Journal of Nonlinear Mathematical Physics (2026) 33:22
https://doi.org/10.1007/s44198-026-00385-x

RESEARCH

Understanding the Non-Relativistic Behavior of the Chern–Simons–Higgs System

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Received: 8 October 2025 / Accepted: 23 January 2026

Abstract

In this paper, we consider a singular limit of the Chern–Simons–Higgs equations that reveals a correspondence between the Chern–Simons–Higgs and Chern–Simons–Schrödinger systems. Specifically, we demonstrate that in the limit as the speed of light goes infinity, known as the *non-relativistic limit*, a family of time-dependent solutions of the modulated Chern–Simons–Higgs system, which depends on the parameter c representing the speed of light, converges to the solution of the Chern–Simons–Schrödinger system, given suitable initial data assumptions.

Keywords Non-relativistic limit · Singular limit · Chern–Simons–Higgs · Chern–Simons–Schrödinger

Mathematics Subject Classification Primary 35B40 · 35L52 · 35L45 · 35Q40

1 Introduction

In this paper, we are interested in a rigorous proof of a singular limit from the Chern–Simons–Higgs (CSH) system to the Chern–Simons–Schrödinger (CSS) system. The dynamics of the $(2+1)$-dimensional CSH system are governed by the following Lagrangian density[1]:

[1] This is a modified version of the original Lagrangian introduced in [12], which we have adjusted to facilitate the discussion of the non-relativistic limit. A detailed derivation of this modification will be provided in Appendix A.

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Published online: 10 February 2026

$$\mathcal{L} = \epsilon^{\rho\mu\nu} A_\rho F_{\mu\nu} + \frac{1}{c^2} D_0\phi\overline{D^0\phi} + D_j\phi\overline{D^j\phi} - \frac{1}{4c^2}|\phi|^2(|\phi|^2 - c^2)^2, \tag{1.1}$$

where $\phi : \mathbb{R}^{2+1} \to \mathbb{C}$ is a matter field, $A_\mu : \mathbb{R}^{2+1} \to \mathbb{R}$ $(\mu = 0, 1, 2)$ are gauge fields, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength tensor of the gauge fields, and $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative. The parameter c represents the speed of light, and $\epsilon^{\rho\mu\nu}$ is the totally skew-symmetric tensor with $\epsilon^{012} = 1$. We consider the $(2+1)$-dimensional Minkowski space $\mathbb{R}^{2+1}$ with the metric $\mathrm{diag}\,(1, -1, -1)$, which is used to raise or lower indices. The summation convention will be used for summing over repeated indices: the Greek indices run over 0, 1, 2, whereas the Latin indices run over 1, 2. We also use the notation $\partial_1 = \partial_x\ \partial_2 = \partial_y$ and $\partial_0 = \partial_t$. Then, by considering the corresponding Euler-Lagrange equations for the Lagrangian (1.1), the governing equations for the CSH system are as follows:

$$\begin{aligned} \frac{1}{c^2} D_0 D_0 \phi - D_1 D_1 \phi - D_2 D_2 \phi + \frac{c^2}{4}\phi &= |\phi|^2\phi - \frac{3}{4c^2}|\phi|^4\phi, \\ F_{01} &= -\ \mathrm{Im}\ (\overline{\phi} D_2 \phi), \\ F_{02} &= \ \mathrm{Im}\ (\overline{\phi} D_1 \phi), \\ F_{12} &= \frac{1}{c^2}\ \mathrm{Im}\ (\overline{\phi} D_0 \phi). \end{aligned} \tag{1.2}$$

The Chern-Simons gauge theory was introduced to explain phenomena in planar physics [7, 12], such as the fractional quantum Hall effect [13] or high T_c superconductivity [7, 22]. Since then, the CSH and CSS systems, proposed within the Chern-Simons gauge theory as relativistic and non-relativistic models respectively, have garnered significant attention in the mathematical community. The initial value problem of CSH system has been extensively studied in various works [4, 6, 11, 19, 20]. Notably, local and global well-posedness in H^2 has been established [4], and global energy solutions in H^1 have been constructed [20]. On the other hand, the initial value problem of CSS system has been studied in [2, 10, 15, 16]. The local and global well-posedness of CSS system in H^2 has been discussed in [2]. Additionally, studies on finite energy solutions in H^1 have been pursued [10, 15], and decay and scattering results have been obtained [18].

Among the various topics related to these systems, our focus is on rigorously deriving the CSS system from the CSH system, which was formally introduced as the *non-relativistic limit* in [12]. Previous mathematical investigations on this topic can be found in [5, 9]. In [9], the authors studied this issue concerning time-independent self-dual solutions, while in [5], it was explored in the context of solutions to the initial value problem. In this paper, we aim to revisit and refine the result presented in [5] from two perspectives, with the goal of deepening our understanding of the asymptotic behavior of solutions to the CSH system in the non-relativistic regime.

(1) In [5], the authors explored with the original version Lagrangian, presented in [12], along with the covariant derivative

$$\mathbf{D}_0 = \frac{1}{c}\partial_t - \frac{i}{c}\mathbf{A}_0 \qquad \text{and} \qquad \mathbf{D}_j = \partial_j - \frac{i}{c}\mathbf{A}_j, \quad \text{for} \quad j = 1, 2,$$

and observed the asymptotic behavior of $c^{-1}\mathbf{A}_j$ in the limit as c goes to infinity, which does not vanish in the limit. This is a remarkable feature when compared to the well-known Maxwell–Klein–Gordon system, where the magnetic field effect disappears in the non-relativistic regime, resulting in the Schrödinger–Poisson equations [1, 17]. However, the Lagrangian used by the authors does not clearly exhibit these characteristics. Thus, we reformulated the Lagrangian density for the CSH model as introduced in (1.1) to address this aspect of the CSH system. This reformulation yields a more concise form of the CSH equations, as shown in (1.2), essentially identical to the equations in [5, 12]. This process facilitates a clearer understanding of the non-relativistic limit of the CSH system. For a detailed discussion, refer to Appendix A.

(2) While conventional derivatives satisfy the inequality

$$\|\nabla_1\nabla_2 u\|_2 + \|\nabla_2\nabla_1 u\|_2 \lesssim \|\nabla_1\nabla_1 u\|_2 + \|\nabla_2\nabla_2 u\|_2,$$

covariant derivatives generally fail to satisfy the following inequality:

$$\|D_1D_2 u\|_2 + \|D_2D_1 u\|_2 \lesssim \|D_1D_1 u\|_2 + \|D_2D_2 u\|_2.$$

As a result of this observation, it was necessary to revise the proof in [5] concerning the uniform bound of $\Phi^\varepsilon(t)$ (where $\varepsilon := c^{-1}$). This revision is addressed in Proposition 3.2, which is a critical step in deriving the main theorem. For details, please refer to *Step 3* of the proof of Proposition 3.2 in this paper. Additionally, we have updated (3) of Lemma 3.2 in [5] to (3) of Lemma 3.4 in this paper to resolve this issue.

To discuss the non-relativistic limit of the CSH system to the CSS system, we apply the standard modulation ansatz introduced in [14, 21], given as

$$u^c(x,t) = \exp\left(i\frac{c^2 t}{2}\right)\phi(x,t)$$

to (1.2). This yields the following modulated CSH system with c-dependency of the solutions:

$$\begin{aligned}
&\frac{1}{c^2}D_0D_0u^c - iD_0u^c - D_1D_1u^c - D_2D_2u^c = |u^c|^2u^c - \frac{3}{4c^2}|u^c|^4u^c,\\
&\partial_0A_1^c - \partial_1A_0^c = -\ \mathrm{Im}\ (\overline{u^c}D_2u^c),\\
&\partial_0A_2^c - \partial_2A_0^c = \ \mathrm{Im}\ (\overline{u^c}D_1u^c),\\
&\partial_1A_2^c - \partial_2A_1^c = \frac{1}{c^2}\ \mathrm{Im}\ (\overline{u^c}D_0u^c) - \frac{1}{2}|u^c|^2.
\end{aligned} \tag{1.3}$$

As $c \to \infty$, the system (1.3) formally converges to the following CSS system:

$$\begin{aligned} &iD_0u + D_1D_1u + D_2D_2u = -|u|^2u, \\ &\partial_0A_1 - \partial_1A_0 = -\ \mathrm{Im}\ (\overline{u}D_2u), \\ &\partial_0A_2 - \partial_2A_0 = \ \mathrm{Im}\ (\overline{u}D_1u), \\ &\partial_1A_2 - \partial_2A_1 = -\frac{1}{2}|u|^2. \end{aligned} \tag{1.4}$$

Therefore, our aim in this study is a rigorous and quantified derivation of the non-relativistic limit from the modulated CSH system (1.3) to the CSS system (1.4) as $c \to \infty$. We defer the precise statement of the main theorem regarding the non-relativistic limit to the end of Section 2, as some prerequisite background is necessary to fully describe the result.

The rest of the paper is organized as follows. Section 2 introduces several properties of the CSH and CSS equations, with a particular focus on establishing the well-posedness of these systems. The main theorem of the paper is also presented in this section. Section 3 presents uniform-in-ε bounds of the solutions, which are crucial estimates in proving the non-relativistic limit. Section 4 offers a complete proof of the main theorem by directly estimating the difference between the solutions of the modulated CSH equations and the CSS equations. Finally, Section 5 summarizes our result and its significance. Appendix A contains a detailed derivation of the Lagrangian density within the context of the non-relativistic limit approach.

We conclude this section by listing the notation used throughout the paper.

Notation. We use the standard Lebesgue spaces $L^p = L^p(\mathbb{R}^2)$, employing the following convenient notation for the corresponding norms:

$$\|f\|_p = \|f\|_{L^p(\mathbb{R}^2)}, \quad 1 \leq p \leq \infty.$$

We also use the standard Sobolev spaces $W^{s,p}(\mathbb{R}^2), \dot{W}^{s,p}(\mathbb{R}^2)$ and $H^s(\mathbb{R}^2) := W^{s,2}(\mathbb{R}^2) (s \geq 0)$ with the norms

$$\|f\|_{W^{s,p}} = \|\mathcal{F}^{-1}\big((1+|\xi|^2)^{s/2}\mathcal{F}f\big)\|_p, \quad \|f\|_{\dot{W}^{s,p}} = \|\mathcal{F}^{-1}\big(|\xi|^s\mathcal{F}f\big)\|_p,$$

where $\mathcal{F}, \mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transforms, respectively. Regarding the L^2-norm of the covariant derivatives, we denote them as follows:

$$\begin{aligned} &\|D_xu(t)\|_2^2 := \sum_{j=1}^{2} \|D_ju(t)\|_2^2, \quad \|D_xD_xu(t)\|_2^2 := \sum_{j,k=1}^{2} \|D_kD_ju(t)\|_2^2, \\ &\|D_0D_xu(t)\|_2^2 := \sum_{j=1}^{2} \|D_0D_ju(t)\|_2^2, \quad \|D_xD_0u(t)\|_2^2 := \sum_{j=1}^{2} \|D_jD_0u(t)\|_2^2. \end{aligned}$$

We denote $\Sigma_\tau := \{(x,t) \in \mathbb{R}^2 \times \mathbb{R} : t = \tau\}$ and use $A \lesssim B$ to indicate an estimate of the form $A \leq CB$, where C denotes a generic constant independent of the speed of light $c := \varepsilon^{-1}$. We also use the notation $g(\varepsilon) = O(\varepsilon^\gamma)$, indicating that $g(\varepsilon) \leq C\varepsilon^\gamma$ to specify the order of dependence on ε.

2 Preliminaries

In this section, we will discuss the basic properties and well-posed results of the CSH and CSS systems, which are essential for the discussion of the non-relativistic limit. Additionally, we will present the main theorem of the paper, which provides rigorous proofs for deriving the CSS equations from the CSH equations.

2.1 Well-Posedness of the CSH and CSS Systems

To discuss the non-relativistic limit, the global-in-time well-posedness of both the CSH system and the CSS system should be guaranteed. We note that the CSH equations (1.2) are invariant under the following gauge transformation:

$$\phi \to \phi e^{i\chi}, \quad A_\mu \to A_\mu + \partial_\mu \chi,$$

where $\chi(x,t)$ is a smooth real-valued function. That is, a solution of the system is given by a class of gauge-equivalent pairs. Therefore, to fix the gauge freedom, an additional gauge-fixing condition, such as the Coulomb gauge, the Lorenz gauge, or the heat gauge conditions, should be imposed. Among many options, we adopt the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$, which provides an elliptic feature for the gauge fields A_μ. We also observe that if the equations $(1.2)_{1,2,3}$ hold, then $(1.2)_4$ is satisfied for all time if it is initially imposed:

$$\partial_t \left(\partial_1 A_2 - \partial_2 A_1 - \frac{1}{c^2} \operatorname{Im} (\overline{\phi} D_0 \phi) \right) = 0.$$

So, we can rewrite the Cauchy problem of CSH system (1.2) with the Coulomb gauge condition as follows:

$$\begin{aligned} &\frac{1}{c^2} D_0 D_0 \phi - D_1 D_1 \phi - D_2 D_2 \phi + \frac{c^2}{4}\phi = |\phi|^2\phi - \frac{3}{4c^2}|\phi|^4\phi, \\ &\Delta A_0 = \partial_1 \operatorname{Im} (\overline{\phi} D_2 \phi) - \partial_2 \operatorname{Im} (\overline{\phi} D_1 \phi), \\ &\Delta A_1 = -\frac{1}{c^2}\partial_2 \operatorname{Im} (\overline{\phi} D_0 \phi), \quad \Delta A_2 = \frac{1}{c^2}\partial_1 \operatorname{Im} (\overline{\phi} D_0 \phi), \end{aligned} \tag{2.1}$$

subject to initial data

$$\phi(x,0) = \phi_0(x), \quad \partial_t \phi(x,0) = \phi_1(x), \quad A_\mu(x,0) = a_\mu(x), \tag{2.2}$$

with the following constraints

$$\partial_1 a_1 + \partial_2 a_2 = 0, \quad \partial_1 a_2 - \partial_2 a_1 = \frac{1}{c^2} \operatorname{Im} (\overline{\phi}_0 \phi_1) - \frac{1}{c^2} a_0 |\phi_0|^2. \tag{2.3}$$

Then, we quote the following theorem that provides the global well-posedness of the CSH system under the Coulomb gauge condtion (2.1)- (2.3).

Theorem 2.1 *[4] For the initial data $(\phi_0, \phi_1, a_1, a_2) \in H^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)^2$ satisfying constraints (2.3), there exists a unigue global solution to (2.1)- (2.2) such that*

$$\phi \in C([0,T_1]; H^2(\mathbb{R}^2)) \cap C^1([0,T_1]; H^1(\mathbb{R}^2)), \quad A_1,\, A_2 \in C([0,T_1]; H^1(\mathbb{R}^2)),$$

for any $T_1 > 0$. Moreover, the solution continuously depends on the initial data.

As for the CSH system, we choose Coulomb gauge condition and reformulate the CSS system (1.4) as follows:

$$\begin{aligned} &iD_0u + D_1D_1u + D_2D_2u = -|u|^2u, \\ &\Delta A_0 = \partial_1 \text{ Im } (\overline{u}D_2u) - \partial_2 \text{ Im } (uD_1u), \\ &\Delta A_1 = \frac{1}{2}\partial_2|u|^2, \quad \Delta A_2 = -\frac{1}{2}\partial_1|u|^2, \end{aligned} \tag{2.4}$$

subject to initial data

$$u(x,0) = u_0(x), \quad A_\mu(x,0) = a_\mu(x), \tag{2.5}$$

with constraints

$$\partial_1 a_1 + \partial_2 a_2 = 0, \quad \partial_1 a_2 - \partial_2 a_1 = \frac{1}{2}|u_0|^2. \tag{2.6}$$

We also note that the CSS system has two conserved quantities: charge and energy, given by

$$Q(t) := \|u(t)\|_2^2, \quad E(t) := \|D_x u(t)\|_2^2 - \frac{1}{2}\|u(t)\|_4^4.$$

The following theorem addresses the local well-posedness of the CSS system (2.4)-(2.6), and also establishes global well-posedness under the assumption of small initial data.

Theorem 2.2 *[2] For the initial data $u_0 \in H^2(\mathbb{R}^2)$, there exists $T_0 = T(\|u_0\|) > 0$ depending only on $\|u_0\|_{H^2}$, such that (2.4)- (2.6) has a unique solution $u(x,t) \in C([0,T_0]; H^2(\mathbb{R}^2))$ that continuously depends on initial data. In addtion, the solution exists globally as long as*

$$\|u_0\|_2^2 \le \frac{1}{2C_4^4},$$

where C_4 is the constant from the covariant Sobolev inequality (3.1).

2.2 Main Theorem

In this subsection, we present the main result concerning the rigorous derivation of (1.4) from (1.3). Given that the global-in-time well-posedness of solutions to (1.3) is assured under the Coulomb gauge condition, we impose this gauge condition on the CSH system (1.3) (see assumption $(\mathcal{A}1)$ below). For the CSS system (1.4), the global-in-time well-posedness with small initial data assumptions is also guaranteed under the same gauge condition. To analyze the non-relativistic limit, we need to impose the following assumptions specifically on the initial data: For each $c > 1$, we consider the following initial data for the CSH system (1.3) given by

$$u^c(x,0) = u_0^c(x), \quad \partial_t u^c(x,0) = u_1^c(x), \quad A_\mu^c(x,0) = a_\mu^c(x),$$

satisfying

$$(\mathcal{A}1) \quad \partial_1 a_1^c + \partial_2 a_2^c = 0, \quad \partial_1 a_2^c - \partial_2 a_1^c = -\frac{1}{2}|u_0^c|^2 + \frac{1}{c^2}\,\mathrm{Im}\,(\overline{u}_0^c u_1^c) - \frac{1}{c^2} a_0^c |u_0^c|^2,$$

$$(\mathcal{A}2) \quad \|u_0^c\|_{H^2} + \|u_1^c\|_{H^1} = O(1), \quad \|a_1^c\|_{H^1} + \|a_2^c\|_{H^1} = O(1),$$

$$(\mathcal{A}3) \quad \exists\, \zeta \in H^2(\mathbb{R}^2) \quad \text{such that} \quad \|u_0^c - \zeta\|_{L^2} = O(\varepsilon^\lambda)\,(\lambda > 0), \quad \text{and} \quad \|\zeta\|_2^2 \le \frac{1}{2C_4^4},$$

$$(\mathcal{A}4) \quad \|u_0^c\|_2^2 + \|D_x u_0^c\|_2^2 \le (2\alpha^c)^{-1}, \quad 0 < \beta^c < (5C_4^4)^{-1},$$

where

$$\begin{aligned} &\alpha^c := C_4^4\Big(\frac{1}{4} + \frac{1}{c^2}\Big), \quad \text{where } C_4 \text{ is the constant from the covariant Sobolev inequality (3.1)}, \\ &\beta^c := \frac{1}{c^2}\|D_0 u_0^c\|_2 + \left\|\frac{2}{c^2} D_0 u_0^c + i u_0^c\right\|_2^2 + \|u_0^c\|_2^2 + \Big(1 + \frac{4}{c^2}\Big)\Big(\|D_x u_0^c\|_2^2 + \frac{1}{4c^2}\|u_0^c\|_6^6 - \frac{1}{2}\|u_0^c\|_4^4\Big). \end{aligned} \tag{2.7}$$

We remark that assumption $(\mathcal{A}4)$ is specifically related to obtaining a uniform bound with respect to c for the solutions, as detailed in the proof of Proposition 3.1 (1).

Theorem 2.3 *For each $c > 1$, let (u^c, A_μ^c) be the unique global solutions to the CSH system (1.3), and let (u, A_μ) be the unique global solution to the CSS system (2.4)-(2.6), subject to the initial data (u_0^c, u_1^c, a_μ^c) and $u_0 = \zeta$, satisfying the assumptions $(\mathcal{A}1)$- $(\mathcal{A}4)$, respectively. Then, as $c \to \infty$, for every $0 < T < \infty$, we have*

$$\|u^c - u\|_{C([0,T];L^2)} \to 0, \quad \|A_j^c - A_j\|_{L^\infty([0,T];\dot W^{1,p}\cap L^q)} \to 0, \quad \|A_0^c - A_0\|_{L^\infty([0,T];L^q)} \to 0,$$

where $j = 1, 2$, $1 < p < \infty$ and $2 < q < \infty$.

3 Uniform-in-ε estimates

First of all, we present some lemmas that will be used to obtain the uniform estimates. We utilize the following covariant Sobolev inequality [8].

Lemma 3.1 *For any $2 < q < \infty$, there exists a constant C_q depending only on q such that for $\psi \in L^2(\mathbb{R}^2)$ with $\partial_j \psi \in L^2(\mathbb{R}^2)$ and $A_j \psi \in L^2(\mathbb{R}^2)$ for $j = 1, 2$, the following inequality holds:*

$$\|\psi\|_q \leq C_q \|\psi\|_2^{\frac{2}{q}} \|D_x \psi\|_2^{1-\frac{2}{q}}. \tag{3.1}$$

We also use the following form of the Brezis–Gallouet inequailty introduced in [3]. It plays an important role in controlling the L^∞-norm of a solution, as seen in [4, 21].

Lemma 3.2 *For any $\psi \in H^2(\mathbb{R}^2)$, the following inequality holds:*

$$\|\psi\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\psi\|_{H^1(\mathbb{R}^2)} \left(1 + \ln(1 + \|\Delta \psi\|_{L^2(\mathbb{R}^2)})\right)^{\frac{1}{2}}.$$

Additionally, we use the following Hardy–Littlewood–Sobolev inequality.

Lemma 3.3 *Let I_1 be the fractional integral operator given by*

$$I_1 f(x) := \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|}\, dy.$$

If $1/q = 1/r - 1/2$, $1 < r < 2$, then

$$\|I_1 f\|_q \lesssim \|f\|_r.$$

From now on, we replace c^{-1} with ε in (1.3) and denote the solutions as $(u^\varepsilon, A_\mu^\varepsilon)$ to indicate dependence on ε. That is, for each $0 < \varepsilon < 1$, let $(u^\varepsilon, A_\mu^\varepsilon)$ be the solution to the following equations:

$$\varepsilon^2 D_0 D_0 u^\varepsilon - i D_0 u^\varepsilon - D_1 D_1 u^\varepsilon - D_2 D_2 u^\varepsilon = |u^\varepsilon|^2 u^\varepsilon - \frac{3}{4}\varepsilon^2 |u^\varepsilon|^4 u^\varepsilon, \tag{3.2}$$

$$\partial_0 A_1^\varepsilon - \partial_1 A_0^\varepsilon = -\ \mathrm{Im}\,(\overline{u^\varepsilon} D_2 u^\varepsilon), \tag{3.3}$$

$$\partial_0 A_2^\varepsilon - \partial_2 A_0^\varepsilon = \ \mathrm{Im}\,(\overline{u^\varepsilon} D_1 u^\varepsilon), \tag{3.4}$$

$$\partial_1 A_2^\varepsilon - \partial_2 A_1^\varepsilon = \varepsilon^2\ \mathrm{Im}\,(\overline{u^\varepsilon} D_0 u^\varepsilon) - \frac{1}{2}|u^\varepsilon|^2, \tag{3.5}$$

subject to the initial data

$$u^\varepsilon(x,0) = u_0^\varepsilon(x), \quad \partial_t u^\varepsilon(x,0) = u_1^\varepsilon(x), \quad A_\mu^\varepsilon(x,0) = a_\mu^\varepsilon(x). \tag{3.6}$$

We note that equations (3.2)- (3.6) exhibit the energy conservation given by

$$\mathcal{E}^\varepsilon(t) := \varepsilon^2\|D_0u^\varepsilon(t)\|_2^2 + \|D_xu^\varepsilon(t)\|_2^2 + \frac{\varepsilon^2}{4}\|u^\varepsilon(t)\|_6^6 - \frac{1}{2}\|u^\varepsilon(t)\|_4^4 = \mathcal{E}^\varepsilon(0) =: \mathcal{E}_0^\varepsilon. \tag{3.7}$$

We then establish several bounds on the solution to (3.2)- (3.6) that are independent of ε and uniform in time.

Proposition 3.1 *Let $(u^\varepsilon, A_\mu^\varepsilon)$ be the solution to (3.2)- (3.6) satisfying the assumptions $(\mathcal{A}1)$- $(\mathcal{A}4)$. Then, for all $t > 0$ and $0 < \varepsilon < 1$, we have*

(1) $\|u^\varepsilon(t)\|_2 + \|D_xu^\varepsilon(t)\|_2 = O(1)$.
(2) $\varepsilon\|D_0u^\varepsilon(t)\|_2 = O(1)$.
(3) $\|A_0^\varepsilon(t)\|_q = O(1), \quad \|A_j^\varepsilon(t)\|_q = O(1)$*, where* $j = 1, 2$ *and* $2 < q < \infty$.

Proof of (1): By energy conservation (3.7), the covariant Sobolev inequality and Young's inequality, we have

$$\begin{aligned}\|D_xu^\varepsilon(t)\|_2^2 &\le \mathcal{E}_0^\varepsilon + \frac{1}{2}\|u^\varepsilon(t)\|_4^4 \le \mathcal{E}_0^\varepsilon + \frac{1}{2}C_4^4\|u^\varepsilon(t)\|_2^2\|D_xu^\varepsilon(t)\|_2^2 \\ &\le \mathcal{E}_0^\varepsilon + \frac{1}{4}C_4^4\big(\|u^\varepsilon(t)\|_2^2 + \|D_xu^\varepsilon(t)\|_2^2\big)^2.\end{aligned} \tag{3.8}$$

On the other hand, multiplying (3.2) by $\overline{u^\varepsilon}$, taking its imaginary part, and integrating it over $\mathbb{R}^2$, we obtain

$$\frac{d}{dt}\int_{\Sigma_t} |u^\varepsilon|^2 - 2\varepsilon^2 \operatorname{Im}\,(\overline{u^\varepsilon}D_0u^\varepsilon)\,dx = 0,$$

which implies

$$\|u^\varepsilon(t)\|_2^2 = 2\varepsilon^2\int_{\Sigma_t} \operatorname{Im}\,(\overline{u^\varepsilon}D_0u^\varepsilon)\,dx + \underbrace{\|u_0^\varepsilon\|_2^2 - 2\varepsilon^2\int_{\Sigma_0} \operatorname{Im}\,(\overline{u^\varepsilon}D_0u^\varepsilon)\,dx}_{:=\mathcal{I}_0}.$$

By Hölder's inequality and Young's inequality, we have

$$\|u^\varepsilon(t)\|_2^2 \le 2\varepsilon^2\|u^\varepsilon(t)\|_2\|D_0u^\varepsilon(t)\|_2 + \mathcal{I}_0 \le \frac{1}{2}\|u^\varepsilon(t)\|_2^2 + 2\varepsilon^4\|D_0u^\varepsilon(t)\|_2^2 + \mathcal{I}_0. \tag{3.9}$$

To control $\|D_0u^\varepsilon(t)\|_2$, we use energy consevation (3.7) and the covariant Sobolev inequality:

$$\varepsilon^2\|D_0u^\varepsilon(t)\|_2^2 \le \mathcal{E}_0^\varepsilon + \frac{1}{2}\|u^\varepsilon(t)\|_4^4 \le \mathcal{E}_0^\varepsilon + \frac{1}{4}C_4^4\left(\|u^\varepsilon(t)\|_2^2 + \|D_xu^\varepsilon(t)\|_2^2\right)^2. \tag{3.10}$$

Applying (3.10) to (3.9) gives

$$\|u^\varepsilon(t)\|_2^2 \le 4\varepsilon^4\|D_0u^\varepsilon(t)\|_2^2 + 2\mathcal{I}_0 \le 4\varepsilon^2\Big(\mathcal{E}_0^\varepsilon + \frac{1}{4}C_4^4\left(\|u^\varepsilon(t)\|_2^2 + \|D_xu^\varepsilon(t)\|_2^2\right)^2\Big) + 2\mathcal{I}_0 \quad (3.11)$$

Now, we add (3.8) and (3.11) to have

$$\|u^\varepsilon(t)\|_2^2 + \|D_xu^\varepsilon(t)\|_2^2 \le (1+4\varepsilon^2)\Big(\mathcal{E}_0^\varepsilon + \frac{1}{4}C_4^4\left(\|u^\varepsilon(t)\|_2^2 + \|D_xu^\varepsilon(t)\|_2^2\right)^2\Big) + 2\mathcal{I}_0$$

Then we have, for $\mathcal{X}^\varepsilon(t) := \|u^\varepsilon(t)\|_2^2 + \|D_xu^\varepsilon(t)\|_2^2$,

$$\alpha^\varepsilon\mathcal{X}^\varepsilon(t)^2 - \mathcal{X}^\varepsilon(t) + \beta^\varepsilon \ge 0,$$

where $\alpha^\varepsilon, \beta^\varepsilon$ are defined in (2.7). Note that our assumption $(\mathcal{A}4)$ on initial data states that

$$\mathcal{X}^\varepsilon(0) \le (2\alpha^\varepsilon)^{-1} \quad \text{and} \quad 0 < 4\alpha^\varepsilon\beta^\varepsilon < 1. \tag{3.12}$$

By the continuity principle with (3.12), we finally have the desired result: for all $t > 0$ and $0 < \varepsilon < 1$,

$$\mathcal{X}^\varepsilon(t) \le \frac{1-\sqrt{1-4\alpha^\varepsilon\beta^\varepsilon}}{2\alpha^\varepsilon} \le C. \tag{3.13}$$

Proof of (2): By combining (3.10) and (3.13), it is easy to have

$$\varepsilon^2\|D_0u^\varepsilon(t)\|_2^2 \le \mathcal{E}_0^\varepsilon + \frac{1}{4}C_4^4\Big(\frac{1-\sqrt{1-4\alpha^\varepsilon\beta^\varepsilon}}{2\alpha^\varepsilon}\Big) \le C.$$

Proof of (3): Under the Coulomb gauge condition, equations (3.3)- (3.5) are equivalent to

$$\begin{aligned} \Delta A_0^\varepsilon &= \partial_1 \text{ Im } (\overline{u^\varepsilon}D_2u^\varepsilon) - \partial_2 \text{ Im } (\overline{u^\varepsilon}D_1u^\varepsilon), \\ \Delta A_j^\varepsilon &= (-1)^j\partial_{j'}\Big(\varepsilon^2 \text{ Im } (\overline{u^\varepsilon}D_0u^\varepsilon) - \frac{1}{2}|u^\varepsilon|^2\Big), \end{aligned} \tag{3.14}$$

where $j' = 2$ if $j = 1$ and $j' = 1$ if $j = 2$. We use Lemma 3.3 with $1/q = 1/r - 1/2$, $2 < q < \infty$ and Lemma 3.1 to show that

$$\begin{aligned} \|A_0^\varepsilon(t)\|_q &\lesssim \|\overline{u^\varepsilon}D_2u^\varepsilon\|_r + \|\overline{u^\varepsilon}D_1u^\varepsilon\|_r \lesssim \|u^\varepsilon\|_q\|D_xu^\varepsilon\|_2 \lesssim \|u^\varepsilon\|_2^{\frac{2}{q}}\|D_xu^\varepsilon\|_2^{2-\frac{2}{q}} \le O(1), \\ \|A_j^\varepsilon(t)\|_q &\lesssim \varepsilon^2\|\overline{u^\varepsilon}D_0u^\varepsilon\|_r + \frac{1}{2}\|u^\varepsilon\|_{2r}^2 \lesssim \varepsilon\|u^\varepsilon\|_q(\varepsilon\|D_0u^\varepsilon\|_2) + O(1) \le O(1), \end{aligned}$$

for all $t > 0$ and $0 < \varepsilon < 1$. □

We need the following energy estimates to derive uniform bounds with respect to ε for the higher-order derivatives of the solutions. To state these estimates, we use the notation for the nonlinear terms $W(u^\varepsilon) := |u^\varepsilon|^2u^\varepsilon - \frac{3}{4}\varepsilon^2|u^\varepsilon|^4u^\varepsilon$ and indicate the ε-dependency in the strengh of the gauge fields as $F^{\varepsilon}_{\mu\nu} := \partial_\mu A^{\varepsilon}_{\nu} - \partial_\nu A^{\varepsilon}_{\mu}$. Additionally, since we are using the Einstein summation convention, we emphasize once again that the repeated Latin indices should be summed over 1 and 2 in the following estimates.

Lemma 3.4 *Let $(u^\varepsilon, A^\varepsilon_\mu)$ be the solution to (3.2)- (3.6) satisfying the assumptions $(\mathcal{A}1)$- $(\mathcal{A}4)$. Then, we have the following:*

$$
\begin{aligned}
(1)\ & \frac{1}{2}\frac{d}{dt}\int_{\Sigma_t} |D_0 u^\varepsilon|^2\,dx - \varepsilon^2 \frac{d}{dt}\int_{\Sigma_t} Im(\overline{D_0 u^\varepsilon} D_0 D_0 u^\varepsilon)\ dx \\
&= -\int_{\Sigma_t} \partial_j F^{\varepsilon}_{j0} Re(u^\varepsilon \overline{D_0 u^\varepsilon}) + 2F^{\varepsilon}_{j0} Re(\overline{D_0 u^\varepsilon} D_j u^\varepsilon)\,dx - \int_{\Sigma_t} Im(\overline{D_0 u^\varepsilon} D_0 W(u^\varepsilon))\,dx. \\
(2)\ & \varepsilon^2 \frac{d}{dt}\int_{\Sigma_t} |D_0 D_0 u^\varepsilon|^2\,dx + \frac{d}{dt}\int_{\Sigma_t} |D_0 D_1 u^\varepsilon|^2 + |D_0 D_2 u^\varepsilon|^2\,dx \\
&= 2\int_{\Sigma_t} \partial_0 F^{\varepsilon}_{0j} Im(u^\varepsilon \overline{D_0 D_j u^\varepsilon}) + 2F^{\varepsilon}_{0j} Im(D_0 u^\varepsilon \overline{D_0 D_j u^\varepsilon}) + F^{\varepsilon}_{0j} Im(\overline{D_0 D_0 u^\varepsilon} D_j u^\varepsilon)\,dx \\
&\quad + 2\int_{\Sigma_t} Re(\overline{D_0 D_0 u^\varepsilon} D_0 W(u^\varepsilon))\,dx. \\
(3)\ & \varepsilon^2 \frac{d}{dt}\int_{\Sigma_t} |D_1 D_0 u^\varepsilon|^2 + |D_2 D_0 u^\varepsilon|^2\,dx + \frac{d}{dt}\int_{\Sigma_t} |D_1 D_1 u^\varepsilon + D_2 D_2 u^\varepsilon|^2\,dx \\
&= 2\varepsilon^2 \int_{\Sigma_t} F^{\varepsilon}_{0j} Im(D_0 u^\varepsilon \overline{D_j D_0 u^\varepsilon}) + \partial_j F^{\varepsilon}_{j0} Im(\overline{u^\varepsilon} D_0 D_0 u^\varepsilon) + 2F^{\varepsilon}_{j0} Im(\overline{D_j u^\varepsilon} D_0 D_0 u^\varepsilon)\,dx \\
&\quad - 2\int_{\Sigma_t} \partial_j F^{\varepsilon}_{j0} Re(\overline{u^\varepsilon} D_0 u^\varepsilon) + 2F^{\varepsilon}_{j0} Re(\overline{D_j u^\varepsilon} D_0 u^\varepsilon)\,dx - 2\int_{\Sigma_t} Re(\overline{D_0 D_j D_j u^\varepsilon} W(u^\varepsilon))\,dx.
\end{aligned}
$$

Proof Essentially, we follow the calculations from [5] with some modifications. In particular, it is crucial to correctly understand the second integral in (3). For this reason, we will provide a detailed proof only for (3), while the remaining parts can be derived in a similar and more straightforward manner.

Let us remind the following calculation laws

$$
\begin{aligned}
\partial_\mu(\overline{\phi}\psi) &= \overline{D_\mu \phi}\psi + \overline{\phi} D_\mu \psi, \\
D_\mu D_\nu \phi &= D_\nu D_\mu \phi + i F_{\nu\mu}\phi.
\end{aligned}
\tag{3.15}
$$

Recall that $(u^\varepsilon, A^\varepsilon_\mu)$ satisfies

$$
\varepsilon^2 D_0 D_0 u^\varepsilon - i D_0 u^\varepsilon - D_k D_k u^\varepsilon - W(u^\varepsilon) = 0. \tag{3.16}
$$

We multiply (3.16) by $\overline{D_0 D_j D_j u^\varepsilon}$ for each $j = 1, 2$, add the resulting equations, take the real part and integrate the resulting equation over $\mathbb{R}^2$ to obtain

$$
\begin{aligned}
0 &= -\varepsilon^2 \int_{\Sigma_t} \text{Re}\,(\overline{D_0 D_j D_j u^\varepsilon} D_0 D_0 u^\varepsilon)\,dx + \int_{\Sigma_t} \text{Re}\,(i\overline{D_0 D_j D_j u^\varepsilon} D_0 u^\varepsilon)\,dx \\
&\quad + \int_{\Sigma_t} \text{Re}\,(\overline{D_0 D_j D_j u^\varepsilon} D_k D_k u^\varepsilon)\,dx + \int_{\Sigma_t} \text{Re}\,(\overline{D_0 D_j D_j u^\varepsilon} W(u^\varepsilon))\,dx =: \sum_{\ell=1}^{4} \mathcal{J}_{1\ell}(t).
\end{aligned}
$$

Applying the calculation law (3.15) carefully, we can derive

$$
\begin{aligned}
\mathcal{J}_{11}(t) &= \frac{\varepsilon^2}{2}\frac{d}{dt}\int_{\Sigma_t} (|D_1 D_0 u^\varepsilon|^2 + |D_2 D_0 u^\varepsilon|^2)\,dx \\
&\quad - \varepsilon^2 \int_{\Sigma_t} F^\varepsilon_{0j} Im(D_0 u^\varepsilon \overline{D_j D_0 u^\varepsilon}) + \partial_j F^\varepsilon_{j0} Im(\overline{u^\varepsilon} D_0 D_0 u^\varepsilon) + 2F^\varepsilon_{j0} Im(\overline{D_j u^\varepsilon} D_0 D_0 u^\varepsilon)\,dx, \\
\mathcal{J}_{12}(t) &= \int_{\Sigma_t} \partial_j F^\varepsilon_{j0} Re(\overline{u^\varepsilon} D_0 u^\varepsilon) + 2F^\varepsilon_{j0} Re(\overline{D_j u^\varepsilon} D_0 u^\varepsilon)\,dx, \\
\mathcal{J}_{13}(t) &= \frac{1}{2}\frac{d}{dt}\int_{\Sigma_t} |D_1 D_1 u^\varepsilon + D_2 D_2 u^\varepsilon|^2\,dx, \qquad \mathcal{J}_{14}(t) = \text{Re}\,(\overline{D_0 D_j D_j u^\varepsilon} W(u^\varepsilon))\,dx,
\end{aligned}
$$

which yield the dersired result. □

We will demonstrate a uniform bound in ε for $\Phi^\varepsilon(t)$, where $\Phi^\varepsilon(t)$ is defined by

$$
\Phi^\varepsilon(t) := \varepsilon^4 \|D_0 D_0 u^\varepsilon(t)\|_2^2 + \varepsilon^2\big(\|D_0 D_x u^\varepsilon(t)\|_2^2 + \|D_x D_0 u^\varepsilon(t)\|_2^2\big) + \|D_x D_x u^\varepsilon(t)\|_2^2 + \frac{1}{4}\|D_0 u^\varepsilon(t)\|_2^2,
$$

which includes the L^2-norms of second derivatives. Unlike Proposition 3.1, this bound depends on the given time T.

Proposition 3.2 *Let $(u^\varepsilon, A^\varepsilon_\mu)$ be the solution to (3.2)- (3.6) satisfying the assumptions $(\mathcal{A}1)$- $(\mathcal{A}4)$. Then, for a given $T > 0$ and for all $0 < \varepsilon < 1$, we have*

$$
\sup_{0 \le t \le T} \Phi^\varepsilon(t) = O(1).
$$

Proof To attain the desired result, we will derive the following Grönwall inequality for $0 \le t \le T$ and for all $0 < \varepsilon < 1$:

$$
\Phi^\varepsilon(t) \le C_0 + \int_0^t C_1(1 + \ln(1 + \Phi^\varepsilon(\tau)^{\frac{1}{2}}))\Phi^\varepsilon(\tau)\,d\tau, \tag{3.17}
$$

where C_0 is a constant depending on the initial data and C_1 is a generic constant, both of which are independent of ε and arise from the following calculations. Applying Grönwall's lemma to (3.17) will then yield a uniform bound in ε for $\Phi^\varepsilon(t)$, depending only on C_0, C_1, and T.

We will derive (3.17) through the following four steps.

Step 1. (Estimates for F^ε_{0j} and $W(u^\varepsilon)$): Applying Hölder's inequality, the covariant Sobolev inequality and Proposition 3.1 to (3.3)- (3.4), for each $j = 1, 2$, we have

$$\begin{aligned}&\|F^{\varepsilon}_{0j}\|_4\le\|u^{\varepsilon}\|_{\infty}\|D_xu^{\varepsilon}\|_4\le O(1)\|u^{\varepsilon}\|_{\infty}\|D_xu^{\varepsilon}\|_2^{\frac12}\|D_xD_xu^{\varepsilon}\|_2^{\frac12}\le O(1)\|u^{\varepsilon}\|_{\infty}\Phi^{\varepsilon}(t)^{\frac14},\\&\|F^{\varepsilon}_{0j}\|_2\le\|u^{\varepsilon}\|_4\|D_xu^{\varepsilon}\|_4\le O(1)\|D_xD_xu^{\varepsilon}\|_2^{\frac12}\le O(1)\Phi^{\varepsilon}(t)^{\frac14}.\end{aligned}\tag{3.18}$$

We apply (3.15) to (3.3)- (3.4) to obtain

$$\begin{aligned}\partial_0F^{\varepsilon}_{0j}&=(-1)^j\ \mathrm{Im}\ (\overline{D_0u^{\varepsilon}}D_{j'}u^{\varepsilon}+\overline{u^{\varepsilon}}D_0D_{j'}u^{\varepsilon}),\\\partial_jF^{\varepsilon}_{0j}&=(-1)^j\ \mathrm{Im}\ (\overline{D_ju^{\varepsilon}}D_{j'}u^{\varepsilon}+\overline{u^{\varepsilon}}D_jD_{j'}u^{\varepsilon}),\end{aligned}\tag{3.19}$$

where $j'=2$ if $j=1$ and $j'=1$ if $j=2$. Simalar calculations as in (3.18) with (3.19) give

$$\begin{aligned}\|\partial_0F^{\varepsilon}_{0j}\|_2&\le\|D_0u^{\varepsilon}\|_4\|D_xu^{\varepsilon}\|_4+\|u^{\varepsilon}\|_{\infty}\|D_0D_xu^{\varepsilon}\|_2\\&\le O(1)\|D_0u^{\varepsilon}\|_2^{\frac12}\|D_xD_0u^{\varepsilon}\|_2^{\frac12}\|D_xD_xu^{\varepsilon}\|_2^{\frac12}+\|u^{\varepsilon}\|_{\infty}\|D_0D_xu^{\varepsilon}\|_2,\\\|\partial_jF^{\varepsilon}_{0j}\|_2&\le\|D_xu^{\varepsilon}\|_4^2+\|u^{\varepsilon}\|_{\infty}\|D_xD_xu^{\varepsilon}\|_2\le O(1)(1+\|u^{\varepsilon}\|_{\infty})\Phi^{\varepsilon}(t)^{\frac12}.\end{aligned}$$

To estimate $W(u^{\varepsilon})=|u^{\varepsilon}|^2u^{\varepsilon}-\frac34\varepsilon^2|u^{\varepsilon}|^4u^{\varepsilon}$, we note that for each $\mu=0,1,2$ and $j=1,2$:

$$\begin{aligned}D_{\mu}(|u^{\varepsilon}|^2u^{\varepsilon})&=2\ \mathrm{Re}\ (\overline{u^{\varepsilon}}D_{\mu}u^{\varepsilon})u^{\varepsilon}+|u^{\varepsilon}|^2D_{\mu}u^{\varepsilon},\\D_{\mu}\Big(\frac34\varepsilon^2|u^{\varepsilon}|^4u^{\varepsilon}\Big)&=3\varepsilon^2\ \mathrm{Re}\ (\overline{u^{\varepsilon}}D_{\mu}u^{\varepsilon})|u^{\varepsilon}|^2u^{\varepsilon}+\frac34\varepsilon^2|u^{\varepsilon}|^4D_{\mu}u^{\varepsilon},\\D_jD_j(|u^{\varepsilon}|^2u^{\varepsilon})&=2|D_ju^{\varepsilon}|^2u^{\varepsilon}+2\ \mathrm{Re}\ (\overline{u^{\varepsilon}}D_jD_ju^{\varepsilon})u^{\varepsilon}+4\ \mathrm{Re}\ (\overline{u^{\varepsilon}}D_ju^{\varepsilon})D_ju^{\varepsilon}+|u^{\varepsilon}|^2D_jD_ju^{\varepsilon}.\end{aligned}$$

The same calculations as in (3.18) show that for each $j=1,2$:

$$\begin{aligned}\|D_0W(u^{\varepsilon})\|_2&\lesssim\|u^{\varepsilon}\|_{\infty}^2(\|D_0u^{\varepsilon}\|_2+\varepsilon^2\||u^{\varepsilon}|^2\|_4\|D_0u^{\varepsilon}\|_4)\\&\lesssim\|u^{\varepsilon}\|_{\infty}^2(\|D_0u^{\varepsilon}\|_2+\varepsilon^{\frac32}O(1)\|D_0u^{\varepsilon}\|_2^{\frac12}\varepsilon^{\frac12}\|D_xD_0u^{\varepsilon}\|_2^{\frac12})\\&\le O(1)\|u^{\varepsilon}\|_{\infty}^2\Phi^{\varepsilon}(t)^{\frac12},\\\|D_jW(u^{\varepsilon})\|_2&\lesssim\||u^{\varepsilon}|^2\|_4\|D_ju^{\varepsilon}\|_4+\varepsilon^2\||u^{\varepsilon}|^4\|_4\|D_ju^{\varepsilon}\|_4\\&\lesssim\|D_xu^{\varepsilon}\|_2^{\frac12}\|D_xD_xu^{\varepsilon}\|_2^{\frac12}\\&\le O(1)\Phi^{\varepsilon}(t)^{\frac14},\\\|D_jD_j(|u^{\varepsilon}|^2u^{\varepsilon})\|_2&\lesssim\|u^{\varepsilon}\|_{\infty}\|D_xu^{\varepsilon}\|_4^2+\|u^{\varepsilon}\|_{\infty}^2\|D_xD_xu^{\varepsilon}\|_2\le O(1)(1+\|u^{\varepsilon}\|_{\infty}^2)\Phi^{\varepsilon}(t)^{\frac12}.\end{aligned}$$

Step 2. (Estimates for the right-hand sides of (1)- (3) in Lemma 3.4): We will show

$$\mathcal{R}_{\ell}(t)\le O(1)(1+\|u^{\varepsilon}(t)\|_{\infty}^2)\Phi^{\varepsilon}(t),\qquad\text{for}\quad\ell=1,2,3,$$

where

$$
\begin{aligned}
\mathcal{R}_1(t) &:= 2\varepsilon^2 \int_{\Sigma_t} \partial_0 F^\varepsilon_{0j} Im(u^\varepsilon \overline{D_0 D_j u^\varepsilon}) + 2F^\varepsilon_{0j} Im(D_0 u^\varepsilon \overline{D_0 D_j u^\varepsilon}) + F^\varepsilon_{0j} Im(\overline{D_0 D_0 u^\varepsilon} D_j u^\varepsilon)\, dx \\
&\quad + 2\varepsilon^2 \int_{\Sigma_t} Re(\overline{D_0 D_0 u^\varepsilon} D_0 W(u^\varepsilon))\, dx, \\
\mathcal{R}_2(t) &:= -\int_{\Sigma_t} \partial_j F^\varepsilon_{j0} Re(u^\varepsilon \overline{D_0 u^\varepsilon}) + 2F^\varepsilon_{j0} Re(\overline{D_0 u^\varepsilon} D_j u^\varepsilon)\, dx - \int_{\Sigma_t} Im(\overline{D_0 u^\varepsilon} D_0 W(u^\varepsilon))\, dx, \\
\mathcal{R}_3(t) &:= 2\varepsilon^2 \int_{\Sigma_t} F^\varepsilon_{0j} Im(D_0 u^\varepsilon \overline{D_j D_0 u^\varepsilon}) + \partial_j F^\varepsilon_{j0} Im(\overline{u^\varepsilon} D_0 D_0 u^\varepsilon) + 2F^\varepsilon_{j0} Im(\overline{D_j u^\varepsilon} D_0 D_0 u^\varepsilon)\, dx \\
&\quad - 2\int_{\Sigma_t} \partial_j F^\varepsilon_{j0} Re(\overline{u^\varepsilon} D_0 u^\varepsilon) + 2F^\varepsilon_{j0} Re(\overline{D_j u^\varepsilon} D_0 u^\varepsilon)\, dx - 2\int_{\Sigma_t} Re(\overline{D_0 D_j D_j u^\varepsilon} W(u^\varepsilon))\, dx.
\end{aligned}
$$

◇ (Estimates of $\mathcal{R}_1$ and $\mathcal{R}_2$): By Hölder's iequality, the covariant Sobolev inequality, and the bounds derived in Proposition 3.1 and Step 1, we have

$$
\begin{aligned}
\mathcal{R}_1(t) &\lesssim \varepsilon^2 \|u^\varepsilon\|_\infty \|\partial_0 F^\varepsilon_{0j}\|_2 \|D_0 D_x u^\varepsilon\|_2 + \varepsilon^2 \|F^\varepsilon_{0j}\|_4 \|D_0 u^\varepsilon\|_4 \|D_0 D_x u^\varepsilon\|_2 \\
&\quad + \varepsilon^2 \|F^\varepsilon_{0j}\|_4 \|D_x u^\varepsilon\|_4 \|D_0 D_0 u^\varepsilon\|_2 + \varepsilon^2 \|D_0 W(u^\varepsilon)\|_2 \|D_0 D_0 u^\varepsilon\|_2 \\
&\lesssim \|u^\varepsilon\|_\infty \Big(\varepsilon^{\frac12} \|D_0 u^\varepsilon\|_2^{\frac12} \varepsilon^{\frac12} \|D_x D_0 u^\varepsilon\|_2^{\frac12} \|D_x D_x u^\varepsilon\|_2^{\frac12} + \|u^\varepsilon\|_\infty \varepsilon \|D_0 D_x u^\varepsilon\|_2 \Big) \varepsilon \|D_0 D_x u^\varepsilon\|_2 \\
&\quad + \|u^\varepsilon\|_\infty \Phi^\varepsilon(t)^{\frac14} \varepsilon^{\frac12} \|D_0 u^\varepsilon\|_2^{\frac12} \varepsilon^{\frac12} \|D_x D_0 u^\varepsilon\|_2^{\frac12} \varepsilon \|D_0 D_x u^\varepsilon\|_2 \\
&\quad + \|u^\varepsilon\|_\infty \Phi^\varepsilon(t)^{\frac14} \|D_x u^\varepsilon\|_2^{\frac12} \|D_x D_x u^\varepsilon\|_2^{\frac12} \varepsilon^2 \|D_0 D_0 u^\varepsilon\|_2 + \|u^\varepsilon\|_\infty^2 \Phi^\varepsilon(t)^{\frac12} \varepsilon^2 \|D_0 D_0 u^\varepsilon\|_2 \\
&\lesssim \|u^\varepsilon\|_\infty \Phi^\varepsilon(t) + \|u^\varepsilon\|_\infty^2 \Phi^\varepsilon(t) \\
&\le O(1)(1 + \|u^\varepsilon\|_\infty^2) \Phi^\varepsilon(t),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}_2(t) &\le \|\partial_j F^\varepsilon_{j0}\|_2 \|u^\varepsilon\|_\infty \|D_0 u^\varepsilon\|_2 + \|F^\varepsilon_{j0}\|_4 \|D_x u^\varepsilon\|_4 \|D_0 u^\varepsilon\|_2 + \|D_0 u^\varepsilon\|_2 \|D_0 W(u^\varepsilon)\|_2 \\
&\le O(1)\|u^\varepsilon\|_\infty (1 + \|u^\varepsilon\|_\infty) \Phi^\varepsilon(t) + O(1)\|u^\varepsilon\|_\infty \Phi^\varepsilon(t)^{\frac14} \|D_x D_x u^\varepsilon\|_2^{\frac12} \Phi^\varepsilon(t)^{\frac12} + O(1)\|u^\varepsilon\|_\infty^2 \Phi^\varepsilon(t) \\
&\le O(1)(1 + \|u^\varepsilon\|_\infty^2) \Phi^\varepsilon(t).
\end{aligned}
$$

◇ (Estimates of $\mathcal{R}_3$): Except for the last term in $\mathcal{R}_3(t)$, all other terms can be bounded by

$$
O(1)\|u^\varepsilon\|_\infty (1 + \|u^\varepsilon\|_\infty) \Phi^\varepsilon(t),
$$

as above. Thus, we omit the details for these terms and focus on estimating the last term in detail.

By (3.15) and integration by parts, we can rewrite it as

$$
\begin{aligned}
&-2\int_{\Sigma_t} \mathrm{Re}\ (\overline{D_0 D_j D_j u^\varepsilon} W(u^\varepsilon))\, dx \\
&= 2\int_{\Sigma_t} F^\varepsilon_{j0}\ \mathrm{Im}\ (D_j u^\varepsilon \overline{W(u^\varepsilon)})\, dx - 2\int_{\Sigma_t} F^\varepsilon_{j0}\ \mathrm{Im}\ (u^\varepsilon \overline{D_j W(u^\varepsilon)})\, dx - 2\int_{\Sigma_t} \mathrm{Re}\ (D_0 u^\varepsilon \overline{D_j D_j W(u^\varepsilon)})\, dx \\
&=: \sum_{\ell=1}^{3} \mathcal{J}_{2\ell}(t).
\end{aligned}
$$

We can then estimate each term as follows:

$$\mathcal{J}_{21}(t) \le \|F^{\varepsilon}_{j0}\|_2 \|D_x u^\varepsilon\|_4 \|W(u^\varepsilon)\|_4 \le O(1)\Phi^\varepsilon(t)^{\frac14} \|D_x D_x u^\varepsilon\|_2^{\frac12} \|u^\varepsilon\|_\infty^2 \le O(1)\|u^\varepsilon\|_\infty^2 \Phi^\varepsilon(t)^{\frac12}.$$

$$\mathcal{J}_{22}(t) \le \|F^{\varepsilon}_{j0}\|_2 \|u^\varepsilon\|_\infty \|D_j W(u^\varepsilon)\|_2 \le O(1)\|u^\varepsilon\|_\infty \Phi^\varepsilon(t)^{\frac12}.$$

$$\begin{aligned}\mathcal{J}_{23}(t) &= -2\int_{\Sigma_t} \mathrm{Re}\left(\overline{D_0 u^\varepsilon} D_j D_j(|u^\varepsilon|^2 u^\varepsilon)\right) dx + \frac32 \varepsilon^2 \int_{\Sigma_t} \mathrm{Re}\left(\overline{D_j D_0 u^\varepsilon} D_j(|u^\varepsilon|^4 u^\varepsilon)\right) \\ &\lesssim \|D_0 u^\varepsilon\|_2 \|D_x D_x(|u^\varepsilon|^2 u^\varepsilon)\|_2 + \varepsilon^2 \|D_x D_0 u^\varepsilon\|_2 \|D_j(|u^\varepsilon|^4 u^\varepsilon)\|_2 \\ &\le O(1)(1+\|u^\varepsilon\|_\infty^2)\Phi^\varepsilon(t) + \varepsilon O(1)\Phi^\varepsilon(t)^{\frac34} \\ &\le O(1)(1+\|u^\varepsilon\|_\infty^2)\Phi^\varepsilon(t).\end{aligned}$$

<u>*Step* 3.</u> (Estimates for Φ^ε): To estimate $\Phi^\varepsilon(t)$, we divide the estimates into two parts: deriving uniform-in-ε estimates of $\|D_0 u^\varepsilon\|_2^2$ and $\|D_x D_x u^\varepsilon\|_2^2$, respectively.

$\diamond$ (Estimates of $\|D_0 u^\varepsilon\|_2^2$): By (2) in Lemma 3.4 and Step 2, we have

$$\frac{d}{dt}\Big(\varepsilon^4 \|D_0 D_0 u^\varepsilon(t)\|_2^2 + \varepsilon^2 \|D_0 D_x u^\varepsilon(t)\|_2^2\Big) \le \mathcal{R}_1(t) \le O(1)(1+\|u^\varepsilon(t)\|_\infty^2)\Phi^\varepsilon(t).$$

We integrate it over the time interval [0, *t*] to derive

$$\varepsilon^4 \|D_0 D_0 u^\varepsilon(t)\| + \varepsilon^2 \|D_0 D_x u^\varepsilon(t)\|_2^2 \le \Phi^\varepsilon(0) + \int_0^t O(1)(1+\|u^\varepsilon(\tau)\|_\infty^2)\Phi^\varepsilon(\tau)\, d\tau. \tag{3.20}$$

On the other hand, by integrating (1) from Lemma 3.4 over the time interval [0, *t*] and applying Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}\frac12 \|D_0 u^\varepsilon(t)\|_2^2 &\le \|D_0 u^\varepsilon(0)\|_2^2 - \varepsilon^2 \int_{\Sigma_0} \mathrm{Im}\,(\overline{D_0 u^\varepsilon} D_0 D_0 u^\varepsilon)\, dx \\ &\qquad + \varepsilon^2 \int_{\Sigma_t} \mathrm{Im}\,(\overline{D_0 u^\varepsilon} D_0 D_0 u^\varepsilon)\, dx + \int_0^t \mathcal{R}_2(\tau)\, d\tau \\ &\le \frac54 \|D_0 u^\varepsilon(0)\|_2^2 + \varepsilon^4 \|D_0 D_0 u^\varepsilon(0)\|_2^2 + \frac14 \|D_0 u^\varepsilon(t)\|_2^2 + \varepsilon^4 \|D_0 D_0 u^\varepsilon(t)\|_2^2 \\ &\qquad + \int_0^t O(1)(1+\|u^\varepsilon(\tau)\|_\infty^2)\Phi^\varepsilon(\tau)\, d\tau.\end{aligned} \tag{3.21}$$

We combine (3.20) and (3.21) to have

$$\frac14 \|D_0 u^\varepsilon(t)\|_2^2 \le O(1)\Phi^\varepsilon(0) + \int_0^t O(1)(1+\|u^\varepsilon(\tau)\|_\infty^2)\Phi^\varepsilon(\tau)\, d\tau. \tag{3.22}$$

$\diamond$ (Estimates of $\|D_x D_x u^\varepsilon\|_2^2$): To rewrite the second integral in (3) from Lemma 3.4, using the calculation law (3.15), we observe that

$$
\begin{aligned}
D_1D_1u^\varepsilon\overline{D_2D_2u^\varepsilon} &= \partial_2(D_1D_1u^\varepsilon\overline{D_2u^\varepsilon}) - \partial_1(D_2D_1u^\varepsilon\overline{D_2u^\varepsilon}) + |D_1D_2u^\varepsilon| \\
&\quad + iF_{12}^\varepsilon u\overline{D_1D_2u^\varepsilon} - iF_{12}^\varepsilon D_1u\overline{D_2u^\varepsilon}, \\
D_2D_2u^\varepsilon\overline{D_1D_1u^\varepsilon} &= \partial_2(D_2D_2u^\varepsilon\overline{D_1u^\varepsilon}) - \partial_1(D_1D_2u^\varepsilon\overline{D_1u^\varepsilon}) + |D_2D_1u^\varepsilon|^2 \\
&\quad + iF_{21}^\varepsilon u\overline{D_2D_1u^\varepsilon} - iF_{21}^\varepsilon D_2u^\varepsilon\overline{D_1u^\varepsilon}, \\
iF_{21}^\varepsilon u^\varepsilon\overline{D_2D_1u^\varepsilon} &= iF_{21}^\varepsilon u(\overline{D_1D_2u^\varepsilon} - iF_{21}^\varepsilon\overline{u^\varepsilon}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\Sigma_t} D_1D_1u^\varepsilon\overline{D_2D_2u^\varepsilon} + D_2D_2u^\varepsilon\overline{D_1D_1u^\varepsilon}\,dx \\
&= \int_{\Sigma_t} |D_1D_2u^\varepsilon|^2 + |D_2D_1u^\varepsilon|^2 + {F_{12}^\varepsilon}^2|u^\varepsilon|^2 + 2F_{12}^\varepsilon\ \mathrm{Im}\ (D_1u^\varepsilon\overline{D_2u^\varepsilon})\,dx.
\end{aligned}
$$

Thus, we can replace the second integral in (3) from Lemma 3.4 with the above equation as

$$
\frac{d}{dt}\Big(\varepsilon^2\|D_xD_0u^\varepsilon(t)\|_2^2 + \|D_xD_xu^\varepsilon(t)\|_2^2 + \int_{\Sigma_t} {F_{12}^\varepsilon}^2|u^\varepsilon|^2\,dx\Big) = -\frac{d}{dt}\int_{\Sigma_t} 2F_{12}^\varepsilon\ \mathrm{Im}\ (D_1u^\varepsilon\overline{D_2u^\varepsilon})\,dx + \mathcal{R}_3(t),
$$

which yields

$$
\begin{aligned}
&\varepsilon^2\|D_xD_0u^\varepsilon(t)\|_2^2 + \|D_xD_xu^\varepsilon(t)\|_2^2 + \int_{\Sigma_t} {F_{12}^\varepsilon}^2|u^\varepsilon|^2\,dx \\
&= \varepsilon^2\|D_xD_0u^\varepsilon(0)\|_2^2 + \|D_xD_xu^\varepsilon(0)\|_2^2 + \int_{\Sigma_0} {F_{12}^\varepsilon}^2|u^\varepsilon|^2\,dx + \int_{\Sigma_0} 2F_{12}^\varepsilon\ \mathrm{Im}\ (D_1u^\varepsilon\overline{D_2u^\varepsilon})\,dx \\
&\quad - \int_{\Sigma_t} 2F_{12}^\varepsilon\ \mathrm{Im}\ (D_1u^\varepsilon\overline{D_2u^\varepsilon})\,dx + \int_0^t \mathcal{R}_3(\tau)\,d\tau.
\end{aligned} \tag{3.23}
$$

Using equation (3.5) for F_{12}^ε and appling Young's inequality, we obtain

$$
\begin{aligned}
&\int_{\Sigma_t} 2F_{12}^\varepsilon\ \mathrm{Im}\ (D_1u^\varepsilon\overline{D_2u^\varepsilon})\,dx \\
&\le 2\|F_{12}^\varepsilon\|_4\|D_xu^\varepsilon\|_4\|D_xu^\varepsilon\|_2 \le C\Big(\||u^\varepsilon|^2\|_4 + \varepsilon^2\|u^\varepsilon D_0u^\varepsilon\|_4\Big)\|D_xD_xu^\varepsilon\|_2^{\frac{1}{2}} \\
&\le C\Big(1 + \varepsilon^2\|u^\varepsilon\|_8\|D_0u^\varepsilon\|_8\Big)\|D_xD_xu^\varepsilon\|_2^{\frac{1}{2}} \le C\Big(1 + \varepsilon^{\frac{7}{4}}\varepsilon^{\frac{1}{4}}\|D_0u^\varepsilon\|_2^{\frac{1}{4}}\|D_xD_0u^\varepsilon\|_2^{\frac{3}{4}}\Big)\|D_xD_xu^\varepsilon\|_2^{\frac{1}{2}} \\
&\le C^{\frac{4}{3}} + \frac{1}{4}\|D_xD_xu^\varepsilon\|_2^2 + C\varepsilon^{\frac{7}{4}}\|D_xD_0u^\varepsilon\|_2^{\frac{3}{4}}\|D_xD_xu^\varepsilon\|_2^{\frac{1}{2}} \\
&\le C + \frac{1}{2}\|D_xD_xu^\varepsilon\|_2^2 + C^{\frac{4}{3}}\varepsilon^{\frac{7}{3}}\|D_xD_0u^\varepsilon\|_2 \le C + \frac{1}{2}\|D_xD_xu^\varepsilon\|_2^2 + \frac{1}{2}\varepsilon^2\|D_xD_0u^\varepsilon\|_2^2 + (C\varepsilon)^{\frac{8}{3}}.
\end{aligned} \tag{3.24}
$$

Combining (3.23) and (3.24), we get

$$
\frac{1}{2}\varepsilon^2\|D_xD_0u^\varepsilon(t)\|_2^2 + \frac{1}{2}\|D_xD_xu^\varepsilon(t)\|_2^2 \le O(1)\Phi^\varepsilon(0) + \int_0^t O(1)(1 + \|u^\varepsilon\|_\infty^2)\Phi^\varepsilon(\tau)\,d\tau. \tag{3.25}
$$

We now combine the estimates from (3.20), (3.22), and (3.25) to conclude:

$$\Phi^\varepsilon(t) \le O(1)\Phi^\varepsilon(0) + \int_0^t O(1)(1 + \|u^\varepsilon(\tau)\|_\infty^2)\Phi^\varepsilon(\tau)\, d\tau. \tag{3.26}$$

<u>*Step* 4.</u> (Estimates for $\|u\|_\infty$): For each $j = 1, 2$, we apply Lemma 3.3 to $(3.14)_2$ to obtain

$$\|A_j^\varepsilon\|_4 \lesssim \||u^\varepsilon|^2\|_{\frac{4}{3}} + \varepsilon^2\|u^\varepsilon D_0 u^\varepsilon\|_{\frac{4}{3}} \le \|u^\varepsilon\|_4\|u^\varepsilon\|_2 + (\varepsilon\|u^\varepsilon\|_4)(\varepsilon\|D_0 u^\varepsilon\|_2) \le O(1).$$

Considering the Coulomb gauge condition, we have

$$\partial_j u^\varepsilon = D_j^\varepsilon u + iA_j^\varepsilon u^\varepsilon, \quad \Delta u^\varepsilon = D_j D_j u^\varepsilon + 2iA_j^\varepsilon D_j u^\varepsilon + A_j^{\varepsilon 2} u^\varepsilon.$$

These yield

$$\|\partial_j u^\varepsilon\|_2 \le \|D_j u^\varepsilon\|_2 + \|A_j^\varepsilon u^\varepsilon\|_2 \le \|D_x u^\varepsilon\|_2 + \|A_j^\varepsilon\|_4\|u^\varepsilon\|_4 \le O(1),$$
$$\|\Delta u^\varepsilon\|_2 \le \|D_x D_x u^\varepsilon\|_2 + \|A_j^\varepsilon\|_4\|D_x u^\varepsilon\|_4 + \|A_j^{\varepsilon 2}\|_4\|u^\varepsilon\|_4 \le O(1) + \Phi^\varepsilon(t)^{\frac{1}{2}}.$$

Applying the Brezis–Gallouet inequality, we obtain

$$\|u^\varepsilon(t)\|_\infty \le O(1)\sqrt{1 + \ln(1 + \Phi^\varepsilon(t)^{\frac{1}{2}})}. \tag{3.27}$$

Finally, we note that, given the uniform-in-ε assumption on the initial data from $(\mathcal{A}2)$, it is straightforward to show that $\Phi^\varepsilon(0)$ is uniformly bounded with respect to ε. We then apply (3.27) to (3.26) to obtain the desired result in (3.17). $\square$

We also establish several bounds for the gauge fields. Since we employ Proposition 3.2 for the proof, the bounds are independent of ε, while they depend on the given time *T*.

Proposition 3.3 *Let $(u^\varepsilon, A_\mu^\varepsilon)$ be the solution to (3.2)- (3.6) satisfying the assumptions $(\mathcal{A}1)$- $(\mathcal{A}4)$. For a given $T > 0$ and for all $0 < \varepsilon < 1$, we have*

$$\sup_{0\le t\le T}\|\nabla A_0^\varepsilon(t)\|_p = O(1), \quad \sup_{0\le t\le T}\|\nabla A_j^\varepsilon(t)\|_p = O(1), \quad \textit{where} \quad j = 1, 2,\ 1 < p < \infty,$$
$$\sup_{0\le t\le T}\|\Delta A_0^\varepsilon(t)\|_2 = O(1), \quad \sup_{0\le t\le T}\|\partial_t A_0^\varepsilon(t)\|_q = O(1), \quad \textit{where} \quad 2 < q < \infty.$$

Proof Direct calculations using $(3.14)_1$ with the bound (3.27) yield:

$$\|\Delta A_0^\varepsilon(t)\|_2 \le \|D_x u^\varepsilon\|_4^2 + \|u^\varepsilon\|_\infty\|D_x D_x u^\varepsilon(t)\|_2 \le O(1)\big(1 + \Phi^\varepsilon(t)^{\frac{3}{2}}\big)^{\frac{1}{2}}.$$

On the other hand, differentiating $(3.14)_1$ with respect to time gives

$$\Delta(\partial_t A_0^\varepsilon) = \partial_1\big(2\ \mathrm{Im}\ (\overline{D_0 u^\varepsilon} D_2 u^\varepsilon) + F_{20}^\varepsilon|u^\varepsilon|^2\big) - \partial_2\big(2\ \mathrm{Im}\ (\overline{D_0 u^\varepsilon} D_1 u^\varepsilon) + F_{10}^\varepsilon|u^\varepsilon|^2\big).$$

Appling Lemma 3.3, for $2 < q < \infty$ and $1/r = 1/q + 1/2$, we get:

$$\begin{aligned}\|\partial_t A_0^\varepsilon\|_q &\le \|\overline{D_0 u^\varepsilon} D_j u^\varepsilon\|_r + \|F_{j0}^\varepsilon |u^\varepsilon|^2\|_r \le \|D_0 u^\varepsilon\|_2 \|D_x u^\varepsilon\|_q + \|F_{j0}^\varepsilon\|_2 \|u^\varepsilon\|_{2q}^2 \\ &\le \Phi^\varepsilon(t)^{\frac{1}{2}} \|D_x u^\varepsilon\|_2^{\frac{2}{q}} \|D_x D_x u^\varepsilon\|_2^{1-\frac{2}{q}} + O(1)\Phi^\varepsilon(t)^{\frac{1}{4}} \le O(1)\Phi^\varepsilon(t)^{1-\frac{1}{q}}.\end{aligned}$$

Lastly, we apply the Calderón–Zygmund inequality to (3.14) to obtain the desired bounds for $\|\nabla A_\mu^\varepsilon\|_p$, for $\mu = 0, 1, 2$ and $1 < p < \infty$. □

4 Proof of the Non-relativistic Limit

In this section, we provide the proof of Theorem 2.3. To do this, we fix $T > 0$ and choose $0 < \varepsilon < 1$ small enough to satisfy

$$\varepsilon \Phi^\varepsilon(t) \le 1, \qquad \text{for all} \quad 0 \le t \le T.$$

We recall the modulated Chern–Simons–Higgs system and the Chern–Simons–Schrödinger system as follows:

$$\begin{aligned}&\varepsilon^2 \partial_{tt} u^\varepsilon - i\partial_t u^\varepsilon - \Delta u^\varepsilon = \varepsilon^2 \mathcal{N}_0^\varepsilon + \mathcal{N}_1^\varepsilon, \\ &\Delta A_0^\varepsilon = \partial_1 \text{ Im } (\overline{u^\varepsilon} D_2 u^\varepsilon) - \partial_2 \text{ Im } (\overline{u^\varepsilon} D_1 u^\varepsilon), \\ &\Delta A_1^\varepsilon = -\partial_2 \Big(\varepsilon^2 \text{ Im } (\overline{u^\varepsilon} D_0 u^\varepsilon) - \frac{1}{2}|u^\varepsilon|^2\Big), \quad \Delta A_2^\varepsilon = \partial_1 \Big(\varepsilon^2 \text{ Im } (\overline{u^\varepsilon} D_0 u^\varepsilon) - \frac{1}{2}|u^\varepsilon|^2\Big),\end{aligned} \tag{4.1}$$

where

$$\mathcal{N}_0^\varepsilon := i\partial_t A_0^\varepsilon u^\varepsilon + 2iA_0^\varepsilon \partial_t u^\varepsilon + {A_0^\varepsilon}^2 u^\varepsilon - \frac{3}{4}|u^\varepsilon|^4 u^\varepsilon, \ \mathcal{N}_1^\varepsilon := -2iA_j^\varepsilon \partial_j u^\varepsilon + A_0^\varepsilon u^\varepsilon - {A_j^\varepsilon}^2 u^\varepsilon + |u^\varepsilon|^2 u^\varepsilon,$$

and

$$\begin{aligned}&i\partial_t u + \Delta u = 2iA_j \partial_j u - A_0 u + A_j^2 u - |u|^2 u =: -\mathcal{N}_1, \\ &\Delta A_0 = \partial_1 \text{ Im } (\overline{u} D_2 u) - \partial_2 \text{ Im } (\overline{u} D_1 u), \\ &\Delta A_1 = \frac{1}{2}\partial_2 |u|^2, \quad \Delta A_2 = -\frac{1}{2}\partial_1 |u|^2.\end{aligned} \tag{4.2}$$

We present the proof in the following four steps.

Step 1. (Estimates for the difference of the matter fields): We begin by adding $(4.1)_1$ and $(4.2)_1$, multiplying the resulting equation by $\overline{u^\varepsilon} - \overline{u}$, taking its imaginary part, and then integrating over $\mathbb{R}^2$. This process yields the following:

$$\frac{d}{dt}\int_{\Sigma_t} |u^\varepsilon - u|^2 \, dx - 2\varepsilon^2 \frac{d}{dt}\int_{\Sigma_t} \text{ Im } \big((\overline{u^\varepsilon} - \overline{u})\partial_t u^\varepsilon\big) \, dx = -2\sum_{\ell=1}^{3} \mathcal{I}_\ell(t), \tag{4.3}$$

where

$$\mathcal{I}_1(t) := \varepsilon^2 \int_{\Sigma_t} \operatorname{Im}\left(\partial_t u^\varepsilon(\partial_t \overline{u^\varepsilon} - \partial_t \overline{u})\right) dx, \quad \mathcal{I}_2(t) := \varepsilon^2 \int_{\Sigma_t} \operatorname{Im}\left(\mathcal{N}_0^\varepsilon(\overline{u^\varepsilon} - \overline{u})\right) dx,$$
$$\mathcal{I}_3(t) := \int_{\Sigma_t} \operatorname{Im}\left((\mathcal{N}_1^\varepsilon - \mathcal{N}_1)(\overline{u^\varepsilon} - \overline{u})\right) dx.$$

The first and second term can be estimated as follows:

$$|\mathcal{I}_1(t)| \le \varepsilon^2 \|\partial_t u^\varepsilon\|_2 \|\partial_t u\|_2 \le O(\varepsilon)(\varepsilon\|D_0 u^\varepsilon\|_2 + \varepsilon\|A_0^\varepsilon\|_4\|u^\varepsilon\|_4) \le O(\varepsilon),$$
$$|\mathcal{I}_2(t)| \lesssim \varepsilon^2(\|\partial_t A_0^\varepsilon u^\varepsilon\|_2 + \|A_0^\varepsilon \partial_t u^\varepsilon\|_2 + \|A_0^{\varepsilon 2} u^\varepsilon\|_2 + \|u^\varepsilon\|_{10}^5) \le O(\varepsilon),$$

where we observed

$$\begin{aligned}
&\varepsilon^2\|\partial_t A_0^\varepsilon u^\varepsilon\|_2 \le \varepsilon^2\|\partial_t A_0^\varepsilon\|_4\|u^\varepsilon\|_4 \le \varepsilon^{\frac{5}{4}}\varepsilon^{\frac{3}{4}}\Phi_\varepsilon(t)^{\frac{3}{4}} \le O(\varepsilon),\\
&\varepsilon^2\|A_0^\varepsilon \partial_t u^\varepsilon\|_2 \le \varepsilon^2\|A_0^\varepsilon\|_4\|\partial_t u^\varepsilon\|_4 \le \varepsilon^2(\|D_0 u^\varepsilon\|_4 + \|A_0^\varepsilon u^\varepsilon\|_4)\\
&\qquad \le \varepsilon^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}\|D_x D_0 u^\varepsilon\|_2^{\frac{1}{2}}\|D_0 u^\varepsilon\|_2^{\frac{1}{2}} + \varepsilon^2\|A_0^\varepsilon\|_8\|u^\varepsilon\|_8 \le O(\varepsilon)(\varepsilon^{\frac{1}{2}}\Phi^\varepsilon(t)^{\frac{1}{2}} + 1) \le O(\varepsilon),
\end{aligned}$$

and we dealt with other terms in a similar manner.

For $0 < t < T$, we integrate (4.3) over $[0, t]$ and use Hölder's inequatility to derive

$$\|(u^\varepsilon - u)(t)\|_2^2 \le \|u_0^\varepsilon - u_0\|_2^2 + O(\varepsilon) + O(\varepsilon)t + 2\int_0^t |\mathcal{I}_3(\tau)|\, d\tau. \tag{4.4}$$

$\underline{Step\,2.}$ (Estimates for the difference of the gauge fields): We subtract the gauge equations in (4.1) and (4.2) to observe their differences. From this point on, we will use vector notation for the spatial gauge fields, denoted as $A^\varepsilon = (A_1^\varepsilon, A_2^\varepsilon)$ and $A = (A_1, A_2)$.

$\diamond$ (Estimate of $A_j^\varepsilon - A_j$): Note that for $j = 1, 2$,

$$\Delta(A_j^\varepsilon - A_j) = (-1)^j \partial_{j'}\Big(\varepsilon^2 \operatorname{Im}\left(\overline{u^\varepsilon} D_0 u^\varepsilon\right) - \frac{1}{2}(|u^\varepsilon|^2 - |u|^2)\Big).$$

By Lemma 3.1 and Lemma 3.3, for $2 < q < \infty$, we have

$$\|A_j^\varepsilon - A_j\|_q \le \varepsilon\|u^\varepsilon\|_q\big(\varepsilon\|D_0 u^\varepsilon\|_2\big) + (\|u^\varepsilon\|_q + \|u\|_q)\|u^\varepsilon - u\|_2 \le O(\varepsilon) + O(1)\|u^\varepsilon - u\|_2. \tag{4.5}$$

Similarly, for $1 < p < \infty$, the Calderón–Zygmund inequality yields

$$\begin{aligned}
\|\nabla A_j^\varepsilon - \nabla A_j\|_p &\le \varepsilon^2\|u^\varepsilon\|_{2p}\|D_0 u^\varepsilon\|_{2p} + (\|u^\varepsilon\|_{2p} + \|u\|_{2p})\|u^\varepsilon - u\|_{2p}\\
&\le O(\varepsilon^{\frac{1}{2}+\frac{1}{p}})\varepsilon^{\frac{1}{2}}\|D_0 u^\varepsilon\|_2^{\frac{1}{p}}\varepsilon^{1-\frac{1}{p}}\|D_x D_0 u^\varepsilon\|_2^{1-\frac{1}{p}} + O(1)\|u^\varepsilon - u\|_2^{\frac{1}{p}}\|(\nabla u^\varepsilon - \nabla u)\|_2^{1-\frac{1}{p}}\\
&\le O(\varepsilon^{\frac{1}{2}+\frac{1}{p}}) + O(1)\|u^\varepsilon - u\|_2^{\frac{1}{p}}.
\end{aligned} \tag{4.6}$$

$\diamond$ (Estimates of $A_0^\varepsilon - A_0$): Direct calculation shows that

$$\Delta(A_0^\varepsilon - A_0) = \partial_1 \operatorname{Im}\big((\overline{u^\varepsilon} - u)(\partial_2 u^\varepsilon + \partial_2 u)\big) - \partial_2 \operatorname{Im}\big((\overline{u^\varepsilon} - u)(\partial_1 u^\varepsilon + \partial_1 u)\big)$$
$$- \partial_1((A_2^\varepsilon - A_2)|u|^2) - \partial_1(A_2^\varepsilon(|u^\varepsilon|^2 - |u|^2)) + \partial_2((A_1^\varepsilon - A_1)|u|^2) + \partial_2(A_1^\varepsilon(|u^\varepsilon|^2 - |u|^2)).$$

Again, using Lemma 3.1 and Lemma 3.3, for $2 < q < \infty$, we obtain

$$\begin{aligned}\|A_0^\varepsilon - A_0\|_q &\le (\|\nabla u^\varepsilon\|_q + \|\nabla u\|_q)\|u^\varepsilon - u\|_2 \\ &\quad + \|A^\varepsilon - A\|_q\|u\|_4^2 + \|A^\varepsilon\|_{2q}(\|u^\varepsilon\|_{2q} + \|u\|_{2q})\|u^\varepsilon - u\|_2 \\ &\le O(\varepsilon) + O(1)\|u^\varepsilon - u\|_2.\end{aligned} \tag{4.7}$$

Step 3. (Estimates for $\mathcal{I}_3$): We expand $\mathcal{I}_3$ as

$$\mathcal{I}_3(t) = \int_{\Sigma_t} \operatorname{Im}\big((\mathcal{N}_1^\varepsilon - \mathcal{N}_1)(\overline{u^\varepsilon} - \overline{u})\big)\,dx =: \sum_{\ell=1}^{4} \mathcal{I}_{3\ell}(t),$$

where $\mathcal{I}_{3\ell}(t)$ for $\ell = 1, 2, 3, 4$ will be defined and estimated separately.

$\diamond$ (Estimate of $\mathcal{I}_{31}$): We observe that

$$\begin{aligned}\mathcal{I}_{31}(t) &:= -2\int_{\Sigma_t} \operatorname{Re}\big((A_j^\varepsilon \partial_j u^\varepsilon - A_j\partial_j u)(\overline{u^\varepsilon} - \overline{u})\big)\,dx \\ &= -2\int_{\Sigma_t} \operatorname{Re}\big((A_j^\varepsilon - A_j)\partial_j u(\overline{u^\varepsilon} - \overline{u})\big)\,dx - 2\int_{\Sigma_t} \operatorname{Re}\big(A_j^\varepsilon \partial_j(u^\varepsilon - u)(\overline{u^\varepsilon} - \overline{u})\big)\,dx \\ &= -2\int_{\Sigma_t} \operatorname{Re}\big((A_j^\varepsilon - A_j)\partial_j u(\overline{u^\varepsilon} - \overline{u})\big)\,dx + 2\int_{\Sigma_t} \operatorname{Re}\big(\partial_j A_j^\varepsilon |u^\varepsilon - u|^2\big)\,dx \\ &= -2\int_{\Sigma_t} \operatorname{Re}\big((A_j^\varepsilon - A_j)\partial_j u(\overline{u^\varepsilon} - \overline{u})\big)\,dx.\end{aligned}$$

By applying (4.5), we have

$$|\mathcal{I}_{31}(t)| \le \|A^\varepsilon - A\|_8\|\nabla u\|_4\|u^\varepsilon - u\|_2 \le O(\varepsilon)\|u^\varepsilon - u\|_2 + O(1)\|u^\varepsilon - u\|_2^2.$$

$\diamond$ (Estimate of $\mathcal{I}_{32}$): Similarly, we rewrite $\mathcal{I}_{32}(t)$ as

$$\mathcal{I}_{32}(t) := \int_{\Sigma_t} \operatorname{Im}\big((A_0^\varepsilon u^\varepsilon - A_0 u)(\overline{u^\varepsilon} - \overline{u})\big)\,dx = \int_{\Sigma_t} \operatorname{Im}\big((A_0^\varepsilon - A_0)u(\overline{u^\varepsilon} - \overline{u})\big)\,dx.$$

Then, applying (4.7) yields

$$|\mathcal{I}_{32}(t)| \le \|A_0^\varepsilon - A_0\|_4\|u\|_4\|u^\varepsilon - u\|_2 \le O(\varepsilon)\|u^\varepsilon - u\|_2 + O(1)\|u^\varepsilon - u\|_2^2.$$

$\diamond$ (Estimate of $\mathcal{I}_{33}$): Similarly, we rewrite $\mathcal{I}_{33}(t)$ as

$$\mathcal{I}_{33}(t) := -\int_{\Sigma_t} \operatorname{Im}\big((A_j^{\varepsilon 2} u^\varepsilon - A_j^2 u)(\overline{u^\varepsilon} - \overline{u})\big)\,dx = -\int_{\Sigma_t} \operatorname{Im}\big((A_j^{\varepsilon 2} - A_j^2)u(\overline{u^\varepsilon} - \overline{u})\big)\,dx.$$

and estimate it as

$$|\mathcal{I}_{33}(t)| \le \|A^\varepsilon - A\|_8(\|A^\varepsilon\|_4 + \|A\|_4)\|u\|_8\|u^\varepsilon - u\|_2 \le O(\varepsilon)\|u^\varepsilon - u\|_2 + O(1)\|u^\varepsilon - u\|_2^2,$$

$\diamond$ (Estimate of $\mathcal{I}_{34}$): Similarly, $\mathcal{I}_{34}(t)$ can be rewritten as

$$\mathcal{I}_{34}(t) := \int_{\Sigma_t} \operatorname{Im}\big((|u^\varepsilon|^2u^\varepsilon - |u|^2u)(\overline{u^\varepsilon} - \overline{u})\big)\,dx = \int_{\Sigma_t} \operatorname{Im}\big((|u^\varepsilon|^2 - |u|^2)u(\overline{u^\varepsilon} - \overline{u})\big)\,dx,$$

which yields

$$|\mathcal{I}_{34}(t)| \le O(1)\|u^\varepsilon - u\|_2^2.$$

Finally, we collect estimates for $\mathcal{I}_{3\ell}(t)$ to obtain

$$|\mathcal{I}_3(t)| \le O(\varepsilon)\|u^\varepsilon - u\|_2 + O(1)\|u^\varepsilon - u\|_2^2. \tag{4.8}$$

<u>*Step* 4.</u> (Passing to the limit): We insert (4.8) into (4.4) to have for $0 \le t \le T$,

$$\|(u^\varepsilon - u)(t)\|_2^2 \le \|u_0^\varepsilon - u_0\|_2^2 + O(\varepsilon) + O(\varepsilon)t + \int_0^t O(\varepsilon)\|(u^\varepsilon - u)(\tau)\|_2 + O(1)\|(u^\varepsilon - u)(\tau)\|_2^2 d\tau. \tag{4.9}$$

Grönwall's lemma and assumption $(\mathcal{A}3)$ yield

$$\|(u^\varepsilon - u)(t)\|_2^2 \le \big(\varepsilon^\lambda + O(\varepsilon) + O(\varepsilon)T\big)e^{O(1)T}.$$

By applying (4.9) to (4.5)- (4.7), we also obtain:

$$\begin{aligned}
\|A_\mu^\varepsilon - A_\mu\|_q &\le O(\varepsilon) + \Big(\big(\varepsilon^\lambda + O(\varepsilon) + O(\varepsilon)T\big)e^{O(1)T}\Big)^{\frac{1}{2}},\\
\|\nabla A_j^\varepsilon - A_j\|_p &\le O(\varepsilon^{\frac{1}{2}+\frac{1}{p}}) + \Big(\big(\varepsilon^\lambda + O(\varepsilon) + O(\varepsilon)T\big)e^{O(1)T}\Big)^{\frac{1}{p}},
\end{aligned}$$

for $\mu = 0, 1, 2$ and $j = 1, 2$, and for all $2 < q < \infty$ and $1 < p < \infty$. Letting $\varepsilon \to 0$, we obtain the desired result:

$$\|u^\varepsilon - u\|_{C([0,T];L^2)} \to 0, \quad \|A_\mu^\varepsilon - A_\mu\|_{L^\infty([0,T];L^q)} \to 0 \quad \text{and} \quad \|\nabla(A_j^\varepsilon - A_j)\|_{L^\infty([0,T];L^p)} \to 0.$$

$\square$

5 Conclusion

In this paper, we revisited the non-relativistic limit of the Chern–Simons–Higgs system previously studied in [5]. Unlike the behavior of Maxwell gauges in the non-relativistic regime, where spatial gauges vanish as the speed of light c tends to infinity, the spatial gauges of the Chern–Simons system remain in the form $c^{-1}A_j$ in this limit, and continue to act as the spatial gauge in the non-relativistic regime. To clarify this distinctive feature, we reformulated the Lagrangian for the non-relativistic limit, which is essentially equivalent to the original version. This reformulation allowed us to correct several inaccuracies in [5] and to provide a more rigorous derivation of the non-relativistic limit with quantified convergence estimates.

Appendix A. Derivation of the Lagrangin Density (1.1)

In [5], the authors utilized relativistic coordinates $(x_0, x_1, x_2) = (ct, x_1, x_2)$, employing $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, and $\partial_0 = c^{-1}\partial_t$. The authors used the original version of the Lagrangian density for the Chern–Simons–Higgs model, as presented in [12], to describe the relativistic-to-non-relativistic correspondence between the CSH and CSS models, which is given by:

$$\mathcal{L}^R = \frac{\kappa'}{2}\epsilon^{\rho\mu\nu}\mathbf{A}_\rho F_{\mu\nu} + \hbar^2\mathbf{D}_\mu\phi\overline{\mathbf{D}^\mu\phi} - \frac{\hbar^2 e^4}{c^4\kappa'^2}|\phi|^2(|\phi|^2-\sigma^2)^2, \tag{A.1}$$

where $\phi : \mathbb{R}^{2+1} \to \mathbb{C}$ the matter field, $\mathbf{A}_\mu : \mathbb{R}^{2+1} \to \mathbb{R}$ the gauge fields and

$$\begin{aligned}
&\mathbf{D}_\mu := \partial_\mu - i\frac{e}{c\hbar}\mathbf{A}_\mu, \quad \text{the covariant derivative},\\
&F_{\mu\nu} := \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu, \quad \text{the field strength},\\
&\epsilon^{\rho\mu\nu}: \ \text{the totally skew-symmetric tensor with } \epsilon^{012}=1,\\
&\kappa' > 0: \ \text{the Chern-Simons coupling constant },\\
&\hbar: \ \text{the Planck constant }, \quad e: \ \text{the charge of electron },\\
&c: \ \text{the velocity of light }, \quad \sigma > 0: \ \text{a real parameter }.
\end{aligned}$$

To simplify the discussion, the parameters e and $\hbar$ were scaled to unity. It was first observed that the quadratic term in the potential in (A.1) defines the mass through its coefficient, which is m^2c^2. This implies

$$m^2c^2 = \frac{\sigma^4}{c^4k'^2}, \quad \text{or equivalently }, \quad m = \frac{\sigma^2}{c^3k'}.$$

To maintain the mass m in the limit $c \to \infty$, the authors set

$$\frac{\sigma^2}{c^2} = 1, \qquad c\kappa' = \kappa,$$

with a fixed κ, considered a new Chern–Simons coupling constatant. Thus, the limit $c \to \infty$ is accompanied by the limits $\sigma = c \to \infty$ and $\kappa' \to 0$.

Reflecting on these observations, the equation (A.1) becomes as follows

$$\mathcal{L}^R = \frac{\kappa}{2c}\epsilon^{\rho\mu\nu}\mathbf{A}_\rho F_{\mu\nu} + \mathbf{D}_\mu\phi\overline{\mathbf{D}^\mu\phi} - \frac{1}{c^2\kappa^2}|\phi|^2(|\phi|^2 - c^2)^2.$$

The remarkable point is that, in the non-relativistic limit, the spatial gauges with the relativistic effect $c^{-1}\mathbf{A}_j$ converges to the spatial gauges of the CSS equations, in contrast to the temporal gauge. Therefore, to make this feature explicit and facilitate analysis, we replace

$$\mathbf{A}_0 \to A_0, \quad \frac{1}{c}\mathbf{A}_j \to A_j,$$

and rewrite (A.1) as

$$\mathcal{L}^R = \frac{\kappa}{2}\epsilon^{\rho\mu\nu}A_\rho F_{\mu\nu} + \frac{1}{c^2}D_0\phi\overline{D^0\phi} + D_j\phi\overline{D^j\phi} - \frac{1}{c^2\kappa^2}|\phi|^2(|\phi|^2 - \sigma^2)^2,$$

where $D_\mu = \partial_\mu - iA_\mu$, and $\partial_0 = \partial_t$, $\partial_1 = \partial_x$, $\partial_2 = \partial_y$. This is the form of the Lagrangian addressed in this paper, particularly focusing on the scaled version with $k = 2$ in Lagrangian (1.1).

Author Contributions H. Huh conceptualized the study and developed the theoretical framework. B. Moon developed and verified the framework, carried out the analytical proofs, and wrote the manuscript. Both authors discussed the results and approved the final version of the manuscript.

Funding The work of H. Huh was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (RS-2025-16067086). The work of B. Moon was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF), funded by the Ministry of Education (RS-2022-NR074808) and by the Korea government(MSIT) (RS-2024-00406821)

Data Availability Not applicable.

Declarations

Ethics Approval and Consent to Participate Not applicable.

Consent for Publication All authors have read and approved the publication of this manuscript.

Competing Interests The authors declare no competing interests.

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