



Decay constant of B_s and B_s^* mesons from $N_f = 2$ lattice QCD

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Abstract We report on a two-flavor lattice QCD estimate of the B_s and B_s^* leptonic decays parameterized by the decay constants f_{B_s} and $f_{B_s^*}$. In addition to their relevance for phenomenology, their extraction has allowed us to investigate whether the “step scaling in mass” strategy is suitable with Wilson–Clover fermions to smoothly extrapolate quantities of the heavy-strange sector up to the bottom scale. From the central value of f_{D_s} quoted by FLAG at $N_f = 2$ and our ratio $\frac{f_{B_s}}{f_{D_s}}$, we obtain $f_{B_s} = 215(10)(2)_{(-5)}^{(+2)}$ MeV and $f_{B_s^*}/f_{B_s} = 1.02(2)_{(-0)}^{(+2)}$.

1 Introduction

In the very active research of new effects in high-energy particle physics, flavour physics does play a key role at the so-called intensity frontier. Indeed, rare events are sensitive probes of New Physics (NP) scenarios with the exchange of extra particles in quantum loops with respect to what is known from the Standard Model (SM). However, theoretical uncertainties on hadronic quantities, for instance hadron decay constants, that encode the dynamics of QCD at large distance, severely weaken the constraints that are derived through the analysis of experimental data. Those hadronic constants cannot be reliably estimated in perturbation theory. b -quark physics is a particularly interesting place to search for NP effects and it has recently regained even stronger attention after experimental signs of several anomalies in B and B_c decays. More precisely several ratios $R_D = \frac{\Gamma(B \rightarrow D \tau \nu_\tau)}{\Gamma(B \rightarrow D \ell \nu_\ell)_{\ell=e,\mu}}$, $R_{D^*} = \frac{\Gamma(B \rightarrow D^* \tau \nu_\tau)}{\Gamma(B \rightarrow D^* \ell \nu_\ell)_{\ell=e,\mu}}$, $R_{J/\psi} = \frac{\Gamma(B_c \rightarrow J/\psi \tau \nu_\tau)}{\Gamma(B_c \rightarrow J/\psi \ell \nu_\ell)_{\ell=e,\mu}}$, $R_K = \frac{\Gamma(B \rightarrow K \mu^+ \mu^-)}{\Gamma(B \rightarrow K e^+ e^-)}$ and $R_{K^*} = \frac{\Gamma(B \rightarrow K^* \mu^+ \mu^-)}{\Gamma(B \rightarrow K^* e^+ e^-)}$ show some discrepancy with SM expectations [1–12]. The three former might bring stringent constraints on $\bar{b}c$ currents, for instance mediated by the exchange of leptoquarks [13]. The fur-

ther ratios $R_{D_s^{(*)}} = \frac{\Gamma(B \rightarrow D_s^{(*)} \tau \nu_\tau)}{\Gamma(B \rightarrow D_s^{(*)} \ell \nu_\ell)_{\ell=e,\mu}}$, under investigation at LHCb, will provide even more informations once, on the theory side, the hadronic matrix elements associated to $B_s \rightarrow D_s^{(*)}$ are under comparable control by means of lattice QCD. Simulating the B_s meson on the lattice is delicate as far as cut-off effects are concerned. Several strategies have been followed in the literature, including simulations of relativistic b -quarks using an action tuned so as to minimize discretization errors [14–17], the use of Non Relativistic QCD [18,19], performing computations in Heavy Quark Effective Theory (HQET) [20] and the extrapolation of simulation results obtained in the region between the charm quark mass m_c and a mass $\sim 3m_c$ to the physical b -quark mass [21,22]. As we plan to employ the latter approach to study B_s decays with $O(a)$ improved Wilson–Clover fermions, an intermediate step is to extract f_{B_s} and $f_{B_s^*}$, in order to validate the method. The lattice QCD community has made a significant effort to compute f_{B_s} with $N_f = 2$ [22,23], $N_f = 2 + 1$ [14,15,18,24,25] and $N_f = 2 + 1 + 1$ [19,26–28]. Recently, the SU(3) symmetry breaking f_{B_s}/f_B has been extracted at the physical point [29]. Concerning the spin-symmetry breaking ratio $f_{B_s^*}/f_{B_s}$ only 2 lattice results are available, both at $N_f = 2 + 1 + 1$ [30,31]. Ratios f_{B^*}/f_B and $f_{B_s^*}/f_{B_s}$ have been investigated with other methods than lattice simulations, i.e. constituent quark models [32,33] and QCD sum rules [34–37].

The paper is organized as follows: in Sect. 2 we recall what is the “step scaling in mass” strategy, in Sect. 3 we present the simulations details and our raw data, and in Sect. 4 we describe our analysis and comment the results. Finally we conclude in Sect. 5.

2 Step scaling in mass with Wilson–Clover fermions

The idea is to extract $f_{B_s} \equiv \frac{f_{B_s}}{f_{D_s}} \times f_{D_s}$ and $f_{B_s^*} \equiv \frac{f_{B_s^*}}{f_{B_s}} \times f_{B_s}$ by separate measurements, of the quantity f_{D_s} on one side,

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the ratio $\frac{f_{B_s}}{f_{D_s}}$ on another side and the ratio $\frac{f_{B_s^*}}{f_{D_s^*}}$ on a third side. In the following we focus on the two latter in the framework of $N_f = 2$ Wilson–Clover fermions. With a given pion mass, through κ_{sea} , the valence strange quark mass κ_s and lattice spacing a , we consider the ratio

$$\frac{[f_P \sqrt{M_P}](a, \kappa_{\text{sea}}, \kappa_{h_{i+1}}, \kappa_s) C_A^{\text{stat}}(\mu_f, M_P(a, \kappa_{\text{sea}}, \kappa_{h_i}, \kappa_s))}{[f_P \sqrt{M_P}](a, \kappa_{\text{sea}}, \kappa_{h_i}, \kappa_s) C_A^{\text{stat}}(\mu_f, M_P(a, \kappa_{\text{sea}}, \kappa_{h_{i+1}}, \kappa_s))} \frac{1}{\sqrt{\lambda}} \equiv r_P(a, \kappa_{\text{sea}}, \kappa_{h_i}, \kappa_{h_{i+1}}, \kappa_s). \quad (1)$$

where $\lambda = \left(\frac{m_{B_s}}{m_{D_s}}\right)^{\frac{1}{K}}$, $i = 0, \dots, K-1$, κ_{h_i} is a valence heavy quark mass, and $M_P(a, \kappa_{\text{sea}}, \kappa_{h_0}, \kappa_s) \equiv m_{D_s}$, up to mistuning effects. For later usage it is convenient to redefine r_P as

$$\begin{aligned} r_P(a, \kappa_{\text{sea}}, \kappa_{h_i}, \kappa_{h_{i+1}}, \kappa_s) &\equiv \frac{C_A^{\text{stat}}(\mu_f, M_P(a, \kappa_{\text{sea}}, \kappa_{h_i}, \kappa_s))}{C_A^{\text{stat}}(\mu_f, M_P(a, \kappa_{\text{sea}}, \kappa_{h_{i+1}}, \kappa_s))} \\ r'_P(a, \kappa_{\text{sea}}, \kappa_{h_i}, \kappa_{h_{i+1}}, \kappa_s) &= \frac{f_P(a, \kappa_{\text{sea}}, \kappa_{h_{i+1}}, \kappa_s)}{f_P(a, \kappa_{\text{sea}}, \kappa_{h_i}, \kappa_s)}. \end{aligned} \quad (2)$$

$C_A^{\text{stat}}(\mu_1, \mu_2)$ and $C_V^{\text{stat}}(\mu_1, \mu_2)$, that will appear later in the paper, are the matching coefficients between the QCD currents $J_{A(V)}^{\text{QCD}} \equiv \bar{Q}\gamma_0\gamma^5 q (\bar{Q}\gamma_1 q)$ and their HQET counterpart $J_{A(V)}^{\text{HQET}} \equiv \bar{h}\gamma_0\gamma^5 q (\bar{h}\gamma_1 q)$ defined at the renormalization scale μ_1 , $J_{A(V)}^{\text{HQET}}(\mu_1) = C_{A(V)}^{\text{stat}-1}(\mu_1, \mu_Q) J_{A(V)}^{\text{QCD}}$, where μ_Q is a scale related to the heavy quark mass m_Q , for instance the heavy-light pseudoscalar meson mass $M_P(Q, q)$. C_A^{stat} and C_V^{stat} are known up to 3-loop of perturbation theory [38–41].¹ r_P is independent of μ_f because it involves the renormalization group equation of C_A^{stat} integrated from $M_P(a, \kappa_{\text{sea}}, \kappa_{h_i}, \kappa_s)$ to $M_P(a, \kappa_{\text{sea}}, \kappa_{h_{i+1}}, \kappa_s)$. Thanks to scaling laws in HQET, it is expected that r_P has a simple expansion in the inverse heavy quark mass defined in a specific renormalization scheme, for instance the pole mass [42]. But, in the case of Wilson–Clover fermions and contrary to the case of twisted-mass fermions, there is so far no straightforward relation between the pole quark mass and the bare quark mass, through the renormalization group invariant (RGI) quark mass, if the quark is significantly heavier than the charm. Indeed, using the series of RGI masses $m_{h_i}^{\text{RGI}}$ such that $m_{h_i}^{\text{RGI}}/m_{h_{i+1}}^{\text{RGI}} = 1/\lambda'$, $m_b^{\text{RGI}} = \lambda'^K m_c^{\text{RGI}}$ with m_b^{RGI} already determined [43] and m_c^{RGI} known after the tuning of κ_c , we define the improved RGI mass $m_{h_i}^{\text{RGI}} \propto (1 + b_m a m_{h_i}) m_{h_i}$, where m_{h_i} is the heavy vector Ward Identity quark mass $m_{h(i)} = \frac{1}{2\kappa_{h_i}} - \frac{1}{2\kappa_{\text{critical}}}$. The problem is that

we can have $1 + b_m a m_{h_i} < 0$ because the improvement coefficient b_m is negative [44]. Negative RGI masses are of course not physical. The issue would be solved by adding the $\mathcal{O}(a^2)$ term in the definition of the RGI mass, which is unfortunately unknown. That is why we have decided to consider the inverse of pseudoscalar heavy-strange pseudoscalar meson masses $M_P(\kappa_{h_i}, \kappa_s) \equiv M_{H_s}(i)$ as the parameter expansion of r_P and we define the steps as sequential ratios $\frac{M_P(\kappa_{h_{i+1}}, \kappa_s)}{M_P(\kappa_{h_i}, \kappa_s)}$ that should be constant with the regulator a and the sea quark mass κ_{sea} . They are the analogous of the ratios of RGI quark masses $\frac{m_{h_{i+1}}^{\text{RGI}}}{m_{h_i}^{\text{RGI}}}$.

3 Lattice calculation details

We have performed our analysis from the CLS ensembles made of $N_f = 2$ nonperturbatively $\mathcal{O}(a)$ -improved Wilson–Clover fermions [45, 46] and the plaquette gauge action [47]. In Table 1 we collect the main informations about the simulations. Three lattice spacings $a_{\beta=5.5} = 0.04831(38)$ fm, $a_{\beta=5.3} = 0.06531(60)$ fm, $a_{\beta=5.2} = 0.07513(79)$ fm, determined from a fit in the chiral sector [48], are considered with pion masses in the range [190, 440] MeV. With respect to the work reported in [49], we have taken the bare strange quark masses at $\beta = 5.2$ from [50] and we have tuned the charm quark mass on those ensembles by imposing $aM_P(a, \kappa_{\text{sea}}, \kappa_c, \kappa_s) = a m_{D_s}^{\text{physical}}$. The values we find for κ_c are close to what is quoted in [50] where the tuning was realised thanks to a constraint on cut-off effect magnitude for the ratio of PCAC masses $m_c^{\text{PCAC}}/m_s^{\text{PCAC}}$. We have used the same procedure as in [49] to compute the statistical error at finite a and in the continuum limit, to compute stochastic all-to-all propagators and to reduce the contamination by excited states on 2-pt correlators by solving a 4×4 Generalized Eigenvalue Problem (GEVP) with one local and 3 Gaussian smeared interpolating fields. In our application of the step scaling in mass strategy we have chosen $K = 6$ steps. Similarly to [49] we have extracted the relevant matrix elements from projected correlators along the fixed ground state generalized eigenvectors $v_p^{(1)}(V)(t_{\text{fix}}, t_0)$. (t_{fix}, t_0) take the values $(4a, 3a)$ at $\beta = 5.2$, $(4a, 3a)$ at $\beta = 5.3$ and $(6a, 5a)$ at $\beta = 5.5$. We collect in Tables 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 of the Appendix the whole set of raw data we need in our analysis, i.e. ratios of pseudoscalar and vector heavy-strange mesons masses for 2 subsequent heavy bare quark masses, ratios of pseudoscalar and vector heavy-strange meson decay constants for 2 subsequent heavy bare quark masses and PCAC quark masses m_{h_s} defined by:

$$m_{h_s}^{\text{PCAC}} = \frac{\frac{\partial_0 + \partial_0^*}{2} C_{A_0 P}(t) - a C_A \partial_0 \partial_0^* C_{P P}(t)}{2 C_{P P}(t)}, \quad (3)$$

¹ In this work we have taken the N²LO formulae for the matching between QCD and HQET at the scale μ_Q .

Table 1 Parameters of the simulations: bare coupling $\beta = 6/g_0^2$, lattice resolution, hopping parameter κ , lattice spacing a in physical units, pion mass, number of gauge configurations, bare strange and charm quark masses

id	β	$(L/a)^3 \times (T/a)$	κ_{sea}	a (fm)	m_π (MeV)	Lm_π	# cfigs	κ_s	κ_c
A5	5.2	$32^3 \times 64$	0.13594	0.0751	333	4.1	198	0.135267	0.12531
B6		$48^3 \times 96$	0.13597		282	5.2	126	0.135257	0.12529
E5	5.3	$32^3 \times 64$	0.13625	0.0653	439	4.7	200	0.135777	0.12724
F6		$48^3 \times 96$	0.13635		313	5	120	0.135741	0.12713
F7		$48^3 \times 96$	0.13638		268	4.3	200	0.135730	0.12713
G8		$64^3 \times 128$	0.13642		194	4.1	176	0.135705	0.12710
N6	5.5	$48^3 \times 96$	0.13667	0.0483	341	4	192	0.136250	0.13026
O7		$64^3 \times 128$	0.13671		269	4.2	160	0.136243	0.13022

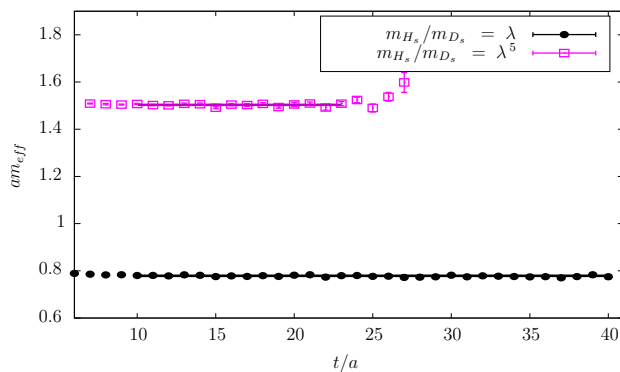


Fig. 1 Effective mass of two pseudoscalar heavy-strange mesons extracted with the projected 2-pt correlation function $\tilde{C}_{A_0P}$ along the the generalised eigenvector $v_P^1(t_{\text{fix}}, t_0)$. The CLS ensemble is F7

where $\frac{\partial_0 + \partial_0^*}{2} f(t) = \frac{f(t+a) - f(t-a)}{2a}$, C_{A_0P} and C_{PP} are axial-pseudoscalar and pseudoscalar-pseudoscalar 2-pt correlation functions of the heavy-strange meson defined by the bare quark masses (κ_h, κ_s) and c_A is the improvement coefficient of the axial bilinear of Wilson–Clover fermions determined in [51].

We show in Fig. 1 the effective mass for two heavy-strange mesons, one at the first step scaling in mass and the other at the next to last step. For very heavy quarks the signal deteriorates quickly. It explains why we fixed shorter interval ranges to extract the hadronic properties, as indicated in Tables 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25.

4 Analysis and discussion

4.1 Extraction of f_{B_s}

We have performed extrapolations to the physical point by doing a global fit analysis. However, in a preparatory stage, we restrict our analysis to a given step in heavy mass i and study the pion mass and the cut-off dependence of $r_P(i)$.

Here we ignore the mistuning effects because our goal is to determine how large are the discretisation effects. We show in Fig. 2 the extrapolation in a^2 and m_π^2 of $r_P(i)$. We observe very big cut-off effects for the 4th ratio (10%) and the 5th ratio (17%). Hence we are not quite confident in using the ensembles at $a = 0.075$ fm and, most probably, those at $a = 0.065$ fm as well, at the final stage of our analysis. That is why we prefer, in the combined fit analysis, to exclude the data at the fourth and the fifth heavy quark mass at the lattice spacings 0.075 fm and 0.065 fm. Then, we have used the following fit ansatz:

$$r_P(a, m_\pi^2, M_{H_s}(i)) = 1 + r_{P_0} + r_{P_1} m_\pi^2 + r_{P_2} (r_{\text{mistune}}(i) - 1) + r_{P_3} / M_{H_s}(i) + r_{P_4} (a/a_{\beta=5.3})^2 + r_{P_5} (a/a_{\beta=5.3})^2 (a M_{H_s}(i))^2, \quad (4)$$

with $r_{\text{mistune}}(i) = \frac{M_{H_s}(i)}{\lambda^i M_{H_s}(0)}$. We collect in Table 2 the fit parameters and $\chi^2/\text{d.o.f.}$ ²

We show in Fig. 3 the dependence of r_P on $1/M_{H_s}$ found with the fit formula (4). We retrieve the parametrically large cut-off effects $\propto (a M_{H_s})^2$ on r_P , justifying our decision to exclude some data of our 2 coarsest lattices in the analysis. It is reassuring that the pion mass dependence is found to be of the order of a few % and that it is numerically a sub-leading effect: by construction of r_P , pion mass effects are expected to vanish. Our way to define r_P is such that, according to the HQET scaling law telling that $\lim_{M_P \rightarrow \infty} f_P \sqrt{M_P} = C_{\text{ste}}$, r_P should tend to $\frac{1}{\sqrt{\lambda}}$. With our value of $\lambda = 1.18$, the limit is expected to be 0.92, in excellent agreement with our fit parameter $1 + r_{P_0} = 0.94(2)$. Then, in the continuum, we interpolate r_P at the 6 points $m_{D_s} \lambda^{i+1}$ to get a set of 6 ratios $r'_P(i)$:

² In lattice QCD data analysis, a $\chi^2/\text{d.o.f.}$ of the order 1 means that the proposed model to describe them is acceptable. In the discussion we have paid more attention to the stability of fit parameters when more terms are added in the fit formula, as well as their compatibility with 0 or not.

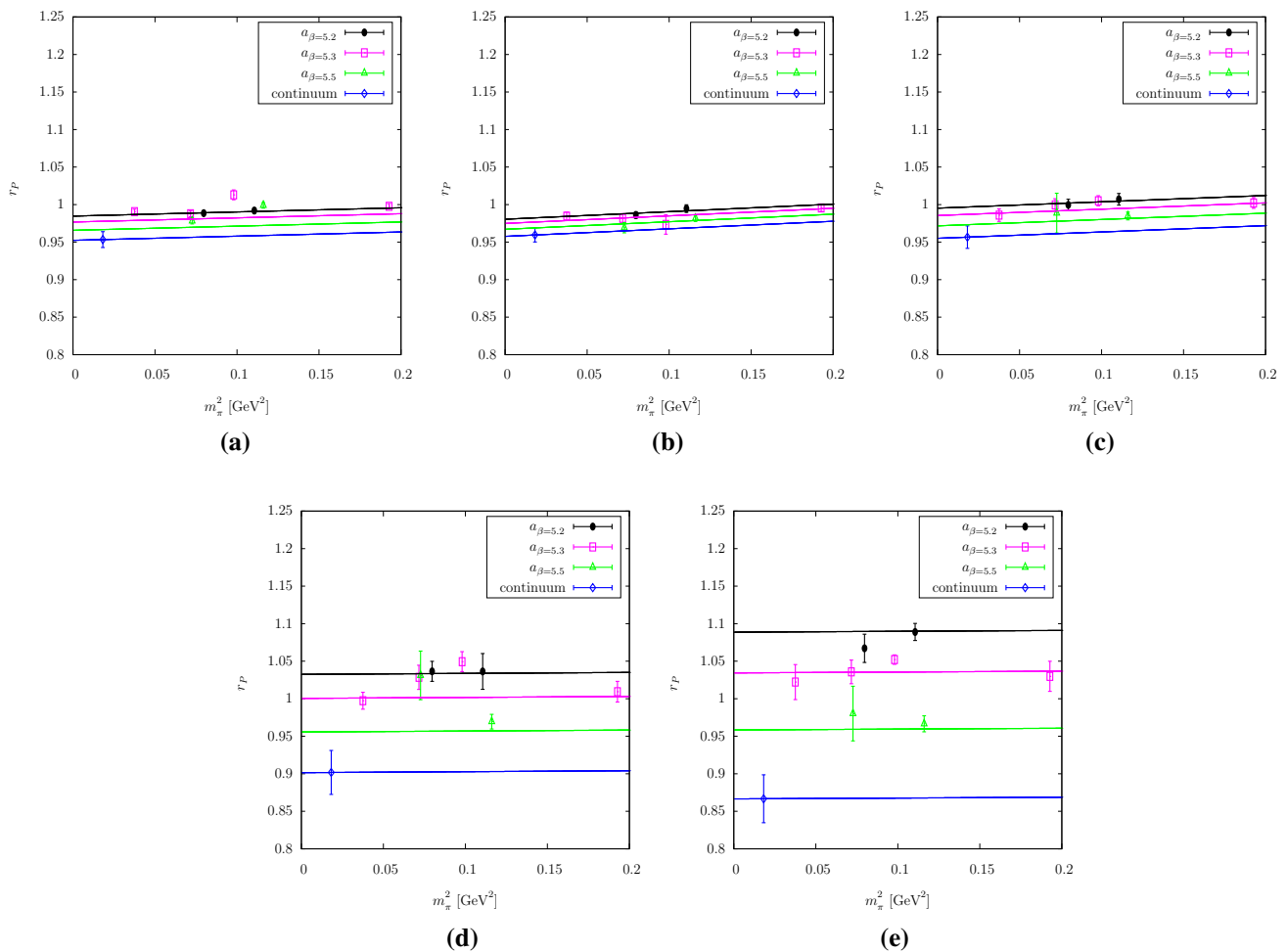


Fig. 2 Continuum and chiral extrapolation of r_P at the heavy masses λm_{D_s} (a), $\dots$, $\lambda^5 m_{D_s}$ (e)

Table 2 Fit parameters of r_P and its $\chi^2/\text{d.o.f.}$

r_{P_0}	$r_{P_1} [\text{GeV}^{-2}]$	r_{P_2}	$r_{P_3} [\text{GeV}]$	r_{P_4}	r_{P_5}	$\chi^2/\text{d.o.f.}$
$-0.06(1)$	$0.08(3)$	$1.1(3)$	$0.08(5)$	$-0.006(3)$	$0.021(9)$	1.4

$$r_P(i) = 1 + r_{P_0} + r_{P_1} m_\pi^{2\text{physical}} + \frac{r_{P_3}}{\lambda^{i+1} m_{D_s}},$$

$$r'_P(i) = \frac{C_A^{\text{stat}}(\mu_f, m_{D_s} \lambda^{i+1})}{C_A^{\text{stat}}(\mu_f, m_{D_s} \lambda^i)} r_P(i), \quad (5)$$

We recall that $r'_P(i)$ is independent of the renormalization scale μ_f and that we have $r'_P(i) = \frac{f_{H_s}(i+1)}{f_{H_s}(i)}$. We collect in Table 3 values of r'_P at the reference points $\frac{1}{\lambda^{i+1} m_{D_s}}$.

The last step is straightforward: f_{B_s}/f_{D_s} is obtained by a series of products:

$$\frac{f_{B_s}}{f_{D_s}} = \prod_{i=0}^5 r'_P(i). \quad (6)$$

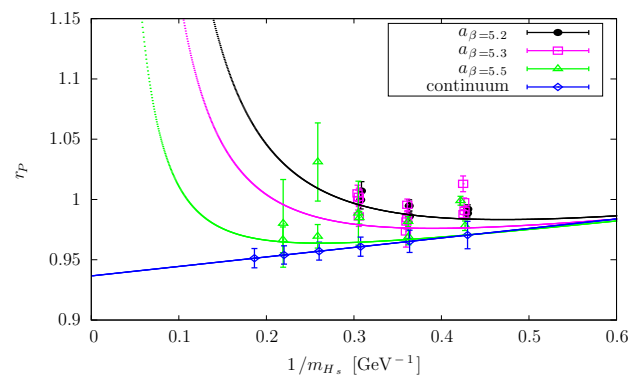


Fig. 3 Extrapolation at the physical point of r_P . The curves correspond to extrapolations at $m_\pi = m_\pi^{\text{physical}}$ and $r_{\text{mistuning}} - 1 = 0$

We get

$$\frac{f_{B_s}}{f_{D_s}} = 0.88(4). \quad (7)$$

The effect on the statistical error of the correlation among the different terms of the product of $r'_p(i)$ is taken into account by the mean of computing the errors described in [49].

To address the systematic error, we have performed two other fits:

- fit(A), adding to (4) a “next to leading order” chiral contribution in $m_\pi^2 \ln(m_\pi^2)$
- fit(B), adding to (4) a contribution in $1/m_{H_s}^2(i)$ to count for a higher order in the heavy quark expansion
- fit(C): fit (4) but using the matching coefficient C_A^{stat} at NLO

Other fits with extra terms in $(aM_{H_s})^2$ or in a^3 give non reliable results. We collect the corresponding fit parameters and $\chi^2/\text{d.o.f.}$ in Table 4 and we get

$$\begin{aligned} \frac{f_{B_s}}{f_{D_s}} &= 0.87(6) \text{ (A)}, \\ \frac{f_{B_s}}{f_{D_s}} &= 0.88(4) \text{ (B)}, \quad \frac{f_{B_s}}{f_{D_s}} = 0.87(4) \text{ (C)}. \end{aligned} \quad (8)$$

A fourth source of systematics can be included by propagating the uncertainty on raw data if we change $t_{\min} \rightarrow t_{\min} + 2a$ to extract plateaus. In this case we get the following result:

$$\frac{f_{B_s}}{f_{D_s}} = 0.89(5). \quad (9)$$

We collect the corresponding fit parameters in Table 5.

Adding together the different sources of systematics, we obtain

$$\frac{f_{B_s}}{f_{D_s}} = 0.88(4)^{(+1)}_{(-2)}. \quad (10)$$

where the first error is statistical and the second error counts for the systematic error.

Concerning the D_s meson decay constant, an update of the analysis reported in [49], now that we have the additional coarsest ensembles A5 and B6, meaning a third lattice spacing at our disposal, gives

$$\begin{aligned} f_{D_s} &= 244(4)(2) \text{ MeV}, \quad f_{D_s^*} = 268(4)(2) \text{ MeV}, \\ f_{D_s^*}/f_{D_s} &= 1.10(2), \end{aligned} \quad (11)$$

where the first error is statistical and the second error comes from the uncertainty on the lattice spacings. As in the previous analysis, a next to leading order contribution to the chiral fit destabilises the fit with chiral fit parameters compatible with zero.

Then we get

$$f_{B_s} = 215(10)(2)^{(+2)}_{(-5)} \text{ MeV}, \quad (12)$$

where the first error is the statistical error, the second one counts for the systematic error on f_{D_s} while the third error corresponds to the systematic error on f_{B_s}/f_{D_s} .

FLAG has recently made an update collection of lattice estimates of f_{B_s} [52]. Our estimate of f_{B_s} using the step scaling in mass strategy is compatible with the value obtained by the ALPHA Collaboration $f_{B_s} = 224(14) \text{ MeV}$ [23] by a

Table 3 Ratio r'_p at the reference points $\frac{1}{\lambda^{j+1}m_{D_s}}$

$1/(m_{D_s}\lambda^{j+1}) [\text{GeV}^{-1}]$	0.4298	0.3637	0.3077	0.2603	0.2202	0.1863
r'_p	0.9945(116)	0.9863(94)	0.9795(81)	0.9739(76)	0.9692(77)	0.9653(81)

Table 4 Fit parameters of r_p and the respective $\chi^2/\text{d.o.f.}$ for the fits (A), (B) and (C). (A) corresponds to adding an NLO term in $m_p^2 i \ln m_p^2 i$ to Eq. (4), (B) corresponds to adding a term in $1/m_{H_s}^2$ to Eq. (4) while (C) is using the expression (4) and the formulae at NLO of the matching coefficient C_A^{stat} to get r_p

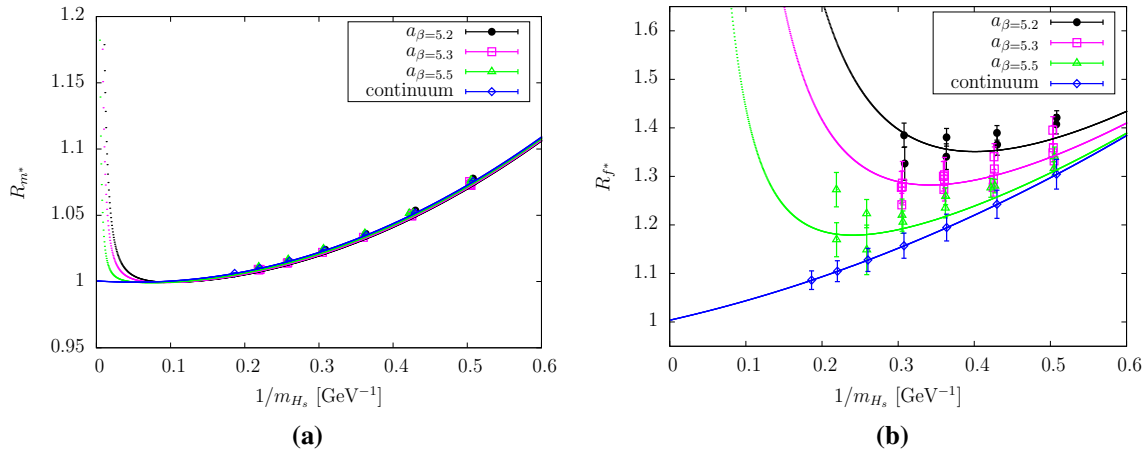
	r_{P_0}	$r_{P_1} [\text{GeV}^{-2}]$	r_{P_2}	$r_{P_3} [\text{GeV}]$	r_{P_4}	r_{P_5}	$r_{P_{\text{fit}}}$	χ^2/dof
A	−0.07(2)	0.4(1.7)	1.1(3)	0.08(5)	−0.01(1)	0.023(9)	−0.03(13) GeV^{-2}	1.5
B	−0.05(6)	0.08(3)	1.1(3)	−0.03(32)	−0.01(1)	0.022(8)	0.2(4) GeV^2	1.5
C	−0.07(2)	0.08(3)	1.1(3)	0.09(5)	−0.01(2)	0.021(9)	—	1.4

Table 5 Fit parameters of r_p and its $\chi^2/\text{d.o.f.}$ when systematic errors on raw data are propagated in the analysis

r_{P_0}	$r_{P_1} [\text{GeV}^{-2}]$	r_{P_2}	$r_{P_3} [\text{GeV}]$	r_{P_4}	r_{P_5}	$\chi^2/\text{d.o.f.}$
−0.06(2)	0.07(3)	1.0(3)	0.07(5)	−0.01(2)	0.021(9)	0.8

Table 6 Fit parameters of R_{m^*} and R_{f^*} and their respective $\chi^2/\text{d.o.f.}$

	$r_{X_0} [\text{GeV}^{-2}]$	$r_{X_1} [\text{GeV}]$	r_{X_2}	$r_{X_3} [\text{GeV}^2]$	r_{X_4}	$\chi^2/\text{d.o.f.}$
$X \equiv m^*$	0.026(3)	-0.007(3)	-0.002(1)	0.364(4)	0.0001(1)	1.7
$X \equiv f^*$	0.2(2)	0.4(2)	-0.01(4)	0.5(2)	0.12(3)	1.2

**Fig. 4** Extrapolations at the physical point of R_{m^*} **a** and R_{f^*} **b**. The curves correspond to extrapolations at $m_{\pi} = m_{\pi}^{\text{physical}}$

computation, performed over almost the same CLS ensembles as in this paper, of hadronic matrix elements in the framework of HQET with a non-perturbative matching of the HQET parameters with QCD. It is 2σ lower than the result reported by the ETM Collaboration [22] with $N_f = 2$ twisted-mass fermions defined at maximal twist. The fact that we cannot constrain the static limit of the ratio r_P to be equal to 1, due to mistuning effects of the heavy quark mass, explains a part of that discrepancy. The second source of discrepancy is the presence of large $a^2(aM_{H_s})^2$ cut-off effects in our data while they are numerically absent in ETMC data. Having to take them into account necessarily increases the uncertainty in extrapolation to the continuum limit because more parameters are required to described the data.

4.2 Extraction of $f_{B_s^*}/f_{B_s}$

To extract $f_{B_s^*}/f_{B_s}$ we have performed an alternative analysis to the one discussed in the previous subsection. We have examined the ratios

$$\begin{aligned}
 R_{m^*} &= \frac{M_V(a, \kappa_{\text{sea}}, \kappa_h, \kappa_s)}{M_P(a, \kappa_{\text{sea}}, \kappa_h, \kappa_s)} \equiv \frac{M_{H_s^*}}{M_{H_s}}, \\
 R_{f^*}' &= \frac{f_V(a, \kappa_{\text{sea}}, \kappa_h, \kappa_s)}{f_P(a, \kappa_{\text{sea}}, \kappa_h, \kappa_s)} \equiv \frac{f_{H_s^*}}{f_{H_s}}, \\
 R_{f^*}(a, m_{\pi}^2, M_{H_s}) &\equiv \frac{C_V^{\text{stat}}(\mu_f, M_{H_s})}{C_A^{\text{stat}}(\mu_f, M_{H_s})} R_{f^*}'. \quad (13)
 \end{aligned}$$

As the HQET anomalous dimension of the axial and vector static-light operator are the same, applying the renormalization group equation makes R_{f^*} independent of the renormal-

ization scale μ_f . To extrapolate to the physical point we have used the following fit ansatz:

$$\begin{aligned}
 R_{m^*}(a, m_{\pi}^2, M_{H_s}) &= 1 + r_{m_0^*} m_{\pi}^2 + r_{m_1^*} / M_{H_s} \\
 &\quad + r_{m_2^*} (a/a_{\beta=5.3})^2 \\
 &\quad + r_{m_3^*} / M_{H_s}^2 + r_{m_4^*} (a/a_{\beta=5.3})^2 (aM_{H_s})^2, \quad (14)
 \end{aligned}$$

$$\begin{aligned}
 R_{f^*}(a, m_{\pi}^2, M_{H_s}) &= 1 + r_{f_0^*} m_{\pi}^2 + r_{f_1^*} / M_{H_s} \\
 &\quad + r_{f_2^*} (a/a_{\beta=5.3})^2 \\
 &\quad + r_{f_3^*} / M_{H_s}^2 + r_{f_4^*} (a/a_{\beta=5.3})^2 (aM_{H_s})^2. \quad (15)
 \end{aligned}$$

We can impose the static limit constraint $\lim_{M_{H_s} \rightarrow \infty} R_{m^*} = \lim_{M_{H_s} \rightarrow \infty} R_{f^*} = 1$ because those ratios are free of heavy quark mistuning effect. We collect in Table 6 the corresponding fit parameters and we obtain

$$\frac{m_{B_s^*}}{m_{B_s}} = 1.0061(4), \quad \left(\frac{m_{B_s^*}}{m_{B_s}} \right)^{\text{exp}} = 1.0091, \quad \frac{f_{B_s^*}}{f_{B_s}} = 1.02(2).$$

We show in Fig. 4 the extrapolation to the physical point of R_{m^*} and R_{f^*} .

To estimate the systematic error, we have performed fits (A') and (B'), that read

- fit(A'): add to (14) and (15) a contribution in $m_{\pi}^2 \ln(m_{\pi}^2)$
- fit(B'): add to (14) a contribution in $1/m_{H_s}^3$
- fit(C'): fit (15) but using matching coefficients C_A^{stat} and C_V^{stat} at NLO

Table 7 Fit parameters of R_{m^*} and its $\chi^2/\text{d.o.f.}$ for the fits (A') and (B'). (A') corresponds to adding an NLO term in $m_p^2 i \ln m_p^2 i$ to Eq. (14) and (B') corresponds to adding a term in $1/m_{H_s}^3$ to Eq. (14)

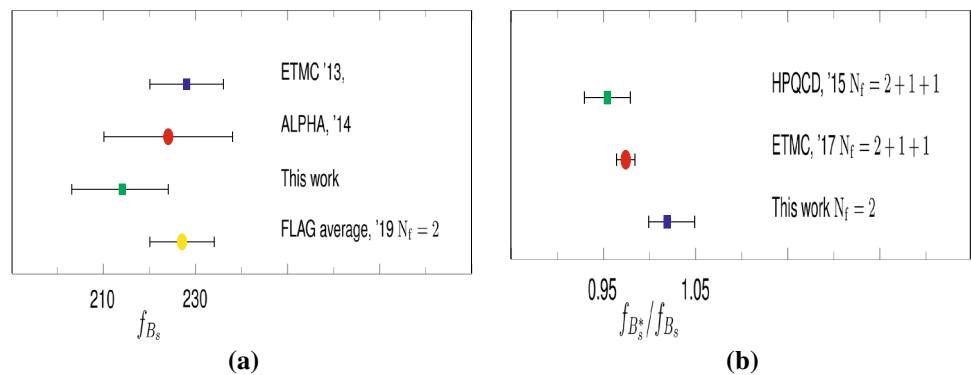
	$r_{m_0^*} [\text{GeV}^{-2}]$	$r_{m_1^*} [\text{GeV}]$	$r_{m_2^*}$	$r_{m_3^*} [\text{GeV}^2]$	$r_{m_4^*}$	$r_{m_{\text{fit}}^*}$	$\frac{\chi^2}{\text{dof}}$
A'	0.1(2)	-0.042(9)	-0.001(1)	0.37(1)	0.0004(10)	0.01(1) GeV^{-2}	1.7
B'	0.027(4)	-0.053(5)	-0.005(1)	0.46(2)	0.0020(1)	-0.11(2) GeV^3	1.4

Table 8 Fit parameters of R_{f^*} and its $\chi^2/\text{d.o.f.}$ for the fits (A') and (C'). (A') corresponds to adding an NLO term in $m_p^2 i \ln m_p^2 i$ to Eq. (15) and (C') is using the expression (15) and the formulae at NLO of the matching coefficient C_A^{stat} and C_V^{stat} to get R_{f^*}

	$r_{f_0^*} [\text{GeV}^{-2}]$	$r_{f_1^*} [\text{GeV}]$	$r_{f_2^*}$	$r_{f_3^*} [\text{GeV}^2]$	$r_{f_4^*}$	$r_{f_{\text{fit}}^*}$	$\frac{\chi^2}{\text{dof}}$
A'	2(10)	0.3(4)	-0.01(4)	0.6(5)	0.12(3)	-0.2(8) GeV^{-2}	1.3
C'	0.2(2)	0.3(1)	-0.01(4)	0.4(2)	0.12(3)	-	1.2

Table 9 Fit parameters of R_{m^*} and R_{f^*} and their respective $\chi^2/\text{d.o.f.}$ when systematic errors on raw data are included

	$r_{X_0} [\text{GeV}^{-2}]$	$r_{X_1} [\text{GeV}]$	r_{X_2}	$r_{X_3} [\text{GeV}^2]$	r_{X_4}	$\chi^2/\text{d.o.f.}$
$X \equiv m^*$	0.02(1)	-0.03(6)	-0.003(1)	0.364(7)	0.001(1)	0.4
$X \equiv f^*$	0.3(2)	0.4(2)	-0.03(5)	0.4(2)	0.13(4)	0.7

Fig. 5 Lattice estimates of f_{B_s} at $N_f = 2$, with the FLAG average of 2016 **a** and of $f_{B_s^*}/f_{B_s}$ **b**

Other terms in the fit lead to unstable and reliable results. We collect the fit parameters in Tables 7 and 8 and we obtain:

$$\frac{m_{B_s^*}}{m_{B_s}} = 1.0059(7), \quad \frac{f_{B_s^*}}{f_{B_s}} = 1.02(2) \quad (A'),$$

$$\frac{m_{B_s^*}}{m_{B_s}} = 1.0058(7) \quad (B'), \quad \frac{f_{B_s^*}}{f_{B_s}} = 1.03(2) \quad (C').$$

As for f_{B_s}/f_{D_s} , we have counted for the systematic error coming from the change $t_{\min} \rightarrow t_{\min} + 2a$ in the plateaus extraction. We get the following results:

$$\frac{m_{B_s^*}}{m_{B_s}} = 1.0066(7), \quad \frac{f_{B_s^*}}{f_{B_s}} = 1.03(2). \quad (16)$$

Fit parameters are collected in Table 9.

Eventually, we quote

$$\frac{m_{B_s^*}}{m_{B_s}} = 1.0061(4)(5), \quad \frac{f_{B_s^*}}{f_{B_s}} = 1.02(2)_{(-0.2)}^{(+2)}, \quad (17)$$

where the first error is statistical and the second error counts for the systematic error estimated by the fits (A') and (B') and contamination from excited states. There is no reason why the ratio $\frac{m_{B_s^*}}{m_{B_s}}$ should correspond to the experimental ratio $\left(\frac{m_{B_s^*}}{m_{B_s}}\right)^{\text{exp}} = 1.0091$ because, in our analysis, the strange quark is quenched. Still, we find a ratio 1σ lower than in [53] (1.0070(6)) where the computation was done in the framework of HQET expanded at $\mathcal{O}(1/m_b)$.

4.3 Comment

We collect in Fig. 5 the lattice QCD estimates of f_{B_s} at $N_f = 2$ [22, 23], with the corresponding FLAG average [52] and those of $f_{B_s^*}/f_{B_s}$ [30], [31]. Of course the fact that we get $f_{B_s^*}/f_{B_s} > 1$ while the 2 other lattice QCD results read $f_{B_s^*}/f_{B_s} < 1$ is puzzling. However, a computation performed by the ETM Collaboration with $N_f = 2$ dynamical quarks

indicated the hierarchy $f_{B^*}/f_B > 1$, $f_{B^*}/f_B = 1.050(16)$ [54]. So, a plausible explanation for the observed tension is the effect of the quenching of the strange quark in the spin-breaking contribution of the heavy quark symmetry to the ratio $f_{B_s^*}/f_{B_s}$. It might be of the same order of magnitude as in $f_{D_s^*}/f_{D_s}$ but with a more important qualitative impact because we examine a region of parameters closer to the symmetric point $f_{H^*}/f_H = 1$. In that respect, studies of this ratio with $N_f = 2 + 1$ ensembles are welcome.

5 Conclusion

In that paper we have reported on a lattice estimate of f_{B_s} and $f_{B_s^*}/f_{B_s}$. The main purpose of the work was testing the step scaling in mass method with Wilson–Clover fermions for which the RGI heavy quark mass can not be used yet as a physical parameter of the heavy quark expansion. Indeed, severe negative $O(am)$ cut-effects need to be balanced by still unknown $O(am)^2$ improvement terms to define safely the RGI mass. Instead, we have chosen the (inverse of) the heavy-strange meson mass as the expansion parameter. We obtain a quite low result for f_{B_s} compared to other lattice QCD estimates at $N_f = 2$, though it is compatible with the one got using the same set of gauge ensembles as here but with a complete different approach to simulate the heavy quark. We have found the hierarchy $f_{B_s^*}/f_{B_s} > 1$, indicating a positive correction to the symmetric point when the strange quark is quenched. A look at the literature leads to the conclusion that this correction becomes negative when the strange quark is taken into account in the sea. The next step of our program is the investigation of the form factors associated to $B_s \rightarrow D_s^{(*)} l \nu$ using CLS $N_f = 2$ ensembles, applying the step scaling in mass method.

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Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: Data discussed in the paper can be provided to anyone who is interested, after a demand directly addressed by mail to the authors.]

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Appendix

In this Appendix we collect all the data (meson masses, PCAC heavy+strange quark masses, meson decay constants), in lattice units, that are used in our analysis. We indicate the time range of the plateaus extraction.

Table 10 Average PCAC heavy and strange quark masses for the ensemble A5

κ_h	$[t_{\min} - t_{\max}]$	am_{hs}^{PCAC}
0.125310	[8–26]	0.1530(6)
0.121344	[8–26]	0.2095(7)
0.116040	[8–24]	0.2910(9)
0.109307	[8–20]	0.4046(10)
0.100407	[8–18]	0.5783(13)
0.089289	[8–14]	0.8484(17)

Table 11 Heavy-strange meson masses and decay constants for the ensemble A5

κ_h	$[t_{\min} - t_{\max}]$	$aM_{H_s}(i)$	$M_{H_s}(i+1)/M_{H_s}(i)$	$aM_{H_s^*}(i)$	$M_{H_s^*}(i+1)/M_{H_s^*}(i)$
0.125310	[10–24]	0.7498(8)	–	0.8081(13)	–
0.121344	[10–24]	0.8851(8)	1.1805(4)	0.9326(13)	1.1541(4)
0.116040	[10–24]	1.0482(8)	1.1843(3)	1.0860(13)	1.1645(4)
0.109307	[10–24]	1.2335(9)	1.1768(3)	1.2631(14)	1.1631(4)
0.100407	[10–20]	1.4572(11)	1.1814(5)	1.4790(17)	1.1709(4)
0.089289	[10–20]	1.7202(15)	1.1804(3)	1.7348(19)	1.1730(3)
κ_h	$[t_{\min} - t_{\max}]$	$af_{H_s}(i)$	$f_{H_s}(i+1)/f_{H_s}(i)$	$af_{H_s^*}(i)$	$f_{H_s^*}(i+1)/f_{H_s^*}(i)$
0.125310	[8–24]	0.0911(15)	–	0.1149(12)	–
0.121344	[8–24]	–	1.017(4)	–	0.999(1)
0.116040	[8–24]	–	1.016(5)	–	1.007(1)
0.109307	[8–24]	–	1.028(8)	–	1.026(2)
0.100407	[8–20]	–	1.055(24)	–	1.088(8)
0.089289	[8–20]	–	1.107(12)	–	1.144(2)

Table 12 Average PCAC heavy and strange quark masses for the ensemble B6

κ_h	$[t_{\min} - t_{\max}]$	am_{hs}^{PCAC}
0.125290	[8–38]	0.1533(5)
0.121313	[8–30]	0.2105(6)
0.116044	[8–24]	0.2911(7)
0.109127	[8–22]	0.4086(8)
0.100172	[8–16]	0.5871(8)
0.088935	[8–14]	0.8690(10)

Table 13 Heavy-strange meson masses and decay constants for the ensemble B6

κ_h	$[t_{\min} - t_{\max}]$	$aM_{H_s}(i)$	$M_{H_s}(i+1)/M_{H_s}(i)$	$aM_{H_s^*}(i)$	$M_{H_s^*}(i+1)/M_{H_s^*}(i)$
0.125290	[10–35]	0.7492(6)	–	0.8053(11)	–
0.121313	[10–35]	0.8858(7)	1.1824(2)	0.9314(10)	1.1565(4)
0.116044	[10–35]	1.0473(8)	1.1823(3)	1.0835(11)	1.1633(3)
0.109127	[10–25]	1.2372(10)	1.1813(3)	1.2655(12)	1.1680(2)
0.100172	[10–20]	1.4614(11)	1.1813(2)	1.4830(14)	1.1719(3)
0.088935	[10–17]	1.7250(14)	1.1803(3)	1.7407(17)	1.1738(3)
κ_h	$[t_{\min} - t_{\max}]$	$af_{H_s}(i)$	$f_{H_s}(i+1)/f_{H_s}(i)$	$af_{H_s^*}(i)$	$f_{H_s^*}(i+1)/f_{H_s^*}(i)$
0.125290	[10–25]	0.0897(9)	–	0.1142(10)	–
0.121313	[10–25]	–	1.013(3)	–	1.003(1)
0.116044	[10–25]	–	1.007(4)	–	1.011(1)
0.109127	[10–25]	–	1.019(8)	–	1.031(2)
0.100172	[10–22]	–	1.055(14)	–	1.081(8)
0.088935	[10–17]	–	1.085(19)	–	1.165(9)

Table 14 PCAC masses for the ensemble E5

κ_h	$[t_{\min} - t_{\max}]$	$am_{\text{PCAC}}(\kappa_h, \kappa_s)$
0.127240	[10–25]	0.1357(9)
0.123874	[10–25]	0.1833(10)
0.119457	[10–25]	0.2484(14)
0.113638	[10–25]	0.3403(17)
0.106031	[10–20]	0.4744(20)
0.096555	[10–18]	0.6713(25)

Table 15 Heavy-strange meson masses and decay constants for the ensemble E5

κ_h	$[t_{\min} - t_{\max}]$	$aM_{H_s}(i)$	$M_{H_s}(i+1)/M_{H_s}(i)$	$aM_{H_s^*}(i)$	$M_{H_s^*}(i+1)/M_{H_s^*}(i)$
0.127240	[10–25]	0.6579(7)	–	0.7075(14)	–
0.123874	[10–25]	0.7770(8)	1.1810(4)	0.8169(14)	1.1546(6)
0.119457	[10–25]	0.9170(10)	1.1803(4)	0.9487(15)	1.1614(5)
0.113638	[10–25]	1.0833(13)	1.1813(3)	1.1081(17)	1.1680(4)
0.106031	[10–20]	1.2826(17)	1.1840(6)	1.3010(18)	1.1741(9)
0.0965545	[10–20]	1.5117(18)	1.1786(2)	1.5248(19)	1.1720(3)
κ_h	$[t_{\min} - t_{\max}]$	$af_{H_s}(i)$	$f_{H_s}(i+1)/f_{H_s}(i)$	$af_{H_s^*}(i)$	$f_{H_s^*}(i+1)/f_{H_s^*}(i)$
0.127240	[10–25]	0.0830(9)	–	0.1025(11)	–
0.123000	[10–25]	–	1.023(4)	–	0.995(2)
0.119457	[10–25]	–	1.018(5)	–	0.999(2)
0.113638	[10–25]	–	1.022(7)	–	1.011(2)
0.106031	[10–20]	–	1.026(14)	–	1.036(10)
0.096555	[10–17]	–	1.047(20)	–	1.094(11)

Table 16 Average PCAC heavy and strange quark masses for the ensemble F6

κ_h	$[t_{\min} - t_{\max}]$	am_{hs}^{PCAC}
0.127130	[16–38]	0.1354(8)
0.123700	[16–38]	0.1837(10)
0.119241	[16–38]	0.2495(13)
0.113382	[16–25]	0.3405(15)
0.105793	[9–18]	0.4824(10)
0.096211	[9–16]	0.6870(11)

Table 17 Heavy-strange meson masses and decay constants for the ensemble F6

κ_h	$[t_{\min} - t_{\max}]$	$aM_{H_s}(i)$	$M_{H_s}(i+1)/M_{H_s}(i)$	$aM_{H_s^*}(i)$	$M_{H_s^*}(i+1)/M_{H_s^*}(i)$
0.127130	[10–42]	0.6577(5)	–	0.7058(9)	–
0.123700	[10–42]	0.7791(5)	1.1847(2)	0.8179(9)	1.1589(5)
0.119241	[10–28]	0.9208(6)	1.1818(3)	0.9516(10)	1.1635(4)
0.113382	[10–26]	1.0883(7)	1.1819(2)	1.1123(10)	1.1688(3)
0.105793	[10–23]	1.2863(9)	1.1819(2)	1.3041(12)	1.1725(4)
0.096211	[10–17]	1.5176(11)	1.1798(3)	1.5316(14)	1.1744(6)
κ_h	$[t_{\min} - t_{\max}]$	$af_{H_s}(i)$	$f_{H_s}(i+1)/f_{H_s}(i)$	$af_{H_s^*}(i)$	$f_{H_s^*}(i+1)/f_{H_s^*}(i)$
0.12713	[10–42]	0.0814(12)	–	0.0987(12)	–
0.123700	[10–42]	–	1.037(7)	–	0.995(4)
0.119241	[10–28]	–	0.995(13)	–	1.022(16)
0.113382	[10–26]	–	1.024(7)	–	1.015(4)
0.105793	[10–23]	–	1.068(13)	–	1.064(10)
0.096211	[10–17]	–	1.070(6)	–	1.088(1)

Table 18 Average PCAC heavy and strange quark masses for the ensemble F7

κ_h	$[t_{\min} - t_{\max}]$	am_{hs}^{PCAC}
0.127130	[16–40]	0.1362(6)
0.123649	[16–40]	0.1847(8)
0.119196	[16–36]	0.2491(9)
0.113350	[16–32]	0.3391(11)
0.105786	[9–27]	0.4686(13)
0.096689	[9–23]	0.6505(17)

Table 19 Heavy-strange meson masses and decay constants for the ensemble F7

κ_h	$[t_{\min} - t_{\max}]$	$aM_{H_s}(i)$	$M_{H_s}(i+1)/M_{H_s}(i)$	$aM_{H_s^*}(i)$	$M_{H_s^*}(i+1)/M_{H_s^*}(i)$
0.127130	[10–41]	0.6557(4)	–	0.7032(10)	–
0.123649	[10–41]	0.7784(5)	1.1872(2)	0.8170(9)	1.1617(3)
0.119196	[10–40]	0.9193(5)	1.1810(2)	0.9500(9)	1.1628(2)
0.113350	[10–35]	1.0861(5)	1.1814(1)	1.1099(9)	1.1683(2)
0.105786	[10–28]	1.2824(6)	1.1808(1)	1.3006(9)	1.1718(2)
0.096689	[10–26]	1.5013(7)	1.1707(1)	1.5149(10)	1.1648(2)

κ_h	$[t_{\min} - t_{\max}]$	$af_{H_s}(i)$	$f_{H_s}(i+1)/f_{H_s}(i)$	$af_{H_s^*}(i)$	$f_{H_s^*}(i+1)/f_{H_s^*}(i)$
0.12713	[10–40]	0.0787(8)	–	0.0971(9)	–
0.123649	[10–40]	–	1.010(4)	–	0.994(2)
0.119196	[10–40]	–	1.003(6)	–	0.997(2)
0.113350	[10–35]	–	1.019(8)	–	1.001(7)
0.105786	[10–28]	–	1.047(16)	–	1.041(8)
0.096689	[10–26]	–	1.056(16)	–	1.083(7)

Table 20 Average PCAC heavy and strange quark masses for the ensemble G8

κ_h	$[t_{\min} - t_{\max}]$	$am_{\text{PCAC}}(\kappa_h, \kappa_s)$
0.127100	[16–46]	0.1374(5)
0.123719	[16–46]	0.1850(6)
0.119260	[16–46]	0.2502(7)
0.113447	[16–36]	0.3405(9)
0.105836	[9–27]	0.4710(11)
0.096143	[9–23]	0.6649(17)

Table 21 Heavy-strange meson masses and decay constants for the ensemble G8

κ_h	$[t_{\min} - t_{\max}]$	$aM_{H_s}(i)$	$M_{H_s}(i+1)/M_{H_s}(i)$	$aM_{H_s^*}(i)$	$M_{H_s^*}(i+1)/M_{H_s^*}(i)$
0.127100	[10–41]	0.6568(4)	–	0.7029(7)	–
0.123719	[10–41]	0.7762(5)	1.1817(1)	0.8140(7)	1.1579(2)
0.119260	[10–38]	0.9174(4)	1.1819(1)	0.9477(6)	1.1644(2)
0.113447	[10–36]	1.0833(6)	1.1809(1)	1.1072(6)	1.1683(2)
0.105836	[10–30]	1.2808(6)	1.1823(1)	1.2993(6)	1.1735(1)
0.096143	[10–27]	1.5141(7)	1.1821(1)	1.5278(7)	1.1759(1)
κ_h	$[t_{\min} - t_{\max}]$	$af_{H_s}(i)$	$f_{H_s}(i+1)/f_{H_s}(i)$	$af_{H_s^*}(i)$	$f_{H_s^*}(i+1)/f_{H_s^*}(i)$
0.12710	[11–41]	0.0800(7)	–	0.0954(6)	–
0.123719	[11–41]	–	1.015(3)	–	0.995(1)
0.119260	[11–38]	–	1.006(5)	–	1.000(3)
0.113447	[11–36]	–	1.005(8)	–	1.007(3)
0.105836	[11–30]	–	1.015(11)	–	1.043(6)
0.096143	[11–21]	–	1.038(24)	–	1.085(6)

Table 22 PCAC masses for the ensemble N6

κ_h	$[t_{\min} - t_{\max}]$	am_{hs}^{PCAC}
0.130260	[16–42]	0.0986(7)
0.127737	[16–42]	0.1336(8)
0.124958	[16–42]	0.1726(10)
0.121051	[16–38]	0.2288(12)
0.115915	[16–36]	0.3060(13)
0.109399	[16–30]	0.4117(15)

Table 23 Heavy-strange meson masses and decay constants for the ensemble N6

κ_h	$[t_{\min} - t_{\max}]$	$aM_{H_s}(i)$	$M_{H_s}(i+1)/M_{H_s}(i)$	$aM_{H_s^*}(i)$	$M_{H_s^*}(i+1)/M_{H_s^*}(i)$
0.130260	[10–42]	0.4845(5)	–	0.5216(6)	–
0.127737	[10–42]	0.5804(6)	1.1978(2)	0.6103(6)	1.1700(3)
0.124958	[10–42]	0.6763(6)	1.1652(2)	0.7008(6)	1.1484(2)
0.121051	[10–40]	0.7994(6)	1.1821(2)	0.8191(7)	1.1688(2)
0.115915	[10–36]	0.9467(6)	1.1842(2)	0.9624(7)	1.1749(2)
0.109399	[10–34]	1.1183(7)	1.1812(2)	1.1306(8)	1.1747(1)
κ_h	$[t_{\min} - t_{\max}]$	$af_{H_s}(i)$	$f_{H_s}(i+1)/f_{H_s}(i)$	$af_{H_s^*}(i)$	$f_{H_s^*}(i+1)/f_{H_s^*}(i)$
0.130260	[13–42]	0.0603(8)	–	0.0714(8)	–
0.127737	[13–42]	–	1.019(4)	–	0.983(2)
0.124958	[13–42]	–	1.008(4)	–	0.984(2)
0.121051	[13–40]	–	1.004(5)	–	0.988(3)
0.115915	[13–36]	–	0.986(10)	–	1.007(4)
0.109399	[13–34]	–	0.982(11)	–	1.028(11)

Table 24 Average PCAC heavy and strange quark masses for the ensemble O7

κ_h	$[t_{\min} - t_{\max}]$	$am_{\text{PCAC}}(\kappa_h, \kappa_s)$
0.130220	[16–46]	0.0993(4)
0.127900	[16–46]	0.1315(5)
0.124944	[16–46]	0.1730(7)
0.120910	[16–42]	0.2309(9)
0.115890	[16–40]	0.3065(10)
0.109400	[16–32]	0.4118(14)

Table 25 Heavy-strange meson masses and decay constants for the ensemble O7

κ_h	$[t_{\min} - t_{\max}]$	$aM_{H_s}(i)$	$M_{H_s}(i+1)/M_{H_s}(i)$	$aM_{H_s^*}(i)$	$M_{H_s^*}(i+1)/M_{H_s^*}(i)$
0.130220	[10–55]	0.4851(4)	–	0.5213(6)	–
0.127900	[10–55]	0.5734(4)	1.1821(2)	0.6029(7)	1.1566(3)
0.124944	[10–55]	0.6756(5)	1.1782(2)	0.6995(7)	1.1602(2)
0.120910	[10–50]	0.8024(5)	1.1877(2)	0.8213(7)	1.1740(2)
0.115890	[10–46]	0.9462(5)	1.1792(2)	0.9609(7)	1.1701(2)
0.109400	[10–34]	1.1170(6)	1.1805(2)	1.1280(7)	1.1739(2)

κ_h	$[t_{\min} - t_{\max}]$	$af_{H_s}(i)$	$f_{H_s}(i+1)/f_{H_s}(i)$	$af_{H_s^*}(i)$	$f_{H_s^*}(i+1)/f_{H_s^*}(i)$
0.130220	[14–55]	0.0589(7)	–	0.0705(10)	–
0.127900	[14–55]	–	1.002(4)	–	0.987(2)
0.124944	[14–55]	–	0.991(7)	–	0.985(2)
0.120910	[14–50]	–	1.006(27)	–	0.983(8)
0.115890	[14–46]	–	1.050(33)	–	0.995(8)
0.109400	[14–34]	–	0.996(37)	–	1.020(6)

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