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PAPER

Classical relativistic electron-field dynamics: Hamiltonian approach to radiation reaction

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Abstract

Motivated by the renewed interest due to the presently available extreme light sources, the dynamics of a single classical relativistic (spinless) extended electron interacting with a classical electromagnetic field (an incoming radiation and the field radiated by the electron) is revisited. The field is treated in Lorentz gauge, with the Lorentz condition. By assumption, there is a crucial finite cut-off $k_{\max}$ on the magnitude of any wavevector contributing to the field (preventing a point electron) and, for a simple formulation, the initial conditions for particle and fields are given in the infinitely remote past. In an infinite three-dimensional vacuum and in an inertial system, Hamilton's dynamical equations for the particle and the complex field amplitudes acting as canonical variables (a's) yield an exact Lorentz force equation for the former, that includes the incoming radiation and an exact radiation reaction force $\mathbf{F}_{RR}$ due to the field radiated by the electron. Uniform motion is obtained as a test of consistency. Based upon numerical computations, some approximations on $\mathbf{F}_{RR}$ are given. A covariant formulation is also presented.

1. Introduction

The problem of radiation reaction may be stated as finding the correct equations of motion that describe, in a consistent way, the interaction of a structureless charged particle and the electromagnetic field, with due account of the radiated field and its back-reaction on the charged particle. The problem has attracted a lot of interest for at least one century, due to its fundamental nature: any charge in accelerated motion must emit radiation, this radiation carries energy, momentum, etc and, forcefully, this has to have an impact on the charge's trajectory. But accounting for this has proven to be difficult, and the problem as it is stated here remains perhaps one of the few for which a completely satisfactory solution is not available, in spite of sustained efforts by many outstanding scientists, like Lorentz, Abraham, Dirac, Landau, Rohrlich, or Schwinger to name a few, see [1, 2] for modern review papers with references to the original literature. Perhaps some of the difficulties arise from the dissimilar nature of the two systems at hand, namely a point particle with a classical trajectory, and an extended object, like the electromagnetic field whose mutual interaction takes the form of singular integrals in a first, naive attempt to account for the radiated field [3–9].

Apart from its theoretical interest, the problem of radiation reaction has recently experienced a vigorous revival due to the availability of extreme light sources: since the discovery of the Chirped Pulse Amplification (CPA) technology [10], laser pulses have increased their intensity in an extraordinary way for almost three decades. Its advantage was clear from the very beginning; now peak intensities of 10^{23} W/cm^2 have been reported [11] and peak powers and peak intensities grow very rapidly particularly for near/mid infrared (IR) laser pulses.

The interaction of electrons with those near IR laser pulses is very peculiar. At intensities of the order of 10^{16} W/cm^2 outer shell electrons become unbound and begin to be driven by the strong laser field. At 10^{18} W/cm^2

electrons move at relativistic speeds and inner electrons can be ionized too, and this happens five orders of magnitude below today's power record.

Those ejected electrons can be considered as electrons that are released with almost zero initial speed and are rapidly accelerated to relativistic speeds and quickly expelled from the high intensity region due to the ponderomotive force. The latter is an ubiquitous effect that can be understood because the electron inside the laser field has to have a quiver motion (oscillating along the laser electric field and moving forward due to the magnetic drift). Such electron, thus, tends to go to lower intensity regions to avoid having this high energy related to its quiver motion.

In the case of electrons accelerated externally to the laser field (using a conventional accelerator or a laser accelerator, for example) the dynamics is different. Highly relativistic electrons (with energies of one GeV or more, in the laboratory frame) that move counter-propagating to the laser field can enter the high energy region because of their inertia. Although we are not going to consider it in the present paper, the dynamics of such electrons can be understood using what is called the Lorentz boost, or the frame moving at the electron's initial velocity. Anyhow, the important point is that such counter-propagating electrons enter the high field part of the pulse and move driven by such field. The dynamics of an electron driven by such extreme fields is not completely understood. We are entering a sort of 'terra incognita' waiting for experiments that show how electrons behave under such extreme circumstances.

There are two effects that can be expected to happen: one is radiation reaction, perhaps the first feature to appear; the other is pair cascading, that is probably expected at even higher intensities.

As pointed above, radiation reaction is one of these effects that are known for many decades but whose description generates endless controversy due to the difficulties in providing self-consistent modeling and due to the direct measurement difficulties. It is clear that an accelerated charged particle is going to radiate, and this is particularly relevant for relativistically driven electrons. From the pioneering description of the laser driven electrons given by [12], and many other works published on that subject, the radiation pattern produced by a laser driven electron (assuming that the loss of energy of this electron is negligibly small) is by now clear. This radiation is useful to determine the motion of the electron and thus the intensity of the laser fields [13]. Therefore the study of the driven electron is a fundamental problem with a large number of applications, see for example [14–16] and references therein for studies on driven electrons by ultra intense or ultrashort structured laser beams, including also the paraxial longitudinal fields. In the case of the onset of relativistic effects (say between 10^{16} and 10^{18} $\mathrm{W/cm^2}$) we can neglect the influence of the emitted radiation to the dynamics of the electron. However, as the intensity increases or when the electron has a large initial speed—radiation reaction has to be taken into account.

The study of radiation reaction is relevant by itself to explore the phenomenology that is opening at intensities between today's limit and the Schwinger critical field ($E_S = m^2c^3/eh = 1.32 \times 10^{18}$ V/m) that corresponds to an intensity around 10^{29} $\mathrm{W/cm^2}$ [17]. This intensity is by far not allowed with current technology. Existing multi-petawatt lasers seem to arrive to a limit that is not much above the current record of 10 PW, although novel schemes are under consideration to bypass the technological difficulties [18] and to arrive well above 50 PW before the end of this decade. Such pulses with a tight focusing could arrive to 10^{24} $\mathrm{W/cm^2}$. In any case, the expected intensity will be at least five orders of magnitude below the Schwinger critical intensity for an electron at rest. However, the situation is completely different if the electron is initially moving against the laser field with a relevant speed (say in the GeV range). In the case of a counter-propagating electron, the intensity seen by the electron (intensity in electron rest frame) can get closer to this limit due to the Lorentz boost. When approaching the Schwinger critical field the dynamics is going to be affected by the creation of pairs [19]. We can say approximately that the incoming electron helps the creation of one pair. The emitted electron-positron help in turn to the creation of new pairs, and we have a cascade. An electron-positron pair cascade in this context has not been observed yet, it is part of this 'terra incognita'. The onset of such cascade [20] will depend on many factors and one of the most relevant will be the slowing down of the incoming electron when entering such extreme laser fields.

Radiation reaction has been studied also using a QED description of the accelerated electron [21] and calculating numerically the driven electron motion. The quantum nature of the electron opens a lot of possible new complexities when describing its deeply relativistic motion. Fortunately solutions of the Dirac equation describing a driven electron are well known from the early times of relativistic quantum mechanics [22] and it is difficult to calculate the electron wave-packet evolution in a laser field [23]. The result is that, in spite of the high electric and magnetic fields involved in extreme laser pulses, many features as the spin dynamics are not relevant for the description of the dynamics [24]. The most relevant quantum feature that remains is the electron-positron pair cascading, the QED fundamental pair creation process. This has been studied too [25] and has been shown that QED effects could be relevant for extremely short pulse durations, maybe we could say unfeasible short. Pulses, laser pulses, of less than one cycle have to be considered apart because the electric field does not average to zero and it is hard to consider them as a propagating field. Today's frontiers in laser technology allow pulses of a few cycles (say two or three complete cycles in the infrared). Those pulses imply a bandwidth covering one octave and extreme amplification techniques of such extreme broadband are not well developed at present time. Talking about extreme lasers (PW and above) todays limits are in the 15 fs duration (for about one micrometer central wavelength) and it is extremely hard to transport and focus such pulses at full power (due to the need of extremely broadband optics combined to very high damage coatings, two properties difficult to be combined). Most of the expected PW or multi-PW systems in operation in construction or in design consider pulsed durations of ten optical cycles or more (of course, we mean durations at focus with short f-number focusing mirrors). Taking all this into account, it is relevant to see how an electron behaves inside an extreme field for the intensities that are expected in the next decade. New effects are going to appear, by sure, and we need to have specific tools developed for that purpose.

The present paper develops a model to analyze classical radiation reaction and the consequent slowing down of a classical electron when entering an extreme laser field (an incoming, non-necessarily monochromatic, classical electromagnetic field) in the framework of the Hamiltonian formalism. Time-honored treatments of the subject (see, for instance, [3–5, 7, 8]) proceed as follows: they start from the full Lorentz force equation for the relativistic electron including the incoming external radiation and the fields generated by the particle. Then, they compute the latter fields (by solving Maxwell's equations with the electron density and current density, as sources) and inject them into the corresponding contribution in the full Lorentz force equation. In so doing, particle and field variables are treated differently from an early stage in the analysis. The Hamiltonian treatment in the present work (even if equivalent to the above standard ones) starts also from first principles and will deal with particle and field variables on an equal footing up to a certain (more advanced) stage, in which the field variables be eliminated in favor of electron ones so as to provide the radiation reaction force $\mathbf{F}_{RR}$. A new form for the latter will be given (to the best of the present authors' knowledge), which will provide a basis for further developments and approximations below. For other treatments of the problem, see for example [26–38].

This work is organized as follows. Section 2 summarizes the essentials of the Hamiltonian description, in an arbitrarily chosen inertial frame in an infinite three-dimensional vacuum, of a classical relativistic electron interacting with the classical electromagnetic field. The field is treated in Lorentz gauge, with the Lorentz condition, and expanded into plane waves with standard complex amplitudes a. Section 3 presents Hamilton's dynamical equations for the particle and the a's. By integrating the a's and substituting into the electron dynamical equation, one arrives directly in section 3 at an exact Lorentz force equation for the latter, which includes the incoming radiation and an exact radiation reaction force $\mathbf{F}_{RR}$ due to the field radiated by the electron, in the considered inertial frame. Section 4 analyzes uniform motion, as a test of consistency. Section 5 discusses briefly Maxwell's equations for the dynamical field radiated by the electron and the description if the radiation gauge were used. Section 6 presents a simpler form of $\mathbf{F}_{RR}$ and employs it for some numerical analysis. Based upon the latter, section 7. analyzes some approximations on $\mathbf{F}_{RR}$. Section 8. deals with a covariant formulation of the exact Lorentz force equation and the exact radiation reaction force $\mathbf{F}_{RR}$ and summarizes the exact solution describing uniform motion of the electron. Section 9 contains some conclusions and discusses several open problems.

2. General formulation

Throughout the work, the MKS system of units [7] is used. c is the velocity of light in vacuum, ϵ_0 and μ_0 are, respectively, the dielectric permittivity and magnetic susceptibility of vacuum. The physical system considered is a single classical relativistic electron with rest mass m and a classical electromagnetic field in vacuum, interacting with each other and in the infinite three-dimensional space, in an arbitrary inertial system. They are supposed not to interact with other systems. $\mathbf{x}$ and t denote a generic position vector and time, respectively. By assumption, the dynamics of the interacting electron-field system will be considered for any $t > -\infty$, that is, for the sake of a simple and neat formulation, the initial conditions for particle and fields are given in the infinitely remote past. Initial conditions at finite time were considered in [36]: in such a case, as the authors emphasize, the initial conditions for the field and for the data cannot be given independently from each other.

At time t, the electron position and momentum are $\mathbf{x}(t)$ and:

$$\mathbf{p} = \mathbf{p}(t) = m\gamma\frac{d\mathbf{x}}{dt} \tag{1}$$

$$\gamma = \gamma(t) = [1 - c^{-2}(d\mathbf{x}/dt)^2]^{-1/2} \tag{2}$$

The electromagnetic (em) field will be described by means of a scalar potential $A^0 = A^0(t, \mathbf{x})$ and a vector potential $\mathbf{A} = \mathbf{A}(t, \mathbf{x})$ (Lorentz gauge: see, for instance, [39]). The standard Lorentz condition will always be assumed:

$$\frac{1}{c^2}\frac{\partial A^0}{\partial t} + \nabla_{\mathbf{x}} \cdot \mathbf{A} = 0 \tag{3}$$

The electron generalized or canonical momentum and Hamiltonian are:

$$\mathbf{P} = \mathbf{p}(t) + e\mathbf{A}(t, \mathbf{x}) \tag{4}$$

$$H_e = [m^2c^4 + c^2(\mathbf{P} - e\mathbf{A})^2]^{1/2} \tag{5}$$

The electric and magnetic fields, in terms of the potentials, are ($\times$denoting vector product):

$$\mathbf{E} = -\nabla_{\mathbf{x}} A^0 - \frac{\partial \mathbf{A}}{\partial t};\ \mathbf{B} = \nabla_{\mathbf{x}} \times \mathbf{A} \tag{6}$$

The scalar and vector potentials admit the following general expansions in terms of plane waves, namely

$$A^0 = \int \frac{d^3\mathbf{k}}{[(2\pi)^3 2\omega_k]^{1/2}} \nu_k [a_{\mathbf{k}}^0 \epsilon_{\mathbf{k}}^0 \exp(i\mathbf{kx}) + a_{\mathbf{k}}^{0*} \epsilon_{\mathbf{k}}^{0*} \exp(-i\mathbf{kx})] \tag{7}$$

$$\mathbf{A} = \sum_{\lambda=1}^{3} \int \frac{d^3\mathbf{k}}{[(2\pi)^3 2\omega_k]^{1/2}} \nu_k [a_{\mathbf{k},\lambda} \boldsymbol{\epsilon}_{\mathbf{k},\lambda} \exp(i\mathbf{kx}) + a_{\mathbf{k},\lambda}^* \boldsymbol{\epsilon}_{\mathbf{k},\lambda}^* \exp(-i\mathbf{kx})] \tag{8}$$

with $\omega_k = ck, k = |\mathbf{k}|$ and * denoting complex conjugate. A real cut-off function $\nu_k = 1$ for $0 \leqslant k \leqslant k_{\max}, \nu_k = 0$ for $k > k_{\max}$ is employed ($k_{\max}$ being the fixed wavevector cut-off). $k_{\max}$, finite by assumption, is a crucial cut-off on the magnitude of any contributing $\mathbf{k}$: this assumption expresses the crucial fact that the electron is extended: would $\boldsymbol{k}_{\mathbf{max}} \to \infty$, then one would be dealing with a point electron.

The ϵ's (polarization vectors) satisfy, by assumption, the following conditions which imply that for every $\mathbf{k}$, $\boldsymbol{\epsilon}_{\mathbf{k},\lambda=1,2,3}$ form an orthonormal basis system

$$\epsilon_{\mathbf{k}}^0 (\epsilon_{\mathbf{k}}^0)^* = c^2 \tag{9}$$

$$\sum_{\lambda=1}^{2} (\boldsymbol{\epsilon}_{\mathbf{k},\lambda})_\alpha (\boldsymbol{\epsilon}_{\mathbf{k},\lambda})_\beta^* = \delta_{\alpha\beta} - \frac{(\mathbf{k})_\alpha (\mathbf{k})_\beta}{\mathbf{k}^2} \tag{10}$$

$$(\boldsymbol{\epsilon}_{\mathbf{k},\lambda=3})_\alpha (\boldsymbol{\epsilon}_{\mathbf{k},\lambda=3})_\beta^* = \frac{(\mathbf{k})_\alpha (\mathbf{k})_\beta}{\mathbf{k}^2} \tag{11}$$

$\alpha, \beta = 1, 2, 3$ denote cartesian components in three-dimensional space, that is, $(\boldsymbol{\epsilon}_{\mathbf{k},\lambda})_\alpha$ is the α-th Cartesian component of the vector $\boldsymbol{\epsilon}_{\mathbf{k},\lambda}$. $\delta_{\alpha\beta}$ is the Kronecker's delta $= 0, 1$ if $\alpha \neq \beta, \alpha = \beta$, respectively. The a's are complex functions which can be regarded as the dynamical variables characterizing the electromagnetic field. The Hamiltonian of the electromagnetic field is:

$$H_{emf} = \epsilon_0 \int d^3\mathbf{k}\ \omega_k \left(\sum_{\lambda=1}^{3} a_{\mathbf{k},\lambda}^* a_{\mathbf{k},\lambda} - a_{\mathbf{k}}^{0*} a_{\mathbf{k}}^0 \right) \tag{12}$$

The total momentum is:

$$\mathbf{P}_{tot} = \mathbf{P} + \mathbf{P}_{emf} \tag{13}$$

$$\mathbf{P}_{emf} = \epsilon_0 \int d^3\mathbf{k}\ \mathbf{k} \left(\sum_{\lambda=1}^{3} a_{\mathbf{k},\lambda}^* a_{\mathbf{k},\lambda} - a_{\mathbf{k}}^{0*} a_{\mathbf{k}}^0 \right) \tag{14}$$

The total Hamiltonian is:

$$H = U_0 + H_e + H_{emf} + eA^0 \tag{15}$$

U_0 is a constant self-energy (irrelevant for the analysis here and which can be discarded if desired).

3. Dynamical equations of motion

Based upon the total Hamiltonian in section 2, Hamilton's equations for field and particle are derived in Subsection 3.1. Hamilton's equations for the field are explicitly integrated in terms of the particle variables and of the incoming field in section 3.2. In section 3.3, the field solution is reshuffled into Hamilton's equation for the electron, thereby arriving at a generalization of the Lorentz force equation which includes the incoming field and the radiation reaction force.

3.1. Hamilton's equations

H depends on the variables $\mathbf{x}$, $\mathbf{P}$ and on the set of all $a_{\mathbf{k}}^0$, $a_{\mathbf{k}}^{0*}$, $a_{\mathbf{k},\lambda}$ and $a_{\mathbf{k},\lambda}^*$, for all $\mathbf{k}$ with $0 \leqslant k \leqslant k_{\max}$ and $\lambda = 1$, 2, 3. Notice that H contains no explicit dependence on t. Hamilton's equations read:

$$\frac{d\mathbf{x}}{dt} = \nabla_{\mathbf{P}} H, \ \frac{d\mathbf{P}}{dt} = -\nabla_{\mathbf{x}} H \tag{16}$$

$$\frac{da^{0*}_{\mathbf{k}}}{dt} = -\frac{i}{\epsilon_0}\frac{\delta H}{\delta a^0_{\mathbf{k}}}, \ \frac{da_{\mathbf{k},\lambda}}{dt} = -\frac{i}{\epsilon_0}\frac{\delta H}{\delta a_{\mathbf{k},\lambda}{}^*} \tag{17}$$

δdenotes variational derivative. Equations (16), (17), (15) become:

$$\frac{d\mathbf{P}}{dt} = e\left[\frac{d\mathbf{x}}{dt}\cdot\nabla_{\mathbf{x}}\right]\mathbf{A} + e\frac{d\mathbf{x}}{dt}\times[\nabla_{\mathbf{x}}\times\mathbf{A}] - e\nabla_{\mathbf{x}} A^0 \tag{18}$$

$$\frac{da^{0*}_{\mathbf{k}}}{dt} = i\omega_k a^{0*}_{\mathbf{k}} - \frac{iev_k\epsilon^0_{\mathbf{k}}}{\epsilon_0[(2\pi)^3 2\omega_k]^{1/2}}\exp[i\mathbf{k}\mathbf{x}(t)] \tag{19}$$

$$\frac{da_{\mathbf{k},\lambda}}{dt} = -i\omega_k a_{\mathbf{k},\lambda} + \frac{iev_k}{\epsilon_0[(2\pi)^3 2\omega_k]^{1/2}}\exp[-i\mathbf{k}\mathbf{x}(t)](\epsilon_{\mathbf{k},\lambda})^*\frac{d\mathbf{x}}{dt} \tag{20}$$

equations (4), (18), (13), (17) and $\frac{d\mathbf{P}_{emf}}{dt} = -\frac{d\mathbf{P}}{dt}$ readily imply the conservation of total energy and momentum in the dynamical evolution of the system:

$$\frac{dH}{dt} = 0, \ \frac{d\mathbf{P}_{tot}}{dt} = 0 \tag{21}$$

3.2. Integration of electromagnetic fields

Another assumption is that the dynamics of the electron-field system will be considered with initial conditions as $t \to -\infty$: then, they are assumed to be non-interacting. Specifically, the incoming electromagnetic field is supposed to be a free one, characterized by the potentials given in equations (7) and equations (8) with $a^{0*}_{\mathbf{k}} = a^{0*}_{\mathbf{k};in}\exp(i\omega_k t)$, $a_{\mathbf{k},\lambda} = a_{\mathbf{k},\lambda;in}\exp(-i\omega_k t)$, with t-independent $a^{0*}_{\mathbf{k};in}$ and $a_{\mathbf{k},\lambda;in}$ and so on for their complex conjugates. For the electron, the initial conditions are $\mathbf{x}_{in}$ and $\mathbf{p}_{in}$.

By integrating equations (19) and (20):

$$\begin{aligned} a^{0*}_{\mathbf{k}} &= a^{0*}_{\mathbf{k};in}\exp[i\omega_k t] \\ &- \int_{-\infty}^{t} dt' \frac{iev_k\epsilon^0_{\mathbf{k}}}{\epsilon_0[(2\pi)^3 2\omega_k]^{1/2}}\exp[i\mathbf{k}\mathbf{x}(t')]\exp[i(\omega_k + i\epsilon)(t - t')] \end{aligned} \tag{22}$$

$$\begin{aligned} a_{\mathbf{k},\lambda} &= a_{\mathbf{k},\lambda;in}\exp[-i\omega_k t] \\ &+ \int_{-\infty}^{t} dt' \frac{iev_k}{\epsilon_0[(2\pi)^3 2\omega_k]^{1/2}}\exp[-i\mathbf{k}\mathbf{x}(t')]\exp[-i(\omega_k - i\epsilon)(t - t')](\epsilon_{\mathbf{k},\lambda})^*\frac{d\mathbf{x}}{dt'} \end{aligned} \tag{23}$$

where $\omega_k \pm i\epsilon$, with a very small $\epsilon > 0$ has been introduced (no confusion with either ϵ_0 or the polarization vectors should arise). This inclusion enables the integrals over t' to converge at $t' \to -\infty$. It is understood that $\epsilon \to 0$ at the end of the computation. And so on for their complex conjugates. Upon replacing equations (22) and (23) and their complex conjugates into equations (7) and (8), one finds:

$$A^0 = A^0_{in} + A^0_{dyn} \tag{24}$$

$$\mathbf{A} = \mathbf{A}_{in} + \mathbf{A}_{dyn} \tag{25}$$

A^0_{in} and $\mathbf{A}_{in}$ are given respectively by the right-hand-sides of equations (7) and (8), with $a^{0*}_{\mathbf{k}} = a^{0*}_{\mathbf{k};in}\exp(i\omega_k t)$, $a_{\mathbf{k},\lambda} = a_{\mathbf{k},\lambda;in}\exp(-i\omega_k t)$ and with their complex conjugates. A^0_{in} and $\mathbf{A}_{in}$ describe free electromagnetic potentials (unaffected by the electron charge): they satisfy the Lorentz condition (equation (3)) and yield through equation (6) the free electromagnetic fields $\mathbf{E}_{in}$, $\mathbf{B}_{in}$ which, in turn, fulfill the source-free Maxwell equations. Let $\Re X$ denote the real part of X. By using equations (22), (23), (7) and (8), one finds the dynamical potentials:

$$\begin{aligned} A^0_{dyn} &= \Re\int\frac{d^3\mathbf{k}|v_k|^2(iec^2)}{(2\pi)^3\omega_k\epsilon_0} \\ &\times\int_{-\infty}^{t} dt'\exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}(t'))]\exp[-i(\omega_k - i\epsilon)(t - t')] \end{aligned} \tag{26}$$

$$\begin{aligned} \mathbf{A}_{dyn} &= \Re\int\frac{d^3\mathbf{k}|v_k|^2 ie}{(2\pi)^3\omega_k\epsilon_0} \\ &\times\int_{-\infty}^{t} dt'\exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}(t'))]\exp[-i(\omega_k - i\epsilon)(t - t')]\frac{d\mathbf{x}}{dt'} \end{aligned} \tag{27}$$

$\mathbf{x}(t')$ is the solution of the electron's dynamical equation, to be given in section 3.3. The dynamical A^0_{dyn} and $\mathbf{A}_{dyn}$ satisfy the Lorentz condition (equation (3)): this is possible due to the inclusion of $-i\epsilon$, enabling to discard contributions at $t' \to -\infty$ (upon performing integrations by parts in t'). The dynamical electric and magnetic fields $\mathbf{E}_{dyn}$ and $\mathbf{B}_{dyn}$, by using equations (6), (26) and (27) are:

$$\mathbf{E}_{dyn} = -\Re \int \frac{d^3\mathbf{k}|v_k|^2 e}{(2\pi)^3 \epsilon_0}$$

$$\times \int_{-\infty}^{t} dt' \exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}(t'))]\exp[-i(\omega_k - i\epsilon)(t - t')]\frac{d\mathbf{x}}{dt'} + \tag{28}$$

$$\Re \int \frac{d^3\mathbf{k}|v_k|^2 ec^2}{(2\pi)^3\omega_k \epsilon_0} \tag{29}$$

$$\times \int_{-\infty}^{t} dt' \exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}(t'))]\exp[-i(\omega_k - i\epsilon)(t - t')]\mathbf{k} \tag{30}$$

$$\mathbf{B}_{dyn} = \Re \int \frac{d^3\mathbf{k}|v_k|^2(-e)}{(2\pi)^3\omega_k \epsilon_0}$$

$$\times \int_{-\infty}^{t} dt' \exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}(t'))]\exp[-i(\omega_k - i\epsilon)(t - t')](\mathbf{k} \times \frac{d\mathbf{x}}{dt'}) \tag{31}$$

The total electric and magnetic fields are

$$\mathbf{E} = \mathbf{E}_{in} + \mathbf{E}_{dyn},\ \mathbf{B} = \mathbf{B}_{in} + \mathbf{B}_{dyn} \tag{32}$$

3.3. Lorentz force equation for the electron: radiation reaction force

From equations (16), (18) (and following, for instance, [4]), one arrives at the Lorentz force equation for the electron:

$$\frac{d}{dt}\left[m\gamma\frac{d\mathbf{x}}{dt}\right] = e\mathbf{E} + e\frac{d\mathbf{x}}{dt} \times \mathbf{B} \tag{33}$$

which, by employing equation (32), becomes:

$$\frac{d}{dt}\left[m\gamma\frac{d\mathbf{x}}{dt}\right] = e\mathbf{E}_{in} + e\frac{d\mathbf{x}}{dt} \times \mathbf{B}_{in} + \mathbf{F}_{RR} \tag{34}$$

where $\mathbf{F}_{RR} = \mathbf{F}_{RR}(t, \mathbf{x}(t))$ is the following radiation reaction (RR) force, when evaluated at the charge's trajectory:

$$\mathbf{F}_{RR} = e\mathbf{E}_{dyn} + e\frac{d\mathbf{x}}{dt} \times \mathbf{B}_{dyn} = \mathbf{F}_{RR1} + \mathbf{F}_{RR2} \tag{35}$$

$$\mathbf{F}_{RR1} = \frac{e^2}{\epsilon_0}\Re \int \frac{d^3\mathbf{k}|v_k|^2}{(2\pi)^3\omega_k}$$

$$\times \int_{-\infty}^{t} dt' \exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}(t'))]\exp[-i(\omega_k - i\epsilon)(t - t')]\left[c^2 - \left(\frac{d\mathbf{x}}{dt'}\right)\cdot\left(\frac{d\mathbf{x}}{dt}\right)\right]\mathbf{k} \tag{36}$$

$$\mathbf{F}_{RR2} = \frac{e^2}{\epsilon_0}\Re \int \frac{d^3\mathbf{k}|v_k|^2}{(2\pi)^3\omega_k}$$

$$\times \int_{-\infty}^{t} dt' \exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}(t'))]\exp[-i(\omega_k - i\epsilon)(t - t')]\left[\left(\mathbf{k}\cdot\frac{d\mathbf{x}}{dt}\right) - ck\right]\frac{d\mathbf{x}}{dt'} \tag{37}$$

The initial conditions for the field as $t \to -\infty$ follow from those embodied in the behavior of $\mathbf{E}_{in}$ and $\mathbf{B}_{in}$ (which in turn follow from those of the chosen $\mathbf{A}_{in}$ and A^0_{in}) in such a limit. The initial conditions for the electron are finite $\mathbf{x}_{in}$ and $\mathbf{p}_{in}$ as $t \to -\infty$.

In [6], a non-relativistic extended classical electron is considered from the outset, and a Lorentz force equation is obtained after integrating Maxwell's equations for the fields in terms of the electron trajectory (thereby, a Hamiltonian approach not being directly implemented in [6]). In such a Lorentz force equation, the actual counterpart of our $\exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}(t'))]$ is approximated by unity, by arguing that the exponent is small in the nonrelativistic limit. So, that Lorentz force equation for the acceleration at time t is a retarded integro-differential one, having in its right-hand-side the acceleration at time $t' \leqslant t$ (referred in [6] as first due to Markov). Equation (34) (from the Hamiltonian approach and in which no such approximations are performed) constitutes a nontrivial generalization of the above developments in [6]. Another interesting feature is that, in such a (Markov) equation, the electron mass m is replaced by the sum $m + \delta m$ (a renormalized mass), with certain electromagnetic mass δm due to the electron-field interaction. In section 8.2, very different approximations on equation (34) will be carried out which will lead to a relativistic effective Lorentz force equation ($k_{\max}$-dependent). Furthermore, in the latter, non-relativistic approximations at a later stage will lead to a nonrelativistic effective Lorentz force equation, with a similar replacement of m by a renormalized mass $m + \delta m$, with the same δm.

4. Uniform motion without incoming field

It is well known that some standard treatments of the radiation reaction effect lead to the unphysical phenomenon of self acceleration, i.e. the particle accelerates even when there is no external electromagnetic field. The aim of this section is to prove that in our case, and in full generality, when there are no external fields, the radiation reaction force exactly cancels out for the uniform motion. Put otherwise, uniform motion in the absence of external fields leads to zero radiation reaction force, hence preventing the appearance of self acceleration.

Let there be no incoming electromagnetic field, that is, $\mathbf{E}_{in} = 0$ and $\mathbf{B}_{in} = 0$ for any $\mathbf{x}$ and any $t > -\infty$. Equivalently, let $a_{\mathbf{k},\lambda;in} = 0$ and $a^{0*}_{\mathbf{k};in} = 0$, for any $\mathbf{k}$, λ. Then: $\mathbf{x}(t) = \mathbf{x}_{in} + \mathbf{v}_0 t$, $\mathbf{p}(t) = \mathbf{p}_0 = m\gamma_0\mathbf{v}_0$, $\gamma_0 = [1 - c^{-2}\mathbf{v}_0^2]^{-1/2}$, with arbitrary constant $\mathbf{v}_0$ such that $1 - c^{-2}\mathbf{v}_0^2 \geqslant 0$ provides an exact solution of the Lorentz equation of motion. It describes a uniform motion with constant velocity $\mathbf{v}_0$ ($\mathbf{v}_0^2$ being allowed as close to c^2 as desired). By using the above $\mathbf{x}(t)$ in equations (22) and (23), and integrating over t' (profiting from the ϵ):

$$a^{0*}_{\mathbf{k}} = \frac{ev_k\epsilon^0_{\mathbf{k}}}{\epsilon_0[(2\pi)^3 2\omega_k]^{1/2}}\frac{\exp[i\mathbf{k}\mathbf{x}(t)]}{\omega_k + i\epsilon - \mathbf{k}\cdot\mathbf{v}_0} \tag{38}$$

$$a_{\mathbf{k},\lambda} = \frac{ev_k(\boldsymbol{\epsilon}_{\mathbf{k},\lambda})^*\cdot\mathbf{v}_0}{\epsilon_0[(2\pi)^3 2\omega_k]^{1/2}}\frac{\exp[-i\mathbf{k}\mathbf{x}(t)]}{\omega_k - i\epsilon - \mathbf{k}\cdot\mathbf{v}_0} \tag{39}$$

The dynamical potentials for a generic $\mathbf{x}$ outside or on the trajectory are:

$$A^0_{dyn} = \Re\int\frac{d^3\mathbf{k}|v_k|^2(ec^2)}{(2\pi)^3\omega_k\epsilon_0}\times\frac{\exp[i\mathbf{k}(\mathbf{x} - \mathbf{v}_0 t)]}{\omega_k - i\epsilon - \mathbf{k}\cdot\mathbf{v}_0} \tag{40}$$

$$\mathbf{A}_{dyn} = \Re\int\frac{d^3\mathbf{k}|v_k|^2 e}{(2\pi)^3\omega_k\epsilon_0}\times\frac{\mathbf{v}_0\exp[i\mathbf{k}(\mathbf{x} - \mathbf{v}_0 t)]}{\omega_k - i\epsilon - \mathbf{k}\cdot\mathbf{v}_0} \tag{41}$$

Clearly, one has $\frac{d}{dt}[m\gamma\frac{d\mathbf{x}}{dt}] = 0$. Next, the radiation reaction force $\mathbf{F}_{RR}$ will be evaluated on the electron trajectory, that is, for $\mathbf{x}(t) = \mathbf{x}_{in} + \mathbf{v}_0 t$. For $\mathbf{F}_{RR1}$, and integrating over t', one finds:

$$\mathbf{F}_{RR1} = \frac{e^2}{\epsilon_0}\Re\int\frac{d^3\mathbf{k}|v_k|^2}{i(2\pi)^3\omega_k}\frac{(c^2 - \mathbf{v}_0\cdot\mathbf{v}_0)\mathbf{k}}{\omega_k - i\epsilon - \mathbf{k}\cdot\mathbf{v}_0} \tag{42}$$

At this stage, for real x, one invokes the formal expression: $1/(x - i\epsilon) = P(1/x) + i\pi\delta(x)$, where P and δ denote Cauchy principal value and the Dirac delta function, respectively. Upon replacing the later formula in equation (42) and employing $\Re(ix) = 0$ for real x, the principal value does not contribute and one gets:

$$\mathbf{F}_{RR1} = \frac{e^2}{\epsilon_0}\Re\int\frac{d^3\mathbf{k}|v_k|^2}{(2\pi)^3\omega_k}(c^2 - \mathbf{v}_0^2)\mathbf{k}\delta(-\omega_k + \mathbf{k}\cdot\mathbf{v}_0) \tag{43}$$

Since $k^2\delta(-\omega_k + \mathbf{k}\cdot\mathbf{v}_0) = 0$, it follows: $\mathbf{F}_{RR1} = 0$. A similar analysis shows that $\mathbf{F}_{RR2} = 0$ and, hence, $\mathbf{F}_{RR} = 0$. The conclusion is that is $\mathbf{x}(t) = \mathbf{x}_{in} + \mathbf{v}_0 t$ is an exact solution of equation (33). The field energy, eA^0 and the total energy (H), evaluated for $\mathbf{x}(t) = \mathbf{x}_{in} + \mathbf{v}_0 t$ are:

$$H_{emf} = -\frac{e^2}{4\pi\epsilon_0}\left[\int_0^{k_{\max}}\frac{dk}{\pi}\right] \tag{44}$$

$$eA^0 = +\frac{e^2}{4\pi\epsilon_0}\left[\int_0^{k_{\max}}\frac{dk}{\pi}\right]\frac{c}{|\mathbf{v}_0|}\ln\left[\frac{c + |\mathbf{v}_0|}{c - |\mathbf{v}_0|}\right] \tag{45}$$

$$\begin{aligned}H = U_0 &+ \frac{mc^2}{[1 - c^{-2}\mathbf{v}_0^2]^{1/2}}\\ &+ \frac{e^2}{4\pi\epsilon_0}\left[\int_0^{k_{\max}}\frac{dk}{\pi}\right]\left[-1 + \frac{c}{|\mathbf{v}_0|}\ln\left[\frac{c + |\mathbf{v}_0|}{c - |\mathbf{v}_0|}\right]\right]\end{aligned} \tag{46}$$

In the non-relativistic limit $|\mathbf{v}_0| \ll c$, equation (46) becomes:

$$H \simeq mc^2 + U_0 + \delta U_0 + \frac{(m + \delta m)\mathbf{v}_0^2}{2} \tag{47}$$

$$\delta U_0 = +\frac{e^2}{4\pi\epsilon_0}\left[\int_0^{k_{\max}}\frac{dk}{\pi}\right] \tag{48}$$

$$\delta m = \frac{4}{3c^2}\frac{e^2}{4\pi\epsilon_0}\left[\int_0^{k_{\max}}\frac{dk}{\pi}\right] = \frac{4}{3c^2}|\delta U_0| \tag{49}$$

Similarly, $e\mathbf{A}$, evaluated for $\mathbf{x}(t)$, reads:

$$e\mathbf{A} = \frac{e^2}{4\pi\epsilon_0}\left[\int_0^{k_{\max}} \frac{dk}{\pi}\right]\frac{\mathbf{v}_0}{|\mathbf{v}_0|}\frac{1}{c}\ln\left[\frac{c+|\mathbf{v}_0|}{c-|\mathbf{v}_0|}\right] \tag{50}$$

Notice that the (effective) electromagnetic mass δm and energy δU_0 turn out to be related by $\delta m = \frac{4}{3c^2}\delta U_0$. The latter agrees with time-honored previous classical computations: see, for instance, [7]. See also [6].

5. Maxwell equations for $\mathbf{E}_{dyn}$ and $\mathbf{B}_{dyn}$. remarks on radiation gauge

5.1. Maxwell equations for $\mathbf{E}_{dyn}$ and $\mathbf{B}_{dyn}$ in Lorentz gauge

Some computations yield (upon performing integrations by parts in t' and discarding certain contributions at $t' \to -\infty$, due to the inclusion of $-i\epsilon$) the following Maxwell equations for the dynamical fields $\mathbf{E}_{dyn}$ and $\mathbf{B}_{dyn}$:

$$\nabla_{\mathbf{x}}\cdot\mathbf{E}_{dyn} = \frac{\rho}{\epsilon_0},\ \rho = \rho(t,\mathbf{x}) = \Re\int\frac{d^3\mathbf{k}|v_k|^2 e}{(2\pi)^3}\exp[i\mathbf{k}(\mathbf{x}-\mathbf{x}(t))] \tag{51}$$

$$\nabla_{\mathbf{x}}\cdot\mathbf{B}_{dyn} = 0 \tag{52}$$

$$\nabla_{\mathbf{x}}\times\mathbf{E}_{dyn} = -\frac{\partial\mathbf{B}_{dyn}}{\partial t} \tag{53}$$

$$\nabla_{\mathbf{x}}\times\mathbf{B}_{dyn} = \frac{1}{c^2}\frac{\partial\mathbf{E}_{dyn}}{\partial t} + \mu_0\mathbf{j} \tag{54}$$

$$\mathbf{j} = \mathbf{j}(t,\mathbf{x}) = \Re\int\frac{d^3\mathbf{k}|v_k|^2 e}{(2\pi)^3}\exp[i\mathbf{k}(\mathbf{x}-\mathbf{x}(t))]\frac{d\mathbf{x}}{dt} = \rho(t,\mathbf{x})\frac{d\mathbf{x}}{dt} \tag{55}$$

Using standard techniques, one derives directly the wave equations for the dynamical potentials:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta_{\mathbf{x}}^2\right)\mathbf{A}_{dyn} = \mu_0\mathbf{j} \tag{56}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta_{\mathbf{x}}^2\right)A_{dyn}^0 = \frac{\rho}{\epsilon_0} \tag{57}$$

equations (56) and (57) lead, by using the standard retarded Green's function, to the Liénard-Wiechert potentials containing ρ and $\mathbf{j}$. In the limit $k_{\max}\to\infty$, $\rho(t,\mathbf{x})$ will tend to Dirac's $e\delta(\mathbf{x}-\mathbf{x}(t))$ and so on for $\mathbf{j}$. Hence, one obtains the physically sound result that the radiation field comes from the EM fields computed from the (singular) charge and current densities located at the trajectory [36].

5.2. Remarks on radiation gauge

In this subsection , some shorts remarks (omitting details) will be made on the radiation gauge, thereby detaching along the former from the discussion in the Lorentz gauge. Adequate discussions on the comparison between both gauges can be seen, for instance, in [40].

In the radiation gauge, $a_{\mathbf{k};in}^{0*} = a_{\mathbf{k}}^{0*} = 0$ (and so on for their complex conjugates), $A^0 = 0$ and $\nabla_{\mathbf{x}}\cdot\mathbf{A} = 0$. Accordingly, the analysis in sections 2, 3 and 4 and in section 5.1 simplify: details will be omitted. $\mathbf{B}_{dyn}$ in Lorentz and radiation gauges coincide, but $\mathbf{E}_{dyn}$ does not. In fact, in the radiation gauge ρ does not appear explicitly and equation (51) is replaced by $\nabla_{\mathbf{x}}\cdot\mathbf{E}_{dyn} = 0$. However, on the electron trajectory $\mathbf{x} = \mathbf{x}(t)$, $\mathbf{E}_{dyn}$ coincide in both Lorentz and radiation gauges (a property which ceases to hold for arbitrary $\mathbf{x}$). Then, the Lorentz equation of motion is the same (namely, equation (34)) in both Lorentz and radiation gauges.

6. Radiation reaction force: integration over k

The integrations over $\mathbf{k}$ can be carried out exactly in both $\mathbf{F}_{RR1}$ and $\mathbf{F}_{RR2}$: firstly, over the whole solid angle in $\mathbf{k}$ and, secondly, in $0 \leqslant k \leqslant k_{\max}$. Let $\mathbf{a} = \mathbf{x} - \mathbf{x}(t')$, $a = |\mathbf{a}|$ and $j_0(x) = \frac{\sin x}{x}$ and $j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$ be the spherical Bessel functions of orders 0 and 1, respectively.

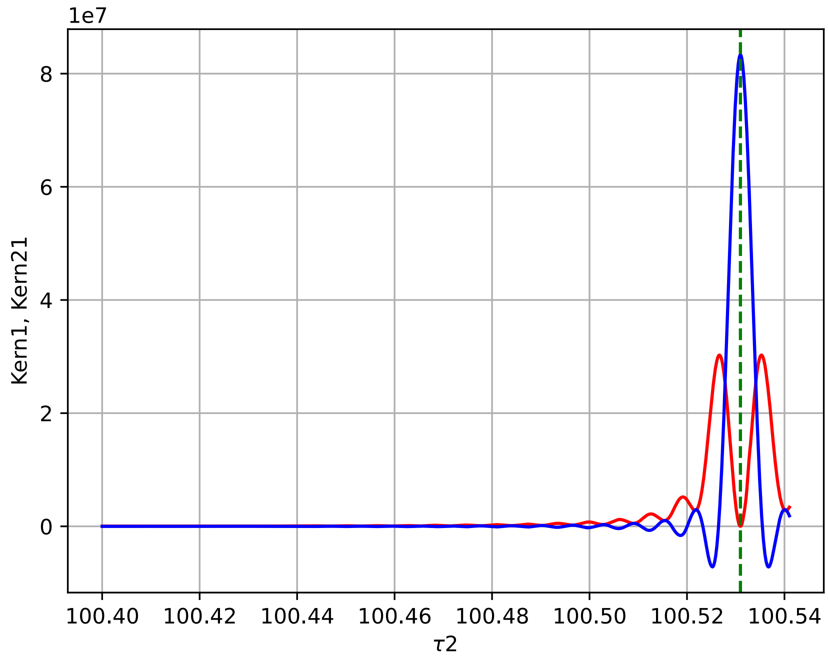

Figure 1. Plot of integrand kernels, i.e, the parts including spherical Bessel functions, in dimensionless variables $\tau' = \tau 2 = \omega_0 t'$, at a fixed value of $\tau = \omega_0 t = 32\pi$. Kernels are rather spiky, but not singular at any τ' value, and their non-zero values are concentrated in a narrow region around $\tau' = \tau$. In red, kernel of equation (58), in blue kernel of equation (60).

The results are:

$$\mathbf{F}_{RR1} = \frac{e^2}{4\pi^2 c\epsilon_0}\int_{-\infty}^{t} dt' \exp\left[-\epsilon(t-t')\right]\left[c^2 - \left(\frac{d\mathbf{x}}{dt'}\right)\cdot\left(\frac{d\mathbf{x}}{dt}\right)\right]\frac{\mathbf{a}}{a} \times \left[\frac{k_{\max}^2 (j_1(k_{\max}(a - c(t-t'))) - j_1(k_{\max}(a + c(t-t')))}{a} + \frac{k_{\max}(j_0(k_{\max}(a - c(t-t'))) - j_0(k_{\max}(a + c(t-t')))}{a^2}\right] \tag{58}$$

$$\mathbf{F}_{RR2} = \mathbf{F}_{RR2,1} + \mathbf{F}_{RR2,2} \tag{59}$$

$$\mathbf{F}_{RR2,1} = -\frac{e^2}{4\pi^2\epsilon_0}\int_{-\infty}^{t} dt' \exp[-\epsilon(t-t')]\frac{d\mathbf{x}}{dt'}\times \frac{k_{\max}^2 (j_1(k_{\max}(a - c(t-t'))) + j_1(k_{\max}(a + c(t-t')))}{a} \tag{60}$$

$$\mathbf{F}_{RR2,2} = \frac{e^2}{4\pi^2 c\epsilon_0}\int_{-\infty}^{t} dt' \exp[-\epsilon(t-t')]\left(\frac{\mathbf{a}}{a}\cdot\frac{d\mathbf{x}}{dt}\right)\frac{d\mathbf{x}}{dt'}\times \left[\frac{k_{\max}^2 (j_1(k_{\max}(a - c(t-t'))) - j_1(k_{\max}(a + c(t-t')))}{a} + \frac{k_{\max}(j_0(k_{\max}(a - c(t-t'))) - j_0(k_{\max}(a + c(t-t')))}{a^2}\right] \tag{61}$$

Equations (58), (60), (61) and (34) constitute an exact integro-differential equation for $\mathbf{x}(t)$: to implement them for effective numerical computations lies outside our scope here. At the present stage, equations (58), (60), (61) display that their integrands containing the spherical Bessel functions oscillate indefinitely, tend to vanish as $a - c(t - t')$ and $a + c(t - t')$ grow and will give rise to interferences and cancellations. So, equations (58), (60), (61) will suggest, at least indirectly and qualitatively, in section 7.1 from which domains receives $\mathbf{F}_{RR}$ its leading contributions: see figure 1 for an example of the behavior of the $[\frac{k_{\max}^2 (j_1(k_{\max}(a - c(t-t'))) - j_1(k_{\max}(a + c(t-t')))}{a} + \frac{k_{\max}(j_0(k_{\max}(a - c(t-t'))) - j_0(k_{\max}(a + c(t-t')))}{a^2}]$ and $\frac{k_{\max}^2 (j_1(k_{\max}(a - c(t-t'))) + j_1(k_{\max}(a + c(t-t')))}{a}$ kernels when computed over a constant velocity trajectory corresponding to a counterpropagating 100 MeV electron, and $k_{\max} = 1000 \times k_0$. $k_0 = 2\pi/\lambda_0 = \omega_0/c$ is chosen having in mind an external field corresponding to an IR ultra intense pulse with central wavelength $\lambda_0 = 800$ nm. The spiky nature of these functions is apparent, specially at large values of k_{max}, and that the whole integral(s) will receive their main contribution from a very narrow region around $t = t'$.

6.1. Exact Cancellation of the Radiation Reaction Force for an Uniform Motion

An interesting property of the integrals in equations (58), (60) and (61) is that they are, without incoming filed, amenable to an exact analytical computation in the case of an uniform motion. Then, the integrands $[c^2 - (\frac{d\mathbf{x}}{dt'}) \cdot (\frac{d\mathbf{x}}{dt})]\frac{\mathbf{a}}{a}$, $\frac{d\mathbf{x}}{dt'}$ and $(\frac{\mathbf{a}}{a} \cdot \frac{d\mathbf{x}}{dt})\frac{d\mathbf{x}}{dt'}$ assume constant values, namely, $c^2\gamma_0^{-2}\boldsymbol{\beta}_0/\beta_0$, $c\boldsymbol{\beta}_0$, $c^2\beta_0\boldsymbol{\beta}_0$ and can be taken out of the integral sign, while the kernels take the simpler forms $[\frac{k_{max}^2(j_1(-ck_{max}(1-\beta_0)(t-t')) - j_1(ck_{max}(1+\beta_0)(t-t')))}{c\beta_0(t-t')} + \frac{k_{max}(j_0(-ck_{max}(1-\beta_0)(t-t')) - j_0(ck_{max}(1+\beta_0)(t-t')))}{(c\beta_0(t-t'))^2}]$ and $\frac{k_{max}^2(j_1(-ck_{max}(1-\beta_0)(t-t')) + j_1(ck_{max}(1+\beta_0)(t-t')))}{c\beta_0(t-t')}$ that can be exactly integrated over t' (as indefinite integrals, hence still depending on t and t'). The obtained integrals, named collectively $I_{unif}(t, t')$, have the property that $I_{unif}(t, t) - I_{unif}(t, -\infty) = 0$, hence the radiation reaction force is exactly zero for the uniform motion, as it should be. This is an extra consistency check, on top of the considerations made in section 4.

7. Approximations for radiation reaction terms (equations (34), (35), (36) and (37))

7.1. A strategy

Here, it will be allowed that the incoming fields do not vanish in principle ($\mathbf{E}_{in} \neq 0$ and $\mathbf{B}_{in} \neq 0$), so that the electron has a non-vanishing acceleration. Inspections at the integrals over t' in equations (36) and (37) and also in equations (58), (60) and (61) suggest that they receive their leading contributions from a suitably small interval $t - \tau_{RR} < t' < t$, with small τ_{RR}, so that $t - \tau_{RR}$ is adequately close to the upper integration limit $t' = t$. In fact, for fixed $\mathbf{k}$, the larger is $t - t'$, the larger the variations of $[\mathbf{k}(\mathbf{x}(t) - \mathbf{x}(t')) - (\omega_k - i\epsilon)(t - t')]$ and, so, the greater the oscillations of $\exp i[\mathbf{k}(\mathbf{x}(t) - \mathbf{x}(t')) - (\omega_k - i\epsilon)(t - t')]$ and, finally, the more important the cancellations of the latter exponential upon integrating over t'. Consequently, one could expect that the dominant contribution to $\int_{-\infty}^{t} dt'$ comes from $\int_{t-\tau_{RR}}^{t} dt'$. Consistently, upon expanding about $t' = t$, one approximates: (i) $\mathbf{x}(t) - \mathbf{x}(t') \simeq \frac{d\mathbf{x}}{dt'}_{t'=t}(t - t') - \frac{1}{2}\frac{d^2\mathbf{x}}{dt'^2}_{t'=t}(t - t')^2$, (ii) $\frac{d\mathbf{x}}{dt'} \simeq \frac{d\mathbf{x}}{dt'}_{t'=t} + \frac{d^2\mathbf{x}}{dt'^2}_{t'=t}(t' - t)$, (iii) $\exp[i\mathbf{k}(\mathbf{x}(t) - \mathbf{x}(t'))]\exp[-i(\omega_k - i\epsilon)(t - t')] \simeq \left[1 - i\frac{1}{2}\left(\mathbf{k}\frac{d^2\mathbf{x}}{dt'^2}_{t'=t}\right)(t - t')^2\right]\exp\left[i\left(\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}_{t'=t}\right) - (\omega_k - i\epsilon)\right)(t - t')\right]$.

Notice that $\exp\left[i(\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}_{t'=t}\right) - (\omega_k - i\epsilon))(t - t')\right]$ in (iii) is neither expanded nor approximated, which will be crucial for the overall approximate consistency of the strategy. It is supposed that $\frac{1}{2}\frac{d^2\mathbf{x}}{dt'^2}_{t'=t}(t - t')^2$ is smaller than $\frac{d\mathbf{x}}{dt'}_{t'=t}(t - t')$, in that integration interval. Then, equations (35), (36) and (37) yield the approximation:

$$\mathbf{F}_{RR} \simeq \mathbf{F}_{RR,app} = \mathbf{F}_{RR,app,0} + \mathbf{F}_{RR,app,1} \tag{62}$$

$$\begin{aligned}\mathbf{F}_{RR,app,0} = \frac{e^2}{\epsilon_0}\Re\int\frac{d^3\mathbf{k}|\nu_k|^2}{(2\pi)^3}\int_{t-\tau_{RR}}^{t} dt' \exp\left[i(\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t} - (\omega_k - i\epsilon))(t - t')\right]\\ \times\left(\left[c^2 - \left(\frac{d\mathbf{x}}{dt}\right)^2\right]\frac{\mathbf{k}}{\omega_k} + \frac{1}{\omega_k}\left[-\omega_k + (\mathbf{k}\cdot\frac{d\mathbf{x}}{dt})\right]\frac{d\mathbf{x}}{dt}\right)\end{aligned} \tag{63}$$

$$\begin{aligned}\mathbf{F}_{RR,app,1} = \frac{e^2}{\epsilon_0}\Re\int\frac{d^3\mathbf{k}|\nu_k|^2}{(2\pi)^3}\int_{t-\tau_{RR}}^{t} dt' \exp\left[i(\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t} - (\omega_k - i\epsilon))(t - t')\right]\\ \times\left[-\left(\left[c^2 - \left(\frac{d\mathbf{x}}{dt}\right)^2\right]\frac{\mathbf{k}}{\omega_k} + \frac{1}{\omega_k}\left[-c|\mathbf{k}| + \left(\mathbf{k}.\frac{d\mathbf{x}}{dt}\right)\right]\frac{d\mathbf{x}}{dt}\right)i\frac{1}{2}\left(\mathbf{k}\cdot\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t}\right)(t - t')^2\right.\\ \left.-\frac{\mathbf{k}}{\omega_k}\left(\frac{d\mathbf{x}}{dt}\cdot\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t}\right)(t' - t) + \frac{1}{\omega_k}\left[-c|\mathbf{k}| + \left(\mathbf{k}\cdot\frac{d\mathbf{x}}{dt}\right)\right]\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t}(t' - t)\right]\end{aligned} \tag{64}$$

The computation for uniform motion in section 4 yielding $\mathbf{F}_{RR} = 0$ provides a strategy towards the computation of $\mathbf{F}_{RR,app,0}$ and this is done. There is an important difference due to the fact that the integration over t' is not carried out for $t' < t - \tau_{RR}$. Then, upon integrating over t': $\int_{t-\tau_{RR}}^{t} dt' \exp\left[i\left(\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t} - (\omega_k - i\epsilon)\right)(t - t')\right] = \frac{i}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t} - (\omega_k - i\epsilon)}\left[1 - \exp(-\epsilon\tau_{RR})\exp i\tau_{RR}(\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t} - \omega_k)\right]$, Then, there is an additional contribution proportional to $\exp(-\epsilon\tau_{RR})$. One has:

$$\mathbf{F}_{RR,app,0} = \frac{e^2}{\epsilon_0}\exp(-\epsilon\tau_{RR})\int\frac{d^3\mathbf{k}|v_k|^2}{(2\pi)^3}$$
$$\times\frac{\sin\tau_{RR}\left(\mathbf{k}\left(\frac{d\mathbf{x}}{dt}\right)-\omega_k\right)}{\left(\mathbf{k}\left(\frac{d\mathbf{x}}{dt}\right)-\omega_k\right)}\left(\left[c^2-\left(\frac{d\mathbf{x}}{dt}\right)^2\right]\frac{\mathbf{k}}{\omega_k}+\frac{1}{\omega_k}\left[-\omega_k+\left(\mathbf{k}\cdot\frac{d\mathbf{x}}{dt}\right)\right]\frac{d\mathbf{x}}{dt}\right) \tag{65}$$

A priori, the magnitude of $\mathbf{F}_{RR,app,0}$ would be an open question, due to two apparently conflicting features: $\exp(-\epsilon\tau_{RR})$ (as $\epsilon) \to 0$ at the end of the computation) and the smallness of τ_{RR} (which is a basis of the above approximation). On the other hand , the net result, after integration over $\mathbf{k}$, is that $\mathbf{F}_{RR,app,0}$ turns out to be nonlinear in the electron velocity: this appears to be an unphysical outcome of the approximation so far, because in section 4 it has been seen that for uniform motion the radiation reaction vanishes exactly. Then, such a controversial magnitude of $\mathbf{F}_{RR,app,0}$ due to those two features appears to be an artifact of the computation, when one pushes the interpretation of those features possibly too far. Then, it does not seem unreasonable to set $\mathbf{F}_{RR,app,0}\simeq 0$ consistently, that is, to regard $\exp(-\epsilon\tau_{RR})\frac{\sin\tau_{RR}(\mathbf{k}\left(\frac{d\mathbf{x}}{dt}\right)-\omega_k)}{\left(\mathbf{k}\left(\frac{d\mathbf{x}}{dt}\right)-\omega_k\right)}$ as implying a negligible result for $\mathbf{F}_{RR,app,0}$ compared to $\mathbf{F}_{RR,app,1}$. The approximate computation of $\mathbf{F}_{RR,app,1}$ is undertaken from now onwards. One now integrates approximately: $\int_{t-\tau_{RR}}^{t}dt'\exp\left[i(\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-(\omega_k-i\epsilon))(t-t')\right]\simeq\frac{i}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-(\omega_k-i\epsilon)}$,having neglected the contribution at $t'=t-\tau_{RR}$. Then:

$$\mathbf{F}_{RR,app}\simeq\mathbf{F}_{RR,app,1}\simeq\mathbf{F}_{RR,app,2} \tag{66}$$

Further analysis of $\mathbf{F}_{RR,app}$ will be considered in appendix and in section 7.2. Notice that, in the present approximations, τ_{RR} no longer appears in $\mathbf{F}_{RR,app,2}$.

7.2. Analysis of radiation reaction force approximation $\mathbf{F}_{RR,app,2}$

One has:

$$\mathbf{F}_{RR,app,2}=\frac{e^2}{\epsilon_0c^2}\frac{k_{\max}}{(2\pi)^3}\mathbf{I} \tag{67}$$

The computation of $\mathbf{I}$ (real) is outlined in appendix. The result is:

$$\mathbf{I}=\mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3+\mathbf{I}_4 \tag{68}$$

$$\boldsymbol{\beta}=\frac{1}{c}\frac{d\mathbf{x}}{dt} \tag{69}$$

$$l_1=\frac{2\pi}{|\boldsymbol{\beta}|}\ln\left[\frac{1-|\boldsymbol{\beta}|}{1+|\boldsymbol{\beta}|}\right] \tag{70}$$

$$l_2=\frac{2\pi}{|\boldsymbol{\beta}|^3}\ln\left[\frac{1-|\boldsymbol{\beta}|}{1+|\boldsymbol{\beta}|}\right]+\frac{4\pi}{|\boldsymbol{\beta}|^2(1-|\boldsymbol{\beta}|)(1+|\boldsymbol{\beta}|)} \tag{71}$$

$$l_3=-\frac{1}{2}l_2 \tag{72}$$

$$l_4=\frac{1}{|\boldsymbol{\beta}|^2}\left[-3l_3-\frac{4\pi}{(1-|\boldsymbol{\beta}|^2)^2}\right] \tag{73}$$

$$\mathbf{I}_1=-(1-|\boldsymbol{\beta}|^2)\left[l_3\frac{d^2\mathbf{x}}{dt'^2}_{t'=t}+l_4\left(\frac{d^2\mathbf{x}}{dt'^2}_{t'=t}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta}\right] \tag{74}$$

$$\mathbf{I}_2=-l_2\left(\frac{d^2\mathbf{x}}{dt'^2}_{t'=t}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta} \tag{75}$$

$$\mathbf{I}_3=\mathbf{I}_2 \tag{76}$$

$$\mathbf{I}_4=l_1\frac{d^2\mathbf{x}}{dt'^2}_{t'=t} \tag{77}$$

$$\mathbf{I}=[l_1-(1-|\boldsymbol{\beta}|^2)l_3]\frac{d^2\mathbf{x}}{dt^2}$$
$$-[(1-|\boldsymbol{\beta}|^2)l_4+2l_2]\left[\left(\frac{d^2\mathbf{x}}{dt^2}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta}\right] \tag{78}$$

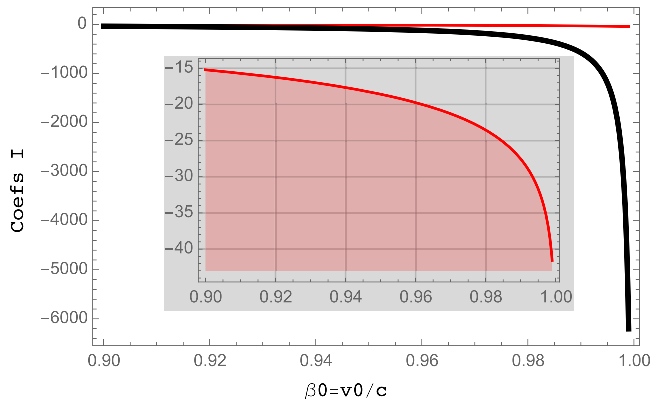

Figure 2. Scaling of radiation reaction force coefficients in the ultra-relativistic limit. The inset shows the behaviour of the coefficient proportional to $\frac{d^2\mathbf{x}}{dt^2}$ (in red); the one proportional to $(\frac{d^2\mathbf{x}}{dt^2}\cdot\boldsymbol{\beta})\boldsymbol{\beta}$ (in black) scales as γ^2.

$$I_1 - (1 - |\boldsymbol{\beta}|^2)I_3 = \frac{2\pi}{|\boldsymbol{\beta}|^2}\left[1 + \frac{1 + |\boldsymbol{\beta}|^2}{2|\boldsymbol{\beta}|}\ln\left[\frac{1 - |\boldsymbol{\beta}|}{1 + |\boldsymbol{\beta}|}\right]\right] \tag{79}$$

$$-(1 - |\boldsymbol{\beta}|^2)I_4 - 2I_2 = \frac{2\pi}{|\boldsymbol{\beta}|^2}\left[-\frac{(3 + |\boldsymbol{\beta}|^2)}{2|\boldsymbol{\beta}|^3}\ln\left[\frac{1 - |\boldsymbol{\beta}|}{1 + |\boldsymbol{\beta}|}\right]\right] + (2\pi/|\boldsymbol{\beta}|^2)\left(-\frac{3}{|\boldsymbol{\beta}|^2} - \frac{2}{1 - |\boldsymbol{\beta}|^2}\right) \tag{80}$$

Then, in the framework of the above approximations: $\mathbf{F}_{RR} \simeq \mathbf{F}_{RR,app,2}$ is linear in $\frac{d^2\mathbf{x}}{dt'^2}_{t'=t}$ (but non-linear in $\frac{d\mathbf{x}}{dt}$). The physical dimension of $\mathbf{F}_{RR,app,2}$ is $\frac{e^2}{\epsilon_0 c^2(2\pi)^3}k_{\max}|\frac{d^2\mathbf{x}}{dt'^2}_{t'=t}|f = \frac{e^2}{\epsilon_0 c^3}|(ck_{\max})\frac{d^2\mathbf{x}}{dt'^2}_{t'=t}|f$. Notice that $|\mathbf{I}| = |\frac{d^2\mathbf{x}}{dt'^2}_{t'=t}|f$. f is a dimensionless (nonlinear) function of $|\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}|/c$ and of the angle σ between $\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}$ and $\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t}$. Figure 2 shows the coefficients multiplying $\frac{d^2\mathbf{x}}{dt^2}$ and $(\frac{d^2\mathbf{x}}{dt^2}\cdot\boldsymbol{\beta})\boldsymbol{\beta}$ in equation (78) as a function of β in the highly relativistic range of velocities. Both coefficients turn out to be negative, although the one multiplying $\left(\frac{d^2\mathbf{x}}{dt^2}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta}$ is substantially larger than the other, scaling in fact as γ^2 in the ultra-relativistic limit.

Summarizing, the approximate equation of motion for the electron follows by combining successively equation (34) (62) (with $\mathbf{F}_{RR,app,0} \simeq 0$), (64), (66), (90), (67), (91), (68)–(77).

An interesting point is the following. Once $\mathbf{F}_{RR,app,2}$ has been obtained as indicated above in the chosen inertial frame (as an approximation to the exact $\mathbf{F}_{RR}$), at a later stage one can replace the acceleration $\frac{d^2\mathbf{x}}{dt^2}$ in the former by what implies the exact Lorentz equation (34) in the same inertial system, but now with the radiation reaction force $\mathbf{F}_{RR}$ eliminated in its right-hand-side. That is, one keeps only $e\mathbf{E}_{in} + e\frac{d\mathbf{x}}{dt} \times \mathbf{B}_{in}$ from the exact Lorentz equation (34), which in general depend on $\mathbf{x}(t)$ and t. This would give rise to a partial differential equation linear in the acceleration but certainly nonlinear in the velocity (and still $k_{\max}$-dependent, as well).

The above approximate $\mathbf{F}_{RR}$ has the following properties:

(1) if $\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t} = 0$, there is no radiation reaction.

(2) Let the electron have an accelerated motion with very small velocity, that is, let $|\boldsymbol{\beta}|$ be very small ($|\boldsymbol{\beta}| \ll 1$). By keeping the leading approximations in equations (68)–(77), one finds:

$$\mathbf{F}_{RR,app,2} \simeq \frac{e^2k_{\max}}{\epsilon_0 c^2(2\pi)^2}\left[-\left(\frac{4}{3} + \frac{8}{15}\beta^2\right)\left(\frac{d^2\mathbf{x}}{dt^2}\right) - \frac{5}{3}\left(\left(\frac{d^2\mathbf{x}}{dt^2}\right)\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta}\right] \tag{81}$$

Then, at very low velocity, the radiation reaction force is opposite to the acceleration at the same t, which is physically reasonable.

(3) Conversely, let the electron have an accelerated motion with very large velocity (very close in magnitude to c). By keeping the leading approximations in equations (68)–(77), one finds for $|\boldsymbol{\beta}| \to 1$ (with $|\boldsymbol{\beta}| < 1$):

$$\mathbf{F}_{RR,app,2} \simeq [\frac{e^2}{\epsilon_0 c^2 (2\pi)^3} k_{\max}] \left[2\pi \ln(1 - |\boldsymbol{\beta}|) \left[\frac{d^2\mathbf{x}}{dt^2} \right] + 4\pi \left(-\ln(1 - |\boldsymbol{\beta}|) - \frac{1}{2(1 - |\boldsymbol{\beta}|)} \right) \left[\left(\frac{d^2\mathbf{x}}{dt^2} \cdot \boldsymbol{\beta} \right) \boldsymbol{\beta} \right] \right] \tag{82}$$

Both contributions to $\mathbf{F}_{RR,app,2}$ grow as the velocity is very close in magnitude to c. The $\left[\frac{d^2\mathbf{x}}{dt^2}\right]$ contribution to the radiation reaction force is opposite to the former (as $\ln(1 - |\boldsymbol{\beta}|) < 0$). The sign of the $(\frac{d^2\mathbf{x}}{dt^2} \cdot \boldsymbol{\beta})\boldsymbol{\beta}$ contribution is negative for sufficiently large velocity, since then $\left(-\ln(1 - |\boldsymbol{\beta}|) - \frac{1}{1-|\boldsymbol{\beta}|}\right) < 0$. Those signs are not physically unreasonable either.

A curious consistency condition *a posteriori* is the following: δm in equation (49) coincides with the coefficient $\frac{e^2 k_{\max}}{\epsilon_0 c^2 (2\pi)^2}\frac{4}{3}$ multiplying $\left(\frac{d^2\mathbf{x}}{dt^2}\right)$ in the right-hand-side of equation (81). That is,the above low velocity approximation in this section also produces the same δm obtained in the exact computation for uniform motion in section 4. Stimulated by the last fact,for $|\boldsymbol{\beta}| \ll 1$ (that is considering only $\mathbf{F}_{RR,app,2} \simeq \frac{e^2 k_{\max}}{\epsilon_0 c^2 (2\pi)^2}\left[-\left(\frac{4}{3}\right)\left(\frac{d^2\mathbf{x}}{dt^2}\right)\right]$), one may regard δm as some sort of mass renormalization counterterm and,so, $m + \delta m$ as some physical or renormalized mass: thus,$k_{\max}$ is absorbed into the renormalized mass. Then,the Lorentz equation equation becomes approximately for $|\boldsymbol{\beta}| \ll 1$ ($\gamma \simeq 1$):

$$(m + \delta m)\frac{d^2\mathbf{x}}{dt^2} \simeq e\mathbf{E}_{in} + e\frac{d\mathbf{x}}{dt} \times \mathbf{B}_{in} \tag{83}$$

in which $k_{\max}$ no longer appears explicitly. Notice that such a complete absorption of $k_{\max}$ into a renormalized mass does not seem to apply in the remaining cases without the non-relativistic approximation, as $\boldsymbol{\beta}$-dependences occur.

8. Covariant formulation

8.1. General aspects

The three-dimensional formulation presented in the previous sections will be recast here as a four-dimensional one in the same inertial reference system. That would enable to transform the description of the electron-field dynamics directly to another arbitrary inertial reference system, just by invoking the standard Lorentz transformations. Only the necessary equations will be given. The metric tensor, enabling to move Lorentz indices up and down, is the standard one (see, for instance, [39]): $g_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, with $g_{\mu\nu} = 0$ for $\mu \neq \nu$ and $g_{00} = +1 = -g_{11} = -g_{22} = -g_{33}$. Let (n^{μ}) be a constant vector such that $n^{\mu}n^{\nu}g_{\mu\nu} = +1$. In the inertial system considered in all previous sections, we understand that: $(n^{\mu}) = (1, 0, 0, 0)$.

The space-time electron coordinates are: $x = (x^{\mu}) = (ct, \mathbf{x})$, $\mu = 0, 1, 2, 3$, with $\mathbf{x} = \mathbf{x}(t)$ along the electron trajectory and $(x_{\mu}) = (ct, -\mathbf{x})$. The differential line interval s along the latter fulfills: $ds = cdt\left[1 - c^{-2}\left(\frac{d\mathbf{x}}{dt}\right)^2\right]^{1/2}$ and, using it, the electron trajectory is: $x^{\mu} = x^{\mu}(s)$. The particle fourvelocity (u^{μ}) and fourmomentum (p^{μ}) are:

$$(u^{\mu}) = \left(\frac{1}{\left[1 - c^{-2}\left(\frac{d\mathbf{x}}{dt}\right)^2\right]^{1/2}}, \frac{c^{-1}\frac{d\mathbf{x}}{dt}}{\left[1 - c^{-2}\left(\frac{d\mathbf{x}}{dt}\right)^2\right]^{1/2}} \right) = \left(\frac{dx^{\mu}}{ds}\right) \tag{84}$$

$$p^{\mu} = mcu^{\mu} \tag{85}$$

with $u^{\mu}u^{\nu}g_{\mu\nu} = +1$. The exact total fourpotential for uniform motion is, with obvious notation and meaning:

$$(A^{\mu}) = (A^{\mu}_{in}) + (A^{\mu}_{dyn}) = (c^{-1}A^0, \mathbf{A}) \tag{86}$$

$$A^{\mu}_{dyn} = A^{\mu}_{dyn}(x) = 2\Re \int \frac{d^4k v_{cov}^2 ie\delta(k^2)\theta(k^0)}{(2\pi)^3 \epsilon_0 c} \times \int_{-\infty}^{s} ds' \exp[-ik(x - x(s'))] \frac{p^{\mu}(s')}{mc} \tag{87}$$

with $k = (c^{-1}(\omega_k - i\epsilon), \mathbf{k}) = (k^{\mu})$, δ denotes Dirac' s delta function, $\theta(k^0)$ is the standard step function, $kx = k^{\mu}x^{\nu}g_{\mu\nu}$. $v_{cov} = v_{cov}(n^{\mu}k^{\nu}g_{\mu\nu})$: in the inertial system considered in the previous sections, $n^{\mu}k^{\nu}g_{\mu\nu} = |\mathbf{k}|$, so that $\nu_{cov} = \nu(|\mathbf{k}|)$, which is just the real cut-off function considered in the previous sections. So, ν_{cov} is a covariant form of writing the cut-off function, due to the introduction of (n^{μ}) .

Also: $(A^{\mu}_{in}) = (c^{-1}A^{0}_{in}, \mathbf{A}_{in})$, $(A^{\mu}_{dyn}) = (c^{-1}A^{0}_{dyn}, \mathbf{A}_{dyn})$.

The (antisymmetric) electromagnetic tensor $F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}$ for uniform motion reduces to, for A^{μ}_{dyn}:

$$F^{\mu\nu}_{dyn} = F^{\mu\nu}_{dyn}(x) = 2\Re \int \frac{d^4k \nu^2_{cov} ie\delta(k^2)\theta(k^0)}{(2\pi)^3\epsilon_0 c} \times \int_{-\infty}^{s} ds' \exp[-ik(x - x(s'))] \frac{k^{\mu}p^{\nu}(s') - k^{\nu}p^{\mu}(s')}{mc} \quad (88)$$

and so on $F^{\mu\nu}_{in}(x(s))$. The Lorentz equation of motion for the electron trajectory $(x = x(s))$ reads, in covariant form:

$$\frac{dp^{\mu}(s)}{ds} = e(F^{\mu\nu}_{in}(x(s))\, u^{\sigma}(s)g_{\nu\sigma} + F^{\mu\nu}_{dyn}(x(s))\, u^{\sigma}(s)g_{\nu\sigma}) \quad (89)$$

8.2. Uniform motion in covariant form

Let there be no incoming electromagnetic field, that is, $F^{\mu\nu}_{in}(x) = 0$ for any $\mathbf{x}$ and any $t > -\infty$. Then, an exact solution of the covariant Lorentz equation is: $x(s)^{\mu} = x^{\mu}_{in} + u^{\mu}_0 s$, with constant x^{μ}_{in} and $(u^{\mu}_0) = (\frac{1}{[1 - c^{-2}\mathbf{v}_0^2]^{1/2}}, \frac{c^{-1}\mathbf{v}_0}{[1 - c^{-2}\mathbf{v}_0^2]^{1/2}})$, the constant $\mathbf{v}_0$ being such that $1 - c^{-2}\mathbf{v}_0^2 \geqslant 0$. In fact, in this case: $F^{\mu\nu}_{dyn} g_{\nu\sigma} u^{\sigma} = 0$: The proof (using $\int_{-\infty}^{s} ds' \exp[-iku_0(s - s')] = 1/(iku_0)$, having recalled $k = (c^{-1}(\omega_k - i\epsilon), \mathbf{k})$) is a direct generalization of the one in section 4 and will be omitted. This solution describes a uniform motion, which is the four-dimensional form of the three-dimensional one in section 4. The actual covariant counterparts of H in section 4 are now more difficult to evaluate directly in an arbitrary inertial system, as the corresponding angular integrals turn out to be somewhat more involved to carry out compactly. The reason is that an additional angular dependence, in an arbitrary inertial system arises, due $\nu_{cov} = \nu_{cov}(n^{\mu}k^{\nu}g_{\mu\nu})$ (arising, in turn, from the existence of the cutt-off) . This discussion will lie outside the scope of this work.

8.3. Comment a posteriori on a covariant approximate solution

The assumptions and approximations in section 7.1 and 7.2 can be readily extended here, with a similar expansion for $x^{\mu}(s')$ up to and including $\frac{d^2x^{\mu}(s)}{ds^2}$, so that $F^{\mu\nu}_{dyn}$ becomes approximately linear in $\frac{d^2x^{\mu}(s)}{ds^2}$. The integration over s' in a small interval below $s' = s$ can be performed similarly. In the resulting approximate (finite) expression for $F^{\mu\nu}_{dyn}$ which is the covariant counterpart of $\mathbf{F}_{RR,app,2}$, the corresponding angular integrals in an arbitrary inertial system are now somewhat more involved to carry out compactly than in section 7.2. The reason is that an additional angular dependence, in an arbitrary inertial system arises, due to $\nu_{cov} = \nu_{cov}(n^{\mu}k^{\nu}g_{\mu\nu})$. This discussion lies outside the scope of this work.

9. Conclusions and further developments

The present investigation is aimed at a consistent treatment of the classical radiation reaction problem of a relativistic extended electron, in which the scientific community has a renewed interest due to the presently available extreme light sources. The scope of our work is mainly focused in the interaction of high-energy electron and ultra intense lasers in vacuum, but it could be worth mentioning that radiation reaction effects are also expected in solids due to high internal intra crystalline fields, see for example [42–44] for suggested effects in diamond, and for reporting high internal fields in semiconductors relevant for modern optoelectronics devices.

The classical electron-field system evolve in an infinite three-dimensional vacuum and in an inertial system. The electromagnetic field is treated in Lorentz gauge, with the Lorentz condition. By assumption, there is a crucial finite cut-off $k_{\max}$ on the magnitude of any wavevector contributing to the field (thereby preventing a point electron) and the initial conditions for particle and fields are given in the infinitely remote past.

The Hamiltonian approach employed here accounts, from first principles, for the radiated field and its effects on the electron dynamics (Radiation Reaction). This Hamiltonian approach has the advantage that particle and field variables are treated on an equal footing up to some stage. Hamilton's dynamical equations for the particle and the field amplitudes a's yield an exact Lorentz force (integro-differential) equation for the former, with includes the incoming radiation and an exact radiation reaction force $\mathbf{F}_{RR}$ due to the field radiated by the electron.

The treatment is relativistic from the outset, and sufficiently general to account for external fields of any intensity and of basically arbitrary spatial and temporal shapes, provided $k_{\max}$ is suitably chosen. The theory is aimed in this sense to contribute to the phenomenon of radiation reaction under the effect of ultra short and ultra intense laser fields, whose current interest has been fostered by the availability of extreme light sources. The only free parameter of the theory is a cutoff $k_{\max}$ in the modes of the total electromagnetic field. At this stage, it is still an open question how to choose this $k_{\max}$ so as to obtain meaningful numerical results from the theory. But as far as one sticks to the condition $k_{\max} < +\infty$, it can be proved that the dynamics is free from pathologies like self-acceleration, etc that have plagued the analytical treatment of radiation reaction since nearly a century ago. A preliminary, and qualitative consideration about $k_{\max}$ is that it should be large enough to account at least for the spacial shape of the external electromagnetic field and for the shortest wavelength photons, if we can share for a moment this concept from quantum theory, that can be emitted from the radiating electron. A possible judicious choice for $k_{\max}$ could be somewhere in the range $k_{max1} \leqslant k_{\max} \leqslant k_{max2}$ with $k_{max1} = 2mc^2/\hbar c$ ($\hbar$being rationalized Planck's constant) and $k_{max2} = 1/r_e$, $r_e = e^2/(4\pi\epsilon_0 mc^2)$ the classical electron radius. $k_{\max 1}$ (about 10^{11} cm^{-1}) corresponds to the energy of a photon producing an electron-positron pair, $k_{\max 2}$ (about $10^{13} cm^{-1}$) is the inverse of the classical electron radius r_e. It is interesting to note that the condition that $k_0 = 2\pi/\lambda_0 < k_{max1} \leqslant k_{\max} \leqslant k_{max2}$ is compatible with ultra high intensity powers in IR lasers, i.e. with central λ_0 typically around 1μ m. Compare with the discussion in section 1, which considered in particular electromagnetic radiations in the infrared or near infrared ranges.

It is shown that an uniform velocity trajectory for the electron, in the absence of incoming external radiation, satisfies exactly the Lorentz force equation, even by including the effect of radiation reaction. Some judicious approximations in the exact Lorentz (retarded integro-differential) equation allow to deal with the radiation reaction term(s). Specifically, based upon numerical computations some approximations on $\mathbf{F}_{RR}$ are given. They have desirable physical properties in both the limits of very small and, particularly, at ultrarelativistic velocities, where a dependence of $\mathbf{F}_{RR}$ on γ^2 has been obtained, in accordance with the main term in the classic Landau-Lifschitz equation.

The theory has also been given in a fully covariant form, so as to be of application in any inertial system, not only in an arbitrarily chosen laboratory system. Related to the approximations in section 7 and to the last item, one can make another remark. If one constructs (following the approximations in section 7) $\mathbf{E}_{dyn,app,2}$ and $\mathbf{B}_{dyn,app,2}$ on the trajectory in the chosen inertial reference system, one could construct directly in the standard way the (antisymmetric) electromagnetic tensor $F_{\mu\nu,app,2}$ (approximate), on the trajectory. Even if such $F_{\mu\nu,app,2}$ is an approximation to the exact $F_{\mu\nu}$, the former, due to its antisymmetry, fulfills exactly $\left(\frac{dx^\mu}{ds}\right)_{t'=t} F_{\mu\nu,app,2}\left(\frac{dx^\nu}{ds}\right)_{t'=t} = 0$, $\left(\frac{dx^\mu}{ds}\right)_{t'=t}$ being the standard four-velocity. Its approximate usefulness in other inertial system is an open issue not to be analyzed here.

Some of the topics that are outside the scope of the paper, but are planned to investigate in further researches are a quantitative comparison of the approximate radiation reaction terms obtained here with other approaches, based for example in the well-established Landau–Lifshitz equation [41]. Similarly, a further development will be to numerically integrate the dynamical equations. Some of the difficulties of this task have to do with the (apparent) arbitrariness of the cutoff limit $k_{\max}$, and also, from a numerical point of view, with the highly oscillatory and spiky behavior of integrand functions, specially in the case of large $k_{\max}$.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix

One has:

$$\mathbf{F}_{RR,app,2} = \frac{e^2}{\epsilon_0}\Re\int\frac{d^3\mathbf{k}|v_k|^2}{(2\pi)^3}$$
$$\left[-\left(\left[c^2-\left(\frac{d\mathbf{x}}{dt}\right)^2\right]\frac{\mathbf{k}}{\omega_k}+\frac{1}{\omega_k}\left[-c|\mathbf{k}|+\left(\mathbf{k}\cdot\frac{d\mathbf{x}}{dt}\right)\right]\frac{d\mathbf{x}}{dt}\right)\right.$$
$$\times\, i\left(\mathbf{k}\cdot\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t}\right)\frac{i}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-(\omega_k-i\epsilon)}\frac{i}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-(\omega_k-i\epsilon)}\frac{i}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-(\omega_k-i\epsilon)}$$
$$+\frac{\mathbf{k}}{\omega_k}\left(\frac{d\mathbf{x}}{dt}\cdot\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t}\right)\frac{i}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-(\omega_k-i\epsilon)}\frac{i}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-(\omega_k-i\epsilon)}$$
$$\left.-\frac{1}{\omega_k}\left[-c|\mathbf{k}|+\left(\mathbf{k}\cdot\frac{d\mathbf{x}}{dt}\right)\right]\right]\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t}\frac{i}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-(\omega_k-i\epsilon)}\frac{i}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-(\omega_k-i\epsilon)}\Bigg] \tag{90}$$

One starts from the above $\mathbf{F}_{RR,app,2}$. One again invokes the formal expression for real x, already employed in section 4: $1/(x-i\epsilon)=P(1/x)+i\pi\delta(x)$, P and δ being Cauchy principal value and the Dirac delta function, respectively. One replaces the later formula in equation (90), and notices that $\Re(ix)=0$ for real x and $k^2\delta(-\omega_k+\mathbf{k}\cdot\frac{d\mathbf{x}}{dt})\delta(-\omega_k+\mathbf{k}\cdot\frac{d\mathbf{x}}{dt})=0$. One recalls that $k_{\max}$ is the upper cut-off in the integration over $\mathbf{k}$ and that v_k only depends on k. One makes use of equation (67), with :

$$\mathbf{I}=\frac{c^2}{k_{\max}}\int d^3\mathbf{k}|v_k|^2\left[-\left(\left[c^2-\left(\frac{d\mathbf{x}}{dt}\right)^2\right]\frac{\mathbf{k}}{\omega_k}+\frac{1}{\omega_k}\left[-c|\mathbf{k}|+\left(\mathbf{k}\cdot\frac{d\mathbf{x}}{dt}\right)\right]\frac{d\mathbf{x}}{dt}\right)\right.$$
$$\times\left(\mathbf{k}\cdot\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t}\right)P\frac{1}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-\omega_k}P\frac{1}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-\omega_k}P\frac{1}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-\omega_k}$$
$$-\frac{\mathbf{k}}{\omega_k}\left(\frac{d\mathbf{x}}{dt}\cdot\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t}\right)P\frac{1}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-\omega_k}P\frac{1}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-\omega_k}$$
$$\left.+\frac{1}{\omega_k}\left[-c|\mathbf{k}|+\left(\mathbf{k}\cdot\frac{d\mathbf{x}}{dt}\right)\right]\right]\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t}P\frac{1}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-\omega_k}P\frac{1}{\mathbf{k}\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}-\omega_k}\Bigg] \tag{91}$$

The $i\epsilon$' s and $i\delta(x)$' s no longer appear and the Cauchy principal value symbols (P) can be safely eliminated in general as long as the electron velocity is smaller than c. All integrals are finite as $k\to 0$ (that is, infrared finite)) and finite at large k as long as $k_{\max}$ is finite (those integrals diverge proportionally to $k_{\max}$, as $k_{\max}\to+\infty$. The integration over $\mathbf{k}$ amounts to carry out two angular integrations. For that purpose, one takes in three-dimensional Cartesian coordinates (x, y, z), the z axis along $\left(\frac{d\mathbf{x}}{dt'}\right)_{t'=t}$ and chooses $\left(\frac{d^2\mathbf{x}}{dt'^2}\right)_{t'=t}$ in the (x,z) plane (forming an angle σ with the z axis). The integrations are direct, but somewhat cumbersome.

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