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Calabi-Yau manifolds in weighted projective space

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Abstract

Candelas et al.[1] made a partial classification of Calabi-Yau manifolds in $\mathbb{WP}^4$. An approximate symmetry was found in the collection of Calabi-Yau manifolds under the interchange of $\chi \to -\chi$. The goal of this thesis is to construct Calabi-Yau manifolds in this class and see if it is possible to extend the list made in[1]. Motivation behind this project is a symmetry of string theory called mirror symmetry that predicts a perfect symmetry in all Calabi-Yau manifolds when interchanging $\chi \to -\chi$. We find a way to extend the results of[1] but also come to the conclusion that a complete classification of orbifolds in weighted projective space has already been made in[2],[4]. We will also discuss these results.

Contents

1 Introduction

String theory is one of the most promising attempts to combine the standard model with gravity, unifying gravity with quantum mechanics, making it a theory of quantum gravity. One of the earliest versions of string theory only contained bosons, this theory is now called bosonic string theory. When people started to add fermions to the theory they also discovered supersymmetry, the resulting string theories where called supersymmetric string theories or superstring theories. String theory is also a theory that is not yet finished, there are a lot of different aspects of string theory that are not yet understood. One of the problems of string theory, that showed itself long ago is that string theory needs to have 26 dimensions (or 10 for superstrings) to be Lorentz invariant. A way to deal with the extra dimensions is to split the theory into a 22 (or 6) dimensional and a 4 dimensional effective theory. This proces is called compactification because the extra dimensions are projected on a very small space.

If we demand that the effective theory is like the standard model than we put restrictions on the space we use to compactify. It was shown[7] that a Calabi-Yau manifold provides an excellent background to compactify string theory on. A new problem immediatly shows itself because there are a lot of Calabi-Yau manifolds. Each manifold giving different physics compared to the next and no way of telling which is the right one.

There are some duality relations of string theory that help us a little bit with this problem. A duality relation is a symmetry between two different theories. One of those dualities is called mirror symmetry, it relates two different superstring theories (Type IIA and Type IIB) to eachother. One of compelling aspects of mirror symmetry is that it interchanges couplings in the two theories, meaning that calculations that are difficult in one theory are easy in the other. The two related theories give the same effective theory when compactified on two different Calabi-Yau manifolds. The two manifolds are related by the interchange of their two Hodge numbers $h^{1,1} \leftrightarrow h^{2,1}$. This interchange of Hodge numbers results in a sign change in the Euler Characteristic $\chi \to -\chi$. Mirror symmetry predicts that every Calabi-Yau manifold has a partner with opposite Euler characteristic.

When people started making a classification of Calabi-Yau manifolds they found this same symmetry in the Euler characteristic. One of the results is published in [1]. The result can be seen in figure (1). A total of 2339 topologically different Calabi-Yau manifolds where found in this class. They where constructed as hypersurfaces in a weighted projective space. The symmetry that is present in the plot is not perfect. Candelas et al. remark that they do not consider all possible hypersurfaces, which could be a reason for an imperfect symmetry.

The goal of this thesis is to try and extend the number of hypersurfaces that are considered and compute their Euler characteristics. In the next section we will introduce string theory and through the use of T-duality try to get a better understanding of mirror symmetry. In the third section we will give a mathematical overview of Calabi-Yau manifolds and projective spaces. We will

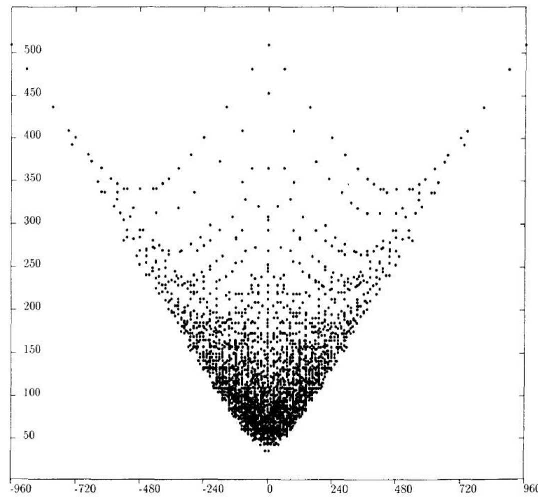


Figure 1: A plot of Euler numbers against $h^{1,1} + h^{2,1}$ as found in[1], each dot represents a manifold

also begin to ask ourselves how to construct a Calabi-Yau manifold in a weighted projective space. In the fourth section we will show how to extend the results obtained by Candelas and we will also discuss the complete classification as obtained by Kreuzer and Skarke[4]. In the last section we will discuss the results and give some ideas on how to proceed. Finally there are three appendices which contain results and also an example of a mathematica script that we used in section 4.

Before we proceed first a short note on notation. We will frequently use $\mathbb{P}^n$ to indicate a projective space with complex variables. $\mathbb{WP}^n$ will be used to indicate a (n+1)-dimensional weighted projective space, where we omit the weights $(k_0, ..., k_n)$. Furthermore we will use the Einstein summation convention, summing over repeated Greek indices.

2 Physical preliminaries

In this section we will introduce string theory with the ultimate goal of describing mirror symmetry. Mirror symmetry is the physical motivation of this project. So it is important to get atleast a feel for the concept. We will start with the very basics by introducing classical bosonic string theory, then we will discuss quantization of this theory. It turns out that this theory needs 26 dimensions to be Lorentz invariant. We already remarked that one of the ways to make a theory with more then 4 dimensions realistic is to compactify the D-4 dimensions on a space different from the 4 dimensional spacetime we live in. We will compactify the bosonic string theory on a circle to illustrate this concept. We will also use this compactification to introduce T-duality, which is a symmetry of string theory. Before we can introduce mirror symmetry we will have to talk about the type IIA and IIB superstring theories. We'll give some remarks about why compactification on Calabi-Yau manifolds is considered to be a realistic way to compactify superstring theories. Finally we will introduce mirror symmetry as a symmetry on the type IIA and IIB superstrings. This part relies heavily on the two books by Polchinski[16], [17] and to some lesser extend on the lectore notes by David Tong[18] and the thesis by K. Stiffler[6].

2.1 String theory

We will begin by discussing a classical string. Just like a particle sweeps out a worldline in Minkowski space, the string sweeps out a surface, the worldsheet (figure: 2), in spacetime. The worldsheet is parametrized by a timelike (τ) and a spacelike (σ) coordinate. The worldsheet defines a parametrization to Minkowski spacetime for all μ (μ= 0,..,D-1). There are two kinds of strings, open and closed. We will focus on the closed string in this study. A closed

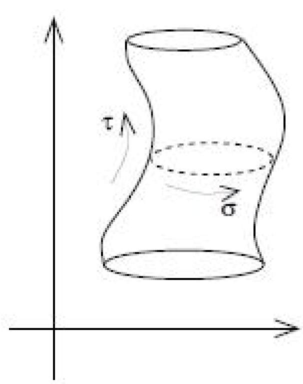


Figure 2: The worldsheet of a string. Time flows upward.

string is defined by a periodicity in the σ coordinate:

$$X^{\mu}(\sigma,\tau) = X^{\mu}(\sigma+2\pi,\tau)$$

We need an action that describes the movement of such a string. There are several ways of doing this, one of the most famous is the Nambu-Goto action**??**:

$$S_{NG} = -T\int d^2\sigma\sqrt{-det(\frac{\partial X^{\mu}}{\partial\sigma^{\alpha}}\frac{\partial X^{\nu}}{\partial\sigma^{\beta}}\eta_{\mu\nu})}$$

Where T is a constant and α and β indicate the worldsheet coordinates. This action is difficult to work with because of the square root. This is one of the reasons why we will use the Polyakov action:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_{\alpha}X^{mu}\partial_{\beta}X^{\nu}\eta_{\mu\nu} \tag{1}$$

α' is the tension of the string, $g_{\alpha\beta}$ is the metric on the worldsheet, $g = det(-g_{\alpha\beta})$ is a new field and α and β are indices that run over σ and τ. The equation of motion for X^{μ}that is obtained from the Polyakov action is:

$$\partial_{\alpha}(\sqrt{-g}g^{\alpha\beta}\partial_{\beta}X^{\mu}) = 0$$

The Polyakov action is equivalent to the Nambu-Goto action. This can be shown by varying $g_{\alpha\beta}$ and putting this into the equation of motion of X^{μ}. This gives the same equation of motion as the Nambu-Goto action did. The equation of motion of the X^{μ} can be put into a simpler form by setting $g_{\alpha\beta} = \eta_{\alpha\beta}$. This simplification is accomplished by playing with the parametrization of the metric and using Weyl invariance of the Polyakov action. Weyl invariance is a symmetry that sends $X^{\mu} \to X^{\mu}$ and $g_{\alpha\beta} \to \Omega^2 g_{\alpha\beta}$. A change like this on $g_{\alpha\beta}$ does not change the Polyakov action because $\sqrt{-g}$ scales as Ω^2 and $g^{\alpha\beta}$ scales as Ω^{-2}. If we write down the equations of motion for X^{μ} of the simplified form we get:

$$\partial_{\alpha}\partial^{\alpha}X^{\mu} = 0 \tag{2}$$

This equation is just the free wave equation. We also have an equation of motion for $g_{\alpha\beta}$ and we have to make sure that these are satisfied. If we set $g_{\alpha\beta} = \eta_{\alpha\beta}$ and define the stress-energy tensor to be:

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\partial S}{\partial g^{\alpha\beta}}$$

Then we find for the equation of motion for $g_{\alpha\beta}$:

$$T_{\alpha\beta} = \partial_{\alpha}X\partial_{\beta}X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\partial_{\rho}X\partial_{\sigma}X = 0$$

From this we find that X has to satisfy these two constrains for the equation of motion of the string:

$$\begin{aligned}\dot{X}X' &= 0\\ \dot{X}^2 + X'^2 &= 0\end{aligned}$$

Where $\dot{X}$ is partial derivation with respect to τ and X' is partial derivation with respect to σ. It is useful to put the equation of motion (2) in lightcone coordinates $s^{\pm} = \tau \pm \sigma$. It then takes the form:

$$\partial_- \partial_+ X^\mu = 0 \tag{3}$$

The solution of this equation of motion can be written in a part that depends only on σ^- (left moving) and in a part that depends only on σ^+ (right moving), and by applying Fourier theory the general solution to the equation of motion (3) is the sum of:

$$\begin{aligned} X_L^\mu(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \widetilde{\alpha}_n^\mu e^{-in\sigma^+} \\ X_R^\mu(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-} \end{aligned}$$

x^μ and p^μ are the position and momentum of the center of mass of the string and $\widetilde{\alpha}_n^\mu$ and α_n^μ are the Fourier coefficients. Even though these equations give the general solution to the free wave equation, we still have to impose the boundary conditions, in lightcone coordinates they read:

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0$$

If we solve one of the boundary conditions we get:

$$\begin{aligned} \partial_+ X^\mu &= \partial_+ X_L^\mu \\ &= \frac{1}{2}\alpha' p^\mu + \sqrt{\alpha'} 2 \sum_{n \neq 0} \widetilde{\alpha}_n^\mu e^{-in\sigma^+} \end{aligned}$$

Notice that if we take the square it means that we actually sum over μ so that:

$$\begin{aligned} (\partial_+ X)^2 &= \alpha' \sum_n \widetilde{L}_n e^{-in\sigma^+} \\ &= 0 \end{aligned}$$

Where we have defined:

$$\begin{aligned} \widetilde{\alpha}_0^\mu &= \sqrt{\frac{\alpha'}{2}} p^\mu \\ \widetilde{L}_n &= \frac{1}{2} \sum_m \widetilde{\alpha}_{n-m} \widetilde{\alpha}_m \end{aligned}$$

Doing the same derivation for the partial derivative in the - direction and setting the appropriate terms to zero we get the following set of constraints:

$$\begin{aligned} \alpha_0^\mu &= \widetilde{\alpha}_0^\mu \\ &= \sqrt{\frac{\alpha'}{2}} p^\mu \\ L_n &= \widetilde{L}_n \\ &= 0 \end{aligned}$$

Where:

$$L_n = \frac{1}{2}\sum_m \alpha_{n-m}\alpha_m$$

L_n and $\widetilde{L}$ are are the Fourier coefficients of the constraints. L_0 and $\widetilde{L}_0$ give us a relation for the mass of the excited oscillator modes. We use the mass-shell condition:

$$p_\mu p^\mu + M^2 = 0 \tag{4}$$

If we set L_0 equal to zero and use $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$ we find the following relation between the oscillators and the momenta:

$$\sum_{m>0}\alpha_{-m}\alpha_m = -\frac{\alpha}{4}p_\mu p^\mu$$

Doing the same for $\widetilde{L}_0$ using the mass-shell condition (4) we get the following relation between the oscillators and the mass:

$$M^2 = \frac{4}{\alpha'}\sum_{m>0}\alpha_m\alpha_{-m} = \frac{4}{\alpha'}\sum_{m>0}\widetilde{\alpha}_m\widetilde{\alpha}_{-m} \tag{5}$$

The fact that L_0 and $\widetilde{L}_0$ give the same mass is called level matching. We will now make the step from the classical string theory to the quantum string theory.

2.1.1 Quantizing the bosonic string

The general idea behind quantization is very simple, we simply promote all X^μ's and their conjugate momenta $\Pi_\mu = \frac{1}{2\pi\alpha'}\dot{X}_\mu$ to operators. This leads to commutator relations for x^μ, p^μ and the α's. This will lead to a quantum theory of the (closed) string, which is complicated enough to fill an entire book. However, we will mostly be interested in a formula for the mass like we found for the classical string (5). The operators for X^μ and Π_μ translate to commutation relations between x^μ, p^μ.

$$[x^\mu, p_\mu] = i\delta^\mu_\nu$$

And similarly for α^μ_n and $\widetilde{\alpha}^\mu_n$.

$$\begin{aligned}[\alpha^\mu_m, \alpha^\nu_n] &= [\widetilde{\alpha}^\mu_m, \widetilde{\alpha}^\nu_n] \\ &= m\eta^{\mu\nu}\delta_{n,-m}\end{aligned}$$

These are the commutation relations for creation and annilation operators. α (and $\widetilde{\alpha}$) can be seen as a creation operator for $n < 0$ and as a annihilation operator for $n > 0$. We want all the oscillators to be normal ordered, this means creation operators to the left of annihilation operators in products. The commutation relations between α and $\widetilde{\alpha}$ give us some problems when we put the operators in L_0 and $\widetilde{L}_0$ in normal order. The reordering of the α's in combination with the commutator relations gives rise to a normal ordering constant: a.

The constraint on the equation of motion for X^μ in the classical theory gave $L_0 = \widetilde{L}_0 = 0$, since L_0 is now an operator we get the following constraint:

$$(L_0 - a)|\phi >= 0$$

We can still compute the formula for the mass of the bosonic string as in the previous section. Taking into account the normal ordering constant a, we find the modified mass formula:

$$\begin{aligned} M^2 &= \frac{4}{\alpha'}(\sum_{m>0} \alpha_{-m}\alpha_m - a) \\ &= \frac{4}{\alpha'}(\sum_{m>0} \widetilde{\alpha}_{-m}\widetilde{\alpha}_m - a) \end{aligned}$$

We can make this equation look more friendly by defining:

$$\begin{aligned} N &= \sum_{m>0} \alpha_{-m}\alpha_m \\ \widetilde{N} &= \sum_{m>0} \widetilde{\alpha}_{-m}\widetilde{\alpha}_m \end{aligned}$$

and setting $a = 1$ we find the final result of this section:

$$M^2 = \frac{4}{\alpha'}(N-1) = \frac{4}{\alpha'}(\widetilde{N}-1) \qquad (6)$$

If we start making a spectrum for this theory then we find that the lowest possible mass is (no oscillators excited):

$$M^2 = -\frac{4}{\alpha'}$$

This is a negative mass squared, so one of the first consequences of the mass formula is the existence of the tachyon, a particle with negative mass squared. Such a particle is normally associated with an unstable ground state. It is possible that the tachyon has some physical interpretation as is discussed in [18] but this is not fully understood. The tachyon disappears in superstring theories, that is theories with fermions.

The Fourier coefficients L_m and $\widetilde{L}_m$ generate the algebra that is associated with states of the theory, it is called the Virasoro algebra. The generators of the Virasoro algebra have their own commutator relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

Where c is the central charge. For $\widetilde{L}_m$ a similar result holds for central charge $\widetilde{c}$. String theory has ghosts, that is particles that are unphysical, that contribute -26 to the central charge of the theory. However, if $c = 0$ then we preserve Lorentz invariance. The way to compensate for the ghosts is to introduce the right number of degrees of freedom, that is require the theory to be 26 dimensional. This also results in the normal ordering constant $a = 1$.

We used L_0 to get the mass-shell condition:

$$0 = \frac{\alpha'}{4}(p_\mu p^\mu + m^2)$$

This is the Klein-Gordon equation. The same result holds for $\widetilde{L}_0$ and, even though we didn't show it, something similar holds for the open string in this theory. The factor $\frac{1}{4}$ is not present for the open string. The Klein-Gordon equation is used in relativistic quantum mechanics to describe bosons. This seems to suggest that we will only find bosons in our theory. The first thing that we will consider when we introduce superstring theory is an extension of the constraint algebra with generators that correspond to the Dirac equation. Before we move towards superstring theories we will first illustrate the concept of compactification by compactifying the bosonic theory on a circle.

2.2 Compactification

The idea of compactification is as old as general relativity. The idea by Kaluza and Klein was to include a 5th dimension in Einstein's field equations which would also describe Maxwell's equations. The 5th dimension was compactified on a circle using the periodicity condition:

$$x^4 \cong x^4 + 2\pi R$$

the $x^\mu(\mu$ =0,..,3) are all noncompact and R is the radius of the compactification circle. The 5-dimensional metric of the theory then seperates into a 4-dimensional metric, a vector and a scalar on the 4-dimensional spacetime. How does this idea apply to the string theory case?

Suppose we have a field theory like string theory with D dimensions (D¿4). The fields of this theory will then be free to move about in these D dimensions. However, spacetime, as we see it, is 4 dimensional. What do we do with all of the D-4 dimensions that we do not see? A solution to this problem is so called compactification of the theory. We project the theory on a 4 dimensional spacetime and a D-4 dimensional internal space. Observers in the 4 dimensional space can not see the internal space because it is to small to see. The fields of the theory are split between the internal space and the 4-dimensional spacetime. The way in which this happens has a direct effect on the physics in the 4 dimensional spacetime, resulting in an effective 4 dimensional theory depending on the type of compactification. Lets see what happens when we compactify our bosonic string theory on a circle.

2.2.1 Compactification of the bosonic string

We consider a closed bosonic string compactified on a circle: $\mathbb{R}^{1,24} \times \mathbb{S}^1$. In the direction of the circle we have the following periodicity requirement:

$$X^{25} \cong X^{25} + 2\pi R$$

This immediatly implies that the momentum in the compactified direction is quantized [18]:

$$p^{25} = \frac{n}{R}$$

This is due to the fact that the string wavefunction includes a factor e^{ipX}. Another effect of compactifying a string on a circle is that we no longer need to require that the string has periodic boundary conditions. Closed strings can wind around the compact dimension, relaxing the boundary condition somewhat. This gives the following boundary condition:

$$X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi mR$$

The number m is called the winding number and tells you many times the string winds around the compact dimension. The winding number is not constant because it can change during string interactions (this is the only time we'll mention string interactions). We can view X^{25} as having a right and a left moving part. We need to introduce a right and a left moving momentum before we can write the general solution for X^{25}:

$$\begin{aligned} p_L &= \frac{n}{R} + \frac{mR}{\alpha'} \\ p_R &= \frac{n}{R} - \frac{mR}{\alpha'} \end{aligned}$$

With the help of these momenta we can write down the left and right moving part of X^{25} in lightcone coordinates:

$$\begin{aligned} X_L^{25}(\sigma) &= \frac{1}{2}x^{25} + \frac{1}{2}\alpha' p_L \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \widetilde{\alpha}_n^{25} e^{-in\sigma^+} \\ X_R^{25}(\sigma) &= \frac{1}{2}x^{25} + \frac{1}{2}\alpha' p_R \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^{25} e^{-in\sigma^-} \end{aligned}$$

The noncompact coordinates are the same as before. The Virasoro generators also change because of the new boundary condition, we get:

$$\begin{aligned} L_0 &= \frac{\alpha' p_{\mu'} p^{\mu'}}{4} + \frac{\alpha' p_L^2}{4} + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n \\ \widetilde{L}_0 &= \frac{\alpha' p_{\mu'} p^{\mu'}}{4} + \frac{\alpha' p_R^2}{4} + \sum_{n=1}^{\infty} \widetilde{\alpha}_{-n}\widetilde{\alpha}_n \end{aligned}$$

Where μ' runs over the noncompact dimensions. We are interested in the effective theory which exists on the noncompact dimensions. The mass of the particles is still given by the mass-shell condition:

$$\begin{aligned} M^2 &= -p^\mu p_\mu \\ &= p_R^2 + \frac{4}{\alpha'}(\widetilde{N} - 1) \\ &= p_L^2 + \frac{4}{\alpha'}(N - 1) \end{aligned}$$

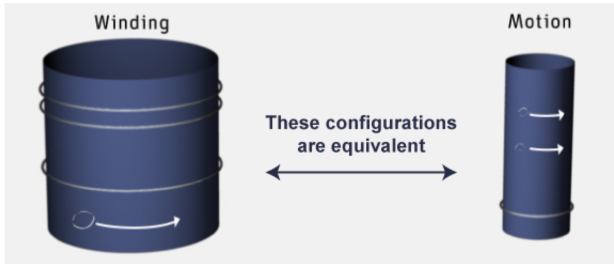


Figure 3: An illustration of two strings compactified on small and large circles whose winding and momentum number have been interchanged

This is the formula that we wanted to obtain in this section. This formula is all that we need to introduce T-duality.

2.2.2 T-duality

Adding the two equations for the mass in the previous section and dividing by two, we get the following mass formula:

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'} + \frac{2}{\alpha'}(N + \widetilde{N} - 2)$$

This formula tells us that a string does not only get a contribution to its mass from its momentum but also from the number of times that it winds around the compact dimension. If we send $R \to 0$ then the compact momentum becomes infinitly massive and the winding states go to a continuous spectrum. If we send $R \to \infty$ then the winding states become infinitly massive and the compact momentum approaches a continuum. This implies that if we change:

$$R \to \frac{\alpha'}{R}, n \leftrightarrow m$$

then the theory will have the same spectrum (Figure 2.2.2). This is called T-duality and it has been shown to be equivalent to mirror symmetry. T-duality is like saying that the string doesn't know the difference between a circle with small radius and one with large radius. This equivalence still holds when we compactify the theory in more dimensions. Another interesting consequence is that the smallest possible scale is given by the self-dual radius $R = \sqrt{\alpha'}$. We should really consider superstring theories next, now that we have seen what T-duality is in the bosonic theory.

2.3 Superstring theories

So far we have only considered non-supersymmetric bosonic string theory. This theory is a nice introduction to some of the more difficult concepts in superstring theory. We have to introduce superstring theory at some point if we want to talk about mirror symmetry, which is something that is not really present in bosonic string theory. Making the theory supersymmetric means that we will have to introduce fermions into our theory. For simplicity we will still only consider closed strings. We used the mass-shell condition to determine the mass of particles:

$$p_\mu p^\mu + M^2 = 0$$

This came in the classical theory from the condition that $L_0|\phi >= 0$. We already remarked that this mass-shell condition is just the Klein-Gordon equation in momentum space, so it should be no surprise that we worked in a theory with only boson [17]. It seems natural that if we enlarge our constraint algebra with some generators that give the Dirac equation:

$$ip_\mu \gamma^\mu + m = 0$$

that we would get a theory that includes fermions. It turns out that the gamma matrices generate the right algebra for an anticommuting worldsheet field ϕ^μ.

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Note the anti-commutator relation. If we include the gamma matrices we have expanded the constraint algebra of the theory. One of the things we find is that this also changes the critical dimension, that is the dimension for which it is Lorentz invariant, of the theory, The dimension goes down to $D = 10$. This is a promising way to start and it is just how Polchinski[**?**] introduces the superstring theories. We will take a slightly different appraoch, follow the thesis by K. Stiffler[6] and look at the Polyakov action (1). Expanding the theory to include fermions means that we also have to make the Polyakov action supersymmetric. We start by introduction fermionic (anti-commuting) fields ψ^μ:

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}$$

We work in the Ramond-Neveu-Schwarz (RNS) formalism adding the right terms to the Polyakov action [6]:

$$S = -\frac{T}{2} \int d^2 z (\partial_\alpha X^\mu \partial^\alpha X_\mu - 2i\psi_- \partial_+ \psi_- - 2i\psi_+ \partial_- \psi_+) \qquad (7)$$

Where $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma$. Varying the action we find the following equations for the fields:

$$\begin{aligned} \partial_\alpha \partial^\alpha X^\mu &= 0 \\ \partial_+ \psi^\mu_- &= 0 \\ \partial_- \psi^\mu_+ &= 0 \end{aligned}$$

Moving in a similar direction as we did for the closed bosonic string we now impose a periodicity condition on X:

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$$

The solution for these fields is the again a sum of a right moving and left moving part like in the bosonic string case. The situation is more complicated for the fermionic fields ψ^μ whose boundary conditions are:

$$\psi_+\partial\psi_+ - \psi_-\partial\psi_-|_{\sigma=0}^{2\pi} = 0$$

There are two ways to satisfy this equation for ψ [6]:

$$\begin{aligned}\psi_\pm(\tau,\sigma) &= \psi_\pm(\tau,\sigma+\pi)\\ \psi_\pm(\tau,\sigma) &= -\psi_\pm(\tau,\sigma+\pi)\end{aligned}$$

The first boundary condition is called the Ramond (R)boundary condition and the second boundary condition is called the Neveu-Scharwz (NS) boundary condition. This means that there are also two possible solutions to the boundary problems for ψ^μ. For the Ramond boundary condition we have:

$$\begin{aligned}\psi_-^\mu(\tau,\sigma) &= \sum_{m\in\mathbb{Z}} d_m^\mu e^{-2im\sigma^-}\\ \psi_+^\mu(\tau,\sigma) &= \sum_{m\in\mathbb{Z}} \widetilde{d}_m^\mu e^{-2im\sigma^+}\end{aligned}$$

and for the Neveu-Schwarz boundary condition:

$$\begin{aligned}\psi_-^\mu(\tau,\sigma) &= \sum_{m\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-2ir\sigma^-}\\ \psi_+^\mu(\tau,\sigma) &= \sum_{m\in\mathbb{Z}+\frac{1}{2}} \widetilde{b}_r^\mu e^{-2ir\sigma^+}\end{aligned}$$

Where d_m^μ, $\widetilde{d}_m^\mu$, $\widetilde{b}_r^\mu$ and b_r^μ are the Fourier coefficients that will take the role of creation and annihilation operators when we quantize. In order to quantize the theory we need to impose (anti-)commutator relations. For the bosonic fields X^μ, these are just the same as for the bosonic theory. For the new fermionic fields we impose:

$$\{\psi_A^\mu(\tau,\sigma), \psi_B^\mu(\tau,\sigma')\} = \pi\eta^{\mu\nu}\delta_{AB}\delta(\sigma-\sigma')$$

where A and B are either + or -. This leads to anti-commutation relations on the oscillators:

$$\begin{aligned}\{b_r^\mu, b_s^\nu\} &= \eta^{\mu\nu}\delta_{r+s,0}\\ \{d_m^\mu, d_m^\nu\} &= \eta^{\mu\nu}\delta_{m+n,0}\end{aligned}$$

The left moving, tilded oscillators, obey the same commutation relations. These oscillators now take the form of creation and annihilation operators on the Fock space. The oscillators with positive m, n, r, s are annihilation operators and the ones with negative m, n, s, r are creation operators. A ground state is given by $|p, 0>_R$ for the R boundary condition and by $|p, 0>_{NS}$ for the NS boundary condition, where p is the center of mass momentum of the string. By working on the ground state with the creation operators we get states such as $|p,|m|>_R$ and $|p,|r|>_{NS}$. Where $|m|$ and $|r|$ are the mass of the states. There are four possibilities for a physical state $|phys>$. These are all the tensor product of a left moving (ground state acted on by a tilded operator) and a right moving state[6]:

$$|phys>= \begin{cases} |p,|\widetilde{m}|>^{\mu}_R & \otimes \quad |p,|n|>^{\nu}_R \quad \text{R-R sector} \\ |p,|\widetilde{r}|>^{\mu}_{NS} & \otimes \quad |p,|s|>^{\nu}_{NS} \quad \text{NS-NS sector} \\ |p,|\widetilde{r}|>^{\mu}_{NS} & \otimes \quad |p,|m|>^{\nu}_R \quad \text{NS-R sector} \\ |p,|\widetilde{m}|>^{\mu}_R & \otimes \quad |p,|r|>^{\nu}_{NS} \quad \text{R-NS sector} \end{cases}$$

Demanding that $(L_0+\widetilde{L}_0)|phys>=0$ we find for the mass of the superstring[6]:

$$\alpha' M^2 = 2(N + \widetilde{N} + a_b + \widetilde{a}_b + a_f + \widetilde{a}_f)$$

Where the constants $a_b, \widetilde{a}_b, a_f, \widetilde{a}_f$ are normal ordering constants. We are now at the point where we will define the type IIA and type IIB superstring theories. The difference lies in the ground state of the theory. Because, even though we have four possibilities for a physical state, there are restrictions we can put on the ground state of the theory. These restrictions give two possibilities for a physical ground state, IIA and IIB. We can define an operator that anticommutes with the full ψ^μ, we define it as[17]:

$$G = e^{i\pi F}$$

Where F is either 1 or 2. This operator can work on different states of the spectrum of the superstring. The ground state of $|p,0>^{\mu}_{NS}$ is the tachyon, an unphysical state. If we work with G on this ground state we get -1 as eigenvalue. If we work with G on the first excited state we get +1 as eigenvalue. Physical states of the NS sector have for G eigenvalue +1. So, even though $|p,0>^{\mu}_{NS}$ is the ground state of the Fock space, it is not the physical ground state for the NS sector of the theory. The physical ground state is the first excited state, $b^{\mu}_{-\frac{1}{2}}|p,o>_{NS}$. This just leaves the R sector. Here we can choose whether we keep the states with negative or with positive parity (-1 or +1 as eigenvalue). This gives two possibilities for the physical ground state, denoted by: $+|p,0>_R \equiv |p,+>_R$ and $-|p,0>_R \equiv |p,->_R$. So, it turns out that we have two possible ground states for a consistent theory. The ground state consists either of (type IIA):

$$\begin{array}{lcl} |p,->_R & \otimes & |p,+>_R \\ \widetilde{b}^{\mu}_{-\frac{1}{2}}|p,0>_{NS} & \otimes & b^{\nu}_{-\frac{1}{2}}|p,0>_{NS} \\ \widetilde{b}^{\mu}_{-\frac{1}{2}}|p,0>_{NS} & \otimes & |p,+>_R \\ |p,->_R & \otimes & b^{\nu}_{-\frac{1}{2}}|p,0>_{NS} \end{array}$$

Or (type IIB):

$$\begin{array}{lcl} |p,+>_R & \otimes & |p,+>_R \\ \widetilde{b}^{\mu}_{-\frac{1}{2}}|p,0>_{NS} & \otimes & b^{\nu}_{-\frac{1}{2}}|p,0>_{NS} \\ \widetilde{b}^{\mu}_{-\frac{1}{2}}|p,0>_{NS} & \otimes & |p,+>_R \\ |p,+>_R & \otimes & b^{\nu}_{-\frac{1}{2}}|p,0>_{NS} \end{array}$$

For type IIA we choose the Ramond ground states to have the same chirality and in type IIB we choose the Ramond ground states to have opposite chirality. The story is of course far from over, we didn't really consider the spectrum of the theory for instance. However, we will not continue this discussion on superstring theory. The last two subsections of this section will deal with Calabi-Yau compactification and mirror symmetry, respectively.

2.4 Calabi-Yau compactification

So far we haven't discussed Calabi-Yau manifolds yet. Now that we have discussed supersymmetric string theory it is time to look at the reason why people want to compactify these on a Calabi-Yau manifold. Superstring theory requires 10-dimensions, however the world we observe is not 10-dimensional. We already explained that compactification is a solution to this problem. There would have to be some small internal space. Experiments have put on upper limit on the size of the internal space at 10^{-18} metres (the TeV scale), meaning that the 6 dimensions, if real, must show themself only at energies higher than the TeV scale. A way to make this a bit more mathematically precise is to say that the spacetime manifold on which the string moves is not a ten-dimensional manifold but rather looks like $M^4 \times N^6$ where M^4 is (presumably something like Minkowskian) 4-dimensional spacetime and N^6 is some 6-dimensional compact manifold. The 4-dimensional spacetime manifold is obviously the one we are free to move around in while the 6-dimensional manifold represents the compact dimensions.

We have already seen that the way we compactify our theory has a direct effect on the spectrum of the theory we compactify. This was the case for the bosonic string on a circle with radius R. The obvious question is what can we say about the space N^6 if we demand the physics on M^4 to be like the physics we see in everyday life? The answer is that N^6 has to be a Calabi-Yau manifold. What are the assumptions we make about physics in everyday life that result in N^6 being a Calabi-Yau manifold? Candelas, Horowitz, Strominger and Witten asked this question, they used the following assumptions[7]:

1. The manifold M^4 is maximally symmetric, i.e. it is Minkowskian, de Sitter or anti-de Sitter.

2. Supersymmetry should be unbroken in the resulting d=4 theory

3. The spectrum of gauge bosons and fermions should bear some resemblence to what we observe in real life.

The first assumption is somewhat like demanding that the theory looks like general relativity in the low energy limit. The third assumption is also obvious because it says that our theory should predict and describe the particles we observe in our accelerators, i.e. the low energy limit of the theory should look like the standard model. Both the first and the third assumption seem like common sense. The second assumption does not.

In fact, there is no proof that nature is supersymmetric. The nice thing about supersymmetry is that it helps solve some theoretical issues like the hierarchy problem of the standard model and it gives a candidate for dark matter. The hierarchy problem concerns quantum corrections of the Higgs mass. Supersymmetry puts a restriction on the corrections making sure they don't run off to infinity. The lightest supersymmetric particle might be stable and thus can be a candidate for dark matter as it is quite massive (more massive then any other particle we have found).

Without going to deep into the technical aspects, we conclude that by looking at the field content of the effective theory and using the assumptions, it can be shown that there must be a covariantly constant spinor field on N^6. This is a strong restriction. If we look at the similar case for a sphere $\mathbb{S}^2$ and try to construct a constant vector field then the hairy ball theorem says that the vector field must vanish on at least one point of $\mathbb{S}^2$ and, being covariantly constant, the vector field must vanish everywhere. Ultimatly, using holonomy theory we find that for a covariantly constant spinor field to exist on N^6, it must be a Kähler manifold with vanishing first Chern class. We will later define this type of manifold as a Calabi-Yau manifold.

2.5 Mirror symmetry

In this final section we will look at T-duality and mirror symmetry of the type II string. We will discuss T-duality by once again compactifying our theory on a circle. Even for something as simple as that, some interesting results follow. Mirror symmetry shows itself when the supersymmetric theory is compactified on a space that has a bit more structure. We will use the simple example of the 2-torus, but the results hold for more general compactifications on Calabi-Yau manifolds.

2.5.1 T-duality of type II strings

Say we compactify a single coordinate X^9 of the type IIA string theory on a circle and take the $R \to 0$ limit. This is equivalent to taking the $R \to \infty$ limit and taking the following reflection[17]:

$$X_R'^9 = -X_R^9$$

Where the $'$ is in the low R limit. This is the same as in the bosonic string theory. However this time we also have to reflect:

$$\psi'^9 = -\psi^9$$

due to an internal invariance of theory. However, this implies that the chirality of the right moving R sector ground state is reversed. Meaning that because we started in a type IIA theory, after T-duality we have a dual type IIB theory. So using T-duality it is possible to switch between type IIA and type IIB theories. Meaning that if we start with type IIA and take the compactification radius small then because the chirality changes we have a theory that is equivalent to a type IIB theory compactified on a circle with radius large. This duality relation holds if we apply the above operations to an odd number of dimensions. If we had done T-duality on an even number of dimensions then the we would end up with the same type II theory [17].

2.5.2 Mirror symmetry of type II strings

We will make use of the term moduli in this section so it necessary to introduce it now. Moduli are parameters that label the geometry of the manifold under consideration[16].

Mirror symmetry has the convenient property that it changes a Type IIA theory in a Type IIB theory while changing the couplings in such a way that an interaction that is difficult to calculate in one theory becomes easy in the other [8]. As an example of mirror symmetry consider the 2-torus defined as $T^2 = \frac{\mathbb{R}^2}{\Gamma}$, where Γ is some two dimensional lattice on $\mathbb{R}^2$. The latice is generated by two basis vectors e_1 and e_2 and we define a metric $G_{ij} = e_i \cdot e_j$ and an antisymmetric tensor $B_{ij} = b\epsilon_{ij}$. The metric has three independent real components and the tensor has 1 independent component, giving four real moduli for strings compactified on the 2-torus. We define the complex structure modulus as follows:

$$\sigma = \frac{|e_1|}{|e_2|} e^{e\phi}$$

where ϕ is the angle between e_1 and e_2. And the Kähler modulus:

$$\tau = 2(b + i\sqrt{det(G)})$$

If we now consider a type IIA theory compactified in two dimensions (on a 2-torus) with compact directions x^8 and x^9. We act with T-duality on the 9-direction. This flips the sign of X_9^μ and also that of ψ^9. T-duality has the effect that the type IIA theory has turned in the type IIB theory. However, by doing the T-duality we interchanged the Kähler modulus ρ with the complex structure modulus τ. This can be seen by taking a look at the metric in the compact dimensions. The metric is invariant if we interchange the Kähler modulus with the complex structure modulus and apply T-duality. This does not change the 2-torus it only changes the values of the moduli.

This result also holds for string theories compactified on Calabi-Yau manifolds. A type IIA theory compactified on a Calabi-Yau manifold is dual to a type IIB theory compactified on another manifold. This is what is called mirror symmetry. The explicit construction is a bit more difficult then the 2-torus example. The difference between between the Calabi-Yau example and the 2-torus example is that the Calabi-Yau does change when applying mirror symmetry. When applying mirror symmetry we have to interchange the two Hodge numbers $h^{2,1}, h^{1,1}$ (we will define them in the next section). Mirror symmetry states that it is always possible to find two Calabi-Yau manifolds with opposite Hodge numbers, these manifolds are called mirror pairs. It is because of this that we suspect that there are manifolds missing in the classification made in[1]. We will start the next section with a mathematical introduction to Calabi-Yau manifolds, Hodge numbers and some results that will help us construct Calabi-Yau manifolds in section 4.

3 Mathematical preliminaries

In the previous section we discussed string theory and Calabi-Yau manifolds. In this section we will give a formal definition of a Calabi-Yau manifold. We will treat De Rham and Dolbeault cohomology, Hodge numbers in the case of Calabi 3-folds, projective spaces, weighted projective spaces and finally we will give some results from the literature that will help us with the construction of Calabi-Yau manifolds in weighted projective space.

3.1 Calabi-Yau manifolds

We will generally be interested in a 3 complex dimensional manifold because of the number of dimensions in superstring theory. A lot of the following is more general though. We will need to choose a definition to define the Calabi-Yau manifolds. There are several (equivalent) definitions of a Calabi-Yau manifold, we will use the one that is best suited for our project, since it is the one used in most related literature. We have already hinted towards the following definition:

Definition 1. *A Calabi-Yau manifold is a compact Kähler manifold with vanishing first Chern class.*

A compact manifold M is a manifold for which every open covering consists of a finite number of open sets. The definition of Calabi-Yau manifold is quite general and as a result there are a lot of them. However, as we already remarked in the introduction, we will only be interested in Calabi-Yau manifolds in $\mathbb{WP}^4$. This definition implies that we will be discussing complex manifolds. A complex manifold of dimension n is a manifold M that at each point on M is isomorphic to a neighborhood of the origin in $\mathbb{C}^n$ where the patch functions are holomorphic functions. The charts of a complex manifold $\mathbb{C}^n \to M$ can also be viewed as charts $\mathbb{R}^{2n} \to M$. This is why sometimes we find that the dimension of a complex manifold is half its real dimension. Some examples of a complex manifold include $\mathbb{C}^n$ and the complex analog of a torus $\mathbb{C}^n/\Gamma$, where Γ is some lattice of $\mathbb{C}^n$

3.1.1 Kähler manifolds

Kähler manifolds are special forms of Hermitian manifolds which are complex manifolds. So, it is necessary to introduce Hermitian manifolds before Kähler manifolds. A Hermitian manifold is the complex analog of a Riemannian manifold [15].

Definition 2. *A Hermitian manifold is a complex manifold equipped with a smooth varying Hermitian inner product on each of its tangent spaces.*

Just like it is possible to make any real manifold a Riemannian manifold by equipping it with an inner product, it is also possible to make any complex manifold a Hermitian manifold by equipping it with a Hermitian inner product.

An inner product on the tangent space of a complex manifold is given in local coordinates by:

$$ds^2 = \sum_{i,j} h_{ij}(z) dz_i \otimes \overline{dz_j}$$

ds^2 is Hermitian if h_{ij} smooth and $h_{ij}(z) = \overline{h_{ji}(z)}$[15]. The real part on the Hermitian metric ds^2 induces a Riemannian metric and the imaginary party of ds^2 defines a differential (1,1)-form $\Omega = -\frac{1}{2} Im(ds^2)$ called the associated form. It can be shown that the associated form can be used to define the Hermitian metric, effectivily going the other way as we just went. The associated form will have the following form:

$$\Omega = \frac{i}{2} \sum_{i,j} h_{ij}(z) dz_i \wedge \overline{dz_j}$$

This can be helpful in determining whether a given manifold is Kähler or not, since it could be easier to find the associated form than it is to find the Hermitian metric. We now have enough information to give the definition of a Kähler manifold[15]:

Definition 3. *A complex manifold M is called a Kähler manifold if it possesses a Kähler metric, which is a Hermitian metric ds^2 such that the associated (1,1)-form Ω is closed: $d\Omega = 0$.*

Both the complex space $\mathbb{C}^n$ and the quotient space $\mathbb{C}^n/\Gamma$ are examples of Kähler manifolds. In addition any complex submanifold of a Kähler manifold is also a Kähler manifold. Another example of a Kähler manifold is the complex projective space $\mathbb{P}^n$, we will show this in some detail later. We noted that every complex manifold can be equipped with a Hermitian metric. It is however not true that every compact complex manifold can be equipped with a Kähler metric, making Kähler manifolds a proper subset of the complex manifolds.

3.1.2 Chern classes

Now that we have taken care of half of the definition of a Calabi-Yau manifold it is time to look at the other half, Chern classes. At the end of this section we will give a condition for the vanishing of the first Chern class that is very easy to use. We will try to give a short introduction to Chern classes here.

Chern classes are topological invariants that are defined over vector bundles of a manifold. A real vector bundle V of rank k over a smooth manifold M is a smoothly varying family of k-dimensional vector spaces. Meaning that at every point on the manifold we have a vector space such that the vector space varies smoothly when we walk over the manifold. This can be made a bit more formal by giving a real vector bundle as a triple (M,V, π) where M and V are smooth manifolds and

$$\pi : V \to M$$

is a smooth map. For each $m \in M$, the fiber $V_m \equiv \pi^{-1}(m)$ of V over m is a real k-dimensional vector space. The vector space structures varies smoothly with m. The spaces M and V are called the base and the total space of the vector bundle (M,V, π). It is customary to call $\pi : V \to M$ a vector bundle and V a vector bundle over M. For completeness we give the definition of a vector bundle[11]:

Definition 4. *A real vector bundle of rank k is a tuple (M,V, π, $\cdot$, $+$)*

1. *M and V are smooth manifolds and $\pi : V \to M$ is a smooth map*
2. *$\cdot$: $\mathbb{R} \times V \to V$ is a map such that $\pi(c \cdot v) = \pi(v)$ for all $(c, v) \in \mathbb{R} \times V$*
3. *$+$: $V \times_M V \to V$ is a map such that $\pi(v_1 + v_2) = \pi(v_1) = \pi(v_2)$ for all $(v_1, v_2) \in V \times_M V$*
4. *For every $m \in M$ there exists a neighborhood U of m in M and a diffeomorphism $h : V_{|U} \to U \times \mathbb{R}^k$ such that*
    - *$\pi_1 \circ h = \pi$ and*
    - *the map $h_{|v_x} : V - x \to x \times \mathbb{R}^k$ is an isomorphism of vector spaces for all $x \in U$*

A complex vector bundle is defined in a similar way replacing all $\mathbb{R}$ with $\mathbb{C}$. Given a base manifold M, how do we know if two vector bundles V and V' over M are isomorphic to each other? Are there any topological invariants asociated with vector bundles that give us information about the structure of the bundle? Chern classes provide an algebraic quantity that gives a partial answer to the question whether two different vector bundles over the same base are isomorphic [5]. We can have a look at the forms defined on V. On every manifold there is a well defined curvature form that gives us just the information we need. We give the following definition of the Chern class [13]:

Definition 5. *Given a vector bundle V over a complex manifold M and a closed curvature (1,1)-form Θ on V, we define the total Chern class:*

$$\begin{aligned} c(\Theta) &\equiv det[I + \frac{i}{2\pi}\Theta] \\ &= I + c_1(\Theta) + c_2(\Theta) + ... \end{aligned} \tag{8}$$

c_i is the i^{th} Chern class

All c_i are closed $(i, i)-$forms and give homology groups. Chern classes are atleast indepedent of parametrization because the determinant is independent under base changes. Vanishing of the first Chern class means that first form in the expansion of equation (8) is equal to zero, $c_1 = 0$. Calculating the Chern class from (8) can be quite hard. At the end of this section we will give a result that will greatly simplify the question whether or not a given hypersurface has vanishing first Chern class. We will now turn our attention towards the Hodge numbers and Euler characteristic.

3.2 Cohomology

In this subsection we will introduce cohomology. This will be necessary so that we can introduce two topological invariants in the next section, namely Hodge numbers and the Euler characteristic. Topological invariants are a valuable tool in the study of surfaces as they hold a lot of information about the surface. We will first look at the De Rham cohomology because is the real analog of the Dolbeault cohomology in which we are ultimatly more interested.

3.2.1 De Rham cohomology

Consider all r-forms on some smooth space M. An r-form ω is called closed if $d\omega = 0$ and an r-form ω is exact if it can be written as the exterior derivative of a (r-1)-form so that $\omega = d\alpha$. Note that all exact forms are closed because $d^2 = 0$. The De Rham cohomology groups are defined as the quotient spaces of closed forms modulo exact forms. Elements of $H^r_{DR}(M)$ are classes of r-forms on M such that[13]:

$$\begin{aligned} d\omega &= 0 \\ \omega \cong \omega' &= \omega + d\alpha \end{aligned}$$

The dimension of H^r_{DR} is the r^{th} Betti number. The Dolbeault cohomology we consider will be defined in a similar way, but we have to deal with the complex nature of the forms first.

3.2.2 Dolbeault cohomology

When we deal with complex manifolds we can split the forms on the manifold in a complex and a real part so that we get a (p,q)-form. Where the p-form is the real part of the form and the q-form is the imaginairy part of the form. Similar we can split the exterior derivative d into ∂ and $\overline{\partial}$ such that:

$$d = \partial + \overline{\partial}$$

∂ acts on the real part of the form taking a (p,q)-form to a (p+1,q)-form and $\overline{\partial}$ acts on the imaginairy part of the form taking a (p,q)-form to a (p,q+1)-form. Both ∂^2 and $\overline{\partial}^2$ are equal to 0. This means that we can again consider closed and exact forms when discussing ∂ and $\overline{\partial}$. This means that we can define two analogs to the De Rham cohomology, one for ∂ and one for $\overline{\partial}$. The Dolbeault cohomology $H^{p,q}_{\overline{\partial}}$ is the quotient space of the $\overline{\partial}$ closed (p,q)-forms modulo the $\overline{\partial}$ exact (p,q)-forms. So, the Dolbeault cohomology is defined as:

$$H^{p,q}_{\overline{\partial}} = \frac{Ker^{p,q}\overline{\partial}}{Im\overline{\partial}^{p,q}}$$

We said that we can also define a similar quotient space based on ∂. However, it can be shown that this new cohomology is equivalent to the Dolbeault cohomology when the forms are defined on a Kähler manifold, so that we would gain no new information. Now that we have introduced the Dolbeault cohomology, it is easy to introduce Hodge numbers. Hodge numbers are defined as the dimension of the Dolbeault cohomology groups $h_{p,q} = dim(H^{p,q})$. Just as the Dolbeault cohomology is the complex analog of the De Rham cohomology, the Hodge numbers are the complex analog of the Betti numbers. The Betti number is related to the Hodge numbers as follows[13]:

$$b^r(M) = \sum_{p=0}^{r} h^{p,r-p}(M) \tag{9}$$

We will now look at the Hodge numbers in case of a 3 complex dimensional Calabi-Yau manifold.

3.3 Hodge diamond and the Euler characteristic

The Hodge numbers are sometimes organized in a structure called the Hodge diamond. In this section we will treat the general shape of the Hodge diamond for Calabi-Yau 3-folds and we will introduce the Euler characterstic. The Euler characteristic will be the only topological invariant used to analyze the Calabi-Yau manifolds in this study.

3.3.1 Hodge diamond for Calabi-Yau 3-folds

We consider the Hodge diamond of a Calabi-Yau 3-fold. There is a lot of structure in the Hodge diamond of a Calabi-Yau 3-fold, due to its properties. Ultimatly this will mean that the number of independent Hodge numbers for a Calabi-Yau 3-fold will be reduced to two[13]. We will give the necessary results here.

We will only consider manifolds that consist of a single connected piece and hence $b_{3,3} = 1$. In addition $b^{p,q} = 0$ if $p + q \geq 7$ when working on a complex 3-fold. The Hodge star ($\star$) is an operator that sends p-forms to (n-p)-forms on an n-dimensional differentiable manifold.

Definition 6. *Let α be an r-form $\alpha = a_{i_1,...,i_r} dx^{i_1} \wedge ... \wedge dx^{i_r}$, then the Hodge star is defined as:*

$$\begin{aligned} \star\alpha &= a^*_{j_1,...,j_{n-r}} dx^{j_1} \wedge ... \wedge dx^{j_{n-r}} \\ a^*_{j_1,...,j_{n-r}} &= \epsilon^{i_1,...,i_r}_{j_1,...,j_{n-r}} a_{i_1,...,i_r} \end{aligned}$$

The Hodge star preserves closed and exact forms, i.e. if α is closed/exact then $\star\alpha$ is closed/exact. The Hodge star leads to following relation on the De Rham cohomology:

$$H^r_{DR} \cong H^{n-r}_{DR}$$

When acting on a complex form, the Hodge star takes $(p,q)-$forms to $(n-p, n-q)-$forms. Together with the previous relation this leads to $H^{p,q} \cong H^{n-q,n-p}$ and thus:

$$h_{p,q} = h_{n-p,n-q}$$

			1			
		$b_{1,0}$		$b_{1,0}$		
	$b_{1,0}$		$b_{1,1}$		$b_{1,0}$	
1		$b_{2,1}$		$b_{2,1}$		1
	$b_{1,0}$		$b_{1,1}$		$b_{1,0}$	
		$b_{1,0}$		$b_{1,0}$		
			1			

Table 1: The Hodge diamond reduced to three independent numbers

For a compact Kähler manifold we have the relation $H^{p,q}(M) = \overline{H^{q,p}(M)}$. Such that complex conjugation of (p,q)-forms implies:

$$h_{p,q} = h_{q,p}$$

Every Calabi-Yau manifold has a unique nowhere vanishing (3,0)-form Ω (so $b^{3,0} = 1$). It can be shown that this leads to the following relation.

$$h^{p,0} = h^{3-p,0}$$

To summerize we have the following Hodge diamond (Table 1). We now need one more result to determine $b_{1,0}$. We use the following theorem from [13]:

Theorem 1. $b_1 = 0$ *on a manifold with a Ricci flat metric*

A Calabi-Yau manifold has a Ricci flat metric. So in particular for any Calabi-Yau manifold $b_1 = 0$. Using equation 9 we find that $b_{1,0} = b_{0,1} = 0$. The Hodge diamond for a Calabi-Yau 3-fold is displayed in Table: 2. We find that the Hodge diamond of any Calabi-Yau manifold has the same basic form. The outer rim of Hodge numbers is completely determined: 1's in the corners and 0's elsewhere. The inside Hodge numbers follow the above relations, so there also is a lot of structure there. The higher the dimension, the more free Hodge numbers. A special case is K3, which is the only CY in 2 complex dimensions besides the torus, it has $b_{1,1} = 20$.

3.3.2 Euler characteristic

Once we have obtained a list of Calabi-Yau manifolds the main topological invariant that we are interested in is the Euler characteristic. This is because

			1			
		0		0		
	0		$b_{1,1}$		0	
1		$b_{2,1}$		$b_{2,1}$		1
	0		$b_{1,1}$		0	
		0		0		
			1			

Table 2: Hodge diamond of a Calabi-Yau 3-fold

of the mirror symmetry we discussed in the previous section. Finding two manifolds with opposite Euler characteristic does not make them mirror pairs so we will not be able to claim to have found a mirror pair at the end of the thesis. We will define the Euler characterstic as follows:

Definition 7. *The Euler characteristic (χ) of a complex manifold M is:*

$$\chi(M) = \sum_{r=0}^{dim(M)} (-1)^r b^r(M)$$

where b^r is the r^{th} betti number.

Using equation 9 and the form of the Hodge diamond for a Calabi-Yau 3-fold, we find that for a Calabi-Yau 3-fold:

$$\chi(M) = 2(b_{2,1} - b_{1,1})$$

We have a similar problem as we did in case of the Chern class, we are not going to calculate the Hodge numbers so we can't use the formula directly. However, like for the Chern class, there are several results in the literature that links the Euler characteristic directly to the space we work in. We will give these results later in this section after we have introduced the (weighted) projective space. First we will link the previous discussion to the discussion on mirror symmetry from the previous section.

We are now able to make the last statement of the previous section more precise, i.e. what we mean with a mirror pair of a Calabi-Yau manifold. The Calabi-Yau manifolds that are mirror pairs are related through their Dolbeault cohomology. In particular:

$$\begin{aligned} H^{2,1}(W) &\cong H^{1,1}(M) \\ H^{1,1}(W) &\cong H^{2,1}(M) \end{aligned}$$

For a mirror pair (M,W) of Calabi-Yau 3-folds. This implies that:

$$\chi_M = -\chi_W$$

We will be particularly interested in any two Calabi-Yau manifolds with opposite Euler characterstic. We already remarked that this does not mean that these two manifolds are a mirror pair, for that we need the Hodge numbers of these manifolds. Even so two manifolds can only be called a mirror pair if they are related through some symmetry (mirror symmetry) of a theory that is compactified on them.

3.4 Projective spaces

In the next section we will introduce weighted projective spaces.Bbefore we do that we will first have a look at ordinary projective spaces and Calabi-Yau manifolds in them.

Definition 8. *A projective space, $\mathbb{P}^n$ is a space that allows, for all constants λ, the following identification on its (complex) coordinates:*

$$(z_1, ..., z_n) \cong \lambda(z_1, ..., z_n)$$

We have already remarked that the projective space itself is a Kähler manifold and it is because of this that Calabi-Yau manifolds are easily constructed in $\mathbb{P}^n$. We will show that $\mathbb{P}^n$ is Kähler by considering the following example which is taken from[15]:

Example 1. *Consider $\mathbb{P}^n$ with homogeneous coördinates $(u_0, ..., u_n)$. Let $z_j = \frac{u_j}{u_0}$ and $u_0 \neq 0$, we consider the following (1,1)-form:*

$$\begin{aligned} \Omega &= \sum_{r,s} \omega_{r,s} dz_r \wedge d\overline{z}_s \\ \omega_{r,s} &= \frac{\partial^2 log(1 + \sum |z_j|^2)}{\partial z_i \overline{\partial} z_s} \end{aligned}$$

$\omega_{r,s}$ is a hermitian matrix and Ω is closed. So this (1,1)-form is the associated form to a Kähler metric on $\mathbb{P}^n$, the Fubini-Study metric[15].

Furthermore, projective spaces are also compact. This implies that any analytical submanifold of $\mathbb{P}$ is also a compact Kähler manifold. By analytical submanifold we mean a manifold M, such that any point of M has a neighborhood that is covered by a finite number of patches that are defined by analytical functions.

3.4.1 Calabi-Yau manifolds in $\mathbb{P}^n$

We will now construct a Calabi-Yau manifold in $\mathbb{P}^n$. We will follow Candelas et al.[3], this means that we consider complete intersection manifolds. That is manifolds that are defined by N polynomials $p_1, ...p_N$ that have a nowhere vanishing N-form: $\Theta = dp_1 \wedge ... \wedge dp_N$. The interesting question is if there are submanifolds of $\mathbb{P}^n$ that have vanishing first Chern class. Candelas uses a theorem that states that the first Chern class of M vanishes if and only if M admits a globally and nowhere vanishing holomorphic three form.

$$\Omega = \frac{1}{3!} \Omega_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho$$

Such that the only non zero part of Ω is (the holomorphic) $\Omega_{\mu\nu\rho}$. After an explicit calculation, they find the condition that Ω is defined globally on M is the same as:

$$\sum_{i=1}^{N} deg(p^i) = N + 4$$

which is then the condition that is used for vanishing first chern class of M. This leads to only five Calabi-Yau manifolds for $N = 1$ (table: 3). When there is more

Space	Degree of p	Euler characterstic
$\mathbb{P}^4$	5	-200
$\mathbb{P}^5$	3,3	-144
$\mathbb{P}^5$	2,4	-176
$\mathbb{P}^6$	2,2,3	-144
$\mathbb{P}^7$	2,2,2,2	-128

Table 3: Calabi-Yau manifolds in $\mathbb{P}^n$

than one degree given it means that the manifold is the complete intersection of that number of polynomials of given degree.
This result is obtained under the assumption that we only use one projective space. If we allow for the product of projective spaces the number of Calabi-Yau manifolds quickly grows. Candelas extends the list that is given in table (3) to atleast a hundred topologically different Calabi-Yau manifolds. It is a remarkeble thing that all of these manifolds have negative Euler characteristic. This is one of the main reasons to want to look beyond this class of Calabi-Yau manifolds. Since it will be impossible to find mirror pairs if we only have manifolds with negative Euler characteristic.

3.5 Weighted projective spaces

It is a natural step after projective spaces to look at weighted projective spaces. These are the spaces that we will work in. Projective spaces may be generalized to weighted projective spaces as follows:

Definition 9. *A space $\mathbb{WP}^n$ is called a weighted projective space with global homogeneous coordinates $z_0, ..., z_1$ if these coordinates allow the following identification:*

$$z_i \cong \lambda^{k_i} z_i$$

for all λ. The constants k_i are called the weights of the space.

It is clear that if all the weights $k_1, ...k_n$ are divisible by some integer such that $k'_i = mk_i$ for all i then $\mathbb{WP}^n_{k'} \cong \mathbb{WP}^n_k$. However, there is an ever more powerful result which is taken from [13]:

Theorem 2. *let $k = (k_1, ..., k_n)$ and*

$$\begin{aligned} d_i &= gcd(k_0, ..., k_{i-1}, k_{i+1}, ..., k_n) \\ m_i &= lcm(d_0, ..., d_{i-1}, d_{i+1}, ..., d_n) \end{aligned}$$

then $\mathbb{WP}^n_k \cong \mathbb{WP}^n_{k'}$ where $k'_i = \frac{k_i}{m_i}$.

In light of the theorem we give the following definition:

Definition 10. *A weighted projective space is called well-formed if each n out of n+1 weights are coprime.*

In light of this it should be clear that we only need to consider well-formed $\mathbb{WP}^n$. We will look at one more aspect of $\mathbb{WP}^n$, before we go to the question of how to define Calabi-Yau manifolds in $\mathbb{WP}^n$.

3.5.1 Singularities

Weighted projective spaces have singularities, this is an important difference between projective spaces and weighted projective spaces. These singularities are due to the weights of the space. To see this consider the following example:

Example 2. *Consider the space* $\mathbb{WP}^4$ *with weights: (1,1,2,3) such that:*

$$(z_0, z_1, z_2, z_3) \cong (\lambda z_0, \lambda z_1, \lambda^2 z_2, \lambda^3 z_3)$$

Consider now $\lambda = -1$ *in the neighborhood of* $(0, 0, 1, 0)$*, this gives:*

$$(z_0, z_1, 1, z_3) \cong (-z_0, -z_1, 1, -z_3)$$

Such that there is a $\mathbb{Z}_2$ *identification on the space corresponding to a cyclic quotient singularity. Similarly, there is a singularity due to* $k_4 = 3$*.*

In fact, we have in general that the weighted projective space is just a quotient of a projective space:

$$\mathbb{WP}^n = \frac{\mathbb{P}^n}{\mathbb{Z}_{k_1} \otimes ... \otimes \mathbb{Z}_{k_n}}$$

A generic surface in $\mathbb{WP}^n$ will intersect these singularities and hence isn't a manifold but an orbifold. We can desingularize these orbifolds if the singularities aren't too bad. [13] gives the following result:

Theorem 3. *The singular set of a well-formed* $\mathbb{WP}^n$ *has dimension at most n-2 and consists of cyclic quotient singularities.*

This result is treated in more detail in [1] for the case n=4. We will not be looking at the actual desingularization proces in this thesis. All results apply to the manifolds that are obtained after proper desingularization. Furthermore, we will frequently call the hypersurface in $\mathbb{WP}^n$ a manifold rather then orbifold. This should not cause confusion. The reason why we were interested in projective spaces was because they are Kähler. We have shown this by giving an associated form on $\mathbb{P}^n$. We can extend this result to weighted projective space because any weighted projective space can be embedded in a large enough $\mathbb{P}^m$, such that any $\mathbb{WP}^n$ is also Kähler. Implying once again that any analytical submanifold is compact Kähler.

3.6 Calabi-Yau manifolds in $\mathbb{WP}^n$

We will define hypersurfaces in $\mathbb{WP}^n$ by using transverse polynomials. We define a transverse polynomial as follows[13]:

Definition 11. *A polynomial p is transverse if* $\nabla p = 0$ *only at the origin*

Furthermore we need the polynomials to be quasihomogeneous due to the nature of the weighted projective space. A quasihomogeneous polynomial is defined as:

Definition 12. *A polynomial is called quasihomogeneous of degree d if the following relation holds:*

$$p(\lambda^{k_0} z_0, ..., \lambda^{k_n} z_n) = \lambda^d p(z_0, ..., z_n) \tag{10}$$

This means that for a given set of weights not any polynomial is quasihomogeneous, as can be seen in the following example:

Example 3. *We consider the space $\mathbb{WP}^1_{2,3}$ and the polynomial:*

$$\begin{aligned} p &= x^2 y + y^2 \\ &= (\lambda^2 x)^2 (\lambda^3 y) + (\lambda^3 y)^2 \\ &= \lambda^{2*2+3*1} x^2 y + \lambda^{3*2} y^2 \\ &\neq \lambda^d p \end{aligned}$$

Clearly there is no degree d to satisfy the relations and hence this polynomial is not quasihomogeneous.

We call the set of all quasihomogeneous polynomials with respect to a set of weights and a certain degree a configuration:

$$\mathbb{WP}_{k_i}[D]$$

A configuration is called transverse if atleast one of the polynomials in the configuration is transverse. In the next section we will consider ways to construct transverse quasihomogeneous polynomials. A necessary condition for a configuration to have a transverse member is that the following expression is a polynomial [2]:

$$P(t) = \prod_i \frac{1 - t^{D-k_i}}{1 - t^{k_i}} \tag{11}$$

This is the Poincaré polynomial. This result is used in[4] to give a classification on configurations. We will not use it, but it should be mentioned because of this classification. We know that any manifold in $\mathbb{WP}^n$ is a compact Kähler manifold. The question is how do we know whether a given manifold has vanishing first Chern class. We already noted that we would not be calculating the first Chern class from(8). We will use the following result that is obtained in [1] and also in [13]:

Theorem 4. *A manifold after desingularization from an orbifold in $\mathbb{WP}^n$ has vanishing first Chern class if*

$$d = \sum_j k_j \tag{12}$$

where d is the degree of the quasihomogeneous polynomial and k_j for $j = 0, .., n$ are the weights of $\mathbb{WP}^n$.

This result completes the relations between the degree and powers of the polynomial and the weights of the space together with (10). The question is whether there are transverse, quasihomogeneous polynomials in $\mathbb{WP}^n$ that satisfy these relations. We will begin the next section with a discussion of the classification of these polynomials in $\mathbb{WP}^1$ and $\mathbb{WP}^2$ a two and three dimensional space respectively. We are concerned with the Euler characteristic of hypersurfaces in $\mathbb{WP}^4$. We have treated cohomology groups and Hodge numbers with the goal of ultimatly defining the Euler characteristic, the downside there was that we defined the Euler characteristic using the Hodge numbers which we will not calculate. We will use a result by Vafa who used a certain type of models compactified on hypersurfaces to obtain a formula for the Euler characterstic [9], which is modified for the case at hand in [4]:

$$\chi = \frac{1}{d} \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} \prod_{(l \cap k) q_i \in \mathbb{Z}} \frac{q_i - 1}{q_i} \tag{13}$$

where $q_i = \frac{k_i}{d}$. Interestingly enough, in the paper by Candelas [1], which motivated this study, Candelas gives another formula for the Euler characteristic giving the paper by Vafa as reference. His formula is:

$$\chi = -\frac{1}{d} \sum_{l,r=0}^{d-1} \prod_{(l \cap k) q_i \in \mathbb{Z}} \frac{d - k_i}{k_i} + \sum_{(l \cap k) q_i \notin \mathbb{Z}} \frac{1}{d} \tag{14}$$

which just seems to be wrong. Not only does it not agree with equation (13), it also seems to be only negative due to the fact that $d \geq k_i$. When calculating Euler numbers it also appears to be unable to give an integer answer. It should be noted that the formula was assumed to have brackets from the first sum to the end of the formula. When we calculated several Euler characteristics from known configurations in [1], all agreed to the given results when we used (13). Whereas when we use (14), we find no sensibel answer. So we will use (13) for the calculation of the Euler characteristic. We will mostly be interested in Calabi-Yau manifolds of complex dimension 3. This means that we will be working in $\mathbb{WP}^{\ntrianglerighteq}$ because we loose one degree of freedom due to the embedding and one degree of freedom due to the weights. It should be clear that we need a classification of transverse polynomials in $\mathbb{WP}^4$. We will discuss this in the next section.

4 Classification and results

The goal of this study is to find all Calabi-Yau 3-folds in a weighted projective space and see if we can find a manifold with an Euler characteristic that is not listen in[1]. The discussion in the previous subsection concluded with the statement that we need to find all transverse quasihomogeneous polynomials in such a space. These polynomials are then used to define the Calabi-Yau manifolds we are after. First we will look at a classification of transverse quasihomogenous polynomials in 2 and 3 variables and we will introduce a method to find transverse polynomials using endomorphisms. We will use this method to extend the list of polynomials from [1] and show some examples of Calabi-Yau manifolds that we constructed. The last part of this section we will be discussing the papers by Kreuzer and Skarke [4] and [2]. A complete classification off all orbifolds in weighted projective space was constructed in these papers.

4.1 Transverse polynomials in 2 and 3 variables

This classification was done by Arnold [12] and we will give an overview of his results here. Two variable case is very easy as can be seen in the next theorem:

Theorem 5. *Any transverse quasihomogeneous function of two variables of corank 2 contains with nonzero coefficients the following polynomials:*

$$\begin{aligned} p_1 &= z_1^a + z_2^b \\ p_2 &= z_1^a + z_2^b z_1 \\ p_3 &= z_1^a z_2 + z_2^b \\ p_4 &= z_1^a z_2 + z_2^b z_1 \end{aligned}$$

The second and third polynomial in this theorem are clearly just the same under changing z_1 and z_2. For the proof of the theorem we invite the reader to read chapter 13 of the book by Arnold. We have (10) the connection between the weights of $\mathbb{WP}^n$ and the degree D of the polynomial. From this we can find conditions on the weights of the space $\mathbb{WP}^1_{k_1,k_2}$. For the second polynomial in the theorem we have for instance:

$$\begin{aligned} D &= k_0 a \\ &= k_1 b + k_0 \end{aligned}$$

From this it is clear that the weights have to satisfy the relation:

$$\begin{aligned} k_0 &= \frac{D}{a} \\ k_1 &= D\frac{a-1}{ab} \end{aligned}$$

Clearly, it is possible to do this for all of the polynomials above. A similar list has been construced for the 3 variable case. It results in a list of seven sets of

#	Monomials	k_1, k_2, k_3
1	z_1^a, z_2^b, z_3^c	$\frac{D}{a}$, $\frac{D}{b}$, $\frac{D}{c}$
2	z_1^a, z_2^b, $z_3^c z_2$	$\frac{D}{a}$, $\frac{D}{b}$, $\frac{D(b-1)}{bc}$
3	z_1^a, $z_2^b z_1$, $z_3^c z_1$	$\frac{D}{a}$, $\frac{D(a-1)}{ab}$, $\frac{D(a-1)}{ac}$
4	z_1^a, $z_2^b z_3$, $z_3^c z_2$	$\frac{D}{a}$, $\frac{D(c-1)}{bc-1}$, $\frac{D(b-1)}{bc-1}$
5	z_1^a, $z_2^b z_3$, $z_3^c z_1$	$\frac{D}{a}$, $\frac{D(ac-a+1)}{abc}$, $\frac{D(a-1)}{ac}$
6	$z_1^a z_2$, $z_2^b z_1$, $z_3^c z_1$	$\frac{D(b-1)}{ab-1}$, $\frac{D(a-1)}{ab-1}$, $\frac{Db(a-1)}{c(ab-1)}$
7	$z_1^a z_2$, $z_2^b z_3$, $z_3^c z_1$	$\frac{D(bc-c+1)}{abc+1}$, $\frac{D(ac-a+1)}{abc+1}$, $\frac{D(ab-b+1)}{(abc+1)}$

Table 4: A list of all the seven monomials in 3 variables

monomials. This list is given in Table (4.1), the relation between the degree of the monomial D and the weights is also given. The proof that these are indeed all monomials that are needed to construct all transverse quasihomogeneous polynomials eventually comes down to a classification of endomorphisms from a set of 3 points to itself (or indeed for n variables to a classification of the endomorphisms off n points). There are $3^3 = 27$ endomorphisms from a set of 3 points to itself and only 7 distinct set of monomials. This difference is due to the fact that we only need to consider endomorphisms modulo renaming the points. It is convenient to draw he endomorphisms schemetically using pictures containing arrows and circles4.1. Any graphical representation of an

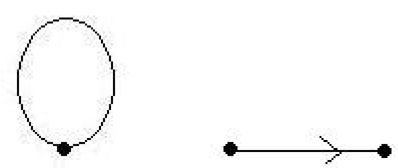


Figure 4: Schematic representation of endomorphisms. The circle means sending a point to itself and an arrow means sending a point to another point

endomorphism is build from these two basic components. A circle corresponds to a term x^a and a line (from x to y) corresponds to $x^a y$. We introduce the following nomenclature. We will call a variable x a root of a polynomial p if p contains a term x^a. A monomial $x^a y$ is called a pointer and a link between two expressions is a monomial that only depends on the variables occuring in these expressions. A link is not necessarily a pointer, $x^a y^b$. We shall call a graph of an endomorphism without links a skeleton graph. The following two results are going to be of great importance to us:

Theorem 6. *If f(z) is a transverse polynomial for some set of variables z and g(y) is also a transverse polynomial for a set of variables y such that $z \neq y$ then f(z)+g(y) is also transverse.*

Theorem 7. *Any transverse quasihomogenous polynomial in 3 variables contains a set of monomials that correspond to an endomorphism of 3 points*

The first result is very straightforward but we will use it quite a lot in the comming sections. The second result implies that we need to check each linear combination of monomials for transversality and in fact if we check table (4.1) we find that both a linear sum of monomials from class 3 and from class 6 is not transverse. We consider the following example using a polynomial based on class 3:

Example 4.

$$p = z_1^a + z_2^b z_1 + z_3^c z_1$$

To check whether this polynomial is transverse or not we have to take a look at the gradient:

$$\nabla p = (a z_1^{a-1} + z_2^b + z_3^c, b z_2^{b-1} z_1, c z_3^{c-1} z_1)$$

Setting $\nabla p = 0$ *and taking* $z_1 = 0$ *we find a relation between* z_2 *and* z_3*, this relation defines a curve. Clearly, on this curve we have both p and* ∇p *equal to zero so this polynomial is not transverse. We need to add an extra link to this polynomial to make sure that it is transverse. In fact, adding a monomial of this form* $z_2^m z_3^n$ *does the trick. We now have the following polynomial:*

$$p = z_1^a + z_2^b z_1 + z_3^c z_1 + z_2^m z_3^n$$

If we look at the gradient of p now we find:

$$\nabla p = (a z_1^{a-1} + z_2^b + z_3^c, b z_2^{b-1} z_1 + m z_2^{m-1} z_3^n, c z_3^{c-1} z_1 + n z_2^p z_3^{n-1})$$

This polynomial is transverse as can be seen by setting the gradient equal to zero and working out the consequences.

In conclusion we see that a skeleton graph need not be transverse but can be made transverse by adding extra links. The only question that remains is whether there are numbers m and n that satisfy the relation $D = mk_2 + nk_3$ such that the polynomial is quasihomogeneous. Adding extra links makes the system of equations that we need to solve bigger and thus adding more constraints to it. We quote the following theorem from [12] to illustrate this:

Theorem 8.

1. *A transverse quasihomogeneous polynomial of class 3 exists if and only if the least common multiple of the numbers b and c is divisible by* $a - 1$*.*

2. *A transverse quasihomogeneous polynomial of class 6 exists if and only if* $(b-1)c$ *is divisible by the product of* $a-1$ *and the greatest common diviser of the numbers b and c.*

3. *Transverse quasihomogenous polynomials of the remaining five classes exist for all a,b,c that satisfy the degree relations.*

The proof of this theorem is just computing the consequences of the example above for all linear combinations of monomials from table (4.1). While this is straightforward it can be quite difficult to see the correct solution in some individual cases. This will be especially true for the cases where there are 5 variables present, where we also need to add extra links. We will prefer to just simply put the condition on the extra monomials as an extra constraint in the computer and let it calculate the solutions for us. We add one final example to this section, where we actually give a solution to the set of equations:

Example 5. *Consider the linear combination of monomials of type 6:*

$$p = z_1^a z_2 + z_2^b z_3 + z_3^c z_1$$

We already remarked that this polynomial is not transverse. Calculating the gradiant we find that we need to add the following monomial: $z_1^m z_2^n$ *to make the polynomial transverse. We find the following set of constraints:*

$$\begin{aligned} D &= ak_1 + k_2 \\ &= k_1 + bk_2 \\ &= k_1 + ck_3 \\ &= mk_1 + nk_2 \end{aligned} \tag{15}$$

A solution to this system of equations is not at all obvious, however we do find the following solution using a computer:

D	a	b	c	m	n	k_1	k_2	k_3
14	5	3	4	3	2	2	4	3

Table 5: Solution to equation (15)

Space	Polynomial
$\mathbb{WP}^2_{2,4,3}[14]$	$z_1^5 z_2 + z_2^3 z_3 + z_3^4 z_1 + z_1^3 z_2^2$

From this example it should be clear that a solution always consists of a polynomial and a corresponding weighted projective space. It is very likely that a given polynomial is transverse in one space but not in another.

4.2 Calabi-Yau manifolds in $\mathbb{WP}^4$

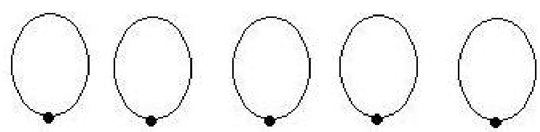
The classification of transverse quasihomogeneous polynomials in 3 variables relied on the classification of endomorphisms. The result is more general however and holds for n variables. In the case of 5 variables we need to classify endomorphisms on 5 points. These endomorphisms give us a minimal set of monomials in 5 variables that need to be present before a polynomial can be transverse. An extension to the list of endomorphisms in 3 variables was given in [1]. The endomorphisms that where used in [1] can be found in Appendix A. Once we have a transverse quasihomogeneous polynomial we still don't know whether it defines a Calabi-Yau manifold or not. For this it needs to satisfy:

$$D = \sum_i k_i$$

putting another constraint on the system. We consider the following examples to illustrate the construction of a Calabi-Yau manifold in $\mathbb{WP}^4$.

4.2.1 Two examples

Example 6. *We start by defining an endomorphism as shown in the figure, this endomorphism corresponds to the following polynomial:*



$$p = z_1^a + z_2^b + z_3^c + z_4^d + z_5^e$$

This polynomial is clearly transverse so we don't need to add extra monomials. We have the following set of constraints to solve:

$$\begin{aligned} D &= ak_1 \\ &= bk_2 \\ &= ck_3 \\ &= dk_4 \\ &= ek_5 \\ &= k_1 + k_2 + k_3 + k_4 + k_5 \end{aligned}$$

Which we solve using the mathematica script in Appendix B. We find the following solution as in table (6), the zero locus of this polynomial is a Calabi-Yau

Space	Polynomial
$\mathbb{WP}^4_{42,258,903,602,1}[1806]$	$z_1^{43} + z_2^7 + z_3^2 + z_4^3 + z_5^{1806}$

manifold after desingularization. The next question is, what is the Euler characteristic of the configuration that we obtained? We use the formula obtained by Vafa (13) and mathematica to find $\chi = 0$.

This example is also treated in [1], where it is remarked that 1806 is the highest power that was found. This raises an interesting question: Is the list we are trying to construct finite? The answer is: yes. If one picks any endomorphism, it can be shown that the constraints on the weights and the constraint coming from the first Chern class limits the number of solutions. We consider again:

$$p = z_1^a + z_2^b + z_3^c + z_4^d + z_5^e$$

Using that $D = k_1 a = ...$ and the condition for vanishing first Chern class we find:

$$\begin{aligned} D &= k_1 + k_2 + k_3 + k_4 + k_5 \\ &= \frac{D}{a} + \frac{D}{b} + \frac{D}{c} + \frac{D}{d} + \frac{D}{e} \end{aligned}$$

If we set $a > b > c > ...$ then biggest value possible for $a = 2$, such that:

$$\frac{1}{2} = \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}$$

And we find for $b = 3$ etc. The links that we need to add from time to time can not increase the number of possibilities. The previous example was the easiest polynomial possible, now we consider an example that is slightly more difficult.

Example 7. *We consider the following endomorphism (Figure 5), which corresponds to this polynomial:*

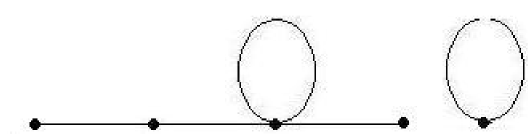


Figure 5: Endomorphism of 5 points

$$p = z_1^a z_2 + z_2^b z_3 + z_3^c + z_4^d z_3 + z_5^e$$

This polynomial is not transverse. We need to add a monomial of the form $z_2^m z_4^n$ *to make it transverse. If we now put the set of constraints into mathematica we find the solution as in the table. If we calculate the Euler characterstic associated*

Space	Polynomial
$\mathbb{WP}^4_{11,3,2,2,1}[132]$	$z_1^{11} z_2 + z_2^3 z_3 + z_3^2 + z_4^2 z_3 + z_5^{132} + z_2^3 z_4^2$

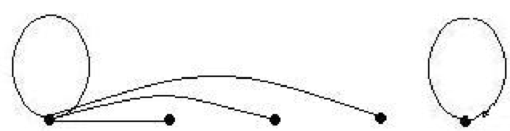
with this configuration then we find $\chi = -288$*, a number that is on the list in [1]*

4.2.2 Extending the list of polynomials

Proceding in the manner of the previous subsection Candelas et al. found 2339 topologically different Calabi-Yau manifolds. The list of endomorphisms that where used is not complete, as can be seen in Appendix A. We have extended the list of endomorphisms and belief to have obtained the complete list of endomorphisms from 5 points to 5 points. It is easy to calculate how many endomorphisms on 5 points there are. It is the same as asking how many ways are there to distribute 5 balls over 5 bowls, which is equal to $5^5 = 3125$. Actually counting the total number of endomorphisms that is represented by a single graph is a bit more difficult. It is here that mistakes could have been made. However, it is possible to construct a total of 47 endomorphisms from 5 points to 5 points with the list in Appendix A, which is a confirmation of the result given in [4]. We conclude that the list in Appendix A is a complete list of endomorphisms from 5 points to 5 points.

All of the polynomials we found are not transverse, meaning that extra monomials have to be added to make the polynomial transverse. This is pretty straightforward as is shown as in the second example of the previous subsection. It can become more and more difficult to find a solution to the set of constraints. We consider this final example to illustrate the construction of a Calabi-Yau manifold using a new endomorphism.

Example 8. *We consider the polynomial based on the following endomorphism:*



$$p = z_1^a + z_2^b z_1 + z_3^c z_1 + z_4^d z_1 + z_5^e$$

This polynomial is clearly not transverse. We can make the polynomial transverse by adding $z_3^m z_4^n$, $z_2^f z_3^g$ and $z_2^x z_4^y$. One can check that the resulting polynomial is transverse, but it already is quite difficult. Then calculating a solution to the equations as before we find the following solution, with corresponding Euler characteristic $\chi = -132$.

Space	Polynomial
$\mathbb{WP}^4_{42,6,14,21,1}[84]$	$z_1^2 + z_2^7 z_1 + z_3^3 z_1 + z_4^2 z_1 + z_5^{84} + z_3^3 z_4^2 + z_2^7 z_4^2 + z_2^7 z_3^3$

We needed to add more monomials to the polynomial in the previous example then we had to do for the earlier examples. This is in general true, the polynomials corresponding to the endomorphisms that we found are not only not transverse, most of them are not transverse in a not so nice way. Meaning that we need to add several monomials to the polynomial to make the polynomial transverse. The reason is the number of pointers that point to a single point. If there are two pointers pointing at a single variable then we also need a link to compensate. We can see that the new endomorphisms have a lot of pointers pointing at the same point in appendix A. We could in principle continue finding more and more solutions for more and more different polynomials. We will not do this, instead we will look at the classification made by Kreuzer and Skarke and discuss their results.

4.3 Classification by Kreuzer and Skarke

We have shown how to construct a Calabi-Yau manifold in a weighted projective space and how to construct its Euler characteristic. A complete classification off the kind we considered has already been made. It is a classification of all orbifolds constructed in a weighted projective space. We will now discuss this classification by Kreuzer and Skarke [2], [4]. So far we have been constructing transverse polynomials in 5 variables. Then we solved the constraints due to the quasihomogeneouity and the vanishing first Chern class conditions to find possible solutions to D, a, b, c, d, e, k_i. We will introduce a new constraint on the weights of the space we consider. In return for this extra constraint we no longer fix the number of variables we consider to 5. The constraint on the weights of a configuration is due to the Virasoro subalgebra of the theory we want to compactify. We start by defining the singularity index of a configuration as:

$$\widehat{c} = \sum_i 1 - 2q_i$$

Where $q_i = k_i/D$. The theory Kreuzer and Skarke consider is a supersymmetric conformal field theory (SCFT). The central charge c of the Virasoro subalgebra of an SCFT is related to the singularity index of the configuration via $c = 3\widehat{c}$ [10]. The number of weights and hence the number of variables is now limited by the central charge of the Virasoro algebra. Kreuzer and Skarke assume $c = 9$ in their papers. It can be shown that the number of configurations for a given singularity index is finite, meaning that the list we try to construct in this way is finite. The first result of [2] is a theorem that tells us whether a polynomial is transverse or not. We have already seen that the endomorphisms gave us a minimal set of monomials that needed to be present in any polynomial. It had to be checked by hand whether the extra terms that we added really made the polynomial transverse. The next theorem makes that unnecessary. A necesarry and sufficient condition for transversality is given by the following theorem:

Theorem 9. *For a polynomial a necessary and sufficient condition to be transverse is:*

1. *Each variable either points to itself or another variable.*
2. *For any pair of variables and/or links pointing at the same variable z there is a link joining the two pointers and not pointing at z or any of the targets of the sublinks which are joined.*

The proof of this theorem is quite long and can be found in [2]. We remarked earlier that a necessary condition for a configuration to be transverse was that (11) is a polynomial. We shall call configurations for which (11) is a polynomial almost transverse. It is a remarkeble result that the formula for the Euler characteristic generates a sensible answer for this class[4]. A configuration is called invertible if it contains a polynomial which does not require any links to be transverse. The results from the previous subsection made it possible to calculate Calabi-Yau manifolds in weighted projective spaces. In this section we added the extra requirement that $\widehat{c} = 3$. The following result is necessary to be able to calculate all transverse configurations with $\widehat{c} = 3$:

# Variables	4	5	6	7	8	9	Total
Transverse	2390	5165	2567	669	47	1	10839
Invervible	2069	4191	2239	568	40	1	9108
Not invertible	321	974	328	101	7	0	1731
Almost transverse	2404	5583	2570	686	47	1	11291

Table 6: Results from [4]

Theorem 10. *For a transverse quasihomogeneous polynomial with* $\widehat{c} = 3$ *the number of exponents* $\alpha_i > 18$ *is smaller then 3 and the number of exponents* $\alpha_i > 84$ *is smaller then 2.*

All but one of the exponents in a skeleton graph have to be smaller then 85. This result makes it possible to compute all possible configurations. Kreuzer and Skarke proceded as follows: start by computing all endomorphisms (they did this for 4 to 9 variables). The next step is to start calculating all inequivalent solutions with exponents smaller then 85, leaving only one free exponent. The free exponent can be calculated by using the singularity index. The second step is to calculate all corresponding configurations. In the third step we check whether the Poincaré polynomial is a polynomial. This seperates the quasihomogeneous polynomials into a non transverse set and an almost transverse set. The almost transverse set also contains the transverse polynomials. The last step is checking which of the almost transverse polynomials can be made transverse by adding appropriate links. The results that were obtained in [4] are listen in table (4.3).

The most remarkable result in the paper is that there is no mirror symmetry present in these models. For all invertible models 92% of the models have mirrors and 69% of all the non invertible models have mirrors. If we combine the invertible and non invertible models to look at all the transverse models, we find that only 77% have mirrors. Also including the Euler characteristic associated with the almost transverse models is not going to improve the symmetry, rather it will give new singles. So there is no mirror symmetry present in this class. There is known way to construct a mirror model given an invertible model by Berglund and Hubsch[4]. The question where to look for the mirror partners of the non invertible models is an open one.

5 Discussion

We have discussed string theory and superstring theory in section 1. We came to the conclusion that every Calabi-Yau manifold has to have a mirror due to mirror symmetry. Another result from the literature seemed to support this. Because a large class of Calabi-Yau manifolds where computed and an approximate symmetry was found in their Euler characteristic. We started section 2 with a short discussion on Calabi-Yau manifolds. We defined projective spaces and weighted projective spaces in order to be able to reproduce the construction of Calabi-Yau manifolds. In section 4 we started the construction of the Calabi-Yau manifolds. We found a way to increase the number of Calabi-Yau manifolds that where constructed. We did not construct all possible manifolds, rather we discussed the results by Kreuzer and Skarke who have found no improvement in the symmetry of the Euler characteristic in the models under consideration even though they did calculate all possible models.

So, the goal of this thesis was to extend the list of transverse polynomials used by Candelas et al. and try to find a Calabi-Yau manifold with Euler characteristic that was missing in the paper by Candelas et al. We have constructed a complete list of endomorphisms from 5 points to 5 points and used that to construct all polynomial types that have the potential to be transverse. Additional links have to be added to make some of them transverse. Theorem 9 is very helpful here. Using these polynomials it is in principle possible to calculate a list of all solutions as discussed in the previous section using the singularity index. We where unable to do this at present. The main restriction we encountered was the Mathematica script that we used. The script, while being able to compute a solution to a given polynomial, is not useful for construction a large numbers of solutions. We recently came acros a software package for C called PALP that has the functionality needed to analyze the models[19]. PALP incorperates the code that Kreuzer and Skarke used in their analysis. It would be interesting to try out this software package on the polynomials that we obtained.

The final conclusion of the previous section was that there is no mirror symmetry present in the interesting models that can be constructed in weighted projective space. This is quite unexpected. This result seems to point to an even larger class of manifolds that need to be considered. We briefly discussed Calabi-Yau manifolds in projective space. We concluded that when looking at only 1 space the total number of Calabi-Yau was 5. Taking the step to complete intersection Calabi-Yau manifolds, that is manifolds defined as an intersection in multiple projective spaces, the total number of manifolds was extended into the hundreds. Another generalization from the projective space was the weighted projective space. This also increased the total number of Calabi-Yau manifolds dramatically. However as we concluded there is no perfect mirror symmetry in this class. It could be interesting to extend the Calabi-Yau manifolds in weighted projective space to Calabi-Yau manifolds in a complete intersection of weighted projective spaces. A first step in this direction is taken in [13]. It could be interesting to see what the effect is on the number of manifolds that are found and also on their Euler characteristics.

One of the strongest constraints on the possible solutions are the extra links that need to be added to make a polynomial transverse. It could be interesting to look what happens if we loosen this restriction a bit. The links have no direct influence on the Euler characteristic but they do limit the number of solutions, it is possible that they are to strict in some cases. The reason weighted projective space is interesting is because it is a Kähler manifold. It is unknown if there are other similar spaces that we did not consider. If there are such spaces then it could be interesting to look at manifolds in those spaces aswell.

Finally we would like to draw attention to the site by Kreuzer and Skarke[19]. All known configurations are on this site for manifolds in weighted projective spaces. Aswell as links to related articles and the software package PALP.

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References

[1] P. Candelas, m. Lynker, R. Schimmrigk *Calabi-Yau manifolds in weighted* $\mathbb{P}_4$, Nuclear physics B341, 1990

[2] M. Kreuzer, H. Skarke, *On the classification of quasihomogeneous functions* Commun. math. physics, 1992

[3] P. Candelas, *Complete intersections Calabi-Yau manifolds*, Nuclear physics B298, 1988

[4] M. Kreuzer, H. Skarke, *No mirror symmetry in Landau-Ginzberg spectra!* Nuclear physics B388, 1992

[5] D. Westra, *Calabi-Yau manifolds for dummies*, http://www.mat.univie.ac.at/ westra/calabiyau3.pdf, 2008

[6] K.M. Stiffler *A walk through suerstring theory with an application to Yang-Mills theory: k-strings and D-branes as gauge/gravity dual objects* dissertation, University of Iowa, 2010 http://ir.uiowa.edu/etd/744

[7] P. Candelas, E. Witten, G. Horowitz, A. Strominger, *Vacuum configurations for superstrings*, Nuclear physics B258, 1985

[8] S. Hosono, A. Klemm, S. Theisen, *Lectures on Mirror Symmetry*, arxiv: 9403096, 1994

[9] C. Vafa, *String Vacua and Orbifoldized LG models*, HUTP-89/A018, 1989

[10] W. Lerche, C. Vafa, N.P. Warner Nuclear Physics B324, 1989

[11] A. Zinger, *Notes on Vector bundles* http://www.math.sunysb.edu/ azinger/mat531-spr10/vectorbundles.pdf, 2010

[12] V.I. Arnold, *Singularities of differentiable maps, Volume 1*, Birkhäuser, 1985

[13] T. Hübsch, *Calabi-Yau manifolds, A bestiary for Physicists*, World scientiefic publishing, 1992

[14] I.R. Shafarevich, *Algebraic Geometry II*, Springer, 1996

[15] A.N. Parshin, I.R. Shafarevich, *Algebraic Geometry III*, Springer, 1998

[16] J. Polchinski, *String theory, Vol I*, Cambridge university press, 1998

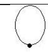
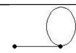
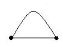
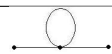
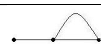
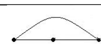
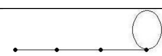
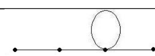
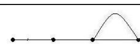
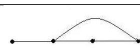
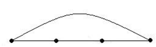
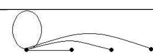
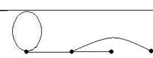
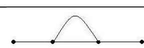
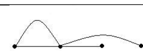
[17] J. Polchinski, *String theory, Vol II*, Cambridge university press, 1998

[18] D. Tong, *String theory*, http://www.damtp.cam.ac.uk/user/tong/string.html, 2009

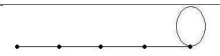
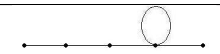
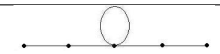
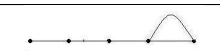
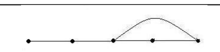
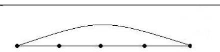
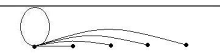
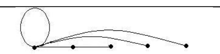
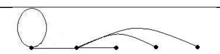
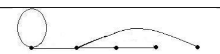
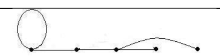
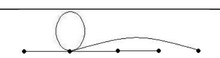
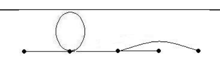
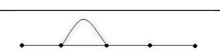
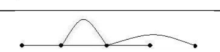
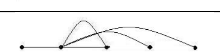
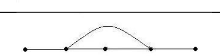
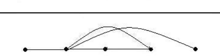
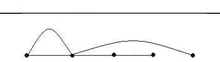
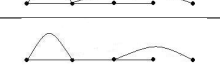
[19] A link to the website of Kreuzer and Skarke, http://hep.itp.tuwien.ac.at/ kreuzer/CY/

A A list of endomorphisms and polynomials

All endomorphisms and corresponding polynomials are listed in this appendix. A star means that it is not in the list given in [1]

Endomorphism	Polynomial	New
	z_1^a	
	$z_1^a z_2 + z_2^b$	
	$z_1^a z_2 + z_2^b z_1$	
	$z_1^a z_2 + z_2^b z_3 + z_3^c$	
	$z_1^a z_2 + z_2^b + z_3^c z_2$	
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_2$	
	$z_1^a z_+ z_2^b z_3 + z_3^c z_1$	
	$z_1^a z^2 + z_2^b z_3 + z_3^c z_4 + z_4^d$	
	$z_1^a z^2 + z_2^b z_3 + z_3^c + z_4^d z_3$	
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_3$	
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_2$	
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_1$	
	$z_1^a + z_2^b z_1 + z_3^c z_1 + z_4^d z_1$	⋆
	$z_1^a + z_2^b z_1 + z_3^c z_2 + z_4^d z_2$	⋆
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_2 + z_4^d z_3$	⋆
	$z_1^a z_2 + z_2^b z_1 + z_3^c z_2 + z_4^d z_2$	⋆

Continuation from the previous page.

Endomorphism	Polynomial	New
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_5 + z_5^e$	
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d + z_5^e z_4$	
	$z_1^a z_2 + z_2^b z_3 + z_3^c + z_4^d z_3 + z_5^e z_4$	
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_5 + z_5^e z_4$	
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_5 + z_5^e z_3$	
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_5 + z_5^e z_2$	
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_5 + z_5^e z_1$	
	$z_1^a + z_2^b z_1 + z_3^c z_1 + z_4^d z_1 + z_5^e z_1$	$\star$
	$z_1^a + z_2^b z_1 + z_3^c z_2 + z_4^d z_1 + z_5^e z_1$	$\star$
	$z_1^a + z_2^b z_1 + z_3^c z_2 + z_4^d z_2 + z_5^e z_2$	$\star$
	$z_1^a + z_2^b z_1 + z_3^c z_2 + z_4^d z_3 + z_5^e z_2$	$\star$
	$z_1^a + z_2^b z_1 + z_3^c z_2 + z_4^d z_3 + z_5^e z_3$	$\star$
	$z_1^a z_2 + z_2^b + z_3^c z_2 + z_4^d z_3 + z_5^e z_2$	$\star$
	$z_1^a z_2 + z_2^b + z_3^c z_2 + z_4^d z_3 + z_5^e z_3$	$\star$
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_2 + z_4^d z_3 + z_5^e z_4$	$\star$
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_2 + z_4^d z_3 + z_5^e z_3$	$\star$
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_2 + z_4^d z_2 + z_5^e z_2$	$\star$
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_2 + z_5^e z_4$	$\star$
	$z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_2 + z_5^e z_2$	$\star$
	$z_1^a z_2 + z_2^b z_1 + z_3^c z_2 + z_4^d z_3 + z_5^2 z_2$	$\star$
	$z_1^a z_2 + z_2^b z_1 + z_3^c z_2 + z_4^d z_3 + z_5^2 z_3$	$\star$

B Mathematica script

Here is an example mathematica script ($p = z_1^a + z_2^b + z_3^c + z_4^d + z_5^e$)

sol = FindInstance[1 == 1/a + 1/b + 1/c + 1/d + 1/e&&a > 0&&b > 0&&c > 0&&d > 0&&e > 0, {a, b, d, c, d, e}, Integers]

{{a → 43, b → 7, d → 3, c → 2, e → 1806}}

sl = Solve[D == 43 * k&&D == 7 * l&&D == 2 * m&&D == 3 * n&&D == 1806 * o&& D == k + l + m + n + o&&D > 0&&k > 0&&m > 0&&n > 0&&l > 0&&o > 0, {D, k, l, m, n, o}, Integers]

{{D → ConditionalExpression[1806C[1], C[1] ∈ Integers&&C[1] ≥ 1], k → ConditionalExpression[42C[1], C[1] ∈ Integers&&C[1] ≥ 1], l → ConditionalExpression[258C[1], C[1] ∈ Integers&&C[1] ≥ 1], m → ConditionalExpression[903C[1], C[1] ∈ Integers&&C[1] ≥ 1], n → ConditionalExpression[602C[1], C[1] ∈ Integers&&C[1] ≥ 1], o → ConditionalExpression[C[1], C[1] ∈ Integers&&C[1] ≥ 1]}}

{D, k, l, m, n, o}/.First[sl]/.Table[{C[1] → i}, {i, 1}]//Simplify

{{1806, 42, 258, 903, 602, 1}}

q = 1/1806 * {42, 258, 903, 602, 1};

chi = 1/1806*
Sum[Product[
If[
Element[
{l * q[[i]], r * q[[i]]}, Integers], (q[[i]] − 1)/q[[i]], 1], {i, 5}],
{l, 0, 1805}, {r, 0, 1805}]

0

C A list of known configurations

Here is a list of all configurations that where calculated. The number in brackets is the degree of the polynomial. A star means that the number is missing in [1], a diamond is a polynomial not in [1] and a wedge is a confirmed result from [1]

#	Space	Polynomial	χ	New
1	$\mathbb{WP}_{42,258,903,602,1}$ [1806]	$z_1^{43} + z_2^7 + z_3^2 + z_4^3 + z_5^{1806}$	0	
2	$\mathbb{WP}_{36,222,777,518,1}$[1554]	$z_1^{37} z_2 + z_2^7 + z_3^2 + z_4^3 + z_5^{1554}$	-60	
3	$\mathbb{WP}_{16,68,85,170,1}$[340]	$z_1^{17} z_2 + z_2^5 + z_3^2 z_4 + z_4^2 + z_5^{340}$	-36	
4	$\mathbb{WP}_{19,81,61,162,1}$[324]	$z_1^{17} z_5 + z_2^2 z_4 + z_3^5 z_1 + z_4^2 + z_5^{324}$	36	
5	$\mathbb{WP}_{10,12,13,15,25}$[75]	$z_1^5 z_5 + z_2^5 z_4 + z_3^5 z_1 + z_4^5 + z_5^3$	6	$\wedge$
6	$\mathbb{WP}_{4,7,9,10,15}$[45]	$z_1^9 z_3 + z_3^5 + z_2^5 z_4 + z_4^3 z_5 + z_5^3$	-6	$\wedge$
7	$\mathbb{WP}_{5,8,12,15,35}$[75]	$z_1^{15} + z_2^5 z_5 + z_3^5 z_4 + z_4^5 + z_5^2 z_1$	6	$\wedge$
8	$\mathbb{WP}_{4,8,2,1,1}$[16]	$z_1^2 z_2 + z_2^2 + z_3^4 z_2 + z_4^{16} + z_5^{16} + z_3^8$	-288	
9	$\mathbb{WP}_{42,6,14,21,1}$[84]	$z_1^2 + z_2^7 z_1 + z_3^3 z_1 + z_4^2 z_1 + z_5^{84} + z_3^3 z_4^2 + z_2^7 z_4^2 + z_2^7 z_3^3$	-132	$\diamond$
10	$\mathbb{WP}_{10,22,66,33,1}$[132]	$z_1^{11} z_2 + z_2^3 z_3 + z_3^2 + z_4^2 z_3 + z_5^{132}$	-84	