Advanced Studies in Theoretical Physics
Vol. 9, 2015, no. 14, 667 - 683
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/astp.2015.5881

Feynman-Cayley Path Integrals Select Chiral Bi-Sedenions with 10-Dimensional Space-Time Propagation

Stephen Winters-Hilt

Connecticut College, Box #5564
270 Mohegan Ave., New London, CT 06320, United States

Abstract

The Feynman-Cayley Path Integral extends the notion of "all paths", with path integration generally involving a real or matrix object, to "all paths with all higher-dimensional Cayley algebras and their propagating sub-algebra (loop) objects". The highest order algebraic structure that can be 'propagated' in the path integral computations is sought for this reason. An 8 dimensional subspace (two versions, so chiral) of the 15dim unit norm sedenions is shown to propagate. A 9 dimensional subspace (chiral again) of the 31dim unit norm bi-sedenions is shown to propagate in a similar manner. The results are verified both computationally and theoretically. The theoretical proof, in turn, indicates where the breakdown in the chiral extensions occurs, which is then verified computationally. This may provide a deeper understanding of why higher order extensions aren't allowed in a number of other mathematical areas, some of which are discussed. At the maximal order of propagate-able Feynman-Cayley Path Integral, the computational and theoretical results indicate a 10-dimensional space-time theory, in agreement with string theory, and yet still clearly indicates how we have spacetime and Lorenz transformation (and all of the standard model embedded for that matter). Chirality at the sedenions level, with two octonions sub-spaces, could be used to describe bosonic and fermionic matter. Chirality at the bi-sedenion level could provide an explanation for (normal) light matter and dark matter.

1 Introduction

For Real numbers unit norm propagation is trivial, consisting of multiplying by +1 or -1. For Complex numbers unit norm propagation involves multiplication by complex numbers on the classic unit circle in the complex plane, which reduces to simple phase addition according to rotations about the center of that circle (motions on S^1). For quaternion numbers unit norm propagation is still straightforward since it's still, in the end, a normed division algebra, where N(xy)=N(x)N(y). Instead of motion on S^1 we now have motion on S^3, the unit hypersphere in four dimensions. This still holds true for Octonions, with unit norm still directly maintained when multiplying unit norm objects in general. Now the motion is that of a point on a seven dimensional hypersphere S^7. Sedenions are not normed division algebras, lacking linear alternativity and the moufang loop identities, thus multiplication of unit norm objects for sedenions (points on S^{15}) will not, generally, remain unit norm, i.e., will leave the S^{15} space.

The question then arises is there is a sub-algebra or loop construct in the sedenions, that is not just trivially the octonions, that can still allow unit norm propagation? If this works for Sedenions, what about Bi-sedenions and higher dimensional Cayley algebras? In this paper it will be shown that there are two Sedenion subspaces, associated with 'left' and 'right' propagation, where the unit norm property is retained. This is found again at the level of the Bi-Sedenions by a similar construction. The results were initially explored computationally, then later established in theoretical proofs. In those proofs a key step fails when attempting to go to higher orders beyond the bi-sedenions and its sub-algebra propagation. (Propagation is taken to mean that a unit norm element of an algebra when multiplied by a unit norm element of an algebra or subalgebra or loop or chiral-loop can be propagated: (unit norm)*(unit norm subalgebra)=(unit norm), where the one-sided multiplication by the special subalgebra results in a product that remains unit norm.)

So, if the Feynman-Cayley construction works on all algebras, it essentially allows a selection argument to be made for the highest order unit norm propagating algebra in devising theories to describe matter. The highest order propagating structure might, thus, be the nine dimensional bi-sedenion elements, that are shown here, that are (chirally) extended sedenions that are themselves made from chirally extended octonions. The nine space dimensionality when paired with the implicit time dimension provides a 10 dim (1,9) spacetime theory, in agreement with string theory. (If the time is augmented to be a complex limit parameter, then we get an 11-dim theory, which shows agreement with M-theory and agreement with a role for Euclideanization related thermodynamics properties.) The core recursive algorithm in the computational work with the Cayley multiplication is shown after the theoretical proofs.

2 Background on Cayley Algebras

The list representation for hypercomplex numbers will make things clearer in what follows so will be introduced here for the first seven Cayley algebras:

Reals: $X_0 \rightarrow (X_0)$.
Complex: $(X_0 + X_1 \boldsymbol{i}) \rightarrow (X_0 , X_1)$.
Quaternions: $(X_0 + X_1 \boldsymbol{i} + X_2 \boldsymbol{j} + X_3 \boldsymbol{k}) \rightarrow (X_0 , X_1 , X_2 , X_3) \rightarrow (X_0 , \ldots , X_3)$.
Octonions: $(X_0 , \ldots , X_7)$ with seven imaginary numbers.
Sedenions: $(X_0 , \ldots , X_{15})$ with fifteen imaginary numbers.
Trigintaduonions (a.k.a Bi-Sedenions): $(X_0 , \ldots , X_{31})$ with 31 imaginary numbers.
Bi-Trigintaduonions: $(X_0 , \ldots , X_{63})$ with 63 types of imaginary number.

Consider how the familiar complex numbers can be generated from two real numbers with the introduction of a single imaginary number '***i***', $\{X_0 , X_1\} \rightarrow (X_0 + X_1 \boldsymbol{i})$. This construction process can be iterated, using two complex numbers, $\{Z_0 , Z_1\}$, and a new imaginary number '***j***':

$$(Z_0 + Z_1 \boldsymbol{j}) = (A+B\boldsymbol{i}) + (C+D\boldsymbol{i})\boldsymbol{j} = A+B\boldsymbol{i} + C\boldsymbol{j} +D\boldsymbol{ij} = A+B\boldsymbol{i} + C\boldsymbol{j} +D\boldsymbol{k},$$

where we have introduced a third imaginary number '***k***' where '***ij=k***'. In list notation this appears as the simple rule ((A,B),(C,D)) = (A,B,C,D). This iterative construction process can be repeated, generating algebras doubling in dimensionality at each iteration, to generate the 1, 2, 4, 8, 16, 32, and 64 dimensional algebras listed above. The process continues indefinitely to higher orders beyond that, doubling in dimension at each iteration, but we will see that the main algebras of interest for physics are those with dimension 1, 2, 4, and 8, and sub-spaces of those with dimension 16 and 32 dimensional algebras.

Addition of hypercomplex numbers is done component-wise, so is straightforward. For hypercomplex multiplication, list notation makes the freedom for group splittings more apparent, where any hypercomplex product ZxQ to be expressed as (U,V)x(R,S) by splitting Z=(U,V) and Q=(R,S). This is important because the product rule, generalized by Cayley, uses the splitting capability. The Cayley algebra multiplication rule is:

$$\mathbf{(A,B)(C,D) = ([AC-D^*B],[BC^*+DA]),}$$

where conjugation of a hypercomplex number flips the signs of all of its imaginary components:

$$(A,B)^* = \mathrm{Conj}(A,B) = (A^*,-B)$$

The specification of new algebras, with addition and multiplication rules as indicated by the constructive process above, is known as the Cayley-Dickson construction, and this gives rise to what is referred to as the Cayley algebras in what follows.

3 Relation of Hypercomplex Formulations to Physics Theories

Physics has a lengthy ‘love-hate’ relationship with hypercomplex numbers. The original formulations of electromagnetism by Maxwell involved quaternionic mathematics, and even at that time this relationship was off to a difficult start. As stated by Maxwell in a manuscript on the application to electromagnetism in November of 1870 [1]: “... The invention of the Calculus of Quaternions by Hamilton is a step towards the knowledge of quantities related to space which can only be compared for its importance with the invention of triple coordinates by Descartes. The limited use which has up to the present time been made of Quaternions ***must be attributed partly to the repugnance of most mature minds to new methods involving the expenditure of thought*** ...” (with emphasis mine). The enthusiasm of Maxwell for use of Quaternionic mathematics did not win over the great physicists of his day, Josiah Willard Gibbs and Oliver Heaviside in particular, who discarded the quaternionic mathematics in favor of a new mathematics (vector calculus) that they invented so as to avoid the ‘foreign’ hypercomplex mathematics. In a biography of Hamilton [2], in a quotation attributed to Gibbs: “My first acquaintance with quaternions was in reading Maxwell's E.&M. where Quaternion notations are considerably used. ... I saw, that although the methods were called quaternionic the idea of the quaternion was quite foreign to the subject.”

The stigma associated with hypercomplex mathematics, and the higher-dimensional physics unification attempts of Maxwell and later Einstein, was still significant decades later when Feynman obtained an unusual proof of the homogeneous Maxwell equations [3-6] in a higher (than 3) dimensional space. Feynman was trying to see if any new theoretical theory would be indicated and the fact that he had obtained a novel new way to explain the existing Maxwell’s equations in higher dimensions was not interesting at the time. The inextricable problems of quantum gravity and the discovery of higher-dimensional string theory, among other things, have changed the focus since that time almost 70 years ago. The accessibility of computational resources makes a big difference too.

It has been shown in numerous papers that the (1, 9) dimensional superstring has a natural parameterization in terms of octonions [7-9]. In [10, 11] the Dirac and Maxwell equations (in vacuum) are derived using octonionic algebras. In [12] a quaternionic equation is described for electromagnetic fields in inhomogenous media. In [13], the D4-D5-E6 model that includes the Standard Model plus Gravity is constructed using octonionic fermion creators and annihilators. In [14] octonionic constructions are shown to be consistent with the $SU(3)_C$ gauge symmetry of QCD. It would appear that there are a number of implementations involving hypercomplex numbers that are consistent with the Standard Model. But there is still the question of why bother? What is shown here is why the bother might be worth it as a critical new link to string theory is provided, that may explain what dimensional reduction will relate to experiments involving the standard model, and the formalism also allows for an explanation for Dark matter, all in a mathematics that can be absorbed into a Lagrangian formulation that could be consistent with a theory of Gravity.

4 The Formulation of the Problem for Sedenion Propagation

Further theoretical details on hypercomplex numbers can be found at [15, 16]. In what follows multiplications involving unit norm Cayley numbers will be done at the various orders using the Cayley algebra multiplication rule described above, that reduces the order of hypercomplex complex multiplication, which when iterated allows all hypercomplex products to reduce to a collection of Real multiplications. Millions of repeated hypercomplex multiplications are done computationally to demonstrate unit norm propagation in the situations that follow, where B denotes a bisedenion, S denotes a sedenion, O a octonion, Q a quaternion, C for complex, and R for a real:

Sedenions have two unit norm propagators of the form:

$S(\text{unit norm}) \times S(\text{unit norm propagator}) = S(\text{unit norm})$
$S(\text{unit norm}) = S_1(O_{Left},O_{Right}) = S_1(O_L,O_R) = (O_{1L},O_{1R})$

If S_1 is unit norm, then $\text{norm}(S_1) = S_1 \times S_1^* = 1$, which for our notation means:
$1 = (O_{1L},O_{1R})\times(O_{1L}^*,-O_{1R}) = ([O_{1L}\times O_{1L}^*+O_{1R}^*\times O_{1R}], [-O_{1R}\times O_{1L}+O_{1R}\times O_{1L}])$
$1 = ([\text{norm}(O_{1L})+\text{norm}(O_{1R})], 0)$
$1 = \text{norm}(O_{1L})+\text{norm}(O_{1R})$

$S(\text{unit norm propagator}) = S_2(O_{Left},O_{Real}) = (O_{2L}, \alpha)$ for the right octonion real, e.g., in list notation have $O_{Real} = (\alpha,0,0,0,0,0,0,0)$, so have $(O_{2L}, (\alpha,0,0,0,0,0,0,0))$ which is abbreviated as (O_{2L}, α) where it is understood that α is real and is the real part of the purely real right octonion. There is another type of unit norm propagator where we have (O_{Real},O_{Right}) where the same results hold, but the example that follows will use the (O_{2L}, α) form.
If S_2 is unit norm, then $\text{norm}(S_2) = S_2 \times S_2^* = 1$, which for our notation means:
$1 = \text{norm}(O_{2L})+\alpha^2$

So we can now ask the question,
Does S(unit norm) × S(unit norm propagator), return a unit norm Sedenion when using the special class of unit norm propagators indicated?

5 Proof that Norm($S_1\times S_2$)=1

$(S_1\times S_2)=(O_{1L},O_{1R})\times(O_{2L}, \alpha) = ([O_{1L}\times O_{2L}- \alpha O_{1R}], [\alpha O_{1L}+O_{1R}\times O_{2L}^*])$
$(S_1\times S_2)^* = ([O_{1L}\times O_{2L}- \alpha O_{1R}]^*, -[\alpha O_{1L}+O_{1R}\times O_{2L}^*])$

$\text{norm}(S_1\times S_2) = (S_1\times S_2) \times (S_1\times S_2)^*$
$= ([O_{1L}\times O_{2L}- \alpha O_{1R}] \times [O_{1L}\times O_{2L}- \alpha O_{1R}]^* +[\alpha O_{1L}+O_{1R}\times O_{2L}^*]^* \times [\alpha O_{1L}+O_{1R}\times O_{2L}^*],$
$-[\alpha O_{1L}+O_{1R}\times O_{2L}^*] \times [O_{1L}\times O_{2L}- \alpha O_{1R}] + [\alpha O_{1L}+O_{1R}\times O_{2L}^*] \times [O_{1L}\times O_{2L}- \alpha O_{1R}])$

$= (\ \mathrm{norm}(O_{1L}\times O_{2L})+\mathrm{norm}(O_{1R}\times O_{2L}{*})+\alpha^2\ \mathrm{norm}(O_{1R})+\alpha^2\ \mathrm{norm}(O_{1L})$
$-\alpha(O_{1L}\times O_{2L})\times O_{1R}{*}-\alpha O_{1R}\times(O_{1L}\times O_{2L}){*}+\alpha O_{1L}{*}\times(O_{1R}\times O_{2L}{*})+\alpha(O_{1R}\times O_{2L}{*}){*}\times O_{1L}, \mathbf{0})$

Multiplying the expressions previously obtained, $1 = \mathrm{norm}(O_{1L})+\mathrm{norm}(O_{1R})$ with $1 = \mathrm{norm}(O_{2L})+\alpha^2$, and making use of the norm property norm(xy)=norm(x)norm(y), we have:

$$\mathrm{norm}(S_1\times S_2) = (1\text{-}\alpha Z, 0), \text{ where,}$$

$Z = +(O_{1L}\times O_{2L})\times O_{1R}{*}+O_{1R}\times(O_{1L}\times O_{2L}){*}-O_{1L}{*}\times(O_{1R}\times O_{2L}{*})-(O_{1R}\times O_{2L}{*}){*}\times O_{1L}.$

Since we are computing the norm, which returns only the real component, we know Z must be real. To work with this expression with a little more clarity, switch to the notation:

$A=O_{1L}$; $B=O_{2L}$; $C=O_{1R}{*}$, then have
Z = (A×B)×C + C*×(A×B)*–A*×(C*×B*)–(C*×B*)*×A
Z = (A×B)×C + C*×(A×B)*–A*×(B×C)*–(B×C)×A

The Cayley algebras up to octionic are also known as the composition algebras for which a number of properties exist. We need the braid laws to proceed, so let's briefly detour to address that. The fundamental composition rule is simply that of the norm of a product being the product of the norms: norm(XY) = norm(X) x normY) Consider the norm of two things added:

Norm(X+Y) = (X+Y)(X+Y)* = XX* +XY*+YX*+YY*
= norm(X) + norm(Y) + 2 real(XY*)

Define [X,Y] = real(XY*) = [norm(X+Y)–norm(X)–norm(Y)]/2, then have another way to express conjugation using norms and real parts:

X* = 2[X,1]-X = 2real(X)-X = (real(X) unchanged, imag(X) negated))

The composition algebras (up to octionic) build from the core norm(XY) = norm(X) x normY) relation to arrive at a number of interesting properties, including the 'braid' laws: [XY,Z] = [Y,X*Z] and [XY,Z]=[X,ZY*]. To arrive at the Braid law (following [15]) you start with the composition law norm(XY)=norm(X)norm(Y), you then prove the scaling law, [XY,XZ]=norm(x)[Y,Z], by substituting Y with Y+Z in the composition law. Then establish the exchange law [XY,UZ] = 2[X,U][Y,Z]-[XZ,UY] by substituting X with X+U in the scaling law. If you put U=1 in the exchange law, it reduces to forms allowing the braid law to be shown.
Let's apply the braid law for the form [XY,Z] to the (B×C)×A term, so let's look at the braid law for [BC,A*] = [C,B*A*], which can be rewritten as:

norm(BC+A*)–norm(BC)–norm(A*) = norm(C+B*A*)–norm(C)–norm(B*A*)
norm(BC+A*)=norm(BC)+norm(A*)+(BC)A+A*(BC)*
norm(C+B*A*)=norm(C)+norm(B*A*)+C(AB)+(AB)*C*

putting this together: (BC)A+A*(BC)*=C(AB)+(AB)*C*. So we can now rewrite the (B×C)×A term as: (B×C)×A = C×(A×B)+(A×B)*×C*–A*×(B×C)*. Substituting this back into Z:

$$Z = (A\times B)\times C + C^*\times(A\times B)^* - C\times(A\times B) - (A\times B)^*\times C^*$$
$$= [(A\times B)\times C - C\times(A\times B)] + [C^*\times(A\times B)^* - (A\times B)^*\times C^*]$$

What is a commutator on the Cayley numbers, is it necessarily non-real?

$XY=(A,B)(C,D) = ([AC-D^*B],[BC^*+DA])$
$YX=(C,D)(A,B)= ([CA-B^*D],[DA^*+BC])$
$\{X,Y\}=XY-YX= ([AC-CA+B^*D-D^*B],[BC^*-BC+DA-DA^*])$
$\{X,Y\}= ([\{A,C\}+2Im(B^*D)], [B\ 2Im(C) + D\ 2Im(A)])$

So the commutator at one order of Cayley number is reduced to an expression involving the commutator at the next lower order Cayley number, plus a bunch of other terms that don't contribute to the real component. This can be iterated to arrive at the real algebra in the commutator, where the commutator is zero, thereby establishing that the commutator on the Cayley numbers must result in a pure imaginary Cayley number. This being the case, we see that since Z consists of two commutator terms, neither of which has a real contribution, and since Z must be real, this proves that Z=0.

This proves the first extension, for unit-norm propagators that are Sedenions of the form $S_{Left}=(O_{Left},\alpha)$ or $S_{Right}=(\alpha,O_{Right})$, where O_{Left} and O_{Right} are any octonion. The next extension is to unit-norm propagators that are Bisedenion by using similar constructions, e.g., Bisedenions, of the form $B=(S_{Left},S_{Real}) = ((O_{Left},\alpha) , \beta)$. (Note that α is a real octonion, while β is a purely real sedenion.)

6 The Formulation of the Problem for Bi-Sedenion Propagation

Bisedenions have two unit norm propagators of the form:
B(unit norm) × B(unit norm propagator) = B(unit norm)
B(unit norm) = $B_1(S_{Left},S_{Right}) = B_1(S_L,S_R) = (S_{1L},S_{1R})$

If B_1 is unit norm, then norm(B_1) = $B_1 \times B_1^* = 1$, which for our notation means:
$1 = (S_{1L},S_{1R})\times(S_{1L}^*,-S_{1R}) = ([S_{1L}\times S_{1L}^*+S_{1R}^*\times S_{1R}], [-S_{1R}\times S_{1L}+S_{1R}\times S_{1L}])$
$1 = ([norm(S_{1L})+norm(S_{1R})], 0)$
$1 = norm(S_{1L})+norm(S_{1R})$

B(unit norm propagator) = $B_2(S_{Left},S_{Real}) = (S_{2L}, \beta)$ for the right sedenion real, e.g., in list notation have S_{Real} = (β,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0), so have (O_{2L}, (β,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)) which is abbreviated as (O_{2L}, β) where it is understood that β is real and is the real part of the purely real right sedenion. There is another type of unit norm propagator where we have (S_{Real},S_{Right}) where the same results hold, but the example that follows will use the (S_{2L}, β) form.

If B_2 is unit norm, then $norm(B_2) = B_2 \times B_2^* = 1$, which for our notation means:
$1 = norm(S_{2L})+\beta^2$

So we can now ask the question:
Does B(unit norm) × B(unit norm propagator), return a unit norm Bisedenion when using the special class of unit norm propagators indicated?

7 Proof that Norm($B_1 \times B_2$)=1

$(B_1 \times B_2)=(S_{1L},S_{1R})\times(S_{2L}, \beta) = ([S_{1L}\times S_{2L}- \beta S_{1R}], [\beta S_{1L}+S_{1R}\times S_{2L}^*])$
$(B_1 \times B_2)^* = ([S_{1L}\times S_{2L}- \beta S_{1R}]^*, -[\beta S_{1L}+S_{1R}\times S_{2L}^*])$

$norm(B_1\times B_2) = (B_1\times B_2) \times (B_1\times B_2)^*$
$= ([S_{1L}\times S_{2L}- \beta S_{1R}] \times [S_{1L}\times S_{2L}- \beta S_{1R}]^* +[\beta S_{1L}+S_{1R}\times S_{2L}^*]^* \times [\beta S_{1L}+S_{1R}\times S_{2L}^*],$
$-[\beta S_{1L}+S_{1R}\times S_{2L}^*] \times [S_{1L}\times S_{2L}- \beta S_{1R}] + [\beta S_{1L}+S_{1R}\times S_{2L}^*] \times [S_{1L}\times S_{2L}- \beta S_{1R}])$

$= (norm(S_{1L}\times S_{2L})+norm(S_{1R}\times S_{2L}^*)+\beta^2\, norm(S_{1R})+\beta^2\, norm(S_{1L})$
$-\beta(S_{1L}\times S_{2L})\times S_{1R}^*-\beta S_{1R}\times(S_{1L}\times S_{2L})^*+\beta S_{1L}^*\times(S_{1R}\times S_{2L}^*)+\beta(S_{1R}\times S_{2L}^*)^*\times S_{1L}, \mathbf{0})$

To proceed as before we need to show that the norm property norm(xy) = norm(x)norm(y) holds for the sedenions when one of them is constrained to be in the form of the sedenion propagator, e.g., does $norm(S_{1L}\times S_{2L})=norm(S_{1L})\times norm(S_{2L})$ where S_{2L} is in the form of the sedenion propagator?

$norm(S_{1L}\times S_{2L}) = (S_{1L}\times S_{2L}) \times (S_{1L}\times S_{2L})^*$
$= ([O_{1LL}\times O_{2LL}- \alpha O_{1LR}] \times [O_{1LL}\times O_{2LL}- \alpha O_{1LR}]^* +$
$[\alpha O_{1LL}+O_{1LR}\times O_{2LL}^*]^* \times [\alpha O_{1LL}+O_{1LR}\times O_{2LL}^*],$
$-[\alpha O_{1LL}+O_{1LR}\times O_{2LL}^*] \times [O_{1LL}\times O_{2LL}- \alpha O_{1LR}] +$
$[\alpha O_{1LL}+O_{1LR}\times O_{2LL}^*] \times [O_{1LL}\times O_{2LL}- \alpha O_{1LR}])$

$= (norm(O_{1LL}\times O_{2LL})+norm(O_{1LR}\times O_{2LL}^*)+\alpha^2\, norm(O_{1LR})+\alpha^2\, norm(O_{1LL})-$
$\alpha(O_{1LL}\times O_{2LL})\times O_{1LR}^*-\alpha O_{1LR}\times(O_{1LL}\times O_{2LL})^*+$
$\alpha O_{1LL}^*\times(O_{1LR}\times O_{2LL}^*)+\alpha(O_{1LR}\times O_{2LL}^*)^*\times O_{1LL}, \quad \mathbf{0})$

Now that we've reduced to this level, we know that the octonions will offer the standard norm property whereby $norm(O_{1LL}\times O_{2LL})=norm(O_{1LL})norm(O_{2LL})$ and we show the other terms are zero since real yet consisting of commutators, the latter arrangements made possible by manipulations according to the braid laws that hold for the composition algebras (including the octonions) without restriction.

So as before, by multiplying the expressions previously obtained, $1 = \mathrm{norm}(S_{1L}) + \mathrm{norm}(S_{1R})$ with $1 = \mathrm{norm}(S_{2L})+\beta^2$, and making use of the norm property norm(xy)=norm(x)norm(y) applicable for the terms of interest, we have:

$$\mathrm{norm}(B_1\times B_2) = (1-\beta Z, 0), \text{ where,}$$

$$Z = +(S_{1L}\times S_{2L})\times S_{1R}^*+S_{1R}\times(S_{1L}\times S_{2L})^*-S_{1L}^*\times(S_{1R}\times S_{2L}^*)-(S_{1R}\times S_{2L}^*)^*\times S_{1L}.$$

Since we are computing the norm, which returns only the real component, we know Z must be real. As with the lower order Cayley extension, we need the braid laws to proceed at this juncture.

What is $(S_{1L}\times S_{2L})\times S_{1R}^*$ when accounting for the special form of $S_{2L}=(O_{2LL}, \alpha)$? First calculate $(S_{1L}\times S_{2L})$:

$$(S_{1L}\times S_{2L})=(O_{1LL},O_{1LR})(O_{2LL}, \alpha) = ([O_{1LL}\times O_{2LL}- \alpha O_{1LR}], [\alpha O_{1LL}+O_{1LR}\times O_{2LL}^*])$$

Then

$$(S_{1L}\times S_{2L})\times S_{1R}^* = ([O_{1LL}\times O_{2LL}- \alpha O_{1LR}], [\alpha O_{1LL}+O_{1LR}\times O_{2LL}^*]) (O_{1RL}^*,-O_{1RR})$$
$$= ([O_{1LL}\times O_{2LL}- \alpha O_{1LR}]O_{1RL}^* + O_{1RR}^*[\alpha O_{1LL}+O_{1LR}\times O_{2LL}^*],$$
$$-O_{1RR} [O_{1LL}\times O_{2LL}- \alpha O_{1LR}] + [\alpha O_{1LL}+O_{1LR}\times O_{2LL}^*]O_{1RL})$$
$$= ((O_{1LL}\times O_{2LL})\times O_{1RL}^*- \alpha O_{1LR}\times O_{1RL}^*+\alpha O_{1RR}^*\times O_{1LL}+ O_{1RR}^*\times(O_{1LR}\times O_{2LL}^*),$$
$$-O_{1RR} \times(O_{1LL}\times O_{2LL})+\alpha O_{1RR}\times O_{1LR}+\alpha O_{1LL}\times O_{1RL}+(O_{1LR}\times O_{2LL}^*)\times O_{1RL})$$

$$S_{1R}\times(S_{1L}\times S_{2L})^* = (O_{1RL},O_{1RR}) ([O_{1LL}\times O_{2LL}- \alpha O_{1LR}]^*, -[\alpha O_{1LL}+O_{1LR}\times O_{2LL}^*])$$
$$= (O_{1RL}\times[(O_{1LL}\times O_{2LL})^*- \alpha O_{1LR}^*]+[\alpha O_{1LL}^*+(O_{1LR}\times O_{2LL}^*)^*]\times O_{1RR},$$
$$-[\alpha O_{1LL}+O_{1LR}\times O_{2LL}^*]\times O_{1RL}+O_{1RR}\times[O_{1LL}\times O_{2LL}- \alpha O_{1LR}])$$
$$= (O_{1RL}\times(O_{1LL}\times O_{2LL})^*- \alpha O_{1RL}\times O_{1LR}^* +\alpha O_{1LL}^*\times O_{1RR}+(O_{1LR}\times O_{2LL}^*)^*\times O_{1RR} ,$$
$$-\alpha O_{1LL}\times O_{1RL}-(O_{1LR}\times O_{2LL}^*)\times O_{1RL}+O_{1RR}\times(O_{1LL}\times O_{2LL})- \alpha O_{1RR}\times O_{1LR})$$

Putting these first two terms together:

$$+(S_{1L}\times S_{2L})\times S_{1R}^*+S_{1R}\times(S_{1L}\times S_{2L})^* =$$
$$((O_{1LL}\times O_{2LL})\times O_{1RL}^*+O_{1RL}\times(O_{1LL}\times O_{2LL})^*$$
$$- \alpha O_{1LR}\times O_{1RL}^*+\alpha O_{1RR}^*\times O_{1LL}- \alpha O_{1RL}\times O_{1LR}^* +\alpha O_{1LL}^*\times O_{1RR}$$
$$+ O_{1RR}^*\times(O_{1LR}\times O_{2LL}^*)+(O_{1LR}\times O_{2LL}^*)^*\times O_{1RR}, \quad 0)$$

For $S_{1L}^*\times(S_{1R}\times S_{2L}^*)$ we have:

$$(S_{1R}\times S_{2L}^*)=(O_{1RL},O_{1RR})(O_{2LL}^*, -\alpha) = ([O_{1RL}\times O_{2LL}^*+ \alpha O_{1RR}], [-\alpha O_{1RL}+O_{1RR}\times O_{2LL}])$$

So, $S_{1L}^*\times(S_{1R}\times S_{2L}^*) = (O_{1LL}^*,-O_{1LR})\times([O_{1RL}\times O_{2LL}^*+\alpha O_{1RR}], [-\alpha O_{1RL}+O_{1RR}\times O_{2LL}])$

$$= (O_{1LL}^*\times(O_{1RL}\times O_{2LL}^*)+\alpha O_{1LL}^*\times O_{1RR}-\alpha O_{1RL}^*\times O_{1LR}+(O_{1RR}\times O_{2LL})^*\times O_{1LR} , \text{term})$$

While for $(S_{1R}\times S_{2L}^*)^*\times S_{1L}$ have

$$(S_{1R}\times S_{2L}^*)^*\times S_{1L}= ([O_{1RL}\times O_{2LL}^*+ \alpha O_{1RR}]^*, [\alpha O_{1RL}-O_{1RR}\times O_{2LL}])\times(O_{1LL},O_{1LR})$$
$$= ([O_{1RL}\times O_{2LL}^*+ \alpha O_{1RR}]^*\times O_{1LL}-O_{1LR}^*\times[\alpha O_{1RL}-O_{1RR}\times O_{2LL}], - \text{term})$$

$S_{1L}^*\times(S_{1R}\times S_{2L}^*)+(S_{1R}\times S_{2L}^*)^*\times S_{1L}=$
$(O_{1LL}^*\times(O_{1RL}\times O_{2LL}^*)+(O_{1RL}\times O_{2LL}^*)^*\times O_{1LL}$
$+\alpha O_{1LL}^*\times O_{1RR}-\alpha O_{1RL}^*\times O_{1LR}+ \alpha O_{1RR}^*\times O_{1LL}-\alpha O_{1LR}^*\times O_{1RL}$
$+(O_{1RR}\times O_{2LL})^*\times O_{1LR}+O_{1LR}^*\times(O_{1RR}\times O_{2LL}), \quad 0)$

So have,
$Z = (\quad \{(O_{1LL}\times O_{2LL})\times O_{1RL}^*+O_{1RL}\times(O_{1LL}\times O_{2LL})^*$
$-O_{1LL}^*\times(O_{1RL}\times O_{2LL}^*)-(O_{1RL}\times O_{2LL}^*)^*\times O_{1LL}\} +$
$\{O_{1RR}^*\times(O_{1LR}\times O_{2LL}^*)+(O_{1LR}\times O_{2LL}^*)^*\times O_{1RR}$
$-(O_{1RR}\times O_{2LL})^*\times O_{1LR}-O_{1LR}^*\times(O_{1RR}\times O_{2LL})\} +$
$- \alpha O_{1LR}\times O_{1RL}^*+\alpha O_{1RR}^*\times O_{1LL}- \alpha O_{1RL}\times O_{1LR}^* +\alpha O_{1LL}^*\times O_{1RR}$
$-\alpha O_{1LL}^*\times O_{1RR}+\alpha O_{1RL}^*\times O_{1LR}-\alpha O_{1RR}^*\times O_{1LL}+\alpha O_{1LR}^*\times O_{1RL} \; , 0)$

$Z = (\{im\}+\alpha\{O_{1RL}^*\times O_{1LR}+\alpha O_{1LR}^*\times O_{1RL}- \alpha O_{1LR}\times O_{1RL}^*- \alpha O_{1RL}\times O_{1LR}^*\} , 0)$
$Z = (\{im\}+\alpha\{2Im\{O_{1RL}^*\times O_{1LR}\} +2Im\{O_{1LR}^*\times O_{1RL}\} , 0)$

So again, have that Z = pure imaginary, and since it must be real, it is thus zero. Thus, we have $norm(B_1\times B_2) = 1$. This proves the second extension, for unit-norm propagators that are Bisedenions of the form $B_{Left}=(S_{Left},\beta)$ or $B_{Right}=(\beta,S_{Right})$, where S_{Left} and S_{Right} are sedenion propagators shown in the first extension, e.g., $S_{Left}=(O_{Left},\alpha)$. (Note that α is a purely real octonion, while β is a purely real sedenion.)

8 The Formulation of the Problem: Bitrigintaduonion Propagation

After the bisedenions (also known as trigintaduonions) come the bitrigintaduonions, the 64-component Cayley algebra (denoted by 'T' in following but later when I reference the RCHO(ST) hypothesis, the 'T' refers to trigintaduonions). Let's try extending further to see if we can have $norm(T_1\times T_2) = 1$, when we build with a similar extension method to define our unit-norm propagator: $T_{Left}=(B_{Left},\gamma)$, $B_{Left}=(S_{Left},\beta)$, and $S_{Left}=(O_{Left},\alpha)$, where, as before, once we get to the octionic Cayley level we are unrestricted (e.g., O_{Left} can be any octonion). Let's see if we can construct, as before, a T unit norm propagators of the form:

T(unit norm) × T(unit norm propagator) = T(unit norm)
T(unit norm) = $T_1(B_{Left},B_{Right}) = T_1(B_L,B_R) = (B_{1L},B_{1R})$

If T_1 is unit norm, then $norm(T_1) = T_1 \times T_1^* = 1$, which for our notation means:
$1 = (B_{1L},B_{1R})\times(B_{1L}^*,-B_{1R}) = ([B_{1L}\times B_{1L}^*+B_{1R}^*\times B_{1R}], [-B_{1R}\times B_{1L}+B_{1R}\times B_{1L}])$
$1 = ([norm(B_{1L})+norm(B_{1R})], 0)$
$1 = norm(B_{1L})+norm(B_{1R})$

T(unit norm propagator) = $T_2(B_{Left},B_{Real}) = (B_{2L}, \gamma)$ for the right bisedenion real γ is real and is the real part of the purely real right bisedenion.

If T_2 is unit norm, then norm(T_2) = $T_2 \times T_2^*$ = 1, which for our notation means:
$1 = \text{norm}(B_{2L})+\gamma^2$

So we can now ask the question,
Does T(unit norm) × T(unit norm propagator), return a unit norm bitrigintaduonion when using the special class of unit norm propagators indicated?

9 Failure of Proof construction for Norm($T_1 \times T_2$)=1, and computational proof of failure of Norm($T_1 \times T_2$)=1

$(T_1\times T_2)=(B_{1L},B_{1R})\times(B_{2L}, \gamma) = ([B_{1L}\times B_{2L}- \gamma B_{1R}], [\gamma B_{1L}+B_{1R}\times B_{2L}^*])$
$(T_1\times T_2)^* = ([B_{1L}\times B_{2L}- \gamma B_{1R}]^*, -[\gamma B_{1L}+B_{1R}\times B_{2L}^*])$

$\text{norm}(T_1\times T_2) = (T_1\times T_2) \times (T_1\times T_2)^*$
$= ([B_{1L}\times B_{2L}- \gamma B_{1R}] \times [B_{1L}\times B_{2L}- \gamma B_{1R}]^* +[\gamma B_{1L}+B_{1R}\times B_{2L}^*]^* \times [\gamma B_{1L}+B_{1R}\times B_{2L}^*],$
$-[\gamma B_{1L}+B_{1R}\times B_{2L}^*] \times [B_{1L}\times B_{2L}- \gamma B_{1R}] + [\gamma B_{1L}+B_{1R}\times B_{2L}^*] \times [B_{1L}\times B_{2L}- \gamma B_{1R}])$

$= (\text{norm}(B_{1L}\times B_{2L})+\text{norm}(B_{1R}\times B_{2L}^*)+\gamma^2 \text{norm}(B_{1R})+\gamma^2 \text{norm}(B_{1L})$
$-\gamma(B_{1L}\times B_{2L})\times B_{1R}^*-\gamma B_{1R}\times(B_{1L}\times B_{2L})^*+\gamma B_{1L}^*\times(B_{1R}\times B_{2L}^*)+\gamma(B_{1R}\times B_{2L}^*)^*\times B_{1L}, \mathbf{0})$

To proceed as before we need to show that the norm property norm(xy) = norm(x)norm(y) holds for the bisedenions when one of them is constrained to be in the form of the bisedenion propagator, e.g., does $\text{norm}(B_{1L}\times B_{2L}) = \text{norm}(B_{1L})\times\text{norm}(B_{2L})$ where B_{2L} is in the form of the bisedenion propagator?

$\text{norm}(B_{1L}\times B_{2L}) = (B_{1L}\times B_{2L}) \times (B_{1L}\times B_{2L})^*$
$= ([S_{1LL}\times S_{2LL}- \beta S_{1LR}] \times [S_{1LL}\times S_{2LL}- \beta S_{1LR}]^* +$
$[\beta S_{1LL}+S_{1LR}\times S_{2LL}^*]^* \times [\beta S_{1LL}+S_{1LR}\times S_{2LL}^*],$
$-[\beta S_{1LL}+S_{1LR}\times S_{2LL}^*] \times [S_{1LL}\times S_{2LL}- \beta S_{1LR}] +$
$[\beta S_{1LL}+S_{1LR}\times S_{2LL}^*] \times [S_{1LL}\times S_{2LL}- \beta S_{1LR}])$

$= (\text{norm}(S_{1LL}\times S_{2LL})+\text{norm}(S_{1LR}\times S_{2LL}^*)+\beta^2 \text{norm}(S_{1LR})+\beta^2 \text{norm}(S_{1LL})-$
$\beta(S_{1LL}\times S_{2LL})\times S_{1LR}^*-\beta S_{1LR}\times(S_{1LL}\times S_{2LL})^*+$
$\beta S_{1LL}^*\times(S_{1LR}\times S_{2LL}^*)+\beta(S_{1LR}\times S_{2LL}^*)^*\times S_{1LL}, \quad \mathbf{0})$

Now that we've reduced to this level we see there is a problem. In the prior reduction we arrived at the variables being octonions at this stage, for which the norm property and braid laws of the octionoic composition algebra allowed $\text{norm}(O_{1LL}\times O_{2LL}) = \text{norm}(O_{1LL})\text{norm}(O_{2LL})$ and showed the non-norm terms were zero by manipulations using the braid laws that hold for the composition algebras. Now that we've moved to the next higher Cayley algebra's in the derivation, and in our extension construction, we now are asking the sedenions to act as a composition algebra to proceed (on an unrestricted part of the Sedenion algebra). The construction fails. Thus, the extension process does not extend past the Bisedenions, it basically requires the Cayley algebra at two Cayley levels lower to still be a composition algebra. It is still possible to extend to the bisedenions because at two levels lower you still have the octonions, which are a composition algebra as needed. Computationally we see a failure to propagate the bitrigintaduonions so this is consistent.

The key software solution to discover/verify the results computationally is a the recursive Cayley definition for multiplication, which avoids use of lookup tables and avoids commutation and associativity issues encountered at higher order. It is shown next. The cayley subroutine takes the references to any pair of Cayley numbers (represented in list form, so represented as simple arrays), and multiplies those Cayley numbers and returns the Cayley number answer (in list form, thus an array). The main usage was with randomly generated unit norm Cayley numbers that were multiplied (from right) against a "running product". Tests on unit norm hold for millions of running product evaluations in cases where there the unit norm propagations are validated, so, like the perfectly meshed gears of a machine, or the perfectly 'braided' threads of a very long string.

10 The Code

---------------------------------- cayley_multiplication.pl ---

```
sub cayley {
    my ($ref1,$ref2)=@_;
    my @input1=@{$ref1};
    my @input2=@{$ref2};
    my $order1=scalar(@input1);
    my $order2=scalar(@input2);
    my @output;
    if ($order1 != $order2) {die;}
    if ($order1 == 1) {
        $output[0]=$input1[0]*$input2[0];
    }
    else{
        my @A=@input1[0..$order1/2-1];
        my @B=@input1[$order1/2..$order1-1];
        my @C=@input2[0..$order1/2-1];
        my @D=@input2[$order1/2..$order1-1];
        my @conjD=conj(\@D);
        my @conjC=conj(\@C);
        my @cay1 = cayley(\@A,\@C);
        my @cay2 = cayley(\@conjD,\@B);
        my @cay3 = cayley(\@D,\@A);
        my @cay4 = cayley(\@B,\@conjC);
        my @left;
        my @right;
        my $length = scalar(@cay1);
        my $index;
        for $index (0..$length-1) {
            $left[$index] = $cay1[$index] - $cay2[$index];
            $right[$index] = $cay3[$index] + $cay4[$index];
```

```
        }
        @output=(@left,@right);
    }
    return @output;
}
```

--------------------------------- cayley_multiplication.pl ---

11 Discussion

Consider the Feynman-Cayley Path Integral, where we extend the notion of all paths with path integration generally involving a real or matrix object to all paths with all higher-dimensional Cayley algebras and their propagating sub-algebra (loop) objects. It is hypothesized that the usual configuration space path integral construction in this larger Cayley space domain can be undertaken in a straightforward manner, allowing many choices to be resolved, or explained, from what is allowed within that formulation. Let's suppose the following was possible:

(1) Reality can be described in terms of a completely multiplicative propagator.

(2) The Path Integral formalism selects ***against*** fields involving algebras with zero divisors, and loss of braid laws, thereby reducing to a RCHO(ST) based theory, where equilibrium and martingale constructs occur asymptotically.

(3) The path integral formalism resuling from the propagotor has two forms, the Feynman-Cayley Path Integral and Weiner Path Integral, according to the introduction of an analytic (complex) time parameter, where the forms are related via Euclideanization.

The central role of path integrals and of time Euclideanization is posited as fundamental in and of itself. The core hypothesis is that complex wavefunctions can be written in a path integral formalism with propagators that involve fields based on Cayley algebras at all orders. The Cayley algebras with no zero divisors include the Real Numbers (R), the Complex Numbers (C), the Quaternions (Q), and the Octonians (O). And what is shown is here is how to extend by two more dimensions beyond the octonions, and in the lower algebras the same methods suggest sub-algebras (e.g., quaternions have four dimensions, if they were extended would possibly have five and six dimensional subalgebras of the octonions and sedenions,).

The stationary phase of a solution, or highly peaked density of states in the Euclideanized time domain, is not possible for fields over the Cayley algebra's that have zero divisors, e.g., that are no longer composition algebras. The zero divisors are posited to disrupt all such higher order Cayley field propagations, thereby eliminating them from path integral considerations except when such a short step is taken that the likelihood of a disruptive (to phase or cohesion) zero divisor occurrence is low. In the extension proofs it was shown how a normed algebra allowed for the braid laws to be valid, which were used repeatedly in the extension proofs. So just as critical as disruption due to zero divisor is disruption in unit norm propagation due to loss of ‘braiding’. (It is conceivable that you might operate in a sector, where you are not free to multiply anything together, and the zero divisor problem is avoided, the braiding failure, however, is still unavoidable.)

A number of embedded Cayley algebra’s arise in a natural way. Recall that embedded in a Cayley algebra at any order above quaternionic, there is a quaternionic sub-algebra. Similarly, within that quaternionic algebra is a complex sub-algebra. The embeddings on sub-algebras works for octonians within higher order Cayley algebras, also, but beyond that the sub-algebras don’t appear to embed (so only have a $C \subset H \subset O$ embedding description within the higher order Cayley algebras).

The RCHO(ST) hypothesis, motivated by Maxwell, Feynman and Cayley, may be able to directly encode the standard model and statistical mechanics, as well as provide a unified framework whereby matter/antimatter (possibly from braiding since there are two types of braiding), boson/fermion (from choice of chiral octionic sector), and light/dark matter or energy (from choice of chiral sedenionic sector), may be encoded. A variational formalism at the level of the10 dim spacetime description may allow quantum gravity by use of the renormalizability of the (10 dim) string theories.

12 Conclusion

The Feynman-Cayley path integral appears to indicate a (1,9) spacetime setting, sufficient to encompass the standard model [10-14,17] and explain other, chiral, structures of matter. The 10-dim spacetime indicated by the RCHO(ST) hypothesis is also consistent with the 10 dimensions indicated by string theory.

Other forms of multiplication rule (than Cayley) are very useful, one being scalar-vector representations that ties into representing physics equations in compact form [17]. Descriptions of the standard model in terms of hypercomplex mathematics is described in [10-14, 17].

Generalization of the N-square theorem is discussed by Rajwade in [20]. Rajwade defines generalized N-square theorems to exist when a triple of integers (P,Q,R) are admissible, where admissible occurs if and only if the product of a sum of P squares and a sum of Q squares is a sum of R squares (over some field). The standard N-square theorem is that (N,N,N) is admissible for N=1,2,4,8, paralleling the dimensionality of the composition algebra in which they operate. So have (1,1,1) for the reals, (2,2,2) for the complex numbers, (4,4,4) for the quaternions, and (8,8,8) for the octonions. Rajwade shows that (16,16,16) is NOT admissible, but that (16,16,32) is admissible over the reals. The results here appear to partly overlap with Rajwade's results and suggests additional admissible triplets:(8,8,16),for products of special sedenion propagator;(16,8,16) for norm one propagation using sedenion propagator; (16,16,32), for products of bisedenion propagator (in agreement with Rajwade); and (32,16,32), for norm one propagation using the bisedenion propagator.

This work appears to relate to efforts to describe a left loop on the 15-sphere and examining sub-loops on the sedenions [21, 22], suggesting that similar subloops may be found in the bisedenions as well (but not in higher order Cayley algebras).

Other than a new link to string theory, consistent with the 10-dim theory, the hypercomplex formulation indicated here may indicate what dimensional reductions will relate to experiments involving the standard model, and the natural chirality of the formalism also allows for an explanation for Dark matter, all in a mathematics that can be absorbed into a Lagrangian formulation that could be consistent with a theory of Gravity.

References

[1] P.M. Harmon, Editor, *The Scientific Letters and Papers of James Clerk Maxwell II*, Cambridge University Press, 1995, 570–576.

[2] T.L. Hankins, *Sir William Rowan Hamilton*, Johns Hopkins Press, 1980, 316–319.

[3] F.J. Dyson, Feynman's proof of the Maxwell equations, *Am. J. Phys*., **58** (1990), 209-211. http://dx.doi.org/10.1119/1.16188

[4] F. J. Dyson, Feynman at Cornell, *Phys. Today*, **42** (1989), no. 2, 32-38. http://dx.doi.org/10.1063/1.881190

[5] R. J. Hughes, On Feynman's proof of the Maxwell equations, *Am. J. Phys*., **60** (1992), no. 4, 301. http://dx.doi.org/10.1119/1.16921

[6] Z.K. Silagadze, Feynman's derivation of Maxwell equations and extra dimensions, *Annales de la Fondation Louis de Broglie*, **27** (2002), no. 2, 241.

[7] C.A. Manogue and J. Schay, Finite Lorentz Transformations, Automorphisms, and Division Algebras, *J. Math. Phys*., **34** (1993), no. 8, 3746-3767. http://dx.doi.org/10.1063/1.530056

[8] C.A. Manogue and A. Sudbery, General solutions of covariant superstring equations of motion, *Phys.*, **40** (1989), 4073.

http://dx.doi.org/10.1103/physrevd.40.4073

[9] M. Cederwall, Octonionic particles and the S7 symmetry, *J. Math. Phys*., **33** (1992), 388. http://dx.doi.org/10.1063/1.529919

[10] M. Gogberashvili, Octonionic electrodynamics, *J. Phys. A: Math. Gen.*, **39** (2006), no. 22, 7099. http://dx.doi.org/10.1088/0305-4470/39/22/020

[11] B.C. Chanyal, P.S. Bisht and O.P.S. Negi, Generalized Octonion Electrodynamics, **49** (2010), no. 6, 1333-1343. http://dx.doi.org/10.1007/s10773-010-0314-5

[12] V. Kravchenko, Quaternionic equation for electromagnetic fields in inhomogenous media, 2003, 361-366. http://dx.doi.org/10.1142/9789812794253_0042

[13] F.D. Smith, Jr., Standard Model plus Gravity from Octonion Creators and Annihilators, Quant-ph/9503009.

[14] Pushpa, P.S. Bisht, T. Li, and O.P.S. Negi, Quaternion Octonion Reformulation of Quantum Chromodynamics, *Int. J. Theor. Phys*., **50** (2011), no. 2, 594-606. http://dx.doi.org/10.1007/s10773-010-0579-8

[15] J.H. Conway and D.A. Smith, *On Quaternions and Octonions: their Geometry, Arithmetic, and Symmetry*, A K Peters, Wellesley, Massachusetts, 2005.

[16] J.C. Baez, The Octonions, *Bull. Amer. Math. Soc*., **39** (2002), 145-205. http://dx.doi.org/10.1090/s0273-0979-01-00934-x

[17] V.L. Mironov, S.V. Mironov, Associative Space-Time Sedenions and Their Application in Relativistic Quantum Mechanics and Field Theory, *Applied Mathematics*, (2015), no. 6, 46-56. http://dx.doi.org/10.4236/am.2015.61006

[18] F.D. Smith, Jr., Standard Model plus Gravity from Octonion Creators and Annihilators, quant-ph/9503009.

[19] V. Chrisitianto, F. Smarandachey, A Derivation of Maxwell Equations in Quaternion Space, April, *Progress In Physics*, **2** (2010), 23-27.

[20] A.R. Rajwade, *Squares*, Cambridge, 1993. http://dx.doi.org/10.1017/cbo9780511566028

[21] J.D.H. Smith, A Left Loop on the 15-sphere, *Journal of Algebr*, **176** (1995), no. 1, 128-138. http://dx.doi.org/10.1006/jabr.1995.1237

[22] B.M. Kivunge, J. D. H Smith, Subloops of sedenions, *Comment. Math. Univ. Carolin.*, **45** (2004), no. 2, 295–302.

Received: August 21, 2015; Published: October 23, 2015