

Matter waves and the detection of Gravitational Waves

P Delva¹, M-C Angonin¹ and Ph Tourenco¹

¹ Université P. et M. Curie, ERGA, case 142, 4, place Jussieu, F-75252, Paris Cedex 05, France

E-mail: pacome.delva@obspm.fr

Abstract. We comment the controversy that followed the article of Chiao & Speliotopoulos [1] about the sensitivity of an atom interferometer to a gravitational wave. Then we report our results [2] on the sensitivity of a Ramsey-Bordé interferometer to a gravitational wave, and we compare it to the one of light wave interferometers. We find that atom interferometers could compete with space based light wave interferometers.

1. Introduction

The wave behavior of a massive particle, as predicted by De Broglie [3], was first shown by a diffraction experiment of electrons [4]. Thirty-five years later the first neutron interferometer was built by Maier-Leibnitz & Springer [5]. In 1991, the first signals of four atom interferometers (hereafter called Matter Waves Interferometers MWIs) were detected [6, 7, 8, 9]. Since then, MWIs have been used for a large variety of studies [10] thanks to the improvements of sensitivity, stability and compactness.

The possibility to detect Gravitational Waves (GWs) with matter wave interferometry has first been explored by Linet & Tourenco [11]. Their calculation is based on a generalisation of the eikonal equation of geometric optics to matter waves. Some years later Stodolsky [12] explored this question with an equivalent approach. Other approaches were considered, based on a generalisation of the Klein-Gordon equation [13], on the Dirac equation on curved spacetime [14, 15, 16] and on the ABCD matrices [17, 18, 19]. These works aimed to describe in a general framework the inertial, quantum and gravitational effects on the MWIs.

Recently a controversy about the sensitivity of MWIs to GWs followed the article of Chiao & Speliotopoulos [1]. Their results were discussed by Foffa et al. [20] and Roura et al. [21]. They all found different results because they studied different physical experiments. It was not clear that the experiments were different because they used various interpretations of the coordinates systems and of the boundary conditions. In a special case, Roura et al. [21] demonstrated explicitly the equivalence of two different descriptions of the same experiment, which is a basic principle of general relativity. In 2006, we discussed two different experiments [2], where fixed and free interferometers were considered in comoving coordinates, and calculated the corresponding sensitivities to GWs.

In this article we find the sensitivity of a fixed interferometer in a Ramsey-Bordé configuration. First we introduce the comoving coordinates of such an experiment : the Fermi Coordinates (FC). Then we explain why we think Foffa et al. [20] are wrong when they claim that the device sensitivity is proportional to the GW frequency. In the third part we describe our method to

calculate the MWI phase difference. Finally, we present our results of the device sensitivity and we compare the MWIs to the light wave interferometers.

2. The Fermi frame

A rigid detector [2] can be studied in a Fermi frame [22, 23, 24]. The spatial origin of the frame is taken at the center of mass of the detector. We will assume here that the worldline of the center of mass is a geodesic, ie. the *apparatus* suffers no acceleration nor rotation. Its dimension L must be very inferior to the GW wavelength Λ ($L \ll \Lambda$). Fortini & Gualdi [25] calculated the GW metric in the Fermi coordinates (FC) $\{X^\alpha\}$ up to any order $\xi \sim L/\Lambda$. Up to the first order, the metric reads

$$\begin{aligned} ds^2 = & \eta_{\mu\nu} dX^\mu dX^\nu + \frac{1}{2} \ddot{h}_{rs} X^r X^s \left(dT^2 + \frac{1}{3} dZ^2 \right) - \frac{2}{3} \ddot{h}_{rs} X^r X^s dT dZ \\ & + \frac{1}{3} \ddot{h}_{rs} Z dX^r \left(2X^s dT - X^s dZ + \frac{1}{2} Z dX^s \right) + O(\xi^3, h^2) \end{aligned} \quad (1)$$

where $r, s = 1, 2$ and the GW goes along $X^3 = Z$. Interferometric detectors are usually studied in the Einstein coordinates (TT gauge), which are well suited to the GWs. $h_{\mu\nu}$ is the metric perturbation in the Einstein coordinates : $g_{\mu\nu}^{TT} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} \ll 1$ and $\eta_{\mu\nu}$ is the Minkowskian metric. The coordinates transformation from the Einstein coordinates $\{x^\alpha\}$ to the FC $\{X^\alpha\}$ is [2]

$$\begin{cases} x^r = X^r - \frac{1}{2} h_s^r X^s - \frac{1}{3} \ddot{h}_s^r X^s Z^2 + O(\xi^4, h^2) \\ x^a = X^a - \frac{1}{4} \dot{h}_{rs} X^r X^s - \frac{1}{6} \ddot{h}_{rs} X^r X^s Z + O(\xi^4, h^2) \end{cases} \quad (2)$$

where $a = 0, 3$. In a Fermi frame, most of the physical laws remain the Special Relativistic ones. For a non relativistic particle, the equation of motion has the standard Newtonian form, plus a driving force due to the GW [26]. The equations of elasticity are the Newtonian ones, with a volume force due to the GW [27]. The Fermi frame is well suited to the lab experiment as it allows a quasi newtonian interpretation of the phenomenons. Three gyroscopes and an accelerometer at the spatial origin allow to control the *apparatus* trajectory and to deduce the relativistic corrections to the newtonian equations [28]. A MWI with a “butterfly” geometry [29] consists itself in such an inertial basis.

3. A criticism of the “rigid” coordinates

Ashby & Dreitlein [30] introduced new coordinates $\{x'^\alpha\}$ by the means of the coordinate transformation

$$\begin{cases} x^r = x'^r - \frac{1}{2} h_s^r x'^s \\ x^a = x'^a \end{cases} \quad (3)$$

where $\{x^\alpha\}$ are the Einstein coordinates. Therefore, with $h_s^r = h_s^r(t - z)$, the metric reads

$$ds^2 = \eta_{\alpha\beta} dx'^\alpha dx'^\beta + \dot{h}_{rs} x'^r dx'^s (dt' - dz') \quad (4)$$

where $\dot{h}_{rs} = \frac{\partial h_{rs}}{\partial t}$.

These coordinates are called “rigid” by Foffa et al. [20] and Roura et al. [21] because the “physical” distance between two world lines with constant spatial coordinates x'^r in the plane $z' = 0$ remains constant. Such coordinates can be considered as comoving coordinates with fixed

devices on an optical bench. This is at least what appears from the article by Foffa et al. [20] for a low Q material (*i.e.* with a high internal damping of elastic waves).

The point is that the “rigid” coordinates system is not the only one that displays such a property. The same property remains true in Fermi coordinates too. Therefore the question has to be discussed in more details.

Let us consider a fundamental element $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$. Two neighboring world lines, A and B , have spatial coordinates x^k and $x^k + dx^k$. The coordinate time necessary for a light signal to fly from A to B and return from B to A is dt . The corresponding proper time measured at point A is $d\tau = \sqrt{g_{00}}dt = 2[(g_{0i}g_{0j}/g_{00} - g_{ij})dx^i dx^j]^{1/2}$. The “physical” distance is $d\ell = \frac{1}{2}d\tau$ (remember that, here, we use units with $c = 1$).

Now with $g_{\alpha\beta} = \eta_{\alpha\beta} + k_{\alpha\beta}$ where $|k_{\alpha\beta}| \ll 1$, one obtains

$$d\ell = d\ell_0 \left(1 - \frac{1}{2}k_{ij}\gamma^i\gamma^j \right) + O(k^2) \quad (5)$$

where $O(k^2)$ is a second order term relatively to $|k_{\alpha\beta}|$, $d\ell_0 = \sqrt{dx^2 + dy^2 + dz^2}$ and $\gamma^k = \frac{dx^k}{d\ell_0}$. Now if we consider points A and B , fixed relatively to the rigid coordinates one gets

$$d\ell = d\ell_0 \left(1 + h_{rs}x^r\gamma^s\gamma^3 \right) \quad (6)$$

For two points A and B in the plane $z = 0$, one finds $\gamma^3 = 0$ and therefore $d\ell = d\ell_0 = cte$. But the general situation (*i.e.* $\gamma^3 \neq 0$) results in

$$d\ell = d\ell_0 \left[1 + O\left(h\frac{L}{\Lambda}\right) \right] \quad (7)$$

where h is the order of magnitude of h_{rs} and Λ is the order of magnitude of the gravitational wavelengths. The size of the experimental setup is L . It is assumed to be much smaller than Λ . A similar calculation for the Fermi coordinates gives

$$d\ell = d\ell_0 \left[1 + O\left(h\left(\frac{L}{\Lambda}\right)^2\right) \right] \quad (8)$$

Now, comparing the behaviors of the two materials comoving with either the “rigid” or the Fermi system, it is clear that the second one is the most rigid material. Actually, the Fermi coordinates are the “best” ones that can be found to describe classical, rigid, comoving materials.

When the GW frequency f is of the order of a mechanical resonance frequency of the bench, and if the mechanical quality factor is high enough, one has to take into account the motion of the mirrors (or gratings). However, we do not consider such a case where the matter waves would be a displacement sensor for a mechanical detector at high frequency (around 10^3 Hz). On the contrary we will see that it is rather at low frequency (10^{-3} Hz) that matter waves are most interesting.

4. The phase difference

We calculate the phase difference between two events with the method described by Linet & Tourrenc [11]. The quantum phase is developed up to the first order relatively to h : $\phi = \phi_o + \delta\phi$ where ϕ_o is the unperturbed quantum phase while $\delta\phi$ is the perturbation of order h . $k_\mu = \partial_\mu\phi_o$ is the unperturbed wave vector of the particle considered. It can be a photon or a massive particle.

The worldlines of the mirrors or the beam splitters A and B are perturbed by the GW. Let us call $x_A^\mu = x_A^\mu(t)$ and $x_B^\mu = x_B^\mu(t)$ their trajectories. Knowing k^μ and t_B (the arrival time of the

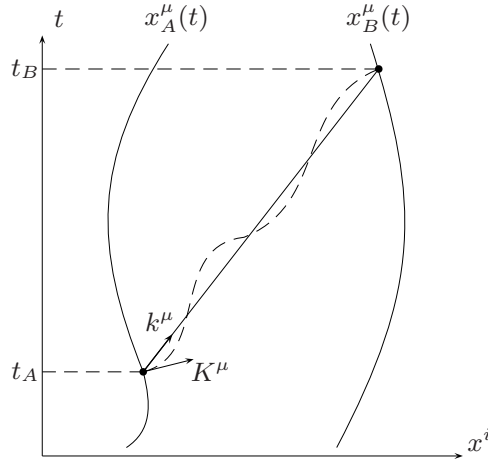


Figure 1. The matter wave propagates between the two optical elements A and B (dashed line). The unperturbed trajectory is the straight line.

atom), we can deduce t_A (its departure time) : $x_B^\mu - x_A^\mu = \alpha k^\mu$, where α is a constant. This is illustrated on Fig. 1. We define $[\phi]_A^B \equiv \phi[x_B^\mu(t_B)] - \phi[x_A^\mu(t_A)]$. The solution reads :

$$[\phi]_A^B = [\phi_o]_A^B + [\delta\phi]_A^B , \quad (9)$$

where

$$[\phi_o]_A^B = k_\mu x_B^\mu - k_\mu x_A^\mu , \quad (10)$$

$$[\delta\phi]_A^B = \frac{\hbar c^2}{2} \int_{t_A}^{t_B} h_{\mu\nu} k^\mu k^\nu \frac{dt}{E} . \quad (11)$$

Usually atoms in matter-wave interferometry are manipulated with lasers. When interacting with atoms, the lasers “imprint” their phase on them. The perturbation of a GW on the propagation of lasers will thus result on a perturbation of the atom phase difference in the interferometer. It is easy to show from (11) that this contribution is negligible. Indeed the phase perturbation is of order $\hbar L/\lambda$, where $\hbar \sim |h_{rs}|$ and λ is the wavelength of the particle considered (the De Broglie wavelength for a massive particle). In a MWI, L is of the same order for the laser and for the atoms, but the laser wavelength ($\sim \mu\text{m}$) is much higher than the atom wavelength ($\sim \text{pm}$ for thermal atoms).

5. The Ramsey-Bordé interferometer

We study now a MWI with a Ramsey-Bordé configuration represented on Fig. 2. We define $X = X^1$ and $Y = X^2$. Atoms move along the X axis and we assume that their initial velocity $v_0 \ll c$. At the output of the interferometer, we calculate from (1) and (9) the phase difference between the two atom beams

$$\Delta\phi = \Delta\phi_+ + \Delta\phi_\times \quad (12)$$

with :

$$\Delta\phi_+ = -4\pi h_+ \frac{L}{\lambda} \sin\psi \tan^2\theta \left[\cos(\Omega t + \psi) + \frac{\sin\psi}{2\psi} \cos(\Omega t) \right] \quad (13)$$

$$\Delta\phi_\times = -4\pi h_\times \frac{L}{\lambda} \cos\psi \tan\theta \left[\sin(\Omega t + \varphi_\times - \psi) - \frac{\sin\psi}{\psi} \tan\psi \cos(\Omega t + \varphi_\times) \right] \quad (14)$$

where θ is the separation angle, $\psi = \frac{\Omega L}{2v_0}$ and Ω is the GW pulsation (we considered a periodic GW).

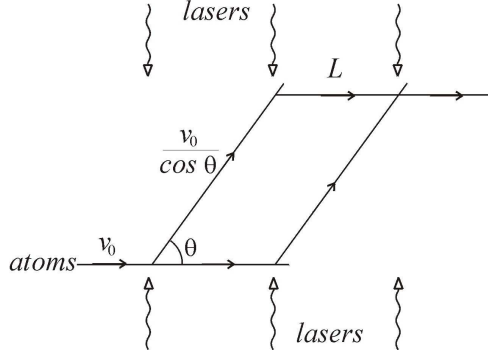


Figure 2. Ramsey-Bordé interferometer in the X-Y plane.

We assume now that $\theta \simeq \pi/4$ so that $\tan \theta \simeq 1$. As for the Michelson configuration [21, 2], two regimes can be considered. If $L \ll \frac{v_0}{c} \cdot \Lambda$ the phase difference reads

$$\Delta\phi \simeq -4\pi h_{\times} \cdot \frac{L}{\lambda} \cdot \sin(\Omega t + \varphi_{\times}) \quad (15)$$

This expression results in the same amplitude than the free Michelson configuration [2], with $h_{+} \rightarrow h_{\times}$. From formula (15), one can compare the sensitivities of matter wave and light wave interferometers when they are limited by the shot noise. On Fig. 3 and 4 are represented the required characteristics¹ of a MWI necessary to reach the sensitivity of Virgo (Fig. 3) and that of LISA (Fig. 4). The cross on each figure corresponds to the MWI described by Gustavson et al. [31].

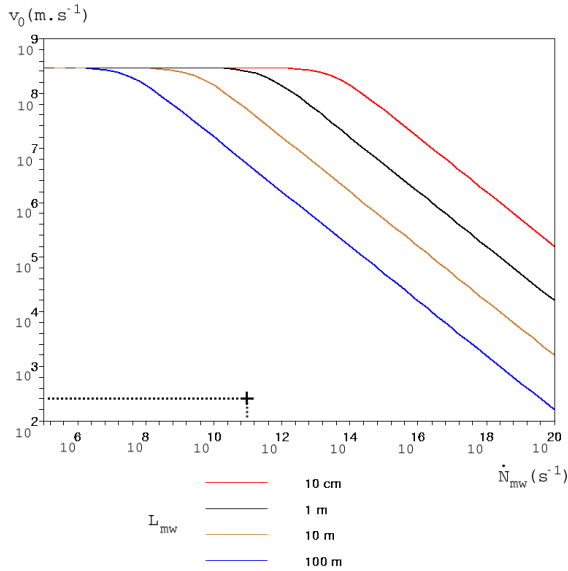


Figure 3. Required characteristics of a MWI necessary to reach the sensitivity of Virgo.

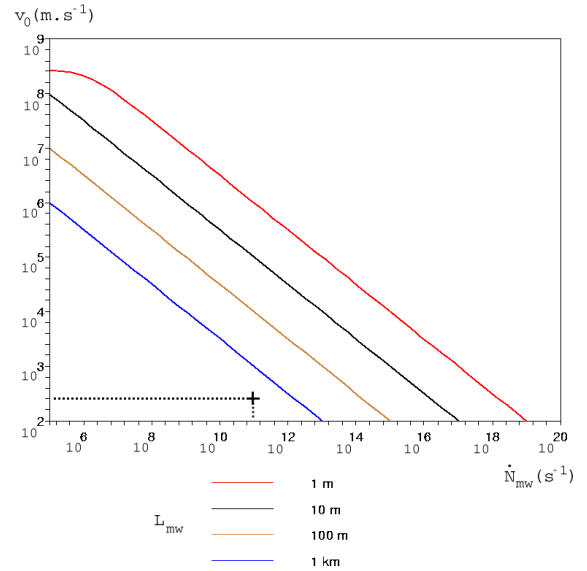


Figure 4. Required characteristics of a MWI necessary to reach the sensitivity of LISA.

¹ The curves are drawn for the caesium mass. v_0 , L_{mw} and \dot{N}_{mw} are respectively the initial atom wave group velocity, the MWI arm length and the atom flux.

To reach the sensitivity of Virgo with the actual atomic fluxes, the atoms need to be relativistic as shown on Fig.3. It would be a real challenge to keep the coherence of the atoms at such a speed. The situation is better for low frequency GWs. Indeed, the photon flux in LISA will be very low compared to the atom flux in the Gustavson et al.'s MWI. On Fig. 4 one can see that a kilometric MWI can reach the sensitivity of LISA at a thermal velocity. This can be compared to the five billions kilometers arms of LISA. The kilometric MWI could even be reduced to a one meter MWI with a matter wave cavity, similar to the Fabry-Perot cavity in Virgo. Cavities where atoms are trapped between two mirrors have been described theoretically by Balykin & Letokhov [32]; but atomic Fabry-Perot cavities coupled to a MWI have never been built (for different views on matter wave cavities see [33, 34, 35]).

The main interest of a compact MWI is that it can be cooled to very low temperatures to reduce the thermal noise.

6. Conclusion

We showed in this proceeding that a correct description of a MWI is easier in the Fermi frame. Indeed, the Fermi coordinates system allows a quasi newtonian description of the *apparatus*. When describing an experiment in the “rigid” coordinates, one has to be very careful: the spatial description of the device is the same but one has to take into account the time distortion. Finally we reported the main result of our paper [2]: the sensitivity of a MWI with a Ramsey-Bordé configuration, compared with the sensitivity of Virgo and LIGO. We found that MWIs would not compete with earth based interferometers in the future because of the very low atomic flux compared to the laser one. On the opposite, they could compete with space based interferometers, as they permit to reach comparable sensitivities in a much more compact way. Of course, major improvements and a serious noise study still remain necessary.

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