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A possible solution to the Hubble constant discrepancy: Cosmology where the local volume expansion is driven by the domain average density

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The Hubble constant problem is the discrepancy between different measurements of the Hubble constant on different scales. We show that this problem can be resolved within the general relativistic framework of the perturbation theory in the inhomogeneous universe, with the help of a spatial averaging procedure over a finite local domain in the $t =$ const. hypersurface. The idea presented in this paper is unique in the sense that it has all of the following properties. a) It is based on the general relativistic perturbation theory, with ordinary dust matter only. No strange matter nor energy components are required. b) The employment of the spatially invariant averaging procedure on the finite domain is essential. c) The key is the first-order effect of the inhomogeneities in the linear perturbation theory. No nonlinear effects are required.

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1. Introduction

Recent high-precision measurements of the Hubble constant H_0 show a large discrepancy. The Planck team's value for H_0 was 67.4 ± 0.5 km/s/Mpc [1], which was reported from the Planck satellite observing the cosmic microwave background at a very distant and large scale. On the other hand, the Supernova H_0 for the Equation of State (SH0ES) Collaboration reported an H_0 value of 73.24 ± 1.74 km/s/Mpc [2], which was based on measurements of the supernovae in our cosmic neighborhood. The result differs from Planck's by more than 3σ, a highly statistically significant discrepancy that cannot be easily explained.

We show that the Hubble constant problem, the discrepancy between the measurements of H_0 on different scales, can be resolved within the general relativistic framework of the perturbation theory in the inhomogeneous universe, with the help of the 3D averaging procedure over a finite domain in the $t =$ const. hypersurface.

The idea presented in this paper is unique in the sense that it has all of the following properties.

a) It is based on the general relativistic perturbation theory, with ordinary dust matter only. No strange matter nor energy components are required.
b) The employment of the spatially invariant averaging procedure on the finite domain is essential.
c) The key is the first-order effect of the inhomogeneities in the linear perturbation theory. No nonlinear effects are required.

2. Basic equations

In this section, we briefly summarize the basic equations [3,4]. We use the following convention: Greek indices $\mu, \nu, \ldots$ run from 0 to 3, Latin indices $i, j, k, \ldots$ run from 1 to 3, and the speed of light is unity, $c = 1$.

We consider the model that contains irrotational dust with density ρ and four-velocity u^μ. In the comoving synchronous gauge, which we adopt throughout the paper, $u^\mu = (1, 0, 0, 0)$ and the line element can be written in the form

$$ds^2 = -dt^2 + g_{ij} dx^i dx^j. \tag{1}$$

The Einstein equations read

$$\frac{1}{2}\left\{(K^i{}_i)^2 - K^i{}_j K^j{}_i + {}^{(3)}R^i{}_i\right\} = 8\pi G\rho, \tag{2}$$

$$K^j{}_{i|j} - K^j{}_{j|i} = 0, \tag{3}$$

$$\dot{K}^i{}_j + K^k{}_k K^i{}_j + {}^{(3)}R^i{}_j = 4\pi G\rho\delta^i{}_j, \tag{4}$$

where an overdot denotes $\partial/\partial t$, $|$ denotes the 3D covariant derivative with respect to g_{ij},

$$K^i{}_j \equiv \frac{1}{2} g^{ik}\dot{g}_{kj} \tag{5}$$

is the extrinsic curvature, and ${}^{(3)}R^i{}_j$ is the Ricci tensor of the 3D space with the spatial metric g_{ij}.

The energy equation is

$$\dot{\rho} + K^i{}_i\,\rho = 0. \tag{6}$$

3. The homogeneous and isotropic "background"

The homogeneous and isotropic "background" is characterized by the isotropic expansion:

$$K^i{}_j = \frac{\dot{a}}{a}\delta^i{}_j, \tag{7}$$

where $a = a(t)$ is the scale factor. Then, the Einstein equations and the energy equation require that the 3D space is of constant curvature with the curvature constant K, i.e.,

$${}^{(3)}R^i{}_j = 2\frac{K}{a^2}\delta^i{}_j, \tag{8}$$

and the density distribution is homogeneous, i.e., $\rho = \rho_b(t)$.

Then, the Einstein equation (2) and the energy equation (6) for the background are

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho_b, \tag{9}$$

$$\dot{\rho}_b + 3\frac{\dot{a}}{a}\rho_b = 0. \tag{10}$$

For the sake of simplicity, hereafter, we restrict ourselves to the case of $K = 0$ background. Generalizations to $K \neq 0$ background cases are straightforward.

4. Weakly perturbed inhomogeneous universe

The universe in reality is neither perfectly homogeneous nor isotropic. We assume that the inhomogeneities are small and briefly summarize the results of linear perturbation theory, only considering the scalar perturbations. We can express the metric and the energy density in the perturbed universe as follows:

$$ds^2 = -dt^2 + a^2 \left(\delta_{ij} + 2E_{,ij} + 2F\delta_{ij}\right) dx^i dx^j, \tag{11}$$

$$\rho = \rho_b(1+\delta). \tag{12}$$

From the linearized Einstein equations and the energy equation, we obtain the second-order differentiation equation for the density contrast δ:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\rho_b\delta = 0. \tag{13}$$

Under the normalization $a(t_0) = 1$ at the present time t_0 and neglecting the decaying mode solution, the growing mode solution for δ, which is proportional to $a(t)$ in the $K = 0$ background, can be written as

$$\delta = \frac{2}{3H_0^2} a(t)\Delta\phi(\boldsymbol{x}), \tag{14}$$

where $\Delta \equiv \delta^{ij}\partial_i\partial_j$ is the Laplace operator. The function $\phi(\boldsymbol{x})$ does not depend on t and can be regarded as the Newtonian potential at the present time t_0 in the sense

$$\Delta\phi(\boldsymbol{x}) = \frac{3H_0^2}{2a(t_0)}\delta(t_0, \boldsymbol{x}) = 4\pi G\rho_b(t_0)\delta(t_0, \boldsymbol{x}). \tag{15}$$

Using $\phi(\boldsymbol{x})$, the solutions for the metric linear perturbations can be written as

$$E = -\frac{2a(t)}{3H_0^2}\phi(\boldsymbol{x}), \quad F = -\frac{5}{3}\phi(\boldsymbol{x}); \tag{16}$$

therefore, the line element is

$$ds^2 = -dt^2 + a^2 \left(\delta_{ij} - \frac{4a(t)}{3H_0^2}\phi(\boldsymbol{x})_{,ij} - \frac{10}{3}\phi(\boldsymbol{x})\delta_{ij}\right) dx^i dx^j. \tag{17}$$

5. The average density and the volume expansion of a finite domain D

In the previous section, we have assumed that the inhomogeneous distribution of the matter density can be decomposed into the homogeneous part, i.e., the "background" density, and the (small) inhomogeneous fluctuation part, i.e., the density contrast. What is the "background" density in the inhomogeneous universe? In the actually inhomogeneous universe, we have to operationally define the "background" density through the averaging procedure.

Let us consider a finite small domain D in the $t =$ const. hypersurface Σ_t. The spatial volume V of the domain D is

$$V \equiv \int_D \sqrt{\det(g_{ij})}\, d^3x. \tag{18}$$

The spatial average of a spatial scalar quantity Q over the domain D is in general defined by

$$\langle Q \rangle \equiv \frac{1}{V} \int_D Q \sqrt{\det(g_{ij})}\, d^3x. \tag{19}$$

The average density in this domain is then

$$\langle \rho \rangle \equiv \frac{1}{V} \int_D \rho \sqrt{\det(g_{ij})}\, d^3x. \tag{20}$$

The "background" density ρ_b is obtained by averaging ρ over any sufficiently large region. Mathematically, it is [3,4]

$$\rho_b \equiv \lim_{D \to \Sigma_t} \langle \rho \rangle, \quad D \subset \Sigma_t. \tag{21}$$

It is assumed that this limit exists. Since we can observe only a finite portion of the entire universe, it is likely that the average density $\langle \rho \rangle$ of the observed domain D is not necessarily equal to the "background" density ρ_b:

$$\langle \rho \rangle \neq \rho_b \quad \text{in general for } D \ll \Sigma_t. \tag{22}$$

From the observational point of view in the domain D, which is sufficiently small compared to the entire universe, the relevant quantity related to the cosmic expansion is not the scale factor a in the "background", but the domain scale factor a_D defined by the volume expansion of the domain D:

$$3\frac{\dot{a}_D}{a_D} \equiv \frac{\dot{V}}{V} = \frac{1}{V}\int_D \frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\, d^3x = \frac{1}{V}\int_D K^i{}_i \sqrt{\det(g_{ij})}\, d^3x = \langle K^i{}_i \rangle. \tag{23}$$

So far the treatment is exact and general. If we use the solutions of the linear perturbation theory (14)–(17), we obtain

$$\frac{\dot{a}_D}{a_D} = \frac{1}{3}\left\langle K^i{}_i \right\rangle = \frac{\dot{a}}{a} + \frac{1}{3}\langle \Delta \dot{E} \rangle = \frac{\dot{a}}{a}\left(1 - \frac{1}{3}\langle \delta \rangle\right). \tag{24}$$

The Friedmann equation for a_D can be obtained by spatially averaging the Einstein equation (2). Again, if we use the linear order solutions, we obtain, up to the linear order of the perturbations,

$$\left(\frac{\dot{a}_D}{a_D}\right)^2 + \frac{K_{\text{eff}}}{a_D^2} = \frac{8\pi G}{3}\langle \rho \rangle, \tag{25}$$

where

$$K_{\text{eff}} \equiv -\frac{2}{3}\langle \Delta F \rangle = \frac{10}{9}\,\langle \Delta \phi(\boldsymbol{x}) \rangle \propto \langle \delta \rangle \tag{26}$$

is a constant that can be regarded as the effective curvature constant in the domain D.

It should be emphasized that the observed part of the domain D, which is weakly inhomogeneous, may behave on average as if it were of constant curvature with $K_{\text{eff}} \neq 0$, even if we have assumed that the "background" is spatially flat, $K = 0$. If $\langle \delta \rangle > 0$, then $K_{\text{eff}} > 0$, and if $\langle \delta \rangle < 0$, then $K_{\text{eff}} < 0$.

The energy equation for the domain average density $\langle \rho \rangle$ is obtained by averaging Eq. (6):

$$\frac{d}{dt}\langle \rho \rangle + 3\frac{\dot{a}_D}{a_D}\langle \rho \rangle = 0. \tag{27}$$

Note that Eq. (27) holds exactly without any approximation, thanks to the following commutation rule:

$$\left\langle \frac{\partial}{\partial t} Q \right\rangle - \frac{d}{dt} \langle Q \rangle = \langle K^i{}_i \rangle \langle Q \rangle - \langle K^i{}_i Q \rangle . \tag{28}$$

6. The cosmological parameters measured in the nearby regions

In spite of the recent progress in observational technology, we can still observe only a finite part of the domain D, which is still small compared to the entire universe. Therefore, the domain average density $\langle \rho \rangle$ plays the important role of driving the cosmic expansion of the observed domain of volume V. The cosmological parameters that are determined from the observations in the nearby regions may not be necessarily equal to those in the "background" universe. Let us clarify this situation.

We define the global Hubble parameter H_0 by

$$H_0 \equiv \left. \frac{\dot{a}}{a} \right|_{t_0} , \tag{29}$$

and the global density parameter Ω_0 by

$$\Omega_0 \equiv \frac{8\pi G \rho_b(t_0)}{3 H_0^2}, \tag{30}$$

which is unity since we have assumed the $K = 0$ background.

On the other hand, the cosmological parameters determined from the local observations in nearby regions of volume V, which are sufficiently small compared to the entire universe, are certainly characterized by a_D, which is driven by the average density $\langle \rho \rangle$ in this region.

Therefore, it is natural to define the local Hubble parameter $\tilde{H}_0$ by

$$\tilde{H}_0 \equiv \left. \frac{\dot{a}_D}{a_D} \right|_{t_0} , \tag{31}$$

and the local density parameter $\tilde{\Omega}_0$ by

$$\tilde{\Omega}_0 \equiv \frac{8\pi G \langle \rho(t_0) \rangle}{3 \tilde{H}_0^2}. \tag{32}$$

From Eqs. (24) and (32), we obtain the relation between the local and global cosmological parameters as

$$\tilde{H}_0 = H_0 \left(1 - \frac{1}{3} \langle \delta \rangle_{t_0} \right), \tag{33}$$

$$\tilde{\Omega}_0 = \frac{8\pi G \rho_b(t_0) \left(1 + \langle \delta \rangle_{t_0} \right)}{3 H_0^2 \left(1 - \frac{1}{3} \langle \delta \rangle_{t_0} \right)^2} = \Omega_0 \left(1 + \frac{5}{3} \langle \delta \rangle_{t_0} \right), \tag{34}$$

up to the linear order of the density perturbation δ.

The local cosmological parameters coincide with the global ones if and only if $\langle \delta \rangle = 0$, i.e., $\langle \rho \rangle = \rho_b$. A rough estimation shows that a 30% under-dense region, i.e., $\langle \delta \rangle_{t_0} = -0.3$, can explain the 10% larger value of the local Hubble parameter $\tilde{H}_0$ compared to the global H_0.

It should also be noted that the density parameter may change the value in different measurements on different scales. For example, if the local Hubble parameter has a higher value than that of the global one, $\tilde{H}_0 > H_0$, then the local region has a lower density parameter, $\tilde{\Omega}_0 < \Omega_0$.

7. Conclusion

We have operationally defined the average behavior of the actual, inhomogeneous universe. Since the observed region is finite and sufficiently small compared to the entire universe, the cosmic expansion of this region is driven by the domain average density $\langle\rho\rangle$, the spatial averaging of the inhomogeneous distribution of matter over this finite region, which is not always coincident with the "background" density ρ_b.

We have also shown that the cosmological parameters determined by the local observations in finite nearby regions may differ from the large-scale, "background" ones, which may be helpful for solving the Hubble constant problem. In particular, about 10% difference between the local and global Hubble parameters may by safely explained within the framework of linear perturbation theory, with the help of the spatial averaging procedure defined over a finite spatial domain in the $t =$ const. hypersurface.

Finally, we would like to mention an interesting possibility of solving the apparent acceleration of cosmic expansion. One of the present authors has reanalyzed the observed magnitude–redshift (m–z) relation of type Ia supernovae (SNe Ia) and has examined the possibility that the apparent acceleration of the cosmic expansion is not caused by dark energy but is instead a consequence of the large-scale inhomogeneities in the universe [5]. He has concluded that, assuming the inhomogeneous Hubble parameter, a larger value of H_0 in the nearby, low-redshift region than that in the distant, high-redshift region may be sufficient to explain the observed m–z relation for SNe Ia, without introducing dark energy. At that time, the author proposed only a phenomenological description of the large-scale inhomogeneities, and did not give a physical explanation why the Hubble parameter can change between the nearby and distant regions.

Now we have a plausible explanation: the value of the local Hubble parameter $\tilde{H}_0$ may be different from that of the global one H_0, if the domain average density $\langle\rho\rangle$ in the locally observed region is different from the "background" one ρ_b.

Therefore, we hope that the idea proposed in this paper may give a simple and interesting tool towards resolving not only the Hubble parameter discrepancy but also the apparent acceleration of the cosmic expansion mystery.

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