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Non-Perturbative Quantum Yang–Mills at Finite Temperature Beyond Lattice: A Dyson–Schwinger Approach

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Abstract: Using a Dyson–Schwinger approach, we perform an analysis of the non-trivial ground state of thermal $SU(N)$ Yang–Mills theory in the non-perturbative regime where chiral symmetry is dynamically broken by a mass gap. Basic thermodynamic observables such as energy density and pressure are derived analytically, using Jacobi elliptic functions. The results are compared with the lattice results. Good agreement is found at low temperatures, providing a viable scenario for a gas of massive glue states populating higher levels of the spectrum of the theory. At high temperatures, a scenario without glue states consistent with a massive scalar field is observed, showing an interesting agreement with lattice data. The possibility is discussed that the results derived in this analysis open up a novel pathway beyond lattice to precision studies of phase transitions with false vacuum and cosmological relics that depend on the equations of state in strong coupled gauge theories of the type of Quantum Chromodynamics (QCD).

Keywords: finite temperature QCD; Yang–Mills theory; Dyson–Schwinger approach



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1. Introduction

Nowadays, only lattice simulations and other very limited theoretical approaches can reliably analyze the key observables in quantum Yang–Mills theories at finite temperatures in the strong coupling regime. Existing analytic methods such as the Polyakov loop model [1,2] still rely on sets of unknown non-perturbative parameters in the effective potential obtained by properly fitting lattice data. The exact form of the coefficients of a thermodynamic potential is found by an appropriate fit to the corresponding lattice data [3], which fully determines this potential for a given theory. Such a lattice-inspired effective model has been extensively employed in view of cosmological studies involving dark sectors (see, e.g., Ref. [4]).

In this work, we develop an approach enabling a fully analytical treatment of non-perturbative quantum field thermodynamics in terms of physical parameters alone. Our technique provides, in principle, a systematic way of incorporating finite temperature corrections to the exact Green function in the framework of a Dyson–Schwinger approach with a non-trivial ground state [5–8]. We consider a pure Yang–Mills theory, i.e., a gauge theory that has no other degrees of freedom than its potentials. It should be noted that Yang–Mills theories appear in nature only in interaction with other fields like fermions and scalars, such as in the Standard Model or QCD. Therefore, the idealized situation without other fields we consider here can only be compared to lattice calculations [9,10].

We demonstrate the robustness of our method by computing the effective action of a pure Yang–Mills quantum thermal theory under the $SU(N)$ symmetry group in the non-perturbative regime. The latter is then used to estimate some of the basic thermodynamic observables such as energy density and pressure. This is achieved by computing the effective energy–momentum tensor of the Yang–Mills theory and evaluating the thermal averages of its components in terms of the correlation functions [11,12]. Comparing the thermodynamic observables with the lattice data available for $N = 3$ or $N = 4$ colors [3] reveals an overall consistency in our results, suggesting that our method captures the most essential features of non-perturbative Yang–Mills dynamics at low temperatures. A recent approach with a free gas of glueballs shows a similar consistency [13]. As pure Yang–Mills theory is not seen in experiments, we emphasize that a proof of principle of the symmetry breaking in Yang–Mills theory can only be obtained by comparison with lattice calculations [3]. Our aim is to show how, from calculations based on the full theory, very good fits can be obtained at low and high temperatures, providing a sound physical interpretation for the lattice results.

To properly frame this work, we study a Yang–Mills theory without interactions with other fields like quarks or scalars. This is done by starting with the full theory and developing an exact solution that is afterwards fitted to the lattice results presented in Ref. [3]. This procedure should prove the soundness of our technique for thermal field theory too. It should be emphasized that, at low temperatures, the results presented in Ref. [3] are too few to obtain a significant result beyond $SU(3)$ and $SU(4)$ Yang–Mills theories. Therefore, we limit our analysis to these cases.

The paper is structured as follows. In Section 2, we present our technique to obtain an exact solution of the quantum Yang–Mills theory. In Section 3, we show how the partition function is obtained in our case. In Section 4, we derive all the thermodynamic variables for the Yang–Mills theory. In Section 5, we evaluate the partition function and present our results in comparison with lattice data. Finally, in Section 6, we give our conclusions.

2. Gaussian Solution of Quantum Yang–Mills Theory

We study an exact solution of the quantum Yang–Mills field theory that provides a Gaussian partition function [5–8,14]. It is worthwhile to study such a solution because exact solutions in quantum field theory are quite rare and are mostly given for non-physical models alone. We expect that the solution found here represents the behavior of the Yang–Mills theory. In order to simplify the presentation, we consider a Yang–Mills theory with a $SU(N)$ gauge group. The action at the classical level is given by

$$S_{YM} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + j_a^\mu A_\mu^a \right) \quad (1)$$

(latin letters a, b, c, \dots represent group indices taking the values 1, 2, 3), where j_a^μ is a generic source and an element of the $su(2)$ algebra. The field strength tensor is given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \quad (2)$$

where g is the coupling constant and f^{abc} are the structure constants of the gauge group, leading to the classical equations of motion given by

$$\partial^\mu F_{\mu\nu}^a + g f^{abc} A^{b\mu} F_{\mu\nu}^c = j_\nu^a. \quad (3)$$

In the following, we will provide a solution to Equation (3). The classical equation of motion generalizes to the quantum case by introducing a gauge fixing term and ghost

fields and by considering the partition function (also known as the generating functional, cf., e.g., Ref. [15])

$$\begin{aligned} \mathcal{Z}[j, \bar{\eta}, \eta] = & \int [dA][d\bar{c}][dc] \exp \left(-S_{YM} + \int d^4x \frac{1}{2\xi} (\partial_\mu A_a^\mu) (\partial_\nu A_a^\nu) \right. \\ & \left. + \int d^4x \left(\bar{c}^a \partial_\mu \partial^\mu c^a + g \bar{c}^a f^{abc} \partial_\mu A^{b\mu} c^c \right) - \int d^4x (\bar{\eta}_a c^a + \bar{c}^a \eta_a) \right). \end{aligned} \quad (4)$$

From the partition function, we can derive the pure correlation functions

$$\begin{aligned} G_{n\mu_1\mu_2\ldots\mu_n}^{a_1a_2\ldots a_n}(x_1, x_2, \ldots, x_n) &= \frac{\delta^n \ln Z[j, \bar{\eta}, \eta]}{\delta j_{a_1}^{\mu_1}(x_1) \delta j_{a_2}^{\mu_2}(x_2) \cdots \delta j_{a_n}^{\mu_n}(x_n)}, \\ P_n^{a_1a_2\ldots a_n}(x_1, x_2, \ldots, x_n) &= \frac{\delta^n \ln Z[j, \bar{\eta}, \eta]}{\delta \bar{\eta}_{a_1}(x_1) \delta \bar{\eta}_{a_2}(x_2) \cdots \delta \bar{\eta}_{a_n}(x_n)}, \\ \bar{P}_n^{a_1a_2\ldots a_n}(x_1, x_2, \ldots, x_n) &= \frac{\delta^n \ln Z[j, \bar{\eta}, \eta]}{\delta \eta_{a_1}(x_1) \delta \eta_{a_2}(x_2) \cdots \delta \eta_{a_n}(x_n)} \end{aligned} \quad (5)$$

and mixed correlation functions, for instance,

$$K_{2\mu}^{ab}(x_1, x_2) = \frac{\delta^2 \ln Z[j, \bar{\eta}, \eta]}{\delta j_\mu^a(x_1) \delta \bar{\eta}^b(x_2)}. \quad (6)$$

Our aim is to find a Gaussian solution where all the correlation functions can be expressed by G_1 and G_2 , given the expectation values of the gauge field, that is, the cumulant expansion of the partition function. It should be emphasized that the first two Dyson–Schwinger equations give the analogues of the classical equations of motion and the propagator of the theory, respectively, if the quantum corrections are properly accounted for.

This aim can be achieved by solving the Dyson–Schwinger Equations [5–8]

$$\begin{aligned} \partial^2 G_{1\nu}^a(x) + g f^{abc} \left(G_{1\mu}^b(x) \partial^\mu G_{1\nu}^c(x) - G_{1\mu}^b(x) \partial_\nu G_1^{\mu c}(x) + \partial^\mu \left(G_{1\mu}^b(x) G_{1\nu}^c(x) \right) \right) \\ + g^2 f^{abc} f^{cde} \left(G_{2\mu}^{\mu bd}(x, x) G_1^{\nu e}(x) + G_{2\nu}^{\mu be}(x, x) G_{1\mu}^d(x) \right. \\ \left. + G_{2\mu\nu}^{de}(x, x) G_1^{\mu b}(x) + G_1^{\mu b}(x) G_{1\mu}^d(x) G_{1\nu}^e(x) \right) = 0, \end{aligned} \quad (7)$$

and

$$\begin{aligned} \partial^2 G_{2\nu\kappa}^{am}(x, y) + g f^{abc} \left(\partial^\mu G_{2\mu\kappa}^{bm}(x, y) G_{1\nu}^c(x) + \partial^\mu G_{1\mu}^b(x) G_{2\nu\kappa}^{cm}(x, y) - \partial_\nu G_{2\mu\kappa}^{bm}(x, y) G_1^{\mu c}(x) \right. \\ \left. - \partial_\nu G_{1\mu}^b(x) G_{2\kappa}^{\mu cm}(x, y) + \partial^\mu \left(G_{2\mu\kappa}^{bm}(x, y) G_{1\nu}^c(x) + G_{1\mu}^b(x) G_{1\nu\kappa}^{cm}(x, y) \right) \right) \\ + g^2 f^{abc} f^{cde} \left(G_{2\mu\nu}^{bd}(x, x) G_{2\kappa}^{\mu em}(x, y) + G_{2\nu\rho}^{eb}(x, x) G_{2\kappa}^{\rho dm}(x, y) + G_{2\kappa}^{\mu bm}(x, y) G_{1\mu}^d(x) G_{1\nu}^e(x) \right. \\ \left. + G_1^{\mu b}(x) G_{2\mu\kappa}^{dm}(x, y) G_{1\nu}^e(x) + G_1^{\mu b}(x) G_{1\mu}^d(x) G_{2\nu\kappa}^{em}(x, y) \right) = \delta_{am} g_{\nu\kappa} \delta^4(x - y). \end{aligned} \quad (8)$$

To solve these equations, we use a mapping theorem on the solutions of the classical Yang–Mills equations of motion [16,17] in order to take all components for the 1P-correlation function as equal,

$$G_{1\mu}^a(x) = \eta_\mu^a \phi(x), \quad (9)$$

where η_μ^a are sets of constants (e.g., $\eta = ((0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1))$ for $SU(2)$), and ϕ is a scalar field to be determined. We also take

$$\tilde{G}_{2\mu\nu}^{ab}(x, y) = \delta^{ab} \eta_{\mu\nu} \Delta(x, y) \quad (10)$$

where we have made explicit the projector, and $\Delta(x, y)$ is a scalar propagator to be determined. Anticipating this *ansatz*, the anti-symmetry of f^{abc} was already used in Equations (7) and (8). In doing so, we are able to obtain both G_1 and G_2 in closed form and, in principle, compute the partition function at any desired order for this Gaussian solution. By direct substitution, we see immediately that

$$\begin{aligned}\partial^2\phi + 2\lambda\Delta(x, x)\phi + \lambda\phi^3 &= 0 \\ \partial^2\Delta(x, y) + 2\lambda\Delta(x, x)\Delta(x, y) + 3\lambda\phi^2(x)\Delta(x, y) &= \delta^4(x - y)\end{aligned}\quad (11)$$

with $\lambda = Ng^2$. As $\Delta(x, x)$ is a constant, we can identify this as a mass correction that arises from quantum effects,

$$\delta m^2 = 2\lambda\Delta(x, x). \quad (12)$$

In the following, we will neglect this correction that, after renormalization, is assumed to have a small effect on the spectrum of the theory [7]. The two-point correlation function we obtain is translation invariant. We show this in Appendix A. Though the description of the Yang–Mills theory by a Lagrange density invariant unveils a symmetry under the $SU(N)$ gauge group that is broken by the introduction of the gauge fixing parameter, this is not the symmetry we concentrate on in this paper. Instead, we investigate the breaking of the chiral symmetry of this theory by a mass gap generated dynamically.

3. Partition Function in the IR Regime

Neglecting the mass correction, the solution for G_1 and G_2 can be obtained by taking

$$G_{1\mu}(x) = \eta_\mu^a \mu \operatorname{sn}(p \cdot x + \theta | -1) \quad (13)$$

in terms of integration constants μ and θ , where the gluon momentum p satisfies the dispersion relation $p^2 = \mu^2\lambda/2$, and $\operatorname{sn}(\zeta|\kappa)$ is the Jacobi elliptic function of the first kind with parameter κ . In our approach, G_1 is treated as a background field or as a specific non-trivial vacuum. For more details on the solution, cf. Appendix B.

In a first approximation that holds in the infrared (IR) limit of the Yang–Mills theory (confined phase), our approach allows us to truncate the functional series derived from Equation (4) at the quadratic term with the exact translationally invariant two-point function $G_{2\mu\nu}^{ab}(x_1, x_2) \equiv \tilde{D}_{\mu\nu}^{ab}(x_1 - x_2)$, while the three- and four-point functions are represented as products of one- and two-point functions evaluated at different points. Because of this, the theory is manifestly translation invariant at the level of observables where the Lehmann–Symanzik–Zimmermann reduction formula is assumed to hold.

In the considered approximation with the partition function truncated at the quadratic order, the IR truncated partition function reads [18]

$$Z_{\text{YM}}[j] \Big|_{\text{IR}} \approx Z_{\text{YM}}[0] \Big|_{\text{IR}} \exp \left\{ - \int d^4x G_{1\mu}^a(x) j^{a\mu}(x) - \frac{1}{2} \int d^4x d^4y j^{a\mu}(x) \tilde{G}_{2\mu\nu}^{ab}(x - y) j^{b\nu}(y) \right\}. \quad (14)$$

The momentum space propagator in the Feynman gauge takes the shape

$$G_{2\mu\nu}^{ab}(p) = \delta_{ab} \eta_{\mu\nu} \sum_{n=0}^{\infty} \frac{B_n}{p^2 - m_n^2 + i\epsilon} \quad (15)$$

(see Appendix A for a derivation), where

$$B_n = (2n + 1)^2 \frac{\pi^3}{4K^3(-1)} \frac{e^{-(n+\frac{1}{2})\pi}}{1 + e^{-(2n+1)\pi}}, \quad (16)$$

and the mass spectrum reads

$$m_n = (2n + 1)m_0, \quad m_0 \equiv \frac{\pi\mu}{2K(-1)}, \quad (17)$$

in terms of the mass of the lowest excitation, m_0 . Note that m_0 is a physical parameter in the same way as the scale of asymptotic freedom is generated dynamically in QCD by perturbation theory. We can identify this parameter with the string tension normally evaluated at $\sigma = (440 \text{ MeV})^2$. Stopping to the two-point correlation function, one obtains

$$\begin{aligned} Z_{\text{YM}}[j] &\propto \exp \left[-\frac{1}{2} \int d^4p j_a^\mu(p) G_{2\mu\nu}^{ab}(p) j_b^\nu(-p) \right] \\ &= \exp \left[-\frac{1}{2} \int d^4p j_a^\mu(p) \sum_{n=0}^{\infty} \frac{B_n}{p^2 - m_n^2 + i\epsilon} j_a^\mu(-p) \right]. \end{aligned} \quad (18)$$

Due to the sum, this is equivalent to a product of partition functions of an infinite set of scalar fields $\{\phi_n\}$ (i.e., glue states of mass m_n), each with weight $n_g = 2(N^2 - 1)$ (the factor 2 arises from the Minkowski metric, assuming all the currents are equal). Such a truncation represents a gas of free massive glue states. For such a gas, we expect a very good agreement at low temperatures, as we will see below. Finally, one can use the functional identity

$$\begin{aligned} &\left\{ \exp \left[-\frac{1}{2} \int d^4p j(p) \sum_{k=0}^{\infty} \frac{B_k}{p^2 + m_k^2} j(-p) \right] \right\}^{n_g} \\ &= \left\{ \prod_{k=0}^{\infty} \int \mathcal{D}\phi_k(p) \exp \left[\frac{-1}{2B_k} \int d^4p \phi_k(p) (p^2 + m_k^2) \phi_k(-p) - \int d^4p j(p) \phi_k(-p) \right] \right\}^{n_g}. \end{aligned} \quad (19)$$

Turning back to the partition function of each of the scalar field separately, one can write the Yang–Mills partition function in Equation (14) as

$$Z_{\text{YM}}[0] \Big|_{\text{IR}} \propto Z_{\text{IR}} \equiv \left[\prod_k \int \mathcal{D}\phi_k \exp \left\{ -\frac{1}{2B_k} \int \frac{d^4p}{(2\pi)^4} \phi_k(p) (p^2 + m_k^2) \phi_k(-p) \right\} \right]^{n_g} \quad (20)$$

up to an overall multiplicative constant that is irrelevant for further considerations. Indeed, the source-free Yang–Mills partition function in the non-perturbative regime can be seen as a partition function of a system of infinitely many free scalar fields, where each field contributes with different weights, depending on its mass m_k determined by Equation (20).

We see that, in the IR limit, the spectrum of the theory is quite different from that of the UV limit, where the true states of the theory are massless gluons and we have asymptotic freedom. In the IR limit, we have an infinite spectrum of massive particles that, in our approximation, are not interacting with each other. If we also use this spectrum for the regime of asymptotic freedom as well, i.e., for high temperatures, this will yield an infinite free energy for the model. We will see that for the high-temperature limit, the omission of the gluon states beyond the lowest one is a fairly good approximation.

4. Thermal Yang–Mills Theory

In what follows, we aim to derive the basic thermodynamic properties of the thermal Yang–Mills theory. Note that Equation (20) is the partition function of an infinite number of free scalar fields. Raised to the exponent, the product over the scalar fields adds a further sum over the scalar field modes. In the imaginary-time formulation, Equation (20) can be rewritten as [19]

$$\ln Z_{\text{IR}} = -\frac{n_g}{2} \sum_k \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} \ln \left(\beta^2 B_k^{-1} (\omega_n^2 + \mathbf{p}^2 + m_k^2) \right), \quad (21)$$

where $\omega_n = 2n\pi/\beta$ (with $\beta \equiv 1/T$) are the Matsubara frequencies. Furthermore, making use of the representation

$$\ln[(2\pi n)^2 + \beta^2 \epsilon_k^2] = \int_1^{\beta^2 \epsilon_k^2} \frac{dx^2}{x^2 + (2\pi n)^2} + \ln[1 + (2\pi n)^2]$$

with $\epsilon_k^2 = \epsilon_k^2(\mathbf{p}) \equiv \mathbf{p}^2 + m_k^2$, performing the summation

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + (x/2\pi)^2} = \frac{2\pi^2}{x} \left(1 + \frac{2}{e^x - 1} \right), \quad (22)$$

and turning to the continuum limit for momentum \mathbf{p} in the system of volume V , we arrive at the final form for the partition function of the thermal Yang–Mills theory, given by

$$\ln Z_{\text{IR}} = n_g V \sum_{k=0}^{\infty} \int \frac{d^3 p}{(2\pi)^3} \left[-\frac{\beta \epsilon_k(\mathbf{p})}{2} - \ln \left(1 - e^{-\beta \epsilon_k(\mathbf{p})} \right) \right]. \quad (23)$$

Here, the first expression in square brackets represents the contribution of the zero-point fluctuations of fields with mass m_k in the ground state. In general, the vacuum term $\ln Z_{\text{IR}}^{\text{vac}}$ does not affect the thermodynamic observables in the thermal equilibrium but effectively renormalizes the physical observables by absorbing divergences emerging in the summation over k as $T \rightarrow \infty$, such that the physics is recovered through a finite, properly regulated, partition function, $\ln Z_{\text{IR}} \rightarrow \ln Z_{\text{R}}^{\text{reg}} \equiv \ln Z_{\text{IR}} - \ln Z_{\text{IR}}^{\text{vac}}$.

For further considerations, it is instructive to turn to a more compact representation in terms of a dimensionless variable $z = p\beta$ (with $p \equiv |\mathbf{p}|$), such that

$$\ln Z_{\text{R}}^{\text{reg}} = -P_0 V \beta g(\alpha), \quad P_0 = \frac{n_g}{\beta^4}, \quad (24)$$

$$g(\alpha) \stackrel{\text{reg}}{=} \sum_{k=0}^{\infty} J(a_k), \quad a_k \equiv (2k+1)\alpha = \frac{m_k}{T}, \quad (25)$$

$$J(a_k) \equiv \frac{1}{2\pi^2} \int_0^{\infty} z^2 dz \ln \left(1 - e^{-\sqrt{z^2 + a_k^2}} \right), \quad (26)$$

where a proper regularization of the summation is implied, $J(a_k)$ is the standard thermal integral for a given state with mass m_k defined in Equation (17), $\alpha \equiv \beta m_0$ is the inverse of the temperature measured in units of the mass gap m_0 , and P_0 is the pressure of the free gluon gas. In what follows, we are interested in analyzing such thermodynamic observables of the quantum Yang–Mills fields like pressure

$$P = \beta^{-1} \frac{\partial}{\partial V} \ln Z_{\text{R}}^{\text{reg}} = -P_0 g(\alpha), \quad (27)$$

which is essentially the (normalized) partition function (cf. Equation (23)), and energy density

$$\epsilon = -\frac{1}{V} \frac{\partial}{\partial \beta} \ln Z_{\text{R}}^{\text{reg}} = \frac{\partial}{\partial \beta} (\beta g(\alpha)) \quad (28)$$

as well as the trace of the energy-momentum tensor (trace anomaly), $\Delta \equiv \epsilon - 3P$.

The characteristic energy scale for the Yang–Mills theory will enter our analysis through the ratio m_0/T_c , where m_0 is the mass gap of the theory and T_c the critical temperature of the phase transition, which will be the only parameter to be fitted against the lattice

data. The critical temperature is known from lattice data being around 271 MeV for $SU(3)$ and around 262 MeV for $SU(4)$, according to Ref. [20].

5. Evaluation of the Partition Function

In order to determine the pressure and energy density, we have to evaluate $g(\alpha)$. This is known in series form as

$$g(\alpha) = -\frac{\alpha^2}{2\pi^2} \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^2} (2k+1)^2 K_2(n(2k+1)\alpha) \quad (29)$$

(see Appendix C for a derivation), where $K_2(x)$ is a modified Bessel function. In the low-temperature limit, no convergence problems arise, and the series can be trusted enough for a comparison with lattice data. On the other hand, for $\alpha \rightarrow 0$, i.e., the high-temperature limit, this is no longer true for the series (29), as it should be expected by the infinite spectrum of free particles we are considering. The reason is that we are using an IR approximate solution to evaluate the UV limit of the Yang–Mills theory. This would be possible only by setting the mass gap of the theory to zero.

In the following plots, we compare our results for the pressure and the energy density for $SU(3)$ and $SU(4)$ with those from the lattice. As a single fitting parameter for both plots, we use the ratio m_0/T_c . For the pressure displayed in Figure 1, we see a fairly good approximation $m'_0/T_c \approx (m_0/T_c)(N/3)^{-1/4}$, where m'_0 is the mass gap for $SU(3)$. Both masses are obtained independently by a fit. Because of this, they already take into account the number of colors. For the energy density, we obtain the result shown in Figure 2. Again, we have good agreement, with the dependency on the number of colors being the same as that for the pressure. We see that the ratio m_0/T_c , both for the pressure and energy density, is very similar within the errors of the corresponding fits.

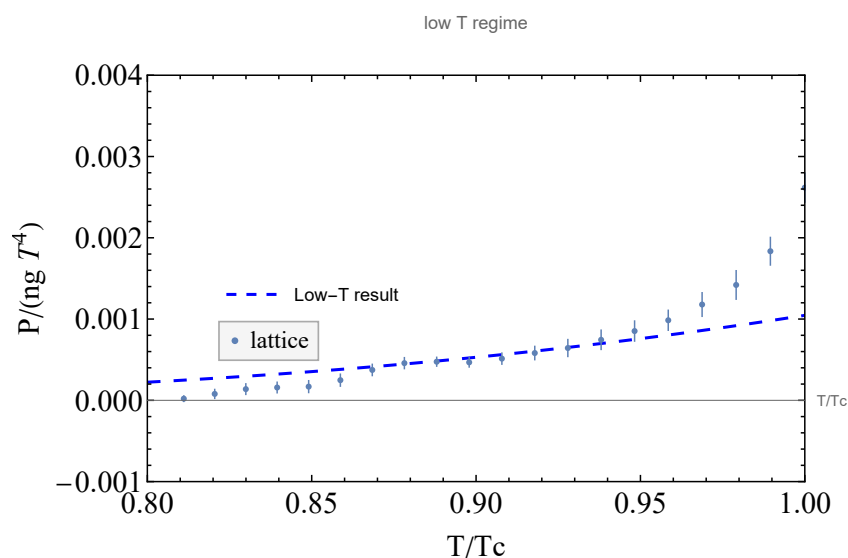


Figure 1. Cont.

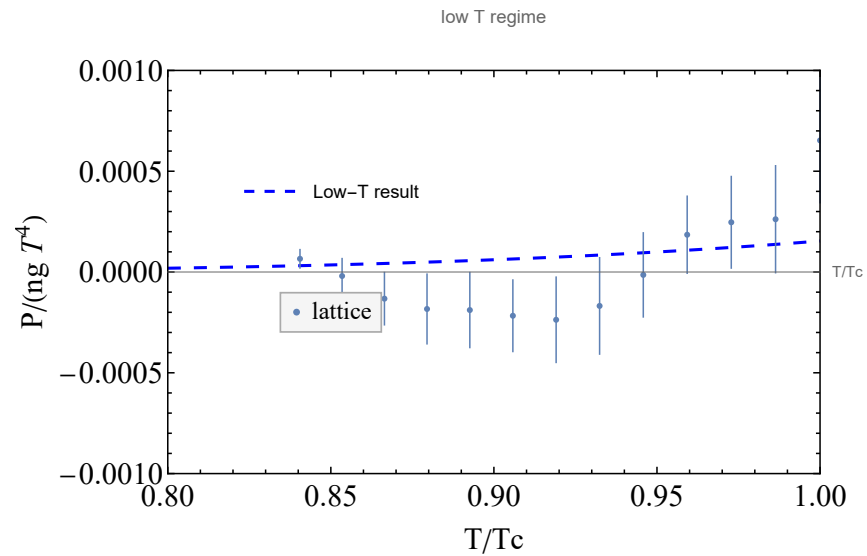


Figure 1. Upper: Plot of the pressure p for $SU(3)$ with the fitted ratio $m_0/T_c \approx 1.61$ compared to the lattice data. **Lower:** The same for $SU(4)$ with the fitted ratio $m_0/T_c \approx 1.95$.

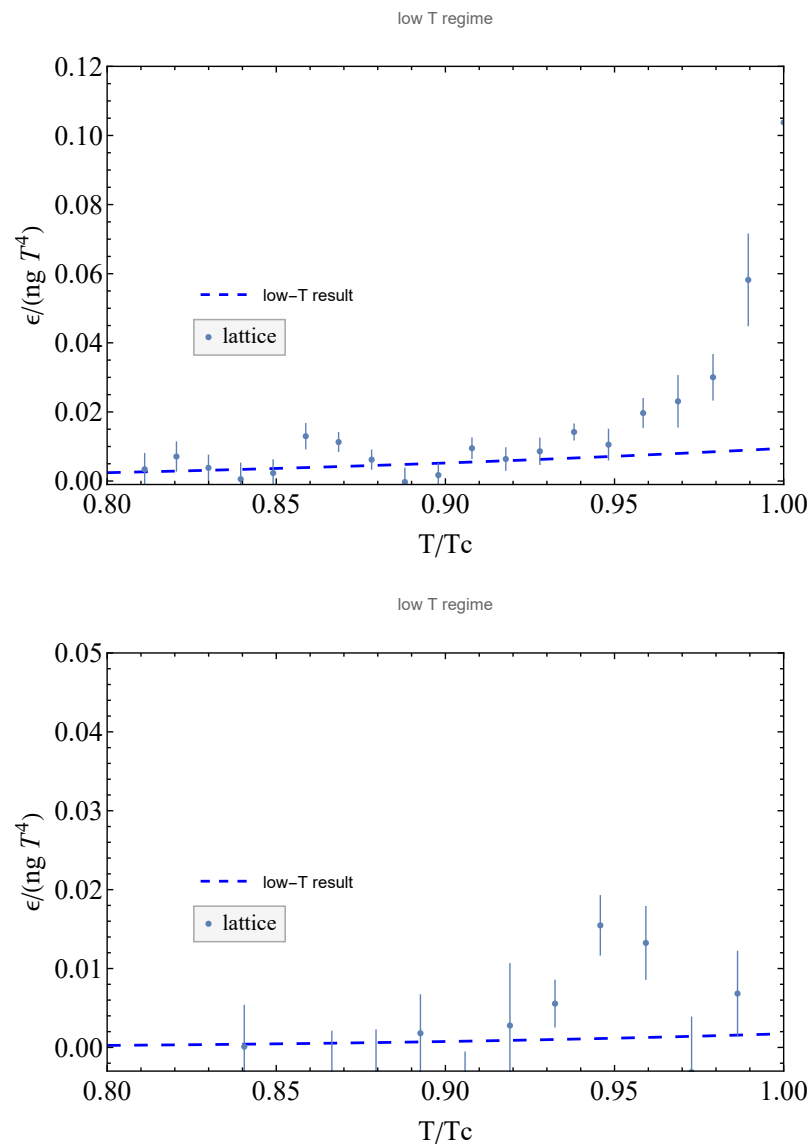


Figure 2. Upper: Plot of the energy density ϵ for $SU(3)$ with the fitted ratio $m_0/T_c \approx 1.61$ compared to the lattice data. **Lower:** The same for $SU(4)$ with the fitted ratio $m_0/T_c \approx 1.95$.

In the high-temperature regime, where perturbative QCD can be satisfactorily applied [21], we achieve the somewhat surprising result that the corresponding thermal series for a scalar field seems to fit very well to the results, mostly for the case of the energy density. We would like to emphasize that the fits we obtain in this regime cannot be successful for the same values of the ratio m_0/T_c . The reason is that we are just extending our approach with a spectrum of massive glue particles to a regime with massless gluons. In addition, the glueball mass can entail some dependence on the temperature that we are not able to catch at this stage [22]. The achievement of satisfactory results, in the high-temperature limit (i.e., for small values of α) too, warrants some extended discussion. For this we consider the expansion

$$g(\alpha) = \frac{\pi^2}{90} - \frac{\alpha^2}{24} + \frac{\alpha^3}{12\pi} + \frac{\alpha^4}{64\pi^2} \left(\log(\alpha^2) - \frac{3}{2} - 2\log(4\pi) + 2\gamma_E \right) + \pi^{\frac{3}{2}} \sum_{k=1}^{\infty} (-1)^k \frac{\zeta(2k+1)}{(k+1)!} \Gamma\left(k + \frac{1}{2}\right) \left(\frac{\alpha^2}{4\pi^2}\right)^{k+2}, \quad (30)$$

where γ_E is the Euler–Mascheroni constant. This corresponds to considering Equation (29) for only a single term and expanding for high temperatures for $\alpha \rightarrow 0$. The corresponding plots are given in Figures 3 and 4. From these plots, it can be easily observed how harsh our approximation is compared to standard perturbative computation. Nevertheless, for the energy density in the high-temperature regime, we obtain Figure 4. The agreement in this case is excellent with the rule for the ratio m_0/T_c and the number of colors satisfied again.

We have fitted each quantity separately, as we cannot expect that the ratio m_0/T_c will stay the same, and for consistency reasons. For the high-temperature limit, the ratio m_0/T_c for the energy density is somewhat distant from the one for the pressure for $SU(4)$. Overall, the agreement is excellent for a single fit parameter. Taking into account that we are working with asymptotic series with zero radius of convergence, we seem to have evaluated the optimal number of terms for a satisfactory agreement with lattice data.

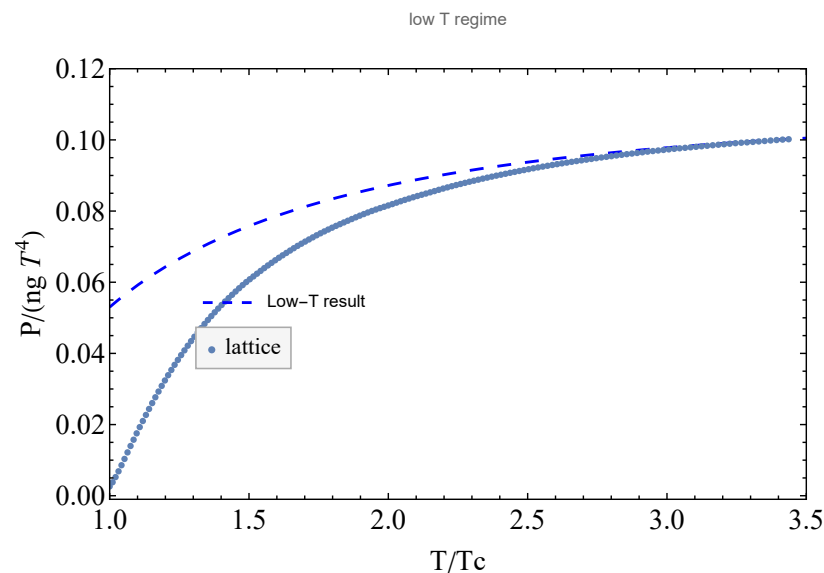


Figure 3. Cont.

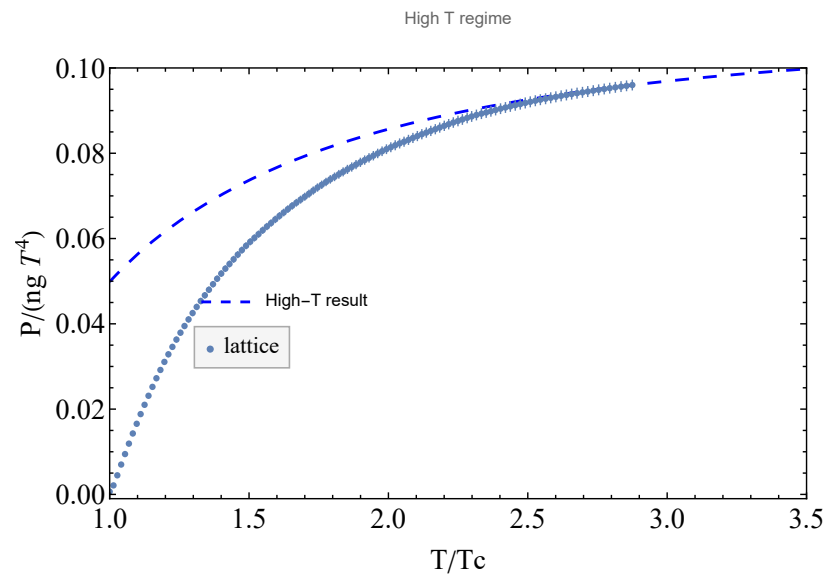


Figure 3. Upper: Plot of the pressure p for $SU(3)$ in the high-temperature limit with the fitted ratio $m_0/T_c \approx 0.53$ compared to the lattice data. **Lower:** The same for $SU(4)$ with the fitted ratio $m_0/T_c \approx 0.41$.

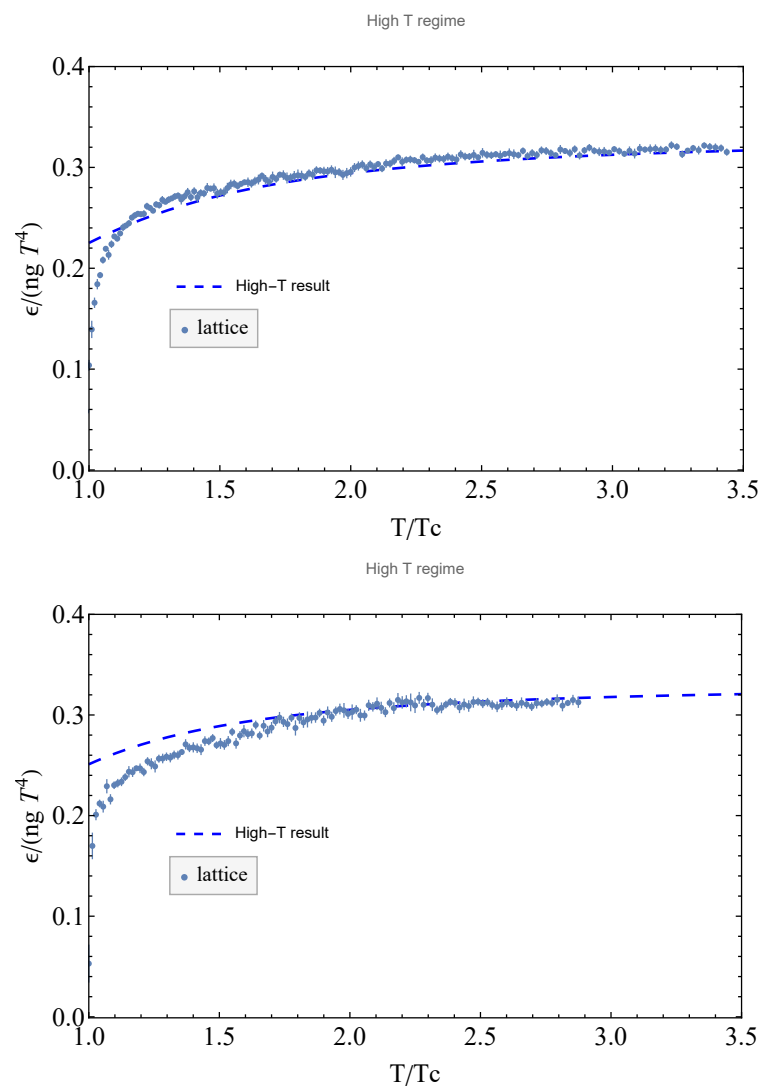


Figure 4. Upper: Plot of the energy density ϵ for $SU(3)$ with the fitted ratio $m_0/T_c \approx 0.53$, as compared to the lattice data. **Right:** The same for $SU(4)$ with the fitted ratio $m_0/T_c \approx 1.59$.

6. Discussion and Conclusions

In this work, we have presented the non-perturbative partition function of the thermal $SU(N)$ Yang–Mills theory in the framework of Dyson–Schwinger equations with a non-trivial ground state in analytic form. Utilizing a novel technique for solving these equation in a differential form, we find very good consistency for the resulting thermodynamical observables—pressure and energy density—with the corresponding lattice results available for $N = 3$ and $N = 4$ colors. Our analytical approach provides a pathway for obtaining thermodynamic characteristics in terms of input parameters to a given action of the Yang–Mills theory. Indeed, in the considered IR limit of the theory, we show that the functional series for the partition function, already truncated at the quadratic term, shows good agreement between our results and the lattice data at low temperatures. While a standard technique could be applied for high temperatures, we considered the solution for the scalar field and found very good agreement for the energy density.

As the quadratic approximation already captures the most essential features of non-perturbative Yang–Mills dynamics at low temperatures, we believe that, with a complete evaluation of the partition function going beyond the quadratic order and accounting for higher-order correlation functions, a more precise behavior could be found, possibly with evidence for a smooth transition from a low-energy spectrum with glue states to that of a pure massless gluon spectrum, as observed in the asymptotic freedom regime.

A scenario with a spectrum of massive glue state excitations in a thermal setting for the Yang–Mills theory, starting from the available lattice data, has already been devised in the literature [23,24]. In our study, we demonstrate that the same behavior can be obtained analytically.

Cosmological first-order phase transitions (FOPT) in the early universe are known to radiate gravitational waves (GWs), which are potentially detectable in upcoming GW missions like LISA [25–28]. If such a signal were to be detected, this would inevitable give hints for new physics, since the electroweak and QCD phase transitions in the standard model are known as cross-overs, not as FOPT. Consequently, studying such GW signals could allow us to probe the dynamics of otherwise completely inaccessible dark or hidden sectors [29–31]. Therefore, we envisage a possible application of our approach to studies of such phase transitions in the early universe, particularly for strong coupling. $SU(N)$ Yang–Mills theories, featuring color confinement string FOPTs [3,32], naturally appear in several well-motivated viable extensions of the standard model; see studies in Refs. [33–41]. Since the only free and independent parameter here is the confinement scale, these scenarios can be deemed minimal, thus serving as suitable benchmark models. Previous attempts to study the GW signal in such models in Refs. [42–49] suffer from uncertainties related to non-perturbative effects, or rely on lattice calculations or methods like AdS/CFT [50] due to the strong coupling involved. We envisage the method developed in this paper to make quantitative estimates for such scenarios. As this is beyond the scope of the present paper, we plan to take up this subject in a future publication, given the timeline of the LISA experiment that is coming very soon and LiGO already taking data at present.

Our results are expected to also have a strong impact on a possible resolution of the long-standing problem of deriving the QCD equation of the state and the precise reconstruction of the QCD phase diagram, critical for many ongoing studies in particle physics (e.g., physics of the quark–gluon plasma) and astrophysics (e.g., the dynamics of neutron stars), and other issues in cosmology (e.g., dark matter, dark energy, and inflation); see Refs. [51–53].

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Appendix A. Two-Point Correlation Function and Spectrum

We have to solve the set of Dyson–Schwinger Equation (11); cf. Refs. [5–8]. The aim is to obtain a Fubini–Lipatov solution to represent the vacuum of the theory and a translation invariant two-point correlation function. Actually, both the equations can be solved analytically using Jacobi elliptic functions [54,55]. In physics, elliptic integrals and functions appear, for instance, for the exact solution of the physical pendulum, in the small angle approximation limited by the mathematical pendulum that obeys a linear differential equation. Without going into too much detail, the period of a physical pendulum is expressed in terms of the incomplete elliptic integral

$$F(\varphi|\kappa) = \int_0^\varphi \frac{du}{\sqrt{1 - \kappa \sin^2 u}} \quad (\text{A1})$$

with parameter κ , where the complete elliptic integral given by $K(\kappa) = F(\pi/2|\kappa)$. The Jacobi elliptic functions are defined via this incomplete elliptic integral by the inversion of the equation $z = F(\varphi|\kappa)$, in the sense that $\text{sn}(z|\kappa) := \sin \varphi$, $\text{cn}(z|\kappa) := \cos \varphi$, and $\text{dn}(z|\kappa) := (1 - \kappa \sin^2 \varphi)^{1/2}$. Using the Jacobi elliptic sine, we can write

$$\phi(x) = \mu \text{sn}(p \cdot x + \theta, \kappa), \quad \kappa = \frac{\delta m^2 - p^2}{p^2}, \quad (\text{A2})$$

where $\delta m^2 = 2\lambda\Delta(x, x)$, and θ and μ are two integration constants. This function is a solution for the nonlinear differential equation shown in the first line of (11), provided the dispersion relation

$$p^2 = \delta m^2 + \mu^2 \frac{\lambda}{2} \quad (\text{A3})$$

holds. It is interesting to point out that $-\phi(x)$ also solves the first equation in (11). Therefore, to choose one or the other solution spontaneously breaks the Z_2 symmetry. Similarly, one can show that the two-point correlation function can be written in the form

$$\Delta(p) = \frac{\pi^3}{4K^3(\kappa)} \sum_{n=0}^{\infty} (-1)^n (2n+1)^2 \frac{q^{n+1/2}}{1 - q^{2n+1}} \frac{1}{p^2 - m_n^2 + i\epsilon}, \quad (\text{A4})$$

where $q = \exp(-\pi K(1 - \kappa)/K(\kappa))$, with $K(\kappa)$ being the complete elliptic integral of the first kind. The technique is shown in Ref. [14] for the classical case but also applies straightforwardly here. The mass spectrum is

$$m_n = (2n+1) \frac{\pi\mu}{2K(\kappa)}. \quad (\text{A5})$$

In order to go back to Equation (16), we set $\delta m = 0$. In this case we obtain $\kappa = -1$ and $q = \exp(-\pi(1-i))$. Inserting this back into Equation (A4), we are able to obtain the coefficients given in the main text.

Appendix B. One-Point Correlation Function

In taking $A_\mu^a(x)$ instead of $G_{1\mu}^a(x)$, Equation (7) for $G_{1\nu}^a(x)$ maps the classical equation of motion very well, given by

$$\partial^\mu (\partial_\mu A_\nu^{a(0)} - \partial_\nu A_\mu^{a(0)} + g f^{abc} A_\mu^{b(0)} A_\nu^{c(0)} + g f^{abc} A^{b(0)\mu} (\partial_\mu A_\nu^{c(0)} - \partial_\nu A_\mu^{c(0)} + g f^{cde} A_\mu^{d(0)} A_\nu^{e(0)}) = 0, \quad (\text{A6})$$

once we select the Feynman gauge [5]. However, there is an important difference as follows: Equation (7) contains a quantum correction of the form $G_{2\mu\nu}^{ab}(x, x)$. After regularizing the divergences, this correction can have the effect of a small shift in the spectrum of the theory [7] that can be neglected. Our aim is to show how the Jacobi elliptic functions are suited to this equation. By taking the mixed symbols, in the simple case of $SU(2)$ ($a = 1, 2, 3$ is the group index), in the form

$$\eta_\mu^1 = (0, 1, 0, 0), \quad \eta_\mu^2 = (0, 0, 1, 0), \quad \eta_\mu^3 = (0, 0, 0, 1), \quad (\text{A7})$$

and using the mapping theorem between a scalar field and the Yang–Mills field [16,17],

$$A_\mu^{a(0)}(x) = \eta_\mu^a \phi_0(x), \quad (\text{A8})$$

$\phi_0(x)$ solves

$$\partial^2 \phi_0 + 2g^2 \phi_0^3 = 0, \quad (\text{A9})$$

that is the first of Equation (11) for $SU(2)$ when the quantum corrections are neglected. Considering a solution of the form $\phi_0(x) = \mu \operatorname{sn}(p \cdot x + \theta | -1)$, we obtain

$$\begin{aligned} \partial_i^2 \phi_0(x) &= \mu p_0^2 (\operatorname{cn}^2(p \cdot x + \theta | -1) - \operatorname{dn}^2(p \cdot x + \theta | -1)) \operatorname{sn}(p \cdot x + \theta | -1), \\ \nabla^2 \phi_0(x) &= \mu |\mathbf{p}|^2 (\operatorname{cn}^2(p \cdot x + \theta | -1) - \operatorname{dn}^2(p \cdot x + \theta | -1)) \operatorname{sn}(p \cdot x + \theta | -1). \end{aligned} \quad (\text{A10})$$

Therefore,

$$\begin{aligned} \partial^2 \phi_0 + 2g^2 \phi_0^3 &= \mu \operatorname{sn}(p \cdot x + \theta | -1) \times \\ &\left[(E^2 - |\mathbf{p}|^2) (\operatorname{cn}^2(p \cdot x + \theta | -1) - \operatorname{dn}^2(p \cdot x + \theta | -1)) + 2g^2 \mu^2 \operatorname{sn}^2(p \cdot x + \theta | -1) \right]. \end{aligned} \quad (\text{A11})$$

Using the identity $\operatorname{cn}^2(z | -1) - \operatorname{dn}^2(z | -1) = -2 \operatorname{sn}^2(z | -1)$ [56], Equation (A9) is solved, provided the dispersion relation $p^2 = \mu^2 g^2$ for $SU(2)$ holds. If the quantum corrections are neglected, we then obtain

$$G_{1\mu}^a(x) = \eta_\mu^a \phi_0(x) = \eta_\mu^a \mu \operatorname{sn}(p \cdot x + \theta | -1), \quad (\text{A12})$$

that is Equation (13).

Appendix C. Evaluation of the Partition Function

Noting that the exponential function grants convergence for $z > 0$, one can expand

$$\ln \left(1 - e^{-\sqrt{z^2 + a_k^2}} \right) = -\frac{1}{n} e^{-n\sqrt{z^2 + a_k^2}}. \quad (\text{A13})$$

Because of the convergence of this series, one can freely interchange the integration and summation to obtain

$$J_k = \int_0^\infty \ln\left(1 - e^{-\sqrt{z^2 + a_k^2}}\right) z^2 dz = - \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty e^{-n\sqrt{z^2 + a_k^2}} z^2 dz. \quad (\text{A14})$$

The integral term can be rewritten in a known form via the change of variables $w^2 = z^2 + a_k^2$. This yields (see Ref. [57], 3.389.4)

$$\int_{a_k}^\infty w(w^2 - a_k^2)^{\frac{1}{2}} e^{-nw} dw = \frac{1}{n} a_k^2 K_2(na_k), \quad (\text{A15})$$

where $K_2(x)$ is a modified Bessel function of the second kind. This yields Equation (29) in the main text. In physics, Bessel functions describe the vibration modes of a circular membrane, being solutions to the Bessel differential equation $x^2 f''(x) + x f'(x) + (x^2 - \alpha^2) f(x) = 0$. Modified Bessel functions are those that obey the modified Bessel differential equation $x^2 f''(x) + x f'(x) + (x^2 + \alpha^2) f(x) = 0$. An integral representation of the modified Bessel function of the second kind (diverging at the origin) is given by

$$K_\alpha(x) = \int_0^\infty e^{-x \cosh t} \cosh(\alpha t) dt, \quad \cosh(z) = \frac{1}{2}(e^z + e^{-z}). \quad (\text{A16})$$

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