p,q}/\Lambda_{p,q}| = N^{(p+q)/2}$ and that $\Gamma_{\Lambda_{p,q}}[P_{abcd}]$ satisfies

$$\Gamma_{\Lambda_{p,q}}[P_{abcd}](\Phi,\tau) = \left(-\mathrm{i}\tau\sqrt{N}\right)^{-4-\frac{p-q}{2}}\ \Gamma_{\Lambda_{p,q}}[P_{abcd}]\left(\sigma\cdot\Phi,-\frac{1}{N\tau}\right) \tag{E.6}$$

where σ is the $O(p,q,\mathbb{R})$ transformation realizing the isomorphism $\Lambda^*_{p,q} \simeq \Lambda_{p,q}[1/N]$. The desired integrals (4.1) are then obtain by taking a limit

$$\begin{aligned} F^{(p,q)}(\Phi) =& \frac{1}{8} \lim_{s\to 1+\frac{k}{2}} \left[F^{p,q}(\Phi,s,1) + F^{(p,q)}(\sigma\cdot\Phi,s,1)\right] \\ F^{(p,q)}_{abcd}(\Phi) =& \lim_{s\to 1+\frac{k}{2}} \left[F^{(p,q)}_{abcd}(\Phi,s,1) + F^{(p,q)}_{abcd}(\sigma\cdot\Phi,s,1)\right]\ . \end{aligned} \tag{E.7}$$

By unfolding the integration domain against the sum over γ, one obtains, for $\mathrm{Re}(s) > \frac{p+q}{4}$,

$$F^{(p,q)}_{abcd}(s,\kappa) = \frac{1}{\Gamma(2s)} \sum_{Q\in\Lambda_{p,q}} \int_{\mathcal{S}} \mathrm{d}\tau_1 \mathrm{d}\tau_2\, \tau_2^{q/2-2}\, P_{abcd}\, e^{\mathrm{i}\pi(\tau p_L^2 - \bar\tau p_R^2)} \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi\mathrm{i}\tau_1\kappa}, \tag{E.8}$$

where $\mathcal{S}$ denotes the strip $-\frac{1}{2} < \tau_1 < \frac{1}{2}, \tau_2 > 0$. The integral over τ_1 enforces the BPS condition $Q^2 = Q_L^2 - Q_R^2 = 2\kappa$. Decomposing

$$P_{abcd}(Q,\tau_2) = \sum_{0\le\ell\le 2} \tilde P_{abcd,\ell}(Q)\, \tau_2^{-\ell}\ , \tag{E.9}$$

where $\tilde P_{abcd,\ell}$ is a polynomial of degree $4-2\ell$ in Q, and integrating over τ_2, we get

$$\begin{aligned} F^{(p,q)}_{abcd}(s,\kappa) =& \frac{1}{\Gamma(2s)} \sum_{0\le\ell\le 2} (4\pi\kappa)^{\ell+1-\frac{q}{2}} \sum_{\substack{Q\in\Lambda_{p,q}\\ Q^2=2\kappa}} \tilde P_{abcd,\ell}(Q) \left(\frac{Q_L^2}{2\kappa}\right)^{\ell+1-s-\frac{q-w}{2}} \\ & \times\Gamma\left(s+\tfrac{q-w}{2}-\ell-1\right)\, {}_2F_1\left(s+\tfrac{w}{2}, s+\tfrac{q-w}{2}-\ell-1; 2s; \frac{2\kappa}{Q_L^2}\right) \\ =& \frac{1}{\Gamma(2s)} \sum_{1\le k\le 3} (4\pi\kappa)^{\ell+1-\frac{q}{2}} \sum_{\substack{Q\in\Lambda_{p,q}\\ Q^2=2\kappa}} \tilde P_{abcd,\ell}(Q) \left(\frac{Q_R^2}{2\kappa}\right)^{\ell+1-s-\frac{q-w}{2}} \\ & \times\Gamma\left(s+\tfrac{q-w}{2}-\ell-1\right)\, {}_2F_1\left(s-\tfrac{w}{2}, s+\tfrac{q-w}{2}-\ell-1; 2s; -\frac{2\kappa}{Q_R^2}\right) \end{aligned} \tag{E.10}$$

where in the second line, we used Pfaff's equality ${}_2F_1(a,b;c;z) = (1-z)^{-b}{}_2F_1(b,c-a;c;\frac{z}{z-1})$. Similarly, for the scalar integral we get

$$F^{(p,q)}(s,\kappa) = \frac{(4\pi\kappa)^{1-\frac{q}{2}}}{\Gamma(2s)} \sum_{\substack{Q\in\Lambda_{p,q}\\ Q^2=2\kappa}} \left(\frac{Q_R^2}{2\kappa}\right)^{1-s-\frac{p+q}{4}} {}_2F_1\left(s+\tfrac{q}{4}, s+\tfrac{p+q}{4}-1; 2s; -\frac{2\kappa}{Q_R^2}\right) \tag{E.11}$$

For $q<6$, the series (E.10) and (E.11) are absolutely convergent at $s=1+\frac{k}{2}$, so the limit (E.7) can be taken term by term. For $q\geq 6$, the limit must be taken after analytically continuing the sum, and subtracting the pole when $q=6$. In either case, the series (E.10) and (E.11) correctly encode the singular behavior of the integral at codimension-q singularities in $G_{p,q}$ where $P_R^2\to 0$ for a norm 2κ in $\Lambda_{p,q}$ or $Q_R^2\to 0$ for a norm $2\kappa/N$ vector in $\Lambda^*_{p,q}$. Near these loci, the leading singular behavior of (E.10) is given, for $\kappa=1$, by

$$F^{(p,q)}_{abcd} \sim \frac{\Gamma\left(\frac{q-2}{2}\right)}{(2\pi)^{\frac{q-2}{2}}}\left[\frac{Q_{L,a}Q_{L,b}Q_{L,c}Q_{L,d}}{(Q_R^2)^{\frac{q-2}{2}}} - \frac{6}{q-4}\frac{\delta_{(ab}Q_{L,c}Q_{L,d)}}{(Q_R^2)^{\frac{q-4}{2}}} - \frac{3}{(q-6)(q-4)}\frac{\delta_{(ab}\delta_{cd)}}{(Q_R^2)^{\frac{q-6}{2}}}\right] \tag{E.12}$$

and similarly for $F^{(p,q)}$.

Using the same argument as in (3.51) and making use of (E.2), it is easy to show that the integrals (E.5) satisfy the differential equation

$$\begin{aligned}\mathcal{D}^2_{ef}\, F^{(p,q)}_{abcd}(s) =& (2-q)\,\delta_{ef}\, F^{(p,q)}_{abcd}(s) + (16-4q)\,\delta_{e)(a}\, F^{(p,q)}_{bcd)(f}(s) + 12\,\delta_{(ab}\, F_{cd)ef}(s) \\ &+ \int_{\Gamma_0(N)\backslash\mathcal{H}} \frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\frac{2(2s+k)}{2\kappa\Gamma(2s)}\,\mathcal{F}_N(s,\kappa,-k-2)\,\Gamma_{\Lambda_{p,q}}[P_{abcdef}]\end{aligned} \tag{E.13}$$

The modular integral on the second line can again by evaluating by the unfolding trick, as a sum over vectors $Q\in\Lambda_{p,q}$ with $Q^2=2\kappa$. For the relevant value $s=1+\frac{k}{2}$ with small $|k|$, such that $\mathcal{F}_N(s,\kappa,-k)$ is weakly holomorphic, $\mathcal{F}_N(s,\kappa,-k-2)$ vanishes so the sum over Q must vanish. We have checked that this is indeed the case in the Euclidean case $q=0, N=1$, such that only a finite number of vectors Q contribute.

E.2 Eisenstein series for $O(p,q,\mathbb{Z})$

While the modular integrals (E.5) result into automorphic forms with singularities on $G_{p,q}$, due to the pole of order κ in the Niebur-Poincaré series $\mathcal{F}_N(s,\kappa,w;\tau)$, it is useful to consider the analogue

$$E^{(p,q)}(\Phi,s) = \int_{\Gamma_0(N)\backslash\mathcal{H}} \frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\,\Gamma_{\Lambda_{p,q}}\, E_N(s,-\tfrac{p-q}{2};\tau)\ , \tag{E.14}$$

where $\mathcal{F}_N(s,\kappa,w;\tau)$ is replaced by the non-holomorphic Eisenstein series for $\Gamma_0(N)$,

$$E_N(s,w;\tau) = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \tau_2^{s-\frac{w}{2}}|_w\gamma\ , \tag{E.15}$$

which can be obtained formally by taking the limit $\kappa\to 0$ in (E.1). The integral converges for $\mathrm{Re}(s) > \frac{p+q-2}{2}$, and can be computed using the unfolding trick, leading to a standard vectorial Eisenstein series for $O(p,q,\mathbb{Z})$, the automorphism group of $\Lambda_{p,q}$,

$$E^{(p,q)}(\Phi,s) = \pi^{-s'}\,\Gamma(s') \sum_{\substack{P\in\Lambda_{p,q}\smallsetminus\{0\}\\ P^2=0}} \frac{1}{(P_L^2+P_R^2)^{s'}} \tag{E.16}$$

with $s' = s + \frac{p+q}{4} - 1$. Another Eisenstein series for the same group is obtained by replacing $E_N(s, w; \tau)$ by its image under the Fricke involution, which amounts to changing $\Phi \mapsto \sigma \cdot \Phi$ in (E.16). Unlike (E.5), both Eisenstein series are smooth automorphic forms on $G_{p,q}$. Their behavior in the degeneration limits $O(p, q) \to O(p - 1, q - 1)$ and $O(p, q) \to O(p - 2, q - 2)$ is easily obtained by applying the same methods as in §4 and §5. In particular, the constant terms proportional to $\tau_2^{s-\frac{w}{2}}$ and to $\tau_2^{1-s-\frac{w}{2}}$ in the Fourier expansion of $E_N(s, w; \tau)$ lead to power-like terms proportional to $R^{2s'}$ and $R^{p+q-2-2s'}$ in the degeneration limit $O(p, q) \to O(p - 1, q - 1)$.

By direct computation, or using the fact that $E_N(s, w; \tau)$ is an eigenmode of the weight w Laplacian on $\mathcal{H}$ with eigenvalue $(s - \frac{w}{2})(s - 1 + \frac{w}{2})$, one sees that

$$\Delta_{G_{p,q}} E^{(p,q)}(\Phi, s) = s'(2s' - p - q + 2)\, E^{(p,q)}(\Phi, s) \ . \tag{E.17}$$

For $s' = \frac{p+4}{2}$, corresponding to $s = 3 + \frac{p-q}{2}$, the eigenvalue coincides with the eigenvalue of $F^{(p,q)}$ in (3.27) (the other value $s' = \frac{q-6}{2}$, $s = -2 - \frac{p-q}{2}$ lies outside the fundamental domain, and is related to the former by the functional equation $s \mapsto 1 - s$). Moreover, using the same methods as in §3.2 it is easy to check that $E^{(p,q)}(\Phi, s)$ satisfies the second constraint in (3.27). It is thus natural to ask if the exact $(\nabla\Phi)^4$ coupling could involve an extra term proportional to $E^{(p,q)}(\Phi, 3 + \frac{p-q}{2})$ in addition to the proposed formula (2.27). However, it turns out that the latter contains terms of order R^{p+4} and R^{q-6} in the degeneration limit $O(p, q) \to O(p - 1, q - 1)$ with a non-zero coefficient, respectively, and the first term R^{p+4} is ruled out by the differential equation (3.22).

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Appendix C [BCHP3]

Exact effective interactions and 1/4-BPS dyons in heterotic CHL orbifolds

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Abstract

Motivated by precision counting of BPS black holes, we analyze six-derivative couplings in the low energy effective action of three-dimensional string vacua with 16 supercharges. Based on perturbative computations up to two-loop, supersymmetry and duality arguments, we conjecture that the exact coefficient of the $\nabla^2(\nabla\phi)^4$ effective interaction is given by a genus-two modular integral of a Siegel theta series for the non-perturbative Narain lattice times a specific meromorphic Siegel modular form. The latter is familiar from the Dijkgraaf-Verlinde-Verlinde (DVV) conjecture on exact degeneracies of 1/4-BPS dyons. We show that this Ansatz reproduces the known perturbative corrections at weak heterotic coupling, including tree-level, one- and two-loop corrections, plus non-perturbative effects of order e^{-1/g_3^2}. We also examine the weak coupling expansions in type I and type II string duals and find agreement with known perturbative results. In the limit where a circle in the internal torus decompactifies, our Ansatz predicts the exact $\nabla^2 F^4$ effective interaction in four-dimensional CHL string vacua, along with infinite series of exponentially suppressed corrections of order e^{-R} from Euclideanized BPS black holes winding around the circle, and further suppressed corrections of order e^{-R^2} from Taub-NUT instantons. We show that instanton corrections from 1/4-BPS black holes are precisely weighted by the BPS index predicted from the DVV formula, including the detailed moduli dependence. We also extract two-instanton corrections from pairs of 1/2-BPS black holes, demonstrating consistency with supersymmetry and wall-crossing, and estimate the size of instanton-anti-instanton contributions.

Contents

1 Introduction

Providing a statistical origin of the thermodynamic entropy of black holes is a key goal for any theory of quantum gravity. More than two decades ago, Strominger and Vafa demonstrated that D-branes of type II string theories provide the correct number of micro-states for supersymmetric black holes in the large charge limit [1]. Since then, much work has gone into performing precise counting of black hole micro-states and comparing with macroscopic supergravity predictions. In vacua with extended supersymmetry, it was found that exact degeneracies of five-dimensional BPS black holes (counted with signs) are given by Fourier coefficients of weak Jacobi forms, giving access to their large charge asymptotics [2, 3, 4]. With hindsight, the modular invariance of the partition function of BPS black holes follows from the existence of an AdS_3 factor in the near-horizon geometry of these extremal black holes.

In a prescient work [5], Dijkgraaf, Verlinde and Verlinde (DVV) conjectured that four-dimensional BPS black holes in type II string theory compactified on $K3\times T^2$ (or equivalently, heterotic string on T^6) are in fact Fourier coefficients of a meromorphic Siegel modular form, invariant under a larger $Sp(4,\mathbb{Z})$ symmetry. This conjecture was subsequently extended to other four-dimensional vacua with 16 supercharges [6], proven using D-brane techniques [7, 8], and refined to properly incorporate the dependence on the moduli at infinity [9], but the origin of the $Sp(4,\mathbb{Z})$ symmetry had remained obscure. In [10, 11, 12], it was noted that a class of 1/4-BPS dyons arises from string networks which lift to M5-branes wrapped on $K3$ times a genus-two curve, but this observation did not lead to a transparent derivation of the DVV formula.

In [13], implementing a strategy advocated earlier in [14], we revisited this problem by analyzing certain protected couplings in the low energy effective action of the four-dimensional string theory compactified on a circle of radius R down to three space-time dimensions. In three-dimensional string vacua with 16 or more supercharges, the massless degrees of freedom are described by a non-linear sigma model on a symmetric manifold G_3/K_3, which contains the four-dimensional moduli space $\mathcal{M}_4 = G_4/K_4$, the holonomies a^i_I of the four-dimensional gauge fields, the NUT potential ψ dual to the Kaluza–Klein vector and the circle radius R. Since stationary solutions with finite energy in dimension 4 yield finite action solutions in dimension 3, it is expected that black holes of mass $\mathcal{M}$ and charge $\Gamma^I_i = (Q^I, P^I)$ in 4 dimensions which break $2r$ supercharges induce instantonic corrections of order $e^{-2\pi R\mathcal{M}+2\pi \mathrm{i} a^i_I \Gamma^I_i}$ to effective couplings with $2r$ fermions (or r derivatives) in dimension 3 (see e.g. [15]); and moreover that these corrections are weighted by the helicity supertrace

$$\Omega_{2r}(\Gamma) = \frac{1}{n!}\mathrm{Tr}_\Gamma[(-1)^F(2h)^n] \ , \tag{1.1}$$

where F is the fermionic parity and h is the helicity in $D = 4$. In addition, there are corrections of order $e^{-2\pi R^2|M_1|+2\pi \mathrm{i} M_1\psi}$ from Euclidean Taub-NUT instantons which asymptote to $\mathbb{R}^3\times S^1$, where the circle is fibered with charge M_1 over the two-sphere at spatial infinity. While the two-derivative effective action is uncorrected and invariant under the full continuous group G_3, higher-derivative couplings need only be invariant under an arithmetic subgroup $G_3(\mathbb{Z})$ known as the U-duality group. For string vacua with 32 supercharges, the $\mathcal{R}^4$, $\nabla^4\mathcal{R}^4$ and $\nabla^6\mathcal{R}^4$ effective interactions are expected to receive instanton corrections from 1/2-BPS, 1/4-BPS and 1/8-BPS black holes, respectively. In [16], two of the present authors demonstrated that the exact $\mathcal{R}^4$, $\nabla^4\mathcal{R}^4$ couplings, given by Eisenstein series for the U-duality group $G_3(\mathbb{Z}) = E_8(\mathbb{Z})$ [17, 18, 19, 20], indeed reproduce the respective helicity supertraces Ω_4 and Ω_6 for 1/2-BPS and 1/4-BPS black holes in dimension 4. At the time of writing, a similar check for the $\nabla^6\mathcal{R}^4$ coupling conjectured in [21] still remains to be performed.

For three-dimensional string vacua with 16 supercharges, the scalar fields span a symmetric space of the form

$$G_3/K_3 = O(2k, 8)/[O(2k) \times O(8)] \ , \tag{1.2}$$

for a model-dependent integer k, which extends the moduli space

$$G_4/K_4 = SL(2)/SO(2) \times O(2k-2, 6)/[O(2k-2) \times O(6)] \tag{1.3}$$

in $D = 4$. The four-derivative scalar couplings of the form $F(\phi)(\nabla\phi)^4$ are expected to receive instanton corrections from 1/2-BPS black holes, along with Taub-NUT instantons, while six-derivative scalar couplings of the form $G(\phi)\nabla^2(\nabla\phi)^4$ receive instanton corrections both from four-dimensional 1/2-BPS and 1/4-BPS black holes, along with Taub-NUT instantons. In [13], we restricted for simplicity to the maximal rank case ($k = 12$) arising in heterotic string compactified on T^7 (or equivalently type II string theory compactified on $K3 \times T^3$). Using low order perturbative computations, supersymmetric Ward identities and invariance under the U-duality group $G_3(\mathbb{Z}) \subset O(24, 8, \mathbb{Z})$, we determined the tensorial coefficients $F_{abcd}(\phi)$ and $G_{ab,cd}(\phi)$ of the above couplings exactly, for all values of the string coupling. In either case, the non-perturbative coupling is given by a U-duality invariant generalization of the genus-one and genus-two contribution, respectively:

$$F^{(24,8)}_{abcd} = \text{R.N.} \int_{SL(2,\mathbb{Z})\backslash\mathcal{H}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\Gamma_{\Lambda_{24,8}}[P_{abcd}]}{\Delta} , \tag{1.4}$$

$$G^{(24,8)}_{ab,cd} = \text{R.N.} \int_{Sp(4,\mathbb{Z})\backslash\mathcal{H}_2} \frac{\mathrm{d}^3\Omega_1 \mathrm{d}^3\Omega_2}{|\Omega_2|^3} \frac{\Gamma^{(2)}_{\Lambda_{24,8}}[P_{ab,cd}]}{\Phi_{10}} , \tag{1.5}$$

where $\mathcal{H}_h$ is the Siegel upper half-plane of degree h, $\Gamma^{(h)}_{\Gamma_{\Lambda_{p,q}}}[P]$ is a genus-h Siegel-Narain theta series for a lattice of signature (p,q) with a polynomial insertion (see (B.4) and (2.32) for the genus-one and two cases), Δ and Φ_{10} are the modular discriminants in genus-one and two, and R.N. denotes a specific regularization prescription. We demonstrated that the Ansätze (1.4) and (1.5) satisfy the relevant supersymmetric Ward identities, and that their asymptotic expansion at weak heterotic string coupling $g_3 \to 0$ reproduces the known perturbative contributions, up to one-loop and two-loop, respectively, plus an infinite series of $\mathcal{O}(e^{-1/g_3^2})$ corrections ascribed to NS5-instantons, Kaluza–Klein (6,1)-branes and H-monopole instantons. We went on to analyze the limit $R \to \infty$ and demonstrated that the $\mathcal{O}(e^{-R})$ corrections to $F^{(24,8)}_{abcd}$ and to $G^{(24,8)}_{ab,cd}$ were proportional to the known helicity supertraces of 1/2- and 1/4-BPS four-dimensional black holes, respectively. In particular, the DVV formula for the index of 1/4-BPS states [5], with the correct contour prescription [9], emerges in a transparent fashion after unfolding the integral over the fundamental domain $Sp(4,\mathbb{Z})\backslash\mathcal{H}_2$ onto the full Siegel upper-half plane $\mathcal{H}_2$.

In [22], we extended the study of the 1/2-BPS saturated coupling $F^{(24,8)}_{abcd}$ to the case of CHL heterotic orbifolds of prime order $N = 2, 3, 5, 7$. All these models have 16 supercharges, and their moduli space in $D = 3$ or 4 is of the form (1.2), (1.3) with $k = 24/(N+1)$. After conjecturing the precise form of the U-duality group $G_3(\mathbb{Z}) \subset O(2k, 8, \mathbb{Z})$ in $D = 3$, we proposed an exact formula for F_{abcd} analogous to (1.4),

$$F^{(2k,8)}_{abcd} = \text{R.N.} \int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\Gamma_{\Lambda_{2k,8}}[P_{abcd}]}{\Delta_k} , \tag{1.6}$$

where $\Gamma_0(N) \subset SL(2,\mathbb{Z})$ is the Hecke congruence subgroup of level N, Δ_k is the unique cusp form of weight k under $\Gamma_0(N)$ and $\Lambda_{2k,8}$ is the 'non-perturbative Narain lattice' of signature $(2k, 8)$. We studied the weak coupling and large radius limits of the Ansatz (1.6), and found that it reproduces correctly the known tree-level and one-loop contributions in the limit $g_3 \to 0$, powerlike corrections in the limit $R \to \infty$, as well as infinite series of instanton corrections consistent with the known helicity supertrace of 1/2-BPS states in $D = 4$, for all orbits of the U-duality group $G_4(\mathbb{Z})$.

The goal of the present work is to provide strong evidence that the tensorial coefficient $G_{ab,cd}$ of the 1/4-BPS saturated coupling $\nabla^2(\nabla\phi)^4$ in the same class of CHL orbifolds is given similarly by the natural extension of (1.5), namely

$$G^{(2k,8)}_{ab,cd} = \text{R.N.} \int_{\Gamma_{2,0}(N)\backslash\mathcal{H}_2} \frac{\mathrm{d}^3\Omega_1 \mathrm{d}^3\Omega_2}{|\Omega_2|^3} \frac{\Gamma^{(2)}_{\Lambda_{2k,8}}[P_{ab,cd}]}{\Phi_{k-2}} , \tag{1.7}$$

where $\Gamma_{2,0}(N)$ is a congruence subgroup of level N inside $Sp(4,\mathbb{Z})$, Φ_{k-2} is a specific meromorphic Siegel modular form of weight $k-2$ and $\Gamma^{(2)}_{\Lambda_{2k,8}}[P_{ab,cd}]$ is a suitable genus-two Siegel-Narain theta series for the same non-perturbative Narain lattice $\Lambda_{2k,8}$ as in (1.6). Using similar techniques as in [22], we shall demonstrate that the Ansatz (1.7) satisfies the relevant supersymmetric Ward identities and produces the correct tree-level, one-loop and two-loop terms in the weak heterotic coupling limit, or powerlike terms in the large radius limit, accompanied by infinite series of instanton corrections consistent with the helicity supertraces of 1/2-BPS and 1/4-BPS states in $D = 4$.

A significant feature and complication of (1.5),(1.7) compared to the 1/2-BPS coupling (1.4), (1.6) is that the integrand $1/\Phi_{k-2}$ has a double pole on the diagonal locus $\Omega_{12} = 0$ and its images under $\Gamma_{2,0}(N)$ (corresponding to the separating degeneration of the genus-two Riemann surface with period matrix Ω). In the context of the DVV formula, these poles are well-known to be responsible for the moduli dependence of the helicity supertrace Ω_6. In the context of the BPS coupling (1.7), these poles are responsible for the fact that the weak coupling and large radius expansions receive infinite series of instanton anti-instanton contributions, as required by the quadratic source term in the differential equation (2.26) for the coefficient $G^{(2k,8)}_{ab,cd}$. A similar phenomenon is encountered in the case of the $\nabla^6\mathcal{R}^4$ couplings in maximal supersymmetric string vacua [23].

Organization

This work is organized as follows. In §2 we recall relevant facts about the moduli space, duality group and BPS spectrum of heterotic CHL models in $D = 4$ and $D = 3$, record the known perturbative contributions to the $\nabla^2 F^4$ and $\nabla^2(\nabla\phi)^4$ couplings in heterotic perturbation theory, and preview our main results. In §3, we derive the differential constraints imposed by supersymmetry on these couplings, and show that they are obeyed by the Ansatz (1.7). In §4, we study the expansion of (1.7) at weak heterotic coupling, and show that it correctly reproduces the known pertubative contributions, along with an infinite series of NS5-brane, Kaluza–Klein (6,1)-branes and H-monopole instanton corrections. In §5, we move to the central topic of this work and study the large radius limit of the Ansatz (1.7). We obtain the exact $\nabla^2 F^4$ and $\mathcal{R}^2 F^2$ couplings in $D = 4$ plus infinite series of $\mathcal{O}(e^{-R})$ and $\mathcal{O}(e^{-R^2})$ corrections. We extract from the former the helicity supertrace of 1/4-BPS black holes with arbitrary charge, and recover the DVV formula and its generalizations. We further analyze two-instanton contributions from pairs of 1/2-BPS black holes and show their consistency with wall-crossing. In §6 we study the weak coupling limit of the $\nabla^2(\nabla\phi)^4$ couplings in CHL orbifolds of type II string on $K3 \times T^3$, and of the related $\nabla^2 H^4$ couplings in type IIB compactified on K3 down to six dimensions.

A number of more technical developments are relegated to appendices. In Appendix A we collect relevant facts about genus-two Siegel modular forms, and the structure of their Fourier and Fourier-Jacobi expansions. In §B we compute the one-loop and two-loop contributions to the $\nabla^2 F^4$ and $\nabla^2(\nabla\phi)^4$ couplings in CHL models, spell out the regularization of the corresponding modular integrals, compute the anomalous terms in the differential constraints due to boundary contributions, and discuss their behavior near points of enhanced gauge symmetry. In §C, we verify that the polar contributions to the Fourier coefficients of $1/\Phi_{k-2}$ are in one-to-one correspondence with the possible splittings $\Gamma = \Gamma_1 + \Gamma_2$ of a 1/4-BPS charge Γ into a pair of 1/2-BPS charges Γ_1, Γ_2. In §D, we use this information to compute the singular contributions to Abelian Fourier coefficients with generic 1/4-BPS charge, and in §E demonstrate that the structure of these coefficients and of the constant terms is consistent with the differential constraint. In §F, we also estimate the corrections to the saddle point value of the Abelian Fourier coefficients, due to the non-constancy of the Fourier coefficients of $1/\Phi_{k-2}$ and show that they are of the size expected for two-instanton effects on the one hand, and Taub-NUT instanton – anti-instanton on the other hand. In §G, we explain how to infer the non-Abelian Fourier coefficients with respect to $O(p-2,q-2)$ from the knowledge of the Abelian coefficients with respect to $O(p-1,q-1)$. Finally, §H collects definitions of various polynomials which enter in the formulae of §4 and §5.1.

Note: The structure of the body of this paper follows that of our previous work [22] on 1/2-BPS couplings, so as to facilitate comparison between our treatments of the genus-one and genus-two modular integrals. The reader is invited to refer to [22] for more details on points discussed cursorily herein.

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2 BPS spectrum and BPS couplings in CHL vacua

In this section, we recall relevant facts about the moduli space, duality group and BPS spectrum of heterotic CHL models in $D=4$ and $D=3$, and summarize the main features of our Ansatz for the exact $\nabla^2(\nabla\phi)^4$ and $\nabla^2 F^4$ couplings in these models.

2.1 Moduli spaces and dualities

Recall that heterotic CHL models are freely acting orbifolds of the heterotic string compactified on a torus, preserving 16 supersymmetries [24, 25] (see [26] for a review of these constructions). We shall be mostly interested in models with $D=4$ non-compact spacetime dimensions, and the reduction of those models on a circle down to $D=3$. Furthermore, for simplicity we restrict to $\mathbb{Z}_N$ orbifolds with $N\in\{1,2,3,5,7\}$ prime, in which case the gauge symmetry in $D=4$ is reduced from $U(1)^{28}$ in the original 'maximal rank' model (namely, heterotic string compactified on T^6) to $U(1)^{2k+4}$ with $k=24/(N+1)$. The lattice of electromagnetic charges in $D=4$ is a direct sum $\Lambda_{em}=\Lambda_e\oplus\Lambda_m$ where Λ_m is an even, lattice of signature $(2k-2,6)$ and $\Lambda_e\equiv\Lambda_m^*$ its dual (see Table 1 in [22]). While Λ_m is not self-dual for $N>1$, it is N-modular in the sense that $\Lambda_m^*=\Lambda_m[1/N]$ [27], in particular we have the chain of inclusions $N\Lambda_m\subset N\Lambda_m^*\subset\Lambda_m\subset\Lambda_m^*$.

The moduli space in $D=4$ is a quotient

$$\mathcal{M}_4 = G_4(\mathbb{Z})\backslash\left[SL(2,\mathbb{R})/SO(2)\times G_{2k-2,6}\right] \ , \tag{2.1}$$

where $SL(2,\mathbb{R})/SO(2)$ is parametrized by the heterotic axiodilaton S and the Grassmannian $G_{2k-2,6}$ is parametrized by the scalars φ in the vector multiplets. Here and elsewere, $G_{p,q}$ will denote the orthogonal Grassmannian $O(p,q)/[O(p)\times O(q)]$ of negative q-planes in $\mathbb{R}^{p,q}$. The U-duality group $G_4(\mathbb{Z})$ in $D=4$ includes the S-duality group $\Gamma_1(N)$ acting on the first factor and the T-duality group $\widetilde{O}(2k-2,6,\mathbb{Z})$ acting on the second (where $\widetilde{O}(2k-2,6,\mathbb{Z})$ denotes the restricted automorphism group of Λ_m, acting trivially on the discriminant group $\Lambda_e/\Lambda_m\sim\mathbb{Z}_N^{k+2}$). As discussed in [27, 22], there are strong indications that BPS observables are invariant under the larger group $\Gamma_0(N)\times O(2k-2,6,\mathbb{Z})$, the automorphism group of Λ_m, along with the Fricke involution acting by $S\mapsto -1/(NS)$ on the first factor, accompanied by a suitable action $\varphi\mapsto\varsigma\cdot\varphi$ of $\varsigma\in O(2k-2,6,\mathbb{R})$ on the second factor, such that $\Lambda_m^*=\varsigma\cdot\Lambda_m/\sqrt{N}$.

After compactification on a circle of radius R down to $D=3$, the moduli space spanned by the scalars $\phi=(R,S,\varphi,a^{Ii},\psi)$ described in the introduction becomes a quotient

$$\mathcal{M}_3 = G_3(\mathbb{Z})\backslash G_{2k,8} \tag{2.2}$$

of the orthogonal Grassmannian $G_{2k,8}$ by the U-duality group in $D=3$. In [22] we provided evidence that the U-duality group includes the restricted automorphism group $\widetilde{O}(2k,8,\mathbb{Z})$ of the 'non-perturbative Narain lattice'

$$\Lambda_{2k,8} = \Lambda_m\oplus I\!I_{1,1}\oplus I\!I_{1,1}[N] \ , \tag{2.3}$$

which is also N-modular. It also includes the U-duality group $G_4(\mathbb{Z})$ as well as the restricted automorphism group $\widetilde{O}(2k-1,7,\mathbb{Z})$ of the perturbative Narain lattice $\Lambda_m\oplus I\!I_{1,1}$. The exact four and six-derivative couplings of interest in this paper will turn out to be invariant under the full automorphism group $O(2k,8,\mathbb{Z})\supset G_3(\mathbb{Z})$, however, this group is not expected to be a symmetry of the full spectrum. In particular, the automorphism group of the perturbative lattice $O(2k-1,7,\mathbb{Z})$ does not preserve the orbifold projection, and does not act consistently on states that are not invariant under the $\mathbb{Z}_N$ action on the circle. Nevertheless, we expect the U-duality group to be larger than $\widetilde{O}(2k,8,\mathbb{Z})$ and to include in particular Fricke duality.

An important consequence of the enhancement of T-duality group $\widetilde{O}(2k-1,7,\mathbb{Z})$ to the U-duality group $\widetilde{O}(2k,8,\mathbb{Z})$ is that singularities in the low energy effective action occur on codimension-8 loci in the full moduli space $\mathcal{M}_3$, partially resolving the singularities which occur at each order in the perturbative expansion on codimension-7 loci where the gauge symmetry is enhanced.

2.2 BPS dyons in $D=4$

We now review relevant facts about helicity supertraces of 1/2-BPS and 1/4-BPS states in heterotic CHL orbifolds. As mentioned above, the lattice of electromagnetic charges $\Gamma=(Q,P)$ decomposes into $\Lambda_{em}=\Lambda_m^*\oplus\Lambda_m$, where on the heterotic side the first factor corresponds to electric charges Q carried by fundamental heterotic strings, while the second factor corresponds to magnetic charges P carried by heterotic five-brane, Kaluza-Klein (6,1)-brane and H-monopoles. The lattices $\Lambda_e=\Lambda_m^*$ and Λ_m carry quadratic forms such that

$$Q^2\in\frac{2}{N}\mathbb{Z}\ ,\quad P^2\in 2\mathbb{Z},\quad P\cdot Q\in\mathbb{Z}\ , \tag{2.4}$$

while Λ_{em} carries the symplectic Dirac pairing $\langle\Gamma,\Gamma'\rangle=Q\cdot P'-Q'\cdot P\in\mathbb{Z}$. A generic BPS state with charge $\Gamma\in\Lambda_{em}$ such that $Q\wedge P\neq 0$ (*i.e.* when Q and P are not collinear) preserves 1/4 of the 16 supercharges, and has mass

$$\mathcal{M}(\Gamma;t)=\sqrt{2\frac{|Q_R+SP_R|^2}{S_2}+4\sqrt{\begin{vmatrix}Q_R^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & P_R^2\end{vmatrix}}} \tag{2.5}$$

where $t=(S,\varphi)$ denote the set of all coordinates on (2.1), and Q_R,P_R are the projections of the charges Q,P on the negative 6-plane parametrized by $\varphi\in G_{2k-2,6}$. When $Q\wedge P=0$, the state preserves half of the 16 supercharges, and the mass formula (2.5) reduces to $\mathcal{M}(\Gamma)^2=2|Q_R+SP_R|^2/S_2$.

In order to describe the helicity traces carried by these states, it is useful to distinguish 'untwisted' 1/2-BPS states, characterized by the fact that their charge vector (Q,P) lies in the sublattice $\Lambda_m\oplus N\Lambda_e\subset\Lambda_e\oplus\Lambda_m$, from 'twisted' 1/2-BPS states where (Q,P) lies in the complement of this sublattice inside Λ_{em}. One can show that twisted 1/2-BPS states lie in two different orbits of the S-duality group $\Gamma_0(N)$: they are either dual to a purely electric state of charge $(Q,0)$ with $Q\in\Lambda_e\smallsetminus\Lambda_m$, or to a purely magnetic state of charge $(0,P)$ with $P\in\Lambda_m\smallsetminus N\Lambda_e$. Similarly, untwisted 1/2-BPS states are either dual to a purely electric state of charge $(Q,0)$ with $Q\in\Lambda_m$, or to a purely magnetic state of charge $(0,P)$ with $P\in N\Lambda_e$. The fourth helicity supertrace is sensitive to 1/2-BPS states only, and is given by

$$\Omega_4(\Gamma)=c_k\left(-\tfrac{\gcd(NQ^2,P^2,Q\cdot P)}{2}\right) \tag{2.6}$$

for twisted electromagnetic charge $\Gamma\in(\Lambda_e\oplus\Lambda_m)\smallsetminus(\Lambda_m\oplus N\Lambda_e)$, and by

$$\Omega_4(\Gamma)=c_k\left(-\tfrac{\gcd(NQ^2,P^2,Q\cdot P)}{2}\right)+c_k\left(-\tfrac{\gcd(NQ^2,P^2,Q\cdot P)}{2N}\right) \tag{2.7}$$

for untwisted charge $\Gamma \in \Lambda_m \oplus N\Lambda_e$. Here, the c_k's are the Fourier coefficient of $1/\Delta_k = \sum_{m\geq 1} c_k(m) q^m = \frac{1}{q} + k + \dots$, where $\Delta_k = \eta^k(\tau)\eta^k(N\tau)$ is the unique cusp form of weight k under $\Gamma_0(N)$. In the maximal rank case $N=1$, we write $c(m) = c_{12}(m)$ for brevity.

In contrast, the helicity supertrace Ω_6 is sensitive to 1/2-BPS and 1/4-BPS states. For 1/4-BPS charge $Q \wedge P \neq 0$, it is given by a Fourier coefficient of a meromorphic Siegel modular form [5, 6, 8]. In the simplest instance corresponding to dyons with 'generic twisted' charge $\Gamma = (Q,P)$ in $(\Lambda_e \smallsetminus \Lambda_m) \oplus (\Lambda_m \smallsetminus N\Lambda_e)$ and unit 'torsion' ($\gcd(Q\wedge P) = 1$), [1]

$$\Omega_6(\Gamma; t) = \frac{(-1)^{Q\cdot P+1}}{N} \int_{\mathcal{C}} \mathrm{d}^3\Omega \, \frac{e^{\mathrm{i}\pi\left[\rho\, Q^2 + \sigma\, P^2 + 2vQ\cdot P\right]}}{\tilde{\Phi}_{k-2}(\rho,\sigma,v)} \tag{2.8}$$

where the contour $\mathcal{C}$ in the Siegel upper half plane $\mathcal{H}_2$ parametrized by $\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$ is given by $0 \leq \rho_1 \leq N$, $0 \leq \sigma_1 \leq 1$, $0 \leq v_1 \leq 1$ with a fixed value Ω_2 of (ρ_2, σ_2, v_2) (see below). The overall sign $(-1)^{Q\cdot P+1}$ ensures that contributions from single-centered black holes are positive [30, 31]. Here, $\tilde{\Phi}_{k-2}(\rho,\sigma,v)$ is a Siegel modular form of weight $k-2$ under a congruence subgroup $\widetilde{\Gamma}_{2,0}(N) \subset Sp(4,\mathbb{Z})$ which is conjugate to the standard Hecke congruence subgroup

$$\Gamma_{2,0}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \subset Sp(4,\mathbb{Z}), C = 0 \bmod N \right\} \tag{2.9}$$

by the transformation S_ρ defined in (A.10). $\tilde{\Phi}_{k-2}(\rho,\sigma,v)$ is the image of a Siegel modular form $\Phi_{k-2}(\rho,\sigma,v)$ of weight $k-2$ under $\Gamma_{2,0}(N)$ under the same transformation,

$$\tilde{\Phi}_{k-2}(\rho,\sigma,v) = (\sqrt{N})^k \, (-\mathrm{i}\rho)^{-(k-2)} \, \Phi_{k-2}\left(-\frac{1}{\rho}, \sigma - \frac{v^2}{\rho}, \frac{v}{\rho} \right) \,. \tag{2.10}$$

Ignoring the choice of contour $\mathcal{C}$, (2.8) is formally invariant under the U-duality group $\Gamma_0(N) \times O(2k-2, 6, \mathbb{Z})$, the first factor acting as the block diagonal subgroup (A.14) of the congruence subgroup $\widetilde{\Gamma}_{2,0}(N)$. Invariance under Fricke S-duality follows from the invariance of $\tilde{\Phi}_{k-2}$ under the genus-two Fricke involution (A.39). Note that the sign $(-1)^{Q\cdot P+1}$ also is invariant under $\Gamma_0(N) \times O(2k-2, 6, \mathbb{Z})$ and Fricke S-duality.

Importantly, both $\Phi_{k-2}(\rho,\sigma,v)$ and $\tilde{\Phi}_{k-2}(\rho,\sigma,v)$ have a double zero on the diagonal divisor $\mathcal{D}$ given by all images of the locus $v=0$ under $\Gamma_{2,0}(N)$. Hence, the right-hand side of (2.8) jumps whenever the contour $\mathcal{C}$ crosses $\mathcal{D}$. As explained in [9, 32], if one chooses the constant part of Ω_2 along the contour $\mathcal{C}$ in terms of the moduli t and charge Γ via

$$\Omega_2^* = \frac{R}{\mathcal{M}(Q,P)} \left[\tfrac{1}{S_2} \begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix} + \tfrac{1}{|Q_R \wedge P_R|} \begin{pmatrix} P_R^2 & -Q_R\cdot P_R \\ -Q_R\cdot P_R & Q_R^2 \end{pmatrix} \right] \,, \tag{2.11}$$

where R is a large positive number (identified in our set-up as the radius of the circle), then $\mathcal{C}$ crosses $\mathcal{D}$ precisely when the moduli allow for the marginal decay of the 1/4-BPS state of charge $\Gamma = \Gamma_1 + \Gamma_2$ into a pair of 1/2-BPS states with charges Γ_1 and Γ_2. The corresponding

[1] Using (A.39), one may rewrite (2.8) in the other common form (see e.g. [28, (5.1.10)]

$$\Omega_6(\Gamma; t) = \frac{(-1)^{Q\cdot P+1}}{N} \int_{\mathcal{C}'} \mathrm{d}^3\Omega \, \frac{e^{\mathrm{i}\pi\left[N\rho\, Q^2 + \sigma\, P^2/N + 2vQ\cdot P\right]}}{\tilde{\Phi}_{k-2}(\sigma,\rho,v)} \,.$$

Note that our $\tilde{\Phi}_{k-2}$ differs from the one in [28] by an exchange of ρ and σ, but agrees with $\Phi_{g,e}(\rho,\sigma,v)$ in [29].

jump in $\Omega_6(Q,P;t)$ can then be shown to agree [33, 34, 35] with the primitive wall-crossing formula [36]

$$\Delta\Omega_6(\Gamma) = -(-1)^{\langle\Gamma_1,\Gamma_2\rangle+1}\,\Omega_4(\Gamma_1)\,\Omega_4(\Gamma_2)\ , \tag{2.12}$$

where $\Delta\Omega_6$ is the index in the chamber where the bound state exists, minus the index in the chamber where it does not.

The formula (2.8) only applies to dyons whose charge is primitive with unit torsion and that is generic, in the sense that it belongs to the highest stratum in the following graph of inclusions [2]

$$\begin{matrix} N\Lambda_e \oplus N\Lambda_e & \subset & & \subset & \Lambda_m \oplus \Lambda_m & \subset & \\ & & \Lambda_m \oplus N\Lambda_e & & & & \Lambda_e \oplus \Lambda_m\,. \\ \Lambda_m \oplus N\Lambda_m & \subset & & \subset & \Lambda_e \oplus N\Lambda_e & \subset & \end{matrix} \tag{2.13}$$

When (Q,P) is primitive and belongs to one of the sublattices above, it may split into pairs of 1/2-BPS charges that are not necessarily 'twisted' nor primitive. As explained in [13], the study of 1/4-BPS couplings in $D=3$ provides a microscopic motivation for the contour prescription (2.11), and gives access to the helicity supertrace for arbitrary charges in (2.13) beyond the special case of the highest stratum for which (2.8) is valid.

Indeed, it will follow from the analysis in the present work that for any primitive charge (Q,P), the helicity supertrace is given by

$$\begin{aligned}(-1)^{Q\cdot P+1}\Omega_6(Q,P;t) = &\sum_{\substack{A\in M_{2,0}(N)/(\mathbb{Z}_2\ltimes\Gamma_0(N))\\ A^{-1}\binom{Q}{P}\in\Lambda_e\oplus\Lambda_m}} |A|\,\widetilde{C}_{k-2}\left[A^{-1}\begin{pmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{pmatrix}A^{-1\intercal};A\Omega_2^\star A^\intercal\right]\\ &+\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda_m\oplus\Lambda_m}} |A|\,C_{k-2}\left[A^{-1}\begin{pmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{pmatrix}A^{-1\intercal};A\Omega_2^\star A^\intercal\right]\\ &+\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P/N}\in\Lambda_e\oplus\Lambda_e}} |A|\,C_{k-2}\left[A^{-1}\begin{pmatrix}-NQ^2 & -Q\cdot P\\ -Q\cdot P & -\frac{1}{N}P^2\end{pmatrix}A^{-1\intercal};A\Omega_2^\star A^\intercal\right]\ .\end{aligned} \tag{2.14}$$

where C_{k-2} and $\widetilde{C}_{k-2}$ are the Fourier coefficients of $1/\Phi_{k-2}$ and $1/\tilde{\Phi}_{k-2}$ evaluated with the same contour prescription as above, and $\Omega_2^\star$ is conjugated by the matrix A. This formula is manifestly invariant under the U-duality group $G_4(\mathbb{Z})$, including Fricke duality that exchanges the last two lines. For primitive 'twisted charges' of $\gcd(Q\wedge P)=1$, only the first line is non-zero and the only allowed matrix A is the identity such that one recovers (2.8). This is also the dominant term in the limit where the charges Q,P are scaled to infinity, since terms with $A\neq 1$ in the sum grow exponentially as $e^{\pi|Q\wedge P|/|\det A|}$, at a much slower rate that the leading term with $A=1$ [2, 8]. It would be interesting to check that the logarithmic corrections to the black hole entropy are consistent with the $\mathcal{R}^2$ coupling in the low energy effective action, generalizing the analysis of [37, 6] to general charges, and to identify the near horizon geometries responsible for the exponentially suppressed contributions, along the lines of [38, 39].

After splitting C_{k-2} and $\widetilde{C}_{k-2}$ into their finite and polar parts, and representing the latter as a Poincaré sum, we shall show that the unfolded sum over matrices A accounts for all

[2]The graph is drawn such that Fricke duality acts by reflection with respect to the horizontal axis.

possible splittings of a charge $(Q,P) = (Q_1,P_1)+(Q_2,P_2)$ into two 1/2-BPS constituents, labeled by $A \sim \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ [33],

$$(Q_1,P_1) = (p,r)\,\frac{sQ-qP}{ps-qr}\ , \qquad (Q_2,P_2) = (q,s)\,\frac{pP-rQ}{ps-qr}\ . \tag{2.15}$$

Generalizing the analysis in [40], we shall show that the discontinuity of $\Omega_6(\Gamma,t)$ for an arbitrary primitive (but possibly torsionful) charge Γ is given by a variant of (2.12) where $\Omega_4(\Gamma)$ on the right-hand side is replaced by

$$\overline{\Omega}_4(\Gamma) = \sum_{\substack{d\geq 1 \\ \Gamma/d\,\in\Lambda_e\oplus\Lambda_m}} \Omega_4(\Gamma/d)\ , \tag{2.16}$$

in agreement with the macroscopic analysis in [35].

2.3 BPS couplings in $D=4$ and $D=3$

In supersymmetric string vacua with 16 supercharges, the low-energy effective action at two-derivative order is exact at tree level, being completely determined by supersymmetry. In contrast, four-derivative and six-derivative couplings may receive quantum corrections from 1/2-BPS and 1/4-BPS states or instantons, respectively. At four-derivative order, the coefficients of the $\mathcal{R}^2+F^4$ and F^4 couplings in $D=4$, which we denote by f and $F^{(2k-2,6)}_{abcd}$, are known exactly, and depend only on the first and second factor in the moduli space (2.1), respectively:

$$\begin{aligned} f(S) &= -\frac{3}{8\pi^2}\log(S_2^k|\Delta_k(S)|^2)\ , & (2.17)\\ F^{(2k-2,6)}_{abcd} &= \text{R.N.}\int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\,\frac{\Gamma_{\Lambda_{2k-2,6}}[P_{abcd}]}{\Delta_k(\rho)}\ , & (2.18)\end{aligned}$$

where $\Gamma_{\Lambda_{2k-2,6}}[P_{abcd}]$ denotes the Siegel–Narain theta series (B.4) for the lattice $\Lambda_m = \Lambda_{2k-2,6}$, with an insertion of the symmetric polynomial

$$P_{abcd}(Q) = Q_{L,a}Q_{L,b}Q_{L,c}Q_{L,d} - \frac{3}{2\pi\rho_2}\delta_{(ab}Q_{L,c}Q_{L,d)} + \frac{3}{16\pi^2\rho_2^2}\delta_{(ab}\delta_{cd)}\ , \tag{2.19}$$

and Δ_k is the same cusp form whose Fourier coefficients enter in the helicity supertrace (2.6),(2.7). Here and elsewhere, we suppress the dependence of $\Gamma_{\Lambda_{p,q}}[P_{abcd}]$, and therefore of the left-hand side of (2.18), on the moduli $\phi \in G_{p,q}$. As explained in [22], both couplings arise as polynomial terms in the large radius limit of the exact $(\nabla\phi)^4$ coupling in $D=3$. The latter is uniquely determined by supersymmetry Ward identities, invariance under U-duality and the tree-level and one-loop corrections in heterotic perturbation string theory to be given by the genus-one modular integral (1.4). In the weak heterotic coupling limit $g_3 \to 0$, (1.4) has an asymptotic expansion

$$g_3^2\,F^{(2k,8)}_{abcd} = \frac{3}{2\pi g_3^2}\delta_{(ab}\delta_{cd)} + F^{(2k-1,7)}_{abcd} + \sum_{Q\in\Lambda_{2k-1,7}}{}' \bar{c}_k(Q)\,e^{-\frac{2\pi\sqrt{2}\,|Q_R|}{g_3^2}+2\pi\mathrm{i}a\cdot Q}\,\mathcal{P}^{(*)}_{abcd}\ , \tag{2.20}$$

reproducing the known tree-level and one-loop corrections, along with an infinite series of $\mathcal{O}(e^{-1/g_3^2})$ corrections ascribable to NS5-brane, Kaluza–Klein (6,1)-brane and H-monopole instantons. Here, $\mathcal{P}^{(*)}_{abcd}$ is a schematic notation for the tensor appearing in front of the exponential, including an infinite series of subleading terms which resum into a Bessel function. In the large radius limit $R \to \infty$, the asymptotic expansion of (1.4) instead gives, schematically,

$$\begin{aligned} F^{(2k,8)}_{abcd}(\phi) =& R^2 \left(f(S) \delta_{(ab}\delta_{cd)} + F^{(2k-2,6)}_{abcd}(\varphi) \right) \\ &+ \sum'_{\substack{(Q,P)\in\Lambda_e\oplus\Lambda_m \\ Q\wedge P=0}} \bar{c}_k(Q,P)\, \mathcal{P}^{(*)}_{abcd}\, e^{-2\pi R \mathcal{M}(Q,P)+2\pi \mathrm{i}(a_1\cdot Q + a_2\cdot P)} \\ &+ \sum_{\substack{M_1\neq 0, M_2\in\mathbb{Z} \\ P\in\Lambda_m}} F^{(\mathrm{TN})}_{abcd\, M_1}\, e^{-2\pi R^2 |M_1| + 2\pi \mathrm{i} M_1} \end{aligned} \tag{2.21}$$

where we used the same schematic notation $\mathcal{P}^{(*)}_{abcd}$ for the tensor appearing in front of the exponential including subleading terms. The first line in (2.21) reproduces the four-dimensional couplings (2.18), while the second line corresponds to $\mathcal{O}(e^{-R})$ corrections from four-dimensional 1/2-BPS states whose wordline winds around the circle. These contributions are weighted by the BPS index $\bar{c}_k(Q,P) = \overline{\Omega}_4(Q,P)$ given in (2.16),

$$\bar{c}_k(Q,P) = \sum_{\substack{d\geq 1 \\ (Q,P)/d\in\Lambda_e\oplus\Lambda_m}} c_k\Big(-\tfrac{\gcd(NQ^2,P^2,Q\cdot P)}{2d^2}\Big) + \sum_{\substack{d\geq 1 \\ (Q,P)/d\in\Lambda_m\oplus N\Lambda_e}} c_k\Big(-\tfrac{\gcd(NQ^2,P^2,Q\cdot P)}{2Nd^2}\Big)\,. \tag{2.22}$$

The last line in (2.21) corresponds to $\mathcal{O}(e^{-R^2})$ corrections from Taub-NUT instantons.

Our interest in this work is on the six-derivative coupling $\nabla^2(\nabla\phi)^4$ in the low energy effective action in $D=3$ (see (3.6) for the precise tensorial structure). The coefficient $G^{(2k,8)}_{ab,cd}(\phi)$ multiplying this coupling is valued in a vector bundle over the Grassmannian $G_{2k,8}$ associated to the representation ⊞ of $O(2k)$. As announced in [13], and proven in §3, supersymmetric Ward identities require that $G^{(2k,8)}_{ab,cd}(\phi)$ satisfies the following differential constraints

$$\mathcal{D}_{[a_1}{}^{\hat{a}} G^{(2k,8)}_{a_2|b|,a_3]c} = 0 \tag{2.23}$$

$$\mathcal{D}_{[a_1}{}^{[\hat{a}_1}\mathcal{D}_{a_2}{}^{\hat{a}_2]} G^{(2k,8)}_{a_3]b,cd} = 0 \tag{2.24}$$

$$\mathcal{D}_{[a_1}{}^{[\hat{a}_1}\mathcal{D}_{a_2}{}^{\hat{a}_2}\mathcal{D}_{a_3]}{}^{\hat{a}_3]} G^{(2k,8)}_{cd,ef} = 0 \tag{2.25}$$

$$\begin{aligned} \mathcal{D}_{(e}{}^{\hat{a}}\mathcal{D}_{f)\hat{a}} G^{(2k,8)}_{ab,cd} =& -\tfrac{5}{2}\delta_{ef} G^{(2k,8)}_{ab,cd} - \big(\delta_{e)(a} G^{(2k,8)}_{b)(f,cd} + \delta_{e)(c} G^{(2k,8)}_{d)(f,ab}\big) \\ &+ \tfrac{3}{2}\delta_{\langle ab,} G^{(2k,8)}_{cd\rangle,ef} - \tfrac{3\pi}{2} F^{(2k,8)}_{|e)\langle ab,}{}^{g} F^{(2k,8)}_{cd\rangle(f|g}\,. \end{aligned} \tag{2.26}$$

Here, for two symmetric tensors A_{ab}, B_{cd}, we denote the projection of their product on the representation ⊞ by

$$A_{\langle ab,}B_{cd\rangle} = \frac{1}{3}\big(A_{ab}B_{cd} + A_{cd}B_{ab} - 2A_{|a)(c}B_{d)(b|}\big)\,. \tag{2.27}$$

The inhomogeneous term in the last equation (2.26), proportional to the square of the $(\nabla\phi)^4$ coupling, originates from higher-derivative corrections to the supersymmetry variations. [3] It

[3]Note that the properly normalized coupling in the Lagrangian is in fact $\frac{1}{\pi}G^{(2k,8)}_{ab,cd}(\phi)$, which accounts for the factor of π on the r.h.s. of (2.26).

follows from (3.16)–(3.20) that in heterotic perturbation string theory, $G^{(2k,8)}_{ab,cd}$ can only receive tree-level, one-loop and two-loop corrections, plus non-perturbative corrections of order e^{-1/g_3^2}. We calculate the one-loop and two-loop contributions in Appendix B using earlier results in the literature [41, 42, 43, 44, 45, 46]. After rescaling to Einstein frame, we find that the perturbative corrections take the form [4]

$$g_3^6\, G^{(2k,8)}_{ab,cd} = -\frac{3}{4\pi g_3^2}\,\delta_{\langle ab,}\delta_{cd\rangle} - \frac{1}{4}\delta_{\langle ab,}\, G^{(2k-1,7)}_{cd\rangle} + g_3^2 G^{(2k-1,7)}_{ab,cd} + \mathcal{O}(e^{-1/g_3^2}) \tag{2.28}$$

where $G^{(p,q)}_{ab}$ denotes the genus-one modular integral

$$G^{(p,q)}_{ab} = \text{R.N.} \int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2}\, \frac{\hat{E}_2\, \Gamma_{\Lambda_{p,q}}[P_{ab}]}{\Delta_k}\ , \tag{2.29}$$

with $P_{ab} = Q_{L,a}Q_{L,b} - \frac{\delta_{ab}}{4\pi\rho_2}$ and $\hat{E}_2 = E_2 - \frac{3}{\pi\rho_2}$ is the almost holomorphic Eisenstein series of weight 2, while $G^{(p,q)}_{ab,cd}$ the genus-two modular integral (of which (1.7) is a special case),

$$G^{(p,q)}_{ab,cd} = \text{R.N.} \int_{\Gamma_{2,0}(N)\backslash\mathcal{H}_2} \frac{\mathrm{d}^3\Omega_1 \mathrm{d}^3\Omega_2}{|\Omega_2|^3}\, \frac{\Gamma^{(2)}_{\Lambda_{p,q}}[P_{ab,cd}]}{\Phi_{k-2}}\ . \tag{2.30}$$

Here $P_{ab,cd}$ is the quartic polynomial

$$\begin{aligned} P_{ab,cd} &= \varepsilon_{rt}\varepsilon_{su} Q^r_{L(a}Q^s_{Lb)}Q^t_{L(c}Q^u_{Ld)} - \frac{3}{4\pi|\Omega_2|}\delta_{\langle ab,}Q^r_{Lc}(\Omega_2)_{rs}Q^s_{Ld\rangle} + \frac{3}{16\pi^2|\Omega_2|}\delta_{\langle ab,}\delta_{cd\rangle}\ , \\ &= \frac{3}{2}\Big(\delta_{\langle rs,}\delta_{tu\rangle}Q^r_{La}Q^s_{Lb}Q^t_{Lc}Q^u_{Ld} - \frac{1}{2\pi|\Omega_2|}\delta_{\langle ab,}Q^r_{Lc}(\Omega_2)_{rs}Q^s_{Ld\rangle} + \frac{1}{8\pi^2|\Omega_2|}\delta_{\langle ab,}\delta_{cd\rangle}\Big)\ , \end{aligned} \tag{2.31}$$

and for any polynomial P in Q^r_{La} and integer lattice $\Lambda_{p,q}$ of signature (p,q), we denote

$$\Gamma^{(2)}_{\Lambda_{p,q}}[P] = |\Omega_2|^{q/2} \sum_{Q\in\Lambda_{p,q}\oplus\Lambda_{p,q}} P(Q_{La})\, e^{\mathrm{i}\pi Q^r_{La}\,\Omega_{rs}\,Q^{s\ a}_{L} - \mathrm{i}\pi Q^r_{R\hat{a}}\,\bar{\Omega}_{rs}\,Q^{s\ \hat{a}}_{R}}\ . \tag{2.32}$$

where $r,s=1,2$ label the choice of A-cycle on the genus-two Riemann surface.

Since the modular integral (2.30) itself satisfies the differential constraints (2.23)–(2.26), as shown in §3.3, it is consistent with supersymmetry to propose that the exact coefficient of the $\nabla^2(\nabla\phi)^4$ coupling be given by (1.7). In §§4 below, we shall demonstrate that the weak coupling expansion of the Ansatz (1.7) indeed reproduces the perturbative corrections (2.28), up to $\mathcal{O}(e^{-1/g_3^2})$ corrections. Unlike the $(\nabla\phi)^4$ couplings (2.20) however, the latter also affect the constant term in the Fourier expansion with respect to the axions a^I, as required by the quadratic source term in the differential equation (3.20). Such corrections can be ascribed to (NS5, KK, H-monopoles) instanton anti-instanton of vanishing total charge.

In the large radius limit, the $\nabla^2(\nabla\phi)^4$ coupling must reduce to the exact $\mathcal{R}^2F^2$ and ∇^2F^4 couplings in $D=4$. Consistently with this expectation, we shall find that the asymptotic

[4] The tree-level term comes from the double-trace contribution in [47]. The relative coefficients of the three contributions are determined by the differential equation required by the supersymmetry Ward identity, which also ensures that there are no contributions at higher loop order.

expansion of (1.7) in the limit $R \to \infty$ takes the form

$$
\begin{aligned}
G^{(2k,8)}_{ab,cd} =& R^4\, G^{(D=4)}_{ab,cd} + \frac{\zeta(3)}{8\pi}(k-12)R^6\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} \\
&+ \sum_{\substack{(Q,P)\in\Lambda_e\oplus\Lambda_m \\ Q\wedge P=0}}{}^{'} \delta_{\langle ab,}\bar{G}^{(2k-1,7)}_{cd\rangle}(Q,P,t)\, e^{-2\pi R\mathcal{M}(Q,P)+2\pi\mathrm{i}(a_1\cdot Q+a_2\cdot P)} \\
&+ \sum_{\substack{(Q,P)\in\Lambda_e\oplus\Lambda_m \\ Q\wedge P\neq 0}}{}^{'} \bar{C}_{k-2}(Q,P;t)\,\mathcal{P}^{(*)}_{abcd}\, e^{-2\pi R\mathcal{M}(Q,P)+2\pi\mathrm{i}(a_1\cdot Q+a_2\cdot P)} \\
&+ \sum_{\substack{(Q_1,P_1)\in\Lambda_e\oplus\Lambda_m \\ Q_1\wedge P_1=0}}{}^{'} \sum_{\substack{(Q_2,P_2)\in\Lambda_e\oplus\Lambda_m \\ Q_2\wedge P_2=0}}{}^{'} \bar{c}_k(Q_1,P_1)\,\bar{c}_k(Q_2,P_2)\,\mathcal{P}^{(*)}_{abcd}(Q_1,P_1;Q_2,P_2;t) \\
&\quad\times\, e^{-2\pi R[\mathcal{M}(Q_1,P_1)+\mathcal{M}(Q_2,P_2)]+2\pi\mathrm{i}(a_1\cdot(Q_1+Q_2)+a_2\cdot(P_1+P_2))} \\
&+ \sum_{M_1\neq 0} G^{(\mathrm{TN})}_{ab,cd\,M_1} e^{-2\pi R^2|M_1|+2\pi\mathrm{i}M_1\psi} + G^{(I\bar{I})}_{ab,cd}\,.
\end{aligned}
\tag{2.33}
$$

In the first line, $G^{(D=4)}_{ab,cd}$ predicts the exact $\mathcal{R}^2F^2$ and ∇^2F^4 couplings in $D=4$, which are exhibited in (5.67),(5.70) below, and involve explicit modular functions of the axio-dilaton S, as well as genus-two and genus-one modular integrals for the lattice Λ_m. These couplings are by construction invariant under the S-duality group $\Gamma_0(N)$ and under Fricke duality.

The second line in (2.33) are the 1/2-BPS Fourier coefficients, weighted by a genus-one modular integral $\bar{G}_{cd}(Q,P;t)$ for the lattice orthogonal to Q,P given in (5.18), (5.46). This weighting is similar to that of 1/2-BPS contributions to the $\nabla^4\mathcal{R}^4$ coupling in maximal supersymmetric vacua [16], and is typical of Fourier coefficients of automorphic representations that do not belong to the maximal orbit in the wavefront set.

The third line corresponds to contributions from 1/4-BPS dyons, weighted by the moduli-dependent helicity supertrace, up to overall sign,

$$
\bar{C}_{k-2}(Q,P;t) = (-1)^{Q\cdot P+1}\,\Omega_6(Q,P;t)
\tag{2.34}
$$

whereas the fourth line corresponds to contributions from two-particle states consisting of two 1/2-BPS dyons that are discussed in detail in Appendices C and D. While the two contributions on the third and fourth line are separately discontinuous as a function of the moduli t, their sum is continuous across walls of marginal stability. In Appendix C we show the non-trivial fact, especially for CHL orbifolds, that for fixed total charge Γ, the sum involves all possible splittings $\Gamma_1+\Gamma_2$, weighted by the respective helicity supertraces (2.22). This complements and extends the consistency checks on the helicity supertrace formulae [29] to arbitrary charges. Moreover, we show in Appendix E that these contributions are consistent with the differential constraint (2.26). The 1/4-BPS Abelian Fourier coefficients of the non-perturbative coupling are the main focus of this paper, and the results are discussed in detail in section 5.3.

The first term $G^{(TN)}_{ab,cd\,M_1}$ on the last line corresponds to non-Abelian Fourier coefficients of order e^{-R^2}, ascribable to Taub-NUT instantons of charge M_1. We compute them in Appendix §G by dualizing the Fourier coefficients in the small coupling limit $g_3 \to 0$ computed in §4, rather than by evaluating them directly from the unfolding method.

Finally $G^{(I\bar{I})}$ contains contributions associated to instanton anti-instantons configurations, which are not captured by the unfolding method but are required by the quadratic source term in the differential equation (2.26). This includes $\mathcal{O}(e^{-R})$ and $\mathcal{O}(e^{-R^2})$ contributions to the constant term, which are independent of the axions a_1, a_2, ψ, and contributions of order $\mathcal{O}(e^{-R^2})$ to the Abelian Fourier coefficients, which depend on the axions a_1, a_2 as $e^{2\pi \mathrm{i}(a_1 \cdot Q + a_2 \cdot P)}$ but are independent of ψ. The latter can be ascribed to Taub-NUT instanton-anti-instantons, and are necessary in order to resolve the ambiguity of the sum over 1/4-BPS instantons [48], which is divergent due to the exponential growth of the measure $\bar{C}_{k-2}(Q, P; \Omega_2^\star) \sim (-1)^{Q\cdot P+1} e^{\pi |Q\wedge P|}$. We do not fully evaluate $G^{(I\bar{I})}$ in this paper, but we identify the origin of the $\mathcal{O}(e^{-R^2})$ corrections as coming from poles of $1/\Phi_{k-2}$ which lie 'deep' in the Siegel upper-half plane $\mathcal{H}_2$ and do not intersect the fundamental domain, becoming relevant only after unfolding. While the precise contributions can in principle be determined by solving the differential equation (2.26), it would be interesting to obtain them via a rigorous version of the unfolding method which applies to meromorphic Siegel modular forms.

In §6, we discuss other pertubative expansions of the exact result (1.7), in the dual type I and type II pictures. In either case, the perturbative limit is dual to a large volume limit on the heterotic side, where either the full 7-torus (in the type I case) of a 4-torus (in the type II case) decompactifies. We find that the corresponding weak coupling expansion is consistent with known perturbative contributions, with non-perturbative effects associated to D-branes, NS5-branes and KK-monopoles wrapped on supersymmetric cycles of the internal space, T^7 in the type I case, or $K3 \times T^3$ on the type II case.

3 Supersymmetric Ward identities

In this section, we establish the supersymmetric Ward identities (3.16)–(3.20), from linearized superspace considerations, and show that the genus-two modular integral (2.30) obeys this identity.

3.1 $\nabla^2(\nabla\Phi)^4$ type invariants in three dimensions

This analysis is a direct generalization of the one provided in [22, §3]. We shall define the linearised superfield $W_{\hat{a}a}$ of half-maximal supergravity in three dimensions that satisfies to the constraints [49, 50, 51]

$$D^i_\alpha W_{\hat{a}a} = (\Gamma_{\hat{a}})^{i\hat{\jmath}}\chi_{\alpha\hat{\jmath}a} \ , \qquad D^i_\alpha \chi_{\beta\hat{\jmath}a} = -\mathrm{i}(\sigma^\mu)_{\alpha\beta}(\Gamma^{\hat{a}})_{\hat{\jmath}}{}^i\partial_\mu W_{\hat{a}a} \ , \tag{3.1}$$

with $\hat{a} = 1$ to 8 for the vector of $O(8)$, $i = 1$ to 8 for the positive chirality Weyl spinor of $Spin(8)$ and $\hat{\imath} = 1$ to 8 for the negative chirality Weyl spinor. The 1/4-BPS linearised invariants are defined using harmonics of $SO(8)/(U(2)\times SO(4))$ parametrizing a $Spin(8)$ group element $u^{r_1}{}_i$, $u^{r_2r_3}{}_i, u_{r_1i}$ in the Weyl spinor representation of positive chirality [52],

$$\begin{gathered} 2u_{r_1(i}u^{r_1}{}_{j)} + \varepsilon_{r_2s_2}\varepsilon_{r_3s_3}u^{r_2r_3}{}_iu^{s_2s_3}{}_j = \delta_{ij} \ , \quad \delta^{ij}u_{r_1i}u^{s_1}{}_j = \delta^{s_1}_{r_1} \ , \quad \delta^{ij}u_{r_1i}u_{s_1j} = 0 \ , \\ \delta^{ij}u_{r_1i}u^{r_2r_3}{}_j = 0 \ , \quad \delta^{ij}u^{r_2r_3}{}_iu^{s_2s_3}{}_j = \varepsilon^{r_2s_2}\varepsilon^{r_3s_3} \ , \quad \delta^{ij}u^{r_1}{}_iu^{r_2r_3}{}_j = 0 \ , \quad \delta^{ij}u^{r_1}{}_iu^{s_1}{}_j = 0 \ , \end{gathered} \tag{3.2}$$

where the r_A indices for $A = 1,2,3$ are associated to the three $SU(2)$ subgroups of $SU(2)_1 \times Spin(4) = SU(2)_1 \times SU(2)_2 \times SU(2)_3$. The harmonic variables parametrize similarly a $Spin(8)$ group element $u^{r_3\hat{a}}$, $u^{r_1r_2\hat{a}}, u_{r_3}{}^{\hat{a}}$ in the vector representation and a group element $u^{r_2}{}_{\hat{\imath}}$, $u^{r_3r_1}{}_{\hat{\imath}}, u_{r_2\hat{\imath}}$ in the Weyl spinor representation of opposite chirality. They satisfy the same relations as (3.2) upon permutation of the three $SU(2)_A$.

The superfield $W^{r_3}_a \equiv u^{r_3\hat{a}}W_{\hat{a}a}$ then satisfies the G-analyticity condition

$$u^{r_1}{}_iD^i_\alpha u^{r_3\hat{a}}W_{\hat{a}a} \equiv D^{r_1}_\alpha W^{r_3}_a = 0 \ . \tag{3.3}$$

One can obtain a linearised invariant from the action of the twelve derivatives $D_{\alpha r_1} \equiv u_{r_1i}D^i_\alpha$ and $D^{r_2r_3}_\alpha \equiv u^{r_2r_3}{}_iD^i_\alpha$ on any homogeneous function of the $W^{r_3}_a$'s. The integral vanishes unless the integrand includes at least the factor $W^1_{[a}W^2_{b]}W^1_{[c}W^2_{d]}$ such that the non-trivial integrands are defined as the homogeneous polynomials of degree $4+2n+m$ in $W^{r_3}_a$ in the representation of $SU(2)$ isospin $m/2$ and in the $SL(p,\mathbb{R}) \supset SO(p)$ representation of Young tableau $[n+2,m]$ ($n+2$ rows of two lines and m of one line) that branches under $SO(p)$ with respect to all possible traces. After integration, the resulting expression is in the same representation of $SO(p)$ and in the irreducible representation of highest weight $m\Lambda_1 + n\Lambda_2$ of $SO(8)$, *i.e.* the traceless component associated to the Young tableau $[n,m]$, with Λ_1,Λ_2 denoting two fundamental weights.

It follows that the non-linear invariant only depends on the scalar fields through the tensor function $F_{ab,cd}$ and its covariant derivatives $\mathcal{D}^nF_{ab,cd}$ and covariant densities $\mathcal{L}_{[n,m]}$ in the corresponding irreducible representation of highest weight $m\Lambda_1 + n\Lambda_2$ of $SO(8)$ that only depend on the scalar fields through the covariant fields

$$P_{\mu\, a\hat{b}} = \partial_\mu\phi^{\mathrm{\mu}}P_{\mathrm{\mu}\, a\hat{b}} \ , \quad \chi_{\alpha\hat{\imath}a} \ , \quad \mathcal{D}_\mu\chi_{\alpha\hat{\imath}a} = \nabla_\mu\chi_{\alpha\hat{\imath}a} + \partial_\mu\phi^{\mathrm{\mu}}\Big(\omega_{\mathrm{\mu}\, a}{}^b\chi_{\alpha\hat{\imath}a} + \frac{1}{4}\omega_{\mathrm{\mu}\,\hat{a}\hat{b}}(\Gamma^{\hat{a}\hat{b}})_{\hat{\imath}}{}^{\hat{\jmath}}\chi_{\alpha\hat{\jmath}a}\Big) \ , \tag{3.4}$$

and the dreibeins and the gravitini fields, and where

$$P_{a\hat{b}} \equiv \mathrm{d}p_{R\hat{b}}{}^I \eta_{IJ} p_{La}{}^J \ , \qquad \omega_{ab} \equiv -\mathrm{d}p_{La}{}^I \eta_{IJ} p_{Lb}{}^J \ , \qquad \omega_{\hat{a}\hat{b}} \equiv \mathrm{d}p_{R\hat{a}}{}^I \eta_{IJ} p_{R\hat{b}}{}^J \ , \tag{3.5}$$

are defined from the Maurer–Cartan form of $SO(p,8)/(SO(p)\times SO(8))$. Using the known structure of the $t_8 \mathrm{tr} \nabla_\mu F \nabla^\mu F \mathrm{tr} FF$ invariant in ten dimensions [47],[5] one computes that the first covariant density $\mathcal{L}_{[0,0]}$ bosonic component is

$$\begin{aligned} \mathcal{L}^{ab,cd} \;=\; & \frac{\sqrt{-g}}{8\pi}\Big(2P_{(\mu}{}^{[a}{}_{\hat{a}} \nabla_\sigma P_{\nu)}{}^{b]\hat{a}} P^{\mu[c}{}_{\hat{b}} \nabla^\sigma P^{\nu|d]\hat{b}} + 2P_\mu{}^{[a}{}_{(\hat{a}} \nabla_\sigma P^{\mu|b]}{}_{\hat{b})} P^{\nu[c|\hat{a}} \nabla^\sigma P_\nu{}^{d]\hat{b}} \\ & - P_\mu{}^{[a}{}_{\hat{a}} \nabla_\sigma P^{\mu\,|b]\hat{a}} P_\nu{}^{[c}{}_{\hat{b}} \nabla^\sigma P^{\nu\,|d]\hat{b}} - 4P_{[\mu}{}^{[a|\hat{a}} \nabla_\sigma P_{\nu]}{}^{b]\hat{b}} P^{\mu[c}{}_{[\hat{a}} \nabla^\sigma P^{\nu\,|d]}{}_{\hat{b}]} + \dots \Big) \ . \end{aligned} \tag{3.6}$$

The factor of π is introduced by convenience for the definition (2.30) to hold. Investigating the possible tensors one can write in this mass dimension, one concludes that the tensor densities $\mathcal{L}_{[n,m]}$ are only non-zero for $0 \le n \le 2$ and $0 \le m \le 4$ and the density $\mathcal{L}_{[2,4]} \sim \chi^{12}$ with open $SO(p)$ indices in the symmetrization $\yng(7,3)$. The invariant admits therefore the decomposition

$$\begin{aligned} \mathcal{L} \;=\; & F_{ab,cd}\mathcal{L}^{ab,cd} + \mathcal{D}_e{}^{\hat{a}} F_{ab,cd} \mathcal{L}^{ab,cd,e}_{\hat{a}} + \mathcal{D}_{(e}{}^{(\hat{a}} \mathcal{D}_{f)}{}^{\hat{b})} F_{ab,cd} \mathcal{L}^{ab,cd,e,f}_{\hat{a},\hat{b}} + \mathcal{D}_{[e}{}^{[\hat{a}} \mathcal{D}_{f]}{}^{\hat{b}]} F_{ab,cd} \mathcal{L}^{ab,cd,ef}_{\hat{a}\hat{b}} \\ & + \dots + \mathcal{D}_{(b_1}{}^{(\hat{b}_1} \dots \mathcal{D}_{b_4)}{}^{\hat{b}_4)} \mathcal{D}_{a_1}{}^{\hat{a}_1} \dots \mathcal{D}_{a_4}{}^{\hat{a}_4} F_{a_5a_6,a_7a_8} \mathcal{L}^{a_1a_2,a_3a_4,a_5a_6,a_7a_8,b_1,b_2,b_3,b_4}_{\hat{a}_1\hat{a}_2,\hat{a}_3\hat{a}_4,\hat{b}_1,\hat{b}_2,\hat{b}_3,\hat{b}_4} \ , \end{aligned} \tag{3.7}$$

where the $\mathcal{L}^{a_1a_2,\dots,a_{2n+3}a_{2n+4},b_1,\dots,b_m}_{\hat{a}_1\hat{a}_2,\dots,\hat{a}_{2n-1}\hat{a}_{2n},\hat{b}_1,\dots,\hat{b}_m}$ are in the irreducible representation of highest weight $m\check{\alpha}_1 + n\check{\alpha}_2$ of $SO(8)$ and admit the symmetry of the Young tableau $[n+2,m]$ with respect to the permutation of the $SO(p)$ indices. In particular, $F_{ab,cd}$ transforms according to $\yng(2,2)$, realized by first symmetrizing along the columns and then antisymmetrizing along the rows $[ab]$, $[cd]$.

Checking the supersymmetry invariance (modulo a total derivative) of $\mathcal{L}$ in this basis, one finds that there is no term to cancel the supersymmetry variation

$$\delta F_{ab,cd} = \big(\bar{\epsilon}_i (\Gamma^{\hat{f}})^{i\jmath} \chi_{\jmath\, e}\big) \mathcal{D}_e{}^{\hat{f}} F_{ab,cd} \tag{3.8}$$

of the tensor $F_{ab,cd}$ and of its derivative when three open $SO(p)$ indices are antisymmetrized, hence the tensor $F_{ab,cd}$ must satisfy the constraints

$$\mathcal{D}_{[a_1}{}^{\hat{a}} F_{a_2a_3],bc} = 0 \ , \quad \mathcal{D}_{[a_1}{}^{[\hat{a}_1} \mathcal{D}_{a_2}{}^{\hat{a}_2]} F_{a_3]b,cd} = 0 \ , \quad \mathcal{D}_{[a_1}{}^{[\hat{a}_1} \mathcal{D}_{a_2}{}^{\hat{a}_2} \mathcal{D}_{a_3]}{}^{\hat{a}_3]} F_{cd,ef} = 0 \ . \tag{3.9}$$

Similarly, because the $\mathcal{L}_{[n,m]}$ are traceless in the $SO(8)$ indices, the $SO(8)$ singlet component of $\delta(\mathcal{D}F)\mathcal{L}_{[0,1]}$ can only be cancelled by terms coming from $F\delta\mathcal{L}_{[0,0]}$, *i.e.*

$$F_{ab,cd}\delta\mathcal{L}^{ab,cd} + \frac{1}{8}\mathcal{D}_e{}^{\hat{a}} \mathcal{D}_{f\hat{a}} F_{ab,cd} (\bar{\epsilon}\,\Gamma^{\hat{c}}\chi^e) \mathcal{L}^{ab,cd,f}_{\hat{c}} \sim 0 \tag{3.10}$$

modulo terms arising from the supercovariantization, so that the covariant components must satisfy

$$\delta\mathcal{L}^{ab,cd} + \frac{b_1}{4}(\bar{\epsilon}\,\Gamma^{\hat{c}}\chi_e)\mathcal{L}^{ab,cd,e}_{\hat{c}} + \frac{b_2}{2}\Big((\bar{\epsilon}\,\Gamma^{\hat{c}}\chi^{[a})\mathcal{L}^{b]e,cd,}_{\hat{c}}{}_e + (\bar{\epsilon}\,\Gamma^{\hat{c}}\chi^{[c})\mathcal{L}^{d]e,ab,}_{\hat{c}}{}_e \Big) = \nabla_\mu(\dots) \ . \tag{3.11}$$

[5] with $t_8 F^4 = F_{\mu\nu}F^{\nu\sigma}F_{\sigma\rho}F^{\rho\mu} - 1/4(F_{\mu\nu}F^{\mu\nu})^2$.

Therefore, the tensor $F_{ab,cd}$ must obey an equation of the form

$$\mathcal{D}_e{}^{\hat a}\mathcal{D}_{f\hat a}F_{ab,cd} = b_1\Big(-\delta_{ef}F_{ab,cd} + \delta_{e[a}F_{b]f,cd} + \delta_{e[c}F_{d]f,ab}\Big) \\ - 3b_2\big(\delta_{f[a}F_{b]e,cd} + \delta_{f[c}F_{d]e,ab}\big) - 4b_2\delta_{c][a}F_{b](e,f)[d}\ , \tag{3.12}$$

for some numerical constants b_1, b_2 which are fixed by consistency. In particular the integrability condition on the component antisymmetric in e and f implies $b_1 = 4 - 3b_2$.

Before determining the constants b_i, it is convenient to generalize $F_{ab,cd}$ to a tensor $F^{(p,q)}_{ab,cd}$ on a general Grassmanian $G_{p,q}$, which would arise by considering a superfield in $D = 10 - q$ dimensions with $4 \leq q \leq 6$, with harmonics parametrizing $SO(q)/(U(2)\times SO(q-4))$ [53]. The same argument leads again to the conclusion that $F^{(p,q)}_{ab,cd}$ satisfies to (3.12) with $b_1 = \frac{q}{2} - 3b_2$. Equivalently, these constraints follow from the general Ansatz preserving the symmetry $\boxplus$ of the indices ab, cd and the two first equations in (3.9). An additional integrability condition comes from the equation

$$\begin{aligned} \mathcal{D}_{[a_1}{}^{\hat a}\mathcal{D}_e{}^{\hat b}\mathcal{D}_{|a_2|\hat b}F^{(p,q)}_{a_3]b,cd} &= [\mathcal{D}_{[a_1}{}^{\hat a},\mathcal{D}_e{}^{\hat b}]\mathcal{D}_{|a_2|\hat b}F^{(p,q)}_{a_3]b,cd} + \frac{1}{2}\mathcal{D}_{e\hat b}[\mathcal{D}_{[a_1}{}^{\hat a},\mathcal{D}_{a_2}{}^{\hat b}]F^{(p,q)}_{a_3]b,cd} \\ = \ & \mathcal{D}_{[a_1}{}^{\hat a}\Big(\tfrac{6b_2-q}{4}\delta_{e|a_2}F^{(p,q)}_{a_3]b,cd} + \tfrac{3b_2}{2}\delta_{b|a_2}F^{(p,q)}_{a_3]e,cd} + 2b_3\delta_{c]|a_2}F^{(p,q)}_{a_3]b,e[d} + b_3\delta_{c]|a_2}F^{(p,q)}_{a_3][d,be}\Big) \\ = \ & \mathcal{D}_{[a_1}{}^{\hat a}\Big(\tfrac{3-q}{4}\delta_{e|a_2}F^{(p,q)}_{a_3]b,cd} + \tfrac{1}{4}\delta_{b|a_2}F^{(p,q)}_{a_3]e,cd} + \tfrac{1}{2}\delta_{c]|a_2}F^{(p,q)}_{a_3]b,e[d}\Big) \\ & + \frac{1}{4}\mathcal{D}_e{}^{\hat a}\big(\delta_{b[a_1}F^{(p,q)}_{a_2a_3],cd} + \delta_{c][a_1}F^{(p,q)}_{a_2a_3],b[d}\big)\ , \end{aligned} \tag{3.13}$$

which is indeed consistent, if and only if $b_2 = \frac{1}{2}$ and so $b_1 = \frac{q-3}{2}$ so that (3.12) reduces to

$$\mathcal{D}_{(e}{}^{\hat a}\mathcal{D}_{f)\hat a}F^{(p,q)}_{ab,cd} = \tfrac{3-q}{2}\delta_{ef}F^{(p,q)}_{ab,cd} + \tfrac{q-6}{2}\big(\delta_{e)[a}F^{(p,q)}_{b](f,cd} + \delta_{e)[c}F^{(p,q)}_{d](f,ab}\big) - 2\delta_{c][a}F^{(p,q)}_{b](e,f)[d}\ . \tag{3.14}$$

Alternatively, one can represent a tensor with the symmetry $\boxplus$ with two pairs of indices that are manifestly symmetric, *i.e.* $G_{ab,cd} = G_{ba,cd} = G_{ab,dc} = G_{cd,ab}$ such that $G_{(ab,c)d} = 0$, such that

$$F_{ab,cd} = G_{c][a,b][d}\ , \qquad G_{ab,cd} = -\tfrac{4}{3}F_{a)(c,d)(b}\ . \tag{3.15}$$

The tensor $G_{ab,cd}$ satisfies the constraints

$$\mathcal{D}_{[a_1}{}^{\hat a}G^{(p,q)}_{a_2|b|,a_3]c} = 0\ , \quad \mathcal{D}_{[a_1}{}^{[\hat a_1}\mathcal{D}_{a_2}{}^{\hat a_2]}G^{(p,q)}_{a_3]b,cd} = 0\ , \quad \mathcal{D}_{[a_1}{}^{[\hat a_1}\mathcal{D}_{a_2}{}^{\hat a_2}\mathcal{D}_{a_3]}{}^{\hat a_3]}G^{(p,q)}_{cd,ef} = 0\ . \tag{3.16}$$

and

$$\mathcal{D}_{(e}{}^{\hat a}\mathcal{D}_{f)\hat a}G^{(p,q)}_{ab,cd} = \tfrac{3-q}{2}\delta_{ef}G^{(p,q)}_{ab,cd} + \tfrac{6-q}{2}\big(\delta_{e)(a}G^{(p,q)}_{b)(f,cd} + \delta_{e)(c}G^{(p,q)}_{d)(f,ab}\big) + \tfrac{3}{2}\delta_{\langle ab,}G^{(p,q)}_{cd\rangle,ef}\ . \tag{3.17}$$

The discussion so far only applies to a supersymmetry invariant modulo the classical equations of motion, whereas one must take into account the first correction in $(\nabla\Phi)^4$. The direct computation of this correction via supersymmetry invariance at the next order is extremely difficult, however, one can determine its form from general arguments. The modification of the supersymmetry Ward identities implies that the corrections to the differential equations must be an additional source term quadratic in the completely symmetric tensor $F^{(p,q)}_{abcd}$ defining the $(\nabla\Phi)^4$ coupling. This correction should preserve the wave-front set associated to the original homogeneous solution, so it is expected that (3.16) is not modified, while the second order equation (3.17) admits a source term quadratic in $F^{(p,q)}_{abcd}$ and consistent with (3.16). Inspection of the various possible tensor structures shows that there is indeed no possible correction to (3.16), because $F^{(p,q)}_{abcd}$ satisfies itself

$$\mathcal{D}_{[a}{}^{\hat{a}} F^{(p,q)}_{b]cde} = 0 \ , \qquad \mathcal{D}_{[a}{}^{[\hat{a}} \mathcal{D}_{b]}{}^{\hat{b}]} F^{(p,q)}_{cdef} = 0 \ . \tag{3.18}$$

Equation (3.17) admits the symmetry associated to the Young tableaux $\yng(4,2)$ and $\yng(3,2,1)$, however it is easy to check that the latter is trivially satisfied

$$\frac{1}{2} \mathcal{D}_{[a_1|}{}^{\hat{a}} \mathcal{D}_{b\hat{a}} F^{(p,q)}_{|a_2 a_3],cd} = -\frac{q}{4} \delta_{b[a_1} F^{(p,q)}_{a_2 a_3],cd} - \frac{q}{4} \delta_{c][a_1} F^{(p,q)}_{a_2 a_3],b[d} \ , \tag{3.19}$$

and therefore cannot be corrected by a source term. The only source term quadratic in $F^{(p,q)}_{abcd}$ with the symmetry structure $\yng(4,2)$ that also satisfies to the constraint (3.16) is $F^{(p,q)}_{|e)\langle ab,}{}^{g} F^{(p,q)}_{cd\rangle(f|g}$. It is indeed straighforward to check that the corresponding combination sourcing (3.17), namely $F^{(p,q)g}_{c]e[a} F^{(p,q)}_{b]fg[d}$, satisfies (3.9) using (3.18), whereas any other combination with the symmetry structure $\yng(4,2)$ involving the Kronecker symbol would not.

We conclude that the correct supersymmetry constraint for $G^{(p,q)}_{ab,cd}$ reads

$$\begin{split} \mathcal{D}_{(e}{}^{\hat{a}} \mathcal{D}_{f)\hat{a}} G^{(p,q)}_{ab,cd} = \tfrac{3-q}{2} \delta_{ef} G^{(p,q)}_{ab,cd} + \tfrac{6-q}{2} \big(\delta_{e)(a} G^{(p,q)}_{b)(f,cd} + \delta_{e)(c} G^{(p,q)}_{d)(f,ab} \big) + \tfrac{3}{2} \delta_{\langle ab,} G^{(p,q)}_{cd\rangle,ef} \\ - \frac{3\varpi}{2} F^{(p,q)}_{|e)\langle ab,}{}^{g} F^{(p,q)}_{cd\rangle(f|g} \ , \end{split} \tag{3.20}$$

where ϖ is an undetermined numerical coefficient at this stage. In §3.3 we shall show that the genus-two modular integral (2.30) satisfies this equation with $\varpi = \pi$.

Let us note that this discussion only applies to the Wilsonian effective action. As we shall see in section B.2.4, the differential Ward identity satisfied by the renormalized coupling $\hat{G}_{ab,cd}$ appearing in the 1PI effective action is expected to be corrected in four dimensions ($q = 6$) by constant terms and by terms linear in $\hat{F}_{abcd}$.

Because of the quadratic source term in (3.20), the tensor $G_{ab,cd}$ does not belong strickly speaking to an automorphic representation of $SO(p,q)$. One can nonetheless define a generalization of the notion of automorphic representation attached to this tensor. The linearised analysis exhibits that the homogeneous differential equation is attached to the $SO(p,q)$ representation associated to the nilpotent orbit of partition $[3^2, 1^{p+q-6}]$ such that the nilpotent elements $Z_{a\hat{b}} \in \mathfrak{so}(p+q)(\mathbb{C}) \ominus (\mathfrak{so}(p)(\mathbb{C}) \oplus \mathfrak{so}(q)(\mathbb{C}))$ satisfy the constraint (cf. (3.9), (3.12))

$$Z_{[a}{}^{[\hat{a}} Z_{b}{}^{\hat{b}} Z_{c]}{}^{\hat{c}]} = 0 \ , \qquad Z_{a\hat{c}} Z_{b}{}^{\hat{c}} = 0 \ . \tag{3.21}$$

For a representative of the nilpotent orbit in the unipotent associated to the maximal parabolic $GL(k) \times SO(p-k, q-k) \ltimes \mathbb{R}^{2(p+q-2k) + \frac{k(k-1)}{2}}$ this gives the constraints [6]

$$Q_{[i}{}^{[m} Q_{j}{}^{n} Q_{k]}{}^{p]} = 0 \ , \qquad Q_{[i}{}^{m} Q_{j}{}^{n} K_{kl]} = 0 \, , \tag{3.22}$$

which admits a subspace of solutions of dimension $2(p+q-k-2)$ for $Q_i{}^m \in SL(k) \times SO(p-k, q-k)/(SO(2) \times SL(k-2) \ltimes \mathbb{R}^{2(k-2)} \times SO(p-k-2, q-k)) \in \mathbb{R}^{2(p+q-2k)}$ and a subspace of

[6] The unipotent being non-Abelian for $k \geq 2$, one cannot generally define the Fourier coefficients for (Q^m_i, K_{ij}), but one must consider separately the Abelian Fourier coefficient with $K_{ij} = 0$, from the non-Abelian Fourier coefficients with K_{ij} and a subset of the charges Q^m_i defining a polarization.

dimension $2k-3$ for $K_{ij} \in \mathbb{R}^{\frac{k(k-1)}{2}}$, and therefore a Kostant–Kirillov dimension $2(p+q-4)+1$ that is exactly saturated by the Fourier coefficients in the maximal parabolic decomposition with $k=2$.

The tensor F_{abcd} is instead in an automorphic representation associated to the nilpotent orbit of partition $[3, 1^{p+q-3}]$ such that the nilpotent elements $Z_{a\hat{b}} \in \mathfrak{so}(p+q)(\mathbb{C}) \ominus (\mathfrak{so}(p)(\mathbb{C}) \oplus \mathfrak{so}(q)(\mathbb{C}))$ satisfy the constraint

$$Z_{[a}{}^{[\hat{a}} Z_{b]}{}^{\hat{b}]} = 0 \ , \qquad Z_{a\hat{c}} Z_b{}^{\hat{c}} = 0 \ . \tag{3.23}$$

For a representative of the nilpotent orbit in the unipotent associated to the maximal parabolic $GL(k) \times SO(p-k, q-k) \ltimes \mathbb{R}^{2(p+q-2k)+\frac{k(k-1)}{2}}$ this gives the constraints

$$Q_{[i}{}^{[m} Q_{j]}{}^{n]} = 0 \ , \qquad Q_{[i}{}^{m} K_{jk]} = 0 \, , \qquad K_{[ij} K_{kl]} = 0 \ , \tag{3.24}$$

which admits a subspace of solutions of dimension $p+q-k-1$ for $Q_i{}^m \in SL(k) \times SO(p-k, q-k)/(SL(k-1) \ltimes \mathbb{R}^{k-1} \times SO(p-k-1, q-k)) \in \mathbb{R}^{2(p+q-2k)}$ and a subspace of dimension $k-1$ for $K_{ij} \in \mathbb{R}^{\frac{k(k-1)}{2}}$, and therefore a Kostant–Kirillov dimension $p+q-2$ that is exactly saturated by the Fourier coefficients in the maximal parabolic decomposition with $k=1$. One easily checks that the sum of two generic elements (Q_i^m, K_{ij}) solving (3.24) always solve (3.22), so that the quadratic source in F_{abcd} sources the Fourier coefficients of the tensor $G_{ab,cd}$ consistently with the automorphic representation associated to the nilpotent orbit of partition $[3^2, 1^{p+q-6}]$.

It is important to note that the 1/4-BPS black hole solutions (single-centered and multi-centered) are solutions of the Euclidean three-dimensional non-linear sigma model over $O(2k, 8)$ $/(O(2k) \times O(8))$ which are themselves associated to a real nilpotent orbit of $O(2k, 8)$ of partition $[3^2, 1^{2+2k}]$ [54, 55]. This is consistent with the property that the Fourier coefficients in the maximal parabolic decomposition $GL(2) \times O(2k-2, 6) \ltimes \mathbb{R}^{2(4+2k)+1}$ saturate the Kostant–Kirillov dimension and are proportional to the helicity supertrace associated with these black holes.

3.2 $\mathcal{R}^2 F^2$ type invariants in four dimensions

In four dimensions, there are two distinct classes of six-derivative supersymmetric invariants. In the linearised approximation, they are defined as harmonic superspace integrals of G-analytic integrands annihilated by a quarter of the fermionic derivatives, and can be promoted to non-linear harmonic superspace integrals [56]. The first class of invariants is the one defined in the preceding section for $q=6$. It includes a $G^{(2k-2,6)}_{ab,cd} \nabla(F^a \bar{F}^b) \nabla(F^c \bar{F}^d)$ coupling with a tensor $G^{(2k-2,6)}_{ab,cd}$ satisfying to (3.16) and (3.20). The second class of invariants is defined as a chiral harmonic superspace integral at the linearised level, as we now explain.

In four dimensional supergravity with half-maximal supersymmetry, the linearised Maxwell superfield $W_{\hat{a}a} \sim W_{ija}$ satisfies the constraints

$$D_{\alpha k} W_{ija} = \varepsilon_{ijkl} \lambda^l_{\alpha a} \ , \qquad \bar{D}^k_{\dot{\alpha}} W_{ija} = 2\delta^k_{[i} \bar{\lambda}_{\dot{\alpha} j]a} \ , \qquad D_{\alpha i} \lambda^j_{\beta a} = \delta^j_i F_{\alpha\beta a} \ , \tag{3.25}$$

whereas the chiral scalar superfield satisfies

$$D_{\alpha i} S = \chi_{\alpha i} \ , \qquad \bar{D}^i_{\dot{\alpha}} S = 0 \ , \qquad D_{\alpha i} \chi_{\beta i} = F_{\alpha\beta ij} \ , \tag{3.26}$$

with $i = 1$ to 4 of $SU(4)$ and α, $\dot\alpha$ the $SL(2,\mathbb{C})$ indices. The chiral 1/4-BPS linearised invariants are defined using harmonics of $SU(4)/S(U(2)\times U(2))$ parametrizing a $SU(4)$ group element ${u_r}^i, {u_{\hat r}}^i$ with r and $\hat r$ the indices of the two respective $SU(2)$ subgroups. The superfield $W_{34a} \equiv {u_3}^i {u_4}^j W_{ija} = \frac{1}{2}\varepsilon^{\hat r\hat s}{u_{\hat r}}^i {u_{\hat s}}^j W_{ija}$ then satisfies the G-analyticity constraints

$$ {u_{\hat r}}^i D_{\alpha i}({u_3}^i {u_4}^j W_{ija}) \equiv D_{\alpha \hat r} W_{34a} = 0 \ , \qquad {u_i}^r \bar D^i_{\dot\alpha}({u_3}^i {u_4}^j W_{ija}) \equiv \bar D^r_{\dot\alpha} W_{34a} = 0 \quad . \tag{3.27} $$

One can obtain a linearised invariant from the action of the eight derivatives $D_{\alpha i}$ and the four derivatives $\bar D^r_{\dot\alpha} \equiv {u_i}^r \bar D^i_{\dot\alpha}$ on any homogeneous function of the G-analytic superfields W_{34a} and S. Using for short $u^{34}_{\hat a} = (\Gamma_{\hat a})^{ij} {u_i}^3 {u_j}^4$ and the projection $(\hat a_1 \dots \hat a_n)'$ on the traceless symmetric component, one gets

$$ \begin{aligned} &\int du\, u^{34}_{\hat a_1} \dots u^{34}_{\hat a_n} [D^8][\bar D^4] \tfrac{1}{(n+2)!(m+2)!} c_{a_1\dots a_{n+2}} W^{a_1}_{34} \dots W^{a_{n+2}}_{34} S^{2+m} \\ =\ & \frac{1}{n!m!} c_{a_1\dots a_n ab} W^{a_1}{}_{(\hat a_1} W^{a_2}{}_{\hat a_2} \dots W^{a_n}{}_{\hat a_n)'} S^m \mathcal{L}^{(0)ab}_{+2} \\ &+ \frac{1}{(n-1)!m!} c_{a_1\dots a_n ab} W^{a_2}{}_{(\hat a_2} W^{a_3}{}_{\hat a_3} \dots W^{a_n}{}_{\hat a_n} S^m \mathcal{L}^{(0)a_1 ab}_{\hat a_1)'+2} \\ &+ \frac{1}{n!(m-1)!} c_{a_1\dots a_n ab} W^{a_1}{}_{(\hat a_1} W^{a_2}{}_{\hat a_2} \dots W^{a_n}{}_{\hat a_n)'} S^{m-1} \mathcal{L}^{(0)ab}_{+4} + \dots \\ &+ \frac{1}{(n-6)!(m-2)!} c_{a_1\dots a_n ab} W^{a_7}{}_{(\hat a_7} W^{a_6}{}_{\hat a_6} \dots W^{a_n}{}_{\hat a_n} S^{m-2} \mathcal{L}^{(0)a_1\dots a_6 ab}_{\hat a_1 \dots \hat a_6)'+6} + \dots \\ &+ \frac{1}{(n-2)!(m-6)!} c_{a_1\dots a_n ab} W^{a_3}{}_{(\hat a_3} W^{a_4}{}_{\hat a_4} \dots W^{a_n}{}_{\hat a_n} S^{m-6} \mathcal{L}^{(0)a_1 a_2 ab}_{\hat a_1 \hat a_2)'+14} + \partial(\dots) \ , \end{aligned} \tag{3.28} $$

where the $\mathcal{L}^{[n+4]}_{[n]+m}$ are symmetric tensors that only depend on the scalar fields through their derivative. One works out in particular that $\mathcal{L}^{(0)ab}_{+2}$ includes a term of type $\mathcal{R}^2 F^2$ as

$$ \mathcal{L}^{(0)ab}_{+2} \sim \bar F^a_{\dot\alpha\dot\beta} \bar F^{\dot\alpha\dot\beta b} C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} + \dots \tag{3.29} $$

with $C_{\alpha\beta\gamma\delta}$ the complex Weyl curvature tensor (which we denote schematically by $\mathcal{R}$), whereas the highest monomials only depend on the fermion fields as

$$ \begin{aligned} \mathcal{L}^{(0)a_1\dots a_6 ab}_{\hat a_1\dots \hat a_6+6} &\sim \bar\lambda^4\lambda^4\chi^4 \ , \qquad \mathcal{L}^{(0)a_1\dots a_5 ab}_{\hat a_1\dots \hat a_5+8} \sim \bar\lambda^4\lambda^3\chi^5 \ , \qquad \mathcal{L}^{(0)a_1\dots a_4 ab}_{\hat a_1\dots \hat a_4+10} \sim \bar\lambda^4\lambda^2\chi^6 \ , \\ \mathcal{L}^{(0)a_1 a_2 a_3 ab}_{\hat a_1 \hat a_2 \hat a_3+12} &\sim \bar\lambda^4\lambda\chi^7 \ , \qquad \mathcal{L}^{(0)a_1 a_2 ab}_{\hat a_1 \hat a_2+14} \sim \bar\lambda^4\chi^8 \ . \end{aligned} \tag{3.30} $$

Note that $\mathcal{L}^{(0)ab}_{+2}$ is of $U(1)$ weight -2, so one can anticipate that it must be multiplied by a modular form of weight 2 at the non-linear level. At the non-linear level, derivatives of the scalar fields only appear through the pull-back of the right-invariant form $P_{a\hat b}$ over the Grassmanian and the covariant derivative $(S-\bar S)^{-1}\partial_\mu S$ of the upper complex half plan field S. One defines in the same way the covariant derivative $\mathcal{D}_{a\hat b}$ on the Grassmanian and the Kähler derivative $\mathcal{D} = (S-\bar S)\frac{\partial}{\partial S} + \frac{w}{2}$ on a weight w form. According to the linearised analysis, the supersymmetry invariant is associated to a tensor $G_{ab}(\phi, S)$, holomorphic in S and function of the Grassmanian coordinates ϕ.

Due to the superconformal symmetry $PSU(2,2|4)$ of the linearised theory in four dimensions, the non-linear invariants are in bijective correspondance with the linearised invariants, themselves determined by harmonic superspace integrals. However, the linearised invariants that combine to define a general class of non-linear invariants are not necessarily defined from the same harmonic superspace. The general $\nabla^2 F^4$ type invariants defined in the preceding section are determined by vector-like harmonic superspace integrals of $SU(4)/S(U(1)\times U(2)\times U(1))$. In contrast the $\mathcal{R}^2F^2$ type invariants described in this section involve both structures, such that the defining function $\mathcal{G}_{ab}(\phi,S)$ is of weight zero, and the terms in the Lagrangian that do not involve its Kähler derivative $\mathcal{D}$ are defined at the linearised level from $SU(4)/S(U(1)\times U(2)\times U(1))$ harmonic superspace integral of a restricted type. These invariants are constructed explicitly in [56] for a $SO(p)$ invariant function on the Grassmannian. One finds that $\mathcal{G}_{ab}(\phi,S)$ must be holomorphic in S, as the linearised analysis suggested. It defines a Lagrange density $\mathcal{L}$ that decomposes naturally as

$$\begin{aligned}\mathcal{L} = \mathcal{G}_{ab}\mathcal{L}^{ab} + \mathcal{D}_{(a}{}^{\hat a}\mathcal{G}_{bc)}\mathcal{L}^{abc}{}_{\hat a} + \mathcal{D}_{(a}{}^{(\hat a}\mathcal{D}_b{}^{\hat b)}\mathcal{G}_{cd)}\mathcal{L}^{abcd}{}_{\hat a\hat b} + \dots\\ + \mathcal{D}\mathcal{G}_{ab}\mathcal{L}^{ab}_{+2} + \dots + \mathcal{D}\mathcal{D}_{(a_1}{}^{\hat a_1}\cdots\mathcal{D}_{a_6}{}^{\hat a_6}\mathcal{G}_{a_7a_8)}\mathcal{L}^{a_1\dots a_8}{}_{\hat a_1\dots\hat a_6+2}\\ + \mathcal{D}^2\mathcal{G}_{ab}\mathcal{L}^{ab}_{+4} + \dots + \mathcal{D}^2\mathcal{D}_{(a_1}{}^{\hat a_1}\cdots\mathcal{D}_{a_5}{}^{\hat a_5}\mathcal{G}_{a_6a_7)}\mathcal{L}^{a_1\dots a_7}{}_{\hat a_1\dots\hat a_5+4}\\ \vdots\\ + \mathcal{D}^7\mathcal{G}_{ab}\mathcal{L}^{ab}_{+14} + \mathcal{D}^7\mathcal{D}_{(a}{}^{\hat a}\mathcal{G}_{bc)}\mathcal{L}^{abc}{}_{\hat a+14} + \mathcal{D}^7\mathcal{D}_{(a}{}^{(\hat a}\mathcal{D}_b{}^{\hat b)}\mathcal{G}_{cd)}\mathcal{L}^{abcd}{}_{\hat a\hat b+14}\ ,\end{aligned} \tag{3.31}$$

where the $\mathcal{L}^{[n+2]}{}_{[n]+m}$ are $SL(2)\times O(2k-2,6)$ invariant polynomial functions of the covariant fields and their derivatives and the vierbeins and the gravitini fields. Because non-linear invariants induce linear invariants by truncation to lowest order in the fields (3.4), the covariant densities $\mathcal{L}^{[n+2]}{}_{[n]+m}$ reduce at lowest order to homogeneous polynomials of degree $n+2$ in the covariant fields (3.4) that coincide with the linearised polynomials $\mathcal{L}^{(0)[n+2]}{}_{[n]+m}$ for $m\geq 2$. For $m=0$, the linearised invariants $\mathcal{L}^{(0)[n+2]}{}_{[n]}$ are the real analytic superspace integrals described in the preceding section $[n+2,m]$ for $n=0$, and where indices are contracted with δ^{ab} to reduce the representation from the Young Tableau $[2,m]$ to $[0,m+1]$. The analysis of the invariant defined as a non-linear harmonic superspace integral indeed shows that the component $\mathcal{L}^{ab}$ is of the type

$$\mathcal{L}^{ab} = \sqrt{-g}\, t_8\big(2\nabla(F^{(a}F^{b)})\nabla(F_cF^c) + \nabla(F_cF^{(a})\nabla(F^{b)}F^c) + \dots\big)\ , \tag{3.32}$$

with $t_8F^4 = F_{\alpha\beta}F^{\alpha\beta}\bar F_{\dot\alpha\dot\beta}\bar F^{\dot\alpha\dot\beta}$, and

$$\mathcal{L}^{ab}_{+2} = \sqrt{-g}\bar F^a_{\dot\alpha\dot\beta}\bar F^{\dot\alpha\dot\beta b}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + \dots\ . \tag{3.33}$$

The complete invariant is the real part of this complex invariant. So the four-photon MHV amplitude gives a contribution to the Wilsonian effective action in $\mathcal{G}_{ab}(\phi,S)+\mathcal{G}_{ab}(\phi,\bar S)$, whereas the amplitude with two gravitons of positive helicity and two photons of negative helicity gives a contribution in $\mathcal{D}\mathcal{G}_{ab}(\phi,S)$. Because $\mathcal{D}\mathcal{G}_{ab}(\phi,\bar S)=0$, we will usually refer to a single function $G^{(0)}_{ab}(\phi,S,\bar S) = \mathcal{G}_{ab}(\phi,S)+\mathcal{G}_{ab}(\phi,\bar S)$.

Similarly to [22], one can show that supersymmetry at the linearised level implies tensorial differential equations of the form

$$\mathcal{D}_d{}^{\hat a}\mathcal{D}_{c\hat a}G^{(0)}_{ab} = \tfrac{3(2-q)}{4}\delta_{d(c}G^{(0)}_{ab)} + \tfrac{3}{2}\,\delta_{(ab}G^{(0)}_{c)d}\ , \qquad \mathcal{D}_{[a}{}^{\hat a}G^{(0)}_{b]c} = 0\ , \tag{3.34}$$

with $q=6$, where the coefficients of the two terms on the right-hand side have been fixed by requiring that these constraints are integrable.

As in the preceding section, this linearized analysis does not take into account the lower order corrections in the effective action and the local terms coming from the explicit decomposition of the effective action into local and non-local components. The coefficient $F_{abcd}(\varphi)$ of the F^4 coupling and the real coefficient $\mathcal{E}(S)$ of the $\mathcal{R}^2$ coupling give rise to source terms in these differential equations, such that we get eventually

$$\begin{aligned}\mathcal{D}_d{}^{\hat{a}}\mathcal{D}_{c\hat{a}}G_{ab}(S,\varphi) &= -3\delta_{d(c}G_{ab)}(S,\varphi)+\tfrac{3}{2}\,\delta_{(ab}G_{c)d}(S,\varphi)+6\mathcal{E}(S)F_{abcd}(\varphi)\ ,\\ \mathcal{D}\bar{\mathcal{D}}G_{ab}(S,\varphi) &= \frac{3}{4\pi}F_{abc}{}^{c}(\varphi)\ .\end{aligned} \tag{3.35}$$

Finally, let us note that the same class of harmonic superspace integrals (3.28) produces higher derivative invariants by integrating instead

$$\int \mathrm{d}u\, u^{34}_{\hat{a}_1}\dots u^{34}_{\hat{a}_n}u^{34}_{\hat{b}_1}\dots u^{34}_{\hat{b}_{2p}}[D^8][\bar{D}^4]\tfrac{1}{n!(m+2)!(p+1)!}c_{a_1\dots a_n}W^{a_1}_{34}\dots W^{a_n}_{34}S^{2+m}(F_{\alpha\beta 34}F^{\alpha\beta}_{34})^{p+1}\ . \tag{3.36}$$

This gives rise to chiral 1/4-BPS-protected invariants of the same class, including couplings of the form

$$\mathcal{G}^{(2p+4)}_{\hat{a}_1\hat{a}_2\dots\hat{a}_{2p}}(S,\varphi)\,C^2\,\nabla^2 S\nabla^2 S(F^{\hat{a}_1}F^{\hat{a}_2})\dots(F^{\hat{a}_{2p-1}}F^{\hat{a}_{2p}})\ . \tag{3.37}$$

Here C is the Weyl tensor and $\mathcal{G}^{(2p)}(S,\varphi)$ is a rank $2p$ $SO(6)$ symmetric traceless tensor, which is a weight $2p+4$ weakly holomorphic modular form in S. It satisfies to a hierarchy of differential equations on the Grassmannian [57]

$$\begin{aligned}\mathcal{D}_a{}^{\hat{a}_{2p}}\mathcal{G}^{(2p+4)}_{\hat{a}_1\dots\hat{a}_{2p}} &= \mathcal{D}_{a[\hat{a}_1}\mathcal{G}^{(2p+4)}_{\hat{a}_2]\hat{a}_3\dots\hat{a}_{2p+1}} = \bar{\mathcal{D}}\mathcal{G}^{(2p+4)}_{\hat{a}_1\dots\hat{a}_{2p}} = 0\ ,\\ \mathcal{D}_a{}^{\hat{c}}\mathcal{D}_{b\hat{c}}\mathcal{G}^{(2p+4)}_{\hat{a}_1\dots\hat{a}_{2p}} &= -2(p+2)\delta_{ab}\mathcal{G}^{(2p+4)}_{\hat{a}_1\dots\hat{a}_{2p}}+\mathcal{D}_{a(\hat{a}_1}\mathcal{D}_{|b|\hat{a}_2}\mathcal{G}^{(2p+2)}_{\hat{a}_3\dots\hat{a}_{2p})}\ .\end{aligned} \tag{3.38}$$

On the type II side these couplings can be computed in topological string theory [58].

3.3 The modular integral satisfies the Ward identities

In this subsection, we shall prove that the modular integral

$$G^{(p,q)}_{ab,cd} = \mathrm{R.N.}\int_{\Gamma_{2,0}(N)\backslash\mathcal{H}_2}\frac{\mathrm{d}^3\Omega_1\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\frac{\Gamma^{(2)}_{\Lambda_{p,q}}[P_{ab,cd}]}{\Phi_{k-2}(\Omega)}\ , \tag{3.39}$$

satisfies the differential equations (3.16) and (3.20), with a specific value of the coefficient ϖ in the quadratic source term. Here, $\Phi_{k-2}(\Omega)$ is the meromorphic Siegel modular form defined in (A.33), and $\Gamma^{(2)}_{\Lambda_{p,q}}[P_{ab,cd}]$ is the genus-two partition function (2.32) for a level N even lattice of signature (p,q), with an insertion of the quartic polynomial $P_{ab,cd}$ defined in (2.31). Since Φ_{k-2} and $\Gamma^{(2)}_{\Lambda_{p,q}}[P_{ab,cd}]$ are modular forms of weight $k-2$ and $\frac{p-q}{2}+2=k-2$ under $\Gamma_{2,0}(N)$, the integrand is well defined on the quotient $\Gamma_{2,0}(N)\backslash\mathcal{H}_2$. The symbol R.N. refers to a regularization procedure which is necessary to make sense of the integral when $q\geq 5$, as discussed in Appendix B.2.4.

In order to derive these results, we shall first establish differential equations for the general class of genus-two Siegel theta series $\Gamma^{(2)}_{\Lambda_{p,q}}[P]$, where the polynomial $P(Q)$ is obtained by acting on a homogeneous polynomial of bidegree (m,n) in $(\varepsilon_{rs}Q^r_L Q^s_L, \varepsilon_{rs}Q^r_R Q_R)^s$ respectively, with the operator $|\Omega_2|^n e^{-\frac{\Delta_2}{8\pi}}$, where ε_{rs} is the rank-two antisymmetric tensor with $\varepsilon_{12}=1$ and Δ_2 is the second order differential operator

$$\Delta_2 \equiv \sum_a \frac{\partial}{\partial Q^r_{La}}(\Omega_2^{-1})^{rs}\frac{\partial}{\partial Q^{sa}_L} + \sum_{\hat a} \frac{\partial}{\partial Q^r_{R\hat a}}(\Omega_2^{-1})^{rs}\frac{\partial}{\partial Q^{s\hat a}_R}\,, \tag{3.40}$$

Under this condition, one can show using Poisson resummation that $\Gamma^{(2)}_{\Lambda_{p,q}}[P]$ satisfies

$$\Gamma^{(2)}_{\Lambda_{p,q}}[P](-\Omega^{-1}) = \frac{(-i)^{p-q}|\Omega|^{\frac{p-q}{2}+m-n}}{|\Lambda^*_{p,q}/\Lambda_{p,q}|}\Gamma^{(2)}_{\Lambda^*_{p,q}}[P](\Omega)\,, \tag{3.41}$$

which implies that $\Gamma^{(2)}_{\Lambda_{p,q}}[P]$ transforms as a modular form of weight $\frac{p-2}{2}+m-n$ under $\Gamma_{2,0}(N)$. For our purposes, it will be sufficient to focus on polynomials of the form, using $(Q_a\,\varepsilon\, Q_b)=\varepsilon_{rs}Q^r_aQ^s_b$,

$$\begin{aligned}&P_{a_1\dots a_m,b_1\dots b_m,\hat c_1\dots\hat c_n,\hat d_1\dots\hat d_n}\\ &\quad= e^{\frac{-\Delta_2}{8\pi}}\Big[\big((Q_{L(a_1|}\varepsilon\, Q_{L|b_1|})\dots(Q_{L|a_m)}\varepsilon\, Q_{Lb_m})\big)\big((Q_{R(\hat c_1|}\varepsilon\, Q_{R|\hat d_1|})\dots(Q_{R|\hat c_n)}\varepsilon\, Q_{R\hat d_n})\big)\Big]\,,\end{aligned} \tag{3.42}$$

where $(b_1\dots b_m)$ denotes all symmetric permutations of $b_1,\dots,b_m$, and similarly for hatted indices. The quadratic polynomial $P=P_{ab,cd}$ arises in the case $(m,n)=(2,0)$ with no contraction among the left-moving indices, as written explicitely in the first line of (2.31).

As in [22, section 3], one can obtain the differential equations satisfied by (3.39) by acting with the covariant derivatives $\mathcal{D}_{a\hat b}$ defined by

$$\mathcal{D}_{a\hat b} = \frac{1}{2}(Q^r_{La}\partial_{r\hat b} + Q^r_{R,\hat b}\partial_{ra})\,, \tag{3.43}$$

where $\partial^a_r = \frac{\partial}{\partial Q^r_{La}}$ and $\partial^{\hat a}_r = \frac{\partial}{\partial Q^r_{R\hat a}}$. Recalling that $p_{L,a}{}^I$ and $p_{R,\hat b}{}^J$ are the left and right orthogonal projectors on the Grassmaniann $G_{p,q}=O(p,q)/[O(p)\times O(q)]$, one can use the effective derivation rules

$$\mathcal{D}_{a\hat b}p_{L,c}{}^I = \frac{1}{2}\delta_{ac}p_{R,\hat b}{}^I\,, \qquad \mathcal{D}_{a\hat b}p_{R,\hat c}{}^I = \frac{1}{2}\delta_{\hat b\hat c}p_{L,a}{}^I\,, \tag{3.44}$$

Acting with $\mathcal{D}_{e\hat g}$ on (2.32) we get

$$\mathcal{D}_{e\hat g}\Gamma^{(2)}_{\Lambda_{p,q}}[P] = \Gamma^{(2)}_{\Lambda_{p,q}}\Big[\big(\mathcal{D}_{e\hat g} - 2\pi(Q_{Le}\Omega_2 Q_{R\hat g})\big)P\Big]\,, \tag{3.45}$$

where $(Q_{L,e}\Omega_2 Q_{R,\hat g}) = (\Omega_2)_{rs}Q^r_{La}Q^s_{R\hat g}$ is a short notation that will be used in the following.

It will prove useful to compute the commutation relations

$$[\Delta_2,\mathcal{D}_{e\hat g}] = 2(\partial_e\Omega_2^{-1}\partial_{\hat g})\,, \qquad [\Delta_2, Q^r_{Le}] = 2(\partial_e\Omega_2^{-1})^r\,, \tag{3.46}$$

$$[\Delta_2, Q^r_{Le}Q^s_{R\hat g}] = 2Q^r_{Le}(\Omega_2^{-1}\partial_{\hat g})^s + 2Q^s_{R\hat g}(\Omega_2^{-1}\partial_e)^r\,, \tag{3.47}$$

$$[\Delta_2, Q^r_{Le}Q^s_{Lf}] = 2\delta_{ef}(\Omega_2^{-1})^{rs} + 4Q^{(r}_{L(e}(\Omega_2^{-1}\partial_{f)})^{s)}\,, \tag{3.48}$$

with the Baker-Campbell-Hausdorff formula

$$e^{\frac{\Delta_2}{8\pi}}\mathcal{O}e^{-\frac{\Delta_2}{8\pi}} = \mathcal{O} + \frac{1}{8\pi}[\Delta_2,\mathcal{O}] + \frac{1}{2!}\frac{1}{(8\pi)^2}[\Delta_2,[\Delta_2,\mathcal{O}]] + \dots\,, \tag{3.49}$$

one obtains

$$\mathcal{D}_{e\hat{g}}\Gamma^{(2)}_{\Lambda_{p,q}}[P] = -2\pi\Gamma^{(2)}_{\Lambda_{p,q}}\Big[e^{-\frac{\Delta_2}{8\pi}}\big((Q_{Le}\Omega_2 Q_{R\hat{g}}) - \tfrac{1}{(4\pi)^2}(\partial_e\Omega_2^{-1}\partial_{\hat{g}})\big)e^{\frac{\Delta_2}{8\pi}}P\Big]\,. \tag{3.50}$$

Note that the derivation rules ensure that the constraints (3.16) are automatically satisfied at the level of the integrand, from the structure of (3.42) with $n = 0$. Antisymmetrizing (3.50) with $n = 0$, one obtains

$$\mathcal{D}_{[e}{}^{\hat{g}}\Gamma^{(2)}_{\Lambda_{p,q}}[P_{a_1|\ldots a_m,|b_1]\ldots b_m}] = \Gamma^{(2)}_{\Lambda_{p,q}}\Big[e^{-\frac{\Delta_2}{8\pi}}\tfrac{1}{8\pi}(\partial_{[e}\Omega_2^{-1}\partial^{\hat{g}})e^{\frac{\Delta_2}{8\pi}}P_{a_1|\ldots a_m,|b_1]\ldots b_m}\Big]\,, \tag{3.51}$$

which vanishes identically since $e^{\frac{\Delta_2}{8\pi}}P_{a_1\ldots a_m,b_1\ldots b_m}$ does not depend on Q_R. The same argument goes for $\mathcal{D}_{[e}{}^{[\hat{e}}\mathcal{D}_f{}^{\hat{f}]}\Gamma^{(2)}_{\Lambda_{p,q}}[P_{a_1]\ldots a_m,b_1\ldots b_m}]$ and $\mathcal{D}_{[e}{}^{[\hat{e}}\mathcal{D}_f{}^{\hat{f}}\mathcal{D}_{g]}{}^{\hat{g}]}\Gamma^{(2)}_{\Lambda_{p,q}}[P_{a_1\ldots a_m,b_1\ldots b_m}]$, and we conclude that for $m = 2$, the modular integral (3.39) satisfies

$$\mathcal{D}_{[e}{}^{\hat{e}}G_{a|b,|c],d} = 0\,,\qquad \mathcal{D}_{[e}{}^{\hat{e}}\mathcal{D}_f{}^{\hat{f}}G_{a]b,cd} = 0\,,\qquad \mathcal{D}_{[e}{}^{[\hat{e}}\mathcal{D}_f{}^{\hat{f}}\mathcal{D}_{g]}{}^{\hat{g}]}G_{ab,cd} = 0\,, \tag{3.52}$$

which thus establishes (3.16). Note that these properties are independent of the details of the function $1/\Phi_{k-2}(\Omega)$.

Now, the main equation (3.20) arises by applying the quadratic operator $\mathcal{D}^2_{ef} \equiv \mathcal{D}_{(e}{}^{\hat{g}}\mathcal{D}_{f)\hat{g}}$ on the lattice partition function with polynomial insertion, and commuting with the summation measure $e^{i\pi Q_L\Omega Q_L - i\pi Q_R\bar{\Omega}Q_R}$ of the partition function

$$\begin{aligned}4\mathcal{D}^2_{ef}\Gamma^{(2)}_{\Lambda_{p,q}}[P] = \Gamma^{(2)}_{\Lambda_{p,q}}\Big[\Big(&4\mathcal{D}^2_{ef} - 8\pi(Q_{L(e}\Omega_2 Q_R{}^{\hat{g}})\mathcal{D}_{f)\hat{g}} - 2q\delta_{ef}\\ &+ 16\pi^2\,\mathrm{tr}\left[\Omega_2\big(Q_{Le}Q_{Lf} - \tfrac{\delta_{ef}}{4\pi}\Omega_2^{-1}\big)\Omega_2\big(Q_R^2 - \tfrac{q}{4\pi}\Omega_2^{-1}\big)\right]\Big)P\Big]\,.\end{aligned} \tag{3.53}$$

Using the commutation relations (3.46), one can re-express it to make modular invariance explicit

$$\begin{aligned}4\mathcal{D}^2_{ef}\Gamma^{(2)}_{\Lambda_{p,q}}[P] = \Gamma^{(2)}_{\Lambda_{p,q}}\Big[e^{-\frac{\Delta_2}{8\pi}}\Big(&16\pi^2(Q_{Le}\Omega_2 Q_R{}^{\hat{g}})(Q_{R\hat{g}}\Omega_2 Q_{Lf}) - \delta_{ef}(qg + (Q_{R\hat{g}}\partial^{\hat{g}}))\\ - q(Q_{L(e}\partial_{f)}) - 2(Q_{L(e}\Omega_2 Q_{R\hat{g}})(\partial^{\hat{g}}\Omega_2^{-1}\partial_f) &+ \frac{1}{16\pi^2}(\partial_e\Omega_2^{-1}\partial_{\hat{g}}\partial^{\hat{g}}\Omega_2^{-1}\partial_f)\Big)e^{\frac{\Delta_2}{8\pi}}P\Big]\,,\end{aligned} \tag{3.54}$$

and notice that all the terms in (3.54) except the first and last one will become linear tensorial combinations of the original partition function $\Gamma^{(2)}_{\Lambda_{p,q}}[P]$. The first term on the r.h.s of (3.54) can be rewritten as the action of the lowering operator for Siegel modular forms,

$$\bar{D}_{rs} = -\mathrm{i}\pi(\Omega_2(\Omega_2\partial_{\bar{\Omega}})^\intercal)_{rs} = -\mathrm{i}\pi(\Omega_2)_{rt}(\Omega_2)_{su}\frac{\partial}{\partial\bar{\Omega}_{tu}}\,, \tag{3.55}$$

which take a weight w representation sym^l modular form to a weight $w-2$ representation $\mathrm{sym}^2\otimes\mathrm{sym}^l$ modular form [59]. Indeed,

$$\begin{aligned}\bar{D}_{rs}\Gamma^{(2)}_{\Lambda_{p,q}}&\Big[e^{-\frac{\Delta_2}{8\pi}}Q^r_{Le}Q^s_{Lf}e^{\frac{\Delta_2}{8\pi}}P\Big]\\ &= -\pi^2\Gamma^{(2)}_{\Lambda_{p,q}}\Big[\mathrm{tr}\left[\Omega_2\big(Q_R^2 - \tfrac{q}{4\pi}\Omega_2^{-1}\big)\Omega_2 e^{-\frac{\Delta_2}{8\pi}}(Q_{Le}Q_{Lf})e^{\frac{\Delta_2}{8\pi}}\right]P\Big]\\ &\qquad + \frac{1}{16}\Gamma^{(2)}_{\Lambda_{p,q}}\Big[(\partial^h_r\partial_{sh} + \partial^{\hat{h}}_r\partial_{s\hat{h}})e^{-\frac{\Delta_2}{8\pi}}Q^r_{Le}Q^s_{Lf}e^{\frac{\Delta_2}{8\pi}}P\Big]\\ &= \Gamma^{(2)}_{\Lambda_{p,q}}\Big[e^{-\frac{\Delta_2}{8\pi}}\big(\tfrac{1}{16}\partial_{rh}\partial^h_s Q^r_{Le}Q^s_{Lf} - \pi^2(Q_{Le}\Omega_2 Q_{R\hat{g}})(Q_R{}^{\hat{g}}\Omega_2 Q_{Lf})\\ &\qquad - \frac{\pi}{2}\big((Q_{L(e}\Omega_2 Q_{R\hat{g}})(\partial^{\hat{g}}Q_{Lf)}) - n(Q_{Le}\Omega_2 Q_{Lf})\big)e^{\frac{\Delta_2}{8\pi}}P\Big]\,,\end{aligned} \tag{3.56}$$

and the r.h.s. of (3.54) can thus be written as

$$\begin{aligned}
4\mathcal{D}_{ef}^2\Gamma^{(2)}_{\Lambda_{p,q}}\big[P\big] = \Gamma^{(2)}_{\Lambda_{p,q}}\Big[e^{-\frac{\Delta_2}{8\pi}}\Big((6-2q-Q_{R\hat{g}}\partial^{\hat{g}})\delta_{ef} + (6-q)(Q_{L(e}\partial_{f)})\\
+ Q_{rLe}Q_{sLf}(\partial_L^2)^{rs} - 2(Q_{L(e}\Omega_2 Q_{R\hat{g}})(\partial^{\hat{g}}\Omega_2^{-1}\partial_f)\\
- 8\pi\big((Q_{L(e}\Omega_2 Q_{R\hat{g}})(\partial^{\hat{g}}Q_{Lf)}) - n(Q_{Le}\Omega_2 Q_{Lf})\big) + \frac{1}{16\pi^2}(\partial_e\Omega_2^{-1}\partial_{\hat{g}}\partial^{\hat{g}}\Omega_2^{-1}\partial_f)\Big)e^{\frac{\Delta_2}{8\pi}}P\Big]\\
- 16\bar{D}_{rs}\Gamma^{(2)}_{\Lambda_{p,q}}\Big[e^{-\frac{\Delta_2}{8\pi}}Q^r_{Le}Q^s_{Lf}e^{\frac{\Delta_2}{8\pi}}P\Big]\,.
\end{aligned} \tag{3.57}$$

The third line contains contributions from partition functions with more or fewer momentum insertions, respectively, and the fourth line is to be computed explicitly. We now specialize to the case of interest and obtain

$$\Delta_{ef}\Gamma^{(2)}_{\Lambda_{p,q}}\big[P_{ab,cd}\big] = -4\bar{D}_{rs}\Gamma^{(2)}_{\Lambda_{p,q}}\big[e^{-\frac{\Delta_2}{8\pi}}Q^r_{Le}Q^s_{Lf}e^{\frac{\Delta_2}{8\pi}}P_{ab,cd}\big]\,, \tag{3.58}$$

where the operator Δ_{ef} is defined as

$$\begin{aligned}
2\Delta_{ef}G_{ab,cd} \equiv & 2\mathcal{D}_{ef}^2 G_{ab,cd} + (q-3)\delta_{ef}G_{ab,cd} + (q-6)\left[\delta_{|e)(a}G_{b)(f|,cd} + \delta_{|e)(c}G_{d)(f|,ab}\right]\\
& - 3\delta_{\langle ab,}G_{cd\rangle,ef}\,.
\end{aligned} \tag{3.59}$$

Let us now return to the modular integral (3.39). In order to regularize the infrared divergences which arise when $q > 5$ (discussed in more detail in Appendix B.2.4), it is useful to first fold the integration domain $\Gamma_{2,0}(N)\backslash\mathcal{H}_2$ onto the fundamental domain $\mathcal{F}_2 = Sp(4,\mathbb{Z})\backslash\mathcal{H}_2$, and restrict the latter to truncated fundamental domain

$$\mathcal{F}_{2,\Lambda,\eta} = \mathcal{F}_2 \cap \{\rho_2 \le \sigma_2 - v_2^2/\rho_2 \le \Lambda\} \cap \{|v| > \eta\} \tag{3.60}$$

excising both the non-separating degeneration at $\Omega_2 = i\infty$ and the separating degeneration at $v = 0$. We thus define

$$G^{(p,q)}_{ab,cd}(\Lambda,\eta) = \int_{\mathcal{F}_{2,\Lambda,\eta}} \frac{d^3\Omega_1 d^3\Omega_2}{|\Omega_2|^3} \sum_{\gamma\in\Gamma_{2,0}(N)\backslash Sp(4,\mathbb{Z})} \left[\frac{\Gamma^{(2)}_{\Lambda_{p,q}}[P_{ab,cd}]}{\Phi_{k-2}(\Omega)}\right]_\gamma\,. \tag{3.61}$$

The renormalized integral (3.39) is defined as the limit of (3.61) as $\Lambda\to\infty$, $\eta\to 0$, possibly after subtracting divergent terms. Acting with the operator Δ_{ef} and using (3.58) one obtains

$$\Delta_{ef}\,G^{(p,q)}_{ab,cd}(\Lambda,\eta) = -4\int_{\mathcal{F}_{2,\Lambda,\eta}} \frac{d^3\Omega_1 d^3\Omega_2}{|\Omega_2|^3} \sum_\gamma \left[\frac{1}{\Phi_{k-2}}\bar{D}_{rs}\Gamma^{(2)}_{\Lambda_{p,q}}\big[e^{-\frac{\Delta_2}{8\pi}}Q^r_{Le}Q^s_{Lf}e^{\frac{\Delta_2}{8\pi}}P_{ab,cd}\big]\right] \tag{3.62}$$

To compute the boundary term, we use Stokes' theorem in the form

$$\int_{\partial\mathcal{F}^\Lambda_{2,\Lambda,\eta}} \frac{d^5\Omega^{rs}}{|\Omega_2|^3}(\Omega_2)_{rt}(\Omega_2)_{su}(f^{tu}g) = \frac{2}{\pi}\int_{\mathcal{F}^\Lambda_{2,\Lambda,\eta}} \frac{d^3\Omega_1 d^3\Omega_2}{|\Omega_2|^3}\big(g\bar{D}_{rs}f^{rs} + f^{rs}\bar{D}_{rs}g\big)\,, \tag{3.63}$$

where f^{rs} and g are modular form of $\Gamma_{2,0}(N)$ respectively of weight w and representation sym^2, and weight $w' = 2 - w$ and trivial representation. The differential operator $\partial_{\bar{\Omega}}$ commutes with factors of Ω_2 because of the natural connection $\bar{D}_{rs}$. Then, since $\bar{D}_{rs}1/\Phi_{k-2}=0$ by holomorphicity, we obtain that the r.h.s. of (3.62) reads

$$-2\pi\int_{\partial\mathcal{F}_{2,\Lambda,\eta}}\frac{d^5\Omega^{rs}}{|\Omega_2|^3}(\Omega_2)_{rt}(\Omega_2)_{su}\sum_{\gamma}\left[\frac{1}{\Phi_{k-2}(\Omega)}\Gamma^{(2)}_{\Lambda_{p,q}}\left[e^{-\frac{\Delta_2}{8\pi}}Q^t_{Le}Q^u_{Lf}e^{\frac{\Delta_2}{8\pi}}P_{ab,cd}\right]\right]_{\gamma}\,. \tag{3.64}$$

The contributions from the Λ-dependent boundary of $\mathcal{F}_{2,\Lambda,\eta}$ lead to powerlike terms in Λ, which cancel in the renormalized integral, except for $q=5$ or $q=6$ where these divergent terms become logarithmic and are responsible for an anomalous term in the differential equation. These anomalous terms are computed in §B.2.5 and will be displayed in the final result below. Here we focus on the contribution from the boundary at $|v|=\eta$ due to the pole of the integrand at $v=0$, which is cut-off independent for any q and can be computed using Cauchy's theorem.

To compute the residue at $v=0$, recall that the function $1/\Phi_{k-2}$ has a second order pole at $v=0$ (cf. (A.44)) and behaves as $\Phi_{k-2}\sim(2\pi i v)^2\Delta_k(\rho)\times\Delta_k(\sigma)+O(v^4)$. The only cosets γ preserving the pole at $v=0$ are those in $\gamma\in(\Gamma_0(N)\backslash SL(2,\mathbb{Z}))_\rho\times(\Gamma_0(N)\backslash SL(2,\mathbb{Z}))_\sigma$. Adding up these contributions, we find that the residue of the integrand at $v=0$ is

$$\frac{1}{\rho_2^2\sigma_2^2}\frac{i}{\Delta_k(\rho)\Delta_k(\sigma)}\sum_{\substack{\gamma\in(\Gamma_0(N)\backslash SL(2,\mathbb{Z}))_\rho\\ \times(\Gamma_0(N)\backslash SL(2,\mathbb{Z}))_\sigma}}\Gamma^{(2)}_{\Lambda_{p,q}}\left[e^{-\frac{\Delta_2}{8\pi}}Q^1_{Lk}Q^{2k}_{L}Q^1_{Le}Q^2_{Lf}e^{\frac{\Delta_2}{8\pi}}P_{ab,cd}\right]\Big|_{\gamma}\left(\Omega=\left(\begin{smallmatrix}\rho&0\\0&\sigma\end{smallmatrix}\right)\right). \tag{3.65}$$

Near the boundary at $|v|=\eta$, the fundamental domain $\mathcal{F}_{2,\Lambda,\eta}$ reduces to $\mathcal{F}_1(\rho)\times\mathcal{F}_1(\sigma)\times\{|v|>\eta\}/\mathbb{Z}_2\times Z_2$ where the first $\mathbb{Z}_2$ exchanges ρ and σ while the second sends $v\mapsto -v$. Thus, the sum in (3.65) factorizes into two genus-one integrands, leading to

$$\begin{aligned}\Delta_{ef}\,G^{(p,q)}_{ab,cd}(\Lambda,\eta)&=-\pi\big(F^{(p,q)}_{abk(e}(\Lambda)F^{(p,q)\,k}_{f)cd}(\Lambda)-F^{(p,q)}_{ak|c)(e}(\Lambda)F^{(p,q)}_{f)(d|b}{}^{k}(\Lambda)\big)+\dots\\&=-\frac{3\pi}{2}F^{(p,q)}_{|e)k\langle ab,}(\Lambda)\,F^{(p,q)\,k}_{cd\rangle}{}_{(f|}(\Lambda)+\dots\,,\end{aligned} \tag{3.66}$$

where the dots denote contributions from the Λ-dependent boundary, discussed in detail in Appendix B.2.4, while $F^{(p,q)}_{abcd}(\Lambda)$ is the genus-one regularized modular integral

$$F^{(p,q)}_{abcd}(\Lambda)=\int_{\mathcal{F}_{1,\Lambda}}\frac{d\rho_1\,d\rho_2}{\rho_2^2}\sum_{\gamma\in\Gamma_0(N)\backslash SL(2,\mathbb{Z})}\left[\frac{1}{\Delta_k}\Gamma_{\Lambda_{p,q}}[P_{abcd}]\right]_{\gamma}\,. \tag{3.67}$$

This establishes (3.20) with $\varpi=\pi$. We show in Appendix B.2.5 that the divergent terms from the Λ-dependent boundary of $\mathcal{F}_{2,\Lambda,\eta}$ combine consistently such that the renormalised coupling satisfies the same differential equation (3.20), but for $q=5$ or $q=6$, for which one gets additional linear source terms. For the perturbative string aplitude, $\upsilon=N$, the additional source term vanishes for $q=5$, and for $q=6$ it can be ascribed to the mixing between the analytic and the non-analytic parts of the amplitude. In this case one obtains (B.96)

$$\Delta_{ef}\,G^{(p,q)}_{ab,cd}=-\frac{3\pi}{2}F^{(p,q)}_{|e)k\langle ab,}F^{(p,q)\,k}_{cd\rangle}{}_{(f|}-\delta_{q,6}\frac{3}{16\pi}\big(\delta_{ef}\delta_{\langle ab,}+2\delta_{e\langle(a}\delta_{b),|f|}\big)F^{(p,q)k}_{cd\rangle k}\,, \tag{3.68}$$

where Δ_{ef} was defined in (3.59).

4 Weak coupling expansion of exact $\nabla^2(\nabla\phi)^4$ couplings

In this section, we study the asymptotic expansion of the proposal (1.5) in the limit where the heterotic string coupling g_3 goes to zero, and show that it reproduces the known tree-level and one-loop amplitudes, along with an infinite series of NS5-brane, Kaluza–Klein monopole and H-monopole instanton corrections. For the sake of generality, we analyze the family of modular integral

$$G^{(p,q)}_{ab,cd} = \text{R.N.} \int_{\Gamma_{2,0}(N)\backslash\mathcal{H}_2} \frac{\mathrm{d}^3\Omega_1 \mathrm{d}^3\Omega_2}{|\Omega_2|^3} \frac{\Gamma^{(2)}_{\Lambda_{p,q}}[P_{ab,cd}]}{\Phi_{k-2}(\Omega)} \,, \tag{4.1}$$

for a level N even lattice $\Lambda_{p,q}$ of arbitrary signature (p,q), in the limit near the cusp where $O(p,q)$ is broken to $O(1,1)\times O(p-1,q-1)$, so that the moduli space decomposes into

$$G_{p,q} \to \mathbb{R}^+ \times G_{p-1,q-1} \ltimes \mathbb{R}^{p+q-2} \,. \tag{4.2}$$

For simplicity, we first discuss the maximal rank case $N=1$, $p-q=16$, where the integrand is invariant under the full Siegel modular group $Sp(4,\mathbb{Z})$, before dealing with the case of N prime, where the integrand is invariant under the congruence subgroup $\Gamma_{2,0}(N)$. The reader uninterested by the details of the derivation may skip to §4.3, where we specialize to the values $(p,q)=(2k,8)$ relevant for the $\nabla^2(\nabla\phi)^4$ couplings in $D=3$ and interpret the various contributions as perturbative and non-perturbative effects in heterotic string theory compactified on T^7. In §6.4 we discuss the case $(p,q)=(21,5)$ relevant for $\nabla^2 H^4$ couplings in type IIB string theory compactified on $K3$.

4.1 $O(p,q)\to O(p-1,q-1)$ for even self-dual lattices

In this subsection we assume that the lattice $\Lambda_{p,q}$ is even self-dual and factorizes in the limit (4.2) as

$$\Lambda_{p,q} \to \Lambda_{p-1,q-1} \oplus I\!I_{1,1} \,. \tag{4.3}$$

We shall denote by R the coordinate on $\mathbb{R}^+$, φ the coordinates on $G_{p-1,q-1}$ and by a^I, $I=1\dots p{+}q{-}2$ the coordinates on $\mathbb{R}^{p+q-2}$. The variable $R>0$ parametrizes a one-parameter subgroup e^{RH_0} in $O(p,q)$, such that the action of the non-compact Cartan generator H_0 on the Lie algebra $\mathfrak{so}_{p,q}$ decomposes into

$$\mathfrak{so}_{p,q} \simeq (\mathbf{p}+\mathbf{q}-\mathbf{2})^{(-2)} \oplus (\mathfrak{gl}_1 \oplus \mathfrak{so}_{p-1,q-1})^{(0)} \oplus (\mathbf{p}+\mathbf{q}-\mathbf{2})^{(2)} \,. \tag{4.4}$$

while the coordinates a^I parametrize the unipotent subgroup obtained by exponentiating the grade 2 component in this decomposition.

The lattice vectors are now labelled according to the choice of A-cycle on the genus-two Riemann surface. They thus take value take value in double copy of the original lattice $\Lambda_{p,q}\oplus\Lambda_{p,q}$. Thus, the generic charge vector $(Q_{1\mathcal{I}},Q_{2\mathcal{I}})\in\Lambda_{p,q}\oplus\Lambda_{p,q}\simeq \mathbf{2}^{(-2)}\oplus(\mathbf{2}\otimes(\mathbf{p}+\mathbf{q}-\mathbf{2}))^{(0)}\oplus\mathbf{2}^{(2)}$,[7] decomposes into

$$(Q^1_{\mathcal{I}},Q^2_{\mathcal{I}}) = (n^1,n^2,\widetilde{Q}^1_I,\widetilde{Q}^2_I,m^1,m^2)\,, \tag{4.5}$$

[7] We use $\mathcal{I}$ to label indices from 1 to $p+q$ in this paragraph to differentiate them from the indices on the sublattice.

where $(n^1, n^2, m^1, m^2) \in I\!I_{1,1} \oplus I\!I_{1,1}$ and $(\widetilde{Q}^1_{\mathcal{I}}, \widetilde{Q}^2_{\mathcal{I}}) \in \Lambda_{p-1,q-1} \oplus \Lambda_{p-1,q-1}$, such that $Q^r \cdot Q^r = -2m^r n^r + \widetilde{Q}^r \widetilde{Q}^r$ (with no summation on r). The orthogonal projectors defined by $Q^r_L \equiv p^I_L Q^r_I$ and $Q^r_R \equiv p^I_R Q^r_I$ decompose according to

$$
\begin{aligned}
p^{\mathcal{I}}_{L,1} Q^r_{\mathcal{I}} =& \frac{1}{R\sqrt{2}} \left(m^r + a \cdot \widetilde{Q}^r + \frac{1}{2} a \cdot a\, n^r \right) - \frac{R}{\sqrt{2}} n^r, \\
p^{\mathcal{I}}_{L,\alpha} Q^r_{\mathcal{I}} =& \widetilde{p}^I_{L,\alpha} (\widetilde{Q}^r_I + n^r a_I), \\
p^{\mathcal{I}}_{R,1} Q^r_{\mathcal{I}} =& \frac{1}{R\sqrt{2}} \left(m^r + a \cdot \widetilde{Q}^r + \frac{1}{2} a \cdot a\, n^r \right) + \frac{R}{\sqrt{2}} n^r, \\
p^{\mathcal{I}}_{R,\hat\alpha} Q^r_{\mathcal{I}} =& \widetilde{p}^I_{R,\hat\alpha} (\widetilde{Q}^r_I + n^r a_I),
\end{aligned}
\tag{4.6}
$$

where $\widetilde{p}^I_{L,\alpha}, \widetilde{p}^I_{R,\hat\alpha}$ ($\alpha = 2 \dots q+16$, $\hat\alpha = 2 \dots q$) are orthogonal projectors in $G_{p-1,q-1}$ satisfying $\widetilde{Q}^r \cdot \widetilde{Q}^s = \widetilde{Q}^r_L \cdot \widetilde{Q}^s_L - \widetilde{Q}^r_R \widetilde{Q}^s_R$. In the following, we shall denote $|\widetilde{Q}^r_R| \equiv \sqrt{\widetilde{p}^I_{R,\hat\alpha} \widetilde{p}^{J\hat\alpha}_R \widetilde{Q}^r_I \widetilde{Q}^r_J}$.

To study the behavior of (4.1) in the limit $R \gg 1$, it is useful to perform a Poisson resummation on the momenta (m_1, m_2). For a lattice partition function $\Gamma^{(2)}_{\Lambda_{p,q}}$ with or without insertion, we must distinguish whether the indices lie along the direction 1 or along the directions α. The result can be obtain by applying the corresponding derivative polynomial with respect to $(y_{r,1}, y_{r,\alpha})$ to the following partition function

$$
\begin{aligned}
&\Gamma^{(2)}_{\Lambda_{p,q}} \left[e^{2\pi \mathrm{i} y_a \cdot \widetilde{Q}^a + \frac{\pi}{2} y_a \cdot \Omega_2^{-1} \cdot y^a} \right] = \\
&\qquad R^2 \sum_{(\mathbf{n},\mathbf{m}) \in \mathbb{Z}^4} e^{-\pi R^2 (\mathbf{n}\, \mathbf{m}) \binom{\Omega}{\mathbb{1}} \cdot \Omega_2^{-1} \cdot \left[(\mathbf{n}\, \mathbf{m}) \binom{\bar\Omega}{\mathbb{1}} \right]^\intercal} e^{\frac{2\pi R}{\sqrt{2}} y_1 \cdot \Omega_2^{-1} \cdot \left[(\mathbf{n}\, \mathbf{m}) \binom{\bar\Omega}{\mathbb{1}} \right]^\intercal} \\
&\qquad\qquad \times \Gamma^{(2)}_{\Lambda_{p-1,q-1}} \left[e^{2\pi \mathrm{i} \mathbf{m}^\intercal \cdot (a^I \widetilde{Q}_I + \frac{1}{2} a^I a_I\, \mathbf{n})} e^{2\pi \mathrm{i} y_\alpha \cdot \widetilde{Q}^\alpha + \frac{\pi}{2} y_\alpha \cdot \Omega_2^{-1} \cdot y^\alpha} \right],
\end{aligned}
\tag{4.7}
$$

where we denote the winding and momenta doublets $\mathbf{n} = (n_1, n_2)$, $\mathbf{m} = (m_1, m_2)$, and we use Einstein summation convention for indices $I = 1, \dots, p+q-2$ and $\alpha = 2, \dots, p$. In this representation, modular invariance is manifest, since a transformation $\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1}$ (A.2) can be compensated by a linear transformation $(\mathbf{n}, \mathbf{m}) \mapsto (\mathbf{n}, \mathbf{m}) \left(\begin{smallmatrix} D^\intercal & -B^\intercal \\ -C^\intercal & D^\intercal \end{smallmatrix} \right)$, $y_1 \mapsto y_1 \cdot (C\Omega + D)$, under which the third line of (4.7) transforms as a weight $\frac{p-q}{2}$ modular form.

We can therefore compute the integral using the orbit method [60, 61, 62, 63, 64], namely decompose the sum over $(\mathbf{n}, \mathbf{m})$ into various orbits under $Sp(4, \mathbb{Z})$, and for each orbit $\mathcal{O}$, retain the contribution of a particular element $\varsigma \in \mathcal{O}$ at the expense of extending the integration domain $\mathcal{F}_2 = Sp(4, \mathbb{Z}) \backslash \mathcal{H}_2$ to $\Gamma_\varsigma \backslash \mathcal{H}_2$, where Γ_ς is the stabilizer of ς in $Sp(4, \mathbb{Z})$. The integration domain is unfolded according to the formula

$$
\bigcup_{\gamma \in \Gamma_\varsigma \backslash Sp(4,\mathbb{Z})} \gamma \cdot \mathcal{F}_2 = \Gamma_\varsigma \backslash \mathcal{H}_2 \,, \tag{4.8}
$$

where one must take into account that $-\mathbb{1} \in Sp(4, \mathbb{Z})$ acts trivially on $\mathcal{H}_2$. The coset representative $\varsigma \in \mathcal{O}$, albeit arbitrary, is usually chosen so as to make the unfolded domain $\Gamma_\varsigma \backslash \mathcal{H}_2$ as simple as possible. In the present case, there are two types of orbits:

The trivial orbit $(\mathbf{n}, \mathbf{m}) = (0, 0, 0, 0)$ produces, up to a factor of R^2, the integrals (4.1) for the lattice $\Lambda_{p-1,q-1}$, provided none of the indices ab, cd lie along the direction 1,

$$
G^{(p,q),0}_{\alpha\beta,\gamma\delta} = R^2\, G^{(p-1,q-1)}_{\alpha\beta,\gamma\delta} \,, \tag{4.9}
$$

while it vanishes otherwise.

The rank-one orbits correspond to terms with $(\mathbf{n},\mathbf{m})\neq(0,0,0,0)$. Setting $(n_1,n_2,m_1,m_2)=k(c_3,c_4,d_3,d_4)$, with $\gcd(c_3,c_4,d_3,d_4)=1$ and $k\neq 0$, the quadruplet (c_3,c_4,d_3,d_4) can always be rotated by an element of $Sp(4,\mathbb{Z})$ into $(0,0,0,1)$, whose stabilizer inside $Sp(4,\mathbb{Z})$ is Γ_1^J (4.10)

$$\Gamma_1^J=\left\{\begin{pmatrix}a&0&b&\mu'\\ \lambda&1&\mu&\kappa\\ c&0&d&-\lambda'\\ 0&0&0&1\end{pmatrix},(\lambda,\mu)=(\lambda',\mu')\left(\begin{smallmatrix}a&b\\ c&d\end{smallmatrix}\right),\ \left(\begin{smallmatrix}a&b\\ c&d\end{smallmatrix}\right)\in SL(2,\mathbb{Z}),(\kappa,\lambda,\mu)\in\mathbb{Z}^3\right\}, \tag{4.10}$$

which is a central extension of the Jacobi group $SL(2,\mathbb{Z})\ltimes\mathbb{Z}^2$ in which the triple $(\kappa,\lambda,\mu)\in\mathbb{Z}^3$ parametrizes the Heisenberg group $H_{2,1}(\mathbb{Z})$.[8]

Thus, quadruplets (c_3,c_4,d_3,d_4) with $\gcd(c_3,c_4,d_3,d_4)=1$ are in one-to-one correspondence with elements of $\Gamma_1^J\backslash Sp(4,\mathbb{Z})$. For each $k\in\mathbb{Z}$, one can therefore unfold the integration domain $Sp(4,\mathbb{Z})\backslash\mathcal{H}_2$ to

$$\Gamma_1^J\backslash\mathcal{H}_2=\mathbb{R}^+_t\times(SL(2,\mathbb{Z})\backslash\mathcal{H}_1)_\rho\times\big((\mathbb{R}/\mathbb{Z})^3/\mathbb{Z}_2\big)_{u_1,u_2,\sigma_1}\,, \tag{4.11}$$

provided one keeps only the term $(c_3,c_4,d_3,d_4)=(0,0,0,1)$ in the sum, and where $\mathbb{Z}_2$ comes from the element $-\mathbb{1}\in SL(2,\mathbb{Z})$ leaving ρ invariant but acting as $(u_1,u_2)\to(-u_1,-u_2)$. In practice, we integrate u_1, u_2 over $\mathbb{R}/\mathbb{Z}$ and multiply the integral by a factor $1/2$. We parametrize the domain $\Gamma_1^J\backslash\mathcal{H}_2$ by $t=\frac{|\Omega_2|}{\rho_2}$, ρ, and $(u_1,u_2,\sigma_1)=(v_1-v_2\rho_1/\rho_2,v_2/\rho_2,\sigma_1)$.

The resulting contribution can be expressed in terms of the y variables (4.7). Changing y_{ra} variables as $(y'_{11},y'_{21},y'_{1\alpha},y'_{2\alpha})=(y_{11},y_{11}u_2-y_{21},y_{1\alpha},y_{1\alpha}u_2-y_{2\alpha})$, we obtain

$$\begin{aligned}G^{(p,q),1}_{ab,cd}=\frac{R^2}{2}\int_0^\infty\frac{\mathrm{d}t}{t^3}\int_{(\mathbb{R}/\mathbb{Z})^3}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}\sigma_1\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\mathcal{P}_{ab,cd}(\frac{\partial}{\partial y'})}{\Phi_{10}}\sum_{k\neq0}e^{-\frac{\pi R^2k^2}{t}}\Gamma^{(2)}_{\Lambda_{p-1,q-1}}\Big[e^{2\pi\mathrm{i}ka^I\widetilde{Q}_{2I}}\\ \times\exp\Big(2\pi\Big(\frac{R}{\sqrt2}\frac{k}{t}y'_{2\,1}+\mathrm{i}y'_{1\,\alpha}(Q_L^{1\alpha}+u_2Q_L^{2\alpha})-\mathrm{i}y'_{2\,\alpha}Q_L^{2\alpha}+\frac{1}{4\rho_2}y'_{1\,\alpha}y_1'^{\,\alpha}+\frac{1}{4t}y'_{2\,\alpha}y_2'^{\,\alpha}\Big)\Big)\Big]\,,\end{aligned} \tag{4.12}$$

where

$$\mathcal{P}_{ab,cd}(\frac{\partial}{\partial y})=\varepsilon_{rt}\varepsilon_{su}\frac{1}{(2\pi\mathrm{i})^4}\frac{\partial}{\partial y_r{}^{(a}}\frac{\partial}{\partial y_s{}^{b)}}\frac{\partial}{\partial y_t{}^{(c}}\frac{\partial}{\partial y_u{}^{d)}}\,. \tag{4.13}$$

The integral over $\Gamma_1^J\backslash\mathcal{H}_2$ can be computed by inserting the Fourier–Jacobi expansion

$$\frac{1}{\Phi_{10}}=\sum_{\substack{m\in\mathbb{Z}\\ m\geq-1}}\psi_m(\rho,v)\,q^m\,. \tag{4.14}$$

The integral over σ_1 picks up the Jacobi form $\psi_m(\rho,v)$ with $m=-\frac12\widetilde{Q}_2^{\,2}$.

For $\widetilde{Q}_2=0$, one has from (A.54), $\psi_0=c(0)\mathcal{P}/\Delta$ where here $\mathcal{P}$ denotes the (rescaled) Weierstrass function (A.55) and $c(0)=24$ is the zero-th Fourier coefficient in $1/\Delta=\sum_{m\geq-1}c(m)q^m$.

[8] They satisfy the group multiplication law $(\lambda,\mu,\kappa)\cdot(\lambda',\mu',\kappa')=(\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\mu'-\lambda'\mu)$.

The integral over σ_1 is trivial while the integral over u_1, u_2 is computed using (A.72),

$$\int_{-1/2}^{1/2} \mathrm{d}u_1 \int_{-1/2}^{1/2} \mathrm{d}u_2\, \psi_0(\rho, u_1 + \rho u_2) = \frac{c(0)\widehat{E}_2}{12\Delta}\,, \tag{4.15}$$

where $\widehat{E}_2(\rho) = E_2(\rho) - \frac{3}{\pi\rho_2}$ is the non-holomorphic completion of the weight 2 Eisenstein series. The contributions with $\widetilde{Q}_2 = 0$ therefore lead to the integral (after exchanging the order of sum and integral)

$$\begin{split} G^{(p,q),1}_{ab,cd} = R^2 \frac{c(0)}{24} \sum_{k\neq 0} \int_0^\infty \frac{\mathrm{d}t}{t} t^{\frac{q-5}{2}} e^{-\frac{\pi R^2 k^2}{t}} \mathcal{P}_{ab,cd}\big(\frac{\partial}{\partial y'}\big) e^{2\pi \mathrm{i}\left(\frac{R}{\mathrm{i}\sqrt{2}}\frac{k}{t} y'_{2\,1} + \frac{1}{4\mathrm{i}t} y'_{2\,\alpha} y'^{\,\alpha}_{2}\right)} \\ \times \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\widehat{E}_2}{\Delta(\rho)} \Gamma_{\Lambda_{p-1,q-1}} \left[e^{2\pi\mathrm{i}\left(y'_{1\,\alpha} Q_L^{1\alpha} + \frac{1}{4\mathrm{i}\rho_2} y'_{1\,\alpha} y'^{\,\alpha}_{1}\right)}\right]\,, \end{split} \tag{4.16}$$

leading to the constant terms in the Fourier expansion of $G^{(p,q)}_{ab,cd}$

$$\begin{split} G^{(p,q),1,0}_{\alpha\beta,\gamma\delta} &= -R^{q-5}\, \xi(q-6) \frac{c(0)}{16\pi} \delta_{\langle\alpha\beta,} G^{(p-1,q-1)}_{\gamma\delta\rangle}\,, \\ G^{(p,q),1,0}_{\alpha\beta,11} &= -R^{q-5}\, \xi(q-6)(7-q) \frac{c(0)}{48\pi} G^{(p-1,q-1)}_{\alpha\beta}\,, \end{split} \tag{4.17}$$

and $G^{(p,q),1,0}_{\alpha\beta,\gamma 1} = 0$. Note that they are the only components by symmetry of the indices ab, cd. Here $G^{(p,q)}_{ab}$ is the genus-one modular integral defined in (2.29) with $N = 1$ and $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s)$ is the completed Riemann zeta function.

The missing constant term: It is clear from the differential equation (3.20) that (4.17) does not give all the power-like terms: indeed, the coupling $F^{(p,q)}$ appearing on the r.h.s. of (3.20) behaves schematically in the same limit as [22, (4.37)]

$$F^{(p,q)} \sim R\, F^{(p-1,q-1)} + \xi(q-6)\, R^{q-6} + \mathcal{O}(e^{-R})\,. \tag{4.18}$$

The power-like terms (4.17) can be checked to satisfy the differential constraint with the source term $R^{q-5}F^{(p-1,q-1)}$ appearing in the square of $F^{(p,q)}$, but the accompanying source term $\xi(q-6)^2 R^{2q-12}$ requires that $G^{(p,q)}_{ab,cd}$ should also include a term proportional to R^{2q-12}. We shall now argue that these terms originate from the intersection of the separating and non-separating degenerations described by the figure-eight supergravity diagram depicted in Figure 1ii). In the region $|\Omega_2| \gg 1$, the fundamental domain asymptotes to the domain $\mathcal{P}_2/GL(2,\mathbb{Z}) \times [0,1]^3$, where Ω_2 parametrizes the first factor. In the case where all external indices are along the subgrassmaniann, the dominant contributions in this limit have $\widetilde{Q}_1 = \widetilde{Q}_2 = 0$ and vanishing winding number (n_1, n_2) along the circle. The sum over dual momenta (m_1, m_2) running in the two loops leads to

$$\frac{3}{16\pi^2} \delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} R^2 \int_{\frac{\mathcal{P}_2}{GL(2,\mathbb{Z})}\times[0,1]^3} \frac{\mathrm{d}^3\Omega_1 \mathrm{d}^3\Omega_2}{|\Omega_2|^{4-\frac{q-1}{2}}} \frac{\sum_{m_r\in\mathbb{Z}^2} e^{-\pi R^2 m_r [\Omega_2^{-1}]^{rs} m_s}}{\Phi_{10}} \tag{4.19}$$

Using (A.90), the integral over Ω_1 leads to a delta function supported at $v_2 = 0$ and its images under the action of $GL(2,\mathbb{Z})$ (modulo the center). After unfolding, the remaining integral then factorizes into two integrals over ρ_2 and σ_2. Assuming that this contribution is accurately computed by this integral by extending the integration domain of ρ_2 and σ_2 to $\mathbb{R}^+$, one obtains the correct power-like term

$$\begin{aligned} G^{(p,q),1,0'}_{\alpha\beta,\gamma\delta} &= -\frac{3}{64\pi^3} R^{2q-12}\,[\xi(q-6)c(0)]^2 \delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\ , \\ G^{(p,q),1,0'}_{\alpha\beta,11} &= -\frac{1}{32\pi^3} R^{2q-12}\,[\xi(q-6)\,c(0)]^2\,(7-q)\,\delta_{\alpha\beta}\ , \end{aligned} \tag{4.20}$$

where the second line — the other non-vanishing polarization — can be deduced in a similar fashion. While the power-like terms (4.20) are not captured by the unfolding trick in the degeneration $(p,q)\to(p-1,q-1)$, we shall be able to recover them below from the degeneration $(p,q)\to(p-2,q-2)$, see (5.26).

The fact that the unfolding method does not give the full result is seemingly due to the non-absolute convergence of the integral near the separating locus. In principle, the missing contributions can be determined by checking the differential equation (3.20). In Appendix E.4 we derive the contributions (4.20) rigorously in this fashion. The same analysis also implies that there exists additional exponentially suppressed corrections to the constant term due to instanton–anti-instanton contributions. For what concerns non-trivial Fourier coefficients, we shall argue in §5.1 (and specifically in Appendix E.1) that the unfolding method is in fact reliable.

Exponentially suppressed corrections: Contributions from non-zero vectors $\widetilde{Q}_2$ lead to exponentially suppressed contributions, which depend on the axions through a phase factor $e^{2\pi i k a^I \widetilde{Q}_{2I}}$. Each Jacobi form $\psi_m(\rho,v)$ in (4.14) can be decomposed as the sum of a finite and polar contributions, $\psi_m = \widehat{\psi}^P_m + \widehat{\psi}^F_m$ (see §A.5), where $\widehat{\psi}^F_m$ is an almost holomorphic Jacobi form, and $\widehat{\psi}^P_m$ is proportional to a completed non-holomorphic Appell–Lerch sum. For $m=-1$, the finite part vanishes and the polar part requires special treatment. In either case, the integral over σ_1 enforces $\widetilde{Q}_2^2=-2m$.

We first treat the finite contributions $\widehat{\psi}^F_m(\rho,v)$ with $m\geq 0$ according to whether $\widetilde{Q}_2^2=0$ or $\widetilde{Q}_2^2\neq 0$, and then consider the polar contributions:

1. In the case $\widetilde{Q}_2^2=0$, since $\widehat{\psi}^F_0=\frac{c(0)\widehat{E}_2}{12\Delta}$ and does not depend on v, the integral over u_1 receives only contributions from vectors $\widetilde{Q}_1$ such that $\widetilde{Q}_1\cdot\widetilde{Q}_2=0$. To express the remaining sum, we choose a second null vector $\widetilde{Q}'_2$ such that $(\widetilde{Q}_2,\widetilde{Q}'_2)=m_2$, where m_2, which we also denote by $\gcd(\widetilde{Q}_2)$, is the largest integer such that $\frac{1}{m_2}\widetilde{Q}_2\in\Lambda_{p-1,q-1}$. The vectors $\widetilde{Q}_1$ orthogonal to $\widetilde{Q}_2$ are then of the form $\widetilde{Q}_1=\widetilde{Q}_1^\perp+\frac{m_1}{m_2}\widetilde{Q}_2$ where $\widetilde{Q}_1^\perp$ is orthogonal to both $\widetilde{Q}_2$ and $\widetilde{Q}'_2$. We denote the resulting lattice by $\Lambda_{p-2,q-2}$. This parametrization is not unique, but the result of the integral will be independent of the choice of $\widetilde{Q}'_2$, in other words it is a function of the Levi subgroup of the stabilizer of $\widetilde{Q}_2$ inside $O(p-1,q-1)$. The sum over $\widetilde{Q}_1$ therefore becomes a sum over $\widetilde{Q}_1^\perp\in\Lambda_{p-2,q-2}$ and $m_1=m_2 s+r$, $s\in\mathbb{Z}\,,r\in\mathbb{Z}_{m_2}$. The sum over s can be used to unfold the integral over $u_2\in[-\frac12,\frac12]$ to the full $\mathbb{R}$ axis, as one can see from (4.12), while the dependence on r can be absorbed by a translation in u_2 and therefore leads to an overall factor m_2. The integral thus becomes, for a given null vector

$\widetilde{Q}_2$,

$$\begin{aligned}
&R^2\,\frac{m_2}{2}\sum_{k\neq 0} e^{2\pi i k\widetilde{Q}_{2I}a^I}\int_{\mathbb{R}^+}\frac{\mathrm{d}t}{t}t^{\frac{q-5}{2}}e^{-\frac{\pi R^2k^2}{t}-2\pi t|\widetilde{Q}_{2R}|^2}\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\,\rho_2^{\frac{q-1}{2}}\,\frac{c(0)\widehat{E}_2}{12\Delta}\\
&\quad\times\int_{\mathbb{R}}\mathrm{d}u_2\sum_{\widetilde{Q}_1^{\perp}\in\Lambda_{p-2,q-2}} q^{\frac12(\widetilde{Q}_1^{\perp}+u_2\widetilde{Q}_2)_L^2}\bar{q}^{\frac12(\widetilde{Q}_1^{\perp}+u_2\widetilde{Q}_2)_R^2} \qquad (4.21)\\
&\quad\times\mathcal{P}_{ab,cd}(\tfrac{\partial}{\partial y'})\;e^{2\pi i\left(\frac{R}{i\sqrt{2}}\frac{k}{t}y'_{21}+y'_{1\alpha}(\widetilde{Q}_1^{\perp}+u_2\widetilde{Q}_2)_L^{\alpha}-y'_{2\,\alpha}\widetilde{Q}_{2L}{}^{\alpha}+\frac{1}{4i\rho_2}y'_{1\alpha}y'_1{}^{\alpha}+\frac{1}{4it}y'_{2\alpha}y'_2{}^{\alpha}\right)}\Big|_{y'=0}\,.
\end{aligned}$$

The Gaussian integral over u_2 removes the dependence on the unipotent part of the stabilizer of $Q=k\widetilde{Q}_2$, leaving a modular integral of a genus-one partition function $G^{(p-1,q-1)\perp}_{\alpha\beta,0}$ for the lattice $\Lambda_{p-2,q-2}$ depending only on the sub-Grassmaniann $G_{p-2,q-2}\subset G_{p-1,q-1}$ parametrizing the Levi component of this stabilizer, given by

$$\begin{aligned}
G^{(p-1,q-1)\perp}_{F,\alpha\beta,0}(Q)&=\frac{\gcd(Q)}{12}\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\widehat{E}_2}{\Delta(\rho)}\rho_2^{\frac{q-2}{2}}\sum_{\widetilde{Q}\in\Lambda_{p-2,q-2}}q^{\frac12\widetilde{Q}_L^2}\bar{q}^{\frac12\widetilde{Q}_R^2}e^{2\pi\rho_2\frac{(\widetilde{Q}_R\cdot Q_R)^2}{Q_R^2}}\\
&\times\left[\left(\widetilde{Q}_{L\alpha}-\frac{\widetilde{Q}_R\cdot Q_R}{Q_R^2}Q_{L\alpha}\right)\left(\widetilde{Q}_{L\beta}-\frac{\widetilde{Q}_R\cdot Q_R}{Q_R^2}Q_{L\beta}\right)-\frac{1}{4\pi\rho_2}\left(\delta_{\alpha\beta}-\frac{Q_{L\alpha}Q_{L\beta}}{Q_R^2}\right)\right]\,, \qquad (4.22)
\end{aligned}$$

where we write $\widetilde{Q}_1$ as $\widetilde{Q}$ for simplicity. Note that the integrand only depends on $\widetilde{Q}$ through $\widetilde{Q}-Q\frac{\widetilde{Q}_R\cdot Q_R}{Q_R^2}$, and so is invariant under $\widetilde{Q}\to\widetilde{Q}+\epsilon Q$ for any $\epsilon\in\mathbb{R}$ such that the sum is defined on the quotient lattice $\Lambda_{p-1,q-1}\mathrm{mod}\frac{Q}{\gcd(Q)}$ with the constraint $Q\cdot\widetilde{Q}=0$, and does not depend on the specific choice of $\Lambda_{p-2,q-2}$.

We find that the Fourier coefficient with charge $Q\in\Lambda_{p-1,q-1}\setminus\{0\}$ for $Q^2=0$, is given by

$$3R^{\frac{q-1}{2}}\,\bar{G}^{(p-1,q-1)\perp}_{F,\langle\alpha\beta,0}(Q,\varphi)\sum_{l=0}^{1}\frac{\tilde{P}^{(l)}_{\gamma\delta\rangle}(Q)}{R^l}\frac{K_{\frac{q-5}{2}-l}\left(2\pi R\sqrt{2|Q_R|^2}\right)}{\sqrt{2|Q_R|^2}^{\frac{q-3}{2}-l}} \qquad (4.23)$$

when all the indices are chosen along the sub-Grassmaniann, where $\tilde{P}^{(l)}_{\gamma\delta}$ are defined in (H.1), and where we defined

$$\bar{G}^{(p-1,q-1)\perp}_{F,\alpha\beta,0}(Q,\varphi)=\sum_{\substack{d\geq 1\\ Q/d\in\Lambda_{p-1,q-1}}}d^{q-6}c(0)\,G^{(p-1,q-1)\perp}_{F,\alpha\beta,0}(\tfrac{Q}{d})\,. \qquad (4.24)$$

The full expression for all polarizations will be given together with the polar contributions in (4.44).

Let us point out that $G^{(p-1,q-1)\perp}_{F,\alpha\beta,0}(Q,\varphi)=\frac{\gcd(Q)}{12}G^{(p-2,q-2)}_{\alpha\beta}(\varphi_Q)$ for the function defined in (2.29) for the lattice $\Lambda_{p-2,q-2}$ orthogonal to Q, where φ_Q parametrizes the Levi subgroup $O(p-2,q-2)$ of the stabilizer of Q in $O(p-1,q-1)$.

2. In the case $\widetilde{Q}_2^2 < 0$, the finite part of the Fourier-Jacobi coefficient has the following expansion in theta series

$$\widehat{\psi}^F_m(\rho, v) = \frac{c(m)}{\Delta(\rho)} \sum_{\ell\in\mathbb{Z}_{2m}} \widehat{h}_{m,\ell}(\rho)\theta_{m,\ell}(\rho, v)\,, \tag{4.25}$$

where $\theta_{m,\ell}$ and $\widehat{h}_{m,\ell}$ are vector-valued modular forms of weight 1/2 and 3/2, respectively defined in (A.62) and (A.67). The integral over σ_1 enforces $\widetilde{Q}_2^2 = -2m$, while the integral over u_1 enforces $\widetilde{Q}_1 \cdot \widetilde{Q}_2 = -\ell$. The summation over $s \in \mathbb{Z}$ in (A.62) can be used to unfold the integral over $u_2 \in [-\frac{1}{2}, \frac{1}{2}]$ to the full real axis, after shifting each term in the lattice sum as $\widetilde{Q}_1 \to \widetilde{Q}_1 + s\widetilde{Q}_2$, since $\widetilde{Q}_1, \widetilde{Q}_2 \in \Lambda_{p-1,q-1}$. One thus obtain Fourier coefficients similar to previous case, using $\widetilde{Q}_2 \to Q/k$,

$$3R^{\frac{q-1}{2}}\, \bar{G}^{(p-1,q-1)}_{F,\langle\alpha\beta,-\frac{Q^2}{2}}(Q,\varphi) \sum_{l=0}^{1} \frac{\tilde{P}^{(l)}_{\gamma\delta\rangle}(Q)}{R^l} \frac{K_{\frac{q-5}{2}-l}\left(2\pi R\sqrt{2|Q_R|^2}\right)}{\sqrt{2|Q_R|^2}^{\frac{q-3}{2}-l}} \tag{4.26}$$

when all the indices are chosen along the sub-Grassmanian, where $\tilde{P}^{(l)}_{\gamma\delta}(Q)$ are defined in (H.1), and where we defined, for $Q^2 \neq 0$

$$\bar{G}^{(p-1,q-1)}_{F,\alpha\beta,-\frac{Q^2}{2}}(Q,\varphi) = \sum_{\substack{d\geq 1\\ Q/d\in\Lambda_{p-1,q-1}}} d^{q-6}\, c\big(-\tfrac{Q^2}{2d^2}\big) G^{(p-1,q-1)\perp}_{F,\alpha\beta,-\frac{Q^2}{2d^2}}\big(\tfrac{Q}{d}\big)\,, \tag{4.27}$$

$$G^{(p-1,q-1)\perp}_{F,\alpha\beta,m}(Q) = \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{1}{\Delta(\rho)} \sum_{\ell\in\mathbb{Z}_{2m}} \widehat{h}_{m,\ell}\Gamma^{m,\ell}_{\alpha\beta}(Q)\,. \tag{4.28}$$

Here $\Gamma^{m,\ell}_{ab}(Q)$ is the lattice partition function (with $\widetilde{Q} = \widetilde{Q}_1 - \frac{\ell}{2m}Q$)

$$\Gamma^{m,\ell}_{\alpha\beta}(Q) = \rho_2^{\frac{q-4}{2}} \sum_{\substack{\widetilde{Q}\in\Lambda_{p-1,q-1}-\frac{\ell}{2m}Q\\ \widetilde{Q}\cdot Q=0}} q^{\frac{1}{2}\widetilde{Q}^2} \phi^F_{\alpha\beta}(\sqrt{2\rho_2}\widetilde{Q}, Q) \tag{4.29}$$

with kernel

$$\phi^F_{\alpha\beta}(\sqrt{2\rho_2}\widetilde{Q}, Q) = e^{-2\pi\rho_2\left(|\widetilde{Q}_R|^2 - \frac{(\widetilde{Q}_R\cdot Q_R)^2}{|Q_R|^2}\right)} \times \left(\rho_2\Big(\widetilde{Q}_{L\alpha} - Q_{L\alpha}\frac{\widetilde{Q}_R\cdot Q_R}{|Q_R|^2}\Big)\Big(\widetilde{Q}_{L\beta} - Q_{L\beta}\frac{\widetilde{Q}_R\cdot Q_R}{|Q_R|^2}\Big) - \frac{1}{4\pi}\Big(\delta_{\alpha\beta} - \frac{Q_{L\alpha}\cdot Q_{L\beta}}{|Q_R|^2}\Big)\right)\,. \tag{4.30}$$

The latter satisfies Vignéras' equation

$$\big(\langle\partial_x, \partial_x\rangle - 2\pi x\partial_x\big)\, \phi^F_{\alpha\beta}(x, Q) = 2\pi(q-4)\phi^F_{\alpha\beta}(x, Q)\,, \tag{4.31}$$

where $\langle\cdot,\cdot\rangle$ is the inverse of the integer norm on the lattice $\Lambda_{p-1,q-1}$, which ensures [65] that (4.29) is a vector-valued modular form of weight $\frac{p-q+5}{2} = \frac{21}{2}$, consistently with the weight of 3/2 of $\widehat{h}_{m,\ell}(\rho)$ (note that the condition $\widetilde{Q}\cdot Q = 0$ in the sum of (4.29) implies that the lattice over which $\widetilde{Q}$ is summed is of dimension $p + q - 3$). The analogue expression for other polarizations will be given along with the polar contributions in (4.44).

3. Let us now consider the contributions arising from the polar part $\widehat{\psi}^P_m$ of the Fourier-Jacobi coefficient ψ_m with $m \geq 0$. According to (A.69), the latter can be written as an indefinite theta series

$$\widehat{\psi}^P_m(\rho, v) = \frac{c(m)}{\Delta(\rho)} \sum_{s,\ell \in \mathbb{Z}} q^{ms^2+s\ell} y^{2ms+\ell} \widehat{b}(s, \ell, m, \rho_2) \tag{4.32}$$

where for $m \geq 1$,

$$\widehat{b}(s, \ell, m, \rho_2) = \frac{1}{2}\ell \left[\mathrm{sgn}(s + u_2) + \mathrm{erf}\left(\ell \sqrt{\frac{\pi \rho_2}{m}} \right) \right] + \frac{\sqrt{m}}{2\pi\sqrt{\rho_2}} e^{-\pi \rho_2 \ell^2/m} - \frac{1}{4\pi\rho_2} \delta(s + u_2) \,, \tag{4.33}$$

whereas

$$\widehat{b}(s, \ell, 0, \rho_2) = \frac{1}{2}\ell \; [\mathrm{sgn}(s + u_2) + \mathrm{sgn}(\ell)] - \frac{1}{4\pi\rho_2} \delta(s + u_2) + \delta_{s,0}\delta_{\ell,0} \frac{1}{4\pi\rho_2} \,. \tag{4.34}$$

As in the previous case, one can shift the charges to $\widetilde{Q}_1 \to \widetilde{Q}_1 + s\widetilde{Q}_2$ since $\widetilde{Q}_1, \widetilde{Q}_2 \in \Lambda_{p-1,q-1}$, and then use the sum over s to unfold the $u_2 \in [-\frac{1}{2}, \frac{1}{2}]$ to $\mathbb{R}$. Then, integrating over $u_1 \in [-\frac{1}{2}, \frac{1}{2}]$ imposes $\widetilde{Q}_1 \cdot \widetilde{Q}_2 = -\ell$. One then carries out the change of variable $u_2 = \frac{u}{\sqrt{2\rho_2 |Q_R|^2}}$. One obtains the Fourier coefficients, using $\widetilde{Q}_2 \to Q/k$,

$$3R^{\frac{q-1}{2}} \bar{G}^{(p-1,q-1)}_{P,\langle\alpha\beta,}(Q) \sum_{l=0}^{1} \frac{\tilde{P}^{(l)}_{\gamma\delta\rangle}(Q)}{R^l} \frac{K_{\frac{q-5}{2}-l}\left(2\pi\, R\sqrt{2|Q_R|^2}\right)}{\sqrt{2|Q_R|^2}^{\frac{q-3}{2}-l}} \tag{4.35}$$

when all indices are chosen along the sub-Grassmanian, and where we define for $Q^2 < 0$

$$\bar{G}^{(p-1,q-1)}_{P,\alpha\beta}(Q, \varphi) = \sum_{\substack{d\geq 1 \\ Q/d \in \Lambda_{p-1,q-1}}} d^{q-6} c(-\tfrac{Q^2}{2d^2})\, G^{(p-1,q-1)}_{P,\alpha\beta}(\tfrac{Q}{d}) \,, \tag{4.36}$$

$$G^{(p-1,q-1)}_{P,\alpha\beta}(Q) = \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\rho_2^{\frac{q-5}{2}}}{\Delta(\rho)} \sum_{\widetilde{Q} \in \Lambda_{p-1,q-1}} q^{\frac{1}{2}\widetilde{Q}^2} \phi_{P,\alpha\beta}(\sqrt{2\rho_2}\widetilde{Q}, Q) \;, \tag{4.37}$$

with the kernel

$$\begin{aligned}\phi_{P,\alpha\beta}(x, Q) = &-\frac{1}{4\sqrt{2}} \int_{\mathbb{R}} \mathrm{d}u\, (x \cdot Q) \left[\mathrm{sgn}(u) + \mathrm{erf}\Big(-\sqrt{\tfrac{\pi}{-Q^2}} x \cdot Q\Big) - \frac{\sqrt{-Q^2}}{\pi x \cdot Q} e^{\frac{\pi (x\cdot Q)^2}{Q^2}} \right] \\ &\times e^{-\pi |x_R|^2 - \pi u^2 - 2\pi u \frac{x_R \cdot Q_R}{|Q_R|}} \left(\Big(x_{L\alpha} + u\frac{Q_{L\alpha}}{|Q_R|}\Big)\Big(x_{L\beta} + u\frac{Q_{L\beta}}{|Q_R|}\Big) - \frac{1}{2\pi}\delta_{\alpha\beta} \right) \\ &- \frac{\sqrt{2|Q_R|^2}}{8\pi} e^{-\pi |x_R|^2} \left(x_{L\alpha} x_{L\beta} - \frac{1}{2\pi} \delta_{\alpha\beta} \right) \;. \end{aligned} \tag{4.38}$$

Using integration by part over u one computes that $\phi_{P,\alpha\beta}(x, Q)$ satisfies the Vignéras equation

$$\big(\langle \partial_x, \partial_x \rangle - 2\pi x \partial_x\big)\, \phi_{P,\alpha\beta}(x, Q) = 2\pi(q - 5)\phi_{P,\alpha\beta}(x, Q) \;, \tag{4.39}$$

therefore the lattice sum in (4.37) is a modular form of weight $\frac{p-q}{2}+4$, and the integral is well defined. For $Q^2=0$, one has instead

$$G^{(p-1,q-1)}_{P,\alpha\beta}(Q)=\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\rho_2^{\frac{q-5}{2}}}{\Delta(\rho)}\Bigg(\sum_{\widetilde{Q}\in\Lambda_{p-1,q-1}}q^{\frac12\widetilde{Q}^2}\phi_{P,\alpha\beta}(\sqrt{2\rho_2}\widetilde{Q},Q)+\rho_2^{-\frac12}\sum_{\widetilde{Q}\in\Lambda_{p-2,q-2}}q^{\frac12\widetilde{Q}^2}\phi'^{\perp}_{P,\alpha\beta}(\sqrt{2\rho_2}\widetilde{Q},Q)\Bigg)\ , \quad (4.40)$$

with

$$\begin{aligned}\phi_{P,\,\alpha\beta}(x,Q)\ =&\ -\frac{1}{4\sqrt2}\int_{\mathbb{R}}\mathrm{d}u\,\big((x\cdot Q)\mathrm{sgn}(u)-|x\cdot Q|\big)e^{-\pi|x_R|^2-\pi u^2-2\pi u\frac{x_R\cdot Q_R}{|Q_R|}}\\&\times\bigg(\Big(x_{L\alpha}+u\frac{Q_{L\alpha}}{|Q_R|}\Big)\Big(x_{L\beta}+u\frac{Q_{L\beta}}{|Q_R|}\Big)-\frac{1}{2\pi}\delta_{\alpha\beta}\bigg)\\&-\frac{\sqrt{2|Q_R|^2}}{8\pi}e^{-\pi|x_R|^2}\Big(x_{L\alpha}x_{L\beta}-\frac{1}{2\pi}\delta_{\alpha\beta}\Big)\ ,\end{aligned} \quad (4.41)$$

$$\phi'^{\perp}_{P,\alpha\beta}(x,Q)\ =\ \frac{e^{-\pi(x_R^2-\frac{(x_R\cdot Q_R)^2}{Q_R^2})}}{8\pi}\bigg(\Big(x_{L\alpha}-\tfrac{x_R\cdot Q_R}{Q_R^2}Q_{L\alpha}\Big)\Big(x_{L\beta}-\tfrac{x_R\cdot Q_R}{Q_R^2}Q_{L\beta}\Big)-\tfrac{1}{2\pi}\Big(\delta_{\alpha\beta}-\tfrac{Q_{L\alpha}Q_{L\beta}}{Q_R^2}\Big)\bigg)\ .$$

The integrand in (4.40) must be modular by construction, but its modularity does not follow directly from Vignéras theorem. In this case $\phi_{P,\,\alpha\beta}(x,Q)$ satisfies Vignéras equation (4.39), but it is a distribution and its second derivative is not square integrable. The function $\phi'^{\perp}_{P,\alpha\beta}(x,Q)$ satisfies Vignéras equation (4.31), but this is not the correct eigenvalue to give the correct modular weight. As the failure of $\phi_{P,\,\alpha\beta}(x,Q)$ to define a modular form comes from its singularity at $(Q\cdot x)=0$, it is somehow natural that its modular anomaly can be compensated by a partition function on the lattice orthogonal to Q.

4. Finally, the case $m=-1$ requires special treatment. The finite part of ψ_{-1} automatically vanishes, but the polar part is proportional to a modified Appell-Lerch sum, as explained in Appendix A.5,

$$\psi_{-1}=-\frac{1}{\Delta}\sum_{s,\ell\in\mathbb{Z}}\bigg[\ell\,\frac{\mathrm{sign}(\ell-2s)+\mathrm{sign}(u_2+s)}{2}-\frac{1}{4\pi\rho_2}\delta(u_2+s)\bigg]\,q^{-s^2+\ell s}\,y^{\ell-2s}\ , \quad (4.42)$$

which differs from the naive Appell-Lerch sum (which diverges when the index is negative) by a replacement $\mathrm{sign}\ell\to\mathrm{sign}(\ell-2s)$. In this case we still get (4.40) with

$$\begin{aligned}\phi_{P,\,\alpha\beta}(x,Q)\ =&\ -\frac{1}{4\sqrt2}\int_{\mathbb{R}}\mathrm{d}u\,(x\cdot Q)\bigg[\mathrm{sgn}(u)-\mathrm{sign}\Big(\tfrac{x\cdot Q}{\sqrt{2\rho_2}}+2\lfloor\tfrac{u}{\sqrt{2\rho_2Q_R^2}}\rfloor\Big)\bigg]\\&\times e^{-\pi|x_R|^2-\pi u^2-2\pi u\frac{x_R\cdot Q_R}{|Q_R|}}\bigg(\Big(x_{L\alpha}+u\frac{Q_{L\alpha}}{|Q_R|}\Big)\Big(x_{L\beta}+u\frac{Q_{L\beta}}{|Q_R|}\Big)-\frac{1}{2\pi}\delta_{\alpha\beta}\bigg)\\&-\frac{\sqrt{2|Q_R|^2}}{8\pi}e^{-\pi|x_R|^2}\Big(x_{L\alpha}x_{L\beta}-\frac{1}{2\pi}\delta_{\alpha\beta}\Big)\ ,\end{aligned} \quad (4.43)$$

$$\phi'^{\perp}_{P,\alpha\beta}(x,Q)\ =\ \frac{e^{-\pi(x_R^2-\frac{(x_R\cdot Q_R)^2}{Q_R^2})}}{8\pi}\bigg(\Big(x_{L\alpha}-\tfrac{x_R\cdot Q_R}{Q_R^2}Q_{L\alpha}\Big)\Big(x_{L\beta}-\tfrac{x_R\cdot Q_R}{Q_R^2}Q_{L\beta}\Big)-\tfrac{1}{2\pi}\Big(\delta_{\alpha\beta}-\tfrac{Q_{L\alpha}Q_{L\beta}}{Q_R^2}\Big)\bigg)\ .$$

Although the modularity of (4.43) no longer follows from Vignéras' theorem, it must hold by construction.

Combining the finite and polar contributions, we finally obtain the full expressions for the exponentially suppressed corrections,

$$
\begin{aligned}
G^{(p,q),1,Q}_{\alpha\beta,\gamma\delta} &= 3R^{\frac{q-1}{2}}\,\bar G^{(p-1,q-1)}_{\langle\alpha\beta,}(Q,\varphi)\sum_{l=0}^{1}\frac{\tilde P^{(l)}_{\gamma\delta\rangle}(Q)}{R^l}\frac{K_{\frac{q-5}{2}-l}\left(2\pi R\sqrt{2Q_R^2}\right)}{(2Q_R^2)^{\frac{q-3-2l}{4}}}\\
G^{(p,q),1,Q}_{\alpha\beta,\gamma 1} &= \frac{3}{2}R^{\frac{q-1}{2}}\,\bar G^{(p-1,q-1)}_{\langle\alpha\beta,}(Q,\varphi)\,\frac{Q_{L\gamma\rangle}}{\mathrm{i}\sqrt{2}}\frac{K_{\frac{q-7}{2}}\left(2\pi R\sqrt{2Q_R^2}\right)}{(2Q_R^2)^{\frac{q-5}{4}}}\\
G^{(p,q),1,Q}_{\alpha\beta,11} &= -R^{\frac{q-1}{2}}\,\bar G^{(p-1,q-1)}_{\alpha\beta}(Q,\varphi)\,\frac{K_{\frac{q-9}{2}}\left(2\pi R\sqrt{2Q_R^2}\right)}{(2Q_R^2)^{\frac{q-7}{4}}}\,,
\end{aligned}
\tag{4.44}
$$

where the polynomials $\tilde P^{(l)}_{\gamma\delta}(Q)$ are given in (H.1), and the coefficient $\bar G^{(p-1,q-1)}_{\alpha\beta}$ is defined by

$$
\bar G^{(p-1,q-1)}_{\alpha\beta}(Q,\varphi) = \sum_{\substack{d\geq 1\\ Q/d\in\Lambda_{p-1,q-1}}} d^{q-6}c(-\tfrac{Q^2}{2d^2})\left(G^{(p-1,q-1)\perp}_{F,\alpha\beta,-\frac{Q^2}{2d^2}}(\tfrac{Q}{d})+G^{(p-1,q-1)}_{P,\alpha\beta}(\tfrac{Q}{d})\right), \tag{4.45}
$$

where $G^{(p-1,q-1)}_F$ and $G^{(p-1,q-1)}_P$ are defined in (4.24), (4.28),(4.36) for $Q^2<0$, in (4.40) for $Q^2=0$ and in (4.43) for $Q^2>0$.

4.2 Extension to $\mathbb{Z}_N$ CHL orbifolds

The degeneration limit (4.2) of the modular integral (2.30) for $\mathbb{Z}_N$ CHL models with $N=2,3,5,7$ can be treated similarly by adapting the orbit method to the case where the integrand is invariant under the congruence subgroup $\Gamma_{2,0}(N)=\{\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in Sp(4,\mathbb{Z}),\,C=0 \bmod N\}$. In (2.30), Φ_{k-2} is the meromorphic Siegel modular form of $\Gamma_{2,0}(N)$ of weight $k-2$ defined in §A.4, and $\Gamma^{(2)}_{\Lambda_{p,q}}$ is the genus-two partition function for a lattice

$$
\Lambda_{p,q}=\Lambda_{p-1,q-1}\oplus I\!I_{1,1}[N]\,, \tag{4.46}
$$

where $\Lambda_{p-1,q-1}$ is a level N even lattice of signature $(p-1,q-1)$. The lattice $I\!I_{1,1}[N]$ is obtained from the usual unimodular lattice $I\!I_{1,1}$ by restricting the winding and momentum to $(n_1,n_2,m_1,m_2)\in N\mathbb{Z}\oplus N\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. After Poisson resummation on m_1,m_2, Eq. (4.7) continues to hold, except for the fact that n_1,n_2 are restricted to run over $N\mathbb{Z}$. The sum over (n_1,n_2,m_1,m_2) can then be decomposed into orbits of $\Gamma_{2,0}(N)$:

Trivial orbit The term $(n_1,n_2,m_1,m_2)=(0,0,0,0)$ produces the same modular integral, up to a factor of R^2,

$$
G^{(p,q),0}_{\alpha\beta,\gamma\delta}=R^2\,G^{(p-1,q-1)}_{\alpha\beta,\gamma\delta}\,, \tag{4.47}
$$

where $G^{(p-1,q-1)}_{\alpha\beta,\gamma\delta}$, is the integral (2.30) for the lattice $\Lambda_{p-1,q-1}$ defined by (4.46).

Rank-one orbits Terms with $(n_1,n_2,m_1,m_2)=k(c_3,c_4,d_3,d_4)$ with $k\neq 0$ and $\gcd(c_3,c_4,d_3,d_4)=1$ fall into two different classes of orbits under $\Gamma_{2,0}(N)$:

1. Quadruplets $k(c_3,c_4,d_3,d_4)$ such that $(c_3,c_4)=(0,0) \bmod N$ and $k\in\mathbb{Z}$ can be rotated by an element of $\Gamma_{2,0}(N)$ into $(0,0,0,1)$, whose stabilizer in $\Gamma_{2,0}(N)$ is $\Gamma_0(N)\ltimes H_{2,1}(\mathbb{Z})\subset\Gamma_1^J$. For these elements, one can unfold the integration domain $\Gamma_{2,0}(N)\backslash\mathcal{H}_2$ into the domain
$$(\Gamma_0(N)\ltimes H_{2,1}(\mathbb{Z}))\backslash\mathcal{H}_2=\mathbb{R}_t^+\times(\Gamma_0(N)\backslash\mathcal{H}_1)_\rho\times\left((\mathbb{R}/\mathbb{Z})^3/\mathbb{Z}_2\right)_{u_1,u_2,\sigma_1} \tag{4.48}$$
where the $\mathbb{Z}_2$ comes from $-\mathbb{1}\in\Gamma_0(N)$ leaving ρ invariant but acting as $(u_1,\,u_2)\to(-u_1,\,u_2)$ on the other moduli.

2. Doublets $k(c_3,c_4,d_3,d_4)$ such that $(c_3,c_4)\neq(0,0)\bmod N$ have $k=0\bmod N$ since $(n_1,n_2)=0\bmod N$. They can be rotated by an element of $\Gamma_{2,0}(N)$ into $(0,1,0,0)$, whose stabilizer in $\Gamma_{2,0}(N)$ is $S_\rho S_\sigma\,(\Gamma^0(N)\ltimes H^{(2)}_{2,1,N}(\mathbb{Z}))\,(S_\rho S_\sigma)^{-1}$, where
$$H^{(2)}_{2,1,N}(\mathbb{Z})=\{(\kappa,\lambda,\mu)\in H_{2,1}(\mathbb{Z}),\,\kappa=\mu=0\bmod N\}\,, \tag{4.49}$$
and the inversion on σ is $S_\sigma:(\rho,\sigma,v)\to(\rho-v^2/\sigma,-1/\sigma,-v/\sigma)$. One can unfold the integration domain $\Gamma_{2,0}(N)\backslash\mathcal{H}_2$ into $S_\rho S_\sigma\,(\Gamma^0(N)\ltimes H^{(2)}_{2,1,N}(\mathbb{Z}))\,(S_\rho S_\sigma)^{-1}\backslash\mathcal{H}_2$, and change variable
$$\Omega\to(S_\rho S_\sigma)\cdot\Omega=-\Omega^{-1}\,, \tag{4.50}$$
so as to reach $(\Gamma^0(N)\ltimes H^{(2)}_{2,1,N}(\mathbb{Z}))\backslash\mathcal{H}_2=\frac{1}{2}\mathbb{R}_t^+\times(\Gamma^0(N)\backslash\mathcal{H}_1)_\rho\times(\mathbb{R}/\mathbb{Z})_{u_2}\times(\mathbb{R}/N\mathbb{Z})^2_{u_1,\sigma_1}$. Under this change of variable, the level-N weight-$(k-2)$ Siegel modular form transforms as
$$\Phi_{k-2}(-\Omega^{-1})=(\mathrm{i}\sqrt{N})^{-2(k-2)}|\Omega|^{k-2}\Phi_{k-2}(\Omega/N)\,, \tag{4.51}$$
while the genus-two partition function for the sublattice $\Lambda_{p-1,q-1}$ transforms as
$$\Gamma^{(2)}_{\tilde\Lambda_{p-1,q-1}}[P_{\alpha\beta,\gamma\delta}](-\Omega^{-1})=\upsilon^2N^{-k-2}(-\mathrm{i})^{p-q}|\Omega|^{k-2}\Gamma^{(2)}_{\tilde\Lambda^*_{p-1,q-1}}[P_{\alpha\beta,\gamma\delta}](\Omega)\,, \tag{4.52}$$
where we denoted $\upsilon^2N^{-k-2}=\left|\Lambda^*_{p-1,q-1}/\Lambda_{p-1,q-1}\right|^{-1}$ the volume factor from Poisson ressummation (Note that $\upsilon^2=N^{2-2\delta_{q,8}}$ for $q\leq 8$ in the cases of interest).

For the function $G^{(p,q),1}_{ab,cd}$, changing y variables as before $(y'_{11},y'_{21},y'_{1\alpha},y'_{2\alpha})=(y_{11},y_{11}u_2-y_{21},y_{1\alpha},y_{1\alpha}u_2-y_{2\alpha})$, the sum of the two classes of orbits then reads
$$\begin{aligned}G^{(p,q),1}_{ab,cd}=&\frac{R^2}{2}\int_{\mathbb{R}^+}\frac{\mathrm{d}t}{t^3}\int_{(\mathbb{R}/\mathbb{Z})^3}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}\sigma_1\int_{\Gamma_0(N)\backslash\mathcal{H}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\mathcal{P}_{ab,cd}(\frac{\partial}{\partial y'})}{\Phi_{k-2}(\Omega)}\\&\qquad\times\sum_{k\neq0}e^{-\frac{\pi R^2k^2}{t}}\Gamma^{(2)}_{\tilde\Lambda_{p-1,q-1}}\left[e^{2\pi\mathrm{i}ka^I\tilde Q_{2I}}\mathcal{Y}(y')\right]\\&+\frac{R^2}{2}\int_{\mathbb{R}^+}\frac{\mathrm{d}t}{t^3}\int_{(\mathbb{R}/N\mathbb{Z})^2}\mathrm{d}u_1\mathrm{d}\sigma_1\int_{\mathbb{R}/\mathbb{Z}}\mathrm{d}u_2\int_{\Gamma^0(N)\backslash\mathcal{H}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\mathcal{P}_{ab,cd}(\frac{\partial}{\partial y'})}{\Phi_{k-2}(\Omega/N)}\\&\qquad\times\frac{\upsilon^2}{N^4}\sum_{\substack{k\neq0\\k=0\bmod N}}e^{-\frac{\pi R^2k^2}{t}}\Gamma^{(2)}_{\tilde\Lambda^*_{p-1,q-1}}\left[e^{2\pi\mathrm{i}ka^I\tilde Q_{2I}}\mathcal{Y}(y')\right]\end{aligned} \tag{4.53}$$
where
$$\mathcal{Y}(y')=e^{2\pi\mathrm{i}\left(\frac{R}{\mathrm{i}\sqrt{2}}\frac{k}{t}y'_{2\,1}+y'_{1\,\alpha}(Q_L^{1\alpha}+u_2Q_L^{2\alpha})-y'_{2\,\alpha}Q_L^{2\alpha}+\frac{1}{4\mathrm{i}\rho_2}y'_{1\,\alpha}y'^{\,\alpha}_1+\frac{1}{4\mathrm{i}t}y'_{2\,\alpha}y'^{\,\alpha}_2\right)}\,. \tag{4.54}$$

As before, we substitute $1/\Phi_{k-2}$ by its Fourier-Jacobi expansion $1/\Phi_{k-2} = \sum_{m\geq -1} \psi_{k-2,m} e^{2\pi i m\sigma}$, so that the integral over σ_1 enforces $\widetilde{Q}_2^2 = -2m$. For $\widetilde{Q}_2^2 = 0$ case, the integral over u_1, u_2 in the first line follows from (A.73),

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}u_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}u_2\, \psi_{k-2,0}(\rho, u_1 + \rho u_2) = \frac{c_k(0)}{12(N-1)} \frac{N^2 \widehat{E}_2(N\rho) - \widehat{E}_2(\rho)}{\Delta_k(\rho)}\,, \tag{4.55}$$

where $N^2 \widehat{E}_2(N\rho) - \widehat{E}_2(\rho)$ is a level-N weight 2 holomorphic modular form. The contribution from the second line in (4.53) is calculated using the transformation properties of the genus-one cusp form and partition function[9]. The transformation $\rho \to -1/\rho$ changes the integration domain from $\Gamma^0(N)\backslash\mathcal{H}_1$ to $\Gamma_0(N)\backslash\mathcal{H}_1$, and one thus obtains, denoting $Q_I = k\widetilde{Q}_{2I}$

$$\begin{aligned} G^{(p,q),1,Q^2=0}_{ab,cd} &= R^2 \int_0^\infty \frac{\mathrm{d}t}{t} t^{\frac{q-5}{2}} \sum_{\substack{\widetilde{Q}_2 \in \Lambda_{p-1,q-1}\\ \widetilde{Q}_2^2=0}} \sum_{k\neq 0} e^{-\frac{\pi R^2 k^2}{t} - 2\pi t Q_R^2/k^2} \frac{c_k(0)}{24(N-1)} \\ \times &\int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{(N^2 - \upsilon N^{q-6})\widehat{E}_2(N\rho) + (\upsilon N^{q-6} - 1)\widehat{E}_2(\rho)}{\Delta_k(\rho)} \Gamma_{\tilde{\Lambda}_{p-1,q-1}} \Big[e^{2\pi i a^I Q_I} \mathcal{P}_{ab,cd}(\tfrac{\partial}{\partial y'}) \mathcal{Y}(y') \Big] \end{aligned} \tag{4.56}$$

The zero mode contribution, $Q = 0$, may be expressed in terms of the genus-one modular integrals

$$G^{(p,q)}_{ab} = \mathrm{R.N.} \int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\hat{E}_2\, \Gamma_{\Lambda_{p,q}}[P_{ab}]}{\Delta_k}\,, \tag{4.57}$$

$$^{\varsigma}G^{(p,q)}_{ab} = \mathrm{R.N.} \int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{N\widehat{E}_2(N\rho)}{\Delta_k(\rho)} \Gamma_{\Lambda_{p,q}}[P_{ab}]\,. \tag{4.58}$$

When $\Lambda_{p,q}$ is N-modular, such that $\Lambda^*_{p,q} = \varsigma \cdot \Lambda_{p,q}/\sqrt{N}$ for $\varsigma \in O(p,q,\mathbb{R})$, then $^{\varsigma}G^{(p,q)}_{ab} = G^{(p,q)}_{ab}(\varsigma \cdot \varphi)$. The zero mode $Q = 0$ thus leads to power-like terms

$$\begin{aligned} G^{(p,q),1,0}_{\alpha\beta,\gamma\delta} &= -R^{q-5}\, \xi(q-6) \frac{c_k(0)}{16\pi} \Big[\frac{\upsilon N^{q-6} - 1}{N-1} \delta_{\langle\alpha\beta,} G^{(p-1,q-1)}_{\gamma\delta\rangle} + \frac{N - \upsilon N^{q-7}}{N-1} \delta_{\langle\alpha\beta,}\, {}^{\varsigma}G^{(p-1,q-1)}_{\gamma\delta\rangle} \Big]\,, \\ G^{(p,q),1,0}_{\alpha\beta,11} &= -R^{q-5}\, \xi(q-6)(7-q) \frac{c_k(0)}{48\pi} \Big[\frac{\upsilon N^{q-6} - 1}{N-1} G^{(p-1,q-1)}_{\alpha\beta} + \frac{N - \upsilon N^{q-7}}{N-1}\, {}^{\varsigma}G^{(p-1,q-1)}_{\alpha\beta} \Big]\,. \end{aligned} \tag{4.59}$$

As in the maximal rank case (4.20), the unfolding trick fails to capture another powerlike term proportional to R^{2q-12}, which is required by the non-homogeneous differential equation (3.20). This term can be seen to arise in the maximal non-separating degeneration, and can be computed as in (4.19), leading to

$$\begin{aligned} G^{(p,q),1,0'}_{\alpha\beta,\gamma\delta} &= -\frac{3}{64\pi^3} R^{2q-12} \left[c_k(0)(1 + \upsilon N^{q-7})\xi(q-6) \right]^2 \delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\,, \\ G^{(p,q),1,0'}_{\alpha\beta,11} &= -\frac{1}{32\pi^3} R^{2q-12} \left[c_k(0)(1 + \upsilon N^{q-7})\xi(q-6) \right]^2 (7-q)\delta_{\alpha\beta}\,. \end{aligned} \tag{4.60}$$

[9] *i.e.* $\Delta_k(-1/N\rho) = N^{\frac{k}{2}}(-i\rho)^k \Delta_k(\rho)$, and $\Gamma_{\Lambda^*_{p-1,q-1}}[P_{ab}](-1/\rho) = \upsilon^{-1} N^{\frac{k}{2}+1} (-i)^k \rho^{k-2} \Gamma_{\Lambda_{p-1,q-1}}[P_{ab}](\rho)$ where $\upsilon = N^{\frac{k}{2}+1} \left| \Lambda^*_{p-1,q-1}/\Lambda_{p-1,q-1} \right|^{-1/2}$

These results can also be obtained by taking the limit $S_2 \to \infty$ from the result (5.60) obtained in the degeneration limit $(p,q) \to (p-2,q-2)$.

The contributions from vectors $Q \neq 0$ lead to exponentially suppressed contributions of the same form as the Fourier modes of null vectors (4.23), non-null vectors (4.26), and the polar contribution (4.36) respectively, with different coefficients:

1. For null Fourier vectors $Q^2 = 0$, the moduli-dependent coefficient coming from the finite part of $1/\Phi_{k-2}(\Omega)$ reads

$$\bar{G}^{(p-1,q-1)}_{F,\alpha\beta,0}(Q,\varphi) = \sum_{\substack{d>0\\ Q/d\in\Lambda_{p-1,q-1}}} d^{q-6} c_k(0) \frac{N\,{}^{\varsigma}G^{(p-1,q-1)\perp}_{F,\alpha\beta,0}\left(\frac{Q}{d}\right) - G^{(p-1,q-1)\perp}_{F,\alpha\beta,0}\left(\frac{Q}{d}\right)}{N-1}$$
$$+\upsilon \sum_{\substack{d>0\\ Q/d\in N\Lambda^*_{p-1,q-1}}} (Nd)^{q-6} c_k(0) \frac{N G^{(p-1,q-1)\perp}_{F,\alpha\beta,0}\left(\frac{Q}{Nd}\right) - {}^{\varsigma}G^{(p-1,q-1)\perp}_{F,\alpha\beta,0}\left(\frac{Q}{Nd}\right)}{N-1}, \quad (4.61)$$

where $G^{(p,q)}_{F,ab,0}(\varphi)$ is defined as in (4.22) with $\hat{E}_2/\Delta$ replaced by $\hat{E}_2/\Delta_k$, and ${}^{\varsigma}G^{(p,q)}_{F,ab,0}(\varphi)$ is defined as in (4.22) with $\hat{E}_2/\Delta$ replaced by $N\hat{E}_2(N\rho)/\Delta_k(\rho)$.

2. For non-null Fourier vectors, $Q^2 \neq 0$, the moduli-dependent coefficient coming from the finite part of $1/\Phi_{k-2}(\Omega)$ is given by

$$\begin{aligned}\bar{G}^{(p-1,q-1)}_{F,\alpha\beta,-\frac{Q^2}{2}}(Q,\varphi) = &\sum_{\substack{d>0\\ Q/d\in\Lambda_{p-1,q-1}}} d^{q-6}\, c_k\left(-\tfrac{Q^2}{2d^2}\right) G^{(p-1,q-1)\perp}_{F,\alpha\beta,-\frac{Q^2}{2d^2}}\left(\tfrac{Q}{d}\right)\\ &+\upsilon \sum_{\substack{d>0\\ Q/d\in N\Lambda^*_{p-1,q-1}}} (Nd)^{q-6}\, c_k\left(-\tfrac{Q^2}{2Nd^2}\right) {}^{\varsigma}G^{(p-1,q-1)\perp}_{F,\alpha\beta,-\frac{Q^2}{2Nd^2}}\left(\tfrac{Q}{Nd}\right),\end{aligned} \quad (4.62)$$

where we defined, similarly to ${}^{\varsigma}G^{(p-1,q-1)}_{F,\alpha\beta,0}(Q)$,

$${}^{\varsigma}G^{(p-1,q-1)}_{F,\alpha\beta,m}(Q) = \int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{d^2\rho}{\rho_2^2} \sum_{l\in\mathbb{Z}_{2m}} \frac{N\widehat{h}_{m,l}(N\rho)}{\Delta_k(\rho)} \Gamma^{m,l}_{\alpha\beta}(Q)\,, \quad (4.63)$$

with $\Gamma^{m,l}_{\alpha\beta}(Q)$ defined in (4.29).

3. For all non-zero vectors $Q \neq 0$, the moduli-dependent coefficient coming from the polar part of $1/\Phi_{k-2}(\Omega)$ is given by

$$\begin{aligned}\bar{G}^{(p-1,q-1)}_{P,\alpha\beta}(Q,\varphi) = &\sum_{\substack{d>0\\ Q/d\in\Lambda_{p-1,q-1}}} d^{q-6}\, c_k\left(-\tfrac{Q^2}{2d^2}\right) G^{(p-1,q-1)}_{P,\alpha\beta}\left(\tfrac{Q}{d}\right)\\ &+\upsilon \sum_{\substack{d>0\\ Q/d\in N\Lambda^*_{p-1,q-1}}} (Nd)^{q-6}\, c_k\left(-\tfrac{Q^2}{2Nd^2}\right) G^{(p-1,q-1)}_{P,\alpha\beta}\left(\tfrac{Q}{Nd}\right).\end{aligned} \quad (4.64)$$

where $G^{(p-1,q-1)}_{P,\alpha\beta}$ is defined as in the previous subsection, upon replacing $\Delta(\rho)$ by $\Delta_k(\rho)$.

Note that the polar part and the finite part of the function $\bar{G}^{(p-1,q-1)}_{\alpha\beta}(Q,\varphi)$ combine for all Q into the same divisor sum of the function $G^{(p-1,q-1)}_{\alpha\beta}(Q) = G^{(p-1,q-1)}_{F\alpha\beta}(Q) + G^{(p-1,q-1)}_{P\alpha\beta}(Q)$ and ${}^{\varsigma}G^{(p-1,q-1)}_{\alpha\beta}(Q) = {}^{\varsigma}G^{(p-1,q-1)}_{F\alpha\beta}(Q) + {}^{\varsigma}G^{(p-1,q-1)}_{P\alpha\beta}(Q)$ as in the maximal rank case (4.45). The only apparent difference is for the finite part of the function (4.61), because we defined the function (4.61) $G^{(p-1,q-1)\perp}_{F,\alpha\beta,0}\big(\frac{Q}{d}\big)$ and ${}^{\varsigma}G^{(p-1,q-1)\perp}_{F,\alpha\beta,0}\big(\frac{Q}{d}\big)$ such that they can be identified to the function $\frac{\gcd(Q)}{12}G^{(p-2,q-2)}_{\alpha\beta}(\varphi_Q)$ and $\frac{\gcd(Q)}{12}\,{}^{\varsigma}G^{(p-2,q-2)}_{\alpha\beta}(\varphi_Q)$ on the quotient of the sublattice of $\Lambda_{p-1,q-1}$ orthogonal to Q by the shift in Q.

4.3 Perturbative limit of exact heterotic $\nabla^2(\nabla\phi)^4$ couplings in $D = 3$

According to our Ansatz (1.7), the exact $\nabla^2(\nabla\phi)^4$ coupling in three-dimensional CHL orbifolds is given by a special case of the family of genus-two modular integrals (4.1) for the 'non-perturbative Narain lattice' (2.3) of signature $(p,q) = (2k,8) = (2k,8)$. The degeneration (4.2) studied in this section corresponds to the limit of weak heterotic coupling $g_3 \to 0$. In this limit, the lattice $\Lambda_{2k,8}$ decomposes into $\Lambda_{2k-1,7} \oplus I\!I_{1,1}[N]$, where the 'radius' of the second factor is related to the heterotic string coupling by $g_3 = 1/\sqrt{R}$, and the U-duality group is broken to $\widetilde{O}(2k-1,7,\mathbb{Z}) \subset \widetilde{O}(2k,8,\mathbb{Z})$, with $\widetilde{O}(2k-1,7,\mathbb{Z})$ the restricted automorphic group of $\Lambda_{2k-1,7} = \Lambda_m \oplus I\!I_{1,1}[N]$. In order to interpret the various power-like terms in the large radius expansion as perturbative contributions to the $\nabla^2(\nabla\phi)^4$ coupling, it is convenient to multiply the coupling by a factor of g_3^6, which arises due to the Weyl rescaling $\gamma_E = \gamma_s/g_3^4$ from the Einstein frame to the string frame [22, Sec 4.3]. The weak coupling expansion can be extracted from section 4.2 upon setting $q = 8$ and $v = 1$, and reads

$$
\begin{aligned}
g_3{}^6 G^{(2k,8)}_{\alpha\beta,\gamma\delta} = & -\frac{3}{4\pi g_3{}^2}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} - \frac{1}{4}\delta_{\langle\alpha\beta,}G^{(2k-1,7)}_{\gamma\delta\rangle}(\varphi) + g_3{}^2\, G^{(2k-1,7)}_{\alpha\beta,\gamma\delta}(\varphi) \\
& + \sum_{Q\in\Lambda^*_{2k-1,7}}{}^{'} \frac{3e^{-\frac{2\pi}{g_3^2}\sqrt{2Q_R^{\,2}}+2\pi i Q\cdot a}}{2Q_R^{\,2}}\,\bar{G}^{(2k-1,7)}_{\langle\alpha\beta,}(Q,\varphi)\Big(Q_{L\gamma}Q_{L\delta\rangle}\Big(\sqrt{2Q_R^{\,2}}+\tfrac{g_3^2}{2\pi}\Big) - \tfrac{g_3^2}{8\pi}\delta_{\gamma\delta\rangle}\Big) \\
& + \sum_{Q\in\Lambda^*_{2k-1,7}}{}^{'} e^{-\frac{4\pi}{g_3^2}\sqrt{2Q_R^{\,2}}}\,G_{\alpha\beta,\gamma\delta}(g_3,Q_L,Q_R)\,.
\end{aligned} \tag{4.65}
$$

The three first terms in (4.65) originate (in reverse order) from the trivial orbit (4.47), the rank one orbit (4.59), and the splitting degeneration contribution (4.60). By construction, the trivial orbit reproduces the two-loop contribution computed in (B.57). More remarkably, the rank one orbit matches the one-loop contribution (B.14), while the splitting degeneration contribution reproduces the tree-level $\nabla^2(\nabla\phi)^4$, obtained by dimensional reduction of the ∇^2F^4 coupling in 10 dimensions.[10]

The exponentially suppressed terms in the second line of (4.65) can be interpreted as instantons from Euclidean NS five-branes wrapped respectively on any possible T^6 inside T^7, KK (6,1)-branes wrapped with any S^1 Taub-NUT fiber in T^7, and H-monopoles wrapped on

[10] As already noted in [13], there also exists a tree-level single trace ∇^2F^4 interaction in ten dimensions, with coefficient proportional to $\zeta(3)$ [47], but the latter vanishes when all gauge bosons belong to an Abelian subalgebra and therefore does not contribute to the $\nabla^2(\nabla\phi)^4$ interaction in three dimensions. Note that the single trace interaction is not protected and receives corrections to all orders in heterotic perturbation theory [66].

T^7. One has similarly for the other components (4.44)

$$\begin{aligned} g_3{}^6 G^{(2k,8),1,Q}_{\alpha\beta,\gamma 1} &= \frac{3}{4\mathrm{i}\sqrt{2}Q_R^2} e^{-\frac{2\pi}{g_3^2}\sqrt{2Q_R^2}} \bar{G}^{(2k-1,7)}_{\langle\alpha\beta,}(Q,\varphi)\, Q_{L\gamma\rangle}\;, \\ g_3{}^6 G^{(2k,8),1,Q}_{\alpha\beta,11} &= -\frac{1}{2\sqrt{2Q_R^2}} e^{-\frac{2\pi}{g_3^2}\sqrt{2Q_R^2}} \bar{G}^{(2k-1,7)}_{\alpha\beta}(Q,\varphi)\,, \end{aligned} \tag{4.66}$$

where $\bar{G}^{(2k-1,7)}_{\alpha\beta,-\frac{Q^2}{2}} = \bar{G}^{(2k-1,7)}_{F,\alpha\beta,-\frac{Q^2}{2}} + \bar{G}^{(2k-1,7)}_{P,\alpha\beta,-\frac{Q^2}{2}}$ and takes the form

$$\begin{aligned} \bar{G}^{(2k-1,7)}_{\alpha\beta,-\frac{Q^2}{2}}(Q,\varphi) = &\sum_{\substack{d>0\\ Q/d\in\Lambda_{2k-1,7}}} d^2\, c_k\big(-\tfrac{Q^2}{2d^2}\big)\, G^{(2k-1,7)}_{\alpha\beta,\,-\frac{Q^2}{2d^2}}\big(\tfrac{Q}{d}\big) \\ &+ \sum_{\substack{d>0\\ Q/d\in N\Lambda^*_{2k-1,7}}} (Nd)^2\, c_k\big(-\tfrac{Q^2}{2Nd^2}\big)\, {}^{\varsigma}G^{(2k-1,7)}_{\alpha\beta,\,-\frac{Q^2}{2Nd^2}}\big(\tfrac{Q}{Nd}\big)\,. \end{aligned} \tag{4.67}$$

For the null charges $Q^2=0$, we write instead the finite contribution as

$$\begin{aligned} \bar{G}^{(2k-1,7)}_{F,\alpha\beta,0}(Q,\varphi) = \frac{k}{N-1} &\sum_{\substack{d>0\\ Q/d\in\Lambda_{2k-1,7}}} d^2\Big[N\, {}^{\varsigma}G^{(2k-1,7)\perp}_{F,\alpha\beta,0}\big(\tfrac{Q}{d}\big) - G^{(2k-1,7)\perp}_{F,\alpha\beta,0}\big(\tfrac{Q}{d}\big)\Big] \\ + \frac{k}{N-1} &\sum_{\substack{d>0\\ Q/d\in N\Lambda^*_{2k-1,7}}} (Nd)^2\Big[N G^{(2k-1,7)\perp}_{F,\alpha\beta,0}\big(\tfrac{Q}{Nd}\big) - {}^{\varsigma}G^{(2k-1,7)\perp}_{F,\alpha\beta,0}\big(\tfrac{Q}{Nd}\big)\Big] \end{aligned} \tag{4.68}$$

In the maximal rank case $N=1$, upon setting ${}^{\varsigma}G^{(p,q)}_{ab} = G^{(p,q)}_{ab}$ and replacing $c_k(m)\to c(m)$, $k\to 12 = c(0)/2$, Eqs. (4.67) and (4.68) simplify to

$$\bar{G}^{(23,7)}_{\alpha\beta,-\frac{Q^2}{2}}(Q,\varphi) = \sum_{\substack{d>0\\ Q/d\in\Lambda_{23,7}}} d^2\, c\big(-\tfrac{Q^2}{2d^2}\big)\, G^{(23,7)}_{\alpha\beta,\,-\frac{Q^2}{2d^2}}\big(\tfrac{Q}{d}\big)\,. \tag{4.69}$$

It is important to note that the orbit method misses exponentially suppressed terms which do not depend on the axions a in the last line of (4.65). The existence of these terms is clear from the differential constraint (3.20), since the $(\nabla\phi)^4$ coupling F_{abcd} appearing on the right-hand side contains both instanton and anti-instanton contributions. Unfortunately, our current tools do not allow us to extract these contributions from the unfolding method at present. One could obtain them by solving the differential equation (E.51) for $Q=0$.

Finally, it is worth stressing that while the perturbative contributions $G^{(2k-1,7)}_{ab}$ and $G^{(2k-1,7)}_{ab,cd}$ have singularities in codimension 7 inside $\mathcal{M}_3$ at points of enhanced gauge symmetry, the full instanton-corrected coupling (1.7) has only singularities in codimension 8. In Appendix B.3, we analyze the structure of the singularities for a general genus-two modular integral of the form (2.30) and find the expected one-loop and two-loop contributions with nearly massless gauge bosons running in the loops.

5 Large radius expansion of exact $\nabla^2(\nabla\phi)^4$ couplings

We now study the asymptotic expansion of the modular integral (1.7) in the limit where the radius R of one circle in the internal space goes to infinity. We show that it reproduces the known ∇^2F^4 and $\mathcal{R}^2F^2$ couplings in $D=4$, along with an infinite series of $\mathcal{O}(e^{-R})$ corrections from 1/2-BPS and 1/4-BPS dyons whose wordline winds around the circle, up to an infinite series of $\mathcal{O}(e^{-R^2})$ corrections with non-zero NUT charge, corresponding to Taub-NUT instantons. We start by analyzing the expansion of genus-two modular integral (2.30) for arbitrary values of (p,q), in the limit near the cusp where $O(p,q)$ is broken to $SL(2,\mathbb{R})\times O(p-2,q-2)$, so that the moduli space decomposes into

$$G_{p,q}\to\mathbb{R}^+\times\left[\frac{SL(2,\mathbb{R})}{SO(2)}\times G_{p-2,q-2}\right]\ltimes\mathbb{R}^{2(p+q-4)}\times\mathbb{R} \tag{5.1}$$

As in the previous section, we first discuss the maximal rank case $N=1$, $p-q=16$, where the integrand is invariant under the full modular group, before dealing with the case of N prime. The reader uninterested by the details of the derivation may skip to §5.3, where we specialize to the values $(p,q)=(2k,8)$ relevant for the $\nabla^2(\nabla\phi)^4$ couplings in $D=3$, and interpret the various contributions arising in the decompactification limit to $D=4$.

5.1 $O(p,q)\to O(p-2,q-2)$ for even self-dual lattices

In this subsection we assume that the lattice $\Lambda_{p,q}$ is even self-dual and factorizes in the limit (5.1) as

$$\Lambda_{p,q}\to\Lambda_{p-2,q-2}\oplus I\!I_{2,2}\ . \tag{5.2}$$

We denote by R,t,a^{Ii},ψ the coordinates for each factors in (5.1) (here $i=1,2$ and $I=3,\dots,p+q-2$). The coordinate R (not to be confused with the one used in §4) parametrizes a one-parameter subgroup e^{RH_1} in $O(p,q)$, such that the action of the non-compact Cartan generator H_1 on the Lie algebra $\mathfrak{so}_{p,q}$ decomposes into

$$\mathfrak{so}_{p,q}\simeq\ \dots\oplus(\mathfrak{gl}_1\oplus\mathfrak{sl}_2\oplus\mathfrak{so}_{p-2,q-2})^{(0)}\oplus(\mathbf{2}\otimes(\mathbf{p}+\mathbf{q}-\mathbf{4}))^{(1)}\oplus\mathbf{1}^{(2)}, \tag{5.3}$$

while (a^{iI},ψ) parametrize the unipotent subgroup obtained by exponentiating the grade 1 and 2 components in this decomposition. We parametrize the $SO(2)\backslash SL(2,\mathbb{R})$ coset representative ${v_\mu}^i$ and the symmetric $SL(2,\mathbb{R})$ element $M\equiv v^Tv$ by the complex upper half-plane coordinate $S=S_1+\mathrm{i}S_2$, such that

$${v_\mu}^i=\frac{1}{\sqrt{S_2}}\begin{pmatrix}1 & S_1\\ 0 & S_2\end{pmatrix}\ ,\quad M^{ij}=\delta^{\mu\nu}{v_\mu}^i{v_\nu}^j=\frac{1}{S_2}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}\ . \tag{5.4}$$

The remaining coordinates in $G_{p-2,q-2}$ will be denoted by φ. As in the weak coupling expansion, lattice vector are labelled according to the choice of A-cycle on the genus-two Riemann surface. A generic charge vector $(Q^1_{\mathcal{I}},Q^2_{\mathcal{I}})\in\Lambda_{p,q}\oplus\Lambda_{p,q}\simeq(\mathbf{2}\otimes\mathbf{2})^{(-1)}\oplus(\mathbf{2}\otimes(\mathbf{p}+\mathbf{q}-\mathbf{4}))^{(0)}\oplus(\mathbf{2}\otimes\mathbf{2})^{(1)}$ decomposes into

$$(Q^1_{\mathcal{I}},Q^2_{\mathcal{I}})=(n^1_i,n^2_i,\widetilde{Q}^1_I,\widetilde{Q}^2_I,m^{1j},m^{2j}) \tag{5.5}$$

where $(n^1_i,n^2_i,m^{1j},m^{2j})\in I\!I_{2,2}\oplus I\!I_{2,2}$ and $(\widetilde{Q}^1_I,\widetilde{Q}^2_I)\in\Lambda_{p-2,q-2}\oplus\Lambda_{p-2,q-2}$ such that $Q^r\cdot Q^s=-m^{ri}n^s_i-m^{si}n^r_i+\widetilde{Q}^r\cdot\widetilde{Q}^s$. The orthogonal projectors defined by $Q^r_L\equiv p^{\mathcal{I}}_LQ^r_{\mathcal{I}}$ and $Q^r_R\equiv p^{\mathcal{I}}_RQ^r_{\mathcal{I}}$

decompose according to

$$
\begin{aligned}
p^{\mathcal{I}}_{L,\mu} Q^r_{\mathcal{I}} =& \frac{v^{-1}_{i\mu}}{R\sqrt{2}} \left(m^{ri} + a^i \cdot \widetilde{Q}^r + (\psi \varepsilon^{ij} + \frac{1}{2} a^i \cdot a^j) n^r_j \right) - \frac{R}{\sqrt{2}} v_\mu{}^i n^r_i \\
p^{\mathcal{I}}_{L,\alpha} Q^r_{\mathcal{I}} =& \tilde{p}^I_{L,\alpha} (\widetilde{Q}^r_I + n^r_i a^i_I) \\
p^{\mathcal{I}}_{R,\mu} Q^r_{\mathcal{I}} =& \frac{v^{-1}_{i\mu}}{R\sqrt{2}} \left(m^{ri} + a^i \cdot \widetilde{Q}^r + (\psi \varepsilon^{ij} + \frac{1}{2} a^i \cdot a^j) n^r_j \right) + \frac{R}{\sqrt{2}} v_\mu{}^i n^r_i \\
p^{\mathcal{I}}_{R,\hat\alpha} Q^r_{\mathcal{I}} =& \tilde{p}^I_{R,\hat\alpha} (\widetilde{Q}^r_I + n^r_i a^i_I)
\end{aligned}
\tag{5.6}
$$

where $\tilde{p}^I_{L,\alpha}, \tilde{p}^I_{R,\hat\alpha}$ ($\alpha = 3 \dots p$, $\hat\alpha = 3 \dots q$) are orthogonal projectors in $G_{p-2,q-2}$ satisfying $\widetilde{Q}^r \widetilde{Q}^s = \widetilde{Q}^r_L \cdot \widetilde{Q}^s_L - \widetilde{Q}^r_R \cdot \widetilde{Q}^s$.

In order to study the region $R \gg 1$ it is useful to perform a Poisson resummation on the momenta m^{ri} along $I\!I_{2,2} \oplus I\!I_{2,2}$. Note that this analysis is in principle valid for a region containing $R > \sqrt{2}$. Insertion of momenta polynomials along the torus or the sublattice can be again obtained using an insertion of a auxiliary variables $(y_{r,\mu}, y_{r,\alpha})$

$$
\begin{aligned}
&\Gamma^{(2)}_{\Lambda_{p,q}} \left[e^{2\pi i y_a \cdot \widetilde{Q}^a + \frac{\pi}{2} y_a \cdot \Omega_2^{-1} \cdot y^a} \right] \\
&\quad = R^4 \sum_{(\mathbf{m}_i, \mathbf{n}_j) \in \mathbb{Z}^8} e^{-\pi R^2 (\mathbf{n}_i\, \mathbf{m}_i) \binom{\Omega}{\mathbb{1}} \cdot \Omega_2^{-1} M^{ij} \cdot \left[(\mathbf{n}_j\, \mathbf{m}_j) \binom{\bar\Omega}{\mathbb{1}} \right]^\intercal} e^{\frac{2\pi R}{i\sqrt{2}} y^\mu \cdot \Omega_2^{-1} \cdot \left[(\mathbf{n}_i\, \mathbf{m}_i) \binom{\bar\Omega}{\mathbb{1}} \right]^\intercal v_\mu{}^i} \\
&\qquad \times \Gamma^{(2)}_{\Lambda_{p-2,q-2}} \left[e^{2\pi i \mathbf{m}_i \cdot (a^i_I \widetilde{Q}^I + \frac{1}{2} a^i_I a^{Ij}\, \mathbf{n}_j)} e^{2\pi i y_{\alpha I} \cdot \widetilde{Q}^{\alpha I} + \frac{\pi}{2} y_{\alpha I} \cdot \Omega_2^{-1} \cdot y^{\alpha I}} \right] ,
\end{aligned}
\tag{5.7}
$$

where the sum over indices $r = 1, 2$ is implicit, we used Einstein summation convention for indices $r = 1, 2$, $\mu = 1, 2$, $i, j = 1, 2$ and $\alpha = 3, ..., p$, and where M^{ij} is defined in (5.4). In this representation, modular invariance is manifest since a transformation $\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1}$ can be compensated by a linear transformation $\begin{pmatrix} \mathbf{n}_1 & \mathbf{m}_1 \\ \mathbf{n}_2 & \mathbf{m}_2 \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{n}_1 & \mathbf{m}_1 \\ \mathbf{n}_2 & \mathbf{m}_2 \end{pmatrix} \begin{pmatrix} D^\intercal & -B^\intercal \\ -C^\intercal & A^\intercal \end{pmatrix}$, $y_\mu \mapsto y_\mu \cdot (C\Omega + D)$, under which the third line of (5.7) transforms as a weight $\frac{p-q}{2}$ modular form. We can therefore decompose charges $(\mathbf{n}_i, \mathbf{m}_j)$ into various orbits under $Sp(4, \mathbb{Z})$ and apply the unfolding trick to each orbit:

The trivial orbit $(\mathbf{n}_i, \mathbf{m}_j) = (0, 0)$ produces the integral (4.1) for the lattice $\Lambda^{\oplus 2}_{p-2,q-2} \equiv \Lambda_{p-2,q-2} \oplus \Lambda_{p-2,q-2}$, up to a factor R^4, and vanishes if one of the indices ab, cd lies along $1, 2$

$$
G^{(p,q),0}_{\alpha\beta,\gamma\delta} = R^4\, G^{(p-2,q-2)}_{\alpha\beta,\gamma\delta} \,. \tag{5.8}
$$

Rank-one orbit This orbit consists of matrices $(\mathbf{n}_i, \mathbf{m}_j) \neq (0, 0)$ where $(\mathbf{n}_1, \mathbf{m}_1)$ and $(\mathbf{n}_2, \mathbf{m}_2)$ are collinear and not simultaneously vanishing. Such matrices can be decomposed as $(\mathbf{n}_i, \mathbf{m}_j) = \binom{j}{p}(c_3, c_4, d_3, d_4)$, $(j, p) \neq (0, 0)$ and $\gcd(c_3, c_4, d_3, d_4) = 1$. Quadruplets (c_3, c_4, d_3, d_4) with $\gcd(c_3, c_4, d_3, d_4) = 1$ can all be rotated to $(0, 0, 0, \pm 1)$ by a $Sp(4, \mathbb{Z})$ element, whose stabilizer is the central extension of the Jacobi group Γ^J_1 (4.10), and are in one-to-one correspondence with elements of $\Gamma^J_1 \backslash Sp(4, \mathbb{Z})$. Thus for each doublet $(j, p) \neq (0, 0)$, one can unfold the integration domain $Sp(4, \mathbb{Z}) \backslash \mathcal{H}_2$ to $\Gamma^J_1 \backslash \mathcal{H}_2 = \mathbb{R}^+_t \times (SL(2, \mathbb{Z}) \backslash \mathcal{H}_1)_\rho \times (T^3/\mathbb{Z}_2)_{u_1, u_2, \sigma_1}$ (for further details, see below (4.11)). We parametrize $\Gamma^J_1 \backslash \mathcal{H}_2$ by $t = \frac{|\Omega_2|}{\rho_2}$, ρ

and $(u_1, u_2, \sigma_1) = (v_1 - u_2\rho_1, v_2/\rho_2, \sigma_1)$, and change the y variables $(y'_{1\mu}, y'_{2\mu}, y'_{1\alpha}, y'_{2\alpha}) = (y_{1\mu}, y_{1\mu}u_1 - y_{2\mu}, y_{1\alpha}, y_{1\alpha}u_2 - y_{2\alpha})$ stabilizing $\mathcal{P}_{ab,cd}$

$$\begin{aligned} G^{(p,q),1}_{ab,cd} &= R^4 \int_0^\infty \frac{\mathrm{d}t}{t^3} \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^3} \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}\sigma_1 \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\mathcal{P}_{ab,cd}(\frac{\partial}{\partial y'})}{\Phi_{10}} \sum_{(j,p)\in\mathbb{Z}^2}{}' e^{-\frac{\pi R^2}{S_2 t}|j+pS|^2} \\ &\times \Gamma^{(2)}_{\Lambda_{p-2,q-2}} \Big[e^{2\pi \mathrm{i}(ja_1^I + pa_2^I)\widetilde{Q}_{2I}} \exp 2\pi\mathrm{i}\Big(\frac{R}{\mathrm{i}\sqrt{2}} y'_{r\mu} (\Omega_2^{-1})^{r2} m_{2i} v^{i\mu} \\ &\qquad + y'_{1\,\alpha}(\widetilde{Q}^{1\alpha}_L + u_2 \widetilde{Q}^{2\alpha}_L) - y'_{2\,\alpha}\widetilde{Q}^{2\alpha}_L + \frac{1}{4\mathrm{i}\rho_2} y'_{1\,\alpha} y'^{\,\alpha}_1 + \frac{1}{4\mathrm{i}t} y'_{2\,\alpha} y'^{\,\alpha}_2 \Big)\Big] , \end{aligned} \tag{5.9}$$

where $m_{2i}v^{i\mu} = \frac{1}{S_2}\begin{pmatrix} 1 & S_1 \\ 0 & S_2 \end{pmatrix}\binom{j}{p}$, and $\mathcal{P}_{ab,cd}(\frac{\partial}{\partial y})$ is derivative polynomial of order four defined in (4.13), and where the Fourier-Jacobi expansion of $1/\Phi_{10}$ is given eq.(4.14).

The integral over σ_1 picks up the Jacobi $\psi_m(\rho, v)$ of index $m = -\frac{1}{2}\widetilde{Q}_2^2$. Contributions from $\widetilde{Q}_2 = 0$ pick up the contribution $c(0)\widehat{E}_2/(12\Delta)$ (4.15), and lead to power-like terms[11]

$$\begin{aligned} G^{(p,q),1,0}_{\alpha\beta,\gamma\delta} &= -R^{q-4} \frac{c(0)}{16\pi} \mathcal{E}^\star(\tfrac{8-q}{2}, S) \delta_{\langle\alpha\beta,} G^{(p-2,q-2)}_{\gamma\delta\rangle} \\ G^{(p,q),1,0}_{\alpha\beta,\mu\nu} &= -R^{q-4} \frac{c(0)}{48\pi} \left[\tfrac{8-q}{2}\delta_{\mu\nu} - 2\mathcal{D}_{\mu\nu}\right] \mathcal{E}^\star(\tfrac{8-q}{2}, S)\, G^{(p-1,q-1)}_{\alpha\beta} , \end{aligned} \tag{5.10}$$

where $\mathcal{E}^\star(s, S)$ is the completed weight 0 non-holomorphic Eisenstein series

$$\mathcal{E}^\star(s,S) = \frac{1}{2}\pi^{-s}\Gamma(s) \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{S_2^s}{|nS+m|^{2s}} \equiv \xi(2s)\mathcal{E}(s,S) , \tag{5.11}$$

with $\xi(2s)$ the reduced zeta function $\xi(2s) = \pi^{-s}\Gamma(s)\zeta(2s)$ and $\mathcal{D}_{\mu\nu}$ is the traceless differential operator on $\frac{SL(2,\mathbb{R})}{SO(2)}$ acting on S and defined in terms of raising and lowering operators of weight w as

$$\mathcal{D}_{\mu\nu} = -\frac{1}{2}\sigma^+_{\mu\nu}\mathcal{D}_w - \frac{1}{2}\sigma^-_{\mu\nu}\bar{\mathcal{D}}_w , \tag{5.12}$$

with $\sigma^\pm = \frac{1}{2}(\sigma_3 \pm i\sigma_1)$ and σ_i the Pauli matrices.

Non-zero vectors $\widetilde{Q}_2$ lead to exponentially suppressed contributions, in a similar fashion as what described for the $O(p,q) \to O(p-1,q-1)$ limit, section 4.1. They depend on the axions through a phase factor $e^{2\pi\mathrm{i}m_{2j}\widetilde{Q}_{2I}a^{Ij}}$. In order to evaluate them, we insert the Fourier-Jacobi expansion (A.54) and decompose each $\psi_m(\rho, v)$ into its finite and polar parts. In either case, the integral over σ_1 imposes $\widetilde{Q}_2^2 = -2m$. As in the previous section, we consider first the contributions of the finite part $\psi^F_m(\rho, v)$, for null and non-null vectors, and then the contributions of the polar part $\psi^P_m(\rho, v)$

1. In the case $\widetilde{Q}_2^2 = 0$, one can make the same decomposition as in section 4.1, using the constraint $\widetilde{Q}_1 \cdot \widetilde{Q}_2 = 0$ from $\widehat{\psi}^F_0(\rho)$. The integral then reads, for a given null vector $\widetilde{Q}_2$ and

[11] Note that (5.10) has a pole at $q = 6$ and $q = 8$, of which the first is substracted by the regularization prescription discussed in §B.2.4, and the second cancels against the pole from the trivial orbit contribution (5.8).

$m_{2j} = \binom{j}{p}$

$$\frac{R^4}{2} \sum_{(j,p)\in\mathbb{Z}^2}' e^{2\pi i m_{2j} \widetilde{Q}_{2I} a^{Ij}} \gcd(\widetilde{Q}_2) \int_{\mathbb{R}^+} \frac{\mathrm{d}t}{t} t^{\frac{q-2}{2}} e^{-\frac{\pi R^2}{S_2 t}|j+pS|^2 - 2\pi t |\widetilde{Q}_{2R}|^2} \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{c(0)\widehat{E}_2}{12\Delta}$$
$$\times \int_{\mathbb{R}} \mathrm{d}u_2\, \rho_2^{\frac{q-1}{2}} \sum_{\widetilde{Q}_1^\perp \in \Lambda_{p-3,q-3}} q^{\frac{1}{2}(\widetilde{Q}_1^\perp + u_2 \widetilde{Q}_2)_L^2} \bar{q}^{\frac{1}{2}(\widetilde{Q}_1^\perp + u_2 \widetilde{Q}_2)_R^2} \mathcal{P}_{ab,cd}(\tfrac{\partial}{\partial y'})$$
$$\times\, e^{2\pi i\left(\frac{R}{i\sqrt{2}} y'_{r\mu} (\Omega_2^{-1})^{r2} m_{2i} v^{i\mu} + y'_{1\alpha}(\widetilde{Q}_L^{1\alpha} + u_2 \widetilde{Q}_L^{2\alpha}) - y'_{2\,\alpha} \widetilde{Q}_L^{2\alpha} + \frac{1}{4i\rho_2} y'_{1\alpha} y_1'^{\,\alpha} + \frac{1}{4it} y'_{2\alpha} y_2'^{\,\alpha}\right)}\Bigg|_{y'=0} , \tag{5.13}$$

where $\gcd(\widetilde{Q}_2)$ comes from unfolding the u_2-integral that uses the component of $\widetilde{Q}_1$ along $\widetilde{Q}_2$, and where $\widetilde{Q}_1^\perp \in \Lambda_{p-3,q-3}$ such that $\Lambda_{p-3,q-3} = \{\widetilde{Q}_1^\perp \in \Lambda_{p-2,q-2},\ \widetilde{Q}_1^\perp \cdot \widetilde{Q}_2 = 0\}/(\mathbb{Z}\frac{\widetilde{Q}_2}{\gcd \widetilde{Q}_2})$ (for further details, see (4.21)). We obtain the a one-loop integral on a sub-Grassmaniann $G_{p-2,q-2}$ parametrizing a space orthogonal to $\widetilde{Q}_2$, labelled $G_{F,\alpha\beta}^{(p-2,q-2)\perp}(\widetilde{Q}_2,\varphi)$, that we define as

$$G_{F,\alpha\beta,0}^{(p-2,q-2)\perp}(Q,\varphi) = \frac{\gcd(Q)}{12} \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\widehat{E}_2}{\Delta_k(\rho)} \rho_2^{\frac{q-3}{2}} \sum_{\widetilde{Q}\in\Lambda_{p-3,q-3}} q^{\frac{1}{2}\widetilde{Q}_L^2} \bar{q}^{\frac{1}{2}\widetilde{Q}_R^2} e^{2\pi\rho_2 \frac{(\widetilde{Q}_R\cdot Q_R)^2}{Q_R^2}}$$
$$\times \left[\left(\widetilde{Q}_{L\alpha} - \frac{\widetilde{Q}_R \cdot Q_R}{Q_R^2} Q_{L\alpha}\right)\left(\widetilde{Q}_{L\beta} - \frac{\widetilde{Q}_R\cdot Q_R}{Q_R^2} Q_{L\beta}\right) - \frac{1}{4\pi\rho_2}\left(\delta_{\alpha\beta} - \frac{Q_{L\alpha}Q_{L\beta}}{Q_R^2}\right)\right] , \tag{5.14}$$

where $\Delta_k = \Delta$ in the case at hand. After defining $\Gamma_i = (Q,P) = m_{2i}\widetilde{Q}_2$, with support on 1/2-BPS states, and covariantizing the expression with the torus vielbein, we find that the Fourier coefficient with support $\Gamma_i \in \Lambda_{p-2,q-2}^{\oplus 2} \setminus \{0\}$, with $\varepsilon^{ij}\Gamma_i\Gamma_j = 0$, and mass $\mathcal{M}(\Gamma) = \sqrt{2M_{ij}\Gamma_R^i \cdot \Gamma_R^j}$, is given by, when $\widetilde{Q}_2^2 = \Gamma_i \cdot \Gamma_j = 0$

$$3R^{\frac{q+2}{2}}\, \bar{G}_{F,\langle\alpha\beta,0}^{(p-2,q-2)}(\Gamma,\varphi) \sum_{l=0}^{1} \frac{\mathcal{P}_{\gamma\delta\rangle}^{(l)}(\Gamma,S)}{R^l} \frac{K_{\frac{q-6}{2}-l}(2\pi R\mathcal{M}(\Gamma))}{\mathcal{M}(\Gamma)^{\frac{q-4}{2}-l}} , \tag{5.15}$$

where the polynomial $\mathcal{P}^{(l)}$ in (4.23) is defined in appendix H.2, and

$$\bar{G}_{F,\alpha\beta,0}^{(p-2,q-2)}(\Gamma,\varphi) = c(0)\Big[\frac{1}{\sqrt{S_2}}|j' + p'S|\Big]^{q-8} \sum_{\substack{d\geq 1\\ \hat{Q}/d\in\Lambda_{p-2,q-2}}} d^{q-8}\, G_{F,\alpha\beta,0}^{(p-2,q-2)\perp}(\tfrac{\hat{Q}}{d},\varphi) , \tag{5.16}$$

and where we defined $\hat{Q}$ and the unique coprimes (j',p') such that $\Gamma = (Q,P) = (j',p')\hat{Q}$. The full expression for all polarizations will be given together with the polar contributions in (5.22).

2. In the case $\widetilde{Q}_2^{\,2} \neq 0$, we replace $\widehat{\psi}_m^F$ by its theta decomposition (4.25). The integral over σ_1 matches $\widetilde{Q}_2^{\,2} = -2m$, while the integral over u_1 imposes the constraint $\widetilde{Q}_1 \cdot \widetilde{Q}_2 = -\ell$. The

variable $s \in \mathbb{Z}$ in (A.62) can be used to unfold the integral over $u_2 \in [-\frac{1}{2}, \frac{1}{2}]$ to $\mathbb{R}$, after shifting each term in the lattice sum as $\widetilde{Q}_1 \to \widetilde{Q}_1 + s\widetilde{Q}_2$, since $\widetilde{Q}_1, \widetilde{Q}_2 \in \Lambda_{p-2,q-2}$. One thus obtain a Fourier coefficient similar to previous case, using $\Gamma_i = m_{2i}\widetilde{Q}_2 = (Q, P)$,

$$3R^{\frac{q+2}{2}}\, \bar{G}^{(p-2,q-2)}_{F,\,\langle\alpha\beta,-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2}}(\Gamma,\varphi) \sum_{l=0}^{1} \frac{\mathcal{P}^{(l)}_{\gamma\delta\rangle}(\Gamma,S)}{R^l} \frac{K_{\frac{q-6}{2}-l}(2\pi\, R\mathcal{M}(\Gamma))}{\mathcal{M}(\Gamma)^{\frac{q-4}{2}-l}} \tag{5.17}$$

where we denoted, by extension, the function

$$\begin{aligned} \bar{G}^{(p-2,q-2)}_{F,\alpha\beta,-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2}}(\Gamma,\varphi) &= \left(M^{ij}\Gamma_i\cdot\Gamma_j\right)^{\frac{q-8}{2}} \\ &\times \sum_{\substack{d\geq 1\\ \Gamma/d\in\Lambda^{\oplus 2}_{p-2,q-2}}} \left(\frac{d^2}{\gcd(\Gamma_i\cdot\Gamma_j)}\right)^{\frac{q-8}{2}} c\Big(-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2d^2}\Big)\, G^{(p-2,q-2)\perp}_{F,\alpha\beta,-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2d^2}}\big(\tfrac{\hat{Q}}{d},\varphi\big)\,, \end{aligned} \tag{5.18}$$

where we introduced the automorphic tensor $G^{(p-2,q-2)\perp}_{F,\alpha\beta,-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2d^2}}(\frac{\hat{Q}}{d},\varphi)$ in (4.28) and the monomials $\mathcal{P}^{(l)}_{\gamma\delta}(\Gamma,S)$ in (H.2). Notice that the function $G^{(p-2,q-2)\perp}_{F,\alpha\beta,-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2d^2}}(\frac{\hat{Q}}{d},\varphi)$ only depends on the direction of $\Gamma = (j',p')\hat{Q}$ in $\Lambda_{p-2,q-2}$, and on the norm $\gcd(\Gamma_i\cdot\Gamma_j)/d^2 = \hat{Q}^2/d^2$.

The full expression for all polarizations will be given together with the polar contributions in (5.22).

3. For the polar contributions, we use the representation

$$\widehat{\psi}^P_m(\rho,v) = \frac{c(m)}{\Delta(\rho)} \sum_{s,\ell\in\mathbb{Z}} q^{ms^2+s\ell} y^{2ms+\ell}\, \widehat{b}(s,\ell,m,\rho_2) \tag{5.19}$$

One can then shift the charges to $\widetilde{Q}_1 \to \widetilde{Q}_1 + s\widetilde{Q}_2$ since $\widetilde{Q}_1$, $\widetilde{Q}_2 \in \Lambda_{p-2,q-2}$, and then use the sum over s to unfold the $u_2 \in [-\frac{1}{2}, \frac{1}{2}]$ to $\mathbb{R}$. Then, integrating over $u_1 \in [-\frac{1}{2}, \frac{1}{2}]$ imposes $\widetilde{Q}_1 \cdot \widetilde{Q}_2 = -\ell$. One obtains the Fourier coefficients, using $\Gamma_i = m_{2i}\widetilde{Q}_2 = (Q, P)$,

$$3R^{\frac{q+2}{2}} \bar{G}^{(p-2,q-2)}_{P,\,\langle\alpha\beta,}(\Gamma,\varphi) \sum_{l=0}^{1} \frac{\mathcal{P}^{(l)}_{\gamma\delta\rangle}(\Gamma)}{R^l} \frac{K_{\frac{q-6}{2}-l}(2\pi\, R\mathcal{M}(\Gamma))}{\mathcal{M}(\Gamma)^{\frac{q-4}{2}-l}}\,, \tag{5.20}$$

where

$$\bar{G}^{(p-2,q-2)}_{P,\alpha\beta}(\Gamma,\varphi) = \left[\tfrac{1}{\sqrt{S_2}}|j'+p'S|\right]^{q-8} \sum_{\substack{d\geq 1\\ \hat{Q}/d\in\Lambda^{\oplus 2}_{p-2,q-2}}} c\Big(-\frac{\hat{Q}^2}{2d^2}\Big) d^{q-8} G^{(p-2,q-2)}_{P,\alpha\beta}\big(\frac{\hat{Q}}{d},\varphi\big)\,. \tag{5.21}$$

Here (j',p') are coprimes such that $\Gamma = (j',p')\hat{Q}$, and where we used the automorphic tensor $G^{(p,q)}_{P,ab}(\hat{Q},\varphi)$ defined in (4.37). Note that the expression above is identical to (5.18), but expressed in a different manner to include the case where the norm of Γ vanishes.

Combining all contributions, the sum of the finite and polar contributions to the rank one Fourier mode are given for all polarizations by

$$\begin{aligned} G^{(p,q),1,\Gamma}_{\alpha\beta,\gamma\delta} &= 3R^{\frac{q+2}{2}}\bar{G}^{(p-2,q-2)}_{\langle\alpha\beta,}(\Gamma,\varphi)\sum_{l=0}^{1}\frac{\mathcal{P}^{(l)}_{\gamma\delta\rangle}(\Gamma)}{R^l}\frac{K_{\frac{q-6}{2}-l}(2\pi R\mathcal{M}(\Gamma))}{\mathcal{M}(\Gamma)^{\frac{q-4}{2}-l}} \\ G^{(p,q),1,\Gamma}_{\alpha\beta,\gamma\mu} &= \frac{3}{2}R^{\frac{q+2}{2}}\bar{G}^{(p-2,q-2)}_{\langle\alpha\beta,}(\Gamma,\varphi)\frac{\Gamma_{L\gamma\rangle\mu}}{\mathrm{i}\sqrt{2}}\frac{K_{\frac{q-8}{2}}(2\pi R\mathcal{M}(\Gamma))}{\mathcal{M}(\Gamma)^{\frac{q-6}{2}}} \\ G^{(p,q),1,\Gamma}_{\alpha\beta,\mu\nu} &= -R^{\frac{q+2}{2}}\bar{G}^{(p-2,q-2)}_{\alpha\beta}(\Gamma,\varphi)\,\Gamma_{R\hat{\alpha}\mu}\Gamma_{R}{}^{\hat{\alpha}}{}_{\nu}\frac{K_{\frac{q-10}{2}}(2\pi R\mathcal{M}(\Gamma))}{\mathcal{M}(\Gamma)^{\frac{q-4}{2}}}\,, \end{aligned} \tag{5.22}$$

where $\bar{G}^{(p-2,q-2)}_{\alpha\beta}(\Gamma,\varphi) = \bar{G}^{(p-2,q-2)}_{F,\alpha\beta,-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2}}(\Gamma,\varphi) + \bar{G}^{(p-2,q-2)}_{P,\alpha\beta}(\Gamma,\varphi)$, $\Gamma_{L\gamma\mu} = v_\mu{}^i\Gamma_{L\gamma i}$, $\Gamma_{R\hat{\alpha}\mu} = v_\mu{}^i\Gamma_{R\hat{\alpha}i}$, and we recall $\Gamma_i = (Q,P)$.

Rank two Abelian orbits These orbits consist of matrices $\left(\begin{smallmatrix}\mathbf{n}_1 & \mathbf{m}_1\\ \mathbf{n}_2 & \mathbf{m}_2\end{smallmatrix}\right)$ where $(\mathbf{n}_1,\mathbf{m}_1)$ and $(\mathbf{n}_2,\mathbf{m}_2)$ are not collinear (in particular, non-zero) but have vanishing symplectic product $\mathbf{n}_1\cdot\mathbf{m}_2-\mathbf{m}_1\cdot\mathbf{n}_2=0$. Such matrices can be decomposed as $\left(\begin{smallmatrix}\mathbf{n}_1 & \mathbf{m}_1\\ \mathbf{n}_2 & \mathbf{m}_2\end{smallmatrix}\right) = \left(\begin{smallmatrix}0 & \mathbf{j}\\ 0 & \mathbf{p}\end{smallmatrix}\right)\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$, where $(\mathbf{j},\mathbf{p})\in M_2(\mathbb{Z})\backslash\{0\}$, and $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in\Gamma_{2,\infty}\backslash Sp(4,\mathbb{Z})$, with $\Gamma_{2,\infty}=GL(2,\mathbb{Z})\ltimes\mathbb{Z}^3$ the residual symmetry at the cusp $\Omega_2\to\infty$, embedded in $Sp(4,\mathbb{Z})$ as

$$\Gamma_{2,\infty} = \{\left(\begin{smallmatrix}\gamma & 0\\ 0 & \gamma^{-\intercal}\end{smallmatrix}\right),\,\gamma\in GL(2,\mathbb{Z})\}\ltimes\{\left(\begin{smallmatrix}\mathbb{1} & M\\ 0 & \mathbb{1}\end{smallmatrix}\right),\,M\in M_2(\mathbb{Z}),\,M=M^\intercal\}. \tag{5.23}$$

Doublets (C,D) can be rotated to $(0,\mathbb{1})$ by an element of $Sp(4,\mathbb{Z})$, and are in one-to-one correspondence with elements $\Gamma_{2,\infty}\backslash Sp(4,\mathbb{Z})$. The fundamental domain can thus be unfolded from $Sp(4,\mathbb{Z})\backslash\mathcal{H}_2$ to $\Gamma_{2,\infty}\backslash\mathcal{H}_2=(GL(2,\mathbb{Z})\backslash\mathcal{P}_2)_{\Omega_2}\ltimes(\mathbb{R}/\mathbb{Z})^3_{\Omega_1}$, where $\mathcal{P}_2$ is the set of positive-definite matrices. Finally, one can restrict the matrices $A=(\mathbf{j},\mathbf{p})\in M_2(\mathbb{Z})$ to $A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})$, in order to unfold $GL(2,\mathbb{Z})\backslash\mathcal{P}_2$ to $\mathcal{P}_2$.

The resulting contribution can be expressed in terms of the auxiliary variables $(y_{r,\mu},y_{r,\alpha})$ (5.7), and we obtain

$$\begin{aligned} G^{(p,q),2\mathrm{Ab}}_{ab,cd} &= 2R^4\int_{\mathcal{P}_2}\frac{\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\int_{[-\frac12,\frac12]^3}\mathrm{d}^3\Omega_1\frac{|\Omega_2|^{\frac{q-2}{2}}}{\Phi_{10}} \\ &\times\sum_{\widetilde{Q}\in\Lambda^{\oplus 2}_{p-2,q-2}} e^{\pi\mathrm{i}\mathrm{Tr}\left[\Omega\widetilde{Q}\cdot\widetilde{Q}^\intercal\right]}\sum_{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})} e^{2\pi\mathrm{i}a^{iI}A_{ij}Q^j_I-\pi\mathrm{Tr}\left[\frac{R^2}{S_2}\Omega_2^{-1}A^\intercal\left(\begin{smallmatrix}1 & S_1\\ S_1 & |S|^2\end{smallmatrix}\right)A+2\Omega_2\widetilde{Q}_R\cdot\widetilde{Q}_R^\intercal\right]} \\ &\quad\times\mathcal{P}_{ab,cd}\left(\tfrac{\partial}{\partial y}\right)e^{2\pi\mathrm{i}\left(\frac{R}{\mathrm{i}\sqrt{2}}y_{r\mu}(\Omega_2^{-1})^{rs}A^\intercal_{si}v^{\intercal\,i\mu}+y_{r\,\alpha}\widetilde{Q}_L{}^{r\alpha}+\frac{1}{4\mathrm{i}}y_{r\,\alpha}(\Omega_2^{-1})^{rs}y_s{}^\alpha\right)}\,, \end{aligned} \tag{5.24}$$

where the factor two comes from the non-trivial center of order 2 of $\mathrm{GL}(2,\mathbb{Z})$ acting on $\mathcal{H}_2$. For sufficiently large $|\Omega_2|$, the integral over $\Omega_1\in[0,1]^3$ selects the Fourier coefficient $C(m,n,L;\Omega_2)$ of $1/\Phi_{10}$, with $\widetilde{Q}_1^2=-2m$, $\widetilde{Q}_2=-2n$, $\widetilde{Q}_1\cdot\widetilde{Q}_2=-L$. As discussed in §A.6, the Fourier coefficient can be decomposed into a finite contribution $C^F(n,m,L)$, independent of Ω_2, and an infinite series of terms associated to the polar part,

$$
\begin{aligned}
C(m,n,L;\Omega_2) &\equiv \int_{\mathcal{C}} \mathrm{d}^3\Omega_1 \frac{e^{\mathrm{i}\pi(Q_1,Q_2)\left(\begin{smallmatrix}\rho & v\\ v & \sigma\end{smallmatrix}\right)\left(\begin{smallmatrix}Q_1\\ Q_2\end{smallmatrix}\right)}}{\Phi_{10}}\\
&= C^F(Q_1^2,Q_2^2,Q_1\cdot Q_2)\\
+ \sum_{\gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4} & c(-\tfrac{(sQ_1-qQ_2)^2}{2})c(-\tfrac{(pQ_2-rQ_1)^2}{2})\Big[-\frac{\delta(\,\mathrm{tr}\,(\left(\begin{smallmatrix}0 & 1/2\\ 1/2 & 0\end{smallmatrix}\right)\gamma^{\intercal}\Omega_2\gamma))}{4\pi}\\
+ \tfrac{(sQ_1-qQ_2)\cdot(pQ_2-rQ_1)}{2} & \big(\mathrm{sign}\big((sQ_1-qQ_2)\cdot(pQ_2-rQ_1)\big)-\mathrm{sign}\big(\,\mathrm{tr}\,(\left(\begin{smallmatrix}0 & 1/2\\ 1/2 & 0\end{smallmatrix}\right)\gamma^{\intercal}\Omega_2\gamma)\big)\big)\Big]
\end{aligned}
\tag{5.25}
$$

where $\gamma = \left(\begin{smallmatrix}p & q\\ r & s\end{smallmatrix}\right)$ and $\mathrm{Dih}_4 \equiv \left\langle \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)\right\rangle$ is the dihedral group of order 8, which stabilizes (up to sign) the matrix $\left(\begin{smallmatrix}0 & 1/2\\ 1/2 & 0\end{smallmatrix}\right)$, or equivalently the locus $v_2 = 0$. As explained in Appendix A.6, this formula holds only when $|\Omega_2| > 1/4$, such that the contour $\mathcal{C} = [0,1]^3 + \mathrm{i}\Omega_2$ avoids the poles of $1/\Phi_{10}$ for generic values of Ω_2. Inserting (5.25) in (5.24), we find the following contributions,

1. The contributions from $(\widetilde{Q}_1,\widetilde{Q}_2) = (0,0)$ produces power-like terms in R^2, from the delta function contribution in (5.25), even though $C^F(0,0,0) = 0$,

$$
\begin{aligned}
G^{(p,q),\,2\mathrm{Ab},0}_{\alpha\beta,\gamma\delta} &= -R^{2q-12}\frac{3c(0)^2}{64\pi^3}\mathcal{E}^\star(\tfrac{8-q}{2},S)^2\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\,,\\
G^{(p,q),\,2\mathrm{Ab},0}_{\alpha\beta,\rho\sigma} &= -R^{2q-12}\frac{c(0)^2}{32\pi^3}\mathcal{E}^\star(\tfrac{8-q}{2},S)\Big[\tfrac{8-q}{2}\delta_{\rho\sigma}-2\mathcal{D}_{\rho\sigma}\Big]\mathcal{E}^\star(\tfrac{8-q}{2},S)\,\delta_{\alpha\beta}\,,\\
G^{(p,q),\,2\mathrm{Ab},0}_{\mu\nu,\rho\sigma} &= -R^{2q-12}\frac{3c(0)^2}{64\pi^3}\Big[\tfrac{8-q}{2}\delta_{\langle\mu\nu,}-2\mathcal{D}_{\langle\mu\nu,}\Big]\mathcal{E}^\star(\tfrac{8-q}{2},S)\Big[\tfrac{8-q}{2}\delta_{\rho\sigma\rangle}-2\mathcal{D}_{\rho\sigma\rangle}\Big]\mathcal{E}^\star(\tfrac{8-q}{2},S)\,,
\end{aligned}
\tag{5.26}
$$

   Here, the non-holomorphic Eisenstein series $\mathcal{E}^\star(s,S)$ and traceless differential operator $\mathcal{D}_{\mu\nu}$ are defined in (5.11) and (5.12). It is worth noting that in the limit $S_2\to\infty$, the constant term proportional to $\xi(q-6)\,S_2^{\frac{q-6}{2}}$ in the Eisenstein series $\mathcal{E}^\star(\frac{8-q}{2},S)$ reproduces the missing constant term in (4.20). Thus, while this term is missed by the unfolding procedure in the degeneration $(p,q)\to(p-1,q-1)$, it is correctly captured by the unfolding procedure in the degeneration $(p,q)\to(p-2,q-2)$.

2. Contributions of non-zero vectors $(\widetilde{Q}_1,\widetilde{Q}_2)\in\Lambda^{\oplus 2}_{p-2,q-2}$ lead to exponentially suppressed contributions. For the finite term $C^F(Q_1^2,Q_2^2,Q_1\cdot Q_2)$ in (5.25), and for the simplest tensorial representation, the unfolded integral leads to

$$
6R^8\delta_{\langle\mu\nu,}\delta_{\rho\sigma\rangle}\sum_{\substack{(\widetilde{Q}_1,\widetilde{Q}_2)\in\Lambda^{\oplus 2}_{p-2,q-2}\\ A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})}}|A|^2\,e^{2\pi\mathrm{i}a^{iI}A_{ij}Q_I^j}\,C^F(Q_1^2,Q_2^2,Q_1\cdot Q_2)\Big(\frac{R^2|A|}{2|\widetilde{Q}_{1R}\wedge\widetilde{Q}_{2R}|}\Big)^{\frac{q-9}{2}}\widetilde{B}_{\frac{q-9}{2}}(Z)\,,
\tag{5.27}
$$

   where

$$
Z = \frac{2R^2}{S_2}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}A\begin{pmatrix}\widetilde{Q}_{1R}^2 & \widetilde{Q}_{1R}\cdot\widetilde{Q}_{2R}\\ \widetilde{Q}_{1R}\cdot\widetilde{Q}_{2R} & \widetilde{Q}_{2R}^2\end{pmatrix}A^{\intercal}\,,
\tag{5.28}
$$

$|Q\wedge P|^2 = \det\begin{pmatrix} Q^2 & Q\cdot P \\ Q\cdot P & P^2\end{pmatrix}$ and $\widetilde{B}_\delta(Z)$ is the matrix-variate Bessel function [67], defined by

$$\widetilde{B}_s(UV) = \frac{1}{2}\left(\frac{|U|}{|V|}\right)^{-s/2} \int_{\mathcal{P}_2} \frac{\mathrm{d}^3\Omega_2}{|\Omega_2|^{\frac{3}{2}-s}}\, e^{-\pi\,\mathrm{tr}\,(\Omega_2^{-1}U+\Omega_2 V)} \tag{5.29}$$

Note that $\widetilde{B}_\delta(Z)$ depends on Z only through its trace and determinant. In the limit $R\to\infty$, or large $|Z|=|UV|$, the integral over Ω_2 is dominated by a saddle point where $\Omega_2^\star V\Omega_2^\star = U$; using the identity $\mathrm{Tr}(UV)\,U - UVU = |U|\,|V|\,V^{-1}$ valid for 2×2 matrices, this is given by

$$\Omega_2^\star = \frac{U+\sqrt{|UV|}\,V^{-1}}{\sqrt{\mathrm{tr}\,(UV)+2\sqrt{|UV|}}}\,. \tag{5.30}$$

For the matrices $U=\frac{R^2}{S_2}A^\intercal\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}A$, $V=2\begin{pmatrix}\widetilde{Q}_{1R}^2 & \widetilde{Q}_{1R}\cdot\widetilde{Q}_{2R}\\ \widetilde{Q}_{1R}\cdot\widetilde{Q}_{2R} & \widetilde{Q}_{2R}^2\end{pmatrix}$, given by (5.24), we obtain

$$\Omega_2^\star = \frac{R}{\mathcal{M}(\Gamma)}A^\intercal\left(\frac{1}{\sqrt{S_2}}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}+\frac{1}{|Q_R\wedge P_R|}\begin{pmatrix}P_R^2 & -Q_R\cdot P_R\\ -Q_R\cdot P_R & Q_R^2\end{pmatrix}\right)A\,, \tag{5.31}$$

where $\mathcal{M}(\Gamma)$ is the mass (2.5) of a 1/4-BPS state with charge $\Gamma=(Q,P)=(\widetilde{Q}_1,\widetilde{Q}_2)A^\intercal$, and $|Q_R\wedge P_R| = \sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}$.

For the contributions on the last line of (5.25), the integral over Ω_2 no longer evaluates to a matrix-variate Bessel integral, since these contributions depend on Ω_2, being discontinuous across the walls where $\mathrm{tr}\left(\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}\gamma^\intercal\Omega_2\gamma\right)$ changes sign. However, as long as (5.31) does not sit on the walls, the integral over Ω_2 is still dominated by the same saddle point, with a prefactor obtained by replacing $C^F(Q_1^2,Q_2^2,Q_1\cdot Q_2)$ by $C(Q_1^2,Q_2^2,Q_1\cdot Q_2;\Omega_2^*)$. In appendix F, we estimate the error made by neglecting the variation of $C(Q_1^2,Q_2^2,Q_1\cdot Q_2;\Omega_2)$ at finite distance away from the saddle point, and find that they are of the order expected for multi-instanton corrections. For the remainder of this section, we ignore these corrections, and perform the above replacement in (5.27).

In order to write the result for more general polarizations, it will be useful to introduce

$$\begin{aligned}\widetilde{B}^{(0)}_{s,\mu\nu}(Z) &= \frac{\delta_{\mu\nu}}{4|Z|^{s/2}}\int \frac{\mathrm{d}^3\Omega_2}{|\Omega_2|^{\frac{3}{2}-s}} e^{-\pi\,\mathrm{tr}\,(\Omega_2^{-1}Z+\Omega_2)}\\ \widetilde{B}^{(1)}_{s,\mu\nu}(Z) &= \frac{1}{2|Z|^{s/2}}\int \frac{\mathrm{d}^3\Omega_2}{|\Omega_2|^{1-s}}(\Omega_2^{-1})_{\mu\nu}\, e^{-\pi\,\mathrm{tr}\,(\Omega_2^{-1}Z+\Omega_2)}\,,\end{aligned} \tag{5.32}$$

such that $\delta^{\mu\nu}\widetilde{B}^{(0)}_{s\,\mu\nu}(Z)=\widetilde{B}_s(Z)$ and $|Z|^{\frac{s}{2}}\widetilde{B}^{(1)}_{s\,\mu\nu}(Z)=\frac{1}{-\pi}\frac{\partial}{\partial Z^{\mu\nu}}\left[\sqrt{|Z|}^{s+\frac{1}{2}}\widetilde{B}_{s+\frac{1}{2}}(Z)\right]$.

Changing variable $\begin{pmatrix}Q\\ P\end{pmatrix}=A\begin{pmatrix}\widetilde{Q}_1\\ \widetilde{Q}_2\end{pmatrix}$, we therefore obtain the Fourier expansion with respect to

(a_1, a_2), with support on $\Gamma = (Q, P) \in \Lambda^{\oplus 2}_{p-2,q-2}$,

$$
\begin{aligned}
G^{(p,q),\,2\mathrm{Ab},\,\Gamma}_{\alpha\beta,\gamma\delta} &\sim 2R^{q-1}\bar{C}(Q,P;\Omega_2^\star)\sum_{l=0}^{2}\frac{P^{(l)\,\mu\nu}_{\alpha\beta,\gamma\delta}(\Gamma)}{R^l}\,\frac{\widetilde{B}^{(l \bmod 2)}_{\frac{q-5-l}{2},\mu\nu}\left[\frac{2R^2}{S_2}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}\begin{pmatrix}Q_R^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & P_R^2\end{pmatrix}\right]}{|2Q_R\wedge P_R|^{\frac{q-5-l}{2}}}\\
G^{(p,q),\,2\mathrm{Ab},\,\Gamma}_{\rho\beta,\gamma\delta} &\sim R^{q-1}\bar{C}(Q,P;\Omega_2^\star)\sum_{l=0}^{1}\frac{P^{(l)\,\mu\nu}_{\rho\beta,\gamma\delta}(\Gamma)}{R^l}\,\frac{\widetilde{B}^{(l+1 \bmod 2)}_{\frac{q-6-l}{2},\mu\nu}\left[\frac{2R^2}{S_2}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}\begin{pmatrix}Q_R^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & P_R^2\end{pmatrix}\right]}{|2Q_R\wedge P_R|^{\frac{q-6-l}{2}}}\\
&\ \vdots\\
G^{(p,q),\,2\mathrm{Ab},\,\Gamma}_{\mu\nu,\rho\sigma} &\sim 2R^{q-1}\bar{C}(Q,P;\Omega_2^\star)\,\frac{\delta_{\langle\mu\nu,}\delta_{\sigma\tau\rangle}}{4}\,\frac{\widetilde{B}_{\frac{q-9}{2}}\left[\frac{2R^2}{S_2}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}\begin{pmatrix}Q_R^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & P_R^2\end{pmatrix}\right]}{|2Q_R\wedge P_R|^{\frac{q-9}{2}}}
\end{aligned}
\tag{5.33}
$$

where the measure factor is given by, for $\Gamma = (Q, P)$

$$
\bar{C}(Q,P;\Omega_2^\star) = \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\Gamma\in\Lambda^{\oplus 2}_{p-2,q-2}}} |A|^{q-7}C\left[A^{-1}\begin{pmatrix}Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{pmatrix}A^{-\intercal};A^\intercal\Omega_2^\star A\right]\ . \tag{5.34}
$$

3. Contributions from the Dirac delta function and sign function in the first line of (5.25) also produce exponentially suppressed contributions to the same Fourier coefficient. These contributions are localized on the walls $\mathrm{tr}\left(\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}\gamma^\intercal\Omega_2\gamma\right)$ associated to the splittings $(Q, P) = (Q_1, P_1) + (Q_2, P_2)$. For the Dirac delta function terms the integral separates into the product of two Bessel functions, with arguments given by the masses $\mathcal{M}(Q_1, P_1)$ and $\mathcal{M}(Q_2, P_2)$ of the 1/2-BPS components, as shown in Appendix D. In Appendix C, we show that he summation measure for these contributions also factorizes into the two respective measures for 1/2-BPS instantons appearing in the genus-one integral (1.4), (1.6). The contributions from the sign functions are estimated in Appendix F.

Rank two non-abelian orbits These orbits consist of matrices $\begin{pmatrix}\mathbf{n}_1 & \mathbf{m}_1\\ \mathbf{n}_2 & \mathbf{m}_2\end{pmatrix}$ where $(\mathbf{n}_1, \mathbf{m}_1)$ and $(\mathbf{n}_2, \mathbf{m}_2)$ have non vanishing symplectic product $M_1 \equiv \mathbf{n}_1\cdot\mathbf{m}_2 - \mathbf{m}_1\cdot\mathbf{n}_2 \neq 0$ (in particular, they are non collinear). Unlike all other orbits considered previously, the contribution of such matrices depend on the scalar ψ corresponding to the top grade component in the decomposition (5.3) via a factor $e^{2i\pi M_1\psi}$, and therefore contribute to the non-Abelian Fourier coefficient. While the classification of the orbits of such matrices under $Sp(4,\mathbb{Z})$ is rather complicated, we show in Appendix G that these contributions can be deduced by a simple change of variables from the already known Fourier coefficients in the degeneration $(p, q) \to (p-1, q-1)$.

5.2 Extension to $\mathbb{Z}_N$ CHL orbifolds

The degeneration limit (5.1) of the modular integral (2.30) for $\mathbb{Z}_N$ CHL models with $N = 2, 3, 5, 7$ can be treated similarly by adapting the orbit method to the case where the integrand is invariant under the congruence subgroup $\Gamma_{2,0}(N) = \{\begin{pmatrix}A & B\\ C & D\end{pmatrix} \in Sp(4,\mathbb{Z}),\ C = 0 \bmod N\}$. In (1.7), Φ_{k-2} is the cusp form of $\Gamma_{2,0}(N)$ of weight $k = \frac{24}{N+1}$ defined in (A.33), and $\Gamma^{(2)}_{\Lambda_{p,q}}[P_{ab,cd}]$

is the genus-two partition function with insertion of $P_{ab,cd}$ for a lattice

$$\Lambda_{p,q} = \Lambda_{p-2,q-2} \oplus I\!I_{1,1} \oplus I\!I_{1,1}[N] \ , \tag{5.35}$$

where $\Lambda_{p-2,q-2}$ is a lattice of level N with signature $(p-2, q-2)$. The lattice $I\!I_{1,1} \oplus I\!I_{1,1}[N]$ is obtained from the usual unimodular lattice $I\!I_{2,2}$ by restricting the windings and momenta to $\begin{pmatrix} \mathbf{n}_1 & \mathbf{m}_1 \\ \mathbf{n}_2 & \mathbf{m}_2 \end{pmatrix} = \begin{pmatrix} n_{11} & n_{12} & m_{11} & m_{12} \\ n_{21} & n_{22} & m_{21} & m_{22} \end{pmatrix} \in \begin{pmatrix} \mathbb{Z}^2 & \mathbb{Z}^2 \\ (N\mathbb{Z})^2 & \mathbb{Z}^2 \end{pmatrix}$, hence breaking the automorphism group $O(2,2,\mathbb{Z})$ to $\sigma_{S\leftrightarrow T} \ltimes [\Gamma_0(N) \times \Gamma_0(N)]$, exactly as in [22]. After Poisson resummation on $\mathbf{m}_1, \mathbf{m}_2$, Eq. (4.7) continues to hold, except for the fact that $\mathbf{n}_2$ are restricted to run over $(N\mathbb{Z})^2$. The sum over $A = \begin{pmatrix} \mathbf{n}_1 & \mathbf{m}_1 \\ \mathbf{n}_2 & \mathbf{m}_2 \end{pmatrix}$ can then be decomposed into orbits of $\Gamma_{2,0}(N)$:

Trivial orbit The term $\begin{pmatrix} \mathbf{n}_1 & \mathbf{m}_1 \\ \mathbf{n}_2 & \mathbf{m}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ produces the same modular integral, up to a factor of R^4,

$$G^{(p,q),0}_{\alpha\beta,\gamma\delta} = R^4\, G^{(p-2,q-2)}_{\alpha\beta,\gamma\delta} \ , \tag{5.36}$$

where $G^{(p-1,q-1)}_{\alpha\beta,\gamma\delta}$ is the integral (4.1) for the lattice $\Lambda_{p-2,q-2}$ defined by (5.35).

Rank-one orbits Matrices A of rank one fall into two different classes of orbits under $\Gamma_{2,0}(N)$. Let us first consider the case where $(\mathbf{n}_2, \mathbf{m}_2) \neq (0,0)$ and denote $(\mathbf{n}_2, \mathbf{m}_2) = p(\mathbf{n}'_2, \mathbf{m}'_2)$ with $p = \gcd(\mathbf{n}_2, \mathbf{m}_2)$:

1. Matrices with $\mathbf{n}'_2 = 0 \bmod N$, as they are required to be rank one, can be decomposed as

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{m}_1 \\ \mathbf{n}_2 & \mathbf{m}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & j \\ 0 & 0 & 0 & p \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{5.37}$$

   with $(j,p) \in \mathbb{Z}^2 \smallsetminus \{(0,0)\}$, $p \neq 0$, and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in (\Gamma_0(N) \ltimes H_{2,1}(\mathbb{Z}))\backslash\Gamma_{2,0}(N)$, with $\mathbb{Z}_2 \times \Gamma_0(N) \ltimes H_{2,1}(N) \subset \Gamma_1^J$. For this class of orbits, one can thus unfold directly the domain $\Gamma_{2,0}(N)\backslash\mathcal{H}_2$ into $(\Gamma_0(N) \ltimes H_{2,1}(\mathbb{Z}))\backslash\mathcal{H}_2 = \mathbb{R}^+_t \times (\Gamma_0(N)\backslash\mathcal{H}_1)_\rho \times \big((\mathbb{R}/\mathbb{Z})^3/\mathbb{Z}_2\big)_{u_1,u_2,\sigma_1}$ (for further details, see (4.48));

2. Matrices with $\mathbf{n}'_2 \neq 0 \bmod N$ can be decomposed as

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{m}_1 \\ \mathbf{n}_2 & \mathbf{m}_2 \end{pmatrix} = \begin{pmatrix} 0 & j & 0 & 0 \\ 0 & p & 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{5.38}$$

   with $(j,p) \in \mathbb{Z} \oplus N\mathbb{Z} \smallsetminus \{(0,0)\}$, $p \neq 0$, since $\mathbf{n}_2 = 0 \bmod N$, and where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_\rho S_\sigma(\Gamma^0(N) \ltimes H^{(2)}_{2,1,N}(\mathbb{Z}))(S_\rho S_\sigma)^{-1}\backslash\Gamma_{2,0}(N)$, recalling the definition

$$H^{(2)}_{2,1,N}(\mathbb{Z}) = \{(\kappa,\lambda,\mu) \in H_{2,1}(\mathbb{Z}),\ \kappa = \mu = 0 \bmod N\} \ , \tag{5.39}$$

   and where S_σ denotes the inversion over σ. One can then unfold the fundamental domain $\Gamma_{2,0}(N)\backslash\mathcal{H}_2$ into $S_\rho S_\sigma\,(\Gamma^0(N) \ltimes H^{(2)}_{2,1,N}(\mathbb{Z}))\,(S_\rho S_\sigma)^{-1}\backslash\mathcal{H}_2$, and change variable $\Omega \to (S_\rho S_\sigma)\cdot\Omega = -\Omega^{-1}$ as in the weak coupling case (4.53) to recover the integration domain $(\Gamma^0(N) \ltimes H^{(2)}_{2,1,N}(\mathbb{Z}))\backslash\mathcal{H}_2 = \mathbb{R}^+_t \times (\Gamma^0(N)\backslash\mathcal{H}_1)_\rho \times (\mathbb{R}/\mathbb{Z})_{u_2} \times (\mathbb{R}/N\mathbb{Z})^2_{u_1,\sigma_1}$. Under this change of variable, the level-N weight-$(k-2)$ cusp form transforms as in (4.51), while the partition function for the sublattice $\Lambda_{p-2,q-2}$ transforms as

$$\Gamma^{(2)}_{\tilde{\Lambda}_{p-2,q-2}}[P_{\alpha\beta,\gamma\delta}](-\Omega^{-1}) = \upsilon^2 N^{-k-2}(-\mathrm{i})^{2k}|\Omega|^{k-2}\Gamma^{(2)}_{\tilde{\Lambda}^*_{p-2,q-2}}[P_{\alpha\beta,\gamma\delta}](\Omega) \ , \tag{5.40}$$

   where we denoted $\upsilon^2 N^{-k-2} = \left|\Lambda^*_{p-2,q-2}/\Lambda_{p-2,q-2}\right|^{-1}$ (Note that $\upsilon^2 = N^{2-2\delta_{q,8}}$ for $q \le 8$ in the cases of interest).

The remaining contributions A with $(\mathbf{n}_2,\mathbf{m}_2)=(0,0)$ can be split in the two classes of orbits above. Given $(\mathbf{n}_1,\mathbf{m}_1)=j(\mathbf{n}_1',\mathbf{m}_1')$, where $j=\gcd(\mathbf{n}_1,\mathbf{m}_1)$ and $j\in\mathbb{Z}$, terms with $\mathbf{n}_1'=0\bmod N$ correspond to cases $(j,p)=(j,0)$ in the first class above, while terms with $\mathbf{n}_1'\neq 0\bmod N$ correspond to $(j,p)=(j,0)$ in the second class above.

For the function $G^{(p,q),1}_{ab,cd}$, changing the y variables as before $(y'_{1\mu},y'_{2\mu},y'_{1\alpha},y'_{2\alpha})=(y_{1\mu},y_{1\mu}u_1-y_{2\mu},y_{1\alpha},y_{1\alpha}u_2-y_{2\alpha})$, the sum of the two classes of orbits then reads (similarly to (4.53))

$$
\begin{aligned}
G^{(p,q),1}_{ab,cd} = R^4\int_{\mathbb{R}^+}\frac{\mathrm{d}t}{t^3}\int_{(\mathbb{R}/\mathbb{Z})^3}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}\sigma_1\int_{\Gamma_0(N)\backslash\mathcal{H}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\mathcal{P}_{ab,cd}(\frac{\partial}{\partial y'})}{\Phi_{k-2}(\Omega)}\\
\times\sideset{}{'}\sum_{(j,p)\in\mathbb{Z}^2}e^{-\frac{\pi R^2}{S_2t}|j+pS|^2}\Gamma^{(2)}_{\Lambda_{p-2,q-2}}\Big[e^{2\pi\mathrm{i}\widetilde{Q}_{2I}(ja_1^I+pa_2^I)}\mathcal{Y}(y')\Big]\\
+R^4\int_{\mathbb{R}^+}\frac{\mathrm{d}t}{t^3}\int_{(\mathbb{R}/N\mathbb{Z})^2}\mathrm{d}u_1\mathrm{d}\sigma_1\int_{\mathbb{R}/\mathbb{Z}}\mathrm{d}u_2\int_{\Gamma^0(N)\backslash\mathcal{H}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\mathcal{P}_{ab,cd}(\frac{\partial}{\partial y'})}{\Phi_{k-2}(\Omega/N)}\\
\times\frac{v^2}{N^4}\sideset{}{'}\sum_{\substack{(j,p)\in\mathbb{Z}^2\\ p=0\bmod N}}e^{-\frac{\pi R^2}{S_2t}|j+pS|^2}\Gamma^{(2)}_{\Lambda^*_{p-2,q-2}}\Big[e^{2\pi\mathrm{i}\widetilde{Q}_{2I}(ja_1^I+pa_2^I)}\mathcal{Y}(y')\Big]\,,
\end{aligned}
\tag{5.41}
$$

where

$$
\mathcal{Y}(y')=e^{2\pi\mathrm{i}\left(\frac{R}{\mathrm{i}\sqrt{2}}\frac{m_iv^i_\mu y_2'^{\,\mu}}{t}+y'_{1\,\alpha}(Q_{1L}{}^\alpha+u_2Q_{2L}{}^\alpha)-y'_{2\,\alpha}Q_{2L}{}^\alpha+\frac{1}{4\mathrm{i}\rho_2}y'_{1\,\alpha}y_1'^{\,\alpha}+\frac{1}{4\mathrm{i}t}y'_{2\,\alpha}y_2'^{\,\alpha}\right)}\,,
\tag{5.42}
$$

with $m_iv^i_\mu=\frac{1}{\sqrt{S_2}}(j+pS_1,pS_2)$. The contributions with $\widetilde{Q}_2^2=0$, after integration over u_1,u_2 (4.55), can be brought back to regular integral over $\Gamma_0(N)\backslash\mathcal{H}_1$ by changing variable $\rho\to-1/\rho$. Similarly to (4.56), the transformation property of the genus-one partition function and the level-N cusp form allows to obtain[12]

$$
\begin{aligned}
G^{(p,q),1,Q^2=0}_{ab,cd}=R^4\int_0^\infty\frac{\mathrm{d}t}{t}t^{\frac{q-6}{2}}\sum_{\substack{\widetilde{Q}_2\in\tilde{\Lambda}_{p-2,q-2}\\ \widetilde{Q}_2^2=0}}e^{-2\pi t\widetilde{Q}^2_{2R}}\frac{c_k(0)}{12(N-1)}\\
\times\Bigg\{\sideset{}{'}\sum_{(j,p)\in\mathbb{Z}^2}e^{-\frac{\pi R^2}{S_2t}|j+pS|^2}\int_{\Gamma_0(N)\backslash\mathcal{H}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{N^2\widehat{E}_2(N\rho)-\widehat{E}_2(\rho)}{\Delta_k}\\
+vN\sideset{}{'}\sum_{\substack{(j,p)\in\mathbb{Z}^2\\ p=0\bmod N}}e^{-\frac{\pi R^2}{S_2t}|j+pS|^2}\int_{\Gamma_0(N)\backslash\mathcal{H}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\widehat{E}_2(\rho)-\widehat{E}_2(N\rho)}{\Delta_k(\rho)}\Bigg\}\\
\times e^{2\pi\mathrm{i}\widetilde{Q}_{2I}(ja_1^I+pa_2^I)}\Gamma_{\Lambda_{p-1,q-1}}\Big[\mathcal{P}_{ab,cd}(\tfrac{\partial}{\partial y'})\mathcal{Y}(y')\Big]
\end{aligned}
\tag{5.43}
$$

[12] Recall that $\Delta_k(-1/N\rho)=N^{\frac{k}{2}}(-i\rho)^k\Delta_k(\rho)$, $\Gamma_{\Lambda^*_{p-2,q-2}}[P_{ab}](-1/\rho)=v^{-1}N^{\frac{k}{2}+1}(-i)^k\rho^{k-2}\Gamma_{\Lambda_{p-2,q-2}}[P_{ab}](\rho)$

The zero mode contribution, $\widetilde{Q}_2 = 0$, lead to power-like terms

$$\begin{aligned}
G^{(p,q),1,0}_{\alpha\beta,\gamma\delta} &= R^{q-4}\,\frac{c_k(0)}{16\pi(N-1)}\Big[\delta_{\langle\alpha\beta,}G^{(p-2,q-2)}_{\gamma\delta\rangle}\big(\mathcal{E}^\star(\tfrac{8-q}{2},S)-\upsilon N^{\frac{q-6}{2}}\mathcal{E}^\star(\tfrac{8-q}{2},NS)\big)\\
&\qquad -\delta_{\langle\alpha\beta,}{}^{\varsigma}G^{(p-2,q-2)}_{\gamma\delta\rangle}\big(N\mathcal{E}^\star(\tfrac{8-q}{2},S)-\upsilon N^{\frac{q-8}{2}}\mathcal{E}^\star(\tfrac{8-q}{2},NS)\big)\Big]\\
G^{(p,q),1,0}_{\alpha\beta,\mu\nu} &= R^{q-4}\,\frac{c_k(0)}{48\pi(N-1)}\Big[\tfrac{8-q}{2}\delta_{\mu\nu}-2\mathcal{D}_{\mu\nu}\Big]\\
&\quad\times\Big[G^{(p-2,q-2)}_{\gamma\delta}\big(\mathcal{E}^\star(\tfrac{8-q}{2},S)-\upsilon N^{\frac{q-6}{2}}\mathcal{E}^\star(\tfrac{8-q}{2},NS)\big)\\
&\qquad -{}^{\varsigma}G^{(p-2,q-2)}_{\gamma\delta}\big(N\mathcal{E}^\star(\tfrac{8-q}{2},S)-\upsilon N^{\frac{q-8}{2}}\mathcal{E}^\star(\tfrac{8-q}{2},NS)\big)\Big]
\end{aligned}\tag{5.44}$$

where we use the genus-one modular integral $G^{(p,q)}_{ab}(\varphi)$ (B.11), with integrand invariant under the Hecke congruence subgroup $\Gamma_0(N)$, as well as ${}^{\varsigma}G^{(p,q)}_{ab}$ (4.57) (Note that the cases of interest satisfy ${}^{\varsigma}G^{(2k-2,6)}_{ab}(\varphi) = G^{(2k-2,6)}_{ab}(\varphi)$, ${}^{\varsigma}G^{(2k-4,4)}_{ab}(\varphi) = G^{(2k-4,4)}_{ab}(\varphi)$).

The terms with non-zero vectors Q lead to exponentially suppressed contributions of the same form as the Fourier modes of null vectors (5.15), non-null vectors (5.17), and the polar contribution (5.20) respectively, with the following changes:

1. In the case of the finite part of $1/\Phi_k(\Omega)$, for null Fourier vectors $Q^2 = 0$, the CHL equivalent of $G^{(p-2,q-2)}_{F,\alpha\beta,0}$ is

$$\begin{aligned}
G^{(p-2,q-2)}_{F\,\alpha\beta,0}(\Gamma_i,S) &= \frac{c_k(0)}{12(N-1)}\Big[\frac{|j'+p'S|}{\sqrt{S_2}}\Big]^{q-8}\\
&\times\Bigg[\Bigg(\sum_{\substack{d\geq 1\\ \Gamma/d\in\Lambda^{\oplus 2}_{p-2,q-2}}} d^{q-8}-\sum_{\substack{d\geq 1\\ \Gamma/d\in\Lambda^*_{p-2,q-2}\oplus N\Lambda^*_{p-2,q-2}}}\upsilon N d^{q-8}\Bigg)G^{(p-2,q-2)\perp}_{F,\alpha\beta,0}(\tfrac{\hat Q}{d},\varphi)\\
&+\Bigg(\sum_{\substack{d\geq 1\\ \Gamma/d\in\Lambda^*_{p-2,q-2}\oplus N\Lambda^*_{p-2,q-2}}}\upsilon d^{q-8}-\sum_{\substack{d\geq 1\\ \Gamma/d\in\Lambda^{\oplus 2}_{p-2,q-2}}} N d^{q-8}\Bigg){}^{\varsigma}G^{(p-2,q-2)\perp}_{F,\alpha\beta,0}(\tfrac{\hat Q}{d},\varphi)\Bigg]\,,
\end{aligned}\tag{5.45}$$

where we defined the coprimes (j',p') such that $\Gamma = (j',p')\hat\Gamma$.

2. For non-null Fourier vectors, $Q^2 \neq 0$, the finite part of $1/\Phi_{k-2}(\Omega)$ contains two terms

$$\begin{aligned}
&G^{(p-2,q-2)}_{\alpha\beta,-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2d^2}}(\Gamma,\varphi) = \big(M^{ij}\Gamma_i\cdot\Gamma_j\big)^{\frac{q-8}{2}}\\
&\Bigg(\sum_{\substack{d\geq 1\\ \Gamma/d\in\Lambda^{\oplus 2}_{p-2,q-2}}} c_k\Big(-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2d^2}\Big)\Big(\frac{d^2}{\gcd(\Gamma_i\cdot\Gamma_j)}\Big)^{\frac{q-8}{2}} G^{(p-2,q-2)\perp}_{\alpha\beta,-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2d^2}}(\tfrac{\hat Q}{d},\varphi)\\
&+\sum_{\substack{d\geq 1\\ \Gamma/d\in\Lambda^*_{p-2,q-2}\oplus N\Lambda^*_{p-2,q-2}}}\upsilon\, c_k\Big(-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2Nd^2}\Big)\Big(\frac{Nd^2}{\gcd(\Gamma_i\cdot\Gamma_j)}\Big)^{\frac{q-8}{2}}{}^{\varsigma}G^{(p-2,q-2)\perp}_{\alpha\beta,-\frac{\gcd(\Gamma_i\cdot\Gamma_j)}{2Nd^2}}(\tfrac{\hat Q}{Nd},\varphi)\Bigg)\,.
\end{aligned}\tag{5.46}$$

with

$$ {}^{\varsigma}G^{(p-2,q-2)\perp}_{\alpha\beta,m}\big(\tfrac{\Gamma}{Nd},\varphi\big) = \int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{\mathrm{d}^2\rho}{\rho_2^2}\frac{N\widehat{h}_{m,l}(N\rho)}{\Delta_k(\rho)}\Gamma^{m,l}_{\alpha\beta}\big(\tfrac{\Gamma}{Nd}\big)\,, \tag{5.47}$$

and $\Gamma^{m,l}_{ab}(Q)$ the vector-valued partition function defined in (4.29).

Rank two abelian orbits Matrices $\big(\begin{smallmatrix}\mathbf{n}_1 & \mathbf{m}_1\\ \mathbf{n}_2 & \mathbf{m}_2\end{smallmatrix}\big) = \big(\begin{smallmatrix}n_{11} & n_{12} & m_{11} & m_{12}\\ n_{21} & n_{22} & m_{21} & m_{22}\end{smallmatrix}\big)$ with vanishing symplectic product $\mathbf{n}_1\cdot\mathbf{m}_2-\mathbf{m}_1\cdot\mathbf{n}_2=0$ but $(\mathbf{n}_i,\mathbf{m}_i)\neq(0,0)$ and $(\mathbf{n}_1,\mathbf{m}_1)$ and $(\mathbf{n}_2,\mathbf{m}_2)$ not aligned, fall into four different classes of orbits. Consider $k_1=\gcd(\mathbf{n}_1,\mathbf{m}_1)$ and $k_2=\gcd(\mathbf{n}_2,\mathbf{m}_2)$, the four classes depend on whether $\mathbf{n}_1/k_1$ and $\mathbf{n}_2/k_2$ are congruent to $0 \bmod N$ or not.

1. When $\mathbf{n}_1/k_1$ and $\mathbf{n}_2/k_2 = 0 \bmod N$, one can rotate the element as $\big(\begin{smallmatrix}\mathbf{n}_1 & \mathbf{m}_1\\ \mathbf{n}_2 & \mathbf{m}_2\end{smallmatrix}\big) = \big(\begin{smallmatrix}0 & \mathbf{p}_1\\ 0 & \mathbf{p}_2\end{smallmatrix}\big)\big(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\big)$, with $(\mathbf{p}_1,\mathbf{p}_2)\in M_2(\mathbb{Z})\smallsetminus\{0\}$ and $\big(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\big)\in\Gamma_{2,\infty}\backslash\Gamma_{2,0}(N)$ (A, C are not independent and the fourth winding entry, say n_{22}, vanishes because of the symplectic contraint). The representative is stabilized by $\Gamma_{2,\infty}=GL(2,\mathbb{Z})\times T^3$, and one can restrict the sum over matrices $A=(\mathbf{j},\mathbf{p})\in M_2(\mathbb{Z})$ to $A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})$ and unfold the fundamental domain from $\Gamma_{2,0}(N)\backslash\mathcal{H}_2$ to $\Gamma_{2,\infty}\backslash\mathcal{H}_2=(\mathcal{P}_2)_{\Omega_2}\times(\mathbb{R}\backslash\mathbb{Z})^3_{\Omega_1}$, with $(Q,P)\in\Lambda_m\oplus\Lambda_m$.

2. The two cases $\mathbf{n}_1/k_1\neq 0 \bmod N$ but $\mathbf{n}_2/k_2 = 0 \bmod N$, and $\mathbf{n}_1/k_1 = 0 \bmod N$ but $\mathbf{n}_2/k_2 \neq 0 \bmod N$, should be considered together. Respectively, the charges can be rotated as $\big(\begin{smallmatrix}\mathbf{n}_1 & \mathbf{m}_1\\ \mathbf{n}_2 & \mathbf{m}_2\end{smallmatrix}\big) = \big(\begin{smallmatrix}k & 0 & 0 & j\\ 0 & 0 & 0 & p\end{smallmatrix}\big)\big(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\big)$, $0\le j<k$, $p\in\mathbb{Z}\smallsetminus\{0\}$, and $\big(\begin{smallmatrix}\mathbf{n}_1 & \mathbf{m}_1\\ \mathbf{n}_2 & \mathbf{m}_2\end{smallmatrix}\big) = \big(\begin{smallmatrix}0 & j & k & 0\\ 0 & p & 0 & 0\end{smallmatrix}\big)\big(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\big)$, $0\le j<Nk$, $p\in N\mathbb{Z}\smallsetminus\{0\}$, by construction of the lattice (5.35). $\big(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\big)\in S_\rho\Gamma^{(1)}_{2,\infty,N}S_\rho^{-1}\backslash\Gamma_{2,0}(N)$ and $\big(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\big)\in S_\sigma\Gamma^{(2)}_{2,\infty,N}S_\sigma^{-1}\backslash\Gamma_{2,0}(N)$ respectively, with

$$\begin{aligned}\Gamma^{(1)}_{2,\infty,N} &= \{\big(\begin{smallmatrix}\mathbb{1} & M\\ 0 & \mathbb{1}\end{smallmatrix}\big), M=\big(\begin{smallmatrix}Nq & r\\ r & s\end{smallmatrix}\big),(q,r,s)\in\mathbb{Z}^3\}\,,\\ \Gamma^{(2)}_{2,\infty,N} &= \{\big(\begin{smallmatrix}\mathbb{1} & M\\ 0 & \mathbb{1}\end{smallmatrix}\big), M=\big(\begin{smallmatrix}q & r\\ r & Ns\end{smallmatrix}\big),(q,r,s)\in\mathbb{Z}^3\}\,,\end{aligned} \tag{5.48}$$

and one can then unfold $\Gamma_{2,0}(N)\backslash\mathcal{H}_2$ to $S_\rho\Gamma^{(1)}_{2,\infty,N}S_\rho^{-1}\backslash\mathcal{H}_2$, $S_\sigma\Gamma^{(2)}_{2,\infty,N}S_\sigma^{-1}\backslash\mathcal{H}_2$, and change variable $\rho\to-1/\rho$, $\sigma\to-1/\sigma$, respectively. After exchanging ρ and σ in the second case[13], the two cases can be assembled together to form the two orbits of the decomposition of

$$M_{2,0}(N)=\{\big(\begin{smallmatrix}p & q\\ r & s\end{smallmatrix}\big)\in M_2(\mathbb{Z}), r=0 \bmod N\}\,, \tag{5.49}$$

over

$$(\mathbb{Z}_2\ltimes\Gamma_0(N))=\{\big(\begin{smallmatrix}p & q\\ r & s\end{smallmatrix}\big)\in GL(2,\mathbb{Z}), r=0 \bmod N\}\,. \tag{5.50}$$

Explicitely,

$$\begin{aligned}M_{2,0}(N)/(\mathbb{Z}_2\ltimes\Gamma_0(N)) = \{&\big(\begin{smallmatrix}k & j\\ 0 & p\end{smallmatrix}\big)\,, 0\le j<k\,, p\in\mathbb{Z}\smallsetminus\{0\}\}\\ &\cup\{\big(\begin{smallmatrix}j & k\\ p & 0\end{smallmatrix}\big)\,, 0\le j<Nk\,, p\in N\mathbb{Z}\smallsetminus\{0\}\}\,.\end{aligned} \tag{5.51}$$

One thus obtains a single sum over matrices $A\in M_{2,0}(N)/(\mathbb{Z}_2\ltimes\Gamma_0(N))$, with a fundamental domain unfolded to $\Gamma^{(1)}_{2,\infty,N}\backslash\mathcal{H}_2=(\mathcal{P}_2)_{\Omega_2}\times(\mathbb{R}\backslash\mathbb{Z})^2_{\sigma_1,v_1}\times(\mathbb{R}\backslash N\mathbb{Z})_{\rho_1}$, with $(Q,P)\in\Lambda_m^*\oplus\Lambda_m$. Under this change of variable, the level-N weight-$(k-2)$ cusp form transforms as

$$\Phi_{k-2}(S_\rho\circ\Omega)=(i\sqrt{N})^{-k}\rho^{k-2}\tilde{\Phi}_{k-2}(\Omega)\,, \tag{5.52}$$

[13] This transformation belongs to $\Gamma_{2,0}(N)$

such that it satifies the splitting degeneration limit (A.44), while the genus-two partition function for the sublattice transforms as

$$\Gamma^{(2)}_{\Lambda_{p-2,q-2}}[P_{ab,cd}](S_\rho\circ\Omega)=\upsilon(i\sqrt{N})^{-k-2}\rho^{k-2}\Gamma^{(2)}_{\Lambda^*_{p-2,q-2}\oplus\Lambda_{p-2,q-2}}[P_{ab,cd}](\Omega)\,, \tag{5.53}$$

where $\upsilon=N^{k/2+1}|\Lambda^*_{p-2,q-2}/\Lambda_{p-2,q-2}|^{-1/2}$ (reducing to $\upsilon=N^{1-\delta_{q,8}}$ for $q\le 8$ in the cases of interest).

3. When $\mathbf{n}_1/k_1,\mathbf{n}_2/k_2\neq 0 \bmod N$, one can rotate the element as $\left(\begin{smallmatrix}\mathbf{n}_1 & \mathbf{m}_1\\ \mathbf{n}_2 & \mathbf{m}_2\end{smallmatrix}\right)=\left(\begin{smallmatrix}j_1 & j_2 & 0 & 0\\ p_1 & p_2 & 0 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$, with $\left(\begin{smallmatrix}j_1 & j_2\\ p_1 & p_2\end{smallmatrix}\right)\in M_{2,00}(N)\smallsetminus\{0\}$,

$$M_{2,00}(N)=\{\left(\begin{smallmatrix}p & q\\ r & s\end{smallmatrix}\right)\in M_2(\mathbb{Z}), r=s=0 \bmod N\} \tag{5.54}$$

by construction of the lattice (5.35), and $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in S_\rho S_\sigma\Gamma^{(3)}_{2,\infty,N}(S_\sigma S_\rho)^{-1}\backslash\Gamma_{2,0}(N)$, with

$$\Gamma^{(3)}_{2,\infty,N}=\{\left(\begin{smallmatrix}\gamma & 0\\ 0 & \gamma^{-\intercal}\end{smallmatrix}\right),\gamma\in GL(2,\mathbb{Z})\}\times\{\left(\begin{smallmatrix}\mathbb{1} & M\\ 0 & \mathbb{1}\end{smallmatrix}\right),M=\left(\begin{smallmatrix}q & r\\ r & s\end{smallmatrix}\right),(q,r,s)\in(N\mathbb{Z})^3\}\,. \tag{5.55}$$

One can then unfold $\Gamma_{2,0}(N)\backslash\mathcal{H}_2$ to $S_\rho S_\sigma\Gamma^{(3)}_{2,\infty,N}(S_\sigma S_\rho)^{-1}\backslash\mathcal{H}_2$, and change variable $\Omega_2\to -\Omega_2^{-1}$ to recover $\Gamma^{(3)}_{2,\infty,N}\backslash\mathcal{H}_2=(GL(2,\mathbb{Z})\backslash\mathcal{P}_2)_{\Omega_2}\times(\mathbb{R}\backslash N\mathbb{Z})^3_{\Omega_1}$. Finally, one can restrict the sum over matrices $A\in M_{2,00}(N)$, $\mathbf{p}=0 \bmod N$ to $A\in M_{2,00}(N)/GL(2,\mathbb{Z})$, in order to unfold $GL(2,\mathbb{Z})\backslash\mathcal{P}_2$ to $\mathcal{P}_2$, with $(Q,P)\in\Lambda^*_m\oplus N\Lambda^*_m$.

After unfolding and changing variables, the result for the simplest component $G^{(p,q),2\,\text{Ab}}_{\alpha\beta,\gamma\delta}$ reads

$$\begin{aligned}G^{(p,q),2\text{Ab}}_{\alpha\beta,\gamma\delta}&=2R^4\int_{\mathcal{P}_2}\frac{d^3\Omega_2}{|\Omega_2|^3}\int_{(\mathbb{R}/\mathbb{Z})^3}\frac{d^3\Omega_1}{\Phi_{k-2}(\Omega)}\sideset{}{'}\sum_{\substack{A\in\\ M_2(\mathbb{Z})/GL(2,\mathbb{Z})}}e^{-\pi\mathrm{Tr}\left[\frac{R^2}{S_2}\Omega_2^{-1}A^\intercal\left(\begin{smallmatrix}1 & S_1\\ S_1 & |S|^2\end{smallmatrix}\right)A\right]}\\&\qquad\times\Gamma^{(2)}_{\Lambda_{p-2,q-2}}[e^{2\pi i a^{iI}A_{ij}\widetilde{Q}^{jI}}P_{\alpha\beta,\gamma\delta}]\\&+2R^4\int_{\mathcal{P}_2}\frac{d^3\Omega_2}{|\Omega_2|^3}\int_{(\mathbb{R}/\mathbb{Z})^2\times(\mathbb{R}/N\mathbb{Z})}\frac{d^3\Omega_1}{\tilde{\Phi}_{k-2}(\Omega)}\frac{\upsilon}{N}\sideset{}{'}\sum_{\substack{A\in\\ M_{2,0}(N)/(\mathbb{Z}_2\ltimes\Gamma_0(N))}}e^{-\pi\mathrm{Tr}\left[\frac{R^2}{S_2}\Omega_2^{-1}A^\intercal\left(\begin{smallmatrix}1 & S_1\\ S_1 & |S|^2\end{smallmatrix}\right)A\right]}\\&\qquad\times\Gamma^{(2)}_{\Lambda^*_{p-2,q-2}\oplus\Lambda_{p-2,q-2}}[e^{2\pi i a^{iI}A_{ij}Q^{jI}}P_{\alpha\beta,\gamma\delta}]\\&+2R^4\int_{\mathcal{P}_2}\frac{d^3\Omega_2}{|\Omega_2|^3}\int_{(\mathbb{R}/N\mathbb{Z})^3}\frac{d^3\Omega_1}{\Phi_{k-2}(\Omega/N)}\frac{\upsilon^2}{N^4}\sideset{}{'}\sum_{\substack{A\in\\ M_{2,00}(N)/GL(2,\mathbb{Z})}}e^{-\pi\mathrm{Tr}\left[\frac{R^2}{S_2}\Omega_2^{-1}A^\intercal\left(\begin{smallmatrix}1 & S_1\\ S_1 & |S|^2\end{smallmatrix}\right)A\right]}\\&\qquad\times\Gamma^{(2)}_{\Lambda^*_{p-2,q-2}}[e^{2\pi i a^{iI}A_{ij}Q^{jI}}P_{\alpha\beta,\gamma\delta}]\,,\end{aligned} \tag{5.56}$$

where $\upsilon^2=N^{k+2}|\Lambda^*_{p-2,q-2}\backslash\Lambda_{p-2,q-2}|^{-1}$ (which reduces to $\upsilon^2=N^{2-2\delta_{q,8}}$ for $q\le 8$ in the cases of interest).

Integrating over Ω_1 selects the Fourier coefficient $C_{k-2}(m,n,l;\Omega_2)$ of $1/\Phi_{k-2}$, and the Fourier coefficient $\widetilde{C}_{k-2}(m,n,l;\Omega_2)$ of $1/\tilde{\Phi}_{k-2}$, with $\widetilde{Q}_1^2=-2m$, $\widetilde{Q}_2=-2n$, $\widetilde{Q}_1\cdot\widetilde{Q}_2=-l$. The first one is invariant under $GL(2,\mathbb{Z})\subset\Gamma_{2,\infty}$, defined in (5.23), and its Fourier coefficients can be written after separating the finite contribution C^F, independent of Ω_2, from the polar ones

$$\int_{[0,1[^3} \mathrm{d}^3\Omega_1 \frac{e^{\mathrm{i}\pi(Q_1,Q_2)\left(\begin{smallmatrix}\rho & v\\ v & \sigma\end{smallmatrix}\right)\left(\begin{smallmatrix}Q_1\\ Q_2\end{smallmatrix}\right)}}{\Phi_{k-2}(\Omega)} = C^F_{k-2}(Q_1^2,Q_2^2,Q_1\cdot Q_2)$$
$$+\sum_{\gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4} c_k\big(-\tfrac{(sQ_1-qQ_2)^2}{2}\big)c_k\big(-\tfrac{(pQ_2-rQ_1)^2}{2}\big)\Big[-\frac{\delta\big(\operatorname{tr}\big(\left(\begin{smallmatrix}0 & 1/2\\ 1/2 & 0\end{smallmatrix}\right)\gamma^\intercal\Omega_2\gamma\big)\big)}{4\pi}$$
$$+\tfrac{(sQ_1-qQ_2)\cdot(pQ_2-rQ_1)}{2}\Big(\operatorname{sign}\big((sQ_1-qQ_2)\cdot(pQ_2-rQ_1)\big)-\operatorname{sign}\big(\operatorname{tr}\big(\left(\begin{smallmatrix}0 & 1/2\\ 1/2 & 0\end{smallmatrix}\right)\gamma^\intercal\Omega_2\gamma\big)\big)\Big)\Big] \tag{5.57}$$

where $\gamma=\left(\begin{smallmatrix}p & q\\ r & s\end{smallmatrix}\right)$, and the finite contributions $C^F_{k-2}(Q_1^2,Q_2^2,Q_1\cdot Q_2)$ are also invariant under $\left(\begin{smallmatrix}\tilde Q_1\\ \tilde Q_2\end{smallmatrix}\right)\to\gamma^{-1}\left(\begin{smallmatrix}\tilde Q_1\\ \tilde Q_2\end{smallmatrix}\right)$. The contributions of $1/\tilde\Phi_{k-2}$ can be written similarly as

$$\frac{1}{N}\int_{[0,N[\times[0,1[^2} \mathrm{d}^3\Omega_1 \frac{e^{\mathrm{i}\pi(Q_1,Q_2)\left(\begin{smallmatrix}\rho & v\\ v & \sigma\end{smallmatrix}\right)\left(\begin{smallmatrix}Q_1\\ Q_2\end{smallmatrix}\right)}}{\tilde\Phi_{k-2}(\Omega)} = \widetilde{C}^F_{k-2}(Q_1^2,Q_2^2,Q_1\cdot Q_2)$$
$$+\sum_{\gamma\in\Gamma_0(N)/\mathbb{Z}_2} c_k\big(-\tfrac{N(sQ_1-qQ_2)^2}{2}\big)c_k\big(-\tfrac{(pQ_2-rQ_1)^2}{2}\big)\Big[-\frac{\delta\big(\operatorname{tr}\big(\left(\begin{smallmatrix}0 & 1/2\\ 1/2 & 0\end{smallmatrix}\right)\gamma^\intercal\Omega_2\gamma\big)\big)}{4\pi}\qquad ,$$
$$+\tfrac{(sQ_1-qQ_2)\cdot(pQ_2-rQ_1)}{2}\Big(\operatorname{sign}\big((sQ_1-qQ_2)\cdot(pQ_2-rQ_1)\big)-\operatorname{sign}\big(\operatorname{tr}\big(\left(\begin{smallmatrix}0 & 1/2\\ 1/2 & 0\end{smallmatrix}\right)\gamma^\intercal\Omega_2\gamma\big)\big)\Big)\Big] \tag{5.58}$$

where $\mathbb{Z}_2\ltimes\Gamma_0(N)$, the symmetry at the cusp, is equivalent to $GL(2,\mathbb{Z})\cap M_{2,0}(N)$, and the stabilizer of $\left(\begin{smallmatrix}0 & 1/2\\ 1/2 & 0\end{smallmatrix}\right)$ inside it is reduced to $\{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right),\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\}$, leading the sum over $\Gamma_0(N)/\mathbb{Z}_2$.

1. The contributions from $(\widetilde Q_1,\widetilde Q_2)=(0,0)$ come in two classes: the ones associated to the zero mode $C^F_{k-2}(0,0,0)=\frac{48N}{N^2-1}$ and $\widetilde C^F_{k-2}(0,0,0)=-\frac{48}{N^2-1}$ (see (A.49) and (A.50)) that were absent for $N=1$, and the ones coming from the delta function contribution in (5.57) and (5.58). The zero mode contribution is proportional to

$$\int_{\mathcal{P}_2}\frac{\mathrm{d}^3\Omega_2}{|\Omega_2|^{\frac{10-q}{2}}}\Bigg(N\sum'_{\substack{A\in\\ M_2(\mathbb{Z})/GL(2,\mathbb{Z})}} - v\sum'_{\substack{A\in\\ M_{2,0}(N)/(\mathbb{Z}_2\ltimes\Gamma_0(N))}} + v^2\sum'_{\substack{A\in\\ M_{2,00}(N)/GL(2,\mathbb{Z})}}\Bigg)e^{-\pi\mathrm{Tr}\left[R^2\Omega_2^{-1}A^\intercal MA\right]}$$
$$= R^{2q-14}\pi^{\frac{2q-13}{2}}\Gamma(\tfrac{7-q}{2})\Gamma(\tfrac{6-q}{2})\Bigg(N\sum'_{\substack{A\in\\ M_2(\mathbb{Z})/GL(2,\mathbb{Z})}} - v\sum'_{\substack{A\in\\ M_{2,0}(N)/(\mathbb{Z}_2\ltimes\Gamma_0(N))}} + v^2\sum'_{\substack{A\in\\ M_{2,00}(N)/GL(2,\mathbb{Z})}}\Bigg)\det A^{q-7}$$
$$= R^{2q-14}\xi(7-q)\xi(6-q)\big(N-v(1+N^{q-6})+v^2N^{q-7}\big)\ , \tag{5.59}$$

where the integral is a matrix-variate Gamma integral [67] and the sums reduce to zeta functions using explicit representatives as (5.51).[14]

[14] Alternatively, the integral can be reduced to a beta integral over $r\in[0,1]$ using the substitution $v=\sqrt{\rho\sigma}r$.

With the same computation as in the preceding section, one obtains

$$
\begin{aligned}
G^{(p,q),\,2\mathrm{Ab}}_{\alpha\beta,\gamma\delta} &= -R^{2q-12}\frac{3c_k(0)^2}{64\pi^3}\big(\mathcal{E}^\star(\tfrac{8-q}{2},S)+\upsilon N^{\frac{q-8}{2}}\mathcal{E}^\star(\tfrac{8-q}{2},NS)\big)^2\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\\
&\quad+\frac{18R^{2q-10}}{\pi^2}\xi(7-q)\xi(6-q)\frac{(N-\upsilon)(1-\upsilon N^{q-7})}{N^2-1}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\,,\\
G^{(p,q),\,2\mathrm{Ab}}_{\alpha\beta,\rho\sigma} &= -R^{2q-12}\frac{c_k(0)^2}{32\pi^3}\big(\mathcal{E}^\star(\tfrac{8-q}{2},S)+\upsilon N^{\frac{q-8}{2}}\mathcal{E}^\star(\tfrac{8-q}{2},NS)\big)\\
&\qquad\times\Big[\tfrac{8-q}{2}\delta_{\rho\sigma}-2\mathcal{D}_{\rho\sigma}\Big]\frac{1}{2}\big(\mathcal{E}^\star(\tfrac{8-q}{2},S)+\upsilon N^{\frac{q-8}{2}}\mathcal{E}^\star(\tfrac{8-q}{2},NS)\big)\,\delta_{\alpha\beta}\\
&\quad+\frac{6(7-q)R^{2q-10}}{\pi^2}\xi(7-q)\xi(6-q)\frac{(N-\upsilon)(1-\upsilon N^{q-7})}{N^2-1}\delta_{\alpha\beta}\delta_{\rho\sigma}\,,\\
G^{(p,q),\,2\mathrm{Ab}}_{\mu\nu,\rho\sigma} &= -R^{2q-12}\frac{3c_k(0)^2}{64\pi^3}\Big[\tfrac{8-q}{2}\delta_{\langle\mu\nu,}-2\mathcal{D}_{\langle\mu\nu,}\Big]\big(\mathcal{E}^\star(\tfrac{8-q}{2},S)+\upsilon N^{\frac{q-8}{2}}\mathcal{E}^\star(\tfrac{8-q}{2},NS)\big)\\
&\qquad\times\Big[\tfrac{8-q}{2}\delta_{\rho\sigma\rangle}-2\mathcal{D}_{\rho\sigma\rangle}\Big]\big(\mathcal{E}^\star(\tfrac{8-q}{2},S)+\upsilon N^{\frac{q-8}{2}}\mathcal{E}^\star(\tfrac{8-q}{2},NS)\big)\\
&\quad+\frac{9(6-q)(7-q)R^{2q-10}}{\pi^2}\xi(7-q)\xi(6-q)\frac{(N-\upsilon)(1-\upsilon N^{q-7})}{N^2-1}\delta_{\langle\mu\nu,}\delta_{\rho\sigma\rangle}\,.
\end{aligned}
\tag{5.60}
$$

Recall that $c_k(0)=\frac{24}{N+1}=k$ is the zero mode of $1/\Delta_k=\sum_m c_k(m)q^m$, and that $\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}=\frac{2}{3}(\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha(\gamma}\delta_{\delta)\beta})$. As in the maximal rank case (5.26), the leading constant term in

$$
\mathcal{E}^\star(\tfrac{8-q}{2},S)\sim\xi(q-6)\,S_2^{\frac{q-6}{2}}+\xi(8-q)\,S_2^{\frac{8-q}{2}}
\tag{5.61}
$$

reproduces the missing constant term in (4.60).

2. Contributions of non-zero vectors $(\widetilde{Q}_1,\widetilde{Q}_2)\in\Lambda^{\oplus 2}_{p-2,q-2}$ lead to the exponentially suppressed contributions written in (5.33). The measure of each Fourier mode will fall in three category, depending on the support of (Q,P) . The simplest one is for the most generic vector $Q\mathrel{\underline{\in}}\Lambda_m^*$, $P\mathrel{\underline{\in}}\Lambda_m$ – where we denote $X\mathrel{\underline{\in}}\Lambda$ the strict inclusion of the vector X in Λ, meaning that $X\in\Lambda$, $X\notin\Lambda[N]$ – for which only the first orbit in (5.51) of the second term in (5.24) contributes

$$
\upsilon\sum_{\substack{A=\left(\begin{smallmatrix}k & j\\ 0 & p\end{smallmatrix}\right),\ \substack{0\le j<k\\ p\neq 0}\\ A^{-1}\left(\begin{smallmatrix}Q\\ P\end{smallmatrix}\right)\in\Lambda_m^*\oplus\Lambda_m}}|A|^{q-7}\widetilde{C}_{k-2}\Big[A^{-1}\left(\begin{smallmatrix}-|Q|^2 & -Q\cdot P\\ -Q\cdot P & -|P|^2\end{smallmatrix}\right)A^{-\intercal};A^\intercal\Omega_2^\star A\Big]\,,
\tag{5.62}
$$

where the N factor comes from the width of the integration domain $(\mathbb{R}/N\mathbb{Z})$.

For less generic vectors $Q\mathrel{\underline{\in}}\Lambda_m^*$, $P\mathrel{\underline{\in}}N\Lambda_m^*$, one must add to (5.62) the second orbit of (5.51), allowing to rewrite the two as a sum over $M_{2,0}(N)/(\mathbb{Z}_2\ltimes\Gamma_0(N))$ defined in (5.51), as well as the contribution from the last term of (5.24). We obtain

$$
\begin{aligned}
&\upsilon\sum_{\substack{A\in M_{2,0}(N)/[\mathbb{Z}_2\ltimes\Gamma_0(N)]\\ A^{-1}\left(\begin{smallmatrix}Q\\ P\end{smallmatrix}\right)\in\Lambda_m^*\oplus\Lambda_m}}|A|^{q-7}\widetilde{C}_{k-2}\Big[A^{-1}\left(\begin{smallmatrix}-|Q|^2 & -Q\cdot P\\ -Q\cdot P & -|P|^2\end{smallmatrix}\right)A^{-\intercal};A^\intercal\Omega_2^\star A\Big]\\
&+\upsilon^2\sum_{\substack{A\in M_2(N)/GL(2,\mathbb{Z})\\ A^{-1}\left(\begin{smallmatrix}Q\\ P/N\end{smallmatrix}\right)\in\Lambda_m^*\oplus\Lambda_m^*}}N^{q-8}|A|^{q-7}C_{k-2}\Big[A^{-1}\left(\begin{smallmatrix}-N|Q|^2 & -Q\cdot P\\ -Q\cdot P & -|P|^2/N\end{smallmatrix}\right)A^{-\intercal};A^\intercal\Omega_2^\star A\Big]\,,
\end{aligned}
\tag{5.63}
$$

where in the second line, N factors come from the width of the integration domain $(\mathbb{R}/N\mathbb{Z})^3$, as well as the argument of $1/\Phi_{k-2}(N\Omega)$, and the magnetic vector is rescaled $P \to P/N$, allowing us to use $M_2(N)/GL(2,\mathbb{Z})$ instead of $M_{2,00}(N)/GL(2,\mathbb{Z})$ (5.54) for simplicity.

Finally, for vectors $Q \in \Lambda_m$, $P \in \Lambda_m$, one must add to (5.62) the contribution from the first term of (5.24). One thus obtain the full measure as

$$\begin{aligned}\bar{C}_{k-2}(Q,P,\Omega^\star) = &\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda_m\oplus\Lambda_m}} |A|^{q-7}C_{k-2}\Big[A^{-1}\begin{pmatrix}-|Q|^2 & -Q\cdot P\\ -Q\cdot P & -|P|^2\end{pmatrix}A^{-\intercal};A^\intercal\Omega_2^\star A\Big]\\ &+\upsilon \sum_{\substack{A\in M_{2,0}(N)/[\mathbb{Z}_2\ltimes\Gamma_0(N)]\\ A^{-1}\binom{Q}{P}\in\Lambda_m^*\oplus\Lambda_m}} |A|^{q-7}\widetilde{C}_{k-2}\Big[A^{-1}\begin{pmatrix}-|Q|^2 & -Q\cdot P\\ -Q\cdot P & -|P|^2\end{pmatrix}A^{-\intercal};A^\intercal\Omega_2^\star A\Big]\,,\\ &+\upsilon^2 \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P/N}\in\Lambda_m^*\oplus\Lambda_m^*}} N^{q-8}|A|^{q-7}C_{k-2}\Big[A^{-1}\begin{pmatrix}-N|Q|^2 & -Q\cdot P\\ -Q\cdot P & -|P|^2/N\end{pmatrix}A^{-\intercal};A^\intercal\Omega_2^\star A\Big]\,.\end{aligned} \tag{5.64}$$

Finally, there are also contributions from rank two non-abelian orbits where the two rows $(\mathbf{n}_1,\mathbf{m}_1)$ and $(\mathbf{n}_2,\mathbf{m}_2)$ have non vanishing symplectic product $\mathbf{n}_1\cdot\mathbf{m}_2-\mathbf{m}_1\cdot\mathbf{n}_2\neq 0$, but as mentioned in the previous subsection, it is more convenient to obtain them from the Fourier coefficients in the degeneration $(p,q)\to(p-1,q-1)$, as explained in Appendix G.

5.3 Large radius limit and BPS dyon counting

We now apply the results in §5.1 and 5.2 for $(p,q)=(2k,8)$ and $\Lambda_{p-2,q-2}=\Lambda_m$, to discuss the limit of the exact $\nabla^2(\nabla\phi)^4$ couplings in three-dimensional CHL orbifolds, in the limit where one circle inside T^7 (orthogonal to the circle involved in the orbifold action) decompactifies. We regularize the coupling coefficient by analytic coninuation of $q=8+2\epsilon$, and we substract the pole at $\epsilon=0$. We find that the conjectured exact $\nabla^2(\nabla\phi)^4$ coupling (1.7) has the large radius expansion

$$G^{(2k,8)}_{\alpha\beta,\gamma\delta} \;=\; G^{(0)}_{\alpha\beta,\gamma\delta}+G^{(1)}_{\alpha\beta,\gamma\delta}+G^{(2)}_{\alpha\beta,\gamma\delta}+G^{(\mathrm{TN})}_{\alpha\beta,\gamma\delta} \tag{5.65}$$

corresponding to the constant term, 1/2-BPS and 1/4-BPS Abelian Fourier modes and finally, the non-Abelian Fourier modes with non-zero Taub-NUT charge discussed in Appendix G.

5.3.1 Effective action in $D=4$

The constant term in (5.65) takes the form

$$G^{(0)}_{\alpha\beta,\gamma\delta}=R^4\,G^{(D=4)}_{\alpha\beta,\gamma\delta}+\frac{\zeta(3)}{8\pi}(k-12)R^6\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}+\mathcal{O}(e^{-2\pi R})\,, \tag{5.66}$$

The first term originates from orbits of rank 0 (5.36), rank-1 (5.44) and Abelian rank-2 (5.60), and combines all terms proportional to R^4 that survive in the decompactification limit. The second term comes from (5.60), and can be ascribed to the 2-loop sunset diagram shown in Figure 1 c), with Kaluza–Klein states running in the loops. Its coefficient vanishes in the maximal rank case. The exponentially suppressed contributions of order e^{-R} and e^{-R^2} are missed by the unfolding procedure, but they must be present because of the differential equation (2.26). We shall return to them in the next subsection.

If our Ansatz (1.7) for the exact $\nabla^2(\nabla\phi)^4$ couplings in $D=3$ is correct, the term proportional to R^4 in (5.66) must reproduce the exact $\nabla^2 F^4$ couplings in four dimensions, up to logarithmic corrections in R due to the mixing between local and non-local couplings in $D=4$. For the maximal rank case, we find

$$G^{(D=4)}_{\alpha\beta,\gamma\delta}(S,\varphi) = \widehat{G}^{(24,6)}_{\alpha\beta,\gamma\delta}(\varphi) - \frac{3}{4\pi}\delta_{\langle\alpha\beta}\delta_{\gamma\delta\rangle}\Big(\hat{\mathcal{E}}_1(S) + \frac{3}{\pi}\log R\Big)^2 - \frac{1}{4}\delta_{\langle\alpha\beta,}\Big(\hat{\mathcal{E}}_1(S) + \frac{3}{\pi}\log R\Big)\widehat{G}^{(24,6)}_{\gamma\delta\rangle}(\varphi)\,, \tag{5.67}$$

where we used the definition (5.11)

$$\mathcal{E}_s(S) = \frac{1}{\xi(2s)}\mathcal{E}^\star(s,S) = S_2^{\,s} + \frac{\xi(2s-1)}{\xi(2s)}S_2^{\,1-s} + \mathcal{O}(e^{-2\pi S_2})\,. \tag{5.68}$$

and the regularized value at $s=1$,

$$\hat{\mathcal{E}}_1(S) = \lim_{s\to 1}\left[\mathcal{E}_s(S) - \frac{3(\frac{A_G^{12}}{4\pi})^{2(s-1)}}{\pi(s-1)}\right] = -\frac{1}{4\pi}\log(S_2^{\,12}|\Delta(S)|)\,,, \tag{5.69}$$

where $A_G = e^{\frac{1}{12}-\zeta'(-1)}$ is the Glaisher-Kinkelin constant.

Recalling that $S_2 = 1/g_4^2$, we see that the first term in (5.67) indeed reproduces the two-loop contribution to the ∇^2F^4 coupling in $D=4$, while the two other terms reproduce the tree-level and one-loop contributions to the same coupling, along with non-perturbative NS5-brane corrections of order $e^{-2\pi S_2}$. Because there is no holomorphic modular form of weight zero for $SL(2,\mathbb{Z})$, supersymmetry Ward identities and U-duality determine uniquely this non-perturbative coupling from its perturbative expansion.

For the CHL orbifolds with $N=2,3,5,7$, we find instead

$$\begin{aligned} G^{(D=4)}_{\alpha\beta,\gamma\delta}(S,\varphi) = \;& \widehat{G}^{(2k-2,6)}_{\alpha\beta,\gamma\delta}(\varphi) - \frac{3}{4\pi}\delta_{\langle\alpha\beta}\delta_{\gamma\delta\rangle}\Big(\frac{\hat{\mathcal{E}}_1(NS)+\hat{\mathcal{E}}_1(S)+\frac{6}{\pi}\log R}{N+1}\Big)^2 \\ & -\frac{1}{4(N+1)}\delta_{\langle\alpha\beta,}\bigg(\Big(\frac{N\hat{\mathcal{E}}_1(NS)-\hat{\mathcal{E}}_1(S)}{N-1}+\frac{6}{\pi}\log R\Big)\widehat{G}^{(2k-2,6)}_{\gamma\delta\rangle}(\varphi) \\ & +\Big(\frac{N\hat{\mathcal{E}}_1(S)-\hat{\mathcal{E}}_1(NS)}{N-1}+\frac{6}{\pi}\log R\Big)\,{}^{\varsigma}\widehat{G}^{(2k-2,6)}_{\gamma\delta\rangle}(\varphi)\bigg)\end{aligned} \tag{5.70}$$

which is manifestly invariant under the Fricke duality $S\mapsto -1/(NS)$, $\varphi\to\varsigma\cdot\varphi$ [27]. In the weak coupling limit $S_2\to+\infty$, this again reproduces the tree-level, one-loop and two-loop contributions to the ∇^2F^4 coupling in $D=4$ (discarding the log terms)

$$G^{(D=4)}_{\alpha\beta,\gamma\delta}(S,\varphi) = \widehat{G}^{(2k-2,6)}_{\alpha\beta,\gamma\delta}(\varphi) - \frac{3}{4\pi}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}S_2{}^2 - \frac{1}{4}\delta_{\langle\alpha\beta,}\widehat{G}^{(2k-2,6)}_{\gamma\delta\rangle}(\varphi)S_2 + \mathcal{O}(e^{-2\pi S_2})$$

This agreement is of course guaranteed by the similar agreement in $D=3$ discussed in §4.3. Since there are no cuspidal forms of weight zero for $\Gamma_0(N)$, (5.70) is in fact the unique non-perturbative completion of the perturbative coupling consistant with supersymmetry Ward identities and U-duality, including Fricke duality.[15]

Other tensorial components $G_{\alpha\beta,\mu\nu}$ correspond instead to $\mathcal{R}^2F^2$ couplings in $D=4$, which we refrain from discussing in detail.

[15] The square of $\frac{\hat{\mathcal{E}}_1(NS)+\hat{\mathcal{E}}_1(S)}{N+1}$ is determined by supersymmetry. The combination $\frac{\hat{\mathcal{E}}_1(NS)+\hat{\mathcal{E}}_1(S)}{N+1}(\widehat{G}^{(2k-2,6)}_{\alpha\beta}(\varphi)+$

5.3.2 Contributions from 1/4-BPS instantons

Exponentially suppressed corrections arise from the rank one orbits (5.22), the Abelian rank two orbits (5.33), and the non-Abelian rank two (G.9). In this section, we focus on the contributions from the the Abelian rank two orbits, which provide the Abelian Fourier coefficients for generic 1/4-BPS charges.[16] These Fourier coefficients can be interpreted as non-perturbative corrections associated to space-time instantons corresponding to 1/4-BPS black holes wrapping the Euclidean time circle.

Decomposing

$$G^{(2)}_{ab,cd} = \sum_{\substack{\Gamma\in\Lambda_m^*\oplus\Lambda_m\\ Q\wedge P\neq 0}} G^{(2,\Gamma)}_{ab,cd}\, e^{2\pi \mathrm{i}(a_1 Q+a_2 P)} \tag{5.71}$$

with $\Gamma=(Q,P)$, using (5.24) and the change of variable $\Omega_2\to A^\intercal\Omega_2 A$, one obtains

$$G^{(2,\Gamma)}_{ab,cd} = 2R^4\int_{\mathcal{P}} \mathrm{d}^3\Omega_2\, \bar{C}_{k-2}(Q,P;\Omega_2)\, P_{ab,cd}(Q_L,P_L,\Omega_2)\, e^{-\pi\mathrm{Tr}\left[\frac{R^2}{S_2}\Omega_2^{-1}\left(\begin{smallmatrix}1 & S_1\\ S_1 & |S|^2\end{smallmatrix}\right)+2\Omega_2\left(\begin{smallmatrix}Q_R^2 & Q_RP_R\\ Q_RP_R & P_R^2\end{smallmatrix}\right)\right]} \tag{5.72}$$

with [17]

$$P_{ab,cd}(Q_L,P_L,\Omega_2) = \left(\mathcal{P}_{ab,cd}(\tfrac{\partial}{\partial y})\, e^{\pi\sqrt{2}R y_{i\mu}(\Omega_2^{-1})^{ij}v^{\intercal\, j\mu}+2\pi\mathrm{i}y_{i\alpha}\Gamma_L^{i\alpha}-\frac{\pi}{2}y_{i\,\alpha}(\Omega_2^{-1})^{ij}y_j{}^{\alpha}}\right)\Big|_{y=0}\,, \tag{5.73}$$

The summation measure $\bar{C}(Q,P,\Omega_2)$ depends both on the charge $\Gamma=(Q,P)$ and on $\Omega_2\in\mathcal{P}$, and is given for the maximal rank model by (cf. (5.34))

$$\bar{C}(Q,P;\Omega_2) = \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\Gamma\in\Lambda_{22,6}\oplus\Lambda_{22,6}}} |A|\, C\left[A^{-1}\left(\begin{smallmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{smallmatrix}\right)A^{-\intercal};A^\intercal\Omega_2 A\right]\,, \tag{5.74}$$

where $\Lambda_{22,6}=\Lambda_m$ is the magnetic lattice of the full rank model, and $C\left[\left(\begin{smallmatrix}2m & l\\ l & 2n\end{smallmatrix}\right);\Omega_2\right]$ are the Fourier coefficients of $1/\Phi_{10}$ defined in (5.25). For CHL models with $N=2,3,5,7$, it is instead given by (cf. (5.64))

$$\begin{split}\bar{C}_{k-2}(Q,P;\Omega_2) = &\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda_m\oplus\Lambda_m}} |A| C_{k-2}\left[A^{-1}\left(\begin{smallmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{smallmatrix}\right)A^{-\intercal};A^\intercal\Omega_2 A\right]\\ &+\sum_{\substack{A\in M_{2,0}(N)/[\mathbb{Z}_2\ltimes\Gamma_0(N)]\\ A^{-1}\binom{Q}{P}\in\Lambda_m^*\oplus\Lambda_m}} |A|\widetilde{C}_{k-2}\left[A^{-1}\left(\begin{smallmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{smallmatrix}\right)A^{-\intercal};A^\intercal\Omega_2 A\right]\\ &+\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P/N}\in\Lambda_m^*\oplus\Lambda_m^*}} |A| C_{k-2}\left[A^{-1}\left(\begin{smallmatrix}-NQ^2 & -Q\cdot P\\ -Q\cdot P & -P^2/N\end{smallmatrix}\right)A^{-\intercal};A^\intercal\Omega_2 A\right]\,,\end{split} \tag{5.75}$$

${}^{\varsigma}\widehat{G}^{(2k-2,6)}_{\alpha\beta}(\varphi)) = \frac{\hat{\mathcal{E}}_1(NS)+\hat{\mathcal{E}}_1(S)}{2}F^{(2k-2,6)\gamma}_{\alpha\beta\gamma}(\varphi)$ is determined with a fixed coefficient by the source term in the differential equation enforced by supersymmetry whereas the coefficient of $\frac{\hat{\mathcal{E}}_1(NS)-\hat{\mathcal{E}}_1(S)}{N-1}(\widehat{G}^{(2k-2,6)}_{\alpha\beta}(\varphi) - {}^{\varsigma}\widehat{G}^{(2k-2,6)}_{\alpha\beta}(\varphi))$ is determined by matching the perturbative expansion.

[16] The dimension of the set of generic 1/4-BPS charges, plus one for the Taub-NUT charge, is equal to the Kostant–Kirillov dimension of the automorphic representation attached to $G_{ab,cd}$, see the end of section 3.1.

[17] When $\alpha\beta\gamma\delta$ lie along the $O(2k-2,6)$ directions, $P_{\alpha\beta,\gamma\delta}(Q_L,P_L,\Omega_2)$ reduces to the polynomial in (2.31).

where $C_{k-2}\big[\big(\begin{smallmatrix}2m & l\\ l & 2n\end{smallmatrix}\big);\Omega_2\big]$ and $\widetilde{C}_{k-2}\big[\big(\begin{smallmatrix}2m & l\\ l & 2n\end{smallmatrix}\big);\Omega_2\big]$ denote the Fourier coefficients of $1/\Phi_{k-2}(\Omega)$ and $1/\widetilde{\Phi}_{k-2}(\Omega)$ given in (5.57), (5.58).

As emphasized earlier, $1/\Phi_{k-2}(\Omega)$ and $1/\widetilde{\Phi}_{k-2}(\Omega)$ are meromorphic functions with poles, so that their Fourier coefficients are piecewise constant functions of Ω_2, with discontinuities as well as delta-function singularities at the boundary between distinct chambers (moreover, they are strictly speaking well-defined only for $|\Omega_2| > \frac{1}{4}$, since the contour $\mathcal{C} = [0,1]^3$ generically crosses the poles for lower values of $|\Omega_2|$). Due to this non-trivial Ω_2-dependence, one cannot compute the integral (5.72) analytically, but one may analyze its asymptotic expansion at large radius.

For generic moduli S and φ, the integral is dominated by a saddle point at $\Omega_2 = \Omega_2^\star$ (5.31), in the neighborhood of which the Fourier coefficients of $1/\Phi_{k-2}(\Omega)$ and $1/\widetilde{\Phi}_{k-2}(\Omega)$ are constant. One can compute the leading contribution in the saddle point approximation by integrating (5.72) with $\bar{C}_{k-2}(Q,P;\Omega_2) \sim \bar{C}_{k-2}(Q,P;\Omega_2^\star)$ kept constant in the integrand. Using (5.33) and the identities [13, (20)]

$$\begin{aligned}\widetilde{B}_{3/2}(Z) &= \frac{\pi K_0\big(2\pi\mathcal{M}(Z)\big)}{\det(Z)^{1/4}} + \frac{\pi\mathcal{M}(Z)K_1\big(2\pi\mathcal{M}(Z)\big)}{2\det(Z)^{3/2}}\\ \widetilde{B}_{1/2}(Z) &= \frac{\pi K_0\big(2\pi\mathcal{M}(Z)\big)}{\det(Z)^{1/4}}\ ,\end{aligned}\tag{5.76}$$

where $\mathcal{M}(Z) = \sqrt{2\sqrt{\det Z} + \operatorname{tr}(Z)}$ (such that $\mathcal{M}(2R^2 v\big(\begin{smallmatrix}Q_R^2 & Q_RP_R\\ Q_RP_R & P_R^2\end{smallmatrix}\big)v^\intercal) = R\mathcal{M}(\Gamma)$), the resulting 1/4-BPS Abelian Fourier coefficients in this approximation can be expressed in terms of the standard modified Bessel functions,

$$\begin{aligned}G^{(2,\Gamma)}_{\alpha\beta,\gamma\delta} &\sim \frac{9}{16}R^5\,\bar{C}_{k-2}(Q,P;\Omega_2^\star)\\ &\times\Bigg(\frac{2\pi}{R^2}\frac{Q_{L\langle\alpha}Q_{L\beta,}P_{L\gamma}P_{L\delta\rangle}}{|Q_R\wedge P_R|^2}\bigg[K_0(2\pi R\mathcal{M}(\Gamma)) + \frac{R\mathcal{M}(\Gamma)}{4R^2|2Q_R\wedge P_R|}K_1(2\pi R\mathcal{M}(\Gamma))\bigg]\\ &+\frac{1}{\pi}\delta_{\langle\alpha\beta,}\frac{\Gamma_{L\gamma}{}^{\kappa}\Gamma_{L\delta\rangle}{}^{\lambda}}{|Q_R\wedge P_R|}\frac{\partial}{\partial Z^{\kappa\lambda}}\bigg[2\sqrt{|Z|}K_0(2\pi\mathcal{M}(Z)) + \mathcal{M}(Z)K_1(2\pi\mathcal{M}(Z))\bigg]\bigg|_{Z=2R^2v\big(\begin{smallmatrix}Q_R^2 & Q_RP_R\\ Q_RP_R & P_R^2\end{smallmatrix}\big)v^\intercal}\\ &+\frac{1}{4\pi|Q_R\wedge P_R|}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}K_0(2\pi R\mathcal{M}(\Gamma))\bigg]\Bigg)\ ,\end{aligned}\tag{5.77}$$

where $\Gamma_{L\gamma}{}^{\kappa} = \frac{1}{\sqrt{S_2}}(Q_{L\gamma}+S_1P_{L\gamma}, S_2P_{L\gamma})$. This leading contribution can be ascribed to instantons of charge Γ associated to 1/4-BPS black holes (including bound states of two 1/2-BPS black holes) wrapping the Euclidean time circle. It is indeed exponentially suppressed in $e^{-2\pi R\mathcal{M}(\Gamma)}$ for $\mathcal{M}(\Gamma)$ (2.5) the BPS mass of a black hole of charge Γ, and it is weighted by the measure factor $\bar{C}_{k-2}(Q,P;\Omega_2^\star)$. For a primitive charge Γ, *i.e.* such that there is no $d\neq 1$ with $d^{-1}\Gamma\in\Lambda_m^*\oplus\Lambda_m$, the only matrix A contributing to the measure is $A=1$ and one can interpret the measure factor (up to an overall sign) as the helicity supertrace counting string theory states of charge Γ, as advocated in the introduction (2.14),

$$\bar{C}_{k-2}(Q,P;\Omega_2^\star) = (-1)^{Q\cdot P+1}\Omega_6(Q,P,S,\varphi)\ .\tag{5.78}$$

The value of Ω_2 at the saddle point (5.31) reproduces the contour prescription of [9, 32] when both electric and magnetic charges are separately primitive in Λ_m^* and Λ_m and $d^{-1}Q\wedge P\in$

$\Lambda_m^* \wedge \Lambda_m$ for $d = 1$ only. More generally, the contour prescription depends on the set of matrices A dividing (Q, P) in the electromagnetic lattice. For example in the maximal rank case, all primitive charges (Q, P) are in the U-duality orbit of a charge of the form [68]

$$Q = e_1 + q\, e_2\ , \qquad P = p\, e_2\ , \qquad Q \wedge P = p\, e_1 \wedge e_2\ , \tag{5.79}$$

with e_1 and e_2 primitive in $\Lambda_{22,6}$. The integer p is sometimes known as the 'torsion'. In that case (5.74) simplifies to

$$\bar{C}(Q, P; \Omega_2^\star) = \sum_{\substack{d\geq 1\\ d|p}} d\, C\Big[\big(\begin{smallmatrix} Q^2 & QP/d \\ QP/d & P^2/d^2 \end{smallmatrix}\big), \big(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\big)\Omega_2^\star \big(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\big)\Big]\ , \tag{5.80}$$

in agreement with the prescription in [40, 69], with additional fineprint on the contour of integration. If we consider the same charge configuration (5.79) in CHL orbifolds for e_1 primitive in Λ_m^* and not in Λ_m, e_2 primitive in Λ_m and not in $N\Lambda_m^*$, and with p not divisible by N, such that it corresponds to a twisted state, only the second line in (5.75) contributes and the result reduces similarly to

$$\bar{C}_{k-2}(Q, P; \Omega_2^\star) = \sum_{\substack{d\geq 1\\ d|p}} d\, \widetilde{C}_{k-2}\Big[\big(\begin{smallmatrix} Q^2 & QP/d \\ QP/d & P^2/d^2 \end{smallmatrix}\big), \big(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\big)\Omega_2^\star \big(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\big)\Big]\ , \tag{5.81}$$

in agreement with [6] for $p = 1$. For general primitive charges such that Q can be in Λ_m and P in $N\Lambda_m^*$, all three terms contribute to the helicity supertrace, and the result is manifestly invariant under U-duality including Fricke duality.

5.3.3 Contributions from pairs of 1/2-BPS instantons

Let us now discuss corrections to the saddle point approximation to (5.72). In Appendix F we estimate the contributions to $G^{(2,\Gamma)}_{\alpha\beta,\gamma\delta}$ due to the deviation of $\bar{C}_{k-2}(Q, P, \Omega_2)$ from its saddle point value $\bar{C}_{k-2}(Q, P, \Omega_2^*)$. In the range[18] $|\Omega_2| > \frac{1}{4}$, the deviation is due to the poles occuring when $n_1\sigma_2 - m^1\rho_2 + jv_2 = 0$ with $4n_1m^1 + j^2 = 0$, resulting in the discontinuities and delta-function singularities of $C_{k-2}(Q, P, \Omega_2^*)$ and $\widetilde{C}_{k-2}(Q, P, \Omega_2^*)$ on $\mathcal{P}$ shown in (5.25), (5.57) and (5.58). In Appendix F.1, we show that these contributions are exponentially suppressed in $e^{-2\pi R(\mathcal{M}(\Gamma_1)+\mathcal{M}(\Gamma_2))}$, and can therefore be ascribed to two-instanton effects associated to two unbounded 1/2-BPS states of charges Γ_1 and Γ_2.

For fixed total charge Γ, we expect contributions from all pairs of 1/2-BPS states with charges Γ_1 and Γ_2 such that $\Gamma = \Gamma_1 + \Gamma_2$. We show in Appendix C that a general such splitting is parametrized by a non-degenerate matrix $B = \big(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\big) \in M_2(\mathbb{Z})$, such that

$$\begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} = \begin{pmatrix} p \\ r \end{pmatrix} \frac{sQ - qP}{ps - qr} = B\pi_1 B^{-1} \begin{pmatrix} Q \\ P \end{pmatrix}\ , \quad \begin{pmatrix} Q_2 \\ P_2 \end{pmatrix} = \begin{pmatrix} q \\ s \end{pmatrix} \frac{pP - rQ}{ps - qr} = B\pi_2 B^{-1} \begin{pmatrix} Q \\ P \end{pmatrix}\ , \tag{5.82}$$

[18] In the range $|\Omega_2| < \frac{1}{4}$, there are additional contributions from 'deep poles' of the form (F.10) with $n_2 \neq 0$ which must be avoided in order to define the Fourier coefficient $\bar{C}(Q, P, \Omega_2)$. In Appendix (F.2), we show that irrespective of the detailed prescription for avoiding these poles, the contribution from the region $|\Omega_2| < \frac{1}{(4n_2^2)}$ is exponentially suppressed in $e^{-2\pi R^2|2n_2|}$, and can be ascribed to pairs of Taub-NUT instanton anti-instantons of charge $\pm n_2$.

where $\pi_1 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $\pi_2 = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$. All splittings of a given charge Γ are in one-to-one correspondence with the matrices $B \in M_2(\mathbb{Z})/\mathrm{Stab}(\pi_i)$ such that $B\pi_1 B^{-1}\Gamma \in \Lambda_m^* \oplus \Lambda_m$ with

$$M_2(\mathbb{Z})/\mathrm{Stab}(\pi_i) = \left\{ \gamma \cdot \begin{pmatrix}1 & j'\\ 0 & k'\end{pmatrix} , \quad \gamma \in GL(2,\mathbb{Z})/\mathrm{Dih}_4 \,, \quad 0 \le j' < k' \,, \quad (j',k') = 1 \right\} . \tag{5.83}$$

In the following it prove convenient to use an equivalent unimodular representative

$$\hat{B} = B\begin{pmatrix}1 & 0\\ 0 & |B|^{-1}\end{pmatrix} = \gamma \cdot \begin{pmatrix}1 & \frac{j'}{k'}\\ 0 & 1\end{pmatrix} , \tag{5.84}$$

in $SL(2,\mathbb{Q})/\mathrm{Stab}(\pi_i,\mathbb{Q})$, where $\mathrm{Stab}(\pi_i,\mathbb{Q})$ is the stabilizer of the doublet π_i in $SL(2,\mathbb{Q})$.

We show in Appendix C that the summation measure (5.74) on the domain $|\Omega_2| > \frac{1}{4}$ (taking into account the discontinuities displayed in (5.25)) reads (focusing on the maximal rank case for simplicity)

$$\begin{aligned}\bar{C}(Q,P;\Omega_2) = & \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\Gamma\in\Lambda_m\oplus\Lambda_m}} |A|\, C^F\left[A^{-1}\begin{pmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{pmatrix}A^{-\intercal}\right] \\ & + \sum_{\substack{\Gamma_i\in\Lambda_m\oplus\Lambda_m\\ Q_i\wedge P_i=0,\,\Gamma_1+\Gamma_2=\Gamma}} \bar{c}(\Gamma_1)\bar{c}(\Gamma_2)\left(-\frac{\delta([\hat{B}^\intercal\Omega_2\hat{B}]_{12})}{4\pi} + \frac{\langle\Gamma_1,\Gamma_2\rangle}{2}\big(\mathrm{sign}(\langle\Gamma_1,\Gamma_2\rangle) - \mathrm{sign}([\hat{B}^\intercal\Omega_2\hat{B}]_{12})\big)\right) ,\end{aligned} \tag{5.85}$$

with $\hat{B} \in SL(2,\mathbb{Q})/\mathrm{Stab}(\pi_i,\mathbb{Q})$ determined such that $\Gamma_i = \hat{B}\pi_i\hat{B}^{-1}\Gamma$ and where $[\hat{B}^\intercal\Omega_2\hat{B}]_{ij}$ denotes the entrises ij of the matrix.

To interpret the second line, recall that the central charge $Z = \frac{2}{\sqrt{S_2}}(Q_R + SP_R)$ for an arbitrary 1/4-BPS state decomposes into orthogonal components $Z = Z_+ + Z_-$ with

$$Z_\pm = \frac{1}{\sqrt{S_2}}\left[(1,S)\cdot\begin{pmatrix}Q_R\\ P_R\end{pmatrix} \pm \frac{\mathrm{i}}{|Q_R\wedge P_R|}(-S,1)\cdot\begin{pmatrix}P_R^2 Q_R - (Q_R\cdot P_R)P_R\\ Q_R^2 P_R - (Q_R\cdot P_R)Q_R\end{pmatrix}\right] \tag{5.86}$$

The BPS mass is $\mathcal{M}(Q,P) = |Z_+|$. It is convenient to write $Z_{+\hat{\alpha}} = (z_1 + \mathrm{i}z_2)_{\hat{\alpha}}\mathcal{M}(Q,P)$ with z_1 and z_2 vectors of $SO(6)$ satisfying

$$z_1^2 + z_2^2 = 1 \,, \quad z_1\cdot\frac{Q_R + S_1 P_R}{\sqrt{S_2}} + z_2\cdot\frac{S_2 P_R}{\sqrt{S_2}} = 2\mathcal{M}(Q,P) \,, \quad z_1\cdot\frac{S_2 P_R}{\sqrt{S_2}} - z_2\cdot\frac{Q_R + S_1 P_R}{\sqrt{S_2}} = 0 \,. \tag{5.87}$$

The matrix $\Omega_2^\star$ at the saddle point determines precisely this decomposition through

$$\begin{pmatrix}z_1\\ z_2\end{pmatrix} = \tfrac{1}{\sqrt{S_2}}\begin{pmatrix}S_2 & 0\\ -S_1 & 1\end{pmatrix}\frac{1}{R}\Omega_2^\star\begin{pmatrix}Q_R\\ P_R\end{pmatrix} . \tag{5.88}$$

A generic two-center 1/4-BPS solution with total charge (Q,P) is written in terms of the harmonic functions [19]

$$(\mathcal{H}^I,\mathcal{K}^I) = \frac{(Q_1^I,P_1^I)}{|x-x_1|} + \frac{(Q_2^I,P_2^I)}{|x-x_2|} - p_{R\hat{\alpha}}{}^I\,\tfrac{1}{\sqrt{S_2}}\begin{pmatrix}S_2 & -S_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}z_1^{\hat{\alpha}}\\ z_2^{\hat{\alpha}}\end{pmatrix} , \tag{5.89}$$

[19] Supersymmetry implies that Q_{iR} and P_{iR} are linear combinations of Q_R and P_R, but this is automatically the case for 1/2-BPS charges such that $Q_i \wedge P_i = 0$.

and is regular away from the points x_1 and x_2 provided the distance $|x_1 - x_2|$ satisfies

$$\frac{\langle \Gamma_1, \Gamma_2\rangle}{|x_1 - x_2|} = -z_1 \cdot \frac{S_2 P_{1R}}{\sqrt{S_2}} + z_2 \cdot \frac{Q_{1R} + S_1 P_{1R}}{\sqrt{S_2}} = -\frac{|Q_R \wedge P_R|}{R}[\hat{B}^\intercal \Omega_2^\star \hat{B}]_{12} , \tag{5.90}$$

which requires that $[\hat{B}^\intercal \Omega_2^\star \hat{B}]_{12}$ and $\langle \Gamma_1, \Gamma_2\rangle$ have opposite sign. Returning to (5.85), we see that when the bound state is allowed, the pair of 1/2-BPS charges contribute to the Fourier coefficient at leading order with measure factor $\bar{c}(\Gamma_1)\bar{c}(\Gamma_2)|\langle \Gamma_1, \Gamma_2\rangle|$.

In contrast, when $[\hat{B}^\intercal \Omega_2^\star \hat{B}]_{12}$ and $\langle \Gamma_1, \Gamma_2\rangle$ have the same sign, the bound state is not allowed and the last term in (5.85) vanishes at the saddle point $\Omega_2 = \Omega_2^\star$ in (5.31). This term still contributes to the integral (5.72), but is exponentially suppressed. At large R, the integral is now dominated by the boundary of the chamber where the sign of $[\hat{B}^\intercal \Omega_2 \hat{B}]_{12}$ flips, as shown in Appendix F.1. On this locus, the argument of the exponential $\mathrm{Tr}\big[\frac{R^2}{S_2}\Omega_2^{-1}\big(\begin{smallmatrix} 1 & S_1 \\ S_1 & |S|^2\end{smallmatrix}\big) + 2\Omega_2\big(\begin{smallmatrix} Q_R^2 & Q_R P_R \\ Q_R P_R & P_R^2\end{smallmatrix}\big)\big]$ in (5.72) decomposes into two pieces associated to Γ_1, Γ_2,

$$\frac{R^2}{\sigma_2 S_2}\big[\hat{B}^\intercal \big(\begin{smallmatrix} 1 & S_1 \\ S_1 & |S|^2\end{smallmatrix}\big)\hat{B}\big]_{11} + 2\sigma_2([\hat{B}^{-1}\Gamma_R]_1)^2 + \frac{R^2}{\rho_2 S_2}\big[\hat{B}^\intercal \big(\begin{smallmatrix} 1 & S_1 \\ S_1 & |S|^2\end{smallmatrix}\big)\hat{B}\big]_{22} + 2\rho_2([\hat{B}^{-1}\Gamma_R]_2)^2 \ . \tag{5.91}$$

The integral is then exponentially suppressed by $e^{-2\pi R(\mathcal{M}(\Gamma_1)+\mathcal{M}(\Gamma_2))}$. The same holds for the contribution of the Dirac delta function which is computed explicitly in Appendix D.

We conclude that (5.72) receives contributions of each possible splitting $\Gamma = \Gamma_1 + \Gamma_2$, weighted by the product of the 1/2-BPS measures $\bar{c}(\Gamma_1)\,\bar{c}(\Gamma_2)$ and further exponentially suppressed by $e^{-2\pi R(\mathcal{M}(\Gamma_1)+\mathcal{M}(\Gamma_2))}$. It is important to distinguish these two-instanton contributions from one-instanton contributions due to bound states of 1/2-BPS states. Due to the triangular inequality $\mathcal{M}(\Gamma_1) + \mathcal{M}(\Gamma_2) \geq \mathcal{M}(\Gamma)$, these contributions are subdominant compared to the one-instanton contributions (5.77) away from the walls of marginal stability. On the wall, the two contributions become comparable and the complete Fourier coefficient is continuous.

This discussion generalizes with some efforts to CHL models with N prime. In Appendix C we show that the measure function for $|\Omega_2| \geq \frac{1}{4}$ decomposes as

$$\begin{aligned}
\bar{C}_{k-2}(Q, P; \Omega_2) = & \sum_{\substack{A \in M_2(\mathbb{Z})/GL(2,\mathbb{Z}) \\ A^{-1}\binom{Q}{P} \in \Lambda_m \oplus \Lambda_m}} |A| C^F_{k-2}\Big[A^{-1}\big(\begin{smallmatrix} -Q^2 & -Q\cdot P \\ -Q\cdot P & -P^2\end{smallmatrix}\big)A^{-\intercal}\Big] \\
& + \sum_{\substack{A \in M_{2,0}(N)/[\mathbb{Z}_2 \ltimes \Gamma_0(N)] \\ A^{-1}\binom{Q}{P} \in \Lambda_m^* \oplus \Lambda_m}} |A| \widetilde{C}^F_{k-2}\Big[A^{-1}\big(\begin{smallmatrix} -Q^2 & -Q\cdot P \\ -Q\cdot P & -P^2\end{smallmatrix}\big)A^{-\intercal}\Big] \\
& + \sum_{\substack{A \in M_2(\mathbb{Z})/GL(2,\mathbb{Z}) \\ A^{-1}\binom{Q}{P/N} \in \Lambda_m^* \oplus \Lambda_m^*}} |A| C^F_{k-2}\Big[A^{-1}\big(\begin{smallmatrix} -NQ^2 & -Q\cdot P \\ -Q\cdot P & -P^2/N\end{smallmatrix}\big)A^{-\intercal}\Big] \\
& + \sum_{\substack{\Gamma_i \in \Lambda_m^* \oplus \Lambda_m \\ Q_i \wedge P_i = 0,\, \Gamma_1 + \Gamma_2 = \Gamma}} \bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2)\bigg(-\frac{\delta([\hat{B}^\intercal \Omega_2 \hat{B}]_{12})}{4\pi} + \frac{\langle \Gamma_1, \Gamma_2\rangle}{2}\big(\mathrm{sign}(\langle \Gamma_1, \Gamma_2\rangle) - \mathrm{sign}([\hat{B}^\intercal \Omega_2 \hat{B}]_{12})\big)\bigg) ,
\end{aligned} \tag{5.92}$$

with $\hat{B} \in SL(2,\mathbb{Q})/\mathrm{Stab}(\pi_i, \mathbb{Q})$ such that $\Gamma_i = \hat{B}\pi_i \hat{B}^{-1}\Gamma$. In this case one must distinguish the charges Γ_1 and Γ_2 that are twisted or untwisted to reproduce the exact measure (2.22).

In Appendix C we analyze all the possible splittings depending on the orbit – electric or magnetic – of the charges Γ_1 and Γ_2 under $\Gamma_0(N)$. The sign $(-1)^{Q\cdot P} = (-1)^{\langle\Gamma_1,\Gamma_2\rangle}$ for all splittings, which ensures that the contribution of the sign function in (5.92) to the helicity supertrace $\Omega_6(Q,P,t)$ satisfies to the wall-crossing formula (2.12) with the correct sign.

It is interesting to understand this property from the differential equation imposed by supersymmetry Ward identities (2.26). We show explicitly in Appendix E.3 that the component of the differential equation with all indices along the decompactified torus is satisfied. In general, one finds that the leading contribution to the Fourier coefficient (5.72) with constant measure $\bar{C}_{k-2}(Q,P;\Omega_2) \sim \bar{C}_{k-2}(Q,P;\Omega_2^\star)$ as in (5.77), solves the homogeneous equation (3.17). The contributions due to the discontinuities of the summation measure $\bar{C}_{k-2}(Q,P;\Omega_2)$ give a particular inhomogeneous solution sourced by the quadratic term in F_{abcd}. For a given 1/4-BPS charge Γ, the Fourier coefficients of F_{abcd} contribute a source term proportional to $\bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2)$ for all possible splittings $\Gamma = \Gamma_1 + \Gamma_2$, which matches the structure of the measure measure in (5.92). In this way, the differential equation constrains the measure function to be consistent with wall crossing, such that the discontinuities must correspond to the sum over all possible splittings weighted by the 1/2-BPS measures of the constituent charges as exhibited in (5.92).

The explicit check of the differential equation in Appendix E.3 demonstrates that the unfolding procedure reproduces the correct Abelian Fourier coefficients, at least up to terms that are exponentially suppressed in $e^{-2\pi R^2}$. This is an important consistency check because the same unfolding procedure fails to reproduce the non-perturbative contributions to the constant terms associated to instanton anti-instantons, which are also required to be present in order for the differential equation to hold . These effects are also necessary in order to resolve the ambiguity of the sum over 1/4-BPS instantons [48], which is divergent due to the exponential growth of the measure $\bar{C}_{k-2}(Q,P;\Omega_2^\star) \sim (-1)^{Q\cdot P+1} e^{\pi|Q\wedge P|}$ [2, 8].

6 Weak coupling expansion in dual string vacua

In section §4.3, we analyzed the weak coupling expansion of the exact $\nabla^2(\nabla\phi)^4$ in $D=3$, in the limit where the heterotic string coupling is small. However, the CHL vacua of interest in this paper also admit dual descriptions in terms of freely acting orbifolds of type II string theory compactified on $K_3 \times T^3$ [70, 71], or of type I strings on T^7 [72]. In this section, we discuss the weak coupling expansion of these exact results on the type II and type I sides. We also include a brief discussion of the $\nabla^2\mathcal{H}^4$ couplings in type IIB string theory compactified on $K3$, whose exact form was conjectured in [46] and involves the same type of genus-two modular integral, albeit with a lattice of signature $(21,5)$.

6.1 Weak coupling limit in CHL orbifolds of type II strings on $K3 \times T^3$

On the type II side, string vacua with 16 supercharges can be obtained by orbifolding the type II string on $K3 \times T^3$ by a symplectic automorphism of K3 combined with a translation on T^3 [70, 71]. In order to keep manifest the four-dimensional origin of these models, we shall assume that the translation acts only on a T^2 inside T^3. In the weak coupling limit $g_6 \to 0$ (where g_6 is the string coupling in type IIA compactified on $K3$), the 'non-perturbative Narain lattice' (2.3) decomposes into [73],

$$\Lambda_{2k,8} \to \Lambda_{2k-4,4} \oplus [I\!I_{1,1} \oplus I\!I_{1,1}[N]] \oplus [I\!I_{1,1} \oplus I\!I_{1,1}[N]] \,, \tag{6.1}$$

where the first summand is the sublattice of the homology lattice $\Lambda_{20,4} = H_{\rm even}(K3)$ which is invariant under the symplectic automorphism, the second is the lattice of windings and momenta along T^2, and the third is the lattice of windings and momenta along S_1 together with the non-perturbative direction. The last two summands can be combined into a lattice $\Lambda_{4,4} = I\!I_{2,2} \oplus I\!I_{2,2}[N]$ which can be thought as the lattice of windings and momenta along a fiducial torus T^4. Assuming for simplicity that flat metric on the torus T^3 is diagonal and the Kalb-Ramond two-form vanishes, the radii of the four circles in this fiducial T^4 are related to the three radii R_5, R_6, R_7 of the physical T^3 by

$$(r_1, r_2, r_3, r_4) = \left(\frac{R_6}{g_6 \ell_{II}}, \frac{R_7}{g_6 \ell_{II}}, \frac{R_5}{g_6 \ell_{II}}, \frac{R_5 R_6 R_7}{g_6 \ell_{II}^3} \right) \tag{6.2}$$

In the limit $g_6 \to 0$, the four radii r_i scale to infinity at the same rate, so the automorphism group $O(\Lambda_{4,4})$ is broken to a congruence subgroup of $SL(4,\mathbb{Z})$, which is identified with the T-duality group $O(\Lambda_{3,3})$ along the three-torus. In order to make T-duality invariance manifest, it is useful to define the type II string coupling in three-dimensions $g_3' = g_6 \sqrt{\ell_{II}^3 / V_3}$ where ℓ_{II} is the type II string length and $V_3 = R_5 R_6 R_7$.

The analysis in §4.1 and §5.1 – and our previous analysis of the one-loop integral in [22] is readily generalized to the case where n radii of a lattice $I\!I_{n-r,n-r} \oplus I\!I_{r,r}[N]$ become large, leading in the maximal rank case $N = 1$ to

$$F^{(p,q)}_{\alpha\beta\gamma\delta} = V_n \, F^{(p-n,q-n)}_{\alpha\beta\gamma\delta} + \frac{3c(0)}{16\pi^2} \, V_n^{\frac{q-6}{n}} \, \Gamma(\tfrac{n+6-q}{2}) \sum_{m^i \in \mathbb{Z}^n}{}' \, (\pi \, m^i U_{ij} m^j)^{\frac{q-n-6}{2}} \, \delta_{(\alpha\beta} \delta_{\gamma\delta)} + \dots \tag{6.3}$$

$$\begin{aligned} G^{(p,q)}_{\alpha\beta,\gamma\delta} = {} & V_n^2 \, G^{(p-n,q-n)}_{\alpha\beta,\gamma\delta} - \frac{c(0)}{32\pi} V_n^{\frac{q+n-6}{n}} \, \Gamma(\tfrac{n+6-q}{2}) \sum_{m^i \in \mathbb{Z}^n}{}' \, (\pi \, m^i U_{ij} m^j)^{\frac{q-n-6}{2}} \, \delta_{\langle \alpha\beta,} G^{(p-n,q-n)}_{\gamma\delta \rangle} \\ & - \frac{3}{4\pi} \left[\frac{c(0)}{8\pi} V_n^{\frac{q-6}{n}} \, \Gamma(\tfrac{n+6-q}{2}) \sum_{m^i \in \mathbb{Z}^n}{}' \, (\pi m^i U_{ij} m^j)^{\frac{q-n-6}{2}} \right]^2 \delta_{\langle \alpha\beta,} \delta_{\gamma\delta \rangle} + \dots \end{aligned} \tag{6.4}$$

or in the case of $N \neq 1$,

$$
\begin{aligned}
F^{(p,q)}_{\alpha\beta\gamma\delta} &= V_n\, F^{(p-n,q-n)}_{\alpha\beta\gamma\delta} + \frac{3c_k(0)}{16\pi^2}\, V_n^{\frac{q-6}{n}}\, \Gamma(\tfrac{n+6-q}{2})\delta_{(\alpha\beta}\delta_{\gamma\delta)}\Big[\sideset{}{'}\sum_{m^i\in\mathbb{Z}^n} (\pi\, m^i U_{ij} m^j)^{\frac{q-n-6}{2}} \\
&\quad + N^{r-1} \sideset{}{'}\sum_{\substack{m^1,\dots,m^{n-r}\in\mathbb{Z}^{n-r}\\ m^{n-r},\dots,m^n\in N\mathbb{Z}^r}} (\pi\, m^i U_{ij} m^j)^{\frac{q-n-6}{2}} \Big] + \dots \\
G^{(p,q)}_{\alpha\beta,\gamma\delta} &= V_n^2\, G^{(p-n,q-n)}_{\alpha\beta,\gamma\delta} - \frac{c_k(0)}{32\pi(N-1)} V_n^{\frac{q+n-6}{n}}\, \Gamma(\tfrac{n+6-q}{2})\delta_{\langle\alpha\beta,} \\
&\quad \times \Big[\Big(N^r \sideset{}{'}\sum_{\substack{m^1,\dots,m^{n-r}\in\mathbb{Z}^{n-r}\\ m^{n-r+1},\dots,m^n\in N\mathbb{Z}^r}} (\pi\, m^i U_{ij} m^j)^{\frac{q-n-6}{2}} - \sideset{}{'}\sum_{m^i\in\mathbb{Z}^n} (\pi\, m^i U_{ij} m^j)^{\frac{q-n-6}{2}} \Big) G^{(p-n,q-n)}_{\gamma\delta\rangle} \\
&\quad + \Big(N \sideset{}{'}\sum_{m^i\in\mathbb{Z}^n} (\pi\, m^i U_{ij} m^j)^{\frac{q-n-6}{2}} - N^{r-1} \sideset{}{'}\sum_{\substack{m^1,\dots,m^{n-r}\in\mathbb{Z}^{n-r}\\ m^{n-r+1},\dots,m^n\in N\mathbb{Z}^r}} (\pi\, m^i U_{ij} m^j)^{\frac{q-n-6}{2}} \Big) \Big] {}^{\varsigma}G^{(p-n,q-n)}_{\gamma\delta\rangle} \\
&\quad - \frac{3c_k(0)^2}{256\pi^3} V_n^{\frac{2q-12}{n}}\, \Gamma(\tfrac{n+6-q}{2})^2 \\
&\qquad \times \Big[\sideset{}{'}\sum_{m^i\in\mathbb{Z}^n} (\pi m^i U_{ij} m^j)^{\frac{q-n-6}{2}} + N^{r-1} \sideset{}{'}\sum_{\substack{m^1,\dots,m^{n-r}\in\mathbb{Z}^{n-r}\\ m^{n-r+1},\dots,m^n\in N\mathbb{Z}^r}} (\pi m^i U_{ij} m^j)^{\frac{q-n-6}{2}} \Big]^2 \delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} \\
&\quad + \frac{18 V_n^{\frac{2q-10}{n}}}{(N^2-1)\pi^{3/2}} \Gamma(\tfrac{n+5-q}{2})\Gamma(\tfrac{n+4-q}{2})\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} \\
&\quad \times \Big(N \sum_{\substack{A\in\\ M_{n,2}(\mathbb{Z})/GL(2,\mathbb{Z})}} - N^{r-1} \sum_{\substack{A\in\\ M_{n,2,0}[N^r]/(\mathbb{Z}_2\ltimes\Gamma_0(N))}} + N^{2r-2} \sum_{\substack{A\in\\ M_{n,2,00}[N^r]/GL(2,\mathbb{Z})}} \Big) \det(\pi A^\intercal U A)^{\frac{q-n-5}{2}} + \dots
\end{aligned}
\tag{6.5}
$$

where the dots denote exponentially suppressed terms and U_{ij} is the metric on the n-torus, normalized to have unit determinant.[20] Here $M_{n,2}(\mathbb{Z})$ is the set of rank two n by 2 matrices over the integers, $M_{n,2,0}[N^r]$ the subset for which the first column last r entries vanish mod N, and $M_{n,2,00}[N^r]$ the subset for which the two columns last r entries vanish mod N.

The sums over $m^i \in \mathbb{Z}^n\backslash\{0\}$ can be expressed in terms of the vector Eisenstein series for the congruence subgroup of $SL(n,\mathbb{Z})$ for which the lower left $r\times(n-r)$ entries vanish mod N in the fundamental matrix representation, which we denote by $SL_n[N^r]$,

$$
\mathcal{E}^{\star SL_n[N^r]}_{s\Lambda_1}(U) = \frac{1}{2}\Gamma(s) \sideset{}{'}\sum_{\substack{m^1,\dots,m^{n-r+1}\in\mathbb{Z}^{n-r}\\ m^{n-r},\dots,m^n\in N\mathbb{Z}^r}} (\pi\, m^i U_{ij} m^j)^{-s} \,. \tag{6.6}
$$

The sums over A can be expressed in terms of rank two tensor Eisenstein series for the same

[20]In the case of a square torus of volume $V_n = r_1 \dots r_n$, $U_{ij} = r_i^2\delta_{ij}/V_n^{\frac{2}{n}}$.

congruence subgroup $SL_n[N^r]$

$$\begin{aligned}
\mathcal{E}^{\star SL_n}_{s\Lambda_2}(U) &= \pi^{\frac12}\Gamma(s)\Gamma(s-\tfrac12) \sum_{\substack{A\in\\ M_{n,2}(\mathbb{Z})/GL(2,\mathbb{Z})}}{}^{'} \det\left(\pi A^\intercal U A\right)^{-s} \,, \\
\mathcal{E}^{\star SL_n[N^r]}_{s\Lambda_2,0}(U) &= \pi^{\frac12}\Gamma(s)\Gamma(s-\tfrac12) \sum_{\substack{A\in\\ M_{n,2,0}[N^r]/(\mathbb{Z}_2\ltimes\Gamma_0(N))}}{}^{'} \det\left(\pi A^\intercal U A\right)^{-s} \,, \\
\mathcal{E}^{\star SL_n[N^r]}_{s\Lambda_2,00}(U) &= \pi^{\frac12}\Gamma(s)\Gamma(s-\tfrac12) \sum_{\substack{A\in\\ M_{n,2,00}[N^r]/GL(2,\mathbb{Z})}}{}^{'} \det\left(\pi A^\intercal U A\right)^{-s} \,.
\end{aligned} \tag{6.7}$$

Note that for $N=1$, $\mathcal{E}^{\star SL_n}_{s\Lambda_k}(U)$ is the standard Langlands Eisenstein series satisfying the functional relation $\mathcal{E}^{\star SL_n}_{(\frac{n}{2}-s)\Lambda_k}(U) = \mathcal{E}^{\star SL_n}_{s\Lambda_k}(U^{-1})$.

For $(n,r)=(1,0)$ and $(n,r)=(2,1)$, (6.3) and (6.5) reduce to the results in §4 and 5 of [22] and the present paper, respectively. The case relevant in the present context is $(n,r)=(4,2)$. Setting $(p,q,n)=(2k,8,4)$, $V_4 = V_3^2/(g_6^4\ell_{II}^6) = 1/g_3'^4$, and multiplying by a suitable power of g_3' for translating to the string frame, we find that the perturbative terms in the $(\nabla\phi)^4$ and $\nabla^2(\nabla\phi)^4$ couplings in the maximal rank case are given by

$$\begin{aligned}
g_3'^2\, F^{(24,8)}_{\alpha\beta\gamma\delta} &= \frac{1}{g_3'^2} F^{(20,4)}_{\alpha\beta\gamma\delta} + \frac{9}{\pi^2}\, \mathcal{E}^{\star SL_4}_{\Lambda_1}(U)\delta_{(\alpha\beta}\delta_{\gamma\delta)} + \dots \\
g_3'^6\, G^{(24,8)}_{\alpha\beta,\gamma\delta} &= \frac{1}{g_3'^2} G^{(20,4)}_{\alpha\beta,\gamma\delta} - \frac{3}{2\pi}\mathcal{E}^{\star SL_4}_{\Lambda_1}(U)\delta_{\langle\alpha\beta,}G^{(20,4)}_{\gamma\delta\rangle} - \frac{27 g_3'^2}{\pi^3}[\mathcal{E}^{\star SL_4}_{\Lambda_1}(U)]^2\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} + \dots
\end{aligned} \tag{6.8}$$

Similarly, for $N>1$ we get

$$g_3'^2\, F^{(2k,8)}_{\alpha\beta\gamma\delta} = \frac{1}{g_3'^2} F^{(2k-4,4)}_{\alpha\beta\gamma\delta} + \frac{9}{\pi^2(N+1)}\left[\mathcal{E}^{\star SL_4}_{\Lambda_1}(U) + N\mathcal{E}^{\star SL_4[N^2]}_{\Lambda_1}(U)\right]\delta_{(\alpha\beta}\delta_{\gamma\delta)} + \dots \tag{6.9}$$

$$\begin{aligned}
g_3'^6\, G^{(2k,8)}_{\alpha\beta,\gamma\delta} &= \frac{1}{g_3'^2} G^{(2k-4,4)}_{\alpha\beta,\gamma\delta} - \frac{3}{2\pi(N^2-1)}\left[N^2\mathcal{E}^{\star SL_4[N^2]}_{\Lambda_1}(U) - \mathcal{E}^{\star SL_4}_{\Lambda_1}(U)\right]\delta_{\langle\alpha\beta,}G^{(2k-4,4)}_{\gamma\delta\rangle} \\
&\quad - \frac{3N}{2\pi(N^2-1)}\left[\mathcal{E}^{\star SL_4}_{\Lambda_1}(U) - \mathcal{E}^{\star SL_4[N^2]}_{\Lambda_1}(U)\right]\delta_{\langle\alpha\beta,}{}^{\varsigma}G^{(2k-4,4)}_{\gamma\delta\rangle} \\
&\quad + \frac{18N}{(N^2-1)\pi^2}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\left[\mathcal{E}^{\star SL_4}_{\frac12\Lambda_2}(U) - \mathcal{E}^{\star SL_4[N^2]}_{\frac12\Lambda_2,0}(U) + N\mathcal{E}^{\star SL_4[N^2]}_{\frac12\Lambda_2,00}(U)\right] \\
&\quad - \frac{27 g_3'^2}{\pi^3(N+1)^2}\left[\mathcal{E}^{\star SL_4}_{\Lambda_1}(U) + N\mathcal{E}^{\star SL_4[N^2]}_{\Lambda_1}(U)\right]^2\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} + \dots
\end{aligned} \tag{6.10}$$

In either case, the rank 0, rank-1 and rank-2 orbits are now interpreted on the type II side as tree-level, one-loop and two-loop contributions, with an additional one-loop contribution in the rank-2 orbit for $N>1$. The tree-level contributions are consistent with the observation in [74] that the tree-level F^4 coupling of four twisted gauge bosons is governed by a genus-one modular integral, and the analogous statement in [75] that the tree-level ∇^2F^4 coupling of four twisted gauge bosons is governed by a genus-two modular integral. For $N=1$, the one-loop contributions are proportional to the vector Eisenstein series of $SL(4,\mathbb{Z})$, or equivalently the spinor Eisenstein series under the T-duality group $O(3,3)$ of the torus T^3, while the two-loop contribution is proportional to the square of the same. For $N>1$ they are similar generalizations of Eisenstein series of $SL_4[N^2]$, and there is an additional contribution at 1-loop in rank two Eisenstein series of $SL_4[N^2]$, that are linear combinations of vector Eisenstein series of the group $O(3,3)$ of automorphisms of $I\!I_{2,2} \oplus I\!I_{1,1}[N]$.[21]

It would be interesting to confirm these predictions by independent one-loop and two-loop computations in type II string theory. Finally, the exponentially suppressed terms in (6.8) can be ascribed to D-brane, NS5-branes and KK (6,1)-brane instantons as explained in more detail in [74].

6.2 Weak coupling limit in type II string theory compactified on $K3 \times T^2$

Let us now consider the expansion of the exact $\nabla^2 F^4$ and $\mathcal{R}^2 F^2$ terms in $D = 4$ obtained in (5.70) at weak coupling on the type II side. Recall that the heterotic axiodilaton S corresponds respectivly to the 2-torus Kähler modulus $T_{\rm A}$ in type IIA, and the 2-torus complex structure modulus $U_{\rm B}$ in type IIB, while the type II axiodilaton $S_{\rm A} = S_{\rm B}$ corresponds to the Kähler modulus T of the 2-torus on the heterotic side, *i.e.*

$$S = T_{\rm A} = U_{\rm B} \ , \qquad T = S_{\rm A} = S_{\rm B} \ , \qquad U = U_{\rm A} = T_{\rm B} \ . \tag{6.11}$$

In order to expand at small type II string coupling, *i.e.* at large T_2, we decompose the lattice $\Lambda_{2k-2,6}$ into $\Lambda_{2k-4,4} \oplus I\!I_{1,1} \oplus I\!I_{1,1}[N]$ as in section 5.2.

For simplicity we shall use the type IIB moduli in this section, and we won't write explicitly the label B. So S is now the type IIB axiodilaton with $S_2 = \frac{1}{g_s{}^2}$. For simplicity we shall only consider the perturbative terms for the Maxwell fields in the RR sector, corresponding to indices $\alpha, \beta, \ldots$ along the sublattice $\Lambda_{2k-4,4}$. Using the results of [22], the perturbative part of the exact F^4 coupling is given by

$$\begin{aligned} \widehat{F}^{(2k-2,6)}_{\alpha\beta\gamma\delta\ \rm II} &= \frac{1}{g_s{}^2} F^{(2k-4,4)}_{\alpha\beta\gamma\delta\ \rm II} + \frac{3}{2\pi}\delta_{(\alpha\beta}\delta_{\gamma\delta)}\Big(\frac{\hat{\mathcal{E}}_1(NT) + \hat{\mathcal{E}}_1(T) + \hat{\mathcal{E}}_1(NU) + \hat{\mathcal{E}}_1(U) + \frac{12}{\pi}\log g_s}{N+1}\Big) \\ &= S_2 F^{(2k-4,4)}_{\alpha\beta\gamma\delta}(t) - \frac{3}{8\pi^2}\delta_{(\alpha\beta}\delta_{\gamma\delta)}\log(S_2{}^k T_2{}^k U_2{}^k |\Delta_k(T)\Delta_k(U)|^2) \ , \end{aligned} \tag{6.12}$$

where the first term matches the tree-level coupling computed in [74], while the second term is related by supersymmetry to the $\mathcal{R}^2$ coupling computed in [76, 77].

The exact $\nabla^2 F^4$ coupling is obtained from (5.70) after dropping the logarithmic terms in R,

$$\begin{aligned} \widehat{G}^{(2k-2,6)}_{ab,cd\ \rm NP}(U,\varphi) = \ & \widehat{G}^{(2k-2,6)}_{ab,cd}(\varphi) - \frac{3}{4\pi}\delta_{\langle ab,}\delta_{cd\rangle}\Big(\frac{\hat{\mathcal{E}}_1(NU) + \hat{\mathcal{E}}_1(U)}{N+1}\Big)^2 \\ & -\frac{1}{4}\delta_{\langle ab,}\Big(\frac{N\hat{\mathcal{E}}_1(NU) - \hat{\mathcal{E}}_1(U)}{N^2-1}\widehat{G}^{(2k-2,6)}_{cd\rangle}(\varphi) + \frac{N\hat{\mathcal{E}}_1(U) - \hat{\mathcal{E}}_1(NU)}{N^2-1}\,{}^{\varsigma}\widehat{G}^{(2k-2,6)}_{cd\rangle}(\varphi)\Big) \ , \end{aligned} \tag{6.13}$$

where U parametrizes $SL(2)/SO(2)$ and φ the Grassmannian on $\Lambda_{2k-2,6}$. The power-behaved term of $\widehat{G}^{(2k-2,6)}_{ab,cd}(\varphi)$ in this limit is given in equations (5.36), (4.59) and (5.60) for $q = 6$,

[21] The condition that $SL(4,\mathbb{Z})$ preserves the lattice $I\!I_{2,2} \oplus I\!I_{1,1}[N]$, so $Q_{34} = 0[N]$, implies that the matrices are either of type $\left(\begin{smallmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{smallmatrix}\right)$ mod N or of type $\left(\begin{smallmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$ mod N, but the condition that the it preserves the dual lattice, *i.e.* $Q_{ij} \in \mathbb{Z}$ for $ij \neq 12$ with $NQ_{12} \in \mathbb{Z}$ forbids the second.

$\upsilon = N$, $R = \sqrt{S_2} = \frac{1}{g_s}$, and $\varphi = t$ the K3 moduli of the Grassmanian $G_{(2k-4,4)}$. After expanding around $q = 6 + 2\epsilon$ and subtracting polar terms,[22] we find

$$\widehat{G}^{(2k-2,6)}_{\alpha\beta,\gamma\delta}(\varphi) \sim \frac{1}{{g_s}^4}\widehat{G}^{(2k-4,4)}_{\alpha\beta,\gamma\delta}(t) - \frac{3}{4\pi}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\Big(\frac{\hat{\mathcal{E}}_1(NT) + \hat{\mathcal{E}}_1(T) + \frac{12}{\pi}\log g_s}{N+1}\Big)^2$$
$$- \frac{1}{4{g_s}^2}\delta_{\langle\alpha\beta,}\Big(\frac{\frac{N\hat{\mathcal{E}}_1(NT)-\hat{\mathcal{E}}_1(T)}{N-1} + \frac{6}{\pi}\log g_s}{N+1}\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t) + \frac{\frac{N\hat{\mathcal{E}}_1(T)-\hat{\mathcal{E}}_1(NT)}{N-1} + \frac{6}{\pi}\log g_s}{N+1}{}^{\varsigma}\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t)\Big) \tag{6.14}$$

To compute the power-like term of $\widehat{G}^{(2k-2,6)}_{ab}(\varphi)$ one proceeds as in [22], and finds after expanding around $q = 6 + 2\epsilon$ and subtracting polar terms

$$\widehat{G}^{(2k-2,6)}_{\alpha\beta}(\varphi) \sim \frac{1}{{g_s}^2}\Big(\widehat{G}^{(2k-4,4)}_{\alpha\beta}(t) + \frac{2N}{N+1}\delta_{\alpha\beta}\big(\hat{\mathcal{E}}_1(T) - \hat{\mathcal{E}}_1(NT)\big)\Big)$$
$$+ \frac{12}{N+1}\frac{1}{2\pi}\delta_{\alpha\beta}\Big(\frac{12}{\pi}\log(g_s) + \hat{\mathcal{E}}_1(T) + \hat{\mathcal{E}}_1(NT)\Big)\ . \tag{6.15}$$

The function ${}^{\varsigma}\widehat{G}^{(2k-2,6)}_{ab}(\varphi)$ is obtained by acting with the involution ς on the K3 moduli t and on the Kähler moduli T by Fricke duality $T \to -\frac{1}{NT}$, so that

$${}^{\varsigma}\widehat{G}^{(2k-2,6)}_{\alpha\beta}(\varphi) \sim \frac{1}{{g_s}^2}\Big({}^{\varsigma}\widehat{G}^{(2k-4,4)}_{\alpha\beta}(t) + \frac{2N}{N+1}\delta_{\alpha\beta}\big(\hat{\mathcal{E}}_1(NT) - \hat{\mathcal{E}}_1(T)\big)\Big)$$
$$+ \frac{12}{N+1}\frac{1}{2\pi}\delta_{\alpha\beta}\Big(\frac{12}{\pi}\log(g_s) + \hat{\mathcal{E}}_1(T) + \hat{\mathcal{E}}_1(NT)\Big)\ . \tag{6.16}$$

Collecting all terms, we obtain the complete perturbative $\nabla^2 F^4$ coupling in $D = 4$,

$$\widehat{G}^{(2k-2,6)}_{\alpha\beta,\gamma\delta\ \mathrm{II}} = \frac{1}{{g_s}^4}\widehat{G}^{(2k-4,4)}_{\alpha\beta,\gamma\delta}(t)$$
$$- \frac{1}{4(N+1){g_s}^2}\delta_{\langle\alpha\beta,}\Big(\Big(\frac{N\hat{\mathcal{E}}_1(NT) - \hat{\mathcal{E}}_1(T) + N\hat{\mathcal{E}}_1(NU) - \hat{\mathcal{E}}_1(U)}{N-1} + \frac{6}{\pi}\log g_s\Big)\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t)$$
$$+ \Big(\frac{N\hat{\mathcal{E}}_1(T) - \hat{\mathcal{E}}_1(NT) + N\hat{\mathcal{E}}_1(U) - \hat{\mathcal{E}}_1(NU)}{N-1} + \frac{6}{\pi}\log g_s\Big)\,{}^{\varsigma}\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t)$$
$$-2N\delta_{\gamma\delta\rangle}\frac{(\hat{\mathcal{E}}_1(T) - \hat{\mathcal{E}}_1(NT))(\hat{\mathcal{E}}_1(U) - \hat{\mathcal{E}}_1(NU))}{N-1}\Big)$$
$$- \frac{3}{4\pi}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\Big(\frac{\hat{\mathcal{E}}_1(NT) + \hat{\mathcal{E}}_1(T) + \hat{\mathcal{E}}_1(NU) + \hat{\mathcal{E}}_1(U) + \frac{12}{\pi}\log g_s}{N+1}\Big)^2\ . \tag{6.17}$$

The terms involving $\log g_s$ originate as usual from the mixing between the local and non-local terms in the effective action [78]. The result (6.17) is manifestly invariant under the exchange of U and T, hence identical in type IIA and type IIB. It is also invariant under the combined Fricke duality $T \to -\frac{1}{NT}$, $U \to -\frac{1}{NU}$, $t \to \varsigma t$ [27], which is built in our conjecture for the

[22] Note that the lattice is fixed to $\Lambda_{2k-2,6}$, and the expansion in $q = 6 + 2\epsilon$ only applies to the numerical value of the various exponents, just like if one introduced a regularizing factor of $|\Omega_2|^\epsilon$ in the genus 2 integral.

non-perturbative amplitude. In the maximal rank case, (6.17) must be replaced by [23]

$$G^{(22,6)}_{\alpha\beta,\gamma\delta\ \mathrm{II}} = \frac{1}{g_s^4}\widehat{G}^{(20,4)}_{\alpha\beta,\gamma\delta}(t) + \frac{3}{4\pi g_s^2}\delta_{\langle\alpha\beta,}\Big(\log(T_2|\eta(T)|^4) + \log(U_2|\eta(U)|^4) - 2\log g_s\Big)G^{(20,4)}_{\gamma\delta\rangle}(t)$$
$$- \frac{27}{4\pi^3}\delta_{\langle\alpha\beta}\delta_{\gamma\delta\rangle}\Big(\log(T_2|\eta(T)|^4) + \log(U_2|\eta(U)|^4) - 2\log g_s\Big)^2 . \quad (6.18)$$

It would be interesting to check these predictions by explicit perturbative computations in type II string theory. Noting that

$$\frac{\hat{\mathcal{E}}_1(NT) + \hat{\mathcal{E}}_1(T)}{N+1} = -\frac{1}{4\pi}\log(T_2^{\,k}|\Delta_k(T)|) , \qquad \hat{\mathcal{E}}_1(T) = -\frac{1}{4\pi}\log(T_2^{\,12}|\Delta(T)|) , \quad (6.19)$$

the 2-loop contribution on the last line of (6.17) takes the suggestive form

$$-\frac{3}{(4\pi)^3}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\big(\log(S_2^{\,k}T_2^{\,k}U_2^{\,k}|\Delta_k(T)\Delta_k(U)|^2)\big)^2 . \quad (6.20)$$

The $(\log g_s)^2$ term is consistent with the 2-loop logarithmic divergence of the four-photon amplitude [79] (recall that the $\log g_s$ can be traced back to the logarithm of the Mandelstam variables in the full amplitude, and therefore to the logarithm supergravity divergences [78, 22]). The term linear in $\log g_s$ in (6.20), corresponding to the t_8F^4 form factor divergence, can be rewritten as

$$-\frac{3k}{4\pi}\log g_s\delta_{\langle\alpha\beta,}\Big(\frac{1}{12g_s^2}\big(\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t) + {}^{\varsigma}\widehat{G}^{(2k-4,4)}_{\gamma\delta\rangle}(t)\big) - \delta_{\gamma\delta\rangle}\frac{1}{8\pi^2}\log(T_2^{\,k}U_2^{\,k}|\Delta_k(T)\Delta_k(U)|^2)\Big)$$
$$= \ -\frac{3}{4\pi}\log g_s\,\delta_{\langle\alpha\beta,}\Big(\frac{1}{g_s^2}F^{(2k-4,4)\eta}_{\gamma\delta\rangle\eta}(t) - \delta_{\gamma\delta\rangle}\frac{2k}{(4\pi)^2}\log(T_2^{\,k}U_2^{\,k}|\Delta_k(T)\Delta_k(U)|^2)\Big)$$
$$= \ -\frac{3}{4\pi}\log g_s\,\delta_{\langle\alpha\beta,}\widehat{F}^{(2k-2,6)c}_{\gamma\delta\rangle c\ \ \mathrm{II}} , \quad (6.21)$$

where one uses integration by part on the definition of $F^{(2k-2,6)}$ with $-\frac{1}{i\pi}\frac{\partial}{\partial\tau}\frac{1}{\Delta_k(\tau)} = \frac{k}{12}(E_2(\tau)+NE_2(N\tau))/\Delta_k(\rho)$, and $\delta_{(ab}\delta_{cd)}\delta^{cd} = \frac{2k}{3}\delta_{ab}$. Ignoring these logarithmic contributions, the two-loop coupling (6.20) does not depend on the K3 moduli, as required by supersymmetry, and might be computable in topological string theory.

The amplitudes with two photons in the Ramond sector and two gravitons can be obtained in the same way. It is non vanishing only when the two photons have the same polarization and the two gravitons have the opposite polarization. In type IIB, the complex amplitude is obtained through the Kähler derivative of the same function (6.17) with respect to U, *e.g.* in the maximal rank case

$$R^{(22,6)}_{\alpha\beta\ \mathrm{II}} = -\frac{9}{2\pi^3}\delta_{\alpha\beta}\hat{E}_2(U)\Big(\log(T_2|\eta(T)|^4) + \log(U_2|\eta(U)|^4) - 2\log g_s\Big) + \frac{1}{4\pi g_s^2}\hat{E}_2(U)\,G^{(20,4)}_{\alpha\beta}(t) , \quad (6.22)$$

or with respect to T in type IIA. The $\log g_s$ term can be interpreted as the divergence of the form factor of the operator $\mathcal{R}\,F_R^2$ (where $F_R^{\hat{\alpha}}$ are the graviphoton field strengths) belonging to the $\mathcal{R}^2$-type supersymmetric invariant.

[23] Note that $G^{(20,4)}_{\alpha\beta}$ is finite for the maximal rank case, whereas $\widehat{G}^{(2k-4,4)}_{\alpha\beta}$ requires in general a regularization due to the 1-loop supergravity divergence in six dimensions.

6.3 Type I string theory

The heterotic string with gauge group $Spin(16)/\mathbb{Z}_2$ is dual to the type I superstring [80]. In ten dimensions, the duality inverts the string coupling $e^{\phi} \to e^{-\phi}$ and identifies the Einstein frame metrics. After compactifying on a torus T^q, the effective string coupling g_s in $10-q$ dimensions and volume V_s in string units are given by

$$g_s = e^{(1-\frac{q}{8})\phi}\, V^{-\frac{1}{2}} \ , \qquad V_s = e^{\frac{q}{4}\phi}\, V \ , \tag{6.23}$$

where V is the volume of the torus T^q measured in ten-dimensional Planck units. It follows that the heterotic/type I duality identifies

$$g_s = {g'_s}^{-1+\frac{q}{4}}\, {V'_s}^{-1+\frac{q}{8}} \ , \quad V_s = {g'_s}^{-\frac{q}{2}}\, {V'_s}^{1-\frac{q}{4}} \ , \tag{6.24}$$

where the unprimed variables refer to the heterotic string while the primed variables refer to the type I string, the unit volume metric U_{ij} being the same on both sides. In particular, the weak coupling regime $g'_s \to 0$ on the type I side corresponds to strong coupling on the heterotic side when $D = 10 - q > 6$, or to weak coupling when $D < 6$. In either case, the volume V'_s in heterotic string units scales to infinity. Furthermore, in dimension $D > 4$ the coefficients of the F^4 and $\nabla^2 F^4$ couplings are purely perturbative on the heterotic side, so their type I dual expansion is obtained by taking the large volume limit. We shall now show that the resulting weak coupling expansion on the type I side has only powers of the form ${g'_s}^{2h+b-2}$, compatible with type I genus expansion where b is the number of boundaries or crosscaps. For simplicity we focus on the maximal rank model and consider only gauge bosons with indices along the D_{16} lattice, but these considerations easily extend to CHL models and gauge bosons with indices along the torus.

Using (6.3) and similar computations using the same method, we find that for $D > 4$, the F^4 coupling at weak type I coupling is given by

$$\begin{aligned}
{g'_s}^{2\frac{q-2}{8-q}} F^I_{\alpha\beta\gamma\delta} &= \frac{{V'_s}^{\frac{1}{2}}}{g'_s} F^{(16,0)}_{\alpha\beta\gamma\delta} + \frac{3}{2\pi} g'_s {V'_s}^{\frac{3}{2}} \delta_{(\alpha\beta}\delta_{\gamma\delta)} + \frac{9}{\pi^2} {g'_s}^2 {V'_s}^{2-\frac{6}{q}} \sum_{m^i\in\mathbb{Z}^n}{}' (\pi\, m^i U_{ij} m^j)^{-3} \delta_{(\alpha\beta}\delta_{\gamma\delta)} \\
&+ \frac{{V'_s}^{1-\frac{2}{q}}}{\pi} \sum_{\substack{Q\in D_{16}\\ Q^2=2}} \sum_{m\in\mathbb{Z}^q}{}' e^{2\pi i m^i Q\cdot a_i} \Big(\frac{Q_\alpha Q_\beta Q_\gamma Q_\delta}{m^i U_{ij} m^j} - \frac{3{V'_s}^{\frac{1}{2}-\frac{2}{q}}}{2\pi^2} g'_s \frac{\delta_{(\alpha\beta} Q_\gamma Q_{\delta)}}{(m^i U_{ij} m^j)^2} + \frac{3{V'_s}^{1-\frac{4}{q}}}{8\pi^4} {g'_s}^2 \frac{\delta_{(\alpha\beta}\delta_{\gamma\delta)}}{(m^i U_{ij} m^j)^3} \Big) \\
&+ \dots
\end{aligned} \tag{6.25}$$

where the dots stand for non-perturbative corrections associated to D1 branes wrapping two-cycles inside T^q. The first term is the expected disk amplitude of 4 open string gauge bosons in type I, while the remaining terms of order ${g'_s}^0, {g'_s}^1, {g'_s}^2$ are contributions from genus 1, 3/2, 2 open Riemann surfaces [81]. Similarly, the $\nabla^2 F^4$ coupling reads

$$\begin{aligned}
{g'_s}^{\frac{2q}{8-q}} G^I_{\alpha\beta,\gamma\delta} &= \frac{1}{{g'_s}^2} G^{(16,0)}_{\alpha\beta,\gamma\delta} - \frac{V'_s}{4}\delta_{\langle\alpha\beta,}G^{(16,0)}_{\gamma\delta\rangle} - \frac{3}{2\pi} g_s {V'_s}^{\frac{3}{2}-\frac{6}{q}} \sideset{}{'}\sum_{m^i\in\mathbb{Z}^q} (\pi\, m^i U_{ij} m^j)^{-3} \delta_{\langle\alpha\beta,}G^{(16,0)}_{\gamma\delta\rangle} \\
&+ \frac{3}{4\pi}\left(-{g'_s}^2 {V'_s}^2 + \frac{2}{\pi^2} {g'_s}^3 {V_s}^{\frac{5}{2}-\frac{6}{q}} - \frac{1}{V'_s}\Big[\frac{6}{\pi} {g'_s}^2 {V'_s}^{2-\frac{6}{q}} \sideset{}{'}\sum_{m^i\in\mathbb{Z}^q} (\pi m^i U_{ij} m^j)^{-3}\Big]^2 \right) \delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} \\
&- g'_s \frac{{V'_s}^{\frac{3}{2}-\frac{2}{q}}}{4\pi} \delta_{\langle\alpha\beta,} \sum_{\substack{Q\in D_{16}\\ Q^2=2 \\ m\in\mathbb{Z}^q\setminus\{0\}}} e^{2\pi i m^i Q\cdot a_i} \left(\frac{Q_\gamma Q_{\delta\rangle}}{m^i U_{ij} m^j} - \frac{g'_s {V'_s}^{\frac{1}{2}-\frac{2}{q}}}{4\pi^2} \frac{12 Q_\gamma Q_{\delta\rangle} - \delta_{\gamma\delta\rangle}}{(m^i U_{ij} m^j)^2} + \frac{{g'_s}^2}{8\pi^4} \frac{3{V'_s}^{1-\frac{4}{q}} \delta_{\gamma\delta\rangle}}{(m^i U_{ij} m^j)^3} \right) \\
&+ 3 \sideset{}{'}\sum_{\substack{Q\in D_{16}\\ Q^2=2}} \bar{G}^{(16,0)}_{\langle\alpha\beta,}(Q) \sideset{}{'}\sum_{m\in\mathbb{Z}^q} e^{2\pi i m^i Q\cdot a_i} \Big(Q_\gamma Q_{\delta\rangle} \frac{{V'_s}^{1-\frac{4}{q}}}{(\pi\, m^i U_{ij} m^j)^2} - \frac{g'_s}{2\pi} \delta_{\gamma\delta\rangle} \frac{{V'_s}^{\frac{3}{2}-\frac{6}{q}}}{(\pi\, m^i U_{ij} m^j)^3} \Big) \\
&+ g'_s \sideset{}{'}\sum_{\substack{Q_i\in D_{16}\oplus D_{16}\\ Q_i^2\le 2}} \int_{\mathcal{P}_2} \frac{d^3\Omega_2}{|\Omega_2|^3}\, C(Q, \tfrac{1}{g'_s}\Omega_2) \sideset{}{'}\sum_{A\in M_{q,2}(\mathbb{Z})/GL(2,\mathbb{Z})} P_{\alpha\beta,\gamma\delta}(Q, \tfrac{1}{g'_s}\Omega_2) e^{2\pi i a\cdot A\cdot Q - \pi {V'_s}^{\frac{2}{q}-\frac{1}{2}} \mathrm{Tr}[A\Omega_2^{-1}A^\intercal U]} \\
&+ \dots \qquad (6.26)
\end{aligned}$$

where the dots stand for non-perturbative corrections associated to D1 branes wrapping two-cycles inside T^q. In the last term, the integral of the constant part $C^F(Q)$ of the Fourier coefficient of $1/\Phi_{10}$ produces a matrix-variate Gamma function and contributes to order g'_s, ${g'_s}^2$, ${g'_s}^3$. The jumps in $C(Q, \frac{1}{g'_s}\Omega_2)$ dues to poles at large $|\Omega_2|$ give terms of order ${g'_s}^\ell$ for $\ell = 0, 1, 2, 3, 4$, which are sourced by the square of the 'Wilson lines corrections' in (6.25) in the differential equation (2.26). The jumps due to deep poles where $|\Omega_2| \le \frac{1}{4}$ lead to further corrections of order $e^{-2\pi/g'_s}$, which can be ascribed to D1-anti-D1 instantons.

The first term $\frac{1}{{g'_s}^2} G^{(16,0)}_{\alpha\beta,\gamma\delta}$ in (6.26) is however apparently inconsistent with type I perturbation theory, since the four-photon amplitude only involves open string vertex operators which cannot couple at genus zero. Fortunately, we can show that this term vanishes for the heterotic $Spin(16)/\mathbb{Z}_2$ string. Indeed, using the same integration by parts argument as in section 3.3 (the boundaries at the cusp do not contribute at $q = 0$) one finds

$$20 G^{(16,0)}_{\alpha\beta,\gamma\delta} + \delta_{\langle\alpha\beta,}G^{(16,0)\epsilon}_{\gamma\delta\rangle,\epsilon} = \pi F^{\epsilon\zeta}{}^{(16,0)}_{\langle\alpha\beta,} F^{(16,0)}_{\gamma\delta\rangle,\epsilon\zeta} = 0 \; , \qquad (6.27)$$

which vanishes because [22, (5.42)]

$$F^{(16,0)}_{\alpha\beta\gamma\delta} = 16\pi \delta_{\alpha\beta\gamma\delta} \; , \qquad (6.28)$$

where $\delta_{\alpha\beta\gamma\delta}$ is equal to one if all for indices are equal and zero otherwise. It follows that

$$G^{(16,0)}_{\alpha\beta,\gamma\delta} = \mathrm{R.N.} \int_{Sp(4,\mathbb{Z})\backslash\mathcal{H}_2} \frac{d^3\Omega_1 d^3\Omega_2}{|\Omega_2|^3} \frac{\Gamma^{(2)}_{D_{16}}[P_{\alpha\beta,\gamma\delta}]}{\Phi_{10}} = 0 \; , \qquad (6.29)$$

so (6.26) is indeed consistent with type I perturbation theory. In particular, the genus-two double trace $\nabla^2(\mathrm{Tr}F^2)^2$ coupling computed in [43] for the ten-dimensional $Spin(16)/\mathbb{Z}_2$

heterotic string vanishes. It is worth stressing that the same genus-two coupling in the $E_8\times E_8$ string does *not* vanish. [24]

Let us now discuss the form of the non-perturbative corrections in some more details. For any $D\geq 3$, the contributions of the non-Abelian rank-2 orbit are non-perturbative on the type I side, with an action given for vanishing gauge charge by

$$S_{D1}=2\pi\frac{V_s'^{\frac{2}{q}}}{g_s'V_s'^{\frac{1}{2}}}\sqrt{\tfrac{1}{2}U_{ik}U_{jl}N^{ij}N^{kl}}+2\pi\mathrm{i}B_{ij}N^{ij}\ , \tag{6.33}$$

where $g_s'V_s'^{\frac{1}{2}}=e^{\phi'}$ is the ten-dimensional type I string coupling. This can be ascribed to Euclidean D1 branes wrapping T^q with charge $N^{ij}\in\mathbb{Z}^q\wedge\mathbb{Z}^q$. For $D=4$, the NS5-brane instantons on the heterotic side translate into D5-brane instantons on the type I side, with action $S_2=\frac{V_s'^{\frac{1}{2}}}{g_s'}$. For $D=3$, the non-perturbative heterotic contributions with vanishing NUT charge translate into type I D5-brane instantons with wrapping number N_i and gauge charge $Q\in D_{16}$, with action

$$\mathrm{Re}[S_{D5}]=2\pi\frac{V_s'^{\frac{6}{7}}}{g_s'V_s'^{\frac{1}{2}}}\sqrt{(U^{-1})^{ij}(N_i+a_i\cdot Q)(N_j+a_j\cdot Q)}\ , \tag{6.34}$$

Finally, non-perturbative heterotic instantons with non-vanishing NUT charge translate into type I Taub-NUT instantons, with action

$$\mathrm{Re}[S_{TN}]=2\pi\frac{V_s'^{\frac{8}{7}}}{g_s'^2V_s'}\sqrt{U_{ij}(k^i+g_s'V_s'^{\frac{3}{14}}(U^{-1})^{ik}\tilde N_k)(k^j+g_s'V_s'^{\frac{3}{14}}(U^{-1})^{jl}\tilde N_l)}\ , \tag{6.35}$$

with

$$\tilde N_i=N_i+a_i\cdot Q+(\tfrac{1}{2}a_i\cdot a_j+B_{ij})k^j\ . \tag{6.36}$$

Thus, all non-perturbative effects on the heterotic side map to expected instanton effects in type I.

6.4 Exact $\nabla^2\mathcal{H}^4$ couplings in type IIB on K3

Finally, let us briefly discuss the couplings of four self-dual three-form field strengths $\mathcal{H}^a_{\mu\nu\rho}$ in type IIB string theory compactified on K3. In [74, 46], it was conjectured that the exact $\mathcal{H}^4$ coupling is given by a genus-one modular integral of the form (1.4) for the non-perturbative Narain lattice $\Lambda_{21,5}$ of signature $(p,q)=(21,5)$. This was later generalized to the case of

[24] For the $E_8\times E_8$ heterotic string, we have instead

$$20G^{(16,0)}_{\alpha\beta,\gamma\delta}+\delta_{\langle\alpha\beta,}G^{(16,0)\epsilon}_{\gamma\delta\rangle,\epsilon}=\pi F^{\epsilon\zeta\,(16,0)}_{\langle\alpha\beta,}F^{(16,0)}_{\gamma\delta\rangle,\epsilon\zeta}=\frac{64\pi^3}{3}(4P^1_{\langle\alpha\beta,}P^1_{\gamma\delta\rangle}+4P^2_{\langle\alpha\beta,}P^2_{\gamma\delta\rangle}-7P^1_{\langle\alpha\beta,}P^2_{\gamma\delta\rangle})\ , \tag{6.30}$$

with

$$F^{(16,0)}_{\alpha\beta\gamma\delta}=8\pi(P^1_{(\alpha\beta}P^1_{\gamma\delta)}+P^2_{(\alpha\beta}P^2_{\gamma\delta)}-P^1_{(\alpha\beta}P^2_{\gamma\delta)})\ , \tag{6.31}$$

and $P^i_{\alpha\beta}$ the two projectors to the eight-dimensional subspaces. One computes that $G^{(16,0)\gamma}_{\alpha\beta,\gamma}=0$, such that

$$G^{(16,0)}_{\alpha\beta,\gamma\delta}=\int_{Sp(4,\mathbb{Z})\backslash\mathcal{H}_2}\frac{\mathrm{d}^3\Omega_1\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\frac{\Gamma^{(2)}_{E_8\oplus E_8}[P_{\alpha\beta,\gamma\delta}]}{\Phi_{10}}=\frac{16\pi^3}{15}(4P^1_{\langle\alpha\beta,}P^1_{\gamma\delta\rangle}+4P^2_{\langle\alpha\beta,}P^2_{\gamma\delta\rangle}-7P^1_{\langle\alpha\beta,}P^2_{\gamma\delta\rangle})\ . \tag{6.32}$$

This reproduces the relative coefficient in [82, (7.4)].

the $\nabla^2\mathcal{H}^4$ couplings, which were conjectured to be given exactly by a genus-two modular integral of the form (1.5) for the same lattice [46]. These conjectures follow from our exact non-perturbative results for the maximal rank model[25] in $D = 3$ by decompactification. Here, we briefly discuss the weak coupling expansion of these results on the type IIB side, using the results of section 4.1.

At weak coupling, the even self-dual lattice $\Lambda_{21,5}$ decomposes into $\Lambda_{20,4} \oplus I\!I_{1,1}$, where the 'radius' associated to the second factor is related to the type IIB string coupling by $g_s = 1/R$. The low energy action in the string frame was recalled in [22, 4.40], after changing the metric for $\gamma = g_s\gamma_E$ and renormalising the Ramond-Ramond field as $\mathcal{H}^a = g_s H^a$. The coefficient of the $\nabla^2\mathcal{H}^4$ coupling in this frame is then given by $G^{(21,5)}_{\alpha\beta,\gamma,\delta}$, without any further power of g_s. The results of section 4.1 then provide its weak coupling expansion,

$$\begin{aligned} G^{(21,5)}_{\alpha\beta,\gamma\delta} &= \frac{1}{g_s^2} G^{(20,4)}_{\alpha\beta,\gamma\delta} - \frac{1}{4}\delta_{\langle\alpha\beta,}G^{(20,4)}_{\gamma\delta\rangle} - \frac{3g_s^2}{4\pi}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} \\ &\quad + \frac{3}{{g_s}^4}\sideset{}{'}\sum_{Q\in\Lambda^*_{21,5}} e^{2\pi \mathrm{i} Q\cdot a}\,\bar{G}^{(20,4)}_{\langle\alpha\beta,}(Q,\varphi)\Big(Q_{L\gamma}Q_{L\delta\rangle}\frac{K_0(\frac{2\pi}{g_s^2}\sqrt{2Q_R^2})}{\sqrt{2Q_R^2}} - \frac{{g_s}^2}{4\pi}\delta_{\gamma\delta\rangle}K_1(\tfrac{2\pi}{g_s^2}\sqrt{2Q_R^2})\Big) \\ &\quad + \sideset{}{'}\sum_{Q\in\Lambda^*_{21,5}} e^{-\frac{4\pi}{{g_s}^2}\sqrt{2Q_R^2}}\,K_{\alpha\beta,\gamma\delta}(g_s,Q_L,Q_R)\,. \end{aligned} \tag{6.37}$$

The first term proportional to $G^{(20,4)}_{\alpha\beta,\gamma\delta}$ is recognized as a tree-level contribution in type IIB on $K3$ [75]. The second and third terms correspond to one-loop and two-loop corrections, and to our knowledge have not been computed independently yet. The second line of (6.37) corresponds to exponentially suppressed terms that originate from D3, D1, D(-1) branes wrapped on K3 [74], or, formally, to Fourier coefficients of the coupling coefficient. The function $\bar{G}^{(21,5)}_{\alpha\beta}$ is the sum of a finite and a polar contribution and reads

$$\bar{G}^{(20,4)}_{\alpha\beta,-\frac{Q^2}{2}}(Q,\varphi) = \sum_{\substack{d>0\\ Q/d\in\Lambda_{21,5}}} d^2\, c_k\big(-\tfrac{Q^2}{2d^2}\big)\, G^{(20,4)}_{\alpha\beta,-\frac{Q^2}{2d^2}}\big(\tfrac{Q}{d}\big)\,, \tag{6.38}$$

where $\bar{G}^{(20,4)}_{\alpha\beta} = \bar{G}^{(20,4)}_{F,\alpha\beta} + \bar{G}^{(20,4)}_{P,\alpha\beta}$ as described in §4. The last line corresponds to instanton anti-instanton corrections that are missed by the unfolding method, and which could be computed by solving (E.51) for $Q = 0$.

[25] Note that CHL models in $D = 3$ all decompactify to the same model in $D = 6$, whose rank is fixed by the constraints of anomaly cancellation.

A Compendium on Siegel modular forms

A.1 Action on $\mathcal{H}_2$

The Siegel's upper half plane $\mathcal{H}_2$ is the space of complex symmetric matrices

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \quad \text{such that} \quad |\Omega_2| > 0, \quad \rho_2 > 0, \quad \sigma_2 > 0 \ , \tag{A.1}$$

where Ω_1 and Ω_2 denote the real and imaginary parts of Ω, similarly for ρ, v, σ, and $|\Omega_2|$ is the determinant of Ω_2. An element $\gamma \in Sp(4,\mathbb{Z})$,

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ , \quad \gamma\varepsilon\gamma^t = \varepsilon \ , \quad \varepsilon = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} \ , \tag{A.2}$$

with

$$A^\intercal C - C^\intercal A = 0, \qquad B^\intercal D - D^\intercal B = 0, \qquad A^\intercal D - C^\intercal B = \mathbf{1}_2 \ , \tag{A.3}$$

acts on $\mathcal{H}_2$ via

$$\Omega \mapsto \tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1} \ . \tag{A.4}$$

A standard fundamental domain for the action of $Sp(4,\mathbb{Z})$ on $\mathcal{H}_2$ is the domain $\mathcal{F}_2$ defined by the conditions [83]

$$-\frac{1}{2} < \rho_1, \sigma_1, v_1 < \frac{1}{2} \ , \quad 0 < 2v_2 \le \rho_2 \le \sigma_2 \ , \quad |C\Omega + D| \ge 1 \tag{A.5}$$

for all $\gamma \in Sp(4,\mathbb{Z})$ (the latter condition needs only to be checked for a finite number of γ's).

The period matrix of a genus-two curve Σ takes values in $\mathcal{H}_2 \backslash S$, where S is the union of the quadratic divisors

$$D(m_i, j, n_i; \Omega) \equiv m^2 - m^1\rho + n_1\sigma + n_2(\rho\sigma - v^2) + jv = 0 \ , \tag{A.6}$$

parametrized by five integers $M = (m^1, m^2, j, n_1, n_2)$. M transform as a vector under $Sp(4) \sim O(3,2)$ such that the signature (2,3) quadratic form

$$\Delta(M) = j^2 + 4(m^1 n_1 + m^2 n_2) \tag{A.7}$$

and the parity of j stay invariant. Under a combined action of γ on Ω and M, the divisor $D(M;\Omega) = 0$ stays invariant,

$$D(\tilde{M}; \tilde{\Omega}) = [\det (C\Omega + D)]^{-1} \, D(M, \Omega) \ . \tag{A.8}$$

The divisor S is the locus where the curve Σ degenerates into the connected sum of two genus-one curves. Its intersection with the fundamental domain $\mathcal{F}_2$ is simply the divisor $v = 0$.

On the other hand, the boundary of the domain $\mathcal{F}_2$ consists of three strata, i) $\sigma_2 \to +\infty$ where Σ degenerates into a one-loop graph, ii) $\rho_2, \sigma_2 \to +\infty$ at the same rate where Σ degenerates into a figure-eight graph, and iii) $v_2, \rho_2, \sigma_2 \to +\infty$ at the same rate where Σ degenerates into a sunset diagram (see Figure 1). In order to discuss these limits, it will be useful to introduce the alternative parametrizations for Ω_2,

$$\Omega_2 = \begin{pmatrix} \rho_2 & \rho_2 u_2 \\ \rho_2 u_2 & t + \rho_2 u_2^2 \end{pmatrix} = \frac{1}{V\tau_2} \begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix} \tag{A.9}$$

such that the limits i) and iii) correspond to $t \to +\infty$ and $V \to 0$, respectively.

We now give the explicit action of some relevant subgroups of $Sp(4,\mathbb{Z})$:

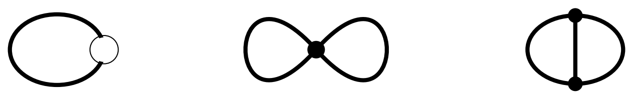

Figure 1: Degenerations of a genus-two Riemann surface corresponding to the boundary strata of the fundamental domain $\mathcal{F}_2$. The white node in i) corresponds to a torus while the black dots in ii), iii) corresponds to a sphere. The 'figure-eight' and 'sunset' diagrams in supergravity are obtained by replacing the black dots in ii) and iii) with supergravity 4-point and 3-point interactions, and attaching four external gauge bosons to the edges.

1. $SL(2)_\rho$ (leaving $t = \sigma_2 - v_2^2/\rho_2$ invariant)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\rho = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad : \quad (\rho, v, \sigma)' = \left(\frac{a\rho + b}{c\rho + d}, \frac{v}{c\rho + d}, \sigma - \frac{cv^2}{c\rho + d} \right) , \qquad \text{(A.10)}$$
$$(m^1, m^2, j, n_1, n_2)' = \big(dm^1 + cm^2, bm^1 + am^2, j, an_1 - bn_2, dn_2 - cn_1 \big)$$

We denote by S_ρ the generator $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_\rho$.

2. $SL(2)_\sigma$ (leaving $t' = \rho_2 - v_2^2/\sigma_2$ invariant):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} \quad : \quad (\rho, v, \sigma)' = \left(\rho - \frac{cv^2}{c\sigma + d}, \frac{v}{c\sigma + d}, \frac{a\sigma + b}{c\sigma + d} \right) , \qquad \text{(A.11)}$$
$$(m^1, m^2, j, n_1, n_2)' = \big(am^1 + bn_2, am^2 - bn_1, j, dn_1 - cm^2, cm^1 + dn_2 \big)$$

We denote by S_σ the generator $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_\sigma$.

3. Heis_ρ (leaving Ω_2 invariant):

$$\begin{aligned} T_{\lambda,\mu,\kappa} &= \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad : \\ (\rho, v, \sigma)' &= \big(\rho, \mu + \lambda\rho + v, \sigma + \kappa + 2\lambda v + \lambda\mu + \lambda^2 \rho \big) \ , \\ (m^1, m^2)' &= \big(m^1 + j\lambda + (\mu n_2 - \lambda n_1)\lambda + \kappa n_2, m^2 - \mu(j - \lambda n_1 + \mu n_2) - \kappa n_1 \big) \ , \\ (j, n_1, n_2)' &= (j - 2\lambda n_1 + 2\mu n_2, n_1, n_2) \end{aligned} \qquad \text{(A.12)}$$

4. Heis_σ (leaving Ω_2 invariant):

$$\tilde{T}_{\lambda,\mu,\kappa} = \begin{pmatrix} 1 & \lambda & \kappa & \mu \\ 0 & 1 & \mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 \end{pmatrix} \tag{A.13}$$

5. $GL(2,\mathbb{Z})_S$ (leaving $V = 1/\sqrt{|\Omega_2|}$ invariant):

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix}_S = \begin{pmatrix} a & -b & 0 & 0 \\ -c & d & 0 & 0 \\ 0 & 0 & d & c \\ 0 & 0 & b & a \end{pmatrix} \quad : \\ &(\rho, v, \sigma)' = \left(a^2\rho - 2ab\, v + b^2\sigma, -ac\rho + (ad+bc)v - bd\,\sigma, c^2\,\rho - 2cd\,v + d^2\,\sigma\right) \ , \\ &(m^1, m^2)' = \left(-c^2\, n_1 - cd\, j + d^2\, m^1, m^2\right) \ , \\ &(j, n_1, n_2)' = \left(j + 2bcj - 2bd\, m^1 + 2ac\, n_1, a^2\, n_1 + ab\, j - b^2\, m^1, n_2\right) \end{aligned} \tag{A.14}$$

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad (ad - bc = 1) \ , \quad \tau \mapsto \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \quad (ad - bc = -1) \ . \tag{A.15}$$

Defining $\Omega_2 = \begin{pmatrix} L_1 + L_2 & L_2 \\ L_2 & L_2 + L_3 \end{pmatrix}$, the permutations of the L_i's correspond to the following elements of $GL(2,\mathbb{Z})_S$:

$$L_1 \leftrightarrow L_2 : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_S \ , \ L_2 \leftrightarrow L_3 : \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}_S \ , \ L_1 \leftrightarrow L_3 : \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}_S \ . \tag{A.16}$$

6. $\mathbb{Z}^3$ (leaving Ω_2 invariant):

$$\begin{aligned} &T_{r_1,r_2,r_3} = \begin{pmatrix} 1 & 0 & r_1 & r_2 \\ 0 & 1 & r_2 & r_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad : \quad (\rho, v, \sigma)' = (\rho + r_1, v + r_2, \sigma + r_3) \ , \\ &(m^1, m^2)' = \left(m^1 + n_2 r_3, m^2 - n_2 r_2^2 - j r_2 + m^1 r_1 - n_1 r_3 + n_2 r_1 r_3\right) \ , \\ &(j, n_1, n_2)' = (j + 2n_2 r_2, n_1 - n_2 r_1, n_2) \end{aligned} \tag{A.17}$$

7. $\sigma_{\rho\leftrightarrow\sigma}$:

$$\begin{aligned} &h_{\rho\leftrightarrow\sigma} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad : \quad (\rho, v, \sigma)' = (\sigma, v, \rho), \\ &(m^1, m^2, j, n_1, n_2)' = (n_1, -m^2, -j, m_1, -n_2) \end{aligned} \tag{A.18}$$

A.2 Siegel modular forms and congruence subgroups

For any $\gamma \in Sp(4,\mathbb{R})$ and integer w, we define the Petersson slash operator

$$(\Phi|_w\gamma)(\Omega) = [\det(C\Omega + D)]^{-w}\, \Phi\left((A\Omega + B)(C\Omega + D)^{-1}\right) \ . \tag{A.19}$$

A Siegel modular form $\Phi(\Omega) = \Phi(\rho, \sigma, v)$ of weight w under a subgroup $\Gamma \subset Sp(4,\mathbb{Z})$ satisfies $\Phi|_w\gamma = \Phi$ for any $\gamma \in \Gamma$. We shall be mostly interested in modular forms with respect to the congruence subgroups of $Sp(4,\mathbb{Z})$ (A.2), denoting its elements by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$,

1. $\Gamma_{2,0}(N)$, restricting to elements with $C = 0 \bmod N$;
2. $\tilde{\Gamma}_{2,0}(N) = S_\rho \cdot \Gamma_{2,0}(N) \cdot S_\rho^{-1}$, its conjugate w.r.t. S_ρ (A.10);
3. $\hat{\Gamma}_{2,0}(N) = S_\sigma \cdot \Gamma_{2,0}(N) \cdot S_\sigma^{-1}$, conjugate of $\Gamma_{2,0}(N)$ w.r.t. S_σ (A.11);
4. $\Gamma_{2,1}(N) \subset \Gamma_{2,0}(N)$, restricting to elements with $A = D = 1 \bmod N$;
5. $\Gamma_2(N) \subset \Gamma_{2,1}(N)$, restricting to elements with $B = 0 \bmod N$;
6. $\Gamma_{2,e_r}(N)$ the subgroup fixing the vector $(0,0,0,r)$ modulo N;
7. $\Gamma_{2,0,e_r}(N) = \Gamma_{2,e_r}(N) \cap \Gamma_{2,0}(N)$.

The indices of these subgroups inside $Sp(4,\mathbb{Z})$ are summarized below:

$$\begin{aligned}
\Big|Sp(4,\mathbb{Z})/\Gamma_2(N)\Big| &= N^{10}\prod_{p|N}\Big(1-\frac{1}{p^2}\Big)\Big(1-\frac{1}{p^4}\Big)\,,\\
\Big|Sp(4,\mathbb{Z})/\Gamma_{2,0}(N)\Big| &= N^{3}\prod_{p|N}\Big(1+\frac{1}{p}\Big)\Big(1+\frac{1}{p^2}\Big)\,,\\
\Big|Sp(4,\mathbb{Z})/\Gamma_{2,1}(N)\Big| &= N^{7}\prod_{p|N}\Big(1-\frac{1}{p^2}\Big)\Big(1-\frac{1}{p^4}\Big)\,,\\
\Big|Sp(4,\mathbb{Z})/\Gamma_{2,0,e_r}(N)\Big| &= \tfrac{N^5}{r^2}\prod_{p|N}\Big(1+\frac{1}{p}\Big)\Big(1+\frac{1}{p^2}\Big)\prod_{p'|\frac{N}{r}}\Big(1-\frac{1}{p'^2}\Big)\,,\\
\Big|Sp(4,\mathbb{Z})/\Gamma_{2,e_1}(N)\Big| &= N^{4}\prod_{p|N}\Big(1-\frac{1}{p^4}\Big)\,,
\end{aligned} \tag{A.20}$$

where p, p' run over primes. Indeed the corresponding quotients can be understood as

$$\begin{aligned}
\Big|\Gamma_{2,1}(N)/\Gamma_2(N)\Big| &= N^3 = \Big(\mathbb{Z}/N\mathbb{Z}\Big)^3_B\,,\\
\Big|\Gamma_{2,0,e_1}(N)/\Gamma_{2,1}(N)\Big| &= N^2\prod_{p|N}\Big(1-\frac{1}{p}\Big) = \Big|\Gamma_1(N)/\Gamma(N)\Big|_D \times \Big|\Gamma_{2,0}(N)/\Gamma_{2,1}(N)\Big|_{\left[\begin{smallmatrix} a_1 b_1 \\ c_1 d_1 \end{smallmatrix}\right]}\,,\\
\Big|\Gamma_{2,0,e_r}(N)/\Gamma_{2,0,e_1}(N)\Big| &= r^2\prod_{\substack{p|N \\ p\nmid\frac{N}{r}}}\Big(1-\frac{1}{p^2}\Big) = \Big|\Gamma_1(\tfrac{N}{r})/\Gamma_1(N)\Big|_D\,,\\
\Big|\Gamma_{2,0}(N)/\Gamma_{2,0,e_r}(N)\Big| &= (\tfrac{N}{r})^2\prod_{p|\frac{N}{r}}\Big(1-\frac{1}{p^2}\Big) = \Big|SL(2,\mathbb{Z})/\Gamma_1(\tfrac{N}{r})\Big|_D\,,
\end{aligned} \tag{A.21}$$

where the subscript indicates the embedding $SL(2,\mathbb{Z}) \subset Sp(4,\mathbb{Z})$ of the coset representatives.

Of special interest is the Hecke congruence subgroup $\Gamma_{2,0}(N)$ and its conjugates $\tilde{\Gamma}_{2,0}(N)$, $\hat{\Gamma}_{2,0}(N)$. The cosets of $Sp(4,\mathbb{Z})/\Gamma_{2,0}(N)$ are in one-to-one correspondence with cosets of

$GSp_N(4,\mathbb{Z})/Sp(4,\mathbb{Z})$, where $GSp_N(4,\mathbb{Z})$ is the group of symplectic similitudes such that $\gamma\varepsilon\gamma^t = N\varepsilon$. For N prime, the $(N+1)(N^2+1) = 1+N+N^2+N^3$ cosets can be chosen as (see e.g. [84, p.6])

$$\begin{pmatrix} N &&& \\ & N && \\ && 1 & \\ &&& 1 \end{pmatrix}, \quad \begin{pmatrix} 1 && a & \\ & N && \\ && N & \\ &&& 1 \end{pmatrix}, \quad \begin{pmatrix} N &&& \\ -a & 1 && b \\ && 1 & a \\ &&& N \end{pmatrix}, \quad \begin{pmatrix} 1 && a & c \\ & 1 & c & b \\ && N & \\ &&& N \end{pmatrix}, \tag{A.22}$$

with $a,b,c = 0\dots N-1$. For $\Phi(\rho,\sigma,v)$ a Siegel modular form of weight w for the full Siegel modular group $Sp(4,\mathbb{Z})$, the sum of the action of these elements on Φ produces again a Siegel modular form for the full Siegel modular group $Sp(4,\mathbb{Z})$, which is the image of Φ under the N-th Hecke operator H_N,

$$\begin{aligned} H_N\Phi(\rho,\sigma,v) =& \Phi(N\rho,N\sigma,Nv) + N^{-w} \sum_{a \bmod N} \Phi\left(\frac{\rho+a}{N}, N\sigma, v\right) \\ &+ N^{-w} \sum_{a,b \bmod N} \Phi\left(N\rho, \tfrac{\sigma-2av+a^2\rho+b}{N}, v-a\rho\right) + N^{-2w} \sum_{a,b,c \bmod N} \Phi\left(\tfrac{\rho+a}{N}, \tfrac{\sigma+b}{N}, \tfrac{v+c}{N}\right) . \end{aligned} \tag{A.23}$$

The first term in this sum, $\Phi(N\rho,N\sigma,Nv)$, is then a Siegel modular form for $\Gamma_{2,0}(N)$. The 'Fricke involution'

$$\Phi \mapsto \Phi|_w \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{N}} \\ 0 & 0 & -\frac{1}{\sqrt{N}} & 0 \\ 0 & -\sqrt{N} & 0 & 0 \\ \sqrt{N} & 0 & 0 & 0 \end{pmatrix} = \Phi|_w \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{N}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0 & 0 & 0 \\ 0 & \sqrt{N} & 0 & 0 \end{pmatrix} = [N|\Omega|]^{-w}\Phi\left(-(N\Omega)^{-1}\right) \tag{A.24}$$

takes a Siegel modular form Φ of weight w under $\Gamma_{2,0}(N)$ into another one. Similarly,

$$\widetilde{\Phi} \mapsto \widetilde{\Phi}|_w \begin{pmatrix} 0 & 1/\sqrt{N} & 0 & 0 \\ \sqrt{N} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{N} \\ 0 & 0 & 1/\sqrt{N} & 0 \end{pmatrix} = \widetilde{\Phi}(\sigma/N, N\rho, v) \tag{A.25}$$

takes a Siegel modular form $\tilde{\Phi}$ of weight w under $\widetilde{\Gamma}_{2,0}(N)$ into another one.

A.3 Genus two theta series

The genus-two even theta series are defined as

$$\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ b_1,b_2 \end{smallmatrix}\right](\Omega,\zeta) = \sum_{p_1,p_2\in\mathbb{Z}} e^{\,i\pi\,\left(p_1+\frac{a_1}{2},p_2+\frac{a_2}{2}\right)^t \Omega \begin{pmatrix} p_1+\frac{a_1}{2} \\ p_2+\frac{a_2}{2} \end{pmatrix} + 2\pi i \left(p_1+\frac{a_1}{2},p_2+\frac{a_2}{2}\right)^t \begin{pmatrix} \zeta_1+\frac{b_1}{2} \\ \zeta_2+\frac{b_2}{2} \end{pmatrix}} \tag{A.26}$$

with $a_i, b_i \in \mathbb{Z}$. It is an even or odd function of $\zeta = (\zeta_1,\zeta_2)^t$ depending on the parity of $a_1b_1+a_2b_2$. When it is even, the value at $\zeta=0$ is the Thetanullwert denoted by $\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ b_1,b_2 \end{smallmatrix}\right](\Omega)$. The value of a_i, b_i modulo two defines a spin structure labelled by the column vector $\kappa =$

$(a_1,a_2,b_1,b_2)^t$, whose parity is that of $a_1b_1+a_2b_2$. Under translations of the characteristics by even integers,

$$\vartheta^{(2)}\left[\begin{smallmatrix} a_1+2a_1', a_2+2a_2' \\ b_1+2b_1', b_2+2b_2' \end{smallmatrix}\right](\Omega,\zeta) = e^{\mathrm{i}\pi(a_1b_1'+a_2b_2')}\,\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ b_1,b_2 \end{smallmatrix}\right](\Omega,\zeta)\ . \tag{A.27}$$

Under $Sp(4,\mathbb{Z})$ transformations,

$$\vartheta^{(2)}[\tilde\kappa](\tilde\Omega,\tilde\zeta) = \epsilon(\kappa,\gamma)\,[\det(C\Omega+D)]^{1/2}\vartheta^{(2)}[\kappa](\Omega,\zeta) \tag{A.28}$$

with $\tilde\Omega=(A\Omega+B)(C\Omega+D)^{-1}$, $\tilde\zeta=(C\Omega+D)^{-t}\zeta$,

$$\tilde\kappa = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}\kappa + \frac{1}{2}\mathrm{diag}\begin{pmatrix} CD^t \\ AB^t \end{pmatrix} \bmod 2 \tag{A.29}$$

and $\epsilon(\kappa,\gamma)$ is an 8-th root of unity. In particular,

$$\begin{aligned}
\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ b_1,b_2 \end{smallmatrix}\right](\rho+1,\sigma,v) &= e^{-\frac{\mathrm{i}\pi}{4}a_1(a_1+2)}\,\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ a_1+b_1+1,b_2 \end{smallmatrix}\right](\rho,\sigma,v)\\
\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ b_1,b_2 \end{smallmatrix}\right](\rho,\sigma+1,v) &= e^{-\frac{\mathrm{i}\pi}{4}a_2(a_2+2)}\,\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ b_1,a_2+b_2+1 \end{smallmatrix}\right](\rho,\sigma,v)\\
\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ b_1,b_2 \end{smallmatrix}\right](\rho,\sigma,v+1) &= e^{-\frac{\mathrm{i}\pi}{2}a_1a_2}\,\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ b_1+a_2,b_2+a_1 \end{smallmatrix}\right](\rho,\sigma,v)\\
\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ b_1,b_2 \end{smallmatrix}\right](\rho,\sigma+\rho-2v,v-\rho) &= \vartheta^{(2)}\left[\begin{smallmatrix} a_1-a_2,a_2 \\ b_1,b_1+b_2 \end{smallmatrix}\right](\rho,\sigma,v)\\
\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ b_1,b_2 \end{smallmatrix}\right](-1/\rho,\sigma-v^2/\rho,v/\rho) &= \sqrt{-\mathrm{i}\rho}\,e^{\frac{\mathrm{i}\pi}{2}a_1b_1}\,\vartheta^{(2)}\left[\begin{smallmatrix} b_1,a_2 \\ -a_1,b_2 \end{smallmatrix}\right](\rho,\sigma,v)\\
\vartheta^{(2)}\left[\begin{smallmatrix} a_1,a_2 \\ b_1,b_2 \end{smallmatrix}\right](\rho-v^2/\rho,-1/\sigma,v/\sigma) &= \sqrt{-\mathrm{i}\sigma}\,e^{\frac{\mathrm{i}\pi}{2}a_2b_2}\,\vartheta^{(2)}\left[\begin{smallmatrix} a_1,b_2 \\ b_1,-a_2 \end{smallmatrix}\right](\rho,\sigma,v)
\end{aligned} \tag{A.30}$$

In the separating degeneration limit,

$$\vartheta^{(2)}\left[\begin{smallmatrix} a_1a_2 \\ b_1b_2 \end{smallmatrix}\right] \overset{v\to 0}{\to} \begin{cases} \vartheta\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right](\rho)\,\vartheta\left[\begin{smallmatrix} a_2 \\ b_2 \end{smallmatrix}\right](\sigma) & \left[\begin{smallmatrix} a_1a_2 \\ b_1b_2 \end{smallmatrix}\right]\neq\left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix}\right] \\ \frac{v}{2\pi\mathrm{i}}\,\vartheta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]'(\rho)\,\vartheta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]'(\sigma) & \left[\begin{smallmatrix} a_1a_2 \\ b_1b_2 \end{smallmatrix}\right]=\left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix}\right] \end{cases} \tag{A.31}$$

where $\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ is the genus-one theta series,

$$\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \sum_{p\in\mathbb{Z}} e^{\mathrm{i}\pi(p+\frac{a}{2})^2\tau)+\mathrm{i}\pi b(p+\frac{a}{2})} \tag{A.32}$$

and $\vartheta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]'(\rho)=2\pi\eta^3\ ,\quad \vartheta_2\vartheta_3\vartheta_4=2\eta^3$.

A.4 Meromorphic Siegel modular forms from Borcherds products

In the context of heterotic CHL orbifolds, two meromorphic Siegel modular forms Φ_{k-2} and $\tilde\Phi_{k-2}$ of weight $k-2$ under $\Gamma_{2,0}(N)$ and $\tilde\Gamma_{2,0}(N)$, respectively play an essential rôle. They are given by infinite products [85] [86, 3.16,3.17] [28, C.18,C.19][26]

$$\Phi_{k-2}(\rho,\sigma,v) = e^{2\pi\mathrm{i}(\rho+\sigma+v)}\prod_{r=0}^{N-1}\prod_{\substack{k',\ell,j\in\mathbb{Z}\\ k',\ell\geq 0,\\ j<0\text{ for }k'=\ell=0}}\left(1-e^{2\pi\mathrm{i}r/N}\,e^{2\pi\mathrm{i}(k'\sigma+\ell\rho+jv)}\right)^{\sum_{s=0}^{N-1}e^{-\frac{2\pi\mathrm{i}rs}{N}}c^{(0,s)}_{j \bmod 2}(4k'\ell-j^2)} \tag{A.33}$$

[26]Note that $\Phi_{k-2}(\rho,\sigma,v)$ and $\tilde\Phi_{k-2}(\rho,\sigma,v)$ are denoted by $\widehat\Phi(\rho,\sigma,v)$ and $\widetilde\Phi(\sigma,\rho,v)$ in [28], while $\widetilde\Phi(\rho,\sigma,v)$ coincides with $\Phi_{g,e}(\rho,\sigma,v)$ in [29].

$$\tilde{\Phi}_{k-2}(\rho,\sigma,v) = e^{2\pi i(\sigma+\frac{1}{N}\rho+v)} \prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},\ell,j\in\mathbb{Z}\\ k',\ell\geq 0,\\ j<0 \text{ for } k'=\ell=0}} \left(1-e^{2\pi i(k'\rho+\ell\sigma+jv)}\right)^{\sum_{s=0}^{N-1} e^{-\frac{2\pi i s\ell}{N}} c^{(r,s)}_{j \bmod 2}(4k'\ell-j^2)} . \tag{A.34}$$

Here, $c_b^{(r,s)}(n)$ with $b \in \mathbb{Z}/(2\mathbb{Z})$ are Fourier coefficients of a family of index 1 weak Jacobi forms

$$F^{(r,s)}(\tau,z) = \sum_{j\in\mathbb{Z}, n\in\frac{\mathbb{Z}}{N}} c^{(r,s)}_{j \bmod 2}(4n-j^2)\, e^{2\pi i(n\tau+jz)} \tag{A.35}$$

obtained as a twining/twisted elliptic genus of the $\mathbb{Z}_N$ orbifold of $K3$. In particular, for $N = 1,2,3,5,7$ and $1 \leq s \leq N-1$,

$$\begin{aligned} F^{(0,0)} = \frac{2}{N}\phi_{0,1} \ , \quad F^{(0,s)} &= \frac{2}{N(N+1)}\phi_{0,1} + \frac{2(E_2(\tau)-NE_2(N\tau))}{(N+1)(N-1)}\phi_{-2,1} \\ F^{(r,s)} &= \frac{2}{N(N+1)}\phi_{0,1} - \frac{2(E_2(\frac{\tau+s/r}{N})-NE_2(\tau))}{N(N+1)(N-1)}\phi_{-2,1} \end{aligned} \tag{A.36}$$

where $\phi_{0,1} = 4\sum_{i=2,3,4}\left(\frac{\vartheta_i(\tau,z)}{\vartheta_i(\tau,0)}\right)^2$, $\phi_{-2,1} = \vartheta_1^2(\tau,z)/\eta^6$ are the standard generators of the ring of weak Jacobi forms, and $s/r = sk$ where $kr = 1 \bmod N$. It is also useful to consider the discrete Fourier transform of the coefficients $c_b^{(r,s)}(n)$ with respect to s,

$$\hat{c}_b^{(r,s)}(n) = \sum_{s'=0}^{N-1} e^{-2\pi i ss'/N} c_b^{(r,s')}(n) \ . \tag{A.37}$$

Using the property $\hat{c}_j^{(r,s)}(n) = \hat{c}_j^{(s,r)}(n)$, one can rewrite $\tilde{\Phi}_{k-2}$ as [29, 5.10]

$$\tilde{\Phi}_{k-2}(\rho,\sigma,v) = e^{2\pi i(\sigma+\frac{\rho}{N}+v)} \prod_{\substack{k'\in\mathbb{Z}/N,\ell,j\in\mathbb{Z}\\ k',\ell\geq 0, j<0 \text{ for } k'=\ell=0}} \left(1-e^{2\pi i(k'\rho+\ell\sigma+jv)}\right)^{\sum_{s=0}^{N-1} e^{-2\pi i sk'} c^{(\ell,s)}_{j \bmod 2}(4k'\ell-j^2)} \tag{A.38}$$

From this relation, it is manifest that $\tilde{\Phi}_{k-2}$ is invariant under the Fricke involution [85, §C],

$$\tilde{\Phi}_{k-2}(\rho,\sigma,v) = \tilde{\Phi}_{k-2}(N\sigma,\rho/N,v) = \tilde{\Phi}_{k-2}|\begin{pmatrix} 0 & \sqrt{N} & 0 & 0 \\ 1/\sqrt{N} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{N} \\ 0 & 0 & \sqrt{N} & 0 \end{pmatrix} , \tag{A.39}$$

and therefore, so is Φ_{k-2},

$$\Phi_{k-2}(\Omega) = (N|\Omega|)^{2-k}\,\Phi_{k-2}(-1/(N\Omega)) = \Phi_{k-2}|\begin{pmatrix} 0 & 0 & 0 & 1/\sqrt{N} \\ 0 & 0 & -1/\sqrt{N} & 0 \\ 0 & -\sqrt{N} & 0 & 0 \\ \sqrt{N} & 0 & 0 & 0 \end{pmatrix} . \tag{A.40}$$

It is worth recalling that the infinite products (A.33) and (A.34) arise as theta liftings of $F^{(r,s)}$, namely

$$\begin{aligned}
\text{R.N.} \int_{\mathcal{F}_1} \mathrm{d}\mu_1 \sum_{\substack{m_1,n_1,j\in\mathbb{Z}\\ m_2\in\mathbb{Z}/N, n_2\in N\mathbb{Z}+r}} q^{\frac{1}{2}p_L^2}\, \bar{q}^{\frac{1}{2}p_R^2}\, e^{2\pi i m_2 s}\, h^{(r,s)}_{j \bmod 2} &= -\,2\log|\Omega_2|^{2(k-2)}|\Phi_{k-2}(\rho,\sigma,v)|^2 \\
\text{R.N.} \int_{\mathcal{F}_1} \mathrm{d}\mu_1 \sum_{\substack{m_1,m_2,n_2,j\in\mathbb{Z}\\ n_1\in\mathbb{Z}+\frac{r}{N}}} q^{\frac{1}{2}p_L^2}\, \bar{q}^{\frac{1}{2}p_R^2}\, e^{2\pi i m_1 s/N}\, h^{(r,s)}_{j \bmod 2} &= -\,2\log|\Omega_2|^{2(k-2)}|\tilde{\Phi}_{k-2}(\sigma,\rho,v)|^2
\end{aligned} \tag{A.41}$$

where $h_b^{(r,s)}$, $b\in\mathbb{Z}/(2\mathbb{Z})$ is the vector valued modular form arising in the theta series decomposition

$$F^{(r,s)}(\tau,z) = h_0^{(r,s)}(\tau)\,\vartheta_3(2\tau,2z) + h_1^{(r,s)}(\tau)\,\vartheta_2(2\tau,2z)\ , \qquad h_b^{(r,s)} = \sum_{n\in\frac{1}{N}\mathbb{Z}-\frac{b^2}{4}} c_b^{(r,s)}(4n) e^{2\pi i n\tau} \tag{A.42}$$

and p_R, p_L are projections of the vector $M=(m_1,n_1,j,m^2,n^2)$ such that

$$p_R^2 = \frac{1}{2|\Omega_2|}\left|m^2 - m^1\rho + n_1\sigma + n_2(\rho\sigma - v^2) + jv\right|^2\ , \quad \frac{1}{2}(p_L^2-p_R^2) = m^1 n_1 + m^2 n_2 + \frac{j^2}{4}\ . \tag{A.43}$$

From the infinite product representation, one can easily read off the location of the zeros and poles which intersect the cusp $\Omega_2 = i\infty$. Such zeros (respectively, poles) arise from the existence of positive (respectively, negative) coefficients $\hat{c}^{(r,s)}(m)$ with $m<0$, known as polar coefficients. For $N=1,2,3,5,7$, the only positive polar term is $\hat{c}_1^{(0,0)}(-1)=2$, which implies that Φ_{k-2} and $\tilde{\Phi}_{k-2}$ have a double zero on the diagonal locus $v=0$, where they behave according to [27]

$$\begin{aligned}
\Phi_{k-2}(\rho,\sigma,v) &\sim -\,4\pi^2 v^2\,\Delta_k(\rho)\,\Delta_k(\sigma)\ ,\\
\tilde{\Phi}_{k-2}(\rho,\sigma,v) &\sim -\,4\pi^2\, v^2\,\Delta_k(\rho/N)\,\Delta_k(\sigma)\ ,
\end{aligned} \tag{A.44}$$

where $\Delta_k(\rho) = \eta^k(\rho)\eta^k(N\rho)$. It can be shown that all zeros of Φ_{k-2} and $\tilde{\Phi}_{k-2}$ occur only on the divisor $v=0$ and its images under the congruence subgroups $\Gamma_{2,0}(N)$ and $\tilde{\Gamma}_{2,0}(N)$, respectively. For $N=1,2,3$, $\hat{c}_1^{(0,0)}(-1)$ is the only polar term, so Φ_{k-2} and $\tilde{\Phi}_{k-2}$ are actually holomorphic Siegel modular forms, corresponding to the Igusa cusp form Φ_{10} for $N=1$, or the cusp forms Φ_6 of level 2 and Φ_4 of level 3 constructed in [87, 88]. In particular, Φ_{10} is proportional to the product of the square of the ten even Thetanullwerte,

$$\Phi_{10} = 2^{-20}\left(\vartheta^{(2)}\begin{bmatrix}00\\00\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}01\\00\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}10\\00\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}00\\10\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}00\\01\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}01\\10\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}10\\01\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}11\\00\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}00\\11\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}11\\11\end{bmatrix}\right)^2\ , \tag{A.45}$$

while Φ_6 is proportional to the product of the square of 6 among the ten even Thetanullwerte,

$$\Phi_6 = 2^{-12}\left(\vartheta^{(2)}\begin{bmatrix}01\\00\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}01\\10\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}10\\00\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}10\\01\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}11\\00\end{bmatrix}\vartheta^{(2)}\begin{bmatrix}11\\11\end{bmatrix}\right)^2\ . \tag{A.46}$$

[27]Note that these two equations are consistent with (2.10) since Δ_k is invariant under the Fricke involution, *i.e.* $\Delta_k(-1/\rho) = (i\sqrt{N})^{-k}\,\rho^k\,\Delta_k(\rho/N)$

For $N = 5$ and $N = 7$, there are additional polar coefficients but they are all negative, implying that Φ_{k-2} and $\tilde{\Phi}_{k-2}$ have poles,

$$\begin{aligned}
&N = 5 : \hat{c}_1^{(1,1)}(-\frac{1}{5}) = \hat{c}_1^{(2,3)}(-\frac{1}{5}) = \hat{c}_1^{(3,2)}(-\frac{1}{5}) = \hat{c}_1^{(4,4)}(-\frac{1}{5}) = -2 \\
&N = 7 : \hat{c}_1^{(1,1)}(-\frac{3}{7}) = \hat{c}_1^{(2,4)}(-\frac{3}{7}) = \hat{c}_1^{(3,5)}(-\frac{3}{7})) = \hat{c}_1^{(4,2)}(\frac{3}{7})) = \hat{c}_1^{(5,3)}(\frac{3}{7})) = \hat{c}_1^{(6,6)}(\frac{3}{7}) = -1 \ .
\end{aligned} \tag{A.47}$$

Note however that the Siegel modular forms relevant for our problem are the inverse of Φ_{k-2} and $\tilde{\Phi}_{k-2}$, which have a double pole on the diagonal locus $v = 0$ for all N.

From the infinite product representation one can also read-off the behavior of $1/\Phi_{k-2}$ and $1/\tilde{\Phi}_{k-2}$ in the maximal non-separating degeneration $\Omega_2 \to \infty$, obtained by setting $e^{2\pi i\rho} = q_1 q_3, e^{2\pi i\sigma} = q_2 q_3, e^{2\pi i v} = q_3$, and Taylor expanding near $q_i \to 0$:

$$\begin{aligned}
\frac{1}{\Phi_{10}} &= \frac{1}{q_1 q_2 q_3} + 2\sum_{i<j} \frac{1}{q_i q_j} + \left[24 \sum_{i=1}^{3} \frac{1}{q_i} + 3 \sum_{i \neq j < k \neq i} \frac{q_i}{q_j q_k} \right] \\
&\quad + \left[0 + 48 \sum_{i \neq j} \frac{q_i}{q_j} + 4 \sum_{i \neq j < k \neq i} \frac{q_i^2}{q_j q_k} \right] + \mathcal{O}(q_i) && \text{(A.48)} \\
\frac{1}{\Phi_{k-2}} &= \frac{1}{q_1 q_2 q_3} + 2\sum_{i<j} \frac{1}{q_i q_j} + \left[\frac{24}{N+1} \sum_{i=1}^{3} \frac{1}{q_i} + 3 \sum_{i \neq j < k \neq i} \frac{q_i}{q_j q_k} \right] \\
&\quad + \left[\frac{48N}{N^2-1} + \frac{48}{N+1} \sum_{i \neq j} \frac{q_i}{q_j} + 4 \sum_{i \neq j < k \neq i} \frac{q_i^2}{q_j q_k} \right] + \mathcal{O}(q_i) && \text{(A.49)} \\
\frac{1}{\tilde{\Phi}_{k-2}} &= \frac{1}{q_1^{1/N} q_2 \, q_3^{1/N}} + \frac{24}{N+1} \frac{1}{q_2} - \frac{48}{N^2-1} + \dots && \text{(A.50)}
\end{aligned}$$

where the dot denotes terms involving positive powers of q_i. Since $Sp(4, \mathbb{Z})$ and its congruence subgroup $\Gamma_{2,0}(N)$ contains $GL(2, \mathbb{Z})_\tau$, the expansion of $1/\Phi_{10}$ and $1/\Phi_{k-2}$ for $N = 2, 3, 5, 7$ are manifestly invariant under permutations of q_1, q_2, q_3. In contrast, the expansion of $1/\tilde{\Phi}_{k-2}$ is only invariant under permutations of q_1 and q_3.

A.5 Fourier-Jacobi coefficients and meromorphic Jacobi forms

Given a meromorphic Siegel modular form $1/\Phi(\rho, \sigma, v)$ of weight $-w$, the Fourier expansion with respect to σ

$$1/\Phi(\rho, \sigma, v) = \sum_{m \gg -\infty} \psi_m(\rho, v) \, e^{2\pi i m\sigma} \tag{A.51}$$

gives rise to an infinite series of meromorphic Jacobi forms $\psi_m(\rho, v)$ of fixed weight w and increasing index m. If Φ is modular under the full Siegel modular group, then $m \in \mathbb{Z}$ and ψ_m is a Jacobi form for the full Jacobi group $SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$, *i.e.* it satisfies

$$\begin{aligned}
\psi_m(\rho, v + \lambda\rho + \mu) &= e^{-2\pi i m(\lambda^2\rho + 2\lambda v)} \, \psi_m(\rho, v) && \text{(A.52)} \\
\psi_m\left(\frac{a\rho + b}{c\rho + d}, \frac{v}{c\rho + d} \right) &= (c\rho + d)^w \, e^{\frac{2\pi i m c v^2}{c\rho + d}} \, \psi_m(\rho, v) && \text{(A.53)}
\end{aligned}$$

for all integers a,b,c,d,λ,μ such that $ad-bc=1$. If Φ is modular under a congruence subgroup $\Gamma\subset Sp(4,\mathbb{Z})$, then

1. for $\Gamma=\Gamma_{2,0}(N)$, then $m\in\mathbb{Z}$ and ψ_m is a Jacobi form for the Jacobi group $\Gamma_0(N)\ltimes\mathbb{Z}^2$, *i.e.* it satisfies (A.52),(A.53) for all integers a,b,c,d,λ,μ such that $ad-bc=1$ and $c=0 \bmod N$
2. For $\Gamma=\tilde\Gamma_{2,0}(N)$, then ψ_m is a Jacobi form for $\Gamma^0(N)\ltimes\mathbb{Z}^2$, *i.e.* it satisfies (A.52),(A.53) for all integers a,b,c,d,λ,μ such that $ad-bc=1$ and $b=0 \bmod N$;
3. For $\Gamma=\hat\Gamma_{2,0}(N)$, then $m\in\mathbb{Z}/N$ and ψ_m is a Jacobi form for $\Gamma_0(N)\ltimes(N\mathbb{Z}\times\mathbb{Z})$ satisfies (A.52),(A.53) for all integers a,b,c,d,λ,μ such that $ad-bc=1$, $c=0 \bmod N$ and $\lambda=0 \bmod N$ (examples of Jacobi forms of index n/N with these periodicity properties are given by $\phi(N\rho,v)$ where $\phi(\rho,v)$ is an ordinary Jacobi form of index n under the full Jacobi group).

In particular, the Fourier-Jacobi expansion of the inverse of the Igusa cusp form is given by [89, (5.16)],

$$\frac{1}{\Phi_{10}}=\frac{1}{\phi_{-2,1}\Delta}q_\sigma^{-1}+24\frac{\mathcal{P}}{\Delta}+\frac{9\phi_{0,1}^2+3E_4\phi_{-2,1}^2}{4\phi_{-2,1}\,\Delta}q_\sigma+\mathcal{O}(q_\sigma^2) \tag{A.54}$$

where

$$\mathcal{P}(\rho,v)=\frac{\phi_{0,1}}{12\phi_{-2,1}}=\frac{1}{(2\pi\mathrm{i})^2}\left[-\partial_v^2\log\vartheta_1(\rho,v)+2\pi\mathrm{i}\partial_\rho\log\eta^2\right] \tag{A.55}$$

is (up to a factor $(2\pi\mathrm{i})^2$) the Weierstrass function, a weak Jacobi form of weight 2 and index 0.

In the case of CHL orbifolds with $N=2,3,5,7$, it will be useful to introduce $\hat\Phi_{k-2}$, the image of Φ_{k-2} under an inversion S_σ,

$$\hat\Phi_{k-2}(\Omega)=(\mathrm{i}\sqrt{N})^k\sigma^{-(k-2)}\Phi_{k-2}(S_\sigma\circ\Omega)=\tilde\Phi_{k-2}(\Omega)\Big|_{\rho\leftrightarrow\sigma}\ . \tag{A.56}$$

where we chose the normalization such that $\hat\Phi_{k-2}\sim-4\pi^2v^2\Delta_k(\rho)\Delta_k(\sigma/N)$ near the divisor $v=0$. The Fourier-Jacobi expansion of Φ_{k-2} and $\hat\Phi_{k-2}$ is given by

$$\frac{1}{\Phi_{k-2}}=\frac{\eta^6(\rho)}{\Delta_k(\rho)\,\vartheta_1^2(\rho,v)}q_\sigma^{-1}+\psi_0+\mathcal{O}(q_\sigma)\ , \tag{A.57}$$

$$\frac{1}{\hat\Phi_{k-2}}=-\frac{\eta^6(N\rho)}{\Delta_k(\rho)\,\vartheta_1^2(N\rho,v)}\,q_\sigma^{-1/N}+\hat\psi_0+\mathcal{O}(q_\sigma^{1/N})\ , \tag{A.58}$$

where $\vartheta_1(\rho,v)=\sum_{n\in\mathbb{Z}}(-1)^n\,q^{\frac12(n-\frac12)^2}y^{n-\frac12}$ (note that it differs from $\vartheta\left[{}^1_1\right](\rho,v)$ by a factor of i) and

$$\begin{aligned}\psi_0=&\frac{k\mathcal{P}(\rho,v)}{\Delta_k(\rho)}+\frac{k}{12(N-1)}\frac{N^2E_2(N\rho)-N\,E_2(\rho)}{\Delta_k(\rho)}\ ,\\ \hat\psi_0=&\frac{k\mathcal{P}(N\rho,v)}{\Delta_k(\rho)}+\frac{k}{12(N-1)}\frac{E_2(\rho)-NE_2(N\rho)}{\Delta_k(\rho)}\end{aligned} \tag{A.59}$$

Now, unlike holomorphic or weak Jacobi forms, a meromorphic Jacobi form $\psi_m(\rho,v)$ of index $m>0$ and weight w in general do not have a theta series decomposition, unless it happens to be holomorphic in the variable v. Instead, it was shown in [90, 89] that it can be decomposed into the sum of a polar part and a finite part,

$$\psi_m(\rho,v) = \psi^F_m(\rho,v) + \psi^P_m(\rho,v)\ , \tag{A.60}$$

where the finite part ψ_F is holomorphic in z and has a theta series decomposition,

$$\psi^F_m(\rho,v) = \frac{c_k(m)}{\Delta_k(\rho)} \sum_{\ell \bmod 2m} h_{m,\ell}(\rho)\, \vartheta_{m,\ell}(\rho,v), \tag{A.61}$$

where

$$\vartheta_{m,\ell}(\rho,v) = \sum_{s\in\mathbb{Z}} q^{(\ell+2ms)^2/4m}\, y^{\ell+2ms}\ , \tag{A.62}$$

are the standard theta series transforming in the Weil representation of dimension $2m$ while the polar part is a linear combination of Appell–Lerch sums which match the poles of $\psi_m(\rho,v)$ in the v variable. Since Appell–Lerch sums transform inhomogeneously under modular transformations, so does the finite part ψ^F_m, which implies that $h_{m,\ell}$ transform as a vector-valued mock modular form of weight $\frac{3}{2}-k$. In the case at hand, it follows from (A.44) that $\psi_m(\rho,v)$ has a double pole at $v = 0 \bmod \mathbb{Z}+\rho\mathbb{Z}$ with coefficient proportional to $c_k(m)/\Delta_k(\rho)$, where $c_k(m)$ are the Fourier coefficients of $1/\Delta_k(\sigma)$, so

$$\psi^P_m(\rho,v) = \frac{c_k(m)}{\Delta_k(\rho)} \mathcal{A}_m(\rho,v) \tag{A.63}$$

where $\mathcal{A}_m(\rho,v)$ is the standard Appell–Lerch sum [89]

$$\mathcal{A}_m(\rho,v) = \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s}\, y^{2ms+1}}{(1-q^s y)^2}\ . \tag{A.64}$$

The latter satisfies the elliptic property (A.52) but not the modular property (A.53). However, it admits a non-holomorphic completion term

$$\mathcal{A}^\star_m(\rho,v) = m \sum_{\ell \bmod 2m} \left[\frac{\overline{\vartheta_{m,\ell}(\rho)}}{2\pi\sqrt{m\rho_2}} - \sum_{\lambda\in\mathbb{Z}+\frac{\ell}{2m}} |\lambda|\, \mathrm{erfc}\big(2|\lambda|\sqrt{\pi m \rho_2}\big)\, q^{-m\lambda^2} \right] \vartheta_{m,\ell}(\rho,v)\ , \tag{A.65}$$

such that $\widehat{\mathcal{A}}_m \equiv \mathcal{A}_m + \mathcal{A}^\star_m$ transforms like a Jacobi form of weight 2 and index m, although it is no longer holomorphic in the ρ and v variables. Consequently, both

$$\widehat{\psi}^P_m(\rho,v) = \psi^P_m + \frac{c_k(m)}{\Delta_k(\rho)} \mathcal{A}^*_m(\rho,v) \qquad \text{and} \qquad \widehat{\psi}^F_m(\rho,v) = \psi^F_m - \frac{c_k(m)}{\Delta_k(\rho)} \mathcal{A}^*_m(\rho,v) \tag{A.66}$$

transform like Jacobi forms of weight $2-k$ and index m, although neither is holomorphic in the ρ and v variables. Moreover, $\widehat{\psi}^F_m(\rho,v)$ has a theta series decomposition similar to (A.61) with coefficients

$$\widehat{h}_{m,\ell}(\rho) = h_{m,\ell}(\rho) - m \left[\frac{\overline{\vartheta_{m,\ell}(\rho)}}{2\pi\sqrt{m\rho_2}} - \sum_{\lambda\in\mathbb{Z}+\frac{\ell}{2m}} |\lambda|\, \mathrm{erfc}\big(2|\lambda|\sqrt{\pi m\rho_2}\big)\, q^{-m\lambda^2} \right] \tag{A.67}$$

transforming as a vector-valued modular form of weight $\frac{3}{2}-k$. By Taylor expanding the denominator, we can rewrite (A.64) as an indefinite theta series of signature $(1,1)$,

$$\mathcal{A}_m(\rho,v)=\frac{1}{2}\sum_{s,\ell\in\mathbb{Z}}\ell\;[\mathrm{sign}(s+u_2)+\mathrm{sign}\,\ell]\;q^{ms^2+\ell s}\,y^{2ms+\ell}\;. \tag{A.68}$$

Similarly, its modular completion can be written as an indefinite theta series,

$$\widehat{\mathcal{A}}_m(\rho,v)=\frac{1}{2}\sum_{s,\ell\in\mathbb{Z}}\ell\left[\mathrm{sign}(s+u_2)+\frac{\sqrt{m}}{\pi\ell\sqrt{\tau_2}}F\left(\ell\sqrt{\frac{\pi\tau_2}{m}}\right)\right]\;q^{ms^2+\ell s}\,y^{2ms+\ell}\;, \tag{A.69}$$

where

$$F(x)=\sqrt{\pi}x\,\mathrm{erf}(x)+e^{-x^2} \tag{A.70}$$

is a smooth function which asymptotes to $\sqrt{\pi}|x|$ at large $|x|$ [91].

For meromorphic Jacobi forms of index $m=0$, the decomposition (A.60) still holds, but the finite part ψ_0^F is now independent of z, while the non-holomorphic completion term of the Appell–Lerch sum $\mathcal{A}_0(\rho,v)$ reduces to $\mathcal{A}_0^*=1/(4\pi\rho_2)$. The simplest example, relevant for the present work, is the (rescaled) Weierstrass function (A.55), which decomposes into

$$\mathcal{P}(\rho,v)=\frac{E_2}{12}+\sum_{s\in\mathbb{Z}}\frac{q^s\,y}{(1-q^sy)^2}=\frac{\hat{E}_2}{12}+\left(\frac{1}{4\pi\rho_2}+\sum_{s\in\mathbb{Z}}\frac{q^sy}{(1-q^sy)^2}\right) \tag{A.71}$$

In particular, it follows from this decomposition and from (A.88) (with $L=0$) that the integral over the elliptic curve $v\in\mathcal{E}$ is given by

$$\int_{\mathcal{E}}\mathcal{P}(\rho,v)\frac{\mathrm{d}v\mathrm{d}\bar{v}}{2\mathrm{i}\rho_2}=\int_{[0,1]^2}\mathrm{d}u_1\mathrm{d}u_2\,\mathcal{P}(\rho,u_1+\rho u_2)=\frac{\hat{E}_2}{12}\;, \tag{A.72}$$

which is non-holomorphic in ρ as a consequence of the pole of $\mathcal{P}(\rho,v)$ at $v=0$. From this, it follows in particular that the average values of the zero-th Fourier-Jacobi modes (A.59) of $1/\Phi_{k-2}$ and $1/\hat{\Phi}_{k-2}$ with respect to v are given by

$$\begin{aligned}\int_{[0,1]^2}\mathrm{d}u_1\,\mathrm{d}u_2\,\psi_0&=\frac{k}{12(N-1)}\frac{N^2\hat{E}_2(N\rho)-\hat{E}_2(\rho)}{\Delta_k(\rho)}\;,\\ \int_{[0,1]\times[0,N]}\mathrm{d}u_1\,\mathrm{d}u_2\,\hat{\psi}_0&=\frac{k}{12(N-1)}\frac{\hat{E}_2(\rho)-\hat{E}_2(N\rho)}{\Delta_k(\rho)}\;.\end{aligned} \tag{A.73}$$

For negative index $m<0$, it turns out that any meromorphic Jacobi form ψ can be expressed as a linear combination of iterated derivative of a modified Appell-Lerch sum, (here $y=e^{2\pi\mathrm{i}z},w=e^{2\pi\mathrm{i}u}$) [92]

$$F_M(z,u;\tau)=(y/w)^M\sum_{s\in\mathbb{Z}}\frac{w^{-2Ms}\,q^{Ms(s+1)}}{1-q^sy/w}\;, \tag{A.74}$$

The latter transforms as a Jacobi form of index $M=-m$ in u and has a simple pole at $u-z\in\mathbb{Z}+\tau\mathbb{Z}$, with residue $1/(2\pi\mathrm{i})$ at $u=z$. If S denotes the set of poles of $\psi(z)$ in a fundamental domain of $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, and $D_{n,u}$ are the Laurent coefficients of ψ at $z = u$, then Theorem 1.1 in [92] states that

$$\psi(z) = -\sum_{u\in S}\sum_{n\geq 0} \frac{D_{n,u}}{(2\pi \mathrm{i})^{n-1}(n-1)!} \left[\partial_v^{n-1} F_{-m}(z,v)\right]_{v=u} \tag{A.75}$$

For the case of interest in this paper, the leading Fourier-Jacobi coefficient $\psi_{-1} = \frac{1}{\eta^{18}\theta_1^2(z)}$ of $1/\Phi_{10}$ has a double pole at $z = 0$ with residue $1/\Delta$, hence

$$\psi_{-1} = \frac{1}{\Delta}\frac{\partial_u}{2\pi \mathrm{i}} F_1(z,u;\tau)|_{u=0} = -\frac{1}{\Delta}\sum_{s\in\mathbb{Z}}\left[\frac{y\, q^{s^2+s}}{(1-q^s y)^2} + \frac{2sy\, q^{s^2+s}}{1-q^s y}\right] \tag{A.76}$$

Note that this plays the role of ψ^P_{-1}, while ψ^F_{-1} vanishes. The modified Appell-Lerch sum can be written as an indefinite theta series,

$$\psi_{-1} = -\frac{1}{\Delta}\sum_{s,\ell\in\mathbb{Z}}\left[(2s+\ell)\,\frac{\mathrm{sign}\ell + \mathrm{sign}(u_2+s)}{2} - \frac{1}{4\pi\rho_2}\delta(u_2+s)\right] q^{s^2+\ell s}\, y^\ell \tag{A.77}$$

where signℓ is interpreted as -1 for $\ell = 0$. To see that this formula is consistent with the quasi-periodicity (A.52), note that under $(y,s,\ell)\to(yq,s-1,\ell+2)$, (A.77) becomes

$$-\frac{1}{\Delta}\sum_{s,\ell\in\mathbb{Z}}\left[(2s+\ell)\,\frac{\mathrm{sign}(\ell+2) + \mathrm{sign}(u_2+s)}{2} - \frac{1}{4\pi\rho_2}\delta(u_2+s)\right] q^{s^2+\ell s+1}\, y^{\ell+2} \tag{A.78}$$

This differs from (A.77) (up to the automorphy factor qy^2) only due to the terms $\ell = 0$ and $\ell = -1$, but those two terms leads to a vanishing contribution,

$$-\frac{1}{2}\sum_{s\in\mathbb{Z}}\left[(2s-1)q^{s(s-1)+1}y + 2s\, q^{s^2+1}y^2\right] = 0\ . \tag{A.79}$$

Shifting ℓ to $\ell - 2s$, (A.77) may be written equivalently as

$$\psi_{-1} = -\frac{1}{\Delta}\sum_{s,\ell\in\mathbb{Z}}\left[\ell\,\frac{\mathrm{sign}(\ell-2s) + \mathrm{sign}(u_2+s)}{2} - \frac{1}{4\pi\rho_2}\delta(u_2+s)\right] q^{-s^2+\ell s}\, y^{\ell-2s} \tag{A.80}$$

which resembles the Appell-Lerch sum (A.68) for $m = -1$, except for the replacement of signℓ by sign$(\ell-2s)$. Of course, the Appell-Lerch sum $\mathcal{A}_{-1}$ would be divergent, while the modified Appell-Lerch sum is absolutely convergent. Similarly, for CHL orbifolds, the leading Fourier-Jacobi coefficient of $1/\Phi_{k-2}$ is given by the same Eq. (A.80) with Δ replaced by Δ_k.

A.6 Fourier coefficients and local modular forms

In this section we shall use the decomposition (A.60) to infer the Fourier coefficients of $1/\Phi_{k-2}$ and $1/\tilde{\Phi}_{k-2}$ in the limit $\Omega_2\to \mathrm{i}\infty$. Starting with the maximal rank case, and assuming that $\sigma_2 \gg \rho_2, v_2$, we find

$$\begin{aligned} C(n,m,L;\Omega_2) =& \int_{[0,1]^3} \mathrm{d}^3\Omega_1\, \frac{e^{-2\pi\mathrm{i}(n\rho+Lv+m\sigma)}}{\Phi_{10}(\rho,\sigma,v)} \\ =& C^F(n,m,L) + c(m)\int_{[0,1]^2} \mathrm{d}\rho_1\, \mathrm{d}v_1\, \frac{e^{-2\pi\mathrm{i}(n\rho+Lv)}}{\Delta(\rho)}\,\mathcal{A}_m(\rho,v) \end{aligned} \tag{A.81}$$

where $C^F(n,m,L) = \int_{[0,1]^2} \mathrm{d}\rho_1\, \mathrm{d}v_1\, \psi^F_m(\rho,v)\, e^{-2\pi \mathrm{i}(n\rho+Lv)}$ are the Fourier coefficients of the finite part of ψ_m. To compute the integral in the second line of (A.81), we Fourier expand $1/\Delta(\rho) = \sum_{M\geq -1} c(m)q^m$ and $\mathcal{A}_m(\rho,v)$ using the representation (A.68), and integrate term by term with respect to v_1, obtaining

$$\frac{1}{2}\sum_{s,\ell\in\mathbb{Z}} c(m)\, c(n-Ls+ms^2)\,(L-2ms)\;[\mathrm{sign}(u_2+s)+\mathrm{sign}(L-2ms)] \tag{A.82}$$

where we have used $\ell = L-2ms$, $M = n-ms^2-\ell s$. However, while this naive manipulation lead to the correct result for generic u_2, it turns out to miss a distributional part localized at $u_2\in\mathbb{Z}$, originating from the poles of $\mathcal{A}_m(\rho,v)$ at $q^s e^{2\pi \mathrm{i} v}=1$.

To compute this distribution, let us first consider the contribution from the term $s=0$ in the sum (A.64). Upon expanding

$$\frac{y}{(1-y)^2} = \begin{cases} \sum_{k\geq 1} k y^k\ , & |y|<1 \\ \sum_{k\geq 1} k y^{-k}\ , & |y|>1 \end{cases}\ , \tag{A.83}$$

one would be tempted to conclude that the integral $\int_0^1 \mathrm{d}v_1\, \frac{y}{(1-y)^2}$ vanishes. However, we claim that instead,

$$\int_0^1 \mathrm{d}v_1\, \frac{y}{(1-y)^2} = -\frac{1}{4\pi}\delta(v_2)\ . \tag{A.84}$$

To see this, we first consider first the single pole function $\frac{1}{2}\frac{y+1}{y-1}$, with Fourier expansion

$$\frac{1}{2}\frac{y+1}{y-1} = -\sum_{\ell\in\mathbb{Z}} \frac{\mathrm{sign}(\ell)+\mathrm{sign}(v_2)}{2} y^\ell \tag{A.85}$$

with the understanding that $\mathrm{sign}(0)=0$. We claim that this identity is valid at the distributional level. As a check, using the Euler formula representation for (A.85) and acting with an anti-holomorphic derivative on each term (recalling that $\partial_{\bar v}\frac{1}{v} = \pi\delta(v_1)\delta(v_2)$), we get

$$-\frac{1}{2\pi \mathrm{i}}\frac{\partial}{\partial\bar v}\Big(\frac{1}{2}\frac{y+1}{y-1}\Big) = -\frac{1}{2\pi \mathrm{i}}\frac{\partial}{\partial\bar v}\Big(\sum_{\ell\in\mathbb{Z}}\frac{1}{2\pi \mathrm{i}(v-\ell)}\Big) = \frac{1}{4\pi}\delta(v_2)\sum_{\ell\in\mathbb{Z}}\delta(v_1-\ell) = \frac{1}{4\pi}\delta(v_2)\sum_{\ell\in\mathbb{Z}} y^\ell\ . \tag{A.86}$$

The right-hand side is also what one gets by acting with $\partial_{\bar v} = \frac{1}{2}(\partial_{v_1}+\mathrm{i}\partial_{v_2})$ on each term in the Fourier series (A.85), noting that $\mathrm{sign}'(v_2) = 2\delta(v_2)$.

The double pole distribution (A.83) is obtained by acting with a holomorphic derivative on (A.85), therefore admits the Fourier expansion

$$\frac{y}{(1-y)^2} = -\frac{1}{2\pi \mathrm{i}}\frac{\partial}{\partial v}\Big(\frac{1}{2}\frac{y+1}{y-1}\Big) = \sum_{\ell\in\mathbb{Z}}\frac{|\ell|+\mathrm{sign}(v_2)\ell}{2}y^\ell - \frac{1}{4\pi}\delta(v_2)\sum_{\ell\in\mathbb{Z}} y^\ell\ . \tag{A.87}$$

In particular, integrating over v_1 we reach (A.84).[28]. More generally, the same argument shows that for any s,

$$\frac{q^s\, y}{(1-q^s y)^2} = \sum_{\ell\in\mathbb{Z}}\Big(\frac{|\ell|+\ell\,\mathrm{sign}(v_2+s\rho_2)}{2} - \frac{1}{4\pi}\delta(v_2+s\rho_2)\Big)\, q^{\ell s} y^\ell \tag{A.88}$$

[28]It is worth cautioning the reader that regularizing the double pole by point splitting would instead produce the same delta distribution with coefficient $-1/(2\pi)$. This however would be inconsistent with modular invariance, e.g. when computing the average of the Weierstrass function in (A.72).

Using this identity, we find that the naive result (A.82) misses an additional term supported at $u_2 = v_2/\rho_2 \in \mathbb{Z}$,

$$-\frac{1}{4\pi}\sum_{s,\ell\in\mathbb{Z}} c(m)\, c(n-Ls+ms^2)\,\delta(v_2+s\rho_2) \ . \tag{A.89}$$

However, this still cannot be the full Fourier coefficient $C(n,m,L;\Omega_2)$, since the latter must be invariant under the action (A.14) of $GL(2,\mathbb{Z})$. Instead, both (A.82) and (A.89) are invariant under the subgroup Γ_∞ which preserves the cusp $\sigma_2=\infty$, where $\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}$ acts by sending $(n,m,L)\to(n-Ls+ms^2, mL-2ms)$. To restore invariance under the full $GL(2,\mathbb{Z})$ group, we may therefore replace the sum over $s\in\mathbb{Z}$ by a sum over all $\gamma\in GL(2,\mathbb{Z})/\mathrm{Di}_4$, obtaining

$$C(n,m,L;\Omega_2) = C^F(n,m,L) + \sum_{\gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4}\left[c(m)\,c(n)\left(\frac{1}{2}L(\mathrm{sgn}L+\mathrm{sgn}v_2)-\frac{1}{4\pi}\delta(v_2)\right)\right]|_\gamma + \dots \tag{A.90}$$

Here, Dih_4 denotes the dihedral group generated by the matrices $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, which stabilizes the locus $v_2=0$, and the dots denotes possible additional contributions which are not visible in the limit $|\Omega_2|\to\infty$. The action of $\gamma=\begin{pmatrix}p & q\\ r & s\end{pmatrix}\in GL(2,\mathbb{Z})$ on the quantities m,n,L,v_2 appearing in the bracket is given by

$$\begin{aligned}
\hat n &\mapsto s^2 n + q^2 m - qsL, \qquad \hat m \mapsto r^2 n + p^2 m - prL, && \text{(A.91)}\\
\hat L &\mapsto -2rs\,n - 2pq\,m + \frac{ps+qr}{2}L, && \text{(A.92)}\\
\hat v_2 &\mapsto \mathrm{tr}\left(\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}\gamma^\intercal\Omega_2\gamma\right) = pq\,\rho_2 + rs\,\sigma_2 + (ps+qr)v_2 \ . && \text{(A.93)}
\end{aligned}$$

Using the same reasoning, we find the Fourier coefficients of $1/\Phi_{k-2}$, which must be invariant under $GL(2,\mathbb{Z})$,

$$\begin{aligned}
C_{k-2}(n,m,L;\Omega_2) &= C^F_{k-2}(n,m,L)\\
&+\sum_{\gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4}\left[c_k(\hat m)\,c_k(\hat n)\left(\frac{1}{2}\hat L\left(\mathrm{sign}\hat L+\mathrm{sign}\hat v_2\right)-\frac{1}{4\pi}\delta(\hat v_2)\right)\right]_\gamma + \dots
\end{aligned} \tag{A.94}$$

For the Fourier coefficients of $1/\tilde\Phi_{k-2}$, which must be invariant under $\Gamma_0(N)$, we find instead

$$\begin{aligned}
\widetilde C_{k-2}(n,m,L;\Omega_2) &= \widetilde C^F_{k-2}(n,m,L)\\
&+\sum_{\gamma\in\Gamma_0(N)/\mathbb{Z}_2}\left[c_k(N\hat m)\,c_k(\hat n)\left[\frac{1}{2}\hat L\left(\mathrm{sign}\hat L+\mathrm{sign}\hat v_2\right)-\frac{1}{4\pi}\delta(\hat v_2)\right]\right]_\gamma + \dots
\end{aligned} \tag{A.95}$$

It is important to note that the identities (A.90),(A.94),(A.95) are only valid when $|\Omega_2|$ is large enough such that the integration contour $[0,1]^3+\mathrm{i}\Omega_2$ does not cross any pole for generic values of Ω_2, and only crosses quadratic divisors (A.6) with $n_2=0$ on real-codimension one loci. When $|\Omega_2|<1/(4n_2^2)$ with $|n_2|\geq 1$, the contour crosses the the quadratic divisor (A.6) for generic values of Ω_2, and the integral on the first line of (A.81) is no longer well-defined. We leave it as an interesting open problem to define the Fourier coefficient $C(n,m,L;\Omega_2)$ of $1/\Phi_{10}$ (or its analogue for $1/\Phi_{k+2}$ and $1/\tilde\Phi_{k-2}$) in the region where $|\Omega_2|\leq 1/4$.

B Perturbative contributions to 1/4-BPS couplings

In this section, we compute the one-loop and two-loop contributions to the coefficient of the $\nabla^2 F^4$ coupling in the low-energy effective action in heterotic CHL orbifolds. In both cases we start with the maximal rank case, *i.e.* heterotic string compactified on a torus T^d, and then turn to the simplest heterotic CHL orbifolds with $N = 2, 3, 5, 7$.

B.1 One-loop $\nabla^2 F^4$ and $\mathcal{R}^2 F^2$ couplings

B.1.1 Maximal rank case

In heterotic string compactified on a torus T^d, the one-loop contribution to the coefficient of the $\nabla^2 F^4$ coupling in the low-energy effective action can be extracted from the four-gauge boson one-loop amplitude, given up to an overal tensorial factor by [41]

$$\begin{aligned}\mathcal{A}^{(1)}_{abcd} =& \frac{1}{(2\pi\mathrm{i})^4}\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{1}{\Delta}\int_{\mathcal{E}^4}\prod_{i=1}^{4}\frac{\mathrm{d}z_i\mathrm{d}\bar{z}_i}{2\mathrm{i}\rho_2}(\chi_{12}\chi_{34})^{\alpha' s}(\chi_{13}\chi_{24})^{\alpha' t}(\chi_{14}\chi_{23})^{\alpha' u}\\ &\times\langle J_a(z_1)\,J_b(z_2)\,J_c(z_3)\,J_d(z_4)\rangle\end{aligned} \tag{B.1}$$

where $\chi_{ij} = e^{g(\rho,z_i-z_j)}$ and $g(\rho,z) = -\log|\theta_1(\rho,z)/\eta|^2 + \frac{2\pi}{\rho_2}(\mathrm{Im}z)^2$ is the scalar Green function on the elliptic curve $\mathcal{E}$ with modulus ρ. The four-point function of the currents evaluates to

$$\begin{aligned}\langle J_a(z_1)\,J_b(z_2)\,J_c(z_3)\,J_d(z_4)\rangle =& \Gamma_{\Lambda_{d+16,d}}[P_{abcd}] - \frac{1}{4\pi^2}\Big(\delta_{ab}\Gamma_{\Lambda_{d+16,d}}[P_{cd}]\,\partial^2 g(z_1-z_2) + 5\,\text{perms}\Big)\\ &+\frac{1}{16\pi^4}\Big(\delta_{ab}\delta_{cd}\,\Gamma_{\Lambda_{d+16,d}}[1]\,\partial^2 g(z_1-z_2)\,\partial^2 g(z_3-z_4) + 2\,\text{perms}\Big)\end{aligned} \tag{B.2}$$

where P_{ab} and P_{abcd} are quadratic and quartic polynomials, respectively, in the projected lattice vector $Q_{La} = p_{La}{}^{\mathcal{I}} Q_{\mathcal{I}} \in \Gamma_{d+16,d}$ arising from the zero-mode of the currents,

$$\begin{aligned}P_{ab} =& Q_{La}Q_{Lb} - \frac{\delta^{ab}}{4\pi\rho_2}\,,\\ P_{abcd} =& Q_{La}Q_{Lb}Q_{Lc}Q_{Ld} - \frac{3}{2\pi\rho_2}\delta_{(ab}Q_{Lc}Q_{Ld)} + \frac{3}{16\pi^2\rho_2^2}\delta_{(ab}\delta_{cd)}\,,\end{aligned} \tag{B.3}$$

and for any polynomial P in Q_{La} and integer lattice $\Lambda_{p,q}$ of signature (p,q), we denote

$$\Gamma_{\Lambda_{p,q}}[P] = \rho_2^{q/2}\sum_{Q\in\Lambda_{p,q}} P(Q_{La})\,e^{\mathrm{i}\pi[\rho Q_L^2 - \bar{\rho}Q_R^2]}\,. \tag{B.4}$$

Upon expanding in powers of α', the leading term reproduces the one-loop contribution to the F^4 coupling,

$$F^{(1)}_{abcd} = \text{R.N.}\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\Gamma_{\Lambda_{d+16,d}}[P_{abcd}]}{\Delta(\rho)}\,, \tag{B.5}$$

where R.N. denotes the regularization procedure introduced in [93, 94, 95], which is needed to make sense of the divergent integral when $d \geq 6$ (we return to this point at the end of this subsection). Equivalently, (B.5) may be written as [46]

$$F^{(1)}_{abcd} = \text{R.N.}\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\partial^4}{(2\pi\mathrm{i})^4\partial y^a\partial y^b\partial y^c\partial y^d}\frac{\Gamma_{\Lambda_{d+16,d}}(y)}{\Delta(\rho)}\Big|_{y=0}\,, \tag{B.6}$$

where $\Gamma_{\Lambda_{p,q}}(y)$ is the partition function of the compact bosons deformed by the current $y_a J^a$ integrated along the A-cycle of the elliptic curve,

$$\Gamma_{\Lambda_{p,q}}(y) = \rho_2^{q/2} \sum_{Q\in\Lambda_{p,q}} e^{\mathrm{i}\pi[\rho Q_L^2 - \bar\rho Q_R^2] + 2\pi\mathrm{i} Q_L\cdot y + \frac{\pi(y\cdot y)}{2\rho_2}} \;. \tag{B.7}$$

At next to leading order in α', the term linear in the Mandelstam variables s, t, u reduces to

$$G^{(1)}_{ab,cd} = \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{1}{\Delta} \int_{\mathcal{E}^4} \prod_{i=1}^{4} \frac{\mathrm{d}z_i \mathrm{d}\bar z_i}{2\mathrm{i}\rho_2} \Big[g(z_1 - z_2)\, \partial^2 g(z_1 - z_2)\, \delta_{ab}\, \Gamma_{\Lambda_{d+16,d}}[P_{cd}] + 5\,\mathrm{perms} \Big] \;, \tag{B.8}$$

since all other terms at this order are total derivatives with respect to z_i. The integral over z can be computed by using the Poincaré series representation of the Green function,

$$g(\rho, z) = \frac{1}{\pi} \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{\rho_2}{|m\rho+n|^2} e^{\frac{\pi}{\rho_2}[\bar z(m\rho+n) - z(m\bar\rho+n)]} \;, \tag{B.9}$$

leading to

$$\int_{\mathcal{E}} \frac{\mathrm{d}z \mathrm{d}\bar z}{2\mathrm{i}\rho_2} g(z-w) \partial^2 g(z-w) = \lim_{s\to 0} \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m\rho+n)^2 |m\rho+n|^{2s}} = \frac{\pi^2}{6} \hat E_2 \;, \tag{B.10}$$

where the sum over (m, n) was regularized à la Kronecker. Up to an overall numerical factor, we therefore find that the one-loop contribution to the coefficient of $\nabla^2 F^4$ coupling for the maximal rank model is given by

$$G^{(1)}_{ab,cd} = \delta_{\langle ab}\, G^{(d+16,d)}_{cd\rangle} \;, \qquad G^{(p,q)}_{ab} = \mathrm{R.N.} \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\hat E_2}{\Delta(\rho)} \Gamma_{\Lambda_{p,q}}[P_{ab}] \;. \tag{B.11}$$

For $d = 0$, corresponding to either of the $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$ heterotic strings in 10 dimensions, one has

$$\Gamma_{\Lambda_{E_8\oplus E_8}}[P_{ab}] = \Gamma_{\Lambda_{D_{16}}}[P_{ab}] = \frac{E_4}{12} \left(\hat E_2 E_4 - E_6 \right) \delta_{ab} \tag{B.12}$$

so $G^{(1)}_{ab,cd}$ becomes proportional to the $\mathrm{Tr}F^2 \mathrm{Tr}\mathcal{R}^2$ coupling computed from the elliptic genus [96, C.5], [97], as required by supersymmetry.

B.1.2 CHL orbifolds

The four-gauge boson amplitude in CHL models with $N = 2, 3, 5, 7$ was obtained in [44, 45]. It was shown in [22, §A] that the one-loop F^4 coupling in these models is given by the simple generalization of (B.5), namely

$$F^{(1)}_{abcd} = F^{(d+r-12,d)}_{abcd}, \qquad F^{(p,q)}_{abcd} = \mathrm{R.N.} \int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\Gamma_{\Lambda_{p,q}}[P_{abcd}]}{\Delta_k} \;, \tag{B.13}$$

where $\Delta_k = [\eta(\rho)\eta(N\rho)]^k$ arises from the partition function in the twisted sectors. The same derivation goes through for the $\nabla^2 F^4$ and $\mathcal{R}^2 F^2$ couplings and yields

$$G^{(1)}_{ab,cd} = \delta_{\langle ab}\, G^{(d+r-12,d)}_{cd\rangle} \;, \qquad G^{(p,q)}_{ab} = \mathrm{R.N.} \int_{\Gamma_0(N)\backslash\mathcal{H}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\hat E_2}{\Delta_k} \Gamma_{\Lambda_{p,q}}[P_{ab}] \;. \tag{B.14}$$

B.1.3 Regularization of the genus-one modular integrals

As indicated above, the modular integrals (B.13) and (B.14) are divergent when $d \geq 6$ and $d \geq 4$, respectively. We follow the same regularization procedure as in [98, 22] and define them by truncating the integration domain to $\mathcal{F}_{N,\Lambda} = \cup_{\gamma \in \Gamma_0(N)\backslash SL(2,\mathbb{Z})} \gamma \cdot \mathcal{F}_{1,\Lambda}$, where $\mathcal{F}_{1,\Lambda} = \{-\frac{1}{2} < \rho_1 < \frac{1}{2}, |\rho| > 1, \rho_2 < \Lambda\}$ is the truncated fundamental domain for $SL(2,\mathbb{Z})$, and minimally subtracting the divergent terms before taking the limit $\Lambda \to \infty$. Using the fact that the constant terms of $1/\Delta_k$ and $\hat{E}_2/\Delta_k$ are equal to k and $k(1-\frac{3}{\pi\rho_2})-24$, the constant terms of their Fricke dual are k and $k(N-\frac{3}{\pi\rho_2})$ and the constant terms of the Fricke dual of the partition function include an extra factor of $\upsilon N^{\frac{q-8}{2}}$, we get

$$F^{(p,q)}_{abcd} = \lim_{\Lambda\to\infty} \left[\int_{\mathcal{F}_{N,\Lambda}} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\Gamma_{\Lambda_{p,q}}[P_{abcd}]}{\Delta_k} - \frac{3k(1+\upsilon N^{\frac{q-8}{2}})}{16\pi^2} \frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}} \delta_{(ab}\delta_{cd)} \right] , \tag{B.15}$$

$$G^{(p,q)}_{ab} = \lim_{\Lambda\to\infty} \left[\int_{\mathcal{F}_{N,\Lambda}} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\hat{E}_2}{\Delta_k} \Gamma_{\Lambda_{p,q}}[P_{ab}] - \frac{3k(1+\upsilon N^{\frac{q-8}{2}})}{4\pi^2} \frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}} \delta_{ab} + \frac{k(1+\upsilon N^{\frac{q-6}{2}})-24}{4\pi} \frac{\Lambda^{\frac{q-4}{2}}}{\frac{q-4}{2}} \delta_{ab} \right] , \tag{B.16}$$

where the terms $\Lambda^{\frac{q-6}{2}}/\frac{q-6}{2}$ and $\Lambda^{\frac{q-4}{2}}/\frac{q-4}{2}$ should be replaced by $\log\Lambda$ when $q=6$ or $q=4$, respectively. Note that the second term in (B.16) cancels in the case of the full rank model where $k=24$. It will be also useful to consider the Fricke dual function to $G^{(p,q)}_{ab}$ for the $N=2,3,5,7$ models, introduced in (4.57) and whose regularization is given by

$$\begin{aligned} {}^{\varsigma}G^{(p,q)}_{ab} = \lim_{\Lambda\to\infty} \Bigg[\int_{\mathcal{F}_{N,\Lambda}} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{N\hat{E}_2(N\rho)}{\Delta_k(\rho)} \Gamma_{\Lambda_{p,q}}[P_{ab}] - \frac{3k(1+\upsilon N^{\frac{q-8}{2}})}{4\pi^2} \frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}} \delta_{ab} \\ + \frac{k(N+\upsilon N^{\frac{q-8}{2}})-24}{4\pi} \frac{\Lambda^{\frac{q-4}{2}}}{\frac{q-4}{2}} \delta_{ab} \Bigg] . \end{aligned} \tag{B.17}$$

B.1.4 Differential identities satisfied by genus-one modular integrals

Like the genus-two modular integral $G^{(p,q)}_{ab,cd}$ discussed in §3.3, the genus-one modular integrals (B.15), (B.16) and (B.17) satisfy differential identities with constant source terms in $q=6$, $q=4$ determined by regularization techniques using the same paramatrization as for section B.1.3. The equation for the modular integral $F^{(p,q)}_{abcd}$ was calculated in [22, (3.57)], which we reproduce below:

$$\mathcal{D}_{(e}{}^{\hat{g}}\mathcal{D}_{f)\hat{g}} F_{abcd} = \tfrac{2-q}{4}\delta_{ef}\, F_{abcd} + (4-q)\delta_{(e|(a}F_{bcd)|f)} + 3\delta_{(ab}F_{cd)ef} + \frac{15k(1+\frac{\upsilon}{N})}{2(4\pi)^2}\delta_{(ab}\delta_{cd}\delta_{ef)}\,\delta_{q,6} \ , \tag{B.18}$$

Here the volume factor υ is either equal to N for the perturbative Narain lattice, or to 1 for the non-perturbative Narain lattice.

The equation satisfied by the genus-one integral $G^{(p,q)}_{ab}$ can be computed using the same techniques described in [22, §3.2] and reads

$$\begin{aligned} \mathcal{D}_{(e}{}^{\hat{g}}\mathcal{D}_{f)\hat{g}} G^{(p,q)}_{ab} = \frac{2-q}{4}\,\delta_{ef}\, G^{(p,q)}_{ab} + \frac{4-q}{2}\,\delta_{e)(a}\, G^{(p,q)}_{b)(f} + \frac{1}{2}\,\delta_{ab}\, G^{(p,q)}_{ef} + 6\, F^{(p,q)}_{efab} \\ - \frac{3\big((1+\frac{\upsilon}{N})k-24\big)}{8\pi}\delta_{(ef}\delta_{ab)}\,\delta_{q,4} + \frac{9(1+\frac{\upsilon}{N})k}{8\pi^2}\delta_{(ef}\delta_{ab)}\,\delta_{q,6} \ , \end{aligned} \tag{B.19}$$

where the term proportional to $F^{(p,q)}_{efab}$ corresponds to the contribution of the non-holomorphic completion in $\hat{E}_2$, and the two constant contributions of the second line correspond to the boundary contribution after integration by part (see [22, (3.54)]). One checks that the divergent contributions cancel each others, so the equation is valid for the renormalized couplings. For the perturbative lattice with $v = N$, these linear corrections are associated to the mixing between the analytic and the non-analytic components of the amplitude, and are indeed proportional to the corresponding 1-loop divergence coefficient in supergravity [79].

The same analysis for ${}^{\varsigma}G^{(p,q)}_{ab}$ gives

$$\begin{aligned}\mathcal{D}_{(e}{}^{\hat{g}}\mathcal{D}_{f)\hat{g}}\,{}^{\varsigma}G^{(p,q)}_{ab} &= \frac{2-q}{4}\,\delta_{ef}\,{}^{\varsigma}G^{(p,q)}_{ab} + \frac{4-q}{2}\,\delta_{e)(a}\,{}^{\varsigma}G^{(p,q)}_{b)(f} + \frac{1}{2}\,\delta_{ab}\,{}^{\varsigma}G^{(p,q)}_{ef} + 6\,F^{(p,q)}_{efab}\\ &\quad - \frac{3\big((N+v)k-24\big)}{8\pi}\delta_{(ef}\delta_{ab)}\,\delta_{q,4} + \frac{9(1+\frac{v}{N})k}{8\pi^2}\delta_{(ef}\delta_{ab)}\,\delta_{q,6}\ .\end{aligned} \tag{B.20}$$

B.2 Two-loop $\nabla^2 F^4$ couplings

B.2.1 Maximal rank case

At two-loop, the scattering amplitude of four gauge bosons in ten-dimensional heterotic string theory was computed in [42, 43]. Upon compactifying on a torus T^d, one obtains

$$\begin{aligned}\mathcal{A}^{(2)}_{abcd} = &\int_{\mathcal{F}_2} \frac{\mathrm{d}^3\Omega_1\,\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\frac{1}{\Phi_{10}}\\ &\times \int_{\Sigma^4} \overline{\mathcal{Y}}_S \prod_{i=1}^{4}\mathrm{d}z_i\,(\chi_{12}\chi_{34})^{\alpha' s}\,(\chi_{13}\chi_{24})^{\alpha' t}\,(\chi_{14}\chi_{23})^{\alpha' u}\,\langle J_a(z_1)\,J_b(z_2)\,J_c(z_3)\,J_d(z_4)\rangle\end{aligned} \tag{B.21}$$

where Σ is a genus-two Riemann surface with period matrix Ω, $\overline{\mathcal{Y}}_S$ is a specific $(1,1)$ form in each of the coordinates z_i on Σ [42, (11.32)],

$$\mathcal{Y}_S = t\,\Delta(1,2)\,\Delta(3,4) - s\,\Delta(1,4)\,\Delta(2,3)\ , \tag{B.22}$$

where $\Delta(z,w) = \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z)$, $\chi_{ij} = e^{G(\Omega, z_i - z_j)}$ and $G(\Omega,z)$ is the scalar Green function on Σ. At leading order in α', χ_{ij} can be set to one, and similarly to (B.6), the integrated current correlator $\int_\Sigma J^a(z)\mathrm{d}z\,\overline{\omega_I z}$ can be expressed as a multiple derivative [46]

$$\langle \int_{\Sigma^4} J^a(z_1)J^b(z_2)J^c(z_3)J^d(z_4)\prod_{i=1}^{4}\mathrm{d}z_i\,\overline{\omega_I(z_i)}\,\rangle = \frac{\frac{1}{3}(\varepsilon_{rr'}\varepsilon_{ss'}+\varepsilon_{rs'}\varepsilon_{sr'})\partial^4}{(2\pi\mathrm{i})^4\partial y^r_a\,\partial y^s_b\,\partial y^{r'}_c\,\partial y^{s'}_d}\Gamma^{(2)}_{\Lambda_{d+16,d}}(y)|_{y=0} \tag{B.23}$$

where $\Gamma^{(2)}_{\Lambda_{d+16,d}}(y)$ is the partition function of the compact bosons deformed by the currents $y^r_a J^a$ integrated along the r-th A-cycle of Σ,

$$\Gamma^{(2)}_{\Lambda_{p,q}}(y) = |\Omega|_2^{q/2}\sum_{Q\in\Lambda^{\otimes 2}_{p,q}} e^{\mathrm{i}\pi Q^r_{La}\,\Omega_{rs}\,Q^{s\,a}_L - \mathrm{i}\pi Q^r_{R\hat{a}}\,\bar{\Omega}_{rs}\,Q^{s\,\hat{a}}_R + 2\pi\mathrm{i}Q^r_{La}y^a_r + \frac{\pi}{2}y^a_r\Omega_2^{rs}y^{as}}\ . \tag{B.24}$$

Evaluating the derivatives explicitly, we obtain the result announced in (2.30) for the two-loop ∇^2F^4 coupling in the maximal rank case,

$$G^{(d,d+16)}_{ab,cd} = \text{R.N.}\int_{\mathcal{F}_2}\frac{\mathrm{d}^3\Omega_1\,\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\frac{\Gamma^{(2)}_{\Lambda_{d,d+16}}[P_{ab,cd}]}{\Phi_{10}} \tag{B.25}$$

where $P_{ab,cd}$ is the quartic polynomial defined in (2.31). The regularization procedure needed to make sense of this modular integral when $d \geq 5$ will be discussed in §B.2.4.

In the special case ($d = 0$) of the $E_8 \times E_8$ heterotic string in 10 dimensions, and for a suitable choice of indices ab, cd, the partition function $\Gamma^{(2)}_{\Lambda_{E_8}\times E_8}[P_{ab;cd}]$ reduces (up to normalization) to $(E_4^{(2)})^2\Psi_2$ where $E_4^{(2)}$ is the holomorphic Eisenstein series of weight 4, which coincides with the Siegel theta series for the lattice E_8, and Ψ_2 is a non-holomorphic modular form of weight (2,0) given by

$$\Psi_2 = \partial_\rho\Phi\partial_\sigma\Phi - \tfrac{1}{4}(\partial_v\Phi)^2 \ , \quad \Phi = \log\left[|\Omega_2|^4 E_4^{(2)}\right] \ , \tag{B.26}$$

in agreement with [43, (5.7)]. This can be viewed as the genus-two counterpart of the genus-one formula (B.12). We shall now discuss the extension of (B.25) to CHL orbifolds, starting with the simplest case $N = 2$.

B.2.2 $\mathbb{Z}_2$ orbifold

The simplest CHL model is obtained by orbifolding the $E_8 \times E_8$ heterotic string on T^d by an involution σ exchanging the two E_8 factors, and translating by half a period along one circle in T^d [25]. This model was studied in more detail in [99, 100] and revisited in [22, §A.1]. Some aspects of the genus-two heterotic amplitude in this model were discussed in [11] in the context of 1/4-BPS dyon counting, which we shall build on.

Following standard rules, the two-loop amplitude is now a sum over all possible twisted or untwisted periodicity conditions $[h_1 h_2]$ and $[g_1 g_2]$ along the A and B cycles of the genus-two curve Σ, respectively,

$$\mathcal{A}^{(2)} = \frac{1}{4}\sum_{\substack{h_1,h_2\in\{0,1\}\\ g_1,g_2\in\{0,1\}}} \mathcal{A}^{(2)}\begin{bmatrix} h_1 h_2 \\ g_1 g_2\end{bmatrix} . \tag{B.27}$$

The untwisted amplitude $\mathcal{A}^{(2)}\begin{bmatrix} 00 \\ 00\end{bmatrix}$ coincides with (B.21), restricted on the locus $G_{d+8,d} \subset G_{d+16,d}$ which is invariant under the involution σ. As in the genus-one case [22, §A.1], it is convenient to further restrict to the locus $G_{d,d} \subset G_{d+8,d}$ where the lattice factorizes as $\Lambda_{d+16,d} = E_8 \oplus E_8 \oplus I\!I_{d,d}$, and retain from $\mathcal{A}^{(2)}\begin{bmatrix} h_1 h_2 \\ g_1 g_2\end{bmatrix}$ the chiral measure for the ten-dimensional string, which we denote by

$$Z_{16}^{(2)}\begin{bmatrix} 00 \\ 00\end{bmatrix} = \frac{\left[\Theta^{(2)}_{E_8}(\Omega)\right]^2}{\Phi_{10}} . \tag{B.28}$$

Now, decomposing $p_1^\alpha + p_2^\alpha = 2\Sigma^\alpha + \mathcal{P}^\alpha, p_1^\alpha - p_2^\alpha = 2\Delta^\alpha - \mathcal{P}^\alpha$ for $p_1^\alpha, p_2^\alpha \in \Lambda_{E_8}$, $\alpha = 1, 2$, the genus-two partition function of the lattice $\Lambda_{E_8\times E_8}$ appearing in the numerator can be decomposed as

$$\left[\Theta^{(2)}_{E_8}(\Omega)\right]^2 = \sum_{(\mathcal{P}_1,\mathcal{P}_2)\in(\Lambda_{E_8}/2\Lambda_{E_8})^{\otimes 2}} \Theta^{(2)}_{E_8[2],(\mathcal{P}_1,\mathcal{P}_2)}(\Omega)\,\Theta^{(2)}_{E_8[2],(\mathcal{P}_1,\mathcal{P}_2)}(\Omega) \tag{B.29}$$

where $\Theta^{(2)}_{E_8[2],(\mathcal{P}_1,\mathcal{P}_2)}$ is the genus-two theta series for $\Lambda_{E_8}[2]$:

$$\Theta^{(2)}_{E_8[2],(\mathcal{P}_1,\mathcal{P}_2)}(\Omega) = \sum_{(\Delta^1,\Delta^2)\in\Lambda_{E_8}^{\otimes 2}} e^{2\pi i(\Delta^r - \frac{1}{2}\mathcal{P}^r)\Omega_{rs}(\Delta^s - \frac{1}{2}\mathcal{P}^s)} \tag{B.30}$$

For $\mathcal{P}_1 = \mathcal{P}_2 = 0$, $\Theta^{(2)}_{E_8[2],(0,0)}(\Omega) = \Theta^{(2)}_{E_8}(2\Omega)$.

As for the twisted sectors $\left[{h \atop g}\right] \equiv \left[{h_1 h_2 \atop g_1 g_2}\right] \neq \left[{00 \atop 00}\right]$, we use the fact that the $\mathbb{Z}_2$ orbifold blocks of d compact scalars on a Riemann surface of genus 2 are given by [101, 102]

$$\left| \frac{\vartheta^{(2)}[\delta_i^+](0,\Omega)\, \vartheta^{(2)}[\delta_i^-](0,\Omega)}{Z_0(\Omega)^2\, \vartheta_i(0,\tau_{h,g})^2} \right|^d \sum_{Q\in\Lambda_{d,d}} e^{\mathrm{i}\pi p_L^2(Q)\tau_{h,g} - \mathrm{i}\pi p_R^2(Q)\bar\tau_{h,g}} \tag{B.31}$$

where $Z_0(\Omega)$ is the inverse of the chiral partition of a (uncompactified, untwisted, unprojected) scalar field on Σ, and $\tau_{h,g}$ is the Prym period, namely the period of the unique even holomorphic form on the double cover of Σ, a Riemann surface $\hat\Sigma$ of genus 3. The Prym period $\tau_{h,g}$ is related to the period matrix Ω by the Schottky-Jung relation [102, (1.6)]

$$\left(\frac{\vartheta_i(0,\tau_{h,g})}{\vartheta_j(0,\tau_{h,g})} \right)^4 = \left(\frac{\vartheta^{(2)}[\delta_i^+](0,\Omega)\, \vartheta^{(2)}[\delta_i^-](0,\Omega)}{\vartheta^{(2)}[\delta_j^+](0,\Omega)\, \vartheta^{(2)}[\delta_j^-](0,\Omega)} \right)^2 \tag{B.32}$$

for any choice of distinct $i,j \in \{1,2,3\}$. Here, $\delta_i^\pm$ are the 6 even spin structures δ such that $\delta + \frac12\left[{h \atop g}\right]$ is also en even spin structure; moreover $\delta_i^- = \delta_i^+ + \frac12\left[{h \atop g}\right]$. The relation (B.32) ensures that (B.31) is independent of the choice of i. Since all 15 non-trivial twists are permuted by $Sp(4,\mathbb{Z})$, it will be convenient to focus on the twisted sector $\left[{h \atop g}\right] = \left[{00 \atop 01}\right]$, in which case the relation (B.32) becomes [102, (6.5)]

$$\frac{\vartheta_4^4(\tau)}{\vartheta_2^4(\tau)} = \left(\frac{\vartheta^{(2)}\left[{01 \atop 00}\right] \vartheta^{(2)}\left[{01 \atop 01}\right]}{\vartheta^{(2)}\left[{10 \atop 00}\right] \vartheta^{(2)}\left[{10 \atop 01}\right]} \right)^2 , \tag{B.33}$$

where $\tau \equiv \tau_{h,g}$. In particular, under $(\rho,\sigma,v) \to (\rho+1,\sigma,v)$, the Prym period transforms as $\tau \to \tau+1$, whereas in the non-separating degeneration $\sigma \to \mathrm{i}\infty$, $\tau \sim \rho \bmod 4\mathbb{Z}$ [102, §7.2].

In our case, we need the orbifold blocks of 16 chiral scalars under exchange $X_i \mapsto X_{i+8 \bmod 16}$. By decomposing X_i into its even and odd components $X_i \pm X_{i+8 \bmod 16}$, we find that the orbifold blocks are given by

$$\frac{\left[\vartheta^{(2)}[\delta_i^+](\Omega)\, \vartheta^{(2)}[\delta_i^-](\Omega)\right]^4}{Z_0^{16}\, \vartheta_i(\tau_{h,g})^8} \times \sum_{\mathcal{P}\in(\Lambda_{E_8}/2\Lambda_{E_8})} \Theta^{(2)}_{E_8[2],(\mathcal{P},0)}(\Omega)\, \Theta_{E_8[2],\mathcal{P}}(\tau_{h,g}) \,. \tag{B.34}$$

As a consistency check on this result (first obtained in [11] from the partition function of the E_8 root lattice on the genus 3 covering surface $\hat\Sigma$), let us consider the maximal non-separating degeneration limit: the imaginary part of the period matrix $\Omega_2 = \begin{pmatrix} L_1+L_2 & L_2 \\ L_2 & L_2+L_3 \end{pmatrix}$ parametrizes Schwinger times along the three edges of the two-loop sunset diagram shown in Figure 1 iii). Assuming that the $\mathbb{Z}_2$ action is inserted along the edge of length L_3, the $E_8 \oplus E_8$ momenta running in the three edges are $(p_1,p_2), (p_1+q, p_2+q), (q,q)$. Decomposing as usual $p_1+p_2 = 2\Sigma + \mathcal{P}, p_1 - p_2 = 2\Delta - \mathcal{P}$, the classical action is

$$\begin{aligned} &L_1(p_1^2+p_2^2) + L_2\left[(p_1+q)^2 + (p_2+q)^2\right] + 2L_3 q^2 \\ =&2(L_1+L_2)\left[\left(\Sigma+\tfrac12\mathcal{P}\right)^2 + \left(\Delta - \tfrac12\mathcal{P}\right)^2\right] + 2(L_2+L_3)q^2 + 4L_2(\Sigma+\tfrac12\mathcal{P})\cdot q \\ =&2\left(\Sigma+\tfrac12\mathcal{P} \quad q\right)\cdot \begin{pmatrix} L_1+L_2 & L_2 \\ L_2 & L_2+L_3 \end{pmatrix} \cdot \begin{pmatrix} \Sigma+\tfrac12\mathcal{P} \\ q \end{pmatrix} + 2(L_1+L_2)\left(\Delta-\tfrac12\mathcal{P}\right)^2 \,, \end{aligned} \tag{B.35}$$

in agreement with the maximal non-separating degeneration limit of the second factor in (B.34), using $\tau_{h,g} \sim \rho$.

The contributions of the other degrees of freedom (spacetime bosons and fermions, ghosts) are unaffected by the orbifolding and, as in the maximal rank, turn the factor $1/Z_0^{16}$ in (B.34) into $1/\Phi_{10}$. In the sector $\left[{h \atop g}\right] = \left[{00 \atop 01}\right]$, the resulting ratio can be written in three equivalent ways [11, (4.29-31)],

$$\begin{aligned}\frac{\left[\vartheta^{(2)}[\delta_i^+](\Omega)\,\vartheta^{(2)}[\delta_i^-](\Omega)\right]^4}{\vartheta_i(\tau_{h,g})^8\Phi_{10}(\Omega)} &= \frac{\vartheta^{(2)}\left[{00 \atop 00}\right]^2\vartheta^{(2)}\left[{00 \atop 01}\right]^2\vartheta^{(2)}\left[{00 \atop 10}\right]^2\vartheta^{(2)}\left[{00 \atop 11}\right]^2}{\vartheta_3^4\vartheta_4^4(\tau)\Phi_{10}(\Omega)} = \frac{1}{\vartheta_3^4\vartheta_4^4(\tau)\,\Phi_{6,0}(\Omega)} \\ &= \frac{\vartheta^{(2)}\left[{00 \atop 00}\right]^2\vartheta^{(2)}\left[{00 \atop 01}\right]^2\vartheta^{(2)}\left[{10 \atop 00}\right]^2\vartheta^{(2)}\left[{10 \atop 01}\right]^2}{\vartheta_3^4\vartheta_2^4(\tau)\Phi_{10}(\Omega)} = \frac{1}{\vartheta_3^4\vartheta_2^4(\tau)\,\Phi_{6,1}(\Omega)} \\ &= \frac{\vartheta^{(2)}\left[{10 \atop 00}\right]^2\vartheta^{(2)}\left[{10 \atop 01}\right]^2\vartheta^{(2)}\left[{00 \atop 10}\right]^2\vartheta^{(2)}\left[{00 \atop 11}\right]^2}{\vartheta_3^4\vartheta_4^4(\tau)\Phi_{10}(\Omega)} = \frac{1}{\vartheta_2^4\vartheta_4^4(\tau)\,\Phi_{6,2}(\Omega)}\end{aligned} \tag{B.36}$$

where $\Phi_{6,0} \equiv \Phi_6$ is the Siegel modular form (A.46) of weight 6 and level 2, and $\Phi_{6,1} \propto \tilde{\Phi}_6$ and $\Phi_{6,2}$ are its images under S_ρ and $T_\rho \cdot S_\rho$, respectively (see (B.44) below). Using the identity

$$\sum_{\mathcal{P}\in(\Lambda_{E_8}/2\Lambda_{E_8})} \Theta^{(2)}_{E_8[2],(\mathcal{P},0)}(\Omega)\,\Theta_{E_8[2],\mathcal{P}}(\tau) = \vartheta_3^4\vartheta_4^4\,\Theta^{(2)}_{E_8}(2\rho,2\sigma,2v)+\frac{1}{16}\vartheta_3^4\vartheta_2^4\,\Theta^{(2)}_{E_8}(\tfrac{\rho}{2},2\sigma,v)+\frac{1}{16}\vartheta_2^4\vartheta_4^4\,\Theta^{(2)}_{E_8}(\tfrac{\rho+1}{2},2\sigma,v) \tag{B.37}$$

we find that the orbifold block in the sector $\left[{h \atop g}\right] = \left[{00 \atop 01}\right]$ is given by [11, (4.38)]

$$Z_8^{(2)}\left[{00 \atop 01}\right] = \frac{\Theta^{(2)}_{E_8}(2\rho,2\sigma,2v)}{\Phi_{6,0}} + \frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},2\sigma,v)}{16\Phi_{6,1}} + \frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},2\sigma,v)}{16\Phi_{6,2}} \tag{B.38}$$

In particular, the dependence on the Prym period τ has disappeared. The result (B.38) is invariant under the index 15 subgroup $\Gamma_{2,e_1}(2)$ of $Sp(4,\mathbb{Z})$ which preserves the twist $\left[{00 \atop 01}\right]$ [102, §6.1]. In fact it can be rewritten as

$$Z_8^{(2)}\left[{00 \atop 01}\right] = \sum_{\gamma\in\Gamma_{2,e_1}(2)/\Gamma_{2,0,e_1}(2)} \left[\frac{\Theta^{(2)}_{E_8}(2\rho,2\sigma,2v)}{\Phi_{6,0}}\right]|_\gamma\ , \tag{B.39}$$

where $\Gamma_{2,0,e_1}(2) \equiv \Gamma_{2,e_1}(2)\cap\Gamma_{2,0}(2)$ has index 3 inside $\Gamma_{2,e_1}(2)$, and 3 inside $\Gamma_{2,0}(2)$. As a consistency check in (B.38), in the separating degeneration limit $v \to 0$ (B.38) becomes

$$Z_8^{(2)}\left[{00 \atop 01}\right] \sim -4\pi^2 v^2\,\frac{E_4(2\sigma)}{\eta\left[{0 \atop 1}\right](\sigma)}\left[\frac{E_4(2\rho)}{\eta\left[{0 \atop 1}\right](\rho)} + \frac{E_4(\frac{\rho}{2})}{\eta\left[{1 \atop 0}\right](\rho)} + \frac{E_4(\frac{\rho+1}{2})}{\eta\left[{1 \atop 1}\right](\rho)}\right] \tag{B.40}$$

where, for N prime and $h \neq 0 \bmod N$ we define

$$\eta\left[{0 \atop g}\right] = \eta^{k+2}(\tau)\,\eta^{k+2}(N\tau)\ ,\quad \eta\left[{h \atop g}\right] = e^{\frac{i\pi a(k+2)}{12}}\,\eta^{k+2}(\tau)\,\eta^{k+2}\left(\frac{\tau+a}{N}\right) \tag{B.41}$$

where $k+2=\ell$, $a=gh^{-1}$, with h^{-1} being the inverse of h in the multiplicative group $\mathbb{Z}/N\mathbb{Z}$. Using [22, Eq.(A.10)], the term in bracket is indeed recognized as the untwisted unprojected one-loop partition function

$$Z_8^{(2)}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \equiv \frac{E_4^2}{\eta^{24}} = \frac{E_4(2\tau)}{\eta^8(\tau)\eta^8(2\tau)} + \frac{E_4(\frac{\tau}{2})}{\eta^8(\tau)\eta^8(\frac{\tau}{2})} + \frac{E_4(\frac{\tau+1}{2})}{e^{2i\pi/3}\eta^8(\tau)\eta^8(\frac{\tau+1}{2})} \,. \tag{B.42}$$

The remaining blocks can be obtained by modular transformations,

$$Z_8^{(2)}[\tilde{\delta}](\tilde{\Omega}) = Z_8[\delta](\Omega) \;, \quad \tilde{\delta} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \delta \bmod 2 \tag{B.43}$$

where $\delta = (h_1, h_2, g_1, g_2)^t$. Using the invariance of $\Theta^{(2)}_{E_8}(\rho,\sigma,v)$ under the full Siegel modular group, and acting with the 15 elements γ of $Sp(4,\mathbb{Z})/\Gamma_{2,e_1}(2)$ on (B.39), we obtain the orbifold blocks shown on Table 1. In this table, $\Phi_{6,1}$ through $\Phi_{6,14}$ are images of $\Phi_{6,0}$ under $\gamma \in Sp(4,\mathbb{Z})/\Gamma_{2,0}(2)$. When γ lies in $SL(2,\mathbb{Z})_\rho \times SL(2,\mathbb{Z})_\sigma \to Sp(4,\mathbb{Z})$ we denote the respective $SL(2,\mathbb{Z})$ generators in subscript:

$$\begin{aligned}
\Phi_{6,1}(\rho,\sigma,v) =& \rho^{-6}\,\Phi_{6,0}(-1/\rho, \sigma - v^2/\rho, v/\rho) = \Phi_6|\begin{pmatrix} 0&0&-1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix} = \Phi_6|_{(S,\mathbb{1})} \;, \\
\Phi_{6,2}(\rho,\sigma,v) =& \Phi_{6,1}(\rho+1,\sigma,v) = \Phi_6|\begin{pmatrix} 1&0&-1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix} = \Phi_6|_{(TS,\mathbb{1})} \\
\Phi_{6,3}(\rho,\sigma,v) =& \sigma^{-6}\,\Phi_{6,0}(\rho - v^2/\sigma, -1/\sigma, v/\sigma) = \Phi_6|\begin{pmatrix} 1&0&0&0\\0&0&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix} = \Phi_6|_{(\mathbb{1},S)} \\
\Phi_{6,4}(\rho,\sigma,v) =& \Phi_{6,3}(\rho,\sigma+1,v) = \Phi_6|\begin{pmatrix} 1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix} = \Phi_6|_{(\mathbb{1},TS)} \\
\Phi_{6,5}(\rho,\sigma,v) =& \sigma^{-6}\,\Phi_{6,1}(\rho - v^2/\sigma, -1/\sigma, v/\sigma) = \Phi_6|\begin{pmatrix} 0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix} = \Phi_6|_{(S,S)} \\
\Phi_{6,6}(\rho,\sigma,v) =& \Phi_{6,5}(\rho+1,\sigma,v) = \Phi_6|\begin{pmatrix} 1&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix} = \Phi_6|_{(TS,S)} \\
\Phi_{6,7}(\rho,\sigma,v) =& \Phi_{6,5}(\rho,\sigma+1,v) = \Phi_6|\begin{pmatrix} 0&0&-1&0\\0&1&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix} = \Phi_6|_{(S,TS)} \\
\Phi_{6,8}(\rho,\sigma,v) =& \Phi_{6,5}(\rho+1,\sigma+1,v) = \Phi_6|\begin{pmatrix} 1&0&-1&0\\0&1&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix} = \Phi_6|_{(TS,TS)}
\end{aligned} \tag{B.44}$$

$\begin{bmatrix}h_1h_2\\g_1g_2\end{bmatrix}$	$Z_8^{(2)}\begin{bmatrix}h_1h_2\\g_1g_2\end{bmatrix}$	$\gamma\in Sp(4,\mathbb{Z})/\Gamma_{2,e_1}(2)$
$\begin{bmatrix}00\\10\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(2\rho,2\sigma,2v)}{\Phi_{6,0}}+\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\sigma}{2},v)}{2^4\Phi_{6,3}}+\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\sigma+1}{2},v)}{2^4\Phi_{6,4}}$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$
$\begin{bmatrix}01\\00\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\sigma}{2},v)}{2^4\Phi_{6,3}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma}{2},\frac{v}{2})}{2^8\Phi_{6,5}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma}{2},\frac{v}{2})}{2^8\Phi_{6,6}}$	$\begin{pmatrix}1&0&0&0\\0&0&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix}$
$\begin{bmatrix}10\\00\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},2\sigma,v)}{2^4\Phi_{6,1}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma}{2},\frac{v}{2})}{2^8\Phi_{6,5}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma+1}{2},\frac{v}{2})}{2^8\Phi_{6,7}}$	$\begin{pmatrix}0&0&0&-1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$
$\begin{bmatrix}11\\00\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma}{2},\frac{v}{2})}{2^8\Phi_{6,5}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma+1}{2},\frac{v+1}{2})}{2^8\Phi_{6,9}}+\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\sigma-2v+\rho}{2},v-\rho)}{2^4\Phi_{6,13}}$	$\begin{pmatrix}1&0&0&0\\-1&0&0&-1\\0&1&1&0\\0&1&0&0\end{pmatrix}$
$\begin{bmatrix}01\\01\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\sigma+1}{2},v)}{2^4\Phi_{6,4}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma+1}{2},\frac{v}{2})}{2^8\Phi_{6,7}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma+1}{2},\frac{v}{2})}{2^8\Phi_{6,8}}$	$\begin{pmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix}$
$\begin{bmatrix}10\\10\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},2\sigma,v)}{2^4\Phi_{6,2}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma}{2},\frac{v}{2})}{2^8\Phi_{6,6}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma+1}{2},\frac{v}{2})}{2^8\Phi_{6,8}}$	$\begin{pmatrix}0&1&0&-1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$
$\begin{bmatrix}01\\10\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\sigma}{2},v)}{2^4\Phi_{6,3}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma}{2},\frac{v+1}{2})}{2^8\Phi_{6,10}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma}{2},\frac{v+1}{2})}{2^8\Phi_{6,11}}$	$\begin{pmatrix}0&1&0&0\\0&0&-1&0\\0&0&1&1\\1&-1&0&0\end{pmatrix}$
$\begin{bmatrix}10\\11\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},2\sigma,v)}{2^4\Phi_{6,2}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma+1}{2},\frac{v+1}{2})}{2^8\Phi_{6,9}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma}{2},\frac{v+1}{2})}{2^8\Phi_{6,11}}$	$\begin{pmatrix}0&1&-1&-1\\1&-1&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$
$\begin{bmatrix}10\\01\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},2\sigma,v)}{2^4\Phi_{6,1}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma}{2},\frac{v+1}{2})}{2^8\Phi_{6,10}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma+1}{2},\frac{v+1}{2})}{2^8\Phi_{6,12}}$	$\begin{pmatrix}0&0&-1&-1\\1&-1&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$
$\begin{bmatrix}01\\11\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\sigma+1}{2},v)}{2^4\Phi_{6,4}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma+1}{2},\frac{v+1}{2})}{2^8\Phi_{6,9}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma+1}{2},\frac{v+1}{2})}{2^8\Phi_{6,12}}$	$\begin{pmatrix}0&1&0&0\\1&-1&-1&0\\0&0&1&1\\1&-1&0&0\end{pmatrix}$
$\begin{bmatrix}00\\11\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(2\rho,2\sigma,2v)}{\Phi_{6,0}}+\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\rho-2v+\sigma}{2},v-\rho)}{2^4\Phi_{6,13}}+\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\rho-2v+\sigma+1}{2},v-\rho)}{2^4\Phi_{6,14}}$	$\begin{pmatrix}0&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&0\end{pmatrix}$
$\begin{bmatrix}11\\01\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma+1}{2},\frac{v}{2})}{2^8\Phi_{6,7}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma}{2},\frac{v+1}{2})}{2^8\Phi_{6,11}}+\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\rho-2v+\sigma+1}{2},v-\rho)}{2^4\Phi_{6,14}}$	$\begin{pmatrix}1&0&0&0\\-1&1&0&-1\\0&1&1&0\\0&1&0&0\end{pmatrix}$
$\begin{bmatrix}11\\10\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma}{2},\frac{v}{2})}{2^8\Phi_{6,6}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma+1}{2},\frac{v+1}{2})}{2^8\Phi_{6,12}}+\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\rho-2v+\sigma+1}{2},v-\rho)}{2^4\Phi_{6,14}}$	$\begin{pmatrix}1&1&1&0\\-1&0&0&-1\\0&1&1&0\\0&1&0&0\end{pmatrix}$
$\begin{bmatrix}11\\11\end{bmatrix}$	$\frac{\Theta^{(2)}_{E_8}(\frac{\rho+1}{2},\frac{\sigma+1}{2},\frac{v}{2})}{2^8\Phi_{6,8}}+\frac{\Theta^{(2)}_{E_8}(\frac{\rho}{2},\frac{\sigma}{2},\frac{v+1}{2})}{2^8\Phi_{6,10}}+\frac{\Theta^{(2)}_{E_8}(2\rho,\frac{\sigma-2v+\rho}{2},v-\rho)}{2^4\Phi_{6,13}}$	$\begin{pmatrix}1&1&1&0\\-1&1&0&-1\\0&1&1&0\\0&1&0&0\end{pmatrix}$

Table 1: List of genus-two orbifold blocks for the $\mathbb{Z}_2$ CHL model

$$\begin{aligned}
\Phi_{6,9}(\rho,\sigma,v) =& \Phi_{6,5}(\rho+1,\sigma+1,v+1) = \Phi_6|\begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\Phi_{6,10}(\rho,\sigma,v) =& \Phi_{6,5}(\rho,\sigma,v+1) = \Phi_6|\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\Phi_{6,11}(\rho,\sigma,v) =& \Phi_{6,5}(\rho+1,\sigma,v+1) = \Phi_6|\begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\Phi_{6,12}(\rho,\sigma,v) =& \Phi_{6,5}(\rho,\sigma+1,v+1) = \Phi_6|\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\Phi_{6,13}(\rho,\sigma,v) =& \Phi_{6,3}(\rho,\sigma-2v+\rho,v-\rho) = \Phi_6|\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\Phi_{6,14}(\rho,\sigma,v) =& \Phi_{6,4}(\rho,\sigma-2v+\rho+1,v-\rho) = \Phi_6|\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{B.45}$$

As a consistency check, using the fact that

$$\Phi_{6,k}(\rho,\sigma,v) \sim \begin{cases} 2^{-8}v^2\eta^{12}\vartheta_i^4(\rho)\,\vartheta_j^4(\sigma) + \mathcal{O}(v^4)\,, & k \le 8 \\ \pm 2^{-4}\,\eta^{12}(\rho)\,\eta^{12}(\sigma) + \mathcal{O}(v^2) & k \ge 9 \end{cases} \tag{B.46}$$

where $(k,i,j) = (0,2,2),(1,4,2),(2,3,2)(3,2,4),(4,2,3),(5,4,4),(6,3,4),(7,4,3),(8,3,3)$ for $k \le 8$, we see that in the separating degeneration limit $v \to 0$,

$$Z_8^{(2)}\big[\begin{smallmatrix} h_1h_2 \\ g_1g_2 \end{smallmatrix}\big](\Omega) \sim -4\pi^2 v^2 Z_8^{(1)}\big[\begin{smallmatrix} h_1 \\ g_1 \end{smallmatrix}\big](\rho)\, Z_8^{(1)}\big[\begin{smallmatrix} h_2 \\ g_2 \end{smallmatrix}\big](\sigma) + \mathcal{O}(v^2)\,, \tag{B.47}$$

where $Z_8^{(1)}\big[\begin{smallmatrix} h \\ g \end{smallmatrix}\big]$ are the genus-one orbifold blocks given in [22, Eq.(A.6)]. Note that each of the numerators appearing in the genus-two orbifold blocks $Z_8^{(2)}\big[\begin{smallmatrix} h_1h_2 \\ g_1g_2 \end{smallmatrix}\big]$ can be interpreted as the genus-two theta series for an Euclidean lattice of rank 8 as follows (here $q^{\frac12 Q^2}$ denotes $e^{\mathrm{i}\pi Q^r\Omega_{rs}Q^s}$)

$$\begin{aligned}
\Theta_{E_8}(2\rho,2v,2\sigma) =& \sum_{\substack{(Q_1,Q_2)\in \\ E_8[2]\oplus E_8[2]}} e^{\mathrm{i}\pi Q^r\Omega_{rs}Q^s}, \\
\Theta_{E_8}(2\rho,v,\tfrac{\sigma}{2}) =& 2^{-4} \sum_{\substack{(Q_1,Q_2)\in \\ E_8[2]\oplus E_8[2]^*}} e^{\mathrm{i}\pi Q^r\Omega_{rs}Q^s} \\
\Theta_{E_8}(2\rho,v,\tfrac{\sigma+1}{2}) =& 2^{-4} \sum_{\substack{(Q_1,Q_2)\in \\ E_8[2]\oplus E_8[2]^*}} (-1)^{Q_2^2}\, e^{\mathrm{i}\pi Q^r\Omega_{rs}Q^s} \\
\Theta_{E_8}(\tfrac{\rho}{2},\tfrac{v}{2},\tfrac{\sigma+1}{2}) =& 2^{-8} \sum_{\substack{(Q_1,Q_2)\in \\ E_8[2]^*\oplus E_8[2]^*}} (-1)^{Q_2^2}\, e^{\mathrm{i}\pi Q^r\Omega_{rs}Q^s}
\end{aligned} \tag{B.48}$$

$$
\begin{aligned}
\Theta_{E_8}(\tfrac{\rho}{2},\tfrac{v+1}{2},\tfrac{\sigma}{2}) &= 2^{-8} \sum_{\substack{(Q_1,Q_2)\in \\ E_8[2]^*\oplus E_8[2]^*}} (-1)^{2Q_1\cdot Q_2}\, e^{\mathrm{i}\pi Q^r\Omega_{rs}Q^s} \\
\Theta_{E_8}(\tfrac{\rho}{2},\tfrac{v+1}{2},\tfrac{\sigma+1}{2}) &= 2^{-8} \sum_{\substack{(Q_1,Q_2)\in \\ E_8[2]^*\oplus E_8[2]^*}} (-1)^{(2Q_1+Q_2)\cdot Q_2}\, e^{\mathrm{i}\pi Q^r\Omega_{rs}Q^s} \\
\Theta_{E_8}(\tfrac{\rho+1}{2},\tfrac{v}{2},\tfrac{\sigma+1}{2}) &= 2^{-8} \sum_{\substack{(Q_1,Q_2)\in \\ E_8[2]^*\oplus E_8[2]^*}} (-1)^{Q_1^2+Q_2^2}\, e^{\mathrm{i}\pi Q^r\Omega_{rs}Q^s} \\
\Theta_{E_8}(\tfrac{\rho+1}{2},\tfrac{v+1}{2},\tfrac{\sigma+1}{2}) &= 2^{-8} \sum_{\substack{(Q_1,Q_2)\in \\ E_8[2]^*\oplus E_8[2]^*}} (-1)^{(Q_1+Q_2)^2}\, e^{\mathrm{i}\pi Q^r\Omega_{rs}Q^s} \\
\Theta_{E_8}(2\rho, v-\rho, \tfrac{\sigma-2v+\rho}{2}) &= 2^{-4} \sum_{\substack{(Q_1,Q_2)\in \\ E_8[2]^*\oplus E_8[2]^*}} \delta_{(Q_1+Q_2)\in E_8[2]}\, e^{\mathrm{i}\pi Q^r\Omega_{rs}Q^s} \\
\Theta_{E_8}(2\rho, v-\rho, \tfrac{\sigma-2v+\rho+1}{2}) &= 2^{-4} \sum_{\substack{(Q_1,Q_2)\in \\ E_8[2]^*\oplus E_8[2]^*}} \delta_{(Q_1+Q_2)\in E_8[2]}\, (-1)^{\frac14(Q_1-Q_2)^2}\, e^{\mathrm{i}\pi Q^r\Omega_{rs}Q^s}
\end{aligned}
\tag{B.49}
$$

Now, as indicated above (B.28), the orbifold blocks $Z_8^{(2)}\left[\begin{smallmatrix}h_1h_2\\g_1g_2\end{smallmatrix}\right]$ only include the contributions from the chiral measure for the ten-dimensional string, and need to be supplemented with the contribution of the bosonic zero-modes of the d compact bosons,

$$
Z_{d,d}^{(2)}\left[\begin{smallmatrix}h_1h_2\\g_1g_2\end{smallmatrix}\right] = |\Omega_2|^{d/2} \sum_{Q\in\Lambda_{d,d}^{\otimes 2}+\frac12(h_1,h_2)\delta} (-1)^{\delta\cdot(g_1Q_1+g_2Q_2)}\, e^{\mathrm{i}\pi Q_L^r\Omega_{rs}Q_L^s-\mathrm{i}\pi Q_R^r\bar\Omega_{rs}Q_R^s}\ , \tag{B.50}
$$

where δ is a null element in $(2I\!I_{d,d})/I\!I_{d,d}$ which depends on the orbifold action on T^d; we shall henceforth restrict to a half-period shift along the d-th circle, so that $\delta=(0^d;0^{d-1}1)$. For this choice, the product of (B.39) and (B.49) can again be written as a sum over images under the stabilizer of the twist,

$$
Z_8^{(2)}\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right] Z_{d,d}^{(2)}\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right] = \sum_{\gamma\in\Gamma_{2,e_1}(2)/\Gamma_{2,0,e_1}(2)} \left.\frac{\Gamma^{(2)}_{\tilde\Lambda_{d+8,d}}\left[(-1)^{\delta\cdot Q_2}\right]}{\Phi_{6,0}}\right|_\gamma\ , \tag{B.51}
$$

where

$$
\tilde\Lambda_{d+8,d} \equiv E_8[2]\oplus I\!I_{d,d}\ . \tag{B.52}
$$

and $\delta\cdot Q_2$ equals the winding of the d-th embedding coordinate along the cycle B_2. Thus, the sum over all the sectors listed in (1), in the case of compactification on T^d at this specific factorization point in the moduli space, can be rewritten as

$$
\begin{aligned}
\sum_{h_r,g_r\in\{0,1\}}{}' Z_8^{(2)}\left[\begin{smallmatrix}h_1h_2\\g_1g_2\end{smallmatrix}\right] Z_{d,d}^{(2)}\left[\begin{smallmatrix}h_1h_2\\g_1g_2\end{smallmatrix}\right] &= \sum_{\gamma\in Sp(4,\mathbb{Z})/\Gamma_{2,e_1}(2)} Z_8^{(2)}\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right] Z_{d,d}^{(2)}\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right]\Big|_\gamma \\
&= \sum_{\gamma\in Sp(4,\mathbb{Z})/\Gamma_{2,0}(2)} \left.\frac{\Gamma^{(2)}_{\tilde\Lambda_{d+8,d}}\left[\left((-1)^{\delta\cdot Q_1}+(-1)^{\delta\cdot Q_2}+(-1)^{\delta\cdot(Q_1+Q_2)}\right)P_{ab,cd}\right]}{\Phi_{6,0}}\right|_\gamma\ ,
\end{aligned}
\tag{B.53}
$$

where for the last equality we expressed $Z_8^{(2)}\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right] Z_{d,d}^{(2)}\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right]$ as a sum over $\Gamma_{2,e_1}(2)/\Gamma_{2,0,e_1}(2)$, similarly to (B.39), and rewrote the two sums as a double sum over $Sp(4,\mathbb{Z})/\Gamma_{2,0}(2)$ and $\Gamma_{2,0}(2)/\Gamma_{2,0,e_1}(2)$.

Including the contribution from the second line in (B.21), and retaining the next-to-leading term in the low energy expansion, we see that the $\nabla^2 F^4$ coupling on the locus $G_{d,d} \subset G_{d+8,d}$ where the lattice $\Lambda_{d+16,d}$ factorizes is given by

$$G^{(2)}_{ab,cd} = \frac{1}{4}\,\text{R.N.} \int_{\mathcal{F}_2} \frac{\mathrm{d}^3\Omega_1\,\mathrm{d}^3\Omega_2}{|\Omega_2|^3} \Big(Z_8^{(2)}\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right] Z_{d,d}^{(2)}\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right] + \sideset{}{'}\sum_{h_r,g_r\in\{0,1\}} Z_8^{(2)}\left[\begin{smallmatrix}h_1h_2\\g_1g_2\end{smallmatrix}\right] Z_{d,d}^{(2)}\left[\begin{smallmatrix}h_1h_2\\g_1g_2\end{smallmatrix}\right]\Big)[P_{ab,cd}]\ , \tag{B.54}$$

where the bracket $[P_{ab,cd}]$ denotes an insertion of the quartic polynomial $P_{ab,cd}$ (2.31) in the sum over the lattice $\tilde\Lambda_{d+8,d}$ and its modular images.

Now, in parallel with the 'Hecke identity' (B.42), observe that the untwisted genus-two chiral partition function satisfies

$$Z_8^{(2)}\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right] \equiv \frac{[\Theta^{(2)}_{E_8}(\Omega)]^2}{\Phi_{10}} = \sideset{}{'}\sum_{h_r,g_r\in\{0,1\}} Z_8^{(2)}\left[\begin{smallmatrix}h_1h_2\\g_1g_2\end{smallmatrix}\right]\ . \tag{B.55}$$

The validity of this identity can for example be checked for the minimal non-separating degeneration using (A.31). Using this identity in the sum over all sectors, as in (B.53), we can rewrite it as a sum over $Sp(4,\mathbb{Z})/\Gamma_{2,0}(2)$, as in the second line of (B.52), to obtain

$$\frac{1}{4}\sum_{h_r,g_r\in\{0,1\}} Z_8^{(2)}\left[\begin{smallmatrix}h_1h_2\\g_1g_2\end{smallmatrix}\right] Z_{d,d}^{(2)}\left[\begin{smallmatrix}h_1h_2\\g_1g_2\end{smallmatrix}\right][P_{ab,cd}] = \sum_{\gamma\in Sp(4,\mathbb{Z})/\Gamma_{2,0}(2)} \frac{\Gamma^{(2)}_{\tilde\Lambda_{d+8,d}}\left[\frac12\big(1+(-1)^{\delta\cdot Q_1}\big)\,\frac12\big(1+(-1)^{\delta\cdot Q_2}\big)\,P_{ab,cd}\right]}{\Phi_{6,0}}\Bigg|_\gamma\ . \tag{B.56}$$

The insertions of $\frac12\big(1+(-1)^{\delta\cdot Q_i}\big)$ can be seen as projectors on the lattice $\tilde\Lambda_{d+8,8}$ to vectors with even entries along one of the cicle designated by δ, such that the resulting sum is recognized as a genus-two partition function, with insertion of $P_{ab,cd}$ only, for the 'magnetic charge lattice' introduced in [22, (A,16)],

$$\Lambda_{d+8,d} = E_8[2] \oplus I\!I_{1,1}[2] \oplus I\!I_{d-1,d-1}\ . \tag{B.57}$$

At this point, we can readily extend the result away from the factorized locus by allowing non-trivial Wilson lines in the lattice partition function. As established in (B.55), the partition function can be written down as a sum over images from under $Sp(4,\mathbb{Z})/\Gamma_{2,0}(2)$, such that the integral can be unfolded from a fundamental domain of $Sp(4,\mathbb{Z})$ to a fundamental domain of $\Gamma_{2,0}(2)$

$$G^{(2)}_{ab,cd} = \text{R.N.}\int_{\Gamma_{2,0}(2)\backslash\mathcal{H}_2} \frac{\mathrm{d}^3\Omega_1\,\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\,\frac{\Gamma^{(2)}_{\Lambda_{d+8,d}}[P_{ab,cd}]}{\Phi_6}\ . \tag{B.58}$$

This concludes the computation of the two-loop $\nabla^2 F^4$ coupling in the $\mathbb{Z}_2$ orbifold.

B.2.3 $\mathbb{Z}_N$ orbifold with $N = 3, 5, 7$

Let us now briefly discuss the genus-two amplitude in heterotic CHL orbifolds with $N = 2, 3, 5, 7$. As in [22, §A.2], we restrict to a locus $G_{d+k-8,d+k-8} \subset G_{d+16,d}$ where the even self-dual lattice $\Lambda_{d+16,d}$ of the heterotic string compactified on T^d factorizes as $\Lambda_{Nk,8-k} \oplus I\!I_{1,1} \oplus I\!I_{d+k-8,d+k-8}$, where the $\mathbb{Z}_N$ action acts by a $\mathbb{Z}_N$ rotation on the first factor and by a translation by $1/N$ period on the second. We denote by $\Lambda_{k,8-k}$ the $\mathbb{Z}_N$-invariant part of $\Lambda_{Nk,8-k}$, and let

$$\tilde{\Lambda}_{d+2k-8,d} = \Lambda_{k,8-k} \oplus I\!I_{1,1} \oplus I\!I_{d+k-9,d+k-9} \; . \tag{B.59}$$

Upon using the Niemeier lattice construction of the $\mathbb{Z}_N$-symmetric lattice outlined in [22], one finds that the invariant lattice $\Lambda_{k,8-k} = D_k[N] \oplus D_{8-k}[-1]$, where the sum is performed with respect to the diagonal glue code $\{(0,0),(s,s),(v,v),(c,c)\}$. For $N = 2$ using the construction in the previous subsection, one has instead $\Lambda_{8,0} = E_8[2]$.

Now, as in (B.27) the genus-two amplitude decomposes into a sum over all possible twisted or untwisted periodicity conditions $\left[{h_1 h_2 \atop g_1 g_2}\right]$ along the A and B cycles of the genus-two curve Σ, with h_r, g_r running over $\mathbb{Z}/(N\mathbb{Z})$. For N prime, all $N^4 - 1$ non-trivial twistings form a single orbit under $Sp(4,\mathbb{Z})$, so it suffices to focus on one of them, say $\epsilon = \left[{00 \atop 01}\right]$. The stabilizer of ϵ under the action (B.43) is $\Gamma_{2,e_1}(N)$ (a subgroup of index $N^4 - 1$ inside $Sp(4,\mathbb{Z})$), so the corresponding orbifold block $\widetilde{Z}^{(2)}_{d+2k-8,d}\left[{00 \atop 01}\right]$ must be a Siegel modular form for $\Gamma_{2,e_1}(N)$, and satisfy

$$\sum_{h_r,g_r\in\mathbb{Z}/(N\mathbb{Z})}{}^{'} \widetilde{Z}^{(2)}_{d+2k-8,d}\left[{h_1 h_2 \atop g_1 g_2}\right] = \sum_{\gamma\in Sp(4,\mathbb{Z})/\Gamma_{2,e_1}(N)} \widetilde{Z}^{(2)}_{d+2k-8,d}\left[{00 \atop 01}\right]\Big|_\gamma \; . \tag{B.60}$$

This orbifold block can in principle be computed using the N-sheeted cover of the genus-two curve Σ, which now has genus $N+1$. Rather than following this route, we instead postulate that it is given by the natural generalization of (B.50), namely

$$\widetilde{Z}^{(2)}_{d+2k-8,d}\left[{00 \atop 01}\right] = \sum_{\gamma\in\Gamma_{2,e_1}(N)/\Gamma_{2,0,e_1}(N)} \frac{\Gamma^{(2)}_{\tilde{\Lambda}_{d+2k-8,d}}\left[e^{\frac{2\pi i \delta\cdot Q_2}{N}}\right]}{\Phi_{k-2}}\Bigg|_\gamma \tag{B.61}$$

where $\Gamma_{2,0,e_1}(N) = \Gamma_{2,e_1}(N) \cap \Gamma_{2,0}(N)$ has index $N+1$ in $\Gamma_{2,e_1}(N)$ and $N^2 - 1$ in $\Gamma_{2,0}(N)$, and $\delta \cdot Q_2 = n_2$ is the winding of the d-th embedding coordinate along the cycle B_2, so that $\Gamma^{(2)}_{\tilde{\Lambda}_{d+2k-8,d}}\left[e^{\frac{2\pi i \delta\cdot Q_2}{N}}\right]$ is a modular form of $\Gamma_{2,0,e_1}(N)$. As a consistency check, one may verify that (B.60) has the correct behavior

$$\widetilde{Z}^{(2)}_{d+2k-8,d}\left[{00 \atop 01}\right](\Omega) \to -4\pi^2 v^2\, \widetilde{Z}^{(1)}_{d+2k-8,d}\left[{0 \atop 0}\right](\rho)\, \widetilde{Z}^{(1)}_{d+2k-8,d}\left[{0 \atop 1}\right](\sigma) \tag{B.62}$$

in the separating degeneration limit $v \to 0$, where

$$\widetilde{Z}^{(1)}_{d+2k-8,d}\left[{0 \atop 0}\right] = \sum_{\gamma\in SL(2,\mathbb{Z})/\Gamma_0(N)} \frac{\Gamma_{\tilde{\Lambda}_{d+2k-8,d}}}{\Delta_k}\Bigg|_\gamma \, , \qquad \widetilde{Z}^{(1)}_{d+2k-8,d}\left[{0 \atop 1}\right] = \frac{\Gamma_{\tilde{\Lambda}_{d+2k-8,d}}\left[e^{\frac{2\pi i \delta\cdot Q}{N}}\right]}{\Delta_k} \, . \tag{B.63}$$

Similarly as in the $N=2$ case, we deduce from (B.59) and (B.60) that the sum over all non-trivial twisted sectors can be rewritten as a sum over images under $\Gamma_{2,0}(N)$,

$$\sum_{h_i,g_i\in\mathbb{Z}/(N\mathbb{Z})}^{\prime} \widetilde{Z}_{d+2k-8,d}\left[{}^{h_1h_2}_{g_1g_2}\right] = \sum_{\gamma\in Sp(4,\mathbb{Z})/\Gamma_{2,e_1}(N)} \sum_{\gamma'\in\Gamma_{2,e_1}(N)/\Gamma_{2,0,e_1}(N)} \frac{\Gamma^{(2)}_{\tilde\Lambda_{d+2k-8,d}}\left[e^{\frac{2\pi i\delta\cdot Q_2}{N}}\right]}{\Phi_{k-2}}\Bigg|_{\gamma'\gamma}$$
$$= \sum_{\gamma\in Sp(4,\mathbb{Z})/\Gamma_{2,0}(N)} \left[\sum_{\gamma'\in\Gamma_{2,0}(N)/\Gamma_{2,0,e_1}(N)} \frac{\Gamma^{(2)}_{\tilde\Lambda_{d+2k-8,d}}\left[e^{\frac{2\pi i\delta\cdot Q_2}{N}}\right]}{\Phi_{k-2}}\Bigg|_{\gamma'}\right]\Bigg|_{\gamma} \tag{B.64}$$

Next, we observe that the untwisted genus-two amplitude also satisfies an Hecke identity generalizing (B.54), namely

$$\widetilde{Z}_{d+2k-8,d}\left[{}^{00}_{00}\right] = \frac{\Gamma^{(2)}_{\Lambda_{d+16,d}}}{\Phi_{10}} = \sum_{\gamma\in Sp(4,\mathbb{Z})/\Gamma_{2,0}(N)} \frac{\Gamma^{(2)}_{\tilde\Lambda_{d+2k-8,d}}}{\Phi_{k-2}}\Bigg|_{\gamma} . \tag{B.65}$$

Combining (B.60) and (B.64), and using

$$\frac{1}{N^2}\left(\Gamma^{(2)}_{\tilde\Lambda_{d+2k-8,d}} + \sum_{\gamma\in\Gamma_{2,0}(N)/\Gamma_{2,0,e_1}(N)} \Gamma^{(2)}_{\tilde\Lambda_{d+2k-8,d}}\left[e^{\frac{2\pi i\delta\cdot Q_2}{N}}\right]\Big|\gamma\right)$$
$$= \Gamma^{(2)}_{\tilde\Lambda_{d+2k-8,d}}\left[\tfrac{1}{N}\left(1+e^{\frac{2\pi i\delta\cdot Q_1}{N}}+\ldots+e^{\frac{2\pi i(N-1)\delta\cdot Q_1}{N}}\right)\tfrac{1}{N}\left(1+e^{\frac{2\pi i\delta\cdot Q_2}{N}}+\ldots+e^{\frac{2\pi i(N-1)\delta\cdot Q_2}{N}}\right)\right], \tag{B.66}$$

we find that the sum over all twisted sectors reduce to a sum over images under $\Gamma_{2,0}(N)$

$$\frac{1}{N^2}\sum_{h_i,g_i\in\mathbb{Z}/(N\mathbb{Z})} Z\left[{}^{h_1h_2}_{g_1g_2}\right] = \sum_{\gamma\in Sp(4,\mathbb{Z})/\Gamma_{2,0}(N)} \frac{\Gamma^{(2)}_{\Lambda_{d+2k-8,d}}}{\Phi_{k-2}}\Bigg|_{\gamma} \tag{B.67}$$

where now the Siegel theta series involves the rescaled lattice

$$\Lambda_{d+2k-8,d} = \Lambda_{k,8-k}\oplus I\!I_{1,1}[N]\oplus I\!I_{d+k-9,d+k-9} . \tag{B.68}$$

After including the contribution from the second line in (B.21), retaining the next-to-leading term in the low energy expansion, and unfolding the integration domain $\mathcal{F}_2$ against the sum over images in (B.66), we conclude that the genus-two ∇^2F^4 coupling is given by

$$G^{(2)}_{ab,cd} = \text{R.N.}\int_{\Gamma_{2,0}(N)\backslash\mathcal{H}_2} \frac{d^3\Omega_1\, d^3\Omega_2}{|\Omega_2|^3}\frac{\Gamma^{(2)}_{\Lambda_{d+2k-8,d}}[P_{ab,cd}]}{\Phi_{k-2}} \tag{B.69}$$

as announced in (2.28).

B.2.4 Regularization of the genus-two modular integral

In order to regulate the genus-two modular integral (2.30), it is easiest to fold the integration domain $\mathcal{H}_2/\Gamma_{2,0}(N)$ back to the standard fundamental domain of $Sp(4,\mathbb{Z})$ defined in (A.5),

$$G^{(p,q)}_{ab,cd} = \text{R.N.}\int_{\mathcal{F}_2}\frac{d^3\Omega_1 d^3\Omega_2}{|\Omega_2|^3}\sum_{\gamma\in\Gamma_{2,0}(N)\backslash Sp(4,\mathbb{Z})} \frac{\Gamma^{(2)}_{\Lambda_{p,q}}[P_{ab,cd}]}{\Phi_{k-2}}\Bigg|_{\gamma} . \tag{B.70}$$

The renormalized modular integral over $\mathcal{F}_2$ can then be defined following the procedure in [103, 64], *i.e.* by truncating the fundamental domain to $\mathcal{F}_2^\Lambda = \mathcal{F}_2 \cap \{t < \Lambda\}$, where the coordinate t on $\mathcal{H}_2$ was defined in (A.9). In order to separate one-loop and primitive two-loop subdivergences, we then decompose $\mathcal{F}_2^\Lambda$ into three subregions,

$$\begin{aligned} \mathcal{F}_2^0 &= \mathcal{F}_2^\Lambda \cap \{\rho_2 \le t + u_2^2\rho_2 \le \Lambda_1\} \\ \mathcal{F}_2^I &= \mathcal{F}_2^\Lambda \cap \{\rho_2 \le \Lambda_1 \le t + u_2^2\rho_2\} \\ \mathcal{F}_2^{II} &= \mathcal{F}_2^\Lambda \cap \{\Lambda_1 \le \rho_2 \le t + u_2^2\rho_2\} \end{aligned} \tag{B.71}$$

where $\Lambda_1 \ll \Lambda$ is a fiducial scale. One-loop subdivergences arise from integration over $\mathcal{F}_2^I$, while primitive divergence arises from integrating over $\mathcal{F}_2^{II}$. In extracting the divergences as $\Lambda \to \infty$, we can safely ignore terms proportional to powers of Λ_1, since they cancel in the sum over the three regions [103].

Let us first consider the divergences from region I. In this region, the variable t is bounded by Λ while ρ is restricted to the fundamental domain $\mathcal{F}_{1,\Lambda_1}$. For the first $1+N$ cosets of $\Gamma_{2,0}(N)\backslash Sp(4,\mathbb{Z})$ listed in (A.22), the charges (Q_1, Q_2) whose contributions are not exponentially suppressed as $t \to \infty$ are those with $Q_2 = 0$. For those, the integral over σ_1 projects $1/\Phi_{k-2}|_\gamma$ to its zero-mode $\psi_0|_\gamma$ in (A.59), while the remaining integral over u_1, u_2 projects the latter to its average value (A.73), with a factor of $1/2$ because of the element of $SL(2,\mathbb{Z})$ permuting them. The divergence from these $N+1$ cosets is then

$$-\frac{k}{32\pi}\int^\Lambda \frac{\mathrm{d}t}{t^3}\, t^{\frac{q}{2}-1}\text{R.N.}\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\sum_{\gamma_\rho\in SL(2,\mathbb{Z})/\Gamma_0(N)}\left[\frac{N^2E_2(N\rho)-E_2(\rho)}{(N-1)\;\Delta_k(\rho)}\Gamma_{\Lambda_{p,q}}[P_{\langle ab,}]\delta_{cd\rangle}\right]\Big|_{\gamma_\rho}\,. \tag{B.72}$$

For the remaining N^2+N^3 cosets, the representative γ includes again the $N+1$ γ_ρ elements again, times the N transformations $\{S_\sigma, T_\sigma S_\sigma, \dots, T_\sigma^{N-1}S_\sigma\}$, which requires a Poisson resummation over Q_2 before setting its dual to 0, and the N shifts b in (A.22). The divergence is then of the same form as above, upon replacing ψ_0 by its image under S_σ, $N^{k/2}\hat{\psi}_0$ (A.59), and including a volume factor $|\Lambda^*_{p,q}/\Lambda_{p,q}|^{-\frac{1}{2}} = \upsilon N^{-\frac{k}{2}-2}$ from the Poisson resummation and a multiplicity factor N^2 from the transformations listed above:

$$-\frac{k\upsilon}{32\pi}\int^\Lambda \frac{\mathrm{d}t}{t^3}\, t^{\frac{q}{2}-1}\text{R.N.}\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\sum_{\gamma_\rho\in SL(2,\mathbb{Z})/\Gamma_0(N)}\left[\frac{E_2(\rho)-E_2(N\rho)}{(N-1)\;\Delta_k(\rho)}\Gamma_{\Lambda_{p,q}}[P_{\langle ab,}]\delta_{cd\rangle}\right]\Big|_{\gamma_\rho} \tag{B.73}$$

For the perturbative ∇^2F^4 coupling in $D = 10-q$ dimensions, the volume factor is $\upsilon = N$. After unfolding the integral to the domain $\mathcal{H}_1/\Gamma_0(N)$, the two contributions (B.71), (B.72) add up to

$$-\frac{k}{32\pi}\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\text{R.N.}\int_{\Gamma_0(N)\backslash\mathcal{H}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{N\hat{E}_2(N\rho)+\hat{E}_2(\rho)}{\Delta_k(\rho)}\Gamma_{\Lambda_{p,q}}[P_{\langle ab,}]\delta_{cd\rangle} = -\frac{3}{8\pi}\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\delta_{\langle ab,}F^{(p,q)\,e}_{cd\rangle e} \tag{B.74}$$

where we recognized the coefficient of the divergence as the renormalized one-loop F^4 coupling by integrating by part, as in [22, §3.2], upon using the identity

$$D_{-k}\Big(\frac{1}{\Delta_k(\rho)}\Big) = \frac{k}{12}\frac{N\hat{E}_2(N\rho)+\hat{E}_2(\rho)}{\Delta_k(\rho)}\,, \tag{B.75}$$

where $D_w = \frac{\mathrm{i}}{\pi}(\partial_\tau - \frac{\mathrm{i}w}{2\tau_2})$ is the raising operator.

We now turn to the primitive two-loop divergence coming from the integral over $\mathcal{F}_2^{II}$. In this region, it is more convenient to use the variables V, τ defined in (A.9). The variable V runs from τ_2/Λ to $1/\tau_2\Lambda_1$, while the variable τ takes values in the standard fundamental domain $\mathcal{F}_1/\mathbb{Z}_2$ of $GL(2,\mathbb{Z})$, truncated at $\tau_2 \leq \sqrt{\Lambda/\Lambda_1}$ [103]. The primitive divergence comes from the region $V \to 0$. For the first coset in (A.22), the contribution of all charge vectors with $Q_1 \neq 0$ or $Q_2 \neq 0$ are exponentially suppressed as $V \to 0$. For $(Q_1, Q_2) = (0,0)$, the polynomial $P_{ab,cd}$ in (2.31) reduces to $3\delta_{\langle ab,}\delta_{cd\rangle}/(16\pi^2|\Omega_2|)$, and the integral over Ω_1 projects $1/\Phi_{k-2}$ to its zero-mode $C_{k-2}(0,0,0) = \frac{48N}{N^2-1}$ in (A.49). For the second and third class of cosets in (A.22), the limit $V \to 0$ requires first performing a Poisson resummation over either Q_1 or Q_2, resulting in a volume factor of $|\Lambda^*_{p,q}/\Lambda_{p,q}|^{-\frac{1}{2}} = \upsilon N^{-\frac{k}{2}-2}$, and the integral over Ω_1 projects $N^{\frac{k}{2}}/\tilde{\Phi}_{k-2}|_\gamma$ to its zero-mode $N^{k/2}\widetilde{C}_{k-2}(0,0,0) = -\frac{48N^{k/2}}{N^2-1}$ from (A.50), for each of the $N(N+1)$ cosets. Finally, for the fourth class of cosets in (A.22), the limit $V \to 0$ requires performing a Poisson resummation over both Q_1 and Q_2, resulting in a volume factor of $|\Lambda^*_{p,q}/\Lambda_{p,q}|^{-1} = \upsilon^2 N^{-k-4}$, and the integral over Ω_1 projects $N^{k-2}/\Phi_{k-2}(\Omega/N)|_\gamma$ to its zero-mode after having used the identity (A.40), for each of the N^3 cosets. Adding up all contributions, we find

$$\frac{3\delta_{\langle ab,}\delta_{cd\rangle}}{16\pi^2}\,\mathrm{R.N.}\int_{\mathcal{F}_1/\mathbb{Z}_2}\frac{48\mathrm{d}\tau_1\mathrm{d}\tau_2}{(N^2-1)\tau_2^2}\int_{\tau_2/\Lambda}^{\tau_2/\Lambda_1} 2V^2\mathrm{d}V\, V^{2-q}\left[N-(N+1)\tfrac{\upsilon}{N}+\tfrac{\upsilon^2}{N^2}\right] \tag{B.76}$$

Setting $\upsilon = N$, the term in square bracket cancels, so the coefficient of the two-loop primitive divergence in fact vanishes.

Finally, it remains to consider a potential divergence from the separating degeneration. For generic values of ρ, σ in $\mathcal{F}_2$, the integral around $v = 0$ is of the form $\int \mathrm{d}v\mathrm{d}\bar{v}/v^2$, which vanishes provided one integrates first over the angular direction in the v-plane. There can however be a divergence from the region $\rho_2, \sigma_2 \to \infty$ while $v \to 0$, where the genus-two curve degenerates into a figure-eight graph. For the first coset in (A.22), the contribution of all charge vectors with $Q_1 \neq 0$ or $Q_2 \neq 0$ are exponentially suppressed as $\rho_2, \sigma_2 \to \infty$. As shown in §A.6, the integral over v_1 gives rise to a delta-function $c_k(0)^2\delta(v_2)$. To integrate this delta distribution it is convenient to unfold the integration domain of Ω_2 near the cusp $|\Omega_2| \to \infty$, $\mathcal{P}_2/GL(2,\mathbb{Z})$ to $\mathcal{P}_2$, using the sum over $GL(2,\mathbb{Z})/\mathrm{Dih}_4$ in (5.25), and taking into account the factor of 4 associated to Dih_4, the stabilizer of the singular locus $v = 0$. Equivalently one can think of the integral over $\mathcal{P}_2/GL(2,\mathbb{Z})$, and simply unfold the order four symmetry permuting σ_2 and ρ_2 and changing the sign of v_2. At $v_2 = 0$, $\sigma_2 = t$ and the integration domain is $\Lambda_1 \leq \rho_2 \leq \sigma_2 < \Lambda$, which after symmetrization gives the divergent contribution

$$-\frac{3k^2}{256\pi^3}\int^\Lambda\frac{\mathrm{d}\rho_2}{\rho_2^3}\int^\Lambda\frac{\mathrm{d}\sigma_2}{\sigma_2^3}(\rho_2\sigma_2)^{\frac{q}{2}}\frac{\delta_{\langle ab}\delta_{cd\rangle}}{\rho_2\sigma_2} \sim -\frac{3k^2}{256\pi^3}\left(\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\right)^2\delta_{\langle ab,}\delta_{cd\rangle} \tag{B.77}$$

For the other cosets in (A.22), the zeroth Fourier-Jacobi coefficient behaves has $N^{\frac{k}{2}}\hat{\psi}_0(\rho,v)$ leading to $N^{\frac{k}{2}}c_k(0)^2\delta(v_2)$, and $N^{k-2}\psi_0(\rho/N, v/N)$ leading to $N^{k-2}c_k(0)^2\delta(v_2/N)$. The first contribution occurs from the trivial coset only; the second from $2N$ cosets because of the symmetry $\rho \leftrightarrow \sigma$, with an overall volume factor $\upsilon N^{-\frac{k}{2}-2}$; and the third from N^3 cosets corresponding to all shifts $(\frac{\rho+a}{N}, \frac{\sigma+b}{N}, \frac{v+c}{N})$, with an overall volume factor $\upsilon^2 N^{-k-4}$. Combining these terms and using $c_k(0)=k$, we find that the divergence from the figure-eight degeneration is

$$-\frac{3k^2}{256\pi^3}\frac{(N^2+2N\upsilon+\upsilon^2)}{N^2}\left(\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\right)^2\delta_{\langle ab,}\delta_{cd\rangle} = -\frac{3k^2(1+\frac{\upsilon}{N})^2}{256\pi^3}\left(\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\right)^2\delta_{\langle ab,}\delta_{cd\rangle}\ . \tag{B.78}$$

For $q=6$, the divergent term $(\Lambda^{\frac{q-6}{2}}/\frac{q-6}{2})^2$ is replaced by $(\log\Lambda)^2$.

Combining these results, we can now define the renormalized integral (2.30) by subtracting all divergent contributions before taking the limit $\Lambda\to\infty$. In the case of the two-loop ∇^2F^4 couplings ($\upsilon=N$), we obtain

$$\begin{aligned}G^{(p,q)}_{ab,cd} = \lim_{\Lambda\to\infty}\Big[\int_{\mathcal{F}_2^\Lambda}\frac{\mathrm{d}^3\Omega_1\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\sum_{\gamma\in\Gamma_{2,0}(N)\backslash Sp(4,\mathbb{Z})}\frac{\Gamma^{(2)}_{\Lambda_{p,q}}[P_{ab,cd}]}{\Phi_{k-2}(\Omega)}\Big|_\gamma + \frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\frac{3}{8\pi}\delta_{\langle ab,}F^{(p,q)\,e}_{cd\rangle e}\\ + \left(\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\right)^2\frac{3k^2}{64\pi^3}\delta_{\langle ab,}\delta_{cd\rangle})\Big]\ .\end{aligned} \tag{B.79}$$

For $q=6$, the $\mathcal{O}(\Lambda^{\frac{q-6}{2}})$ and $\mathcal{O}(\Lambda^{q-6})$ divergences become logarithmic and doubly logarithmic,

$$\begin{aligned}\widehat{G}^{(2k-2,6)}_{ab,cd} = \lim_{\Lambda\to\infty}\Big[\int_{\mathcal{F}_2^\Lambda}\frac{\mathrm{d}^3\Omega_1\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\sum_{\gamma\in\Gamma_{2,0}(N)\backslash Sp(4,\mathbb{Z})}\frac{\Gamma^{(2)}_{\Lambda_{2k\text{-}2,6}}[P_{ab,cd}]}{\Phi_{k-2}(\Omega)}\Big|_\gamma + \log\Lambda\,\frac{3}{8\pi}\delta_{\langle ab,}\widehat{F}^{(2k-2,6)\,e}_{cd\rangle e}\\ + (\log\Lambda)^2\frac{3k^2}{64\pi^3}\delta_{\langle ab,}\delta_{cd\rangle}\Big]\ ,\end{aligned} \tag{B.80}$$

where $F^{(p,q)}_{abcd}$ is the regularized integral (B.13).

The renormalization of the couplings F_{abcd} and $G_{ab,cd}$ is in fact consistent with supergravity computations [79], as we now explain. Recall that the complete string theory amplitude can be obtained by performing a functional integral over the fields of $\mathcal{N}=4$ supergravity with $2k-2$ vector multiplets, weighted by the Wilsonian effective action computed in string theory. This Wilsonian action can be defined by imposing an infrared cutoff Λ on the moduli space of complex structures, identified with the ultra-violet cutoff in supergravity. It follows that the Λ-dependent couplings

$$\begin{aligned}F^{(2k-2,6)}_{abcd}(\Lambda) &= F^{(2k-2,6)}_{abcd} + \frac{3k}{8\pi^2}\log\Lambda\delta_{(ab}\delta_{cd)}\ ,\\ G^{(2k-2,6)}_{ab,cd}(\Lambda) &= G^{(2k-2,6)}_{ab,cd} - \log\Lambda\,\frac{3}{8\pi}\delta_{\langle ab,}F^{(2k-2,6)\,e}_{cd\rangle e} - (\log\Lambda)^2\frac{3k^2}{64\pi^3}\delta_{\langle ab,}\delta_{cd\rangle}\ ,\end{aligned} \tag{B.81}$$

define a bare Lagrangian

$$\begin{aligned}\mathcal{L}(\Lambda) =& \frac{2}{\kappa^2}\mathcal{R} - \frac{1}{4}\delta_{ab}F^aF^b + \tfrac{1}{8}(\tfrac{\kappa}{2})^4F^{(2k-2,6)}_{abcd}(\Lambda)t_8F^aF^bF^cF^d\\ &+ \tfrac{1}{8\pi}(\tfrac{\kappa}{2})^6G^{(2k-2,6)}_{ab,cd}(\Lambda)\,t_8\nabla F^a\nabla F^bF^cF^d + \dots\end{aligned} \tag{B.82}$$

such that the UV divergences in the path integral cancel at this order. These divergences cancel for any functions $F^{(2k-2,6)}_{abcd}$ and $G^{(2k-2,6)}_{ab,cd}$ satisfying their respective differential constraints.

Upon setting $F^{(2k-2,6)}_{abcd}$ and $G^{(2k-2,6)}_{ab,cd}$ to zero in (B.80), one reproduces precisely the counter-terms computed in [79] in four dimensions. The variation of $\mathcal{L}(\Lambda)$ with respect to $F^{(2k-2,6)}_{abcd}$ is interpreted in supergravity as the form factor for the operator $t_8 F^4$ (at zero momentum and properly supersymmetrized). Similarly, the variation of $\mathcal{L}(\Lambda)$ with respect to $G^{(2k-2,6)}_{ab,cd}$ is the form factor for the operator $t_8\nabla^2 F^4$. Because (3.20) does not admit a constant homogeneous solution for $q=6$, there cannot be any genuine 2-loop divergence proportional to $\delta_{\langle ab,}\delta_{cd\rangle}$ in $\mathcal{N}=4$ supergravity. The 2-loop divergence proportional to $(\log\Lambda)^2$ in (B.80) is therefore a consequence of the 1-loop divergence, via the renormalization group equation

$$\Lambda\frac{\mathrm{d}}{\mathrm{d}\Lambda}G^{(2k-2,6)}_{ab,cd}(\Lambda) = -\frac{3}{4\pi}\delta_{\langle ab,}F^{(2k-2,6)\ e}_{cd\rangle e}(\Lambda)\ . \tag{B.83}$$

This is consistent with the supergravity analysis in [79, §5.A], where the two-loop divergence originates entirely from figure-eight supergravity diagrams (shown in Figure 1ii), for which the subdivergence is proportional to the 1-loop counter-term form factor.

Let us now briefly discuss the regularization of the integral (B.69) in the case where the lattice $\Lambda_{p,q}$ is the non-perturbative Narain lattice (2.3). In this case, the volume factor υ is equal to 1. In this case, the cancellation in (B.75) still takes place in the maximal rank case since the zero-th Fourier coefficient of $1/\Phi_{10}$ vanishes from (A.48), but it no longer holds for CHL models with $N=2,3,5,7$. Setting $\upsilon=1$ in the previous computations, we now get

$$\begin{aligned} G^{(p,q)}_{ab,cd} = \lim_{\Lambda\to\infty}\Big[\int_{\mathcal{F}_2^\Lambda}\frac{\mathrm{d}^3\Omega_1\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\sum_{\gamma\in\Gamma_{2,0}(N)\backslash Sp(4,\mathbb{Z})}\frac{\Gamma^{(2)}_{\Lambda_{p,q}}[P_{ab,cd}]}{\Phi_{k-2}(\Omega)}\Big|_\gamma + \frac{27}{\pi^2N^2}\frac{\Lambda^{q-6}}{(q-6)^2}\delta_{\langle ab,}\delta_{cd\rangle} \\ -\frac{9(N-1)}{\pi^2N^2}\frac{\Lambda^{q-5}}{q-5}\delta_{\langle ab,}\delta_{cd\rangle}\,\mathrm{R.N.}\int_{\mathcal{F}_1}\frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\tau_2^{5-q} + \frac{3}{2\pi N}\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\,{}^{\varsigma}G^{(p,q)}_{\langle ab,}\delta_{cd\rangle}\Big)\Big]\ , \end{aligned} \tag{B.84}$$

where ${}^{\varsigma}G^{(p,q)}_{ab}$ denotes the regularized integral (B.17). The maximal rank case is obtained by setting $N=1$, and ${}^{\varsigma}G^{(p,q)}_{ab} = G^{(p,q)}_{ab}$. Of course, the case relevant for the non-perturbative $\nabla^2(\nabla\phi)^4$ coupling in $D=3$ corresponds to $q=8$, in which case there are power-like divergences but no logarithmic divergence.

B.2.5 Anomalous terms in the differential equation for $G_{ab,cd}$

In section 3.3 we established that the renormalized integral $G^{(p,q)}_{ab,cd}$ satisfies the differential equation (3.20), with a quadratic source term originating from the separating degeneration locus $\upsilon=0$. In this section we take into account the boundary of the regularized domain $\mathcal{F}_2^\Lambda$ and show that the equation indeed holds for the renormalized couplings at generic values of q. For $q=5$ with $\upsilon\neq N$ and $q=6$ we find additional linear source terms from the non-separating degeneration. For the perturbative amplitude in four dimensions, $q=6$, $\upsilon=N$, these linear term originate from the mixing between the analytic and the non-analytic components of the amplitude. Our analysis parallels that of the $D^6\mathcal{R}^4$ couplings in [103, §3.3].

From the $t=\Lambda$ boundary of the region $\mathcal{F}_2^I$ defined in (B.70), the leading contribution of the polynomial insertion is given by

$$\Omega_2^{2r}\Omega_2^{2s}e^{-\frac{\Delta_2}{8\pi}}Q_{Ler}Q_{Lfs}e^{\frac{\Delta_2}{8\pi}}P_{ab,cd}\Big|_{Q_2=0} = \frac{3}{16\pi^2}\left(\delta_{ef}\delta_{\langle ab,}P_{cd\rangle} + 2\delta_{e\langle b}\delta_{|f|b,}P_{cd\rangle}\right) + \mathcal{O}(t^{-1})\ , \tag{B.85}$$

so using (A.73), with a factor 1/2 due to the $\mathbb{Z}_2$ symmetry $(u_1, u_2) \to (-u_1, -u_2)$ at the cusp, we find that the right-hand side of (3.62) receives an additional contribution given by

$$- \frac{k\Lambda^{\frac{q-6}{2}}}{64\pi} \text{R.N.} \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \sum_{\gamma_\rho \in SL(2,\mathbb{Z})/\Gamma_0(N)} \left[\frac{N^2 \hat{E}_2(N\rho) - \hat{E}_2(\rho)}{(N-1)\ \Delta_k(\rho)} \Gamma_{\Lambda_{p,q}}[P_{\langle ab,}](\delta_{cd\rangle}\delta_{ef} + 2\delta_{c|e|}\delta_{d\rangle f}) \right] \Big|_{\gamma_\rho}$$
$$- \frac{k\upsilon\Lambda^{\frac{q-6}{2}}}{64\pi} \text{R.N.} \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \sum_{\gamma_\rho \in SL(2,\mathbb{Z})/\Gamma_0(N)} \left[\frac{\hat{E}_2(\rho) - \hat{E}_2(N\rho)}{(N-1)\ \Delta_k(\rho)} \Gamma_{\Lambda_{p,q}}[P_{\langle ab,}](\delta_{cd\rangle}\delta_{ef} + 2\delta_{c|e|}\delta_{d\rangle f}) \right] \Big|_{\gamma_\rho} \tag{B.86}$$

where the first and second line results respectively from cosets elements $(\gamma, 1)$ and $(\gamma, S_\sigma) \in (SL(2,\mathbb{Z})/\Gamma_0(N))_\rho \times (SL(2\mathbb{Z})/\Gamma_0(N))_\sigma$, while other terms in the coset sum are annihilated by integration over σ_1, $\upsilon_1 \in [-\frac{1}{2}, \frac{1}{2}[$. The sum (B.85) can be rewritten in terms of the regularized integral $G^{(p,q)}_{ab}$ as

$$- \frac{k(\upsilon-1)\Lambda^{\frac{q-6}{2}}}{64\pi(N-1)} \left(\delta_{ef}\delta_{\langle ab} G^{(p,q)}_{cd\rangle} + 2\delta_{e\langle a}\delta_{b,|f|} G^{(p,q)}_{cd\rangle} \right)$$
$$- \frac{k(N^2-\upsilon)\Lambda^{\frac{q-6}{2}}}{64\pi N(N-1)} \left(\delta_{ef}\delta_{\langle ab}\, {}^{\mathrm{S}}G^{(p,q)}_{cd\rangle} + 2\delta_{e\langle a}\delta_{b,|f|}\, {}^{\mathrm{S}}G^{(p,q)}_{cd\rangle} \right) . \tag{B.87}$$

This terms gives a finite correction to the differential equation for $q = 6$.

The right-hand side of (3.62) also receives contributions from the boundary of region $\mathcal{F}_2^{II}$ in (B.70), where the leading contribution of the polynomial insertion is

$$(\Omega_2)_{rs} e^{-\frac{\Delta_2}{8\pi}} Q^r_{Le} Q^s_{Lf} e^{\frac{\Delta_2}{8\pi}} P_{ab,cd} \Big|_{Q_1=Q_2=0} = -\frac{3}{32\pi^3 |\Omega_2|} \left(\delta_{ef}\delta_{\langle ab,}\delta_{cd\rangle} + 2\delta_{e\langle a}\delta_{|f|b,}\delta_{cd\rangle} \right) + \mathcal{O}(\Omega_2^{-1}) . \tag{B.88}$$

Its contribution to the right hand side of (3.62) thus reduces to [29]

$$-\frac{3}{32\pi^2} \text{R.N.} \int_{\mathcal{F}_1/\mathbb{Z}_2} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2} \int_{\frac{\tau_2}{\Lambda}} 2dV \frac{\partial}{\partial V} \frac{1}{V^3} \left(\frac{|\Omega_2|^{\frac{q}{2}}}{|\Omega_2|^3} \frac{\delta_{ef}\delta_{\langle ab}\delta_{cd\rangle} + 2\delta_{e\langle a}\delta_{|f|b,}\delta_{cd\rangle}}{|\Omega_2|} \right)$$
$$\times \left(\frac{2k}{N-1} \left[N - \frac{\upsilon}{N}(N+1) + \frac{\upsilon^2}{N^2} \right] - \frac{1}{4\pi} k^2 \delta\Big(\frac{\tau_1}{V\tau_2} \Big) \left[1 + 2\frac{\upsilon}{N} + \frac{\upsilon^2}{N^2} \right] \right) . \tag{B.89}$$

In (B.88) we kept the constant term in the Fourier expansions of $1/\Phi_{k-2}$ and we used $\partial/\partial\bar{\Omega} \sim -\frac{\mathrm{i}}{4} V \Omega_2^{-1} \partial/\partial V$. On the boundary at $V = \tau_2/\Lambda$, the first term in (B.88) gives

$$\Lambda^{q-5} \frac{3k(N-\upsilon)(1-\frac{\upsilon}{N^2})}{8\pi^2(N-1)} (\delta_{ef}\delta_{\langle ab}\delta_{cd\rangle} + 2\delta_{e\langle a}\delta_{b,|f|}\delta_{cd\rangle}) \text{R.N.} \int_{\mathcal{F}_1/\mathbb{Z}_2} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2} \tau_2^{5-q} \tag{B.90}$$

which vanishes in the perturbative case, $\upsilon = N$. The second term in (B.88) integrates to

$$-\frac{\Lambda^{q-6}}{q-6} \frac{3k^2(\delta_{ef}\delta_{\langle ab}\delta_{cd\rangle} + 2\delta_{e\langle a}\delta_{b,|f|}\delta_{cd\rangle})}{128\pi^3} \left(1 + \frac{\upsilon}{N} \right)^2 . \tag{B.91}$$

[29] Where one uses $2\mathrm{i}d^3\Omega_2 \frac{\partial}{\partial \Omega_{rs}} \big((\Omega_2)_{rt} (\Omega_2)_{su} (\Omega_2^{-1})^{tu} X(\Omega_2) \big) = \frac{2dV d\tau_1 d\tau_2}{\tau_2^2} \frac{\partial}{\partial V} \frac{X(\Omega_2)}{V^3}$ at the boundary $V = \frac{\tau_2}{\Lambda}$.

The case $q = 6$ must be computed separately and turns out to give zero. Finally, the quadratic term in the second line of (3.66) can be written using the regularized genus-one integral $F^{(p,q)}_{abcd}$ (B.15) as

$$-\frac{3\pi}{2}F^{(p,q)}_{|e)k\langle ab,}(\Lambda)F^{(p,q)\,k}_{cd\rangle}{}_{(f|}(\Lambda) = -\frac{3\pi}{2}F^{(p,q)}_{|e)k\langle ab,}F^{(p,q)\,k}_{cd\rangle}{}_{(f|} - \frac{3k(1+\frac{\upsilon}{N})}{16\pi}\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\delta_{\langle ab,}F^{(p,q)}_{cd\rangle ef} \quad \text{(B.92)}$$
$$-\frac{3k^2(1+\frac{\upsilon}{N})^2}{512\pi^3}\left(\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\right)^2\left(\delta_{ef}\delta_{\langle ab,}\delta_{cd\rangle} + 2\delta_{e\langle a}\delta_{b,|f|}\delta_{cd\rangle}\right) .$$

Using the action of the operator (3.59) on the tensor defining the counter-terms of $G^{(p,q)}_{ab,cd}$,

$$\Delta_{ef}\delta_{\langle ab}G^{(p,q)}_{cd\rangle} = \tfrac{q-6}{4}\left(\delta_{ef}\delta_{\langle ab,}G^{(p,q)}_{cd\rangle} + 2\delta_{e\langle(a}\delta_{b),|f|}G^{(p,q)}_{cd\rangle}\right) + 6\delta_{\langle ab,}F^{(p,q)}_{cd\rangle ef} \quad \text{(B.93)}$$
$$\Delta_{ef}\delta_{\langle ab}{}^{\varsigma}G^{(p,q)}_{cd\rangle} = \tfrac{q-6}{4}\left(\delta_{ef}\delta_{\langle ab,}{}^{\varsigma}G^{(p,q)}_{cd\rangle} + 2\delta_{e\langle(a}\delta_{b),|f|}{}^{\varsigma}G^{(p,q)}_{cd\rangle}\right) + 6\delta_{\langle ab,}F^{(p,q)}_{cd\rangle ef} \quad \text{(B.94)}$$
$$\Delta_{ef}\delta_{\langle ab,}\delta_{cd\rangle} = \tfrac{q-5}{2}\left(\delta_{ef}\delta_{\langle ab,}\delta_{cd\rangle} + 2\delta_{e\langle(a}\delta_{b),|f|}\delta_{cd\rangle}\right) , \quad \text{(B.95)}$$

one finds that all Λ dependent terms cancel in the differential equation for the renormalized coupling, such that for generic q,

$$\Delta_{ef}\Bigg(G^{(p,q)}_{ab,cd}(\Lambda) + \frac{k}{32\pi}\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\delta_{\langle ab}\left(\tfrac{\upsilon-1}{N-1}G^{(p,q)}_{cd\rangle} + \tfrac{N-\frac{\upsilon}{N}}{N-1}{}^{\varsigma}G^{(p,q)}_{cd\rangle}\right) + \frac{3k^2(1+\frac{\upsilon}{N})^2}{256\pi^3}\left(\frac{\Lambda^{\frac{q-6}{2}}}{\frac{q-6}{2}}\right)^2\delta_{\langle ab,}\delta_{cd\rangle}$$
$$-\frac{3k}{4\pi}\frac{\Lambda^{q-5}}{q-5}\frac{(N-\upsilon)(1-\frac{\upsilon}{N^2})}{N-1}\delta_{\langle ab,}\delta_{cd\rangle}\text{R.N.}\int_{\mathcal{F}_1/\mathbb{Z}_2}\frac{d\tau_1 d\tau_2}{\tau_2^2}\tau_2^{5-q}\Bigg)$$
$$= -\frac{3\pi}{2}F^{(p,q)}_{|e)k\langle ab,}F^{(p,q)\,k}_{cd\rangle}{}_{(f|} . \quad \text{(B.96)}$$

The cases featuring logs must be treated separately. Here we shall only discuss the case of the perturbative lattice in four dimensions, *i.e.* $\upsilon = N$ and $q = 6$, which is physically relevant.

Because the first term proportional to $q - 6$ in (B.92) vanishes at $q = 6$, it does not cancel the finite contribution from (B.86) and one gets an additional linear source term in the equation. The computation of the anomalous terms from the counter-term in $G^{(2k-2,6)}_{ab} + {}^{\varsigma}G^{(2k-2,6)}_{ab}$ involves the detailed analysis of the integration by part in the boundary between regions $\mathcal{F}^I_2$ and $\mathcal{F}^{II}_2$. Since this boundary is artificial, these anomalous terms must cancel other contributions from (B.85) and (B.88), such that one can assume that $G^{(2k-2,6)}_{ab} + {}^{\varsigma}G^{(2k-2,6)}_{ab}$ satisfies the naive differential equation (B.92), ignoring the anomalous source term in (B.19). This prescription is in fact necessary for the differential equation to be well defined on the renormalized couplings. In this way we obtain

$$\Delta_{ef}\,\widehat{G}^{(2k-2,6)}_{ab,cd} = -\frac{3\pi}{2}\widehat{F}^{(2k-2,6)}_{|e)k\langle ab,}\widehat{F}^{(2k-2,6)\,k}_{cd\rangle}{}_{(f|} - \frac{3}{16\pi}\left(\delta_{ef}\delta_{\langle ab,}\widehat{F}^{(2k-2,6)k}_{cd\rangle k} + 2\delta_{e\langle(a}\delta_{b),|f|}\widehat{F}^{(2k-2,6)k}_{cd\rangle k}\right) , \quad \text{(B.97)}$$

where we recall that Δ_{ef} is a shorthand for the operator in (3.59).

B.3 Loci of enhanced gauge symmetry

Even after regulating infrared divergences occurring at generic points on $G_{p,q}$, further divergences may occur on loci of enhanced gauge symmetry, where perturbative 1/2-BPS states become massless. Divergences from region $\mathcal{F}_2^I$ in (B.70) occur from contributions of lattice vectors $Q_2 \in \Lambda$ such that $Q_2^2 = 2$. For such vectors, the integral over $\sigma_1 \in [0,1]$ picks up the polar term in the Fourier-Jacobi expansion (A.57) of $1/\Phi_{k-2}$, contributing a term of the form

$$\begin{aligned}\int^{\infty} \mathrm{d}t\, t^{\frac{q}{2}-3} e^{-2\pi t Q_{2R}^2} \times \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \int_{[0,1]^2} \mathrm{d}u_1\, \mathrm{d}u_2\, |\rho_2|^{q/2} \\ \times \sum_{Q_1 \in \Lambda_{p,q}} P_{ab,cd}\, q^{\frac{1}{2}Q_{1L}^2}\, \bar{q}^{\frac{1}{2}Q_{1R}^2}\, e^{-\pi\rho_2^2 Q_{2R}^2 + 2\pi\mathrm{i}(vQ_{1L}\cdot Q_{2L} - \bar{v}Q_{1R}\cdot Q_{2R})} \frac{\eta^6}{\Delta_k(\rho)\,\theta_1^2(\rho,v)}\end{aligned} \tag{B.98}$$

to the modular integral $G_{ab,cd}^{(p,q)}$. The integral over t diverges on the codimension q locus where $|Q_{2R}| \to 0$, corresponding to 1/2-BPS states with charge $\pm Q_2$ becoming massless. This is a familiar phenomenon in perturbative heterotic string theory, where such BPS states can be viewed as W-bosons for a $SU(2)$ gauge symmetry which spontaneously broken away from the locus where $|Q_{2R}| = 0$. Near the singular locus, the genus-two integral diverges as a sum of powers of the mass $\mathcal{M} = \sqrt{2}|Q_{2R}|$, weighted by the genus-one modular integral appearing in (B.97), which can interpreted as the four-point amplitude with two massless and two massive gauge bosons. Note that this genus-one integral does not suffer from any divergence from the lattice vector $Q_1 = Q_2$, since the polynomial $P_{ab,cd}$ in representation ⊞ vanishes when Q_1 and Q_2 are collinear. Of course, similar gauge symmetry enhancements arise from vectors $Q_2 \in \Lambda_{p,q}$ with $Q_2^2 = 2/N$, due to the polar term in the Fourier-Jacobi expansion of the images of $1/\Phi_{k-2}$ under $\Gamma_{2,0}(N)\backslash Sp(4,\mathbb{Z})$.

In addition, the modular integral $G_{ab,cd}^{(p,q)}$ has further singularities from region $\mathcal{F}_2^{II}$, due to polar terms of the form $q_1^{-N_1} q_2^{-N_2} q_3^{-N_3}$ in the Fourier expansion (A.49) of $1/\Phi_{k-2}$, with $N_1, N_2, N_3 < 0$. The integral over Ω_1 picks up contributions of pairs of vectors $(Q_1, Q_2) \in \Lambda_{p,q} \oplus \Lambda_{p,q}$ satisfying the level-matching conditions

$$Q_1^2 - 2N_1 = Q_2^2 - 2N_2 = Q_3^2 - 2N_3 = 0 \tag{B.99}$$

where we denote $Q_3 = Q_1 + Q_2$. The remaining integral over Ω_2 is of then the form

$$\int \frac{\mathrm{d}L_1 \mathrm{d}L_2 \mathrm{d}L_3}{(L_1L_2 + L_2L_3 + L_3L_1)^{\frac{6-q}{2}}} P_{ab,cd}\, e^{-2\pi\left(L_1 Q_{1R}^2 + L_2 Q_{2R}^2 + L_3 Q_{3R}^2\right)}\,, \tag{B.100}$$

which for $q = 6$ has a leading singularity in

$$\int \mathrm{d}L_1 \mathrm{d}L_2 \mathrm{d}L_3 P_{ab,cd}\, e^{-2\pi\left(L_1 Q_{1R}^2 + L_2 Q_{2R}^2 + L_3 Q_{3R}^2\right)} \sim \frac{\varepsilon_{rt}\varepsilon_{su} Q^r_{L(a} Q^s_{Lb)} Q^t_{L(c} Q^u_{Ld)}}{8\pi^3 Q_{1R}^2 Q_{2R}^2 Q_{3R}^2}\,. \tag{B.101}$$

This integral is singular on the codimension q locus where $Q_{iR}^2 = 0$ for one index $i \in \{1,2,3\}$, but the corresponding divergence is covered by region I. Genuine new divergences occur in codimension $2q$ where $Q_{1R}^2 = Q_{2R}^2 = 0$ for two distinct indices, in which case Q_{3R}^2 automatically vanishes. The latter occurs for $(N_1, N_2, N_3) = (1,1,1)$ and corresponds to a $SU(3)$ gauge symmetry enhancement. Of course, similar divergences arise from pairs of vectors $(Q_1, Q_2) \in \Lambda_{p,q}^* \oplus \Lambda_{p,q}^*$ due to the polar terms in the Fourier expansion of the images of $1/\Phi_{k-2}$ under $\Gamma_{2,0}(N)\backslash Sp(4,\mathbb{Z})$. It would be interesting to recover (B.100) from a two-loop computation in a super-Yang-Mills theory with $SU(3)$ gauge group.

C Composite 1/4-BPS states, and instanton measure

In this Appendix our main aim is to prove Eqs (5.85) and (5.92), which play a central role in our analysis of the decompactification limit in §5. In particular, they ensure the consistency of the 1/4-BPS Abelian Fourier coefficients of $G_{ab,cd}$ with the differential equation (2.26), (3.20), and the consistency of the helicity supertrace (2.14) with wall-crossing, generalizing the consistency checks of [29] to arbitrary charges Γ. Specifically, we show that the summation measure $\bar{c}(Q,P;\Omega_2)$ for 1/4-BPS Abelian Fourier coefficients of $G_{ab,cd}$ decomposes into an Ω_2-independent part associated to single-centered 1/4-BPS black holes, and a sum over all possible splittings of a 1/4-BPS charge vector $\Gamma=\Gamma_1+\Gamma_2$ into 1/2-BPS charges, Γ_1 and Γ_2, weighted by the product $\bar{c}(\Gamma_1)\bar{c}(\Gamma_2)$ of the summation measures for 1/2-BPS black holes.

We start by describing the possible splittings of a 1/4-BPS charge $\Gamma=(Q,P)$ into 1/2-BPS constituents. Assuming an Ansatz of the form $\Gamma_1=(p',r')(sQ-qP+tR)$ and $\Gamma_2=(q',s')(pP-rQ+uR)$ for rational coefficients and linearly independent charges (Q,P,R), with R an arbitrary auxiliary charge, it is easy to find that the condition $\Gamma=\Gamma_1+\Gamma_2$ fixes $t=u=0$ and p', r' q', s' such that

$$\begin{pmatrix}Q_1\\P_1\end{pmatrix}=\begin{pmatrix}p\\r\end{pmatrix}\frac{sQ-qP}{ps-qr}\,,\quad \begin{pmatrix}Q_2\\P_2\end{pmatrix}=\begin{pmatrix}q\\s\end{pmatrix}\frac{pP-rQ}{ps-qr}\,. \tag{C.1}$$

This splitting is conveniently parametrized by the a non-degenerate matrix $B=\left(\begin{smallmatrix}p&q\\r&s\end{smallmatrix}\right)\in M_2(\mathbb{Z})$, such that

$$\begin{pmatrix}Q_1\\P_1\end{pmatrix}=B\pi_1B^{-1}\begin{pmatrix}Q\\P\end{pmatrix}\,,\quad \begin{pmatrix}Q_2\\P_2\end{pmatrix}=B\pi_2B^{-1}\begin{pmatrix}Q\\P\end{pmatrix}\,, \tag{C.2}$$

where $\pi_1=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $\pi_2=\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$. To parametrize the possible splittings bijectively one must factorize out the stabilizer $\mathrm{Stab}(\pi_i)$ of π_1 and π_2 in $M_2(\mathbb{Z})$ up to permutation, *i.e.*

$$\mathrm{Stab}(\pi_i)=\left\{\left(\begin{smallmatrix}d_1&0\\0&d_2\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right\}\,. \tag{C.3}$$

All splittings of a charge Γ are therefore classified by the set of matrices $B\in M_2(\mathbb{Z})/\mathrm{Stab}(\pi_i)$. Decomposing the matrix B as

$$\begin{aligned}\begin{pmatrix}p&q\\r&s\end{pmatrix}&=\gamma\cdot\begin{pmatrix}p'&j\\0&k\end{pmatrix}\,, \qquad \gamma\in GL(2,\mathbb{Z})\,,\quad p'>0\quad 0\le j<k\,,\\ &=\gamma\cdot\begin{pmatrix}1&\frac{j}{\gcd(j,k)}\\0&\frac{k}{\gcd(j,k)}\end{pmatrix}\begin{pmatrix}p'&0\\0&\gcd(j,k)\end{pmatrix}\,,\end{aligned} \tag{C.4}$$

and using $\mathrm{Stab}(\pi_i)\cap GL(2,\mathbb{Z})=\mathrm{Dih}_4$ one can always choose $\gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4$.[30] We conclude that the possible splittings are in one-to-one correspondence with the elements of

$$M_2(\mathbb{Z})/\mathrm{Stab}(\pi_i)=\left\{\gamma\cdot\begin{pmatrix}1&j'\\0&k'\end{pmatrix}\,,\qquad \gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4\,,\quad 0\le j'<k'\,,\quad (j',k')=1\right\}\,. \tag{C.5}$$

such that the quantization condition $B\pi_iB^{-1}\Gamma\in\Lambda_m^*\oplus\Lambda_m$, $i=1,2$ on the charges of the two constituents is obeyed. It suffices to check this condition for $i=1$, since the sum of the two is by assumption in $\Lambda_m^*\oplus\Lambda_m$.

[30] One checks indeed that the quotient by Dih_4 passes to the right of γ, by changing the representatives γ and $j/\gcd(j,k)$ for $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\in\mathrm{Dih}_4$.

C.1 Maximal rank

In the maximal rank case the condition $B\pi_1 B^{-1}\Gamma \in \Lambda_m^* \oplus \Lambda_m$ reduces to $\frac{aP-cQ}{k'} \in \Lambda_m$, with

$$(Q_1,P_1)=(a,c)\Big(dQ-bP-\frac{j'}{k'}(aP-cQ)\Big)\,,\qquad (Q_2,P_2)=\Big(\frac{j'}{k'}(a,c)+(b,d)\Big)(aP-cQ)\,, \quad \text{(C.6)}$$

These splittings are all related by $GL(2,\mathbb{Z})$ to a canonical splitting

$$\binom{Q}{P}=\binom{1}{0}\Big(Q-\frac{j'}{k'}P\Big)+\binom{j'}{k'}\tfrac{1}{k'}P\,,\qquad P/k'\in\Lambda_m\,. \quad \text{(C.7)}$$

Denoting by

$$\Delta\bar{C}(Q,P;\Omega_2)=\bar{C}(Q,P;\Omega_2)-\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\Gamma\in\Lambda_{22,6}\oplus\Lambda_{22,6}}}|A|\,C^F\left[A^{-1}\begin{pmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{pmatrix}A^{-\intercal}\right] \quad \text{(C.8)}$$

the contribution from the poles of $1/\Phi_{10}$ on the second line of (5.25) to the measure factor (5.74) we thus find

$$\begin{aligned}\Delta\bar{C}(Q,P;\Omega_2) = &\sum_{\substack{A\in M_2(\mathbb{Z})/\mathrm{Dih}_4\\ A^{-1}\Gamma\in\Lambda_{22,6}\oplus\Lambda_{22,6}}}|A|\,c\big(-\tfrac{([A^{-1}\Gamma]_1)^2}{2}\big)\,c\big(-\tfrac{([A^{-1}\Gamma]_2)^2}{2}\big) \quad \text{(C.9)}\\ &\times\Big(-\frac{\delta([A^\intercal\Omega_2A]_{12})}{4\pi}+\frac{[A^{-1}\Gamma]_1\cdot[A^{-1}\Gamma]_2}{2}\big(\mathrm{sign}([A^{-1}\Gamma]_1\cdot[A^{-1}\Gamma]_2)-\mathrm{sign}([A^\intercal\Omega_2A]_{12})\big)\Big)\,.\end{aligned}$$

where we combined the sum over $A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})$ and the sum over $\gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4$ into the sum over $A\gamma\in M_2(\mathbb{Z})/\mathrm{Dih}_4$ that we call A again, Further decomposing the sum over A as

$$A=\gamma\cdot\begin{pmatrix}1 & \frac{j'}{k'}\\ 0 & 1\end{pmatrix}\begin{pmatrix}d_1 & 0\\ 0 & d_2\end{pmatrix}=\hat{B}\begin{pmatrix}d_1 & 0\\ 0 & d_2\end{pmatrix}\,, \quad \text{(C.10)}$$

with $k'|d_2$, and $\hat{B}=B\begin{pmatrix}1 & 0\\ 0 & |B|^{-1}\end{pmatrix}$ parametrizing the splittings, one obtains

$$\begin{aligned}\Delta\bar{C}(Q,P;\Omega_2) = &\sum_{\substack{B\in M_2(\mathbb{Z})/\mathrm{Dih}_4\\ \hat{B}^{-1}\Gamma\in\Lambda_m\oplus\Lambda_m}}\ \sum_{\substack{d_1\geq 1\\ \Gamma_1/d_1\in\Lambda_m\oplus\Lambda_m}} c\Big(-\frac{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\sum_{\substack{d_2\geq 1\\ \Gamma_2/d_2\in\Lambda_m\oplus\Lambda_m}} c\Big(-\frac{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\Big)\\ &\times\Big(-\frac{\delta([\hat{B}^\intercal\Omega_2\hat{B}]_{12})}{4\pi}+\frac{\langle\Gamma_1,\Gamma_2\rangle}{2}\big(\mathrm{sign}(\langle\Gamma_1,\Gamma_2\rangle)-\mathrm{sign}([\hat{B}^\intercal\Omega_2\hat{B}]_{12})\big)\Big) \quad \text{(C.11)}\end{aligned}$$

with $\Gamma_i=B\pi_iB^{-1}\Gamma=\hat{B}\pi_i\hat{B}^{-1}\Gamma$.

C.2 $\Gamma_0(N)$ orbits of splittings

For CHL orbifolds the charge quantization condition $B\pi_iB^{-1}\Gamma\in\Lambda_m^*\oplus\Lambda_m$ for the splitting (C.6) does not reduce to a single condition. They will depend on the charge orbit, as well as on its twistedness, and only if $\gamma\in\mathbb{Z}_2\ltimes\Gamma_0(N)\subset GL(2,\mathbb{Z})$, the quantization condition $B\pi_iB^{-1}\Gamma\in\Lambda_m^*\oplus\Lambda_m$ reduces to $\frac{aP-cQ}{k'}\in\Lambda_m^*$. Therefore it will be more convenient to decompose $M_2(\mathbb{Z})/\mathrm{Stab}(\pi_i)$ into orbits of $\gamma \in \Gamma_0(N)/\mathbb{Z}_2$ acting on the left.[31] Therefore we choose to decompose the splitting matrix as

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p' & j \\ 0 & k \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}_2 \ltimes \Gamma_0(N), \quad p' > 0\,, \quad 0 \le j < k\,, \tag{C.12}$$

if $\frac{(p,r)}{\gcd(p,r)} = (*, 0) \bmod N$, and

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & k \\ p' & j \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}_2 \ltimes \Gamma_0(N), \quad p' > 0\,, \quad 0 \le j < Nk \tag{C.13}$$

otherwise. In the former case the splitting can be rotated under $\Gamma_0(N)$ to the canonical splitting (C.7), such that Γ_1 is in the $\Gamma_0(N)$ orbit of a purely electric charge. In this case we say that Γ_1 is of electric type and we call (C.7) 'splitting of electric type'. This splitting exists if and only if $P/k' \in \Lambda_m^*$. In contrast, the splitting (C.13) can be rotated under $\Gamma_0(N)$ to the canonical form

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big(P - \frac{j'}{k'} Q \Big) + \begin{pmatrix} k' \\ j' \end{pmatrix} \frac{1}{k'} Q\,, \tag{C.14}$$

such that Γ_1 is in the $\Gamma_0(N)$ orbit of a purely magnetic charge. We then way Γ_1 is of magnetic type and we call (C.14) a 'splitting of magnetic type'. This splitting exists if and only if $Q/k' \in \Lambda_m$. Note that the second charge Γ_2 can be either of electric or of magnetic type in both types of splitting. In fact, we shall see that a splitting of mixed type, such that one charge is of electric type and the other of magnetic type, can be rotated by a suitable $\gamma \in \Gamma_0(N)$ into either type of splittings.

We drop the primes on (j', k') in this discussion to simplify the notation, with the understanding that k and j are now relative prime. In the electric type, a splitting matrix with $k = 0 \bmod N$, such that $\binom{j}{k}\frac{1}{k}P$ is of electric type, can be rotated by a $\Gamma_0(N)$ element to another splitting of electric type

$$\begin{pmatrix} 1 & j \\ 0 & k \end{pmatrix} = \begin{pmatrix} j & b \\ k & -\tilde{\jmath} \end{pmatrix} \begin{pmatrix} \tilde{\jmath} & 1 \\ k & 0 \end{pmatrix}, \tag{C.15}$$

with $0 \le \tilde{\jmath} < k$, $j\tilde{\jmath} + bk = 1$. In the case where $k \neq 0 \bmod N$, such that $\binom{j}{k}\frac{1}{k}P$ is of magnetic type, an element of $\Gamma_0(N)$ rotates it to a splitting of magnetic type

$$\begin{pmatrix} 1 & j \\ 0 & k \end{pmatrix} = \begin{pmatrix} a & j \\ -\tilde{\jmath} & k \end{pmatrix} \begin{pmatrix} k & 0 \\ \tilde{\jmath} & 1 \end{pmatrix}, \tag{C.16}$$

with $\tilde{\jmath} = 0 \bmod N$, $\tilde{\jmath} < Nk$. This can be understood as follows: in (C.15), the second charge in the splitting is also electric since $k = 0 \bmod N$, and thus exchanging (Q_1, P_1) with (Q_2, P_2) preserves the type of the splitting; in (C.16), the second charge is magnetic since $k \neq 0 \bmod N$, and thus exchanging the two charges of the splitting sends the splitting of electric type to a splitting of magnetic type. The same reasoning applies to the splitting of magnetic types: when $j = 0 \bmod N$, such that $\binom{k}{j}\frac{1}{k}Q$ is of electric type, one has

$$\begin{pmatrix} 0 & k \\ 1 & j \end{pmatrix} = \begin{pmatrix} k & -\tilde{\jmath} \\ j & d \end{pmatrix} \begin{pmatrix} \tilde{\jmath} & 1 \\ k & 0 \end{pmatrix}, \tag{C.17}$$

with $d \neq 0 \bmod N$, $0 \le \tilde{\jmath} < k$, and when $j \neq 0 \bmod N$, such that $\binom{k}{j}\frac{1}{k}Q$ is of magnetic type,

$$\begin{pmatrix} 0 & k \\ 1 & j \end{pmatrix} = \begin{pmatrix} -\tilde{\jmath} & k \\ c & j \end{pmatrix} \begin{pmatrix} k & 0 \\ \tilde{\jmath} & 1 \end{pmatrix}, \tag{C.18}$$

[31] Note that $\mathrm{Dih}_4 \cap \mathbb{Z}_2 \ltimes \Gamma_0(N) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and the corresponding quotient $\mathbb{Z}_2 \ltimes \Gamma_0(N)/[\mathbb{Z}_2 \times \mathbb{Z}_2] = \Gamma_0(N)/\mathbb{Z}_2$.

with $c = 0 \bmod N$, $0 \leq \tilde{j} < Nk$ and $j\tilde{j} + ck = -1$.

It follows from this discussion that the splittings are in one-to-one correspondence with the cosets

$$\begin{aligned} M_2(\mathbb{Z})/\mathrm{Stab}(\pi_i) \;=\; & \Big\{\gamma\cdot\begin{pmatrix}1 & j'\\ 0 & k'\end{pmatrix},\ \gamma\in\Gamma_0(N)/\mathbb{Z}_2\,,\ 0\leq j'<k'\,,\ (j',k')=1\Big\} \\ & \cup\Big\{\gamma\cdot\begin{pmatrix}0 & k'\\ 1 & j'\end{pmatrix},\ \gamma\in\Gamma_0(N)/\mathbb{Z}_2\,,\ 0\leq j'<Nk'\,,\ j'\neq 0 \bmod N\,,\ (j',k')=1\Big\} \\ = & \Big\{\gamma\cdot\begin{pmatrix}1 & j'\\ 0 & k'\end{pmatrix},\ \gamma\in\Gamma_0(N)/\mathbb{Z}_2\,,\ 0\leq j'<k'\,,\ k'\neq 0 \bmod N\,,\ (j',k')=1\Big\} \\ & \cup\Big\{\gamma\cdot\begin{pmatrix}0 & k'\\ 1 & j'\end{pmatrix},\ \gamma\in\Gamma_0(N)/\mathbb{Z}_2\,,\ 0\leq j'<Nk'\,,\ (j',k')=1\Big\} \end{aligned} \tag{C.19}$$

where the splittings of mixed type are included either in the electric type or the magnetic type. In the following we shall consider both representatives, keeping in mind that we systematically double-count the splittings of mixed type in this way.

It is worth noting that the sign $(-1)^{\langle\Gamma_1,\Gamma_2\rangle}$ appearing in the wall-crossing formula (2.12) does not depend on the type of splitting. For an electric-type splitting

$$\langle\Gamma_1,\Gamma_2\rangle = (Q - \tfrac{j'}{k'}P)\cdot P = Q\cdot P \bmod 2\ . \tag{C.20}$$

To prove this, note that either $P \notin N\Lambda^*$ and $\frac{1}{k'}P \in \Lambda$ so $(\frac{1}{k'}P)^2 = 0 \bmod 2$, or $P \in N\Lambda^*$ and $\frac{1}{k'}P \in \Lambda^*$ so $(\frac{1}{k'}P)\cdot P = 0 \bmod 2$. The same reasoning shows for a magnetic-type splitting

$$\langle\Gamma_1,\Gamma_2\rangle = Q\cdot(P - \tfrac{j'}{k'}Q) = Q\cdot P \bmod 2\ . \tag{C.21}$$

Moreover, under $\Gamma_0(N)$ the parity of $Q\cdot P$ is preserved:

$$(aQ+bP)\cdot(cQ+dP) = Q\cdot P + ac\,Q^2 + 2bc\,Q\cdot P + bd\,P^2 = Q\cdot P \bmod 2\ . \tag{C.22}$$

C.3 Factorization of the measure factor

We now discuss the factorization of the measure factor associated to the poles of $1/\Phi_{k-2}$ and $1/\tilde{\Phi}_{k-2}$ for $|\Omega_2| > \frac{1}{4}$ displayed in (5.75). In this subsection we show that whenever a term in the measure associated to the charge Γ factorizes, it produces the correct measure factor of the corresponding 1/2-BPS charges Γ_i.

- For the first term in (5.75), we combine the sum over $A \in M_2(\mathbb{Z})/GL(2,\mathbb{Z})$ and the sum over $\gamma \in GL(2,\mathbb{Z})/\mathrm{Dih}_4$ in (5.57) as in (C.9), and use the decomposition (C.10) to get

$$\begin{aligned}
&\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda_m\oplus\Lambda_m}} |A|\Big(C_{k-2}\Big[A^{-1}\big(\begin{smallmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{smallmatrix}\big)A^{-\intercal};A^\intercal\Omega_2 A\Big]-C^F_{k-2}\Big[A^{-1}\big(\begin{smallmatrix}-Q^2 & -Q\cdot P\\ -Q\cdot P & -P^2\end{smallmatrix}\big)A^{-\intercal}\Big]\Big)\\
&=\sum_{\substack{A\in M_2(\mathbb{Z})/\mathrm{Dih}_4\\ A^{-1}\Gamma\in\Lambda_m\oplus\Lambda_m}}|A|c_k(-\tfrac{([A^{-1}\Gamma]_1)^2}{2})c_k(-\tfrac{([A^{-1}\Gamma]_2)^2}{2})\\
&\quad\times\Big(-\frac{\delta([A^\intercal\Omega_2A]_{12})}{4\pi}+\frac{[A^{-1}\Gamma]_1\cdot[A^{-1}\Gamma]_2}{2}\big(\mathrm{sign}([A^{-1}\Gamma]_1\cdot[A^{-1}\Gamma]_2)-\mathrm{sign}([A^\intercal\Omega_2A]_{12})\big)\Big)\\
&=\sum_{\substack{B\in M_2(\mathbb{Z})/\mathrm{Dih}_4\\ \hat{B}^{-1}\Gamma\in\Lambda_m\oplus\Lambda_m}}\ \sum_{\substack{d_1\geq1\\ \Gamma_1/d_1\in\Lambda_m\oplus\Lambda_m}} c_k\big(-\tfrac{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\big)\sum_{\substack{d_2\geq1\\ \Gamma_2/d_2\in\Lambda_m\oplus\Lambda_m}} c_k\big(-\tfrac{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\big)\\
&\quad\times\Big(-\frac{\delta([\hat{B}^\intercal\Omega_2\hat{B}]_{12})}{4\pi}+\frac{\langle\Gamma_1,\Gamma_2\rangle}{2}\big(\mathrm{sign}(\langle\Gamma_1,\Gamma_2\rangle)-\mathrm{sign}([\hat{B}^\intercal\Omega_2\hat{B}]_{12})\big)\Big)\ ,
\end{aligned}\tag{C.23}$$

where B determines a splitting $\Gamma=\Gamma_1+\Gamma_2$. In this sum, the only non-trivial contributions arise when Γ_1 is of electric type, such that $\gcd(Q_1^2,P_1^2,Q_1P_1)=\frac{\gcd(NQ_1^2,P_1^2,Q_1P_1)}{N}$, and because it is electric in Λ_m, Γ_1/d_1 is untwisted. Whereas, when Γ_1/d_1 is of magnetic type, $\gcd(Q_1^2,P_1^2,Q_1P_1)=\gcd(NQ_1^2,P_1^2,Q_1P_1)$, and because it is magnetic in Λ_m, Γ_1 can be either twisted or untwisted. Therefore we get the correct contribution to the measure for 1/2-BPS displayed in (2.22).

- For the third term in the measure (5.75), it is convenient to consider instead $\tilde{A}=\big(\begin{smallmatrix}1&0\\0&N\end{smallmatrix}\big)A\in M_{2,00}(N)$ such that

$$\begin{aligned}
&\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P/N}\in\Lambda_m^*\oplus\Lambda_m^*}} |A|\ \Big(C_{k-2}\Big[A^{-1}\big(\begin{smallmatrix}-NQ^2 & -Q\cdot P\\ -Q\cdot P & -P^2/N\end{smallmatrix}\big)A^{-\intercal};A^\intercal\Omega_2 A\Big]-C^F_{k-2}\Big[A^{-1}\big(\begin{smallmatrix}-NQ^2 & -Q\cdot P\\ -Q\cdot P & -P^2/N\end{smallmatrix}\big)A^{-\intercal}\Big]\Big)\\
&=\sum_{\substack{\tilde{A}\in M_{2,00}(N)/\mathrm{Dih}_4\\ \tilde{A}^{-1}\Gamma\in\Lambda_m^*\oplus\Lambda_m^*}}|\tilde{A}|c_k(-N\tfrac{([\tilde{A}^{-1}\Gamma]_1)^2}{2})\,c_k(-N\tfrac{([\tilde{A}^{-1}\Gamma]_2)^2}{2})\\
&\quad\times\Big(-\frac{\delta([\tilde{A}^\intercal\Omega_2\tilde{A}]_{12})}{4\pi}+\frac{[\tilde{A}^{-1}\Gamma]_1\cdot[\tilde{A}^{-1}\Gamma]_2}{2}\big(\mathrm{sign}([\tilde{A}^{-1}\Gamma]_1\cdot[\tilde{A}^{-1}\Gamma]_2)-\mathrm{sign}([\tilde{A}^\intercal\Omega_2\tilde{A}]_{12})\big)\Big)
\end{aligned}\tag{C.24}$$

A matrix $\tilde{A}\in M_{2,00}(N)$ admits either a decomposition with $\gamma\in\Gamma_0(N)$ such that

$$\tilde{A}=\gamma\cdot\begin{pmatrix}p'&j\\0&Nk\end{pmatrix}=\gamma\cdot\begin{pmatrix}1&\frac{j}{Nk}\\0&1\end{pmatrix}\begin{pmatrix}p'&0\\0&Nk\end{pmatrix}\ ,\tag{C.25}$$

and $\Gamma_\gamma=\gamma^{-1}\Gamma$ satisfies

$$\frac{P_\gamma}{k}\in N\Lambda_m^*\ ,\qquad \frac{Q_\gamma-\frac{j}{kN}P_\gamma}{p'}\in\Lambda_m^*\ ,\tag{C.26}$$

or a decomposition with $\gamma\in\Gamma_0(N)$ such that

$$\tilde{A}=\gamma\cdot\begin{pmatrix}0&k\\Np'&Nj\end{pmatrix}=\gamma\cdot\begin{pmatrix}0&1\\1&\frac{Nj}{k}\end{pmatrix}\begin{pmatrix}Np'&0\\0&k\end{pmatrix}\ ,\tag{C.27}$$

and

$$\frac{Q_\gamma}{k} \in \Lambda_m^* , \qquad \frac{P_\gamma - \frac{Nj}{k} Q_\gamma}{p'} \in N\Lambda_m^* . \tag{C.28}$$

For the splitting matrix of electric type (C.25), the charge Γ_1 is of electric type with

$$N([\tilde{A}^{-1}\Gamma]_1)^2 = \frac{\gcd(NQ_1^2, P_1^2, Q_1 \cdot P_1)}{p'^2} , \tag{C.29}$$

with the divisor integer $d_1 = p'$; and either the second charge Γ_2 is of electric type, with $\frac{kN}{\gcd(j,kN)} = 0 \bmod N$ and

$$N([\tilde{A}^{-1}\Gamma]_2)^2 = \frac{\gcd(NQ_2^2, P_2^2, Q_2 P_2)}{\gcd(j, Nk)^2} , \tag{C.30}$$

with the divisor integer $d_2 = \gcd(j, Nk)$, or Γ_2 is of twisted magnetic type with $\frac{kN}{\gcd(j,kN)} \neq 0 \bmod N$ and

$$N([\tilde{A}^{-1}\Gamma]_2)^2 = \frac{\gcd(NQ_2^2, P_2^2, Q_2 P_2)}{N(\gcd(j, Nk)/N)^2} , \tag{C.31}$$

with the divisor integer $d_2 = \gcd(j, Nk)/N$.

For the splitting matrix of magnetic type (C.27) the first charge Γ_1 is of untwisted magnetic type with

$$N([\tilde{A}^{-1}\Gamma]_1)^2 = \frac{\gcd(NQ_1^2, P_1^2, Q_1 \cdot P_1)}{Np'^2} , \tag{C.32}$$

with the divisor integer $d_1 = p'$; and either the second charge Γ_2 is of electric type, with $\frac{Nj}{\gcd(Nj,k)} = 0 \bmod N$ and

$$N([\tilde{A}^{-1}\Gamma]_2)^2 = \frac{\gcd(NQ_2^2, P_2^2, Q_2 P_2)}{\gcd(Nj, k)^2} \tag{C.33}$$

with the divisor integer $d_2 = \gcd(Nj, k)$, or Γ_2 is of twisted magnetic type with $\frac{Nj}{\gcd(Nj,k)} \neq 0 \bmod N$ and

$$N([\tilde{A}^{-1}\Gamma]_2)^2 = \frac{\gcd(NQ_2^2, P_2^2, Q_2 P_2)}{N(\gcd(Nj, k)/N)^2} \tag{C.34}$$

with the divisor integer $d_2 = \gcd(Nj, k)/N$.

- At last we consider the second term in (5.75), which is a combination . We combine the sum over $A \in M_{2,0}(\mathbb{Z})/[\mathbb{Z}_2 \ltimes \Gamma_0(N)]$ and the sum over $\gamma \in \Gamma_0(N)/\mathbb{Z}_2$ in (5.58) to get

$$\begin{aligned}
&\sum_{\substack{A \in M_{2,0}(N)/[\mathbb{Z}_2 \ltimes \Gamma_0(N)] \\ A^{-1}\binom{Q}{P} \in \Lambda_m^* \oplus \Lambda_m}} |A| \left(\widetilde{C}_{k-2}\Big[A^{-1}\big(\begin{smallmatrix} -Q^2 & -Q\cdot P \\ -Q\cdot P & -P^2 \end{smallmatrix}\big)A^{-\intercal}; A^\intercal \Omega_2 A\Big] - \widetilde{C}^F_{k-2}\Big[A^{-1}\big(\begin{smallmatrix} -Q^2 & -Q\cdot P \\ -Q\cdot P & -P^2 \end{smallmatrix}\big)A^{-\intercal}\Big] \right) \\
&+ \sum_{\substack{A \in M_{2,0}(N)/[\mathbb{Z}_2 \times \mathbb{Z}_2] \\ A^{-1}\Gamma \in \Lambda_m^* \oplus \Lambda_m}} |A| c_k(-N\tfrac{([A^{-1}\Gamma]_1)^2}{2}) c_k(-\tfrac{([A^{-1}\Gamma]_2)^2}{2}) \\
&\times \left(-\frac{\delta([A^\intercal \Omega_2 A]_{12})}{4\pi} + \frac{[A^{-1}\Gamma]_1 \cdot [A^{-1}\Gamma]_2}{2} \big(\mathrm{sign}([A^{-1}\Gamma]_1 \cdot [A^{-1}\Gamma]_2) - \mathrm{sign}([A^\intercal \Omega_2 A]_{12}) \big) \right) .
\end{aligned} \tag{C.35}$$

A matrix in $A \in M_{2,0}(N)$ admits one of the following decompositions with respect to $\gamma \in \Gamma_0(N)$:

1.

$$A=\gamma\cdot\begin{pmatrix}1&\frac{j}{k}\\0&1\end{pmatrix}\begin{pmatrix}p'&0\\0&k\end{pmatrix}\quad\Rightarrow\quad\frac{Q_\gamma-\frac{j}{k}P_\gamma}{p'}\in\Lambda_m^*\,,\quad\frac{P_\gamma}{k}\in\Lambda_m\,,\tag{C.36}$$

Γ_1 is always of electric type and $N([A^{-1}\Gamma]_1)^2=\frac{\gcd(NQ_1^2,P_1^2,Q_1P_1)}{p'^2}$, and Γ_2 is either of untwisted electric type with $\frac{k}{\gcd(j,k)}=0\bmod N$ with $([A^{-1}\Gamma]_2)^2=\frac{\gcd(NQ_2^2,P_2^2,Q_2P_2)}{N\gcd(j,k)^2}$ or of magnetic type with $\frac{k}{\gcd(j,k)}\neq 0\bmod N$ with $([A^{-1}\Gamma]_2)^2=\frac{\gcd(NQ_2^2,P_2^2,Q_2P_2)}{\gcd(j,k)^2}$.

2.

$$A=\gamma\cdot\begin{pmatrix}0&1\\N&\frac{j}{k}\end{pmatrix}\begin{pmatrix}p'&0\\0&k\end{pmatrix}\quad\Rightarrow\quad\frac{P_\gamma-\frac{j}{k}Q_\gamma}{p'}\in N\Lambda_m^*\,,\quad\frac{Q_\gamma}{k}\in\Lambda_m\,,\tag{C.37}$$

Γ_1 is always of untwisted magnetic type and $N([A^{-1}\Gamma]_1)^2=\frac{\gcd(NQ_1^2,P_1^2,Q_1P_1)}{Np'^2}$, and Γ_2 is either of untwisted electric type with $\frac{j}{\gcd(j,k)}=0\bmod N$ with $([A^{-1}\Gamma]_2)^2=\frac{\gcd(NQ_2^2,P_2^2,Q_2P_2)}{N\gcd(j,k)^2}$ or of magnetic type with $\frac{j}{\gcd(j,k)}\neq 0\bmod N$ with $([A^{-1}\Gamma]_2)^2=\frac{\gcd(NQ_2^2,P_2^2,Q_2P_2)}{\gcd(j,k)^2}$.

3.

$$A=\gamma\cdot\begin{pmatrix}1&\frac{j}{Nk}\\0&1\end{pmatrix}\begin{pmatrix}0&p'\\Nk&0\end{pmatrix}\quad\Rightarrow\quad\frac{Q_\gamma-\frac{j}{Nk}P_\gamma}{p'}\in\Lambda_m\,,\quad\frac{P_\gamma}{k}\in N\Lambda_m^*\,,\tag{C.38}$$

Γ_2 is always of untwisted electric type and $([A^{-1}\Gamma]_2)^2=\frac{\gcd(NQ_2^2,P_2^2,Q_2P_2)}{Np'^2}$, and Γ_1 is either of electric type with $\frac{Nk}{\gcd(j,Nk)}=0\bmod N$ with $N([A^{-1}\Gamma]_1)^2=\frac{\gcd(NQ_1^2,P_1^2,Q_1P_1)}{\gcd(j,Nk)^2}$ or of untwisted magnetic type with $\frac{Nk}{\gcd(j,Nk)}\neq 0\bmod N$ with $N([A^{-1}\Gamma]_1)^2=\frac{\gcd(NQ_1^2,P_1^2,Q_1P_1)}{N(\gcd(j,Nk)/N)^2}$.

4.

$$A=\gamma\cdot\begin{pmatrix}0&1\\1&\frac{Nj}{k}\end{pmatrix}\begin{pmatrix}0&p'\\k&0\end{pmatrix}\quad\Rightarrow\quad\frac{P_\gamma-\frac{Nj}{k}Q_\gamma}{p'}\in\Lambda_m\,,\quad\frac{Q_\gamma}{k}\in\Lambda_m^*\,,\tag{C.39}$$

Γ_2 is always of magnetic type and $([A^{-1}\Gamma]_2)^2=\frac{\gcd(NQ_2^2,P_2^2,Q_2P_2)}{p'^2}$, and Γ_1 is either of electric type with $\frac{Nj}{\gcd(Nj,k)}=0\bmod N$ with $N([A^{-1}\Gamma]_1)^2=\frac{\gcd(NQ_1^2,P_1^2,Q_1P_1)}{\gcd(Nj,k)^2}$ or of untwisted magnetic type with $\frac{Nj}{\gcd(Nj,k)}\neq 0\bmod N$ with $N([A^{-1}\Gamma]_1)^2=\frac{\gcd(NQ_1^2,P_1^2,Q_1P_1)}{N(\gcd(Nj,k)/N)^2}$.

We conclude that after trading each of the sums over A as sums over splitting matrices B, the contribution from (5.75) gives a term of the form

$$\left(-\frac{\delta([\hat{B}^\intercal\Omega_2\hat{B}]_{12})}{4\pi}+\frac{\langle\Gamma_1,\Gamma_2\rangle}{2}\Big(\mathrm{sign}(\langle\Gamma_1,\Gamma_2\rangle)-\mathrm{sign}([\hat{B}^\intercal\Omega_2\hat{B}]_{12})\Big)\right)c'(\Gamma_1)\,c'(\Gamma_2)\tag{C.40}$$

to the last line in (5.92), where $c'(\Gamma_i)$ is either

$$c_U(\Gamma_i) = \sum_{\substack{d_i>1 \\ d_i^{-1}\Gamma_i \in \Lambda_m \oplus N\Lambda_m^*}} c_k\left(-\frac{\gcd(NQ_i^2, P_i^2, Q_i \cdot P_i)}{2Nd_i^2}\right) , \tag{C.41}$$

when the contribution is only non-vanishing for untwisted charge Γ_i, or

$$c_T(\Gamma_i) = \sum_{\substack{d_i>1 \\ d_i^{-1}\Gamma_i \in \Lambda_m^* \oplus \Lambda_m}} c_k\left(-\frac{\gcd(NQ_i^2, P_i^2, Q_i \cdot P_i)}{2d_i^2}\right) , \tag{C.42}$$

for generic contribution such the charge Γ_i is either twisted or untwisted.

It remains to show that the three terms in the measure count all the possible splittings with the correct multiplicity, so as to reproduce the product of the summation factors of formula (2.22) for the two charges Γ_i.

C.4 Electric-magnetic type of splittings

We summarize the conditions from the three terms in (5.75) to contribute to a given splitting in Table 2, where for the second term we distinguish the cases where $B^{-1}(Q,P) \in \Lambda_m^* \oplus \Lambda_m$ or $B^{-1}(Q,P) \in \Lambda_m \oplus \Lambda_m^*$.

$\mathcal{O}_{ij}$	Electric type		Magnetic type		Counted by
$\Lambda_m \oplus \Lambda_m$	$Q - \frac{j}{k}P \in \Lambda_m$,	$\frac{1}{k}P \in \Lambda_m$	$P - \frac{j}{k}Q \in \Lambda_m$,	$\frac{1}{k}Q \in \Lambda_m$	$\Phi_{k-2}^{-1}(\Omega)$
$\Lambda_m^* \oplus \Lambda_m$	$Q - \frac{j}{k}P \in \Lambda_m^*$,	$\frac{1}{k}P \in \Lambda_m$	$P - \frac{j}{k}Q \in N\Lambda_m^*$,	$\frac{1}{k}Q \in \Lambda_m$	$\tilde{\Phi}_{k-2}^{-1}(\Omega)$
$\Lambda_m \oplus \Lambda_m^*$	$Q - \frac{j}{Nk'}P \in \Lambda_m$,	$\frac{1}{Nk'}P \in \Lambda_m^*$	$P - \frac{Nj'}{k}Q \in \Lambda_m$,	$\frac{1}{k}Q \in \Lambda_m^*$	$\tilde{\Phi}_{k-2}^{-1}(\Omega)$
$\Lambda_m^* \oplus \Lambda_m^*$	$Q - \frac{j}{Nk'}P \in \Lambda_m^*$,	$\frac{1}{Nk'}P \in \Lambda_m^*$	$P - \frac{Nj'}{k}Q \in N\Lambda_m^*$,	$\frac{1}{k}Q \in \Lambda_m^*$	$\Phi_{k-2}^{-1}(\Omega/N)$

Table 2: $\Gamma_0(N)$ orbits of splittings from the three terms in (5.75). The first column indicates the support of $B^{-1}(Q,P)$. The second and third columns give the corresponding constraints on Γ_1, Γ_2, for each of the two possible splittings (C.7) and (C.14). The last column records the counting function. We write $k = Nk'$ and $j = Nj'$ whenever k or j are forced to be multiple of N. $\mathcal{O}_{ij}$ is used in the text to denote in the table above contribution from row i and column j.

For this purpose we enumerate the possible 1/4-BPS charges Γ and the type of 1/2-BPS charges they can possibly split into, *i.e.* twisted or untwisted, electric or magnetic. It will be convenient to introduce some notation for classifying pairs of 1/2-BPS charges: for each type of splitting we define a 2-component vector which first component accounts for the electric type charges and the second for the magnetic type charges, with a U for untwisted and a T for twisted. *e.g.*

1. $(TT, \emptyset)$, (T, T) and $(\emptyset, TT)$ stand for electric-twisted electric-twisted, electric-twisted magnetic-twisted, and magnetic-twisted magnetic-twisted splittings, respectively.

2. $(TU,\emptyset)$, (T,U), (U,T) and $(\emptyset,TU)$ stand for electric-twisted electric-untwisted, electric-twisted magnetic-untwisted, electric-untwisted magnetic-twisted and magnetic-twisted magnetic-untwisted splittings, respectively.

3. $(UU,\emptyset)$, (U,U) and $(\emptyset,UU)$ stand for electric-untwisted electric-untwisted, electric-untwisted magnetic-untwisted, and magnetic-untwisted magnetic-untwisted splittings, respectively.

We shall enumerate the possible splittings according to the following graph of inclusions,

$$N\Lambda_e\oplus N\Lambda_m \begin{array}{c}\subset\\ \subset\end{array} \begin{array}{c}N\Lambda_e\oplus N\Lambda_e\\ \Lambda_m\oplus N\Lambda_m\end{array} \begin{array}{c}\subset\\ \subset\end{array} \Lambda_m\oplus N\Lambda_e \begin{array}{c}\subset\\ \subset\end{array} \begin{array}{c}\Lambda_m\oplus\Lambda_m\\ \Lambda_e\oplus N\Lambda_e\end{array} \begin{array}{c}\subset\\ \subset\end{array} \Lambda_e\oplus\Lambda_m\,, \tag{C.43}$$

We will denote $X\,\underline{\in}\,\Lambda$ the strict inclusion of the vector X in Λ, meaning that X is a generic vector in Λ and does not belong to a smaller lattice $\tilde{\Lambda}$ in this sequence

$$\ldots\subset N^k\Lambda_m\subset N^k\Lambda_m^*\subset\ldots\subset N\Lambda_m\subset N\Lambda_m^*\subset\Lambda_m\subset\Lambda_m^*\,. \tag{C.44}$$

In the following, it will be convenient to recall the generating function whose Fourier coefficients give the contribution to the measure. According to table 2, the factorizations (A.44) imply that when the condition is $\frac{1}{d_i}[B^{-1}\Gamma]_i\in\Lambda_m$, the corresponding measure factor for the 1/2-BPS charge $B\pi_iB^{-1}\Gamma$ is a Fourier coefficient of $\Delta_k(\tau)^{-1}$, whereas when $\frac{1}{d_i}[B^{-1}\Gamma]_i\in\Lambda_m^*$ it is a Fourier coefficient of $\Delta_k(\tau/N)^{-1}$. For a magnetic type charge $B\pi_iB^{-1}\Gamma$, $\Delta_k(\tau)^{-1}$ gives a contribution $c_T(\Gamma)$ and $\Delta_k(\tau/N)^{-1}$ a contribution $c_U(\Gamma)$. On the contrary for an eletric type charge, $\Delta_k(\tau)^{-1}$ gives a contribution $c_U(\Gamma)$ and $\Delta_k(\tau/N)^{-1}$ a contribution $c_T(\Gamma)$.

There are seven cases of interest:

1. $Q\,\underline{\in}\,\Lambda_m^*\,,P\,\underline{\in}\,\Lambda_m$: the only contributions are from $\mathcal{O}_{21}$ and $\mathcal{O}_{32}$. These two contributions give (T,T) splittings with Fourier contributions in $[\Delta_k(\rho/N)\Delta_k(\sigma)]^{-1}$. We thus obtain a single contribution in $c_T(\Gamma_1)c_T(\Gamma_2)$ (with (C.42)), as expected for twisted 1/2-BPS charges.

2. $Q\,\underline{\in}\,\Lambda_m\,,P\,\underline{\in}\,\Lambda_m$: contributions from $\mathcal{O}_{11}$, $\mathcal{O}_{21}$, $\mathcal{O}_{12}$, $\mathcal{O}_{22}$, and $\mathcal{O}_{32}$ fall in (U,T), $(\emptyset,UT)$, and $(\emptyset,TT)$ splitting sectors.

   <u>Electric-type splitting</u> : the first charge in (C.7) is purely electric and thus untwisted in both $\mathcal{O}_{11}$ and $\mathcal{O}_{21}$, the second one is congruent to an electric charge for $k=0\,\mathrm{mod}\,N$, and a magnetic one otherwise. But since $P\,\underline{\in}\,\Lambda_m$, $\frac{1}{k}P\in\Lambda_m$ implies that $k\neq 0\,\mathrm{mod}\,N$, and thus the second charge in (C.7) is magnetic-twisted. $\mathcal{O}_{11}$ and $\mathcal{O}_{21}$ combine together to give (U,T) splittings with measure factor $(c_T(\Gamma_1)+c_U(\Gamma_1))c_T(\Gamma_2)$ coming from Fourier coefficients of $\big(\Delta_k(\rho)^{-1}+\Delta_k(\rho/N)^{-1}\big)\Delta_k(\sigma)^{-1}$.

   <u>Magnetic-type splitting</u> : the first charge in (C.14) is purely magnetic, and thus twisted for $\mathcal{O}_{32}$, untwisted for $\mathcal{O}_{22}$, and can be either twisted or untwisted for $\mathcal{O}_{12}$, the second 1/2-BPS charge is congruent to a magnetic-untwisted charge for $j\neq 0\,\mathrm{mod}\,N$, and electric-twisted otherwise.

   When $P-\frac{j}{k}Q\in N\Lambda_m^*$, $\mathcal{O}_{12}$ contribute only when $j\neq 0\,\mathrm{mod}\,N$, thus combining with $\mathcal{O}_{22}$ to give $(\emptyset,TU)$ splittings with the measure $(c_T(\Gamma_1)+c_U(\Gamma_1))c_T(\Gamma_2)$ coming from Fourier coefficients of $\big(\Delta_k(\rho)^{-1}+\Delta_k(\rho/N)^{-1}\big)\Delta_k(\sigma)^{-1}$.

   When $P-\frac{j}{k}Q\,\underline{\in}\,\Lambda_m$ with $j\neq 0\,\mathrm{mod}\,N$, $\mathcal{O}_{12}$ gives $(\emptyset,TT)$ splittings with measure $c_T(\Gamma_1)$ $c_T(\Gamma_2)$ from Fourier coefficients of $[\Delta_k(\rho)\Delta_k(\sigma)]^{-1}$. Finally, when $j=0\,\mathrm{mod}\,N$, $\mathcal{O}_{12}$

combines with $\mathcal{O}_{32}$ — for which $j = 0 \bmod N$ by construction — to give (U,T) splittings with measure $c_T(\Gamma_1)(c_T(\Gamma_2)+c_U(\Gamma_2))$ from Fourier coefficients of $(\Delta_k(\rho)^{-1}+\Delta_k(\rho/N)^{-1})$ $\Delta_k(\sigma)^{-1}$. Recall that we double-count this last splitting, which is the same as the one defined above from $\mathcal{O}_{11}$ and $\mathcal{O}_{21}$ with Γ_1 and Γ_2 exchanged, according to (C.19).

3. $Q \underline{\in} \Lambda_m^*$, $P \underline{\in} N\Lambda_m^*$: contribution from $\mathcal{O}_{21}$, $\mathcal{O}_{31}$, $\mathcal{O}_{41}$, $\mathcal{O}_{32}$ and $\mathcal{O}_{42}$ fall in $(TT,\emptyset)$, $(TU,\emptyset)$, and (T,U).

   Electric-type splitting : the contributions $\mathcal{O}_{31}$, $\mathcal{O}_{41}$ are both constrained to $k = 0 \bmod N$, imposing the second charge in (C.7) to be electric-twisted.

   If $j \neq 0 \bmod N$ and $Q - \frac{j}{k}P \underline{\in} \Lambda_m^*$, the splitting is $(TT,\emptyset)$ and only $\mathcal{O}_{41}$ contributes accordingly, with measure $c_T(\Gamma_1)c_T(\Gamma_2)$ from $[\Delta_k(\rho/N)\Delta_k(\sigma/N)]^{-1}$.

   When $Q - \frac{j}{k}P \in \Lambda_m$, the splitting is $(TU,\emptyset)$ and both $\mathcal{O}_{31}$, $\mathcal{O}_{41}$ contribute with measure $(c_T(\Gamma_1)+c_U(\Gamma_1))c_T(\Gamma_2)$ from Fourier coefficients of $\Delta_k(\rho/N)^{-1}\big(\Delta_k(\sigma)^{-1}+\Delta_k(\sigma/N)^{-1}\big)$.

   If instead $j = Nj'$, contributions from $\mathcal{O}_{41}$, whose condition rewrites $Q-\frac{j'}{k'}P \in \Lambda_m^*$, $\frac{1}{k'}P \in N\Lambda_m^*$, combine with $\mathcal{O}_{21}$ to (T,U) splittings with measure $c_T(\Gamma_1)(c_T(\Gamma_2)+c_U(\Gamma_2))$ from Fourier coefficients of $\Delta_k(\rho/N)^{-1}\big(\Delta_k(\sigma)^{-1}+\Delta_k(\sigma/N)^{-1}\big)$ — note that their second 1/2-BPS charge in (C.7) is congruent to a magnetic-untwisted one since $P \underline{\in} N\Lambda_m^*$ implies $k' \neq 0 \bmod N$ in $\mathcal{O}_{41}$.

   Magnetic-type splitting : contributions from $\mathcal{O}_{32}$, $\mathcal{O}_{42}$ have $j = 0 \bmod N$ by construction, imposing their second 1/2-BPS charge in (C.14) to be congruent to an electric-twisted one, as well as $P - \frac{Nj'}{k}Q \in N\Lambda^*$, implying the first 1/2-BPS charge to be magnetic-untwisted for both of them. They thus combine to give (T,U) splittings with measure $(c_T(\Gamma_1)+c_U(\Gamma_2))c_T(\Gamma_2)$ from Fourier coefficients in $[\Delta_k(\rho/N)\big(\Delta_k(\sigma)+\Delta_k(\sigma/N)\big)]^{-1}$. Recall that we couble-count this last splitting, which is the same as the one defined above from $\mathcal{O}_{41}$ and $\mathcal{O}_{21}$ with Γ_1 and Γ_2 exchanged, according to (C.19).

4. $Q \underline{\in} \Lambda_m$, $P \underline{\in} N\Lambda_m^*$: contribution from $\mathcal{O}_{11}$, $\mathcal{O}_{21}$, $\mathcal{O}_{31}$, $\mathcal{O}_{41}$, $\mathcal{O}_{12}$, $\mathcal{O}_{22}$, $\mathcal{O}_{32}$, and $\mathcal{O}_{42}$ fall symmetrically in (U,U), $(TT,\emptyset)$, and $(\emptyset,TT)$.

   Electric-type splitting : $P \underline{\in} N\Lambda_m^*$ imposes $k \neq 0 \bmod N$ for $\mathcal{O}_{11}$, $\mathcal{O}_{21}$, for which the conditions rewrite $\frac{1}{k'}P \in N\Lambda_m^*$, with $k' \neq 0 \bmod N$, thus implying that the second 1/2-BPS charge in (C.7) is congruent to magnetic-untwisted one.

   Given $j = 0 \bmod N$ in $\mathcal{O}_{31}$, $\mathcal{O}_{41}$, one can rewrite their conditions as $Q - \frac{j'}{k'}P \in \Lambda_m$ and $\frac{1}{k'}P \in N\Lambda_m^*$, and these combine with $\mathcal{O}_{11}$, $\mathcal{O}_{21}$ to give (U,U) splittings with measure $(c_T(\Gamma_1)+c_U(\Gamma_1))(c_T(\Gamma_2)+c_U(\Gamma_2))$ from Fourier coefficients of all factors $\big(\Delta_k(\rho)^{-1}+\Delta_k(\rho/N)^{-1}\big)\big(\Delta_k(\sigma)^{-1}+\Delta_k(\sigma/N)^{-1}\big)$. These are the only contributions from $\mathcal{O}_{11}$ and $\mathcal{O}_{12}$, because $k \neq 0 \bmod N$.

   For $j \neq 0 \bmod N$, one has $Q - \frac{j}{Nk'}P \underline{\in} \Lambda_m^*$, and $\mathcal{O}_{31}$ is empty while $\mathcal{O}_{41}$ contributes alone to $(TT,\emptyset)$ splittings with measure $c_T(\Gamma_1)c_T(\Gamma_2)$ from $[\Delta_k(\rho/N)\Delta_k(\sigma/N))]^{-1}$.

   Magnetic-type splitting : in the case where $j = 0 \bmod N$, all $\mathcal{O}_{12}$, $\mathcal{O}_{22}$, $\mathcal{O}_{32}$ and $\mathcal{O}_{42}$ combine, with $k \neq 0 \bmod N$ for each, to give (U,U) splittings measure from Fourier coefficients of $\big(\Delta_k(\rho)^{-1}+\Delta_k(\rho/N)^{-1}\big)\big(\Delta_k(\sigma)^{-1}+\Delta_k(\sigma/N)^{-1}\big)$, double-counting the electric type (U,U) splittings describe above. For $j \neq 0 \bmod N$, and when $P - \frac{j}{k}Q \underline{\in} \Lambda_m$, $\mathcal{O}_{12}$ contribute alone to $(\emptyset,TT)$ splittings, with measure $c_T(\Gamma_1)c_T(\Gamma_2)$ from Fourier coefficients of $[\Delta_k(\rho)\Delta_k(\sigma)]^{-1}$.

5. $Q \in \Lambda_m\,, P \in N\Lambda_m$: contributions from all $\mathcal{O}_{ij}$ fall in (U,U), $(UU,\emptyset)$, and $(\emptyset,TT)$ splitting sectors.

   Electric-type splitting : cases with $k \neq 0 \bmod N$ and $Q - \frac{j}{k}P \in \Lambda_m$ appear in $\mathcal{O}_{11}$ and $\mathcal{O}_{21}$, together with $\mathcal{O}_{31}$ and $\mathcal{O}_{41}$ when $j = 0 \bmod N$, corresponding to (U,U) splittings with the generic measure from Fourier coefficients of $\big(\Delta_k(\rho)^{-1} + \Delta_k(\rho/N)^{-1}\big)\big(\Delta_k(\sigma)^{-1} + \Delta_k(\sigma/N)^{-1}\big)$.

   When $k = 0 \bmod N$, cases with $j \neq 0 \bmod N$ get contributions from $\mathcal{O}_{11}$, $\mathcal{O}_{21}$, $\mathcal{O}_{31}$ and $\mathcal{O}_{41}$, corresponding to $(UU,\emptyset)$ splittings with the generic measure from Fourier coefficients of $\big(\Delta_k(\rho)^{-1} + \Delta_k(\rho/N)^{-1}\big)\big(\Delta_k(\sigma)^{-1} + \Delta_k(\sigma/N)^{-1}\big)$.

   Magnetic-type splitting : we obtain that $k \neq 0 \bmod N$ in all cases. When $j \neq 0 \bmod N$, one has $P - \frac{j}{k}Q \in \Lambda_m$ and only $\mathcal{O}_{12}$ contributes, giving $(\emptyset,TT)$ splittings with measure $c_T(\Gamma_1)c_T(\Gamma_2)$ from $[\Delta_k(\rho)\Delta_k(\sigma)]^{-1}$.

   When $j = 0 \bmod N$, $P - \frac{j}{k}Q \in N\Lambda_m$ and $\mathcal{O}_{12}$, $\mathcal{O}_{22}$, $\mathcal{O}_{32}$, $\mathcal{O}_{42}$ contribute, giving (U,U) splittings with the generic measure from Fourier coefficients of all four factors in $\big(\Delta_k(\rho)^{-1} + \Delta_k(\rho/N)^{-1}\big)\big(\Delta_k(\sigma)^{-1} + \Delta_k(\sigma/N)^{-1}\big)$. These splittings are the same as the electric type splittings of the same (U,U) type.

6. $Q \in N\Lambda_m^*$, $P \in N\Lambda_m^*$: all $\mathcal{O}_{ij}$ contribute and fall in (U,U), $(\emptyset,UU)$, and $(TT,\emptyset)$ splitting sectors.

   Electric-type splitting : when $k \neq 0 \bmod N$, $Q - \frac{j}{k}P \in N\Lambda_m^*$ and $\mathcal{O}_{11}$, $\mathcal{O}_{21}$, together with $\mathcal{O}_{31}$ and $\mathcal{O}_{41}$ when $j = 0 \bmod N$, contribute to (U,U) splittings, with measure contributions from Fourier coefficients of all four factors in $\big(\Delta_k(\rho)^{-1} + \Delta_k(\rho/N)^{-1}\big)\big(\Delta_k(\sigma)^{-1} + \Delta_k(\sigma/N)^{-1}\big)$.

   When $k = 0 \bmod N$, only the two last orbits can contribute, and when $j \neq 0 \bmod N$ there is no other contribution than $\mathcal{O}_{41}$, leading to $(TT,\emptyset)$ splittings with measure $c_T(\Gamma_1)c_T(\Gamma_2)$ from Fourier coefficients of $[\Delta_k(\rho/N)\Delta_k(\sigma/N)]^{-1}$.

   Magnetic-type splitting : when $k \neq 0 \bmod N$, $\mathcal{O}_{12}$ and $\mathcal{O}_{22}$ with $j \neq 0 \bmod N$ contribute, together with $\mathcal{O}_{32}$ and $\mathcal{O}_{42}$ when $k = 0 \bmod N$, to $(\emptyset,UU)$ splittings with the generic measure from Fourier coefficients of $\big(\Delta_k(\rho)^{-1} + \Delta_k(\rho/N)^{-1}\big)\big(\Delta_k(\sigma)^{-1} + \Delta_k(\sigma/N)^{-1}\big)$.

   When $k \neq 0 \bmod N$ and $j = 0 \bmod N$, $\mathcal{O}_{12}$, $\mathcal{O}_{22}$, $\mathcal{O}_{32}$ and $\mathcal{O}_{42}$ contribute to (U,U) splittings with the generic measure from Fourier coefficients of $\big(\Delta_k(\rho)^{-1} + \Delta_k(\rho/N)^{-1}\big)\big(\Delta_k(\sigma)^{-1} + \Delta_k(\sigma/N)^{-1}\big)$, associated to the same splittings of electric type (U,U) described above.

7. $Q \in N\Lambda_m^*$, $P \in N\Lambda_m$: all $\mathcal{O}_{ij}$ contribute and fall in (U,U), $(\emptyset,UU)$, and $(UU,\emptyset)$ splitting sectors.

   Electric-type splitting : when $k \neq 0 \bmod N$, $Q - \frac{j}{k}P \in N\Lambda_m^*$ and $\mathcal{O}_{11}$, $\mathcal{O}_{21}$, together with $\mathcal{O}_{31}$ and $\mathcal{O}_{41}$ when $j = 0 \bmod N$, contribute to (U,U) splittings, with the generic measure from Fourier coefficients of $\big(\Delta_k(\rho)^{-1} + \Delta_k(\rho/N)^{-1}\big)\big(\Delta_k(\sigma)^{-1} + \Delta_k(\sigma/N)^{-1}\big)$.

   When $k = 0 \bmod N$, all the four orbits can contribute and $j \neq 0 \bmod N$. They lead to $(UU,\emptyset)$ splittings with generic measure from Fourier coefficients of $\big(\Delta_k(\rho)^{-1}+\Delta_k(\rho/N)^{-1}\big) \times\big(\Delta_k(\sigma)^{-1} + \Delta_k(\sigma/N)^{-1}\big)$.

   Magnetic-type splitting : when $k \neq 0 \bmod N$, $\mathcal{O}_{12}$ and $\mathcal{O}_{22}$ with $j \neq 0 \bmod N$ contribute, together with $\mathcal{O}_{32}$ and $\mathcal{O}_{42}$ when $k = 0 \bmod N$, to $(\emptyset,UU)$ splittings with the generic measure from Fourier coefficients of $\big(\Delta_k(\rho)^{-1} + \Delta_k(\rho/N)^{-1}\big)\big(\Delta_k(\sigma)^{-1} + \Delta_k(\sigma/N)^{-1}\big)$.

When $k \neq 0 \bmod N$ and $j = 0 \bmod N$, $\mathcal{O}_{12}$, $\mathcal{O}_{22}$, $\mathcal{O}_{32}$ and $\mathcal{O}_{42}$ contribute to (U,U) splittings with the generic measure $(c_T(\Gamma_1)+c_U(\Gamma_1))(c_T(\Gamma_2)+c_U(\Gamma_2))$ from Fourier coefficients of $\big(\Delta_k(\rho)^{-1}+\Delta_k(\rho/N)^{-1}\big)\big(\Delta_k(\sigma)^{-1}+\Delta_k(\sigma/N)^{-1}\big)$, which count the same splitting of electric type described above.

This concludes the proof of formula (5.92). As a consistency check, we note that these results are consistent with Fricke duality. Namely, for 1/4-BPS charges belonging to Fricke-invariant subsets, such as $(Q,P)\,\underline{\in}\,\Lambda_m^*\oplus\Lambda_m$ or $(Q,P)\,\underline{\in}\,\Lambda_m\oplus N\Lambda_m^*$, the possible splittings are invariant under the exchange of electric and magnetic type; whereas for charges in subsets that are exchanged under Fricke duality, as $(Q,P)\,\underline{\in}\,\Lambda_m\oplus N\Lambda_m$ and $(Q,P)\,\underline{\in}\,N\Lambda^*\oplus N\Lambda^*$, the possible splittings are themselves exchanged under Fricke duality. Moreover, we find that all the splittings of electric-magnetic type are correctly double-counted through the splitting matrices of electric and magnetic type, consistently with (C.19).

D Two-instanton singular contributions to Abelian Fourier coefficients

In this section, we extract the contributions to the rank-2 Abelian Fourier modes from the Dirac delta functions in the Poincaré series representation (5.25), (5.57) of the Fourier coefficients of $\frac{1}{\Phi_{k-2}}$.

D.1 Maximal rank

Starting from (5.24), the sum over $\gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4$ can be unfolded against the integration domain,[32] by changing variables as $\Omega_2\to\gamma^{-\intercal}\Omega_2\gamma^{-1}$. The contribution of the delta functions in (5.25) then leads to

$$
\begin{aligned}
-\frac{R^4}{2\pi}\sum_{\widetilde{Q}\in\Lambda_{p-2,q-2}^{\oplus 2}}\;\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ \gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4}} & e^{2\pi\mathrm{i}a^{iI}A_{ij}Q_I^j}\int_{\mathcal{P}_2}\frac{\mathrm{d}^3\Omega_2}{\Omega_2^{3/2}}|\Omega_2|^{\frac{q-5}{2}}c\big(-\tfrac{(s\widetilde{Q}_1-q\widetilde{Q}_2)^2}{2}\big)c\big(-\tfrac{(p\widetilde{Q}_2-r\widetilde{Q}_1)^2}{2}\big)\\
&\times\delta\big(\mathrm{tr}\begin{pmatrix}0&1/2\\1/2&0\end{pmatrix}\Omega_2\big)\,e^{-\pi\mathrm{Tr}\left[\frac{R^2}{S_2}\Omega_2^{-1}\gamma^\intercal A^\intercal\begin{pmatrix}1&S_1\\S_1&|S|^2\end{pmatrix}A\gamma+2\Omega_2\gamma^{-1}\widetilde{Q}\cdot\widetilde{Q}^\intercal\gamma^{-\intercal}\right]}\\
&\quad\times\mathcal{P}_{ab,cd}\big(\tfrac{\partial}{\partial y}\big)\,e^{2\pi\mathrm{i}\left(\frac{R}{\mathrm{i}\sqrt{2}}y_{r\mu}(\gamma\Omega_2^{-1}\gamma^\intercal)^{rs}A^\intercal_{si}v^{\intercal\, i\mu}+y_{r\,\alpha}\widetilde{Q}_L{}^{r\alpha}+\frac{1}{4\mathrm{i}}y_{r\,\alpha}(\gamma\Omega_2^{-1}\gamma^\intercal)^{rs}y_s{}^\alpha\right)}\,,
\end{aligned}
\tag{D.1}
$$

where a factor 2 comes from the center of order 2 of $GL(2,\mathbb{Z})$ acting on $\mathcal{H}_2$, $\gamma=\begin{pmatrix}p&q\\r&s\end{pmatrix}$. The integral over positive definite matrices $\mathcal{P}_2$ splits into two Bessel-type integrals, using the

[32] Recall that Dih_4 is the dihedral group of order 8 generated by the matrices $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $\begin{pmatrix}0&1\\1&0\end{pmatrix}$, which stabilize $\begin{pmatrix}0&1/2\\1/2&0\end{pmatrix}$.

projectors $\pi_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, $\pi_2 = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$

$$\begin{aligned}
-\frac{R^4}{\pi} & \sum_{\widetilde{Q}\in\Lambda^{\oplus 2}_{p-2,q-2}} \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ \gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4}} e^{2\pi \mathrm{i} a^{iI} A_{ij}\widetilde{Q}^j_I} c\big(-\tfrac{(s\widetilde{Q}_1-q\widetilde{Q}_2)^2}{2}\big) c\big(-\tfrac{(p\widetilde{Q}_2-r\widetilde{Q}_1)^2}{2}\big) \\
& \times \int_0^\infty \frac{\mathrm{d}\rho_2}{\rho_2}\rho_2^{\frac{q-6}{2}} e^{-\pi \mathrm{Tr}\left[\pi_1\left(\frac{R^2}{\rho_2 S_2}\gamma^\intercal A^\intercal \left(\begin{smallmatrix}1 & S_1\\ S_1 & |S|^2\end{smallmatrix}\right) A\gamma + 2\rho_2\gamma^{-1}\widetilde{Q}\cdot\widetilde{Q}^\intercal\gamma^{-\intercal}\right)\right]} \\
& \times \int_0^\infty \frac{\mathrm{d}\sigma_2}{\sigma_2}\sigma_2^{\frac{q-6}{2}} e^{-\pi \mathrm{Tr}\left[\pi_2\left(\frac{R^2}{\sigma_2 S_2}\gamma^\intercal A^\intercal \left(\begin{smallmatrix}1 & S_1\\ S_1 & |S|^2\end{smallmatrix}\right) A\gamma + 2\sigma_2\gamma^{-1}\widetilde{Q}\cdot\widetilde{Q}^\intercal\gamma^{-\intercal}\right)\right]} \\
& \times \mathcal{P}_{ab,cd}\big(\tfrac{\partial}{\partial y}\big)\, e^{2\pi \mathrm{i}\left(\frac{R}{\rho_2 \mathrm{i}\sqrt{2}}(y_\mu\gamma\pi_1\gamma^\intercal A^\intercal v^{\intercal\,\mu}) + y_\alpha\gamma\,\gamma^{-1}\widetilde{Q}_L{}^\alpha + \frac{1}{4\mathrm{i}\rho_2}(y_\alpha\gamma\pi_1\gamma^\intercal y^\alpha)\right)} \\
& \times e^{2\pi \mathrm{i}\left(\frac{R}{\sigma_2 \mathrm{i}\sqrt{2}}(y_\mu\gamma\pi_2\gamma^\intercal A^\intercal v^{\intercal\mu}) + \frac{1}{4\mathrm{i}\sigma_2}(y_\alpha\gamma\pi_2\gamma^\intercal y^\alpha)\right)} .
\end{aligned} \tag{D.2}$$

The matrices $\gamma \in GL(2,\mathbb{Z})/\mathrm{Dih}_4$ in the last two rows can be absorbed by a change of variable $(y_\mu, y_\alpha) \to (y_\mu\gamma^{-1}, y_\alpha\gamma^{-1})$. After relabelling the summation variable as $\left(\begin{smallmatrix}Q\\ P\end{smallmatrix}\right) = A\left(\begin{smallmatrix}\widetilde{Q}_1\\ \widetilde{Q}_2\end{smallmatrix}\right)$, one obtains a sum over all splittings $\Gamma = (Q,P) = \Gamma_1 + \Gamma_2$ in the lattice $\Lambda^{\oplus 2}_{p-2,q-2}$,

$$\begin{aligned}
& \sum_{\widetilde{Q}_i\in\Lambda^{\oplus 2}_{p-2,q-2}} \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ \gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4}} e^{2\pi \mathrm{i} a^{iI} A_{ij}\widetilde{Q}^j_I} f(A\gamma\pi_i\gamma^{-1}\widetilde{Q})\, g((\pi_1\gamma^{-1}\widetilde{Q})^2) g((\pi_2\gamma^{-1}\widetilde{Q})^2) \\
&= \sum_{\Gamma\in\Lambda^{\oplus 2}_{p-2,q-2}} e^{2\pi \mathrm{i}(a^1\cdot Q + a^2\cdot P)} \\
&\qquad \times \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ \gamma\in GL(2,\mathbb{Z})/\mathrm{Dih}_4\\ A^{-1}\Gamma\in\Lambda^{\oplus 2}_{p-2,q-2}}} f(A\gamma\,\pi_i(\gamma A)^{-1}\Gamma)\, g\big(-\tfrac{(\pi_1(\gamma A)^{-1}\Gamma)^2}{2}\big) g\big(-\tfrac{(\pi_2(\gamma A)^{-1}\Gamma)^2}{2}\big) \\
&= \sum_{\Gamma\in\Lambda^{\oplus 2}_{p-2,q-2}} e^{2\pi \mathrm{i}(a^1\cdot Q + a^2\cdot P)} \sum_{\substack{B\in M_2(\mathbb{Z})/\mathrm{Stab}(\pi_i)\\ \Gamma_1,\Gamma_2\in\Lambda_{p-2,q-2}}} f(\Gamma_1,\Gamma_2) \\
&\qquad \times \sum_{\Gamma_1/d_1\in\Lambda_{p-2,q-2}} g\big(-\tfrac{(B^{-1}\Gamma_1)^2}{2d_1^2}\big) \sum_{\Gamma_2/d_2\in\Lambda_{p-2,q-2}} g\big(-\tfrac{(B^{-1}\Gamma_2)^2}{2d_2^2}\big) .
\end{aligned} \tag{D.3}$$

where $\Gamma_i = B\pi_i B^{-1}\Gamma = (Q_i, P_i)$, such that $\Gamma_1 + \Gamma_2 = \Gamma$, and where $\mathrm{Stab}(\pi_i)$ is the stabilizer of $\pi_i = \left(\begin{smallmatrix}\delta_{1,i} & 0\\ 0 & \delta_{2,i}\end{smallmatrix}\right)$ inside $M(2,\mathbb{Z})$. The rearrangement (D.3) holds for arbitrary functions $f(Q)$, $g(x)$, in particular for the product of Bessel integrals and the measure factors $c(x)$ in

(D.2). The singular contributions to the Fourier modes are thus

$$\begin{aligned}
G^{(p,q),\,2\text{Ab},\,\Gamma}_{\alpha\beta,\gamma\delta} &= -\frac{R^4}{\pi} \sum_{\substack{B\in M_2(\mathbb{Z})/\text{Stab}(\pi_i)\\ B\pi_i B^{-1}\Gamma\in\Lambda^{\oplus 2}_{p-2,q-2}}} \bar{c}(\Gamma_1)\,\bar{c}(\Gamma_2) \sum_{l_1,l_2=0}^{2} \frac{P^{(l_1,l_2)}_{\alpha\beta,\gamma\delta}(\Gamma_1,\Gamma_2)}{R^{l_1+l_2}} \\
&\qquad \times \frac{K_{\frac{q-6}{2}-l_1}(2\pi R\mathcal{M}(\Gamma_1))}{\mathcal{M}(\Gamma_1)^{\frac{q-6}{2}-l_1}} \frac{K_{\frac{q-6}{2}-l_2}(2\pi R\mathcal{M}(\Gamma_2))}{\mathcal{M}(\Gamma_2)^{\frac{q-6}{2}-l_2}} \\
G^{(p,q),\,2\text{Ab},\,\Gamma}_{\alpha\beta,\gamma\upsilon} &= -\frac{R^4}{\pi} \sum_{\substack{B\in M_2(\mathbb{Z})/\text{Stab}(\pi_i)\\ B\pi_i B^{-1}\Gamma\in\Lambda^{\oplus 2}_{p-2,q-2}}} \bar{c}(\Gamma_1)\,\bar{c}(\Gamma_2) \sum_{l_1,l_2=0}^{1} \frac{P^{(l_1,l_2)}_{\alpha\beta,\gamma\upsilon}(\Gamma_1,\Gamma_2)}{\mathrm{i}\sqrt{2}R^{l_1+l_2}} \\
&\qquad \times \frac{K_{\frac{q-6}{2}-l_1}(2\pi R\mathcal{M}(\Gamma_1))}{\mathcal{M}(\Gamma_1)^{\frac{q-6}{2}-l_1}} \frac{K_{\frac{q-6}{2}-l_2}(2\pi R\mathcal{M}(\Gamma_2))}{\mathcal{M}(\Gamma_2)^{\frac{q-6}{2}-l_2}} \qquad (D.4) \\
&\vdots \\
G^{(p,q),\,2\text{Ab},\,\Gamma}_{\rho\sigma,\tau\upsilon} &= -\frac{R^4}{\pi} \sum_{\substack{B\in M_2(\mathbb{Z})/\text{Stab}(\pi_i)\\ B\pi_i B^{-1}\Gamma\in\Lambda^{\oplus 2}_{p-2,q-2}}} \bar{c}(\Gamma_1)\,\bar{c}(\Gamma_2) \sum_{l_1,l_2=0}^{1} \frac{P^{(l_1,l_2)}_{\rho\sigma,\tau\upsilon}(\Gamma_1,\Gamma_2)}{4R^{l_1+l_2}} \\
&\qquad \times \frac{K_{\frac{q-6}{2}-l_1}(2\pi R\mathcal{M}(\Gamma_1))}{\mathcal{M}(\Gamma_1)^{\frac{q-6}{2}-l_1}} \frac{K_{\frac{q-6}{2}-l_2}(2\pi R\mathcal{M}(\Gamma_2)}{\mathcal{M}(\Gamma_2)^{\frac{q-6}{2}-l_2}}
\end{aligned}$$

where the measure $\bar{c}(\Gamma_i)$ is defined by

$$\bar{c}(\Gamma) = \sum_{\substack{d_i>0\\ \Gamma/d_i\in\Lambda^{\oplus 2}_{p-2,q-2}}} c\Big(-\frac{\gcd(Q^2,P^2,Q\cdot P)}{2d_i^2}\Big)\Big(\frac{d^2}{\gcd(Q^2,P^2,Q\cdot P)}\Big)^{\frac{q-8}{2}}\,. \qquad (D.5)$$

The factorized form of these singular contributions is indeed consistent with the differential equation (3.20), as discussed in §E.3.

D.2 Measure factorization in CHL orbifolds

For CHL orbifolds, the contributions from the Dirac delta functions in (5.57) and (5.58) to the Fourier mode (5.56) can be computed similarly to the full rank case (D.4) by using the results of Appendix C. Here we explain the factorization of the measure for a general lattice $\Lambda_{p-2,q-2}$ of signature $(p-2,q-2)$, which we denote by Λ for short. When the lattice is N-modular, as in the case of the magnetic lattice Λ_m discussed in section C, one can rewrite the measure in a form manifestly invariant under Fricke electro-magnetic duality. However, this is not the case in generic signature. In this section we use the results of the previous section to write the 1/2-BPS charge measure factors coming from the different orbit terms in (5.64). By abuse of language we shall refer to the charges $(Q,P)\in\Lambda^*\oplus\Lambda$ components as electric and magnetic, although this terminology is only accurate when $q=8$.

For the most generic lattice vectors, namely $(Q,P)\,\underline{\in}\,\Lambda^*\oplus\Lambda$, the only matrices A which contribute belong either to the electric first orbit of the second set of splittings (C.36), or the magnetic second orbit (C.39) contribute. They both lead to the factorized measure

$$\bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2) = \upsilon \sum_{\substack{d_1>0 \\ (Q_1,P_1)/d_1\in\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\Big(\frac{d_1^2}{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}}$$
$$\times \sum_{\substack{d_2>0 \\ (Q_2,P_2)/d_2\in\Lambda\oplus\Lambda}} c_k\Big(-\frac{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\Big)\Big(\frac{d_2^2}{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}}, \tag{D.6}$$

where Γ_1 is of electric type and Γ_2 of magnetic type. As explained in appendix C, this measure is consistent with splittings into pairs of 1/2-BPS charges of (T,T) type.

For less generic vectors $(Q,P)\in\Lambda\oplus\Lambda$, the measure receives additional contributions from the first term of (5.56), as well as from the first magnetic orbit from the second set (C.37). Unlike the previous case Γ_1 can be either of electric or magnetic type, while Γ_2 is always of magnetic type. When Γ_1 is of electric type, the resulting measure is given by

$$\bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2) = \Bigg[\sum_{\substack{d_1>0 \\ (Q_1,P_1)/d_1\in\Lambda\oplus N\Lambda}} c_k\Big(-\frac{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\Big(\frac{d_1^2}{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}}$$
$$+\,\upsilon \sum_{\substack{d_1>0 \\ (Q_1,P_1)/d_1\in\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\Big(\frac{d_1^2}{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}}\Bigg]$$
$$\times \sum_{\substack{d_2>0 \\ (Q_2,P_2)/d_2\in\Lambda\oplus\Lambda}} c_k\Big(-\frac{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\Big)\Big(\frac{d_2^2}{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}} \tag{D.7}$$

where only untwisted states can contribute in this case. This result is consistent with splittings of type (U,T), as explained in Appendix C. When Γ_1 is of magnetic type, the measure is instead given by

$$\bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2) = \Bigg[\sum_{\substack{d_1>0 \\ (Q_1,P_1)/d_1\in\Lambda\oplus\Lambda}} c_k\Big(-\frac{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\Big(\frac{d_1^2}{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}}$$
$$+\,\upsilon \sum_{\substack{d_1>0 \\ (Q_1,P_1)/d_1\in N\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}{2Nd_1^2}\Big)\Big(\frac{Nd_1^2}{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}}\Bigg]$$
$$\times \sum_{\substack{d_2>0 \\ (Q_2,P_2)/d_2\in\Lambda\oplus\Lambda}} c_k\Big(-\frac{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\Big)\Big(\frac{d_2^2}{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}} \tag{D.8}$$

where both twisted and untwisted states can contribute, and where the former only get contributions form the second term in the bracket, while the latter get contributions from both, which is consistent with splittings into doublets of 1/2-BPS of $(\emptyset,UT)$ and $(\emptyset,TT)$ type.

For the vectors $(Q,P) \underline{\in} \Lambda^* \oplus N\Lambda^*$, one must add to (D.6) the contribution from the last term of (5.56), *i.e.* both electric and magnetic orbits of the third set of contributions (C.26), (C.28). In this case, Γ_1 can only be of electric type, while Γ_2 can either be of electric or magnetic type. For electric Γ_2 one obtains

$$\begin{aligned}\bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2) = \upsilon \sum_{\substack{d_1>0\\(Q_1,P_1)/d_1\in\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\Big(\frac{d_1^2}{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}}\\ \times\Big[\sum_{\substack{d_2>0\\(Q_2,P_2)/d_2\in\Lambda\oplus N\Lambda}} c_k\Big(-\frac{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\Big)\Big(\frac{d_2^2}{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}}\\ +\upsilon\sum_{\substack{d_2>0\\(Q_2,P_2)/d_2\in\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\Big)\Big(\frac{d_2^2}{\gcd(NQ_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}}\Big]\end{aligned} \tag{D.9}$$

where both twisted and untwisted states can contribute in this case, consistently with splittings of type $(TU,)$ and $(TT,)$. For magnetic Γ_2, one obtains

$$\begin{aligned}\bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2) = \upsilon \sum_{\substack{d_1>0\\(Q_1,P_1)/d_1\in\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\Big(\frac{d_1^2}{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}}\\ \times\Big[\sum_{\substack{d_2>0\\(Q_2,P_2)/d_2\in\Lambda\oplus\Lambda}} c_k\Big(-\frac{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\Big)\Big(\frac{d_2^2}{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}}\\ +\upsilon\sum_{\substack{d_2>0\\(Q_2,P_2)/d_2\in N\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_2^2,P_2^2,Q_2\cdot P_2)}{2Nd_2^2}\Big)\Big(\frac{Nd_2^2}{\gcd(NQ_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}}\Big]\end{aligned} \tag{D.10}$$

where only untwisted states can contribute, consistently with splittings of (T,U) type. In both cases, the factors of N come from the width of the integration domain $(\mathbb{R}/N\mathbb{Z})^3$.

Finally, for vectors $Q\in\Lambda\,,\ P\in N\Lambda^*$, one must add each contribution specific to the two last cases as well as the contribution from the second type of orbit of (5.51). Each 1/2-BPS state Γ_1, Γ_2 can be either electric or magnetic. When both of them are electric, we obtain

$$\begin{aligned}\bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2) = \Bigg[&\sum_{\substack{d_1>0\\(Q_1,P_1)/d_1\in\Lambda\oplus N\Lambda}} c_k\Big(-\frac{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\Big(\frac{d_1^2}{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}} \\ &+ \upsilon \sum_{\substack{d_1>0\\(Q_1,P_1)/d_1\in\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\Big(\frac{d_1^2}{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}}\Bigg] \\ &\times \Bigg[\sum_{\substack{d_2>0\\(Q_2,P_2)/d_2\in\Lambda\oplus N\Lambda}} c_k\Big(-\frac{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\Big)\Big(\frac{d_2^2}{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}} \\ &+ \upsilon \sum_{\substack{d_2>0\\(Q_2,P_2)/d_2\in\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\Big)\Big(\frac{d_2^2}{\gcd(NQ_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}}\Bigg] ,\end{aligned} \tag{D.11}$$

with constraints on the possible splittings, as explained in appendix C, selecting splittings of type $(TT,)$ only. When both states magnetic, one obtains

$$\begin{aligned}\bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2) = \Bigg[&\sum_{\substack{d_1>0\\(Q_1,P_1)/d_1\in\Lambda\oplus\Lambda}} c_k\Big(-\frac{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\Big(\frac{d_1^2}{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}} \\ &+ \upsilon \sum_{\substack{d_1>0\\(Q_1,P_1)/d_1\in N\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}{2Nd_1^2}\Big)\Big(\frac{Nd_1^2}{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}}\Bigg] \\ &\times \Bigg[\sum_{\substack{d_2>0\\(Q_2,P_2)/d_2\in\Lambda\oplus\Lambda}} c_k\Big(-\frac{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\Big)\Big(\frac{d_2^2}{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}} \\ &+ \upsilon \sum_{\substack{d_2>0\\(Q_2,P_2)/d_2\in N\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_2^2,P_2^2,Q_2\cdot P_2)}{2Nd_2^2}\Big)\Big(\frac{Nd_2^2}{\gcd(NQ_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}}\Bigg] ,\end{aligned} \tag{D.12}$$

with again constraints on the possible splittings, selecting splittings of type $(\emptyset, TT)$ only.

When one state, say Γ_1, is electric, and the other magnetic, one obtains

$$\begin{aligned}\bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2) = \Bigg[& \sum_{\substack{d_1>0 \\ (Q_1,P_1)/d_1\in\Lambda\oplus N\Lambda}} c_k\Big(-\frac{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\Big(\frac{d_1^2}{\gcd(Q_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}} \\ & + v \sum_{\substack{d_1>0 \\ (Q_1,P_1)/d_1\in\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}{2d_1^2}\Big)\Big(\frac{d_1^2}{\gcd(NQ_1^2,P_1^2,Q_1\cdot P_1)}\Big)^{\frac{q-8}{2}} \Bigg] \\ & \times \Bigg[\sum_{\substack{d_2>0 \\ (Q_2,P_2)/d_2\in\Lambda\oplus\Lambda}} c_k\Big(-\frac{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}{2d_2^2}\Big)\Big(\frac{d_2^2}{\gcd(Q_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}} \\ & + v \sum_{\substack{d_2>0 \\ (Q_2,P_2)/d_2\in N\Lambda^*\oplus N\Lambda^*}} c_k\Big(-\frac{\gcd(NQ_2^2,P_2^2,Q_2\cdot P_2)}{2Nd_2^2}\Big)\Big(\frac{Nd_2^2}{\gcd(NQ_2^2,P_2^2,Q_2\cdot P_2)}\Big)^{\frac{q-8}{2}} \Bigg] ,\end{aligned} \tag{D.13}$$

where the constraints on the possible splitting here select (U,U) only.

When the charge vectors(Q,P) lies in an even finer sublattice, such as $\Lambda\oplus N\Lambda$, $N\Lambda^*\oplus N\Lambda^*$, and so on, the measure is still given by (D.13), but it includes less generic type of splittings like $(UU,)$ or $(\emptyset,UU)$, as explained in appendix C.

Thus, we have established that the delta function contributions to the Abelian Fourier coefficients factorize into the product $\bar{c}_k(\Gamma_1)\bar{c}_k(\Gamma_2)$ of the measures associated with each 1/2-BPS component for all splittings $\Gamma = \Gamma_1 + \Gamma_2$ of an arbitrary 1/4-BPS charge Γ in CHL models with $N = 2, 3, 5, 7$. This factorization is required for consistency with the differential equation (3.20), as further discussed in §E.3.

E Consistency with differential constraints

In this section we analyze the consistency of the asymptotic expansion of the the two-loop modular integral $G_{ab,cd}$ near the degenerations $O(p,q) \to O(p-2,q-2)$ and degeneration $O(p,q) \to O(p-1,q-1)$ with the differential equation (3.3). In the first case we consider both the constant terms and generic rank-2 Abelian Fourier coefficients, and show consistency with the quadratic source term in (3.3). In the second case for brevity we restrict to the constant terms.

E.1 Differential equation under the degeneration $O(p,q) \to O(p-2,q-2)$

Here we write explicitly the differential equation 3.3 in the variables relevant to the degeneration limit $O(p,q) \to O(p-2,q-2)$. Using the decomposition (5.3), and changing variable $R = e^{-\phi}$, the metric on the moduli space reads

$$2P_{a\hat{b}}P^{a\hat{b}} = 4\mathrm{d}\phi^2 + 2P_{\mu\nu}P^{\mu\nu} + 2P_{\alpha\hat{\beta}}P^{\alpha\hat{\beta}} + e^{2\phi}M_{ij}g^{IJ}\mathrm{d}a_I^i\mathrm{d}a_J^j + e^{4\phi}\nabla\psi\nabla\psi\,, \tag{E.1}$$

with

$$\nabla\psi = \mathrm{d}\psi - \tfrac{1}{2}\varepsilon_{ij}a^i\cdot\mathrm{d}a^j\ , \tag{E.2}$$

and the Maurer–Cartan coset component

$$P = \begin{pmatrix} d\phi\, \delta^{\nu}_{\mu} - P_{\mu}{}^{\nu} & \frac{e^{\phi}}{\sqrt{2}} v^{-1}_{i\mu} p_L^{\beta\, I}\, da^i_I & -\frac{e^{\phi}}{\sqrt{2}} v^{-1}_{i\mu} p_R^{\hat\beta\, I}\, da^i_I & \frac{1}{2} e^{2\phi} \varepsilon_{\mu}{}^{\hat\nu} \nabla\psi \\ \frac{e^{\phi}}{\sqrt{2}} v_i^{-1\nu} p_{L\alpha}{}^{I}\, da^i_I & 0 & P_{\alpha}{}^{\hat\beta} & \frac{e^{\phi}}{\sqrt{2}} v_i^{-1\hat\nu} p_{L\alpha}{}^{I}\, da^i_I \\ -\frac{e^{\phi}}{\sqrt{2}} v_i^{-1\nu} p_{R\hat\alpha}{}^{I}\, da^i_I & P^{\beta}{}_{\hat\alpha} & 0 & \frac{e^{\phi}}{\sqrt{2}} v_i^{-1\hat\nu} p_{R\hat\alpha}{}^{I}\, da^i_I \\ \frac{1}{2} e^{2\phi} \varepsilon^{\nu}{}_{\hat\mu} \nabla\psi & \frac{e^{\phi}}{\sqrt{2}} v^{-1}_{i\hat\mu} p_L^{\beta I}\, da^i_I & \frac{e^{\phi}}{\sqrt{2}} v^{-1}_{i\hat\mu} p_R^{\hat\beta I}\, da^i_I & -d\phi \delta^{\hat\nu}_{\hat\mu} + P_{\hat\mu}{}^{\hat\nu} \end{pmatrix} . \tag{E.3}$$

Beware that in this section we use the symbols p_L and p_R for the $G_{p-2,q-2} = O(p-2,q-2)/[O(p-2)\times O(q-2)]$ projection $p_{L\alpha I} Q^I$, and not for the $G_{p,q} = O(p,q)/[O(p)\times O(q)]$ projection $p_{La\mathcal{I}} Q^{\mathcal{I}}$ as in the body of the paper. We use Greek letters of the beginning of the alphabet, *i.e.* $\{\alpha,\beta,\gamma,\delta,\epsilon,\eta,\theta\}$, to denote local indices along $G_{p-2,q-2}$, and Greek letters of the middle of the alphabet, *i.e.* $\{\kappa,\lambda,\mu,\nu,\rho,\sigma,\tau\}$, to denote indices along $SO(2)\backslash SL(2,\mathbb{R})$.

The covariant derivative of a vector Z_a in the tangent frame must obey the usual equation

$$dZ_a = 2P^{b\hat c}\partial_{b\hat c} Z_a = 2P^{b\hat c}\Big(D_{b\hat c} Z_a - B_{b\hat c a}{}^{d} Z_d\Big) , \tag{E.4}$$

allowing us to write down its action, for any vector $Z_a = (Z_\sigma, Z_\gamma)$

$$\begin{aligned} D_{\mu\hat\nu} Z_a &= \big(\frac{1}{4}\delta_{\mu\hat\nu}\partial_\phi - \mathcal{D}_{\mu\hat\nu} + \frac{1}{2}e^{-2\phi}\varepsilon_{\mu\hat\nu}\partial_\psi\big) Z_a + \frac{1}{2}(\delta_{\sigma[\mu}\delta^{\rho}_{\hat\nu]} Z_\rho, 0) , \\ D_{\alpha\hat\nu} Z_a &= \frac{1}{\sqrt{2}} e^{-\phi} v_{\hat\nu}{}^{i} p_{L\alpha}{}^{I}\Big(\frac{\partial}{\partial a^{iI}} - \tfrac{1}{2}\varepsilon_{ij} a^j_I \partial_\psi\Big) Z_a + \frac{1}{2}(-\delta_{\hat\nu\sigma} Z_\alpha, \delta_{\alpha\gamma}\delta^{\nu}_{\hat\nu} Z_\nu) , \\ D_{\mu\hat\alpha} Z_a &= \frac{1}{\sqrt{2}} e^{-\phi} v_{\mu}{}^{i} p_{R\hat\alpha}{}^{I}\Big(\frac{\partial}{\partial a^{iI}} - \tfrac{1}{2}\varepsilon_{ij} a^j_I \partial_\psi\Big) Z_a , \end{aligned} \tag{E.5}$$

and on any vector $Z_{\hat a} = (Z_{\hat\alpha}, Z_{\hat\sigma})$ as

$$\begin{aligned} D_{\mu\hat\nu} Z_{\hat a} &= \big(\frac{1}{4}\delta_{\mu\hat\nu}\partial_\phi - \mathcal{D}_{\mu\hat\nu} + \frac{1}{2}e^{-2\phi}\varepsilon_{\mu\hat\nu}\partial_\psi\big) Z_{\hat a} + \frac{1}{2}(0, -\delta_{\hat\sigma[\mu}\delta^{\hat\rho}_{\hat\nu]} Z_{\hat\rho}) , \\ D_{\alpha\hat\nu} Z_{\hat a} &= \frac{1}{\sqrt{2}} e^{-\phi} v_{\hat\nu}{}^{i} p_{L\alpha}{}^{I}\Big(\frac{\partial}{\partial a^{iI}} - \tfrac{1}{2}\varepsilon_{ij} a^j_I \partial_\psi\Big) Z_{\hat a} , \\ D_{\mu\hat\alpha} Z_{\hat a} &= \frac{1}{\sqrt{2}} e^{-\phi} v_{\mu}{}^{i} p_{R\hat\alpha}{}^{I}\Big(\frac{\partial}{\partial a^{iI}} - \tfrac{1}{2}\varepsilon_{ij} a^j_I \partial_\psi\Big) Z_{\hat a} + \frac{1}{2}(\delta_{\hat\alpha\hat\alpha}\delta^{\hat\mu}_{\mu} Z_{\hat\mu}, -\delta_{\mu\hat\sigma} Z_{\hat\alpha}) , \end{aligned} \tag{E.6}$$

where $v_\mu{}^i \in SO(2)\backslash SL(2,\mathbb{R})$ such that

$$\mathcal{D}_{\mu\nu} v_\rho{}^i = \frac{1}{2}\delta_{\rho(\mu} v_{\nu)}{}^i - \frac{1}{4}\delta_{\mu\nu} v_\rho{}^i , \tag{E.7}$$

and finally, the operator $D_{\alpha\hat\beta} = \mathcal{D}_{\alpha\hat\beta}$ the differential operator on the Grassmanian $O(p-2,q-2)$, which acts on the projectors $p_{L\gamma}{}^I$, $p_{R\hat\alpha}{}^I$ as

$$\mathcal{D}_{\alpha\hat\beta} p_{L\gamma}{}^{I} = \frac{1}{2}\delta_{\alpha\gamma} p_{L\hat\beta}{}^{I} \quad , \qquad \mathcal{D}_{\alpha\hat\beta} p_{R\hat\alpha}{}^{I} = \frac{1}{2}\delta_{\hat\beta\hat\alpha} p_{L\alpha}{}^{I} . \tag{E.8}$$

In this decomposition, the tensor $G_{ab,cd}$ admits six independent components

$$\begin{aligned} &G_{\mu\nu,\sigma\rho} = \frac{3}{4}\delta_{\langle\mu\nu,}\delta_{\sigma\rho\rangle} G_{\lambda}{}^{\lambda}{}_{,\kappa}{}^{\kappa} , \qquad G_{\mu\nu,\sigma\delta} = \delta_{\mu\nu} G_{\sigma\delta,\lambda}{}^{\lambda} - \delta_{\sigma(\mu} G_{\nu)\delta,\lambda}{}^{\lambda} , \quad G_{\mu\nu,\gamma\delta} , \\ &G_{\mu\beta,\nu\delta} = G_{\beta[\mu,\nu]\delta} - \frac{1}{2} G_{\mu\nu,\beta\delta} , \qquad\qquad G_{\mu\beta,\gamma\delta} , \quad G_{\alpha\beta,\gamma\delta} , \end{aligned} \tag{E.9}$$

but for simplicity we shall only consider the components $G_{\mu\nu,\sigma\rho}$, $G_{\mu\nu,\gamma\delta}$ and $G_{\alpha\beta,\gamma\delta}$ that admit a non-trivial constant term. The differential operator $D_{(\mu}{}^{\hat{c}}D_{\nu)\hat{c}}G_{ab,cd}$ acts diagonally on the various components of fixed number of indices along the Grassmanian, so it is consistant to only consider the components with an even number of indices along the sub-Grassmaniann in the differential equation (3.20). Using the Fourier decompositions

$$\begin{aligned} G_{ab,cd} &= \sum_{\Gamma\in\Lambda^*\oplus\Lambda} G^{\Gamma}_{ab,cd}e^{2\pi \mathrm{i}(\Gamma,a)} + \sum_{n\neq 0} G^{\mathrm{TN}\,n}_{ab,cd}e^{2\pi \mathrm{i}n\psi} \ , \\ F_{abcd} &= \sum_{\Gamma\in\Lambda^*\oplus\Lambda} F^{\Gamma}_{abcd}e^{2\pi \mathrm{i}(\Gamma,a)} + \sum_{n\neq 0} F^{\mathrm{TN}\,n}_{abcd}e^{2\pi \mathrm{i}n\psi} \ , \end{aligned} \tag{E.10}$$

one obtains from (3.20)

$$\begin{aligned} &\Big(2\mathcal{D}_{(\mu}{}^{\tau}\mathcal{D}_{\nu)\tau} - (\partial_\phi + q - 2)\mathcal{D}_{\mu\nu} + \tfrac{1}{8}(\partial_\phi+8)(\partial_\phi+2q-10)\delta_{\mu\nu} - 4\pi e^{-2\phi}\Gamma_{R\mu}\cdot\Gamma_{R\nu}\Big)G^{\Gamma}_{\sigma\rho,\kappa\lambda} \\ =\ & -\frac{3\pi}{4}\delta_{\langle\sigma\rho,}\delta_{\kappa\lambda\rangle}\sum_{\Gamma_1\in\Lambda^*\oplus\Lambda}\big(F^{\Gamma_1}_{\kappa d(\mu}{}^{\kappa}F^{\Gamma-\Gamma_1\,\lambda d}_{\nu)\lambda} - F^{\Gamma_1}_{\kappa d(\mu}{}^{\lambda}F^{\Gamma-\Gamma_1\,\kappa d}_{\nu)\lambda}\big) - 3\pi F^{\Gamma}_{\mu\nu,\sigma\rho,\kappa\lambda}\ , \end{aligned} \tag{E.11}$$

$$\begin{aligned} &\Big(2\mathcal{D}_{(\mu}{}^{\tau}\mathcal{D}_{\nu)\tau} - (\partial_\phi + q - 2)\mathcal{D}_{\mu\nu} + \tfrac{1}{8}(\partial_\phi+6)(\partial_\phi+2q-8)\delta_{\mu\nu} - 4\pi e^{-2\phi}\Gamma_{R\mu}\cdot\Gamma_{R\nu}\Big)G^{\Gamma}_{\sigma\rho,\gamma\delta} \\ =\ & \frac{1}{2}\delta_{\mu\nu}G^{\Gamma}_{\sigma\rho,\gamma\delta} + \frac{8-q}{2}\delta_{\sigma\rho}G^{\Gamma}_{\mu\nu\gamma\delta} + \frac{6-q}{2}\delta_{\sigma(\mu}\delta_{\nu)\rho}G^{\Gamma}_{\alpha\beta,\lambda}{}^{\lambda} + \frac{2q-13}{2}\delta_{\mu\nu}\delta_{\sigma\rho}G^{\Gamma}_{\alpha\beta,\lambda}{}^{\lambda} \\ &+\mathcal{D}_{(\mu}{}^{\lambda}\delta_{\nu)(\sigma}G^{\Gamma}_{\rho)\lambda,\alpha\beta} - \mathcal{D}_{\sigma)(\mu}G^{\Gamma}_{\nu)(\rho|,\alpha\beta} + \delta_{\alpha\beta}G^{\Gamma}_{\mu\nu,\sigma\rho} \\ &-2\pi\sum_{\Gamma_1\in\Lambda^*\oplus\Lambda}\big(F^{\Gamma_1}_{\sigma\rho d(\mu}{}^{\kappa}F^{\Gamma-\Gamma_1\,d}_{\nu)\gamma\delta} - F^{\Gamma_1}_{\sigma)\gamma d(\mu}F^{\Gamma-\Gamma_1\,d}_{\nu)\delta(\rho}\big) - 3\pi F^{\Gamma}_{\mu\nu,\sigma\rho,\gamma\delta}\ , \end{aligned} \tag{E.12}$$

and

$$\begin{aligned} &\Big(2\mathcal{D}_{(\mu}{}^{\tau}\mathcal{D}_{\nu)\tau} - (\partial_\phi + q - 2)\mathcal{D}_{\mu\nu} + \tfrac{1}{8}(\partial_\phi+4)(\partial_\phi+2q-6)\delta_{\mu\nu} - 4\pi e^{-2\phi}\Gamma_{R\mu}\cdot\Gamma_{R\nu}\Big)G^{\Gamma}_{\alpha\beta,\gamma\delta} \\ =\ & 3\delta_{\langle\alpha\beta,}G^{\Gamma}_{\gamma\delta\rangle,\mu\nu} - 3\pi\sum_{\Gamma_1\in\Lambda^*\oplus\Lambda}F^{\Gamma_1}_{\mu)d\langle\alpha\beta,}F^{\Gamma-\Gamma_1\,d}_{\gamma\delta\rangle(\nu} - 3\pi F^{\Gamma}_{\mu\nu,\alpha\beta,\gamma\delta}\ , \end{aligned} \tag{E.13}$$

where the additional term of order $\mathcal{O}(e^{-R^2})$ comes from the Abelian Fourier coefficients of the quadratic source in F_{abcd} involving nonzero Taub-NUT charge,

$$\sum_{n\neq 0} F^{\mathrm{TN}\,n}_{eg\langle ab,}F^{\mathrm{TN}\,-n\,g}_{cd\rangle f} = \sum_{\Gamma\in\Lambda^*\oplus\Lambda} F^{\Gamma}_{ef,ab,cd}e^{2\pi\mathrm{i}(\Gamma,a)}\ . \tag{E.14}$$

It is a non-trivial task to compute these Fourier coefficients from the explicit non-Abelian Fourier coefficients of the tensor F_{abcd}, which we shall attempt to carry out in this paper.

Introducing for brevity the vector $\vec{G}_\Gamma$

$$\vec{G}_\Gamma = (G^{\Gamma}_{\rho\sigma,\tau\upsilon}, G^{\Gamma}_{\rho\sigma,\gamma\delta}, G^{\Gamma}_{\alpha\beta,\gamma\delta})\,, \tag{E.15}$$

we find that the differential operator with two indices along the sub-Grassmaniann acts on

$\vec{G}_\Gamma$ according to

$$\begin{aligned}
4D_{(\eta}{}^{\hat\eta}D_{\theta)\hat\eta}\vec{G}_\Gamma &= (4\mathcal{D}_{(\eta}{}^{\hat\eta}\mathcal{D}_{\theta)\hat\eta} + \delta_{\eta\theta}\partial_\phi - 8\pi^2 e^{-2\phi}\Gamma_{L\eta}{}^{\kappa}\Gamma_{L\theta\kappa})\vec{G}_\Gamma \\
&\quad + 8\mathrm{i}\pi\sqrt{2}e^{-\phi}\Gamma_{L(\eta|}{}^{\kappa}\begin{pmatrix} -\delta_{\kappa(\rho}G^\Gamma_{\sigma)|\theta),\tau\upsilon} - \delta_{\kappa(\tau}G^\Gamma_{\upsilon)|\theta),\rho\sigma} \\ -\delta_{\kappa(\rho}G^\Gamma_{\sigma)|\theta),\gamma\delta} + \delta_{\theta)(\gamma}G^\Gamma_{\delta)\kappa,\rho\sigma} \\ \delta_{\theta)(\alpha}G^\Gamma_{\beta)\kappa,\gamma\delta} + \delta_{\theta)(\gamma}G^\Gamma_{\delta)\kappa,\alpha\beta}\end{pmatrix}, \\
&\quad + \begin{pmatrix} 6\delta_{\langle\rho\sigma,}G^\Gamma_{\tau\upsilon\rangle,\gamma\delta} - 4\delta_{\eta\theta}G^\Gamma_{\rho\sigma,\tau\upsilon} \\ 2\delta_{\rho\sigma}G^\Gamma_{\eta\theta,\gamma\delta} - 2\delta_{\eta\theta}G^\Gamma_{\rho\sigma,\gamma\delta} + 2\delta_{\eta(\gamma}\delta_{\delta)\theta}G^\Gamma_{\rho\sigma,\kappa}{}^{\kappa} \\ 6\delta_{\eta\langle(\alpha}\delta_{\beta),|\theta|}G^\Gamma_{\gamma\delta\rangle,\kappa}{}^{\kappa} - 8\delta_{(\eta|\langle(\alpha}G^\Gamma_{\beta),|\theta),|\gamma\delta\rangle}\end{pmatrix},
\end{aligned} \tag{E.16}$$

where the term linear in Γ involves the components of G^Γ with an odd number of indices along the sub-Grassmaniann. Using this action, we find the differential equation obeyed by the components with two indices along the sub-Grassmaniann $G_{p-2,q-2}$,

$$\begin{aligned}
&\big(2\mathcal{D}_{(\eta}{}^{\hat\alpha}\mathcal{D}_{\alpha)\hat\zeta} + \tfrac{1}{2}\delta_{\eta\theta}\partial_\phi - 4\pi^2 e^{-2\phi}\Gamma_{L\eta}{}^{\kappa}\Gamma_{L\theta\kappa}\big)\vec{G}_\Gamma \\
&\quad + 4\mathrm{i}\pi\sqrt{2}e^{-\phi}\Gamma_{L(\eta|}{}^{\kappa}\begin{pmatrix} -\delta_{\kappa(\rho}G^\Gamma_{\sigma)|\theta),\tau\upsilon} - \delta_{\kappa(\tau}G^\Gamma_{\upsilon)|\theta),\rho\sigma} \\ -\delta_{\kappa(\rho}G^\Gamma_{\sigma)|\theta),\gamma\delta} + \delta_{\theta)(\gamma}G^\Gamma_{\delta)\kappa,\rho\sigma} \\ \delta_{|\theta)(\alpha}G^\Gamma_{\beta)\kappa,\gamma\delta} + \delta_{\theta)(\gamma}G^\Gamma_{\delta)\kappa,\alpha\beta}\end{pmatrix} \\
&= \begin{pmatrix} (5-q)\delta_{\eta\theta}G^\Gamma_{\rho\sigma,\tau\upsilon} \\ (4-q)\delta_{\eta\theta}G^\Gamma_{\rho\sigma,\gamma\delta} + (6-q)\delta_{|\eta)(\gamma}G^\Gamma_{\delta)(\theta|,\rho\sigma} + \delta_{\gamma\delta}G^\Gamma_{\eta\theta,\rho\sigma} - \delta_{\theta(\gamma}\delta_{\delta)\eta}G^\Gamma_{\rho\sigma,\lambda}{}^{\lambda} \\ (3-q)\delta_{\eta\theta}G^\Gamma_{\alpha\beta,\gamma\delta} + 2(8-q)\delta_{|\eta)\langle(\alpha}G^\Gamma_{\beta),(\theta|,|\gamma\delta\rangle} + 3\delta_{\langle\alpha\beta,}G^\Gamma_{\gamma\delta\rangle,\eta\theta} - 3\delta_{\eta\langle(\alpha}\delta_{\beta),|\theta|}G^\Gamma_{\gamma\delta\rangle,\kappa}{}^{\kappa}\end{pmatrix} \\
&\quad - \pi\sum_{\Gamma_1\in\Lambda^*\oplus\Lambda}\begin{pmatrix} 3F^{\Gamma_1}_{\eta d\langle\rho\sigma,}F^{\Gamma-\Gamma_1\,d}_{\tau\upsilon\rangle\theta} \\ 2F^{\Gamma_1}_{\rho\sigma d(\eta}{}^{\kappa}F^{\Gamma-\Gamma_1\,d}_{\theta)\gamma\delta} - 2F^{\Gamma_1}_{\sigma)\gamma d(\eta}F^{\Gamma-\Gamma_1\,d}_{\theta)\delta(\rho} \\ 3F^{\Gamma_1}_{\eta d\langle\alpha\beta,}F^{\Gamma-\Gamma_1\,d}_{\gamma\delta\rangle\theta}\end{pmatrix} - 3\pi\vec{F}_{\eta\theta\,\Gamma}\,.
\end{aligned} \tag{E.17}$$

E.2 Zero mode equations

In this subsection we analyze the consistency of the differential equations (E.11), (E.12), (E.13), (E.17) with the results in §5 for the constant term $G^0_{ab,cd}$. As mentioned earlier, the unfolding method fails to capture exponentially suppressed corrections to the constant term, which are sourced by the quadratic terms $\sum_{\Gamma_1\neq 0}F^{\Gamma_1}F^{-\Gamma_1}$ and $F^0_{ef,ab,cd}$ defined in (E.14) on the right-hand side of the differential equation, and can be ascribed to instanton anti-instanton configurations. These terms can in principle be computed by solving the differential equation. Here we concentrate on the perturbative part of $G^0_{ab,cd}$, which is sourced by the square of the perturbative part of F_{abcd}.The latter is given by [22, (5.29)]

$$\begin{aligned}
F^0_{\mu\nu\rho\sigma} &= 4e^{(6-q)\phi}\Big(\mathcal{D}_{(\mu\nu}\mathcal{D}_{\rho\sigma)} + \tfrac{q-10}{2}\delta_{(\mu\nu}\mathcal{D}_{\sigma\rho)} + \tfrac{(8-q)(12-q)}{16}\delta_{(\mu\nu}\delta_{\rho\sigma)}\Big)\mathcal{E}(S)\,, \\
F^0_{\mu\nu\gamma\delta} &= e^{(6-q)\phi}\delta_{\gamma\delta}\Big(\tfrac{8-q}{2}\delta_{\mu\nu} - 2\mathcal{D}_{\mu\nu}\Big)\mathcal{E}(S)\,, \\
F^0_{\alpha\beta\gamma\delta} &= e^{-2\phi}\mathcal{F}_{\alpha\beta\gamma\delta}(\varphi) + 3e^{(6-q)\phi}\delta_{(\alpha\beta}\delta_{\gamma\delta)}\mathcal{E}(S)\,,
\end{aligned} \tag{E.18}$$

where

$$\mathcal{E}(S) = \frac{3}{(N+1)\pi^2}\big(\mathcal{E}^\star(\tfrac{8-q}{2},S) + \upsilon N^{\frac{8-q}{2}}\mathcal{E}^\star(\tfrac{8-q}{2},NS)\big) \tag{E.19}$$

is a specific solution of the Laplace equation

$$2\mathcal{D}_{\rho\sigma}\mathcal{D}^{\rho\sigma}\mathcal{E}(S) = \frac{1}{2}(D_{-2}\bar{D}_0 + \bar{D}_2 D_0)\mathcal{E}(S) = \frac{(8-q)(6-q)}{4}\mathcal{E}(S) \ . \tag{E.20}$$

It is then straightforward to find a particular solution to Eq. (E.17)

$$\begin{aligned}
G^0_{\mu\nu,\rho\sigma} &= -3\pi e^{2(6-q)\phi}\delta_{\langle\mu\nu,}\delta_{\rho\sigma\rangle}\Big(\big(\tfrac{8-q}{2}\big)^2\mathcal{E}(S)^2 - 2\mathcal{D}_{\kappa\lambda}\mathcal{E}(S)\mathcal{D}^{\kappa\lambda}\mathcal{E}(S)\Big) \ ,\\
G^0_{\mu\nu,\gamma\delta} &= -\tfrac{\pi}{6}e^{(4-q)\phi}\Big(\tfrac{8-q}{2}\delta_{\mu\nu} - 2\mathcal{D}_{\mu\nu}\Big)\mathcal{E}(S)\mathcal{G}_{\gamma\delta}(\varphi) - 2\pi e^{2(6-q)\phi}\delta_{\gamma\delta}\mathcal{E}(S)\Big(\tfrac{8-q}{2}\delta_{\mu\nu} - 2\mathcal{D}_{\mu\nu}\Big)\mathcal{E}(S) \ ,\\
G^0_{\alpha\beta,\gamma\delta} &= e^{-4\phi}\mathcal{G}_{\alpha\beta,\gamma\delta}(\varphi) - \tfrac{\pi}{2}e^{(4-q)\phi}\mathcal{E}(S)\delta_{\langle\alpha\beta,}\mathcal{G}_{\gamma\delta\rangle}(\varphi) - 3\pi e^{2(6-q)\phi}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\mathcal{E}(S)^2 \ ,
\end{aligned} \tag{E.21}$$

with $\mathcal{G}_{\alpha\beta,\gamma\delta}(\varphi)$ solution to an equation analogue to (3.20) with source term quadratic in $\mathcal{F}_{\alpha\beta\gamma\delta}(\varphi)$, and $\mathcal{G}_{\alpha\beta}(\varphi)$ solution to the equation on the sub-Grassmaniann $G_{p-2,q-2}$

$$2\mathcal{D}_{(\gamma}{}^{\hat\alpha}\mathcal{D}_{\delta)\hat\alpha}\mathcal{G}_{\alpha\beta} = \tfrac{4-q}{2}\delta_{\gamma\delta}\mathcal{G}_{\alpha\beta} + (6-q)\delta_{\gamma)(\alpha}\mathcal{G}_{\beta)(\delta} + \delta_{\alpha\beta}\mathcal{G}_{\gamma\delta} + 12\mathcal{F}_{\alpha\beta\gamma\delta} \ . \tag{E.22}$$

One can then check that $G^0_{ab,cd}$ is also a solution to (E.11), (E.12) and (E.13), using the identity

$$\begin{aligned}
F^0_{\kappa d(\mu}{}^{\kappa}F^0_{\nu)\lambda}{}^{\lambda d} - F^0_{\kappa d(\mu}{}^{\lambda}F^0_{\nu)\lambda}{}^{\kappa d} &= 2(8-q)^2\Big(\Big(\frac{8-q}{2}\Big)^2 + 1\Big)\delta_{\mu\nu}\mathcal{E}^2 - (8-q)^2(10-q)\mathcal{E}\mathcal{D}_{\mu\nu}\mathcal{E}\\
&\quad + 8(10-q)\mathcal{D}_{\mu\nu}\mathcal{D}_{\rho\sigma}\mathcal{E}\mathcal{D}^{\rho\sigma}\mathcal{E} - 16\mathcal{D}_{\mu}{}^{\lambda}\mathcal{D}_{\rho\sigma}\mathcal{E}\mathcal{D}_{\nu\lambda}\mathcal{D}^{\rho\sigma}\mathcal{E}
\end{aligned} \tag{E.23}$$

and the fact that for any two symmetric tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$, one has

$$X_{\langle\mu\nu,}Y_{\rho\sigma\rangle} = \tfrac{1}{2}\delta_{\langle\mu\nu,}\delta_{\rho\sigma\rangle}(X_\lambda{}^\lambda Y_\kappa{}^\kappa - X^{\kappa\lambda}Y_{\kappa\lambda}) \ . \tag{E.24}$$

The most general solution is obtained by adding a solution of the homogeneous equation without source term, given by

$$\begin{aligned}
\tilde G^0_{\mu\nu,\rho\sigma} &= \tfrac{(6-q)(7-q)}{2}c\, e^{2(5-q)\phi}\delta_{\langle\mu\nu,}\delta_{\rho\sigma\rangle} \ ,\\
\tilde G^0_{\mu\nu,\gamma\delta} &= -\tfrac{\pi}{6}e^{(4-q)\phi}\Big(\tfrac{8-q}{2}\delta_{\mu\nu} - 2\mathcal{D}_{\mu\nu}\Big)\tilde{\mathcal{E}}(S)\tilde{\mathcal{G}}_{\gamma\delta}(\varphi) + \tfrac{7-q}{3}c\, e^{2(5-q)\phi}\delta_{\mu\nu}\delta_{\gamma\delta} \ ,\\
\tilde G^0_{\alpha\beta,\gamma\delta} &= e^{-4\phi}\tilde{\mathcal{G}}_{\alpha\beta,\gamma\delta}(\varphi) - \tfrac{\pi}{2}e^{(4-q)\phi}\tilde{\mathcal{E}}(S)\delta_{\langle\alpha\beta,}\tilde{\mathcal{G}}_{\gamma\delta\rangle}(\varphi) + c\, e^{2(5-q)\phi}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle} \ ,
\end{aligned} \tag{E.25}$$

with c a numerical constant, $\tilde{\mathcal{G}}_{\alpha\beta,\gamma\delta}(\varphi)$ a solution to the homogeneous equation (3.17) on the sub-Grassmanian, $\tilde{\mathcal{E}}$ a solution to (E.20) and $\tilde{\mathcal{G}}_{\alpha\beta}(\varphi)$ solution to the homogeneous equation (3.34) on the sub-Grassmanian. The explicit results (5.44), (5.60) for the constant term $G^0_{ab,cd}$ obtained by unfolding method for generic values of q indeed lie in this class, upon setting

$$\begin{aligned}
\tilde{\mathcal{E}}(S) &= \frac{3}{(N-1)\pi^2}\big(-\mathcal{E}^\star(\tfrac{8-q}{2},S) + \upsilon N^{\frac{8-q}{2}}\mathcal{E}^\star(\tfrac{8-q}{2},NS)\big) \ ,\\
\mathcal{G}_{\alpha\beta}(\varphi) &= \tfrac{1}{2}\big(G^{(p-2,q-2)}_{\alpha\beta}(\varphi) + {}^{\varsigma}G^{(p-2,q-2)}_{\alpha\beta}(\varphi)\big) \ ,\\
\tilde{\mathcal{G}}_{\alpha\beta}(\varphi) &= \tfrac{1}{2}\big(G^{(p-2,q-2)}_{\alpha\beta}(\varphi) - {}^{\varsigma}G^{(p-2,q-2)}_{\alpha\beta}(\varphi)\big) \ ,\\
c &= \frac{18}{\pi^2}\xi(7-q)\xi(6-q)\frac{(N-\upsilon)(1-\upsilon N^{q-7})}{N^2-1} \ .
\end{aligned} \tag{E.26}$$

For special values of q one must take into account additional source terms due to logarithmic divergences. For example for $q=8$, one has instead

$$
\begin{aligned}
F^0_{\mu\nu\rho\sigma} &= e^{-2\phi}\Big(4(\mathcal{D}_{(\mu\nu}\mathcal{D}_{\rho\sigma)} - \delta_{(\mu\nu}\mathcal{D}_{\sigma\rho)})\hat{\mathcal{E}}(S) - 2\kappa\delta_{(\mu\nu}\delta_{\rho\sigma)}\Big)\ ,\\
F^0_{\mu\nu\gamma\delta} &= -e^{-2\phi}\delta_{\gamma\delta}\left(\kappa\delta_{\mu\nu} + 2\mathcal{D}_{\mu\nu}\hat{\mathcal{E}}(S)\right)\ ,\\
F^0_{\alpha\beta\gamma\delta} &= e^{-2\phi}\big(\hat{\mathcal{F}}_{\alpha\beta\gamma\delta}(\varphi) + 3\delta_{(\alpha\beta}\delta_{\gamma\delta)}(\hat{\mathcal{E}}(S) - 2\kappa\phi)\big)\ ,
\end{aligned}
\tag{E.27}
$$

where $\hat{\mathcal{E}}(S) = \frac{1}{2\pi(N+1)}\big(\hat{\mathcal{E}}_1(S) + \hat{\mathcal{E}}_1(NS)\big)$ satisfies a Poisson equation with a constant source term,

$$
2\mathcal{D}_{\rho\sigma}\mathcal{D}^{\rho\sigma}\hat{\mathcal{E}}(S) = \frac{1}{2}(D_{-2}\bar{D}_0 + \bar{D}_2 D_0)\hat{\mathcal{E}}(S) = \kappa\ . \tag{E.28}
$$

One finds the particular solution to E. (E.17),

$$
\begin{aligned}
G^0_{\mu\nu,\rho\sigma} &= -3\pi e^{-4\phi}\delta_{\langle\mu\nu,}\delta_{\rho\sigma\rangle}\big(\kappa^2 - 2\mathcal{D}_{\kappa\lambda}\hat{\mathcal{E}}(S)\mathcal{D}^{\kappa\lambda}\hat{\mathcal{E}}(S)\big)\ , \qquad (E.29)\\
G^0_{\mu\nu,\gamma\delta} &= e^{-4\phi}\Big(\kappa\delta_{\mu\nu} + 2\mathcal{D}_{\mu\nu}\hat{\mathcal{E}}(S)\Big)\Big(\tfrac{\pi}{6}\hat{\mathcal{G}}_{\gamma\delta}(\varphi) + 2\pi\delta_{\gamma\delta}(\hat{\mathcal{E}}(S) - 2\kappa\phi)\Big)\ ,\\
G^0_{\alpha\beta,\gamma\delta} &= e^{-4\phi}\Big(\hat{\mathcal{G}}_{\alpha\beta,\gamma\delta}(\varphi) - \tfrac{\pi}{2}(\hat{\mathcal{E}}(S) - 2\kappa\phi)\delta_{\langle\alpha\beta,}\hat{\mathcal{G}}_{\gamma\delta\rangle}(\varphi) - 3\pi\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\big(\hat{\mathcal{E}}(S) - 2\kappa\phi\big)^2\Big)\ ,
\end{aligned}
$$

with

$$
\begin{aligned}
2\mathcal{D}_{(\gamma}{}^{\hat{\alpha}}\mathcal{D}_{\delta)\hat{\alpha}}\hat{\mathcal{G}}_{\alpha\beta} &= -2\delta_{\gamma\delta}\hat{\mathcal{G}}_{\alpha\beta} - 2\delta_{\gamma)(\alpha}\hat{\mathcal{G}}_{\beta)(\delta} + \delta_{\alpha\beta}\hat{\mathcal{G}}_{\gamma\delta} + 12\hat{\mathcal{F}}_{\alpha\beta\gamma\delta} + 36\kappa\delta_{(\alpha\beta}\delta_{\gamma\delta)}\ ,\\
2\mathcal{D}_{(\eta}{}^{\hat{\alpha}}\mathcal{D}_{\theta)\hat{\alpha}}\hat{\mathcal{G}}_{\alpha\beta,\gamma\delta} &= -3\delta_{\eta\theta}\hat{\mathcal{G}}_{\alpha\beta,\gamma\delta} + 3\delta_{\langle\alpha\beta,}\hat{\mathcal{G}}_{\gamma\delta\rangle,\eta\theta} - \tfrac{\kappa}{2}(\delta_{\eta\theta}\delta_{\langle\alpha\beta,}\hat{\mathcal{G}}_{\gamma\delta\rangle} + 2\delta_{\eta\langle\alpha}\delta_{\beta,|\theta|}\hat{\mathcal{G}}_{\gamma\delta\rangle})\\
&\quad -3\pi\hat{\mathcal{F}}_{\eta)\epsilon\langle\alpha\beta,}\hat{\mathcal{F}}_{\gamma\delta\rangle(\theta}{}^{\epsilon}\ .
\end{aligned}
\tag{E.30}
$$

This is indeed consistent with the result (5.70) from the unfolding method, upon setting $\kappa = \frac{3}{\pi^2(N+1)} = \frac{k}{8\pi^2}$.

E.3 Abelian Fourier coefficients

In this subsection we show that the generic Abelian Fourier coefficients of the tensor $G_{ab,cd}$ computed in section 5 satisfy the differential equation (E.11), including the quadratic source term.

For simplicity we shall only consider the component of the Fourier coefficient with all indices along the decompactified torus $G^{(p,q),\,2\mathrm{Ab},\,(Q,P)}_{\mu\nu,\sigma\rho} = -\frac{1}{2}\varepsilon_{\mu(\sigma}\varepsilon_{\rho)\nu}G^{(p,q)}(Q,P)$. The latter is proportional to the scalar function

$$
G^{(p,q)}(Q,P) = R^8 \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda^{\oplus 2}_{p-2,q-2}}} \int_{\mathcal{P}_2} \frac{\mathrm{d}^3\Omega_2}{|\Omega_2|^{\frac{12-q}{2}}}|A|^2 C\big[A^{-1}\big(\begin{smallmatrix}Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\big)A^{-\intercal};\Omega_2\big]L(A^{-\intercal}\Omega_2A^{-1}) \tag{E.31}
$$

with

$$
L(A^{-\intercal}\Omega_2A^{-1}) = e^{-\pi R^2\,\mathrm{tr}\left[vA\Omega_2{}^{-1}A^\intercal v^\intercal\right] - 2\pi\,\mathrm{tr}\left[\Omega_2A^{-1}\left(\begin{smallmatrix}Q_R^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & P_R^2\end{smallmatrix}\right)A^{-\intercal}\right]}\ . \tag{E.32}
$$

One can rewrite the differential operator in (E.11) as

$$
\begin{aligned}
&\Big(\mathcal{D}_\mu{}^{\hat{a}}\mathcal{D}_{\nu\hat{a}} + (q-5)\delta_{\mu\nu}\Big)G^{(p,q)}(Q,P)e^{2\pi\mathrm{i}(Qa^1+Pa^2)}\\
&= \Big(\big(\tfrac{1}{16}(-R\partial_R)^2 - \tfrac{q-1}{8}R\partial_R + q - 5\big)\delta_{\mu\nu} - \tfrac{1}{2}\mathcal{D}_{\mu\nu}(-R\partial_R + q - 2) + \mathcal{D}_{(\mu}{}^{\sigma}\mathcal{D}_{\nu)\sigma}\\
&\qquad - 2\pi^2R^2\big(v\big(\begin{smallmatrix}Q_R^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & P_R^2\end{smallmatrix}\big)v^\intercal\big)\Big)G^{(p,q)}(Q,P)e^{2\pi\mathrm{i}(Qa^1+Pa^2)}\ .
\end{aligned}
\tag{E.33}
$$

Acting with this differential operator on $R^8 L(A^{-\intercal}\Omega_2 A^{-1})$ one obtains

$$\Big(\pi R^2 (vA\Omega_2^{-1}A^\intercal v^\intercal)^2 + \tfrac{q-12}{2} vA\Omega_2^{-1}A^\intercal v^\intercal - 2\pi v\left(\begin{smallmatrix} Q_R^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & P_R^2\end{smallmatrix}\right) v^\intercal\Big)_{\mu\nu} \pi R^{10}|\Omega_2|^{\frac{q-12}{2}} L(A^{-\intercal}\Omega_2 A^{-1})\ , \tag{E.34}$$

which allows to rewrite (E.34) as a total derivative in Ω_2,

$$\begin{aligned}\Big(\mathcal{D}_\mu{}^{\hat a}\mathcal{D}_{\nu\hat a} + (q-5)\delta_{\mu\nu}\Big) R^8|\Omega_2|^{\frac{q-12}{2}} L(A^{-\intercal}\Omega_2A^{-1}) e^{2\pi \mathrm{i}(Qa^1+Pa^2)}\\ = (\pi R^2 vA\frac{\partial}{\partial\Omega_2}A^\intercal v^\intercal)_{\mu\nu} R^8 |\Omega_2|^{\frac{q-12}{2}} L(A^{-\intercal}\Omega_2 A^{-1}) e^{2\pi\mathrm{i}(Qa^1+Pa^2)}\ .\end{aligned} \tag{E.35}$$

By integration by parts, it follows that the Fourier coefficient would satisfy the homogeneous differential equation *if* the Fourier coefficient $C\big[A^{-1}\left(\begin{smallmatrix} Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal};\Omega_2\big]$ did not depend on Ω_2.

We shall now show that the dependence of the Fourier coefficients of $1/\Phi_{10}$ in Ω_2, due to the poles at large $|\Omega_2|$ accounts for the appearance of the quadratic source term in the differential equation. Using (5.25) and (A.90), we obtain

$$\begin{aligned} G^{(p,q)}(Q,P) &= R^8 \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda^{\oplus 2}_{p-2,q-2}}} \int_{\mathcal{P}_2} \frac{d^3\Omega_2}{|\Omega_2|^{\frac{12-q}{2}}} |A|^2 C^F\big[A^{-1}\left(\begin{smallmatrix} Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal}\big] L(A^{-\intercal}\Omega_2A^{-1})\\ &\quad + \frac{R^8}{2}\sum_{\substack{A\in M_2(\mathbb{Z})/\mathrm{Dih}_4\\ A^{-1}\binom{Q}{P}\in\Lambda^{\oplus 2}_{p-2,q-2}}} |A|^2 c\Big(-\frac{(A^{-1}\left(\begin{smallmatrix} Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{11}}{2}\Big) c\Big(-\frac{(A^{-1}\left(\begin{smallmatrix} Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{22}}{2}\Big)\\ &\times\int_{\mathcal{P}_2}\frac{d^3\Omega_2}{|\Omega_2|^{\frac{12-q}{2}}}\Big(-\frac{1}{2\pi}\delta(v_2) - (A^{-1}\left(\begin{smallmatrix} Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{12}\mathrm{sign}(v_2) + \big|(A^{-1}\left(\begin{smallmatrix} Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{12}\big|\Big) L(A^{-\intercal}\Omega_2A^{-1})\\ &\quad +\mathcal{O}(e^{-R^2})\ .\end{aligned} \tag{E.36}$$

The differential operator (E.35) annihilates the finite part of the Fourier coefficient, and gives

$$\begin{aligned} &\Big(\mathcal{D}_\mu{}^{\hat a}\mathcal{D}_{\nu\hat a} + (q-5)\delta_{\mu\nu}\Big) G^{(p,q)}(Q,P) e^{2\pi\mathrm{i}(Qa^1+Pa^2)}\\ &= -\frac{1}{2}\sum_{\substack{A\in M_2(\mathbb{Z})/\mathrm{Dih}_4\\ A^{-1}\binom{Q}{P}\in\Lambda^{\oplus 2}_{p-2,q-2}}} |A|^2 c\Big(-\frac{(A^{-1}\left(\begin{smallmatrix} Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{11}}{2}\Big) c\Big(-\frac{(A^{-1}\left(\begin{smallmatrix} Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{22}}{2}\Big)\\ &\quad\times\left(\frac{1}{2\pi}\Big(\mathcal{D}_\mu{}^{\hat a}\mathcal{D}_{\nu\hat a} + (d-5)\delta_{\mu\nu}\Big) - 2\pi R^2\Big(vA\pi_{(1}A^{-1}\begin{pmatrix} Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{pmatrix}A^{-\intercal}\pi_{2)}A^\intercal v^\intercal\Big)_{\mu\nu}\right)\\ &\qquad\times\int_0^\infty\frac{d\rho_2}{\rho_2^{\frac{12-q}{2}}}\int_0^\infty\frac{d\sigma_2}{\sigma_2^{\frac{12-q}{2}}} R^8 L(A^{-\intercal}\left(\begin{smallmatrix}\rho_2 & 0\\ 0 & \sigma_2\end{smallmatrix}\right)A^{-1}) e^{2\pi\mathrm{i}(Qa^1+Pa^2)} + \mathcal{O}(e^{-R^2})\end{aligned} \tag{E.37}$$

where the differential operator acting on the first term in (E.36) gives a total derivative, while the second term factorizes after integrating the Dirac delta function, and the third is integrated by part using $\frac{d}{d\Omega_2}\mathrm{sign}(\mathrm{tr}\left(\begin{smallmatrix} 0 & 1/2\\ 1/2 & 0\end{smallmatrix}\right)\Omega_2) = \delta(v_2)\left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)$ and

$$vA\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}A^\intercal v^\intercal\Big[A^{-1}\begin{pmatrix} Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{pmatrix}A^{-\intercal}\Big]_{12} = 2vA\pi_{(1}A^{-1}\begin{pmatrix} Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{pmatrix}A^{-\intercal}\pi_{2)}A^\intercal v^\intercal\ . \tag{E.38}$$

Further using (E.34) to express $\Big(\mathcal{D}_\mu{}^{\hat a}\mathcal{D}_{\nu\hat a}+(d-5)\delta_{\mu\nu}\Big)L(A^{-\intercal}\left(\begin{smallmatrix}\rho_2 & 0\\ 0 & \sigma_2\end{smallmatrix}\right)A^{-1})e^{2\pi\mathrm{i}(Qa^1+Pa^2)}$, and inserting $\pi_1+\pi_2=\mathbb{1}$ on both sides of (E.35), we see that the terms which involve two powers of π_1 or two powers of π_2 cancel out since they are total derivatives with respect to ρ_2 or to σ_2, leaving only terms involving one factor of π_1 and one factor of π_2:

$$\begin{aligned}
&\Big(\mathcal{D}_\mu{}^{\hat a}\mathcal{D}_{\nu\hat a}+(q-5)\delta_{\mu\nu}\Big)G^{(p,q)}(Q,P)e^{2\pi\mathrm{i}(Qa^1+Pa^2)}\\
=&-\frac{\pi}{2}\sum_{\substack{A\in M_2(\mathbb{Z})/\mathrm{Dih}_4\\ A^{-1}\left(\begin{smallmatrix}Q\\P\end{smallmatrix}\right)\in\Lambda^{\oplus 2}_{p-2,q-2}}}|A|^2 c\Big(-\frac{(A^{-1}\left(\begin{smallmatrix}Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{11}}{2}\Big)c\Big(-\frac{(A^{-1}\left(\begin{smallmatrix}Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{22}}{2}\Big)\int_0^\infty\frac{\mathrm{d}\rho_2}{\rho_2^{\frac{12-q}{2}}}\int_0^\infty\frac{\mathrm{d}\sigma_2}{\sigma_2^{\frac{12-q}{2}}}\\
&\times\Big(vA\pi_{(1}\Big(\frac{R^2}{\sigma_2\rho_2}A^\intercal v^\intercal vA-2A^{-1}\left(\begin{smallmatrix}Q_R^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & P_R^2\end{smallmatrix}\right)A^{-\intercal}-2A^{-1}\left(\begin{smallmatrix}Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal}\Big)\pi_{2)}A^\intercal v^\intercal\Big)_{\mu\nu}\\
&\qquad\times R^{10}L(A^{-\intercal}\left(\begin{smallmatrix}\rho_2 & 0\\ 0 & \sigma_2\end{smallmatrix}\right)A^{-1})e^{2\pi\mathrm{i}(Qa^1+Pa^2)}+\mathcal{O}(e^{-R^2})\\
=&-\frac{\pi}{2}\sum_{\substack{A\in M_2(\mathbb{Z})/\mathrm{Dih}_4\\ A^{-1}\left(\begin{smallmatrix}Q\\P\end{smallmatrix}\right)\in\Lambda^{\oplus 2}_{p-2,q-2}}}|A|^2 c\Big(-\frac{(A^{-1}\left(\begin{smallmatrix}Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{11}}{2}\Big)c\Big(-\frac{(A^{-1}\left(\begin{smallmatrix}Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{22}}{2}\Big)\\
&\times\int_0^\infty\frac{\mathrm{d}\rho_2}{\rho_2^{\frac{12-q}{2}}}\int_0^\infty\frac{\mathrm{d}\sigma_2}{\sigma_2^{\frac{12-q}{2}}}\Big(vA\pi_1\Big(\frac{R^2}{\sigma_2\rho_2}A^\intercal v^\intercal vA-2A^{-1}\left(\begin{smallmatrix}Q_L^2 & Q_L\cdot P_L\\ Q_L\cdot P_L & P_L^2\end{smallmatrix}\right)A^{-\intercal}\Big)\pi_2A^\intercal v^\intercal\Big)_{(\mu\nu)}\\
&\qquad\times R^{10}L(A^{-\intercal}\left(\begin{smallmatrix}\rho_2 & 0\\ 0 & \sigma_2\end{smallmatrix}\right)A^{-1})e^{2\pi\mathrm{i}(Qa^1+Pa^2)}+\mathcal{O}(e^{-R^2})\ ,
\end{aligned}\tag{E.39}$$

where in the last step we recognized $Q^2+Q_R^2=Q_L^2$. Defining for $i=1$ or 2 the tensors

$$\begin{aligned}
L^i_{\mu\nu\sigma\rho}(\rho_2)&=R^6(vA)_\mu{}^i(vA)_\nu{}^i(vA)_\sigma{}^i(vA)_\rho{}^ie^{-\frac{\pi}{\rho_2}R^2(vA)_\mu{}^i(vA)^{\mu i}-2\pi\rho_2(A^{-1}\left(\begin{smallmatrix}Q_R^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & P_R^2\end{smallmatrix}\right)A^{-\intercal})_{ii}}\\
L^i_{\mu\nu\sigma\alpha}(\rho_2)&=\mathrm{i}R^5(vA)_\mu{}^i(vA)_\nu{}^i(vA)_\sigma{}^i\big(A^{-1}\left(\begin{smallmatrix}Q_L\\P_L\end{smallmatrix}\right)\big)_{i\alpha}e^{-\frac{\pi}{\rho_2}R^2(vA)_\mu{}^i(vA)^{\mu i}-2\pi\rho_2(A^{-1}\left(\begin{smallmatrix}Q_R^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & P_R^2\end{smallmatrix}\right)A^{-\intercal})_{ii}}
\end{aligned}\tag{E.40}$$

one obtains

$$\begin{aligned}
&\Big(\mathcal{D}_\mu{}^{\hat a}\mathcal{D}_{\nu\hat a}+(d-5)\delta_{\mu\nu}\Big)G^{(p,q)}(Q,P)\,e^{2\pi\mathrm{i}(Qa^1+Pa^2)}+\mathcal{O}(e^{-R^2})\\
=&-\frac{\pi}{2}\sum_{\substack{A\in M_2(\mathbb{Z})/\mathrm{Dih}_4\\ A^{-1}\left(\begin{smallmatrix}Q\\P\end{smallmatrix}\right)\in\Lambda^{\oplus 2}_{p-2,q-2}}}c\Big(-\frac{(A^{-1}\left(\begin{smallmatrix}Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{11}}{2}\Big)c\Big(-\frac{(A^{-1}\left(\begin{smallmatrix}Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{smallmatrix}\right)A^{-\intercal})_{22}}{2}\Big)\varepsilon^{\sigma\kappa}\varepsilon^{\rho\lambda}e^{2\pi\mathrm{i}(Qa^1+Pa^2)}\\
&\Bigg(\int_0^\infty\frac{\mathrm{d}\rho_2}{\rho_2^{\frac{14-q}{2}}}L^1_{\sigma\rho\vartheta(\mu}(\rho_2)\int_0^\infty\frac{\mathrm{d}\sigma_2}{\sigma_2^{\frac{14-q}{2}}}L^2_{\nu)\kappa\lambda}{}^\vartheta(\sigma_2)+2\int_0^\infty\frac{\mathrm{d}\rho_2}{\rho_2^{\frac{12-q}{2}}}L^1_{\sigma\rho(\mu|\alpha}(\rho_2)\int_0^\infty\frac{\mathrm{d}\sigma_2}{\sigma_2^{\frac{12-q}{2}}}L^2_{\nu)\kappa\lambda}{}^\alpha(\sigma_2)\Bigg)\\
=&-2\pi\varepsilon^{\sigma\kappa}\varepsilon^{\rho\lambda}\delta^{ab}e^{2\pi\mathrm{i}(Qa^1+Pa^2)}\sum_{\substack{B\in M_2(\mathbb{Z})/\mathrm{Stab}\\ B\pi_1B^{-1}\left(\begin{smallmatrix}Q\\P\end{smallmatrix}\right)\in\Lambda^{\oplus 2}_{p-2,q-2}}}F^{(p,q),\,B\pi_1B^{-1}\left(\begin{smallmatrix}Q\\P\end{smallmatrix}\right)}_{\sigma\rho(\mu|a}F^{(p,q),\,B\pi_2B^{-1}\left(\begin{smallmatrix}Q\\P\end{smallmatrix}\right)}_{\nu)\kappa\lambda b}
\end{aligned}\tag{E.41}$$

that indeed recovers (3.20),

$$\begin{aligned}
\Big(\mathcal{D}_\mu{}^{\hat a}\mathcal{D}_{\nu\hat a}+(d-5)\delta_{\mu\nu}\Big)G^{(p,q)\Gamma}_{\sigma\rho,\kappa\lambda}&=\frac{\pi}{2}\varepsilon_{\kappa(\sigma}\varepsilon_{\rho)\lambda}\varepsilon^{\vartheta\upsilon}\varepsilon^{\tau\iota}\sum_{\Gamma_1\in\Lambda^*\oplus\Lambda}F^{(p,q)\Gamma_1}_{a\vartheta\tau(\mu}F^{(p,q)\Gamma-\Gamma_1\,a}_{\nu)\upsilon\iota}\\
&=-\frac{3\pi}{2}\sum_{\Gamma_1\in\Lambda^*\oplus\Lambda}F^{(p,q)\Gamma_1}_{a\mu\langle\sigma\rho,}F^{(p,q)\Gamma-\Gamma_1\,a}_{\kappa\lambda\rangle\nu}\ .
\end{aligned}\tag{E.42}$$

Thus, we have shown that the abelian Fourier coefficients with generic 1/4-BPS charge are consistent with the differential constraint (3.20). This is a strong consistency check on the validity of the unfolding method in this sector.

E.4 Differential equation in the degeneration $O(p,q) \to O(p-1,q-1)$

We now briefly discuss the consistency of the constant terms (4.20) computed in §4 with the differential equation (3.20). We follow [22, §B] for the parametrization of the Grassmannian and of the decomposition of the covariant derivative operators.

The operator $\mathcal{D}_{a\hat{b}}$ decomposes into $\mathcal{D}_{1\hat{1}}$, $\mathcal{D}_{1\hat{\beta}}$, $\mathcal{D}_{\alpha\hat{1}}$, $\mathcal{D}_{\alpha\hat{\beta}}$ acting on any vector $F_a = (F_1,\, F_\alpha)$,

$$\begin{aligned}
\mathcal{D}_{1\hat{1}} F_a &= -\frac{1}{2}\frac{\partial}{\partial\phi} F_a \ , \\
\mathcal{D}_{\alpha\hat{1}} F_a &= \frac{1}{\sqrt{2}} e^{-\phi} v^{\text{-}1I}{}_\alpha \frac{\partial}{\partial a^I} F_a + \frac{1}{2}\left(F_\alpha, -\delta_{\alpha\beta} F_1\right) \\
\mathcal{D}_{1\hat{\alpha}} F_a &= \frac{1}{\sqrt{2}} e^{-\phi} v^{\text{-}1I}{}_{\hat{\alpha}} \frac{\partial}{\partial a^I} F_b \ ,
\end{aligned} \tag{E.43}$$

and on any vector $F_{\hat{b}} = (F_{\hat{\beta}},\, F_{\hat{1}})$, as

$$\begin{aligned}
\mathcal{D}_{1\hat{1}} F_{\hat{b}} &= -\frac{1}{2}\frac{\partial}{\partial\phi} F_{\hat{b}} \ , \\
\mathcal{D}_{\alpha\hat{1}} F_{\hat{b}} &= \frac{1}{\sqrt{2}} e^{-\phi} v^{\text{-}1I}{}_\alpha \frac{\partial}{\partial a^I} F_{\hat{b}} \\
\mathcal{D}_{1\hat{\alpha}} F_{\hat{b}} &= \frac{1}{\sqrt{2}} e^{-\phi} v^{\text{-}1I}{}_{\hat{\alpha}} \frac{\partial}{\partial a^I} F_{\hat{b}} + \frac{1}{2}\left(-\delta_{\alpha\beta} F_{\hat{1}}, F_{\hat{\alpha}}\right) \ ,
\end{aligned} \tag{E.44}$$

whereas $\mathcal{D}_{\alpha\hat{\beta}}$ reduce to the differential operators on the sub-Grassmannian that act on the projectors $p^I_{L\,\gamma}$, $p^I_{R\,\hat{\alpha}}$:

$$\mathcal{D}_{\alpha\hat{\beta}} p^I_{L\gamma} = \tfrac{1}{2}\delta_{\alpha\gamma} p^I_{R\hat{\beta}} \ , \qquad \mathcal{D}_{\alpha\hat{\beta}} p^I_{L\hat{\alpha}} = \tfrac{1}{2}\delta_{\hat{\beta}\hat{\alpha}} p^I_{R\alpha} \ . \tag{E.45}$$

In this decomposition, the tensor $G_{ab,cd}$ admits 3 independent components $G_{11,\gamma\delta}$, $G_{1\beta,\gamma\delta}$ and $G_{\alpha\beta,\gamma\delta}$, but only the first and last have a non-trivial constant term. Using the Fourier decomposition

$$G_{ab,cd} = \sum_{Q\in\Lambda^*} G^Q_{ab,cd}\, e^{2\pi \mathrm{i} Q\cdot a} \ , \qquad F_{abcd} = \sum_{Q\in\Lambda^*} F^Q_{abcd}\, e^{2\pi \mathrm{i} Q\cdot a} \ , \tag{E.46}$$

we obtain that the first component of (3.20) with $(e,f)=(1,1)$ reads

$$\begin{aligned}
&\big((\partial_\phi+4)(\partial_\phi+q-5) - 8\pi^2 e^{-2\phi} Q_R^2\big) G^Q_{\alpha\beta,11} = -4\pi \sum_{Q_1\in\Lambda} \Big(F^{Q_1}_{1k\alpha\beta} F^{Q-Q_1\,k}_{111} - F^{Q_1}_{11k(\alpha} F^{Q-Q_1\,k}_{\beta)11} \Big) \\
&\big((\partial_\phi+2)(\partial_\phi+q-3) - 8\pi^2 e^{-2\phi} Q_R^2\big) G^Q_{\alpha\beta,\gamma\delta} = 6\delta_{\langle\alpha\beta,} G^Q_{\gamma\delta\rangle,11} - 6\pi \sum_{Q_1\in\Lambda} F^{Q_1}_{1k\langle\alpha\beta,} F^{Q-Q_1\,k}_{\gamma\delta\rangle 1} \ ,
\end{aligned} \tag{E.47}$$

where the sum over k in the r.h.s. runs over all indices α and 1.

Introducing for brievity the vector $\vec{G}^Q$

$$\vec{G}^Q = (G^Q_{\alpha\beta,11}, G^Q_{\alpha\beta,\gamma\delta})\,, \tag{E.48}$$

the differential operator $\mathcal{D}^{\hat\alpha}_{(1}\mathcal{D}_{\eta)\hat\alpha}$ acts on $\vec{G}^Q$ as

$$\begin{aligned}
&2\mathcal{D}_1{}^{\hat c}\mathcal{D}_{\eta\hat c}\vec{G}_Q + 2\mathcal{D}_\eta{}^{\hat c}\mathcal{D}_{1\hat c}\vec{G}^Q \\
&= \mathcal{D}^{(Q)}_\eta \vec{G}^Q - (\partial_\phi + \tfrac{q-2}{2})\begin{pmatrix} 2G^Q_{1\eta,\gamma\delta} \\ -2\delta_{\eta(\alpha}G^Q_{1\beta),\gamma\delta} - 2\delta_{\eta(\gamma}G^Q_{1\delta),\alpha\beta}\end{pmatrix}\,,
\end{aligned} \tag{E.49}$$

where we define for short

$$\mathcal{D}^{(Q)}_\eta \equiv -\mathrm{i}\sqrt{2}e^{-\phi}(Q_{L\eta}(\partial_\phi + q - 2) + 2Q_{R\hat\alpha}\mathcal{D}^{\hat\alpha}_\eta)\ . \tag{E.50}$$

The off-diagonal component of the differential equation (3.20) with $(e,f) = (1,\eta)$ take the form

$$\begin{aligned}
\mathcal{D}^{(Q)}_\eta G^Q_{11,\gamma\delta} &= 2(\partial_\phi + 3)\,G^Q_{1\eta,\gamma\delta} - \pi\sum_{Q_1\in\Lambda^*}\Big(F^{Q_1}_{111k}F^{Q-Q_1\,k}_{\eta\gamma\delta} + F^{Q_1}_{\eta 11k}F^{Q-Q_1\,k}_{1\gamma\delta} - 2F^{Q_1}_{11k(\gamma}F^{Q-Q_1\,k}_{\delta)1\eta}\Big)\\
\mathcal{D}^{(Q)}_\eta G^Q_{1\beta,\gamma\delta} &= (\partial_\phi + 2)\,G^Q_{\eta\beta,\gamma\delta} - (\partial_\phi + q - 4)\,\big(\delta_{\eta\beta}G^Q_{\gamma\delta,11} - \delta_{\eta(\gamma}G^Q_{\delta)\beta,11}\big) - \delta_{\gamma\delta}G^Q_{\eta\beta,11}\\
&+\delta_{\beta(\gamma}G^Q_{\delta)\eta,11} - 2\pi\sum_{Q_1\in\Lambda^*}\Big(F^{Q_1}_{1k1\beta}F^{Q-Q_1\,k}_{\gamma\delta\eta} + F^{Q_1}_{1k\gamma\delta}F^{Q-Q_1\,k}_{1\beta\eta} - F^{Q_1}_{1k\beta(\gamma}F^{Q-Q_1\,k}_{\delta)1\eta} - F^{Q_1}_{1k1(\gamma}F^{Q-Q_1\,k}_{\delta)\beta\eta}\Big)\\
\mathcal{D}^{(Q)}_\eta G^Q_{\alpha\beta,\gamma\delta} &= -2(\partial_\phi + q - 4)\,\big(\delta_{\eta(\alpha}G^Q_{\beta)1,\gamma\delta} + \delta_{\eta(\gamma}G^Q_{\delta)1,\alpha\beta}\big) + 3\delta_{\langle\alpha\beta}G^Q_{\gamma\delta\rangle,1\eta}\\
&\quad -6\pi\sum_{Q_1\in\Lambda^*}F^{Q_1}_{1k\langle\alpha\beta,}F^{Q-Q_1\,k}_{\gamma\delta\rangle\eta}.
\end{aligned} \tag{E.51}$$

The component of the differential operator with two indices along the subgrassmaniann $(e,f) = (\eta,\vartheta)$, acts as

$$\begin{aligned}
4D_{(\eta}{}^{\hat c}D_{\theta)\hat c}\vec{G}^Q &= (4\mathcal{D}_{(\eta}{}^{\hat\alpha}\mathcal{D}_{\theta)\hat\alpha} + \delta_{\eta\theta}\partial_\phi - 8\pi^2e^{-2\phi}Q_{L\eta}Q_{L\theta})\vec{G}^Q\\
&\quad + 8\pi\mathrm{i}\sqrt{2}e^{-\phi}Q_{L(\eta|}\begin{pmatrix} G^Q_{1|\theta),\gamma\delta} \\ -\delta_{|\theta)(\alpha}G^Q_{\beta)1,\gamma\delta} - \delta_{|\theta)(\gamma}G^Q_{\delta)1,\alpha\beta}\end{pmatrix}\,,\\
&\quad + \begin{pmatrix} 2G^Q_{\eta\theta,\gamma\delta} - 2\delta_{\eta\theta}G^Q_{11,\gamma\delta} + 2\delta_{|\eta)(\gamma}G^Q_{\delta)(\theta|,11} \\ 6\delta_{\eta\langle(\alpha}\delta_{\beta),|\theta|}G^Q_{\gamma\delta\rangle,11} - 4\delta_{(\eta|\langle(\alpha}G^Q_{\beta),|\theta),|\gamma\delta\rangle}\end{pmatrix}\,,
\end{aligned} \tag{E.52}$$

and thus we obtain the differential equation on the sub-Grassmaniann $G_{p-2,q-2}$

$$\begin{aligned}
&\big(2\mathcal{D}_{(\eta}{}^{\hat\alpha}\mathcal{D}_{\theta)\hat\alpha} + \tfrac{1}{2}\delta_{\eta\theta}\partial_\phi - 4\pi^2e^{-2\phi}Q_{L\eta}Q_{L\theta}\big)\vec{G}^Q\\
&\qquad + 4\pi\mathrm{i}\sqrt{2}e^{-\phi}Q_{L(\eta}\begin{pmatrix} G^Q_{\theta)1,\gamma\delta} \\ -\delta_{\theta)(\alpha}G^Q_{\beta)1,\gamma\delta} - \delta_{\theta)(\gamma}G^Q_{\delta)1,\alpha\beta}\end{pmatrix}\\
&= \begin{pmatrix} (4-q)\delta_{\eta\theta}G^Q_{11,\gamma\delta} + (5-q)\delta_{|\eta)(\gamma}G^Q_{\delta)(\theta|,11} + \delta_{\gamma\delta}G^Q_{\eta\theta,11} \\ (3-q)\delta_{\eta\theta}G^Q_{\alpha\beta,\gamma\delta} + 2(6-q)\delta_{|\eta)\langle(\alpha}G^Q_{\beta),(\theta|,|\gamma\delta\rangle} + 3\delta_{\langle\alpha\beta,}G^Q_{\gamma\delta\rangle,\eta\theta} - 3\delta_{\eta\langle(\alpha}\delta_{\beta),|\theta|}G^Q_{\gamma\delta\rangle,11}\end{pmatrix}\\
&\qquad - 2\pi\sum_{Q_1\in\Lambda^*}\begin{pmatrix} F^{Q_1}_{11d(\eta}F^{Q-Q_1\,d}_{\theta)\gamma\delta} - F^{Q_1}_{1\gamma d(\eta}F^{Q-Q_1\,d}_{\theta)\delta 1} \\ \tfrac{3}{2}F^{Q_1}_{\eta d\langle\alpha\beta,}F^{Q-Q_1\,d}_{\gamma\delta\rangle\theta}\end{pmatrix}\,,
\end{aligned} \tag{E.53}$$

The constant terms sourcing the perturbative part of $G^0_{ab,cd}$ are given by [22, (4.16)]

$$\begin{aligned} F^0_{1111} &= a\, e^{-(q-6)\phi}\xi(q-6) \\ F^0_{11\gamma\delta} &= b\, e^{-(q-6)\phi}\xi(q-6)\delta_{\delta\gamma} \\ F^0_{\alpha\beta\gamma\delta} &= e^{-\phi}\mathcal{F}_{\alpha\beta\gamma\delta} + c\, e^{-(q-6)\phi}\xi(q-6)\delta_{(\alpha\beta}\delta_{\gamma\delta)}\ , \end{aligned} \tag{E.54}$$

with a, b, c constants which were computed in [22] ($a = \frac{3(7-q)(9-q)}{\pi^2}$, $b = \frac{3(7-q)}{\pi^2}$, $c = \frac{9}{\pi^2}$). As mentioned in the previous section, the unfolding method fails to capture exponentially suppressed corrections to the constant term, which are sourced by the instanton anti-instanton quadratic terms $\sum_{Q_1\neq 0} F^{Q_1}F^{-Q_1}$ on the right-hand side of the differential equations (E.47), (E.51), (E.53). These terms can in principle be computed by solving the differential equation. Here we focus on the perturbative part sourced by the constant terms in (E.54).

We find the particular solution to equations (E.53)

$$\begin{aligned} G^0_{11,\gamma\delta} &= -\frac{\pi c}{18}e^{-(q-5)\phi}\xi(q-6)(7-q)\mathcal{G}_{\gamma\delta}(\varphi) - \frac{2\pi c^2}{9}(7-q)e^{-2(q-6)\phi}\xi(q-6)^2\delta_{\alpha\beta} \\ G^0_{\alpha\beta,\gamma\delta} &= e^{-2\phi}\mathcal{G}_{\alpha\beta,\gamma\delta}(\varphi) - \frac{\pi c}{6}\, e^{-(q-5)\phi}\xi(q-6)\, \delta_{\langle\alpha\beta,}\mathcal{G}_{\gamma\delta\rangle}(\varphi) - \frac{\pi c^2}{3}e^{-2(q-6)\phi}\xi(q-6)^2\delta_{\langle\alpha\beta}\delta_{\gamma\delta\rangle}\ , \end{aligned} \tag{E.55}$$

and $b = \frac{7-q}{3}c$ in (E.54), which matches the result obtained in [22, (4.16)]. $\mathcal{G}_{\alpha\beta,\gamma\delta}(\varphi)$ is a solution to the equation (3.20) on the sub-Grassmaniann $G_{p-1,q-1}$ with source term quadratic in $\mathcal{F}_{\alpha\beta\gamma\delta}(\varphi)$, and $\mathcal{G}_{\alpha\beta}(\varphi)$ satisfies the equation (B.19) along $G_{p-1,q-1}$

$$2\mathcal{D}_{(\eta}{}^{\hat\alpha}\mathcal{D}_{\theta)\hat\alpha}\,\mathcal{G}_{\alpha\beta} = \frac{3-q}{2}\delta_{\eta\theta}\mathcal{G}_{\alpha\beta} + (5-q)\delta_{|\eta)(\alpha}\mathcal{G}_{\beta)(\theta|} + \delta_{\alpha\beta}\mathcal{G}_{\eta\theta} + 12\mathcal{F}_{\alpha\beta\eta\theta}\,. \tag{E.56}$$

One can check that $G^0_{ab,cd}$ is also solution to (E.47), setting $a = \frac{(7-q)(9-q)c}{3}$ which matches the results [22, (4.16)].

The most general solution to (E.53) is obtained by adding a solution of the homogeneous equation without source term

$$\begin{aligned} \tilde G^0_{11,\gamma\delta} &= -\frac{\pi c}{18}(7-q)e^{-(q-5)\phi}\xi(q-6)\tilde{\mathcal{G}}_{\gamma\delta}(\varphi) \\ \tilde G^0_{\alpha\beta,\gamma\delta} &= e^{-2\phi}\tilde{\mathcal{G}}_{\alpha\beta,\gamma\delta}(\varphi) - \frac{\pi c}{6}\, e^{-(q-5)\phi}\xi(q-6)\, \delta_{\langle\alpha\beta,}\tilde{\mathcal{G}}_{\gamma\delta\rangle}(\varphi)\,, \end{aligned} \tag{E.57}$$

with $\tilde{\mathcal{G}}_{\alpha\beta,\gamma\delta}(\varphi)$ a solution to the homogeneous equation (3.17) on the sub-Grassmaniann $G_{p-1,q-1}$, and $\tilde{\mathcal{G}}_{\alpha\beta}(\varphi)$ solution to the homogeneous equation (3.34) on $G_{p-1,q-1}$. The results (4.59) and (4.60) for the constant term $G^0_{ab,cd}$ obtained by the unfolding method in this decomposition lie in this class, upon setting for generic q

$$\begin{aligned} \mathcal{G}_{\alpha\beta} &= \frac{\upsilon N^{q-7}+1}{N+1}\tfrac{1}{2}(G^{(p-1,q-1)}_{\alpha\beta}(\varphi) + {}^{\varsigma}G^{(p-1,q-1)}_{\alpha\beta}(\varphi)) \\ \tilde{\mathcal{G}}_{\alpha\beta} &= \frac{\upsilon N^{q-7}-1}{N-1}\tfrac{1}{2}(G^{(p-1,q-1)}_{\alpha\beta}(\varphi) - {}^{\varsigma}G^{(p-1,q-1)}_{\alpha\beta}(\varphi))\,, \end{aligned} \tag{E.58}$$

with $c = \frac{9}{\pi^2}$ as in [22].

F Beyond the saddle point approximation

In the analysis of the large radius limit of the genus-two modular integral in §5.1, we neglected the dependence of the Fourier coefficients $C_{k-2}(n,m,L;\Omega_2)$ of the meromorphic Siegel modular form $1/\Phi_{k-2}$ on Ω_2, and evaluated the integral over Ω_2 arising in the Abelian rank-two orbit in terms of a matrix variate Bessel function. Since the integral over Ω_2 is dominated by a saddle point at large R, and since $C_{k-2}(n,m,L;\Omega_2)$ is constant in the vicinity of the saddle point (at least at generic point in the moduli space G_4/K_4) , this approximation correctly captures the leading behavior of order $e^{-2\pi R\mathcal{M}(Q,P)}$ at large R, as well as the infinite series of perturbative corrections around the saddle point. As a result of the poles in $1/\Phi_{k-2}$ however, the Fourier coefficient $C_{k-2}(n,m,L;\Omega_2)$ is only locally constant, and this approximation misses contributions from the region where this Fourier coefficient differs from its saddle point value. Here we shall estimate these effects and find that

1. poles occuring at large $|\Omega_2|$ give rise to contributions of order $e^{-2\pi R(\mathcal{M}(Q_1,P_1)+\mathcal{M}(Q_2,P_2))}$ for all possible splittings $(Q,P)\to(Q_1,P_1)+(Q_2,P_2)$ of the total charge into a pair of 1/2-BPS charges; these contributions are subleading away from the walls of marginal stability, but crucial for the smoothness of the physical couplings across the wall;

2. deep poles occuring at $|\Omega_2|\le\frac{1}{(2n_2)^2}$ give rises to subleading contributions exponentially suppressed in $e^{-4\pi|n_2|R^2}$ which can be interpreted as $|n_2|$ pairs of Taub-NUT instantons and anti-instantons.

In either case, the gist of the argument is as follows: one decomposes the integral

$$\int_{\mathcal{P}_2}\frac{\mathrm{d}^3\Omega_2}{|\Omega_2|^{\frac{3}{2}-s}}\,C[\Omega_2]e^{-2\pi S[\Omega_2]}=\sum_k C_k\int_{W_k}\frac{\mathrm{d}^3\Omega_2}{|\Omega_2|^{\frac{3}{2}-s}}\,e^{-2\pi S[\Omega_2]}\ , \tag{F.1}$$

with a locally constant insertion $C[\Omega_2]$ into a sum over chambers W_k where $C[\Omega_2]=C_k$ is constant. Applying the saddle point approximation, one can bound the integral over W_k at large R as

$$-\frac{1}{2\pi}\log\left(\int_{W_k}\frac{\mathrm{d}^3\Omega_2}{|\Omega_2|^{\frac{3}{2}-s}}\,e^{-2\pi S[\Omega_2]}\right)=S[\Omega_2^*(W_k)]+o(R)\ , \tag{F.2}$$

where $\Omega_2^*(W_k)$ is the minimum of $S[\Omega]$ on W_k.

F.1 Poles at large $|\Omega_2|$

Recall that the saddle point lies at

$$\Omega_2^*=\frac{R}{\mathcal{M}(Q,P)}A^\intercal\Big[\tfrac{1}{S_2}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}+\tfrac{1}{|Q_R\wedge P_R|}\begin{pmatrix}P_R^2 & -Q_R\cdot P_R\\ -Q_R\cdot P_R & Q_R^2\end{pmatrix}\Big]A\,, \tag{F.3}$$

where A is a non-generate integer matrix, which we decompose as $A=\begin{pmatrix}1 & \frac{j}{k}\\ 0 & 1\end{pmatrix}\begin{pmatrix}d_1 & 0\\ 0 & d_2\end{pmatrix}\gamma$ for $\gamma\in SL(2,\mathbb{Z})$. We consider the component of the diagonal divisor $\mathcal{D}$ where the matrix $\begin{pmatrix}1 & 0\\ \frac{j}{k} & 1\end{pmatrix}A^{-\intercal}\Omega_2A^{-1}\begin{pmatrix}1 & \frac{j}{k}\\ 0 & 1\end{pmatrix}$ becomes diagonal. On this divisor, we parametrize Ω_2 as

$$\Omega_2=RA^\intercal\begin{pmatrix}1 & 0\\ -\frac{j}{k} & 1\end{pmatrix}\begin{pmatrix}\rho_2 & 0\\ 0 & \sigma_2\end{pmatrix}\begin{pmatrix}1 & -\frac{j}{k}\\ 0 & 1\end{pmatrix}A\ . \tag{F.4}$$

It is straightforward to compute the minimum of the action

$$S[\Omega_2]=\frac{R^2}{2}\,\mathrm{tr}\left[\Omega_2^{-1}A^\intercal\tfrac{1}{S_2}\begin{pmatrix}1 & S_1\\ S_1 & |S|^2\end{pmatrix}A\right]+\,\mathrm{tr}\left[\Omega_2A^{-1}\begin{pmatrix}Q_R^2 & Q_R\cdot P_R\\ Q_R\cdot P_R & P_R^2\end{pmatrix}A^{-\intercal}\right] \tag{F.5}$$

on the surface parametrized by σ_2 and ρ_2, because the matrices in the traces are then diagonal. The minimum is reached at

$$\Omega_2'=RA^\intercal\begin{pmatrix}1 & 0\\ -\frac{j}{k} & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2S_2(Q_R-\frac{j}{k}P_R)^2}} & 0\\ 0 & \frac{|S+\frac{j}{k}|}{\sqrt{2S_2P_R^2}}\end{pmatrix}\begin{pmatrix}1 & -\frac{j}{k}\\ 0 & 1\end{pmatrix}A\,, \tag{F.6}$$

with

$$S[\Omega_2']=R\Big(\sqrt{\tfrac{2}{S_2}(Q_R-\tfrac{j}{k}P_R)^2}+\sqrt{\tfrac{2|S+\frac{j}{k}|^2}{S_2}P_R^2}\Big)=R\big(\mathcal{M}(Q-\tfrac{j}{k}P,0)+\mathcal{M}(\tfrac{j}{k}P,P)\big)\ , \tag{F.7}$$

which we recognize as the sums of the actions associated to 1/2-BPS states with charge $(Q_1,P_1)=(Q-\frac{j}{k}P,0)$ and $(Q_2,P_2)=(\frac{j}{k}P,P)$, as announced. Taking $j=0$ for simplicity and parametrizing the distance away from the divisor $v=0$ by ϵ such that

$$\frac{S_1}{S_2}-\frac{Q_R\cdot P_R}{|Q_R\wedge P_R|}=\epsilon\qquad\Rightarrow\qquad Q_R\cdot P_R=\sqrt{\frac{Q_R^2P_R^2}{S_2^2+(S_1-\epsilon S_2)^2}}(S_1-\epsilon S_2)\ , \tag{F.8}$$

the perturbation of the action at small v_2 gives

$$\begin{aligned} S\Big[RA^\intercal\begin{pmatrix}\frac{1}{\sqrt{2S_2Q_R^2}} & v_2\\ v_2 & \frac{|S|}{\sqrt{2S_2P_R^2}}\end{pmatrix}A\Big] &= S[\Omega_2']+2Rv_2\sqrt{Q_R^2P_R^2}\Big(\frac{S_1-\epsilon S_2}{\sqrt{S_2^2+(S_1-\epsilon S_2)^2}}-\frac{S_1}{|S|}\Big)+\mathcal{O}(v_2^2)\\ &= S[\Omega_2']-2Rv_2\sqrt{Q_R^2P_R^2}\frac{S_2^3}{|S|^3}\epsilon+\mathcal{O}(\epsilon^2)+\mathcal{O}(v_2^2)\ . \end{aligned} \tag{F.9}$$

For ϵ small enough, *i.e.* Ω_2^* close enough to the wall $v_2=0$, one sees indeed that the action increases monotonically away from the wall, and therefore, the minimum of the action in the neighboring chamber must indeed be reached along the wall. All the other cases are then determined from this one by $SL(2,\mathbb{Z})$.

F.2 Deep poles

When the determinant $|\Omega_2|$ becomes sufficiently small, the contour $\mathcal{C}=[0,1]^3+\mathrm{i}\Omega_2$ starts intersecting additional poles of the form

$$m^2-m^1\rho+n_1\sigma+n_2(\rho\sigma-v^2)+jv=0 \tag{F.10}$$

with $n_2\neq 0$, $4(m^1n_1+m^2n_2)+j^2=1$. This intersection occurs for generic values of Ω_2, which make the Fourier coefficient $C(m,n,p;\Omega_2)$ itself ill-defined. In this section it is convenient to parametrize Ω_2 as

$$\Omega_2=\frac{1}{V\tau_2}\begin{pmatrix}1 & \tau_1\\ \tau_1 & |\tau|^2\end{pmatrix}\ . \tag{F.11}$$

Eq. (F.10) can be solved for τ_1, v_1 as a function of $\tau_2, V, \rho_1, \sigma_1$,

$$\begin{aligned} v_1 =&\frac{1}{2n_2}\left(j \pm \sqrt{1-4(n_1+n_2\rho_1)(m_1-n_2\sigma_1)-\frac{4n_2^2}{V^2}}\right) \\ \tau_1 =&\frac{1}{2(n_1+n_2\rho_1)}\left[\sqrt{1-4(n_1+n_2\rho_1)(m_1-n_2\sigma_1)-\frac{4n_2^2}{V^2}}-\sqrt{1-4(n_1+n_2\rho_1)^2\tau_2^2-\frac{4n_2^2}{V^2}}\right] \end{aligned} \tag{F.12}$$

The solution is real only if $V^2-4V^2(n_1+n_2\rho_1)^2\tau_2^2-4n_2^2 \geq 0$, which requires $V^2 \geq 4n_2^2$, *i.e.* that $|\Omega_2| < 1/(2n_2)^2$.

In order to bound the contribution from this region, we shall look for the minimum of the action (F.5) on the domain $\mathcal{P}_2$ with $|\Omega_2| < \frac{1}{(2n_2)^2}$. For simplicity we consider the case $A=1$, but the argument is general. Extremizing over τ in the parametrization (F.11) one obtains the solution

$$\tau^* = S_1 + S_2\frac{-P_R\cdot(Q_R+S_1P_R)+\mathrm{i}\sqrt{|Q_R\wedge P_R|^2+\frac{R^2V^2}{2}\frac{|Q_R+SP_R|^2}{S_2}+\frac{R^4V^4}{4}}}{P_R^2+\frac{R^2V^2}{2}}\,, \tag{F.13}$$

at which point the action becomes

$$S[\tau^*,V] = \sqrt{R^4V^2+2R^2\frac{|Q_R+SP_R|^2}{S_2}+\frac{4}{V^2}|Q_R\wedge P_R|^2}\,. \tag{F.14}$$

At large R the action grows monotonically in V, so the minimum of the action on the domain $V \geq 2|n_2|$ is reached on the boundary at $V = 2|n_2|$, where it evaluates to

$$S[\tau^*,2|n_2|] = \sqrt{(2n_2R^2)^2+2R^2\frac{|Q_R+SP_R|^2}{S_2}+\frac{1}{n_2^2}|Q_R\wedge P_R|^2}\,. \tag{F.15}$$

The correction in this domain are therefore exponentially suppressed as $e^{-4\pi R^2|n_2|}$, which is the expected magnitude of a contribution for $|n_2|$ pairs of Taub-NUT instanton anti-instantons.

G Non-Abelian Fourier coefficients

In this section we show that the non-Abelian Fourier coefficients in the degeneration $(p,q)\to(p-2,q-2)$ can be deduced from the (Abelian) Fourier coefficients in the degeneration $(p,q)\to(p-1,q-1)$. First, recall that the Fourier expansion of an automorphic form F on $G_{p,q}$ with respect to the maximal parabolic subgroup with Levi $GL(1)\times O(p-2,q-2)$, corresponding to the grading (5.3), which we copy for convenience,

$$\mathfrak{so}_{p,q} \simeq \ldots \oplus (\mathfrak{gl}_1 \oplus \mathfrak{sl}_2 \oplus \mathfrak{so}_{p-2,q-2})^{(0)} \oplus (\mathbf{2}\otimes(\mathbf{p}+\mathbf{q}-\mathbf{4}))^{(1)} \oplus \mathbf{1}^{(2)}, \tag{G.1}$$

consists of three parts:

1. the constant term $F_0(R,t)$, defined as the average of F with respect to (a^{iI},ψ) parametrizing the grade (1) and (2) components in (G.1);

2. the Abelian Fourier coefficients $F_{Q,P}(R,t)$, defined as the average of the product of F by a character $e^{-2\pi \mathrm{i}(Qa_1+Pa_2)}$ with (Q,P) in the lattice $\Lambda_{p-2,q-2}^{\oplus 2}$;
3. the non-Abelian Fourier coefficients $F_{M_1}(R,t,a)$ for $M_1\in\mathbb{Z}\backslash\{0\}$, defined as the average of F times $e^{-2\pi \mathrm{i} M_1\psi}$ over $\psi\in[0,1]$.

The non-Abelian Fourier coefficient can be further decomposed by diagonalizing a half-dimensional Lagrangian subspace in the grade (1) space, e.g. dual to the lattice $\Lambda_{p-2,q-2}$ of magnetic charges. This leads to the 'wave function representation'

$$F_{M_1}(R,t,a^1,a^2)=\sum_{\mu\in\frac{\Lambda_{p-2,q-2}}{M_1\Lambda_{p-2,q-2}}}\ \sum_{P\in M_1\Lambda_{p-2,q-2}+\mu} F_{M_1,\mu}(R,t;P-M_1a_1)\,e^{2\pi\mathrm{i}(P\cdot a_2-\frac12 M_1 a_1\cdot a_2)} \tag{G.2}$$

However, an alternative representation of the same non-Abelian Fourier coefficient can be obtained by diagonalizing not only translations in ψ and a_2 but also in S_1, corresponding to the positive root in the $\mathfrak{sl}(2)$ factor appearing in the grade 0 component of (G.1). This amounts to performing the (Abelian) Fourier decomposition with respect to a different maximal parabolic subgroup associated to the decomposition (4.4),

$$\mathfrak{so}_{p,q}\simeq(\mathbf{p}+\mathbf{q}-\mathbf{2})^{(-2)}\oplus(\mathfrak{gl}_1\oplus\mathfrak{so}_{p-1,q-1})^{(0)}\oplus(\mathbf{p}+\mathbf{q}-\mathbf{2})^{(2)}\ . \tag{G.3}$$

The only task is to relate the coordinates (R,S,φ,a^1,a^2) appropriate to (G.1) to the coordinates (R',φ',a') appropriate to (G.3). To this aim, let us parametrize the $(SO(p)\times SO(q))\backslash SO(p,q)$ Grassmannian in the parabolic gauge as

$$g(R,S_2,S_1,\varphi,a_1,a_2,\psi)=L(R,S_2,\varphi)U_2(S_1)U_{\text{e.m.}}(a_1,a_2)U_1(\psi)\ , \tag{G.4}$$

with $L(1,1,\varphi)\subset SO(p-2,q-2)$, $L(1,S_2,0)U_2(S_1)\subset SL(2,\mathbb{R})$ and $U_{\text{e.m.}}(a_1,a_2)U_1(\psi)$ in the unipotent radical. One straightforwardly computes that

$$\begin{aligned}
&\big[L(R,S_2,\varphi)U_2(S_1)\big]U_{\text{e.m.}}(a_1,a_2)U_1(\psi) \\
=\ & L(R,S_2,\varphi)U_2(S_1)U_{\text{e.m.}}(a_1,0)U_{\text{e.m.}}(0,a_2)U_1(\psi-\tfrac12 a_1\cdot a_2)\\
=\ & \big[L(R,S_2,\varphi)U_{\text{e.m.}}(a_1,0)\big]U_2(S_1)U_{\text{e.m.}}(0,a_2-S_1a_1)U_1(\psi-\tfrac12 a_1\cdot a_2+\tfrac12 S_1{a_1}^2)\ ,
\end{aligned} \tag{G.5}$$

where $L(R,S_2,\varphi)U_{\text{e.m.}}(a_1,0)\in\mathbb{R}^+\times SO(p-1,q-1)$ and $U_2(S_1)U_{\text{e.m.}}(0,a')U_1(\psi')$ belongs to the corresponding abelian unipotent radical. Using this parametrization, the non-abelian Fourier coefficients can simply be obtained from the Fourier coefficients (4.44) by substituting

$$\begin{aligned}
R' &= R\sqrt{S_2}\\
Q'_{R\hat\alpha} &= \begin{cases}\frac{1}{\sqrt{2S_2}}\Big(RM_1+\frac{S_2}{R}(M_2-a_1\cdot P+\frac12{a_1}^2M_1)\Big) & \text{if } \hat\alpha=q-1\\ P_{R\hat\alpha}-a_{1R\hat\alpha}M_1 & \text{if } \hat\alpha<q-1\end{cases}\\
Q'_{L\alpha} &= \begin{cases}\frac{1}{\sqrt{2S_2}}\Big(RM_1-\frac{S_2}{R}(M_2-a_1\cdot P+\frac12{a_1}^2M_1)\Big) & \text{if } \alpha=p-1\\ P_{L\alpha}-a_{1L\alpha}M_1 & \text{if } \alpha<p-1\end{cases}\\
Q\cdot a' &\to M_1(\psi-\tfrac12 a_1\cdot a_2+\tfrac12 S_1{a_1}^2)+P\cdot(a_2-S_1a_1)+M_2S_1\ ,
\end{aligned} \tag{G.6}$$

where $Q=(P,M_1,M_2)\in\Lambda_{p-1,q-1}$ split into $P\in\Lambda_{p-2,q-2}$ and $(M_1,M_2)\in I\!I_{1,1}$, such that $Q^2=P^2-2M_1M_2$. The index $\mu=1$ is combined with the index α ranging from 1 to $p-2$ of

$SO(p-2)$ to give the index α ranging from 1 to $p-1$ of $SO(p-1)$, whereas the index $\mu=2$ corresponds to the index 1 in the decomposition (4.44). The non-Abelian Fourier coefficient $F_{M_1}(R,t,a^1,a^2)$ of $G^{(p,q)}_{abcd}$ is then

$$G^{(p,q),\,2\text{nab},M_1}_{ab,cd} = \sum_{P\in\Lambda_{p-2,q-2}} \Big(\sum_{M_2\in\mathbb{Z}} G^{(p,q),(P,M_1,M_2)}_{ab,cd} e^{2\pi \mathrm{i}(M_2 - a_1\cdot P + \frac12 a_1{}^2 M_1)S_1}\Big) e^{2\pi\mathrm{i}(P\cdot a_2 - \frac12 M_1 a_1\cdot a_2)} \tag{G.7}$$

with the classical action

$$S_{\rm cl}(M_1,M_2,P) = \sqrt{\big(R^2 M_1 + S_2(M_2 - a_1\cdot P + \tfrac12 a_1{}^2 M_1)\big)^2 + 2R^2 S_2 (P_R - a_{1R}M_1)^2}\ , \tag{G.8}$$

and

$$\begin{aligned}
G^{(p,q),(P,M_1,M_2)}_{\alpha\beta,\gamma\delta} &= 6\,\bar G^{(p-1,q-1)}_{\langle\alpha\beta,}(P,M_1,M_2;\tfrac{R^2}{S_2},\varphi,a_1) \sum_{l=0}^{1} \frac{\tilde P^{(l)}_{\gamma\delta\rangle}(P_L - a_{1L}M_1)}{(R^2S_2)^{l-\frac{q-2}{2}}}\, \frac{K_{\frac{q-5}{2}-\ell}(2\pi S_{\rm cl})}{S_{\rm cl}^{\frac{q-3}{2}-l}} \\
G^{(p,q),(P,M_1,M_2)}_{\alpha\beta,\gamma 2} &= 3(R^2S_2)^{\frac{q-3}{2}}\bar G^{(p-1,q-1)}_{\langle\alpha\beta,}(P,M_1,M_2;\tfrac{R^2}{S_2},\varphi,a_1)\, \frac{P_{L\gamma\rangle} - a_{1L\gamma\rangle}M_1}{\mathrm{i}\sqrt2}\, \frac{K_{\frac{q-7}{2}}(2\pi S_{\rm cl})}{S_{\rm cl}^{\frac{q-5}{2}}} \\
G^{(p,q),(P,M_1,M_2)}_{\alpha\beta,22} &= -(R^2S_2)^{\frac{q-4}{2}}\bar G^{(p-1,q-1)}_{\alpha\beta}(P,M_1,M_2;\tfrac{R^2}{S_2},\varphi,a_1)\, \frac{K_{\frac{q-9}{2}}(2\pi S_{\rm cl})}{S_{\rm cl}^{\frac{q-7}{2}}}\ ,
\end{aligned} \tag{G.9}$$

whereas the components with $\mu=1$ are obtained by replacing $Q'_{L\alpha} = P_L - a_{1L}M_1$ for $\alpha = 1, p-2$ by Q'_{Lp-1} in (G.6). The tensor $\bar G^{(p-1,q-1)}_{\alpha\beta}(P,M_1,M_2)$ is defined on $SO(p-1,q-1)$ from (4.45) in the parabolic gauge in which $g(\frac{R^2}{S_2},t,a_1) = L((\frac{R^2}{S_2})^{\frac14},(\frac{S_2}{R^2})^{\frac12},\varphi)U_{\rm e.m.}(a_1,0)$, with $Q=(P,M_1,M_2)$, and with the index $\alpha=p-1$ interpreted as $\mu=1$, according to (G.6). Note that the tensor $\bar G^{(p-1,q-1)}_{\alpha\beta}(P,M_1,M_2)$ is not invariant under the shift of a_1 by a vector $\epsilon\in\Lambda_{p-2,q-2}$, but satisfies

$$\bar G^{(p-1,q-1)}_{\alpha\beta}(P,M_1,M_2;\tfrac{R^2}{S_2},t,a_1+\epsilon) = \bar G^{(p-1,q-1)}_{\alpha\beta}(P-\epsilon M_1, M_1, M_2 - \epsilon\cdot P + \tfrac12\epsilon^2 M_1;\tfrac{R^2}{S_2},\varphi,a_1)\ , \tag{G.10}$$

which ensures that the decomposition (G.7) is consistent with the action of the Heisenberg group generated by the grade 1 and 2 components in (G.1). Note that the wave function representation (G.2) of the non-abelian Fourier coefficient can be recovered from (G.7) by a Poisson resummation on M_2.

H Covariantized Polynomials

In the degeneration limit $O(p,q)\to O(p-1,q-1)$ studied in §4, the monomials $\tilde P^{(l)}_{\alpha_1\dots\alpha_i}(Q)$ with $l\ge 0$ are of degree $i-2l$, and defined by

$$\begin{aligned}
\sum_{l=0}^{1}\tilde P^{(l)}_{\gamma\delta}(Q) &= Q_{L\gamma}Q_{L\delta} - \frac{1}{4\pi}\delta_{\gamma\delta} \\
\tilde P^{(0)}_{\delta}(Q) &= Q_{L\delta} \\
\tilde P^{(0)}(Q) &= 1,
\end{aligned} \tag{H.1}$$

In the degeneration limit $O(p,q) \to O(p-2,q-2)$ studied in §5.1, the monomials $\mathcal{P}^{(l)}_{\alpha_1\ldots\alpha_i}$ with $l \geq 0$ are of degree $i-2l$, and defined by

$$\begin{aligned}
&\sum_{l=0}^{1} \mathcal{P}^{(l)}_{\gamma\delta}(\Gamma_i, S) = \Gamma_{L,\gamma\tau}\Gamma_{L,\delta}{}^{\tau} - \frac{1}{4\pi}\delta_{\gamma\delta} \\
&\mathcal{P}^{(0)}_{\delta\tau}(\Gamma) = Q_{L\delta\tau} \\
&\mathcal{P}^{(0)}(\Gamma) = 1,
\end{aligned} \tag{H.2}$$

For the Abelian rank-2 orbits (5.33), the polynomial are contracted with their matrix-variate Bessel function as

$$\begin{aligned}
\sum_{l=0}^{2} \mathcal{P}^{(l)\,\mu\nu}_{\alpha\beta,\gamma\delta}(\Gamma_i, S)\widetilde{B}^{(l \,\mathrm{mod}\, 2)}_{\frac{q-5-l}{2}\;\mu\nu}(Z) &= \delta_{\langle\lambda\kappa,}\delta_{\tau\epsilon\rangle}\,\Gamma_{L\alpha}{}^{\lambda}\Gamma_{L\beta}{}^{\kappa}\Gamma_{L\gamma}{}^{\tau}\Gamma_{L\delta}{}^{\epsilon}\,\delta^{\mu\nu}\widetilde{B}^{(0)}_{\frac{q-5}{2}\;\mu\nu}(Z) \\
&\quad - \frac{3}{4\pi}\delta_{\langle\alpha\beta,}(\Gamma_{L\gamma}{}^{\kappa}\Gamma_{L\delta\rangle}{}^{\lambda})\widetilde{B}^{(1)}_{\frac{q-6}{2}\;\kappa\lambda}(Z) \\
&\quad + \frac{3}{16\pi^2}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}\delta^{\mu\nu}\widetilde{B}^{(0)}_{\frac{q-7}{2}\;\mu\nu}(Z) \\
\sum_{l=0,1} \mathcal{P}^{(l)\,\mu\nu}_{\rho\beta,\gamma\delta}(\Gamma_i, S)\widetilde{B}^{(l+1 \,\mathrm{mod}\, 2)}_{\frac{q-6-l}{2}\;\mu\nu}(Z) &= \Gamma_{L\,\langle\beta,\rho}\Gamma_{L\,\gamma}{}^{\tau}\Gamma_{L\,\delta\rangle}{}^{\epsilon}\widetilde{B}^{(1)}_{\frac{q-6}{2}\;\tau\epsilon}(Z) \\
&\quad - \frac{3}{4\pi}\delta_{\langle\gamma\delta,}\Gamma_{L\beta\rangle}{}^{\kappa}\widetilde{B}^{(0)}_{\frac{q-7}{2}\;\kappa\rho}(Z) \\
\sum_{l=0,1} \mathcal{P}^{(l)\,\mu\nu}_{\rho\sigma,\gamma\delta}(\Gamma_i, S)\widetilde{B}^{(l \,\mathrm{mod}\, 2)}_{\frac{q-7-l}{2}\;\mu\nu}(Z) &= \Gamma_{L\,\gamma,\rho}\Gamma_{L\,\delta,\sigma}\delta^{\mu\nu}\widetilde{B}^{(0)}_{\frac{q-7}{2}\;\mu\nu}(Z) \\
&\quad - \frac{1}{4\pi}\delta_{\gamma\delta}\widetilde{B}^{(1)}_{\frac{q-8}{2}\;\rho\sigma}(Z) \\
\mathcal{P}^{(0)\,\mu\nu}_{\rho\sigma,\tau\delta}(\Gamma_i, S)\widetilde{B}^{(1)}_{\frac{q-8}{2}\;\mu\nu}(Z) &= \Gamma_{L\,\delta,\langle\tau}\widetilde{B}^{(1)}_{\frac{q-8}{2}\;\rho\sigma\rangle}(Z) \\
\mathcal{P}^{(0)\,\mu\nu}_{\rho\sigma,\tau\upsilon}B^{(0)}_{\frac{q-9}{2}\;\mu\nu}(Z) &= \delta_{\langle\rho\sigma,}\delta_{\tau\upsilon\rangle}B^{(0)}_{\frac{q-9}{2}\;\mu}{}^{\mu}(Z)
\end{aligned} \tag{H.3}$$

For the singular contribution (D.4), the monomials $\mathcal{P}^{(l_1,l_2)}_{\alpha_1\ldots\alpha_i}$ with $l_1, l_2 \geq 0$ are of degree

$i - 2l_1 - 2l_2$, and defined by

$$\sum_{l_1=0}^{2}\sum_{l_2=0}^{2}\mathcal{P}^{(l_1,l_2)}_{\alpha\beta,\gamma\delta}(\Gamma_1,\Gamma_2,S) = \delta_{\langle\lambda\kappa,}\delta_{\tau\epsilon\rangle}\,\Gamma_{1L\alpha}{}^{\lambda}\Gamma_{1L\beta}{}^{\kappa}\Gamma_{2L\gamma}{}^{\tau}\Gamma_{2L\delta}{}^{\epsilon} + \delta_{\langle\lambda\kappa,}\delta_{\tau\epsilon\rangle}\,\Gamma_{2L\alpha}{}^{\lambda}\Gamma_{2L\beta}{}^{\kappa}\Gamma_{1L\gamma}{}^{\tau}\Gamma_{1L\delta}{}^{\epsilon}$$
$$- \tfrac{3}{4\pi}\left(\delta_{\langle\alpha\beta,}\Gamma_{L1\gamma}\Gamma_{L1\delta\rangle} + \delta_{\langle\alpha\beta,}\Gamma_{L2\gamma}\Gamma_{L2\delta\rangle}\right) - \frac{3}{8\pi^2}\delta_{\langle\alpha\beta,}\delta_{\gamma\delta\rangle}$$
$$\sum_{l_1=0}^{1}\sum_{l_2=0}^{1}\mathcal{P}^{(l_1,l_2)}_{\rho\beta,\gamma\delta}(\Gamma_1,\Gamma_2,S) = \Gamma_{1L\langle\beta,\rho}\Gamma_{2L\gamma}{}^{\tau}\Gamma_{2L\delta\rangle\tau} + \Gamma_{2L\langle\beta,\rho}\Gamma_{1L\gamma}{}^{\tau}\Gamma_{1L\delta\rangle\tau}$$
$$- \tfrac{3}{4\pi}\left(\delta_{\langle\gamma\delta,}\Gamma_{L1\beta\rangle} + \delta_{\langle\gamma\delta,}\Gamma_{L2\beta\rangle}\right)$$
$$\sum_{l_1=0}^{1}\sum_{l_2=0}^{1}\mathcal{P}^{(l_1,l_2)}_{\rho\sigma,\gamma\delta}(\Gamma_1,\Gamma_2,S) = \Gamma_{1L\gamma\rho}\Gamma_{1L\delta\sigma} + \Gamma_{2L\gamma\rho}\Gamma_{2L\delta\sigma} - \tfrac{1}{2\pi}\delta_{\rho\sigma}\delta_{\gamma\delta}$$
$$\mathcal{P}^{(0,0)}_{\rho\sigma,\tau\delta}(\Gamma_1,\Gamma_2,S) = \delta_{\langle\rho\sigma,}(\Gamma_{1L\delta\tau\rangle} + \Gamma_{2L\delta\tau\rangle})$$
$$\mathcal{P}^{(0,0)}_{\rho\sigma,\tau\upsilon}(\Gamma_1,\Gamma_2,S) = \delta_{\langle\rho\sigma,}\delta_{\tau\upsilon\rangle} \tag{H.4}$$

where $\Gamma = \Gamma_1 + \Gamma_2$.

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