The $U(1)^3$ Model of Euclidean Quantum Gravity

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The beauty of mathematics only shows itself to more patient followers.

Maryam Mirzakhani (1977 - 2017)

زیبایی ریاضیات خود را تنها برای پیروان شکیباتر نمایان می‌کند.

مریم میرزاخانی (۱۳۹۶–۱۳۵۶)

Dedicated to my beloved parents

and in memory of my two grandfathers

Abstract

Loop quantum gravity is one of the leading candidates to construct a mathematically well-defined, non-perturbative and background independent quantisation of general relativity. In spite of the well developed kinematical structure, loop quantum gravity is plagued by the lack of an unambiguous formulation of its dynamics. This long-standing issue of dynamics is one of the main open problems yet to be addressed.
Finding a way to improve the situation requires rumination and a deeper understanding of the problem which can be provided by applying LQG techniques to simpler theories. So far, studying such toy models has given us a new perception of the difficulties and ambiguities of dynamics. The weak coupling limit of Euclidean gravity serves as a toy model which captures the significant structure of GR. It is simply constructed by replacing the constraints of the Ashtekar-Barbero $SU(2)$ gauge theory of Euclidean gravity with their $U(1)^3$ version. In particular, it demonstrates a non-trivial realisation of the hypersurface deformation algebra which proves it can be an interesting testing ground for (Euclidean) quantum gravity.
In this thesis, we study different approaches for the quantisation of the $U(1)^3$ model. Since important levels of Dirac quantisation have already been processed and analysed by Varadarajan et al, our key results mainly concern establishing a suitable framework for the other two approaches; i.e. the reduced phase space and path integral quantisations. This is especially attractive because, in the end, one can compare outcomes of different methods of quantisation to gain fascinating insights into the problem of dynamics.
More precisely, we show that in suitable gauges, it is feasible to find a closed and explicit formula for the physical Hamiltonian governing dynamics of physical observables. The rather simple expression of the physical Hamiltonian promises that its quantisation may not be as difficult as one expects. It turns out that analysis of the reduced phase space needs the asymptotic behaviours of the canonical variables, we hence investigate in detail the asymptotically flat boundary conditions of the $U(1)^3$ model, leading to well-defined generators for temporal and spatial asymptotic translations. Furthermore, the covariant origin of the $U(1)^3$ model is found which is considered as the main foundation of path integral quantisation.

چکیده

گرانش کوانتومی حلقه‌ای یکی از کاندیداهای پیشرو برای ساخت یک نظریه کوانتومی خوش‌تعریف غیراختلالی و مستقل از پس‌زمینه است. علیرغم داشتن ساختار سینماتیکی به خوبی توسعه‌یافته، گرانش کوانتومی حلقه‌ای گرفتار فقدان یک فرمول روشن از دینامیک آن است. این مشکل دیرینه دینامیک یکی از مسائل باز اصلی است که هنوز باید به آن پرداخته شود.

یافتن راهی برای بهبود وضعیت نیاز به درکی عمیق‌تر از این مشکل دارد که می‌تواند با به‌کار بستن تکنیک‌های LQG در نظریه‌های ساده‌تر به دست آید. تاکنون مطالعه این گونه مدل‌ها درک جدیدی از مشکلات و ابهامات دینامیک برای ما فراهم کرده است. حد ضعیف گرانش اقلیدسی یکی از این مدل‌ها تلقی می‌شود که ساختار بااهمیت GR را نمایش می‌دهد. این نظریه به سادگی با جایگزینی قیود نظریه گرانش اقلیدسی، که برحسب متغیرهای $SU(2)$ اشتکار– باربرو نوشته شده‌اند، با نسخه $U(1)^3$ آنها ساخته می‌شود. به طور خاص، یک تحقق غیربدیهی از جبر دگردیسی ابررویه‌ای را نشان می‌دهد که ثابت می‌کند این نظریه می‌تواند یک زمینه آزمایشی جالب برای گرانش کوانتومی (اقلیدسی) باشد.

در این پایان‌نامه، ما رویکردهای مختلف را برای کوانتش مدل $U(1)^3$ مطالعه می‌کنیم. از آن‌جایی که فرآیند کوانتش دیراک قبلاً توسط وارادراجان و همکارانش تکمیل شده است، نتایج کلیدی ما عمدتاً مربوط به ایجاد یک چارچوب مناسب برای دو رویکرد دیگر است، یعنی کوانتش فضای فاز تقلیل‌یافته و کوانتش انتگرال مسیر. جذابیت نهفته در این مطالعات در این است که، در پایان، می‌توان نتایج روش‌های مختلف کوانتش را با هم مقایسه کرد تا به بینش‌های بهتری در مورد مسئله دینامیک دست یافت.

به‌طور دقیق‌تر، نشان می‌دهیم که در پیمانه‌های مناسب، یافتن یک فرمول صریح برای هامیلتونی فیزیکی حاکم بر دینامیک مشاهده‌پذیرهای فیزیکی امکان‌پذیر است. عبارت نسبتاً ساده هامیلتونی فیزیکی این نوید را می‌دهد که کوانتش آن احتمالاً آن‌گونه که انتظار می‌رود دشوار نباشد. در این مسیر به این نتیجه رسیدیم که بررسی کامل فضای فاز تقلیل‌یافته به چگونگی رفتارهای مجانبی متغیرهای کانونی نیازمند است، از این‌رو مجبور شدیم شرایط مرزی مجانبی مدل $U(1)^3$ را با جزئیات بررسی کنیم، که منجر به مولدهای خوش‌تعریف برای انتقال‌های فضایی و زمانی شد. علاوه بر این، منشاء هموردای مدل $U(1)^3$ را هم یافتیم که به عنوان پایه اصلی کوانتش انتگرال مسیر در نظر گرفته می‌شود.

Zusammenfassung

Die Schleifenquantengravitation ist eine der führenden Kandidatinnen für eine mathematisch wohldefinierte, nicht-perturbative und hintergrundunabhängige Quantisierung der Allgemeinen Relativitätstheorie (ART). Trotz der weit entwickelten kinematischen Struktur dieser Theorie bestehen noch einige Defizite hinsichtlich einer eindeutigen Formulierung ihrer Dynamik. Dieses seit langem bestehende Problem der Dynamik ist eines der wichtigsten offenen Probleme der Schleifenquantengravitation, die noch angegangen werden müssen.
Um einen Weg zu finden, die Situation zu verbessern ist ein tieferes Verständnis des Problems erforderlich, das durch die Anwendung von LQG-Techniken auf einfachere Theorien gewonnen werden kann. Bisher hat das Studium solcher Spielzeugmodelle eine bessere Einsicht in die Schwierigkeiten und Mehrdeutigkeiten der Dynamik ermöglicht. Der Grenzfall einer schwachen Kopplung der euklidischen Gravitation kann hierbei als Spielzeugmodell dienen, das die Struktur der ART erfasst. Sie wird konstruiert, indem die Randbedingungen der Ashtekar-Barbero $SU(2)$ Eichtheorie der euklidischen Gravitation durch ihre $U(1)^3$ Version ersetzt werden. Insbesondere zeigt es eine nichttriviale Realisierung der Hyperflächen-Deformationsalgebra, die beweist, dass sie ein interessantes Testfeld für die (euklidische) Quantengravitation sein kann.
In der vorliegenden Dissertationsschrift untersuchen wir deshalb verschiedene Ansätze zur Quantisierung des $U(1)^3$ Modells. Da wichtige Stufen der Dirac Quantisierung bereits von Varadarajan et al bearbeitet und analysiert wurde, beziehen sich unsere wichtigsten Ergebnisse hauptsächlich auf die Schaffung eines geeigneten Rahmens für die anderen beiden Ansätze, insbesondere auf die reduzierten Phasenraum- und Pfadintegralquantisierungen. Letztere sind besonders attraktiv, da man nun die Ergebnisse verschiedener Quantisierungsmethoden vergleichen kann, um tiefere Einblicke in das Problem der Dynamik zu erhalten.
Genauer gesagt zeigen wir, dass es bei einer geeigneten Wahl der Eichung möglich ist, eine geschlossene und explizite Formel für den physikalischen Hamilton-Operator zu finden, der die Dynamik physikalischer Observablen bestimmt. Die eher einfache Form dieses physikalischen Hamilton-Operators weist darauf hin, dass dessen Quantisierung möglicherweise nicht so schwierig ist, wie man es erwartet. Es stellt sich heraus, dass die Analyse des reduzierten Phasenraums das asymptotische Verhalten der kanonischen Variablen benötigt. Daher untersuchen wir im Detail die asymptotisch flachen Randbedingungen des $U(1)^3$-Modells, was zu wohldefinierten Generatoren für zeitliche und räumliche asymptotische Translationen führt. Außerdem wird der kovariante Ursprung des $U(1)^3$-Modells gefunden, das als Hauptgrundlage der Pfadintegralquantisierung angesehen wird.

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Independently, I appreciate the author of the book [1] which was the source of my interest and passion on loop quantum gravity. This book provides a perfect guidance with all mathematical and conceptual details for those who wish to work on LQG. I remember nights when I could not sleep because of the excitement of reading this book, it has been my best teacher!
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Contents

Introduction

0.1. Motivation and Historical Review

0.1.1. Quantum Gravity

Quantum field theory (QFT) [3] and general relativity (GR) [4] are broadly deemed to be two successful and well-verified fundamental theories describing nature. Defining a theory of quantum gravity, i.e., a theory that consistently combines GR and QFT has been one of the main challenges in theoretical physics in the past decades. In addition to the interest in constructing a unified theory to describe the world, the significance of having a quantum gravity theory becomes crucial in circumstances where both quantum mechanics and strong gravity are relatively important. Also, due to the interactions between the quantum world and the gravitational field itself, gravity is automatically required to be understood in a quantum language. However, at first glance, these two theories do not seem to be conciliable because they portray the world in discrepant ways: while GR works with smooth structures and fully deterministic systems, where space-time is a dynamical object like any other, QFT uses a probabilistic one where space-time variables are just parameters. Moreover, as the sought-after theory of quantum gravity is expected to manifest itself mainly at the Plank scale which is so far away from the scope of our experimental instruments, the theory cannot be constructed based on a wealth of physical evidence. Nevertheless, the hope is that in the pursuit of a conceptual and theoretical unification, not only might new testable predictions spring up but shortcomings of both theories might also be addressed. In QFTs, divergences are often treated by introducing a cutoff and casting away terms including this cutoff. It is envisioned that if gravity can be consistently taken into account, this problem will be cured naturally as GR provides a cutoff in terms of the Planck length. On the other hand, GR suffers from inevitable singularities in the framework of gravitational collapse under reasonable physical conditions on energy and causality [5, 6, 7]. These singularities rule the internal structure of black holes and to the initial singularity of the cosmos. Hence, in presence of strong gravitational fields, GR loses its predictability and cannot be considered as a valid theory. Fundamentally, such singularities are unphysical and considering quantum effects is expected to resolve them [8].

Despite the fact that a consensual theory for quantum gravity is yet to be discovered, there are several promising approaches available with impressive progress, e.g. string theory [9], Loop Quantum Gravity (LQG) [1, 10], group field theory [11], Causal sets [12], asymptotic safety [13], causal dynamical triangulation [14], etc. A key objective that needs to be addressed in all tentative theories of quantum gravity, is their ability to produce clear observational predictions [15] that can be tested against data to confirm the validity of the theory. In this regard, there is reasonable hope that primordial gravitational waves may soon be observed through B-modes in the CMB: this would be the first direct observation of a quantum gravity phenomenon, albeit

at a linear level. Nonetheless, alongside working on the observational predictions, it is of critical importance to work on the completion of their theoretical aspects simultaneously, to gain a better and deeper understanding of the sector where gravity and quantum both play notable roles.

The first endeavours to quantise gravity date back almost to the years of the invention of quantum theory, when in the context of the linearised GR, the metric perturbations propagating on a fixed Minkowskian spacetime was quantised and the graviton was introduced [16, 17]. Afterwards, the Feynman rules of GR were founded [18] and it was realised that there are two loop divergences in the theory that make it non-renormalisable. This is rooted in the "dimensionful" coupling constant of graviton that gives rise to a perturbatively non-renormalizable QFT [19, 20]. In spite of that supergravity [21] and supersymmetric string theory emerges in the hope that one can still think of GR as the low energy limit of a more fundamental QFT such that additional higher energy terms in the Lagrangian may resolve the intrinsic divergencies of GR. But, so far, there is no proof that string theory cures the whole divergencies of the four-dimensional quantum linearised version of GR [1]. Although important efforts were made also in the direction of non-perturbative string theory (i.e. M-theory), a full background independent formulation of the theory is still elusive. This justifies investigating other non-perturbative and background independent approaches to quantum gravity such as loop quantum gravity[1].

In parallel with Feynman's quantisation approach, attempts to tackle the quantum gravity problem from the canonical point of view was also developed. Dirac worked out the formalism of constrained systems and applied it to GR to obtain the canonical structure of the theory that was simplified by Arnowitt, Deser and Misner [23, 24], i.e. in the so-called ADM approach. It turned out that GR is a fully constrained system with two families of constraints, namely diffeomorphism and Hamiltonian constraints. According to Dirac's algorithm, one first has to find a kinematical Hilbert space that supports the quantum operators corresponding to the constraints, and then the kernel of the operators, which defines the physical Hilbert space, is supposed to be determined. The first quantum form of the Hamiltonian constraint (that was the representative of the dynamics) was written down formally by Wheeler and DeWitt, which became known as the Wheeler-DeWitt equation. This could not be the end of the story as this equation turned out to be too ill-defined and the early attempts [8, 25, 26, 27] to rigorously define it, were doomed to failure. In fact, the highly complicated and non-polynomial structure of the Hamiltonian constraint would preclude any progress in even defining the quantum operator, let alone constructing a kinematical Hilbert space. The solution to this problem remained obscure for almost twenty years until the building blocks of "loop quantum gravity" were gradually established.

0.1.2. Loop Quantum Gravity

LQG is a very favourable approach to quantise gravity that is non-perturbative and manifestly background-independent. Moreover, in contrast to string theory, it does not rely on additional dimensions or the existence of supersymmetric particles, albeit extra dimensions or supersymmetry can be assimilated into LQG, as shown in [28, 29, 30, 31, 32, 33, 34]. In principle, LQG strives to quantise GR directly by employing the least possible number of extra structures.

The starting point of LQG traces back to Ashtekar's papers [35, 36] in which, inspired by [37], he introduced new variables for GR and provided a canonical formulation of GR similar to that of Yang-Mills theory. In other words, he provides tools to describe the phase space of GR in such a way that is feasible to convert it into a quantum language. Instead of the spatial metric

[1] For more information on the history of quantum gravity see [22]. Moreover, in his book [1], Thiemann provided a large amount of historical and physical reasons pushing us to work on LQG.

and extrinsic curvature, Ashtekar used triads and complex connections as canonical variables in terms of which the Hamiltonian constraint turned to a fourth-order polynomial that was a drastic simplification compared to the highly non-polynomial Wheeler-DeWitt equation. He introduced the new variables at the price of enlarging the phase space and, hence, having an additional constraint known as the Gauß constraint. Since Ashtekar's original connection was a $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form, there were two obstacles to accessing a Hilbert space representation: 1) non-compactness feature of the group $SL(2,\mathbb{C})$: a mathematically rigorous quantisation for non-compact gauge groups invokes severe technical difficulties, and 2) non-real canonical variables that drag along the reality conditions in order to recover real GR: the quantum implementation of the reality conditions is intricate. Neither of these two problems has been overcome as yet.

After about nine years, Barbero and later Immirzi [38, 39, 40] generalised Ashtekar's original idea by introducing a free parameter γ named after them, such that $\gamma = i$ recovers Ashtekar's original proposal. Furthermore, for a real non-zero Barbero-Immirzi parameter not only is the connection real but also it is valued in the Lie algebra of the "compact" group $SU(2)$. However, this was achieved at the price of losing the simple structure of the Hamiltonian constraint; again one had to deal with a highly non-polynomial constraint!

Although the Hamiltonian constraint was again complicated, the virtue of working with Barbero's real variables was that thanks to the compactness of $SU(2)$, one could now rigorously construct a kinematical Hilbert space. In fact, a great deal of progress had been made in the front line of background-independent quantisation of gauge theories with compact groups [41, 42, 43, 44, 45, 46, 47, 48]. Applying this quantisation procedure to the $SU(2)$ Ashtekar-Barbero formulation nowadays is called canonical loop quantum gravity. An orthogonal basis for the resulting kinematical Hilbert space was introduced by Rovelli and Smolin [49]. They realised that the Hilbert space is spanned by spin network functions that are functions of connections defined on finite graphs in the spatial hypersurface, whose edges are labelled by irreducible representations of $SU(2)$. This recovered an independent construction of spin networks due to Penrose [50]. The next step toward completing the quantisation was the implementation of the constraints. The Gauß constraint was the most straightforward to incorporate in the kinematical Hilbert space. Indeed, by placing an intertwiner at each vertex of the graph the spin network is based on, one can naturally obtain gauge-invariant spin network functions. The incorporation of the diffeomorphism constraint turned out to be more subtle but was achieved using group averaging methods in [51, 52]. Therefore, the general quantum solutions to two of three types of constraints were attained [48]. Afterwards, this notable theorem was proven that, under mild assumptions, the chosen representation actually is unique [53, 54]. What remained is dealing with the Hamiltonian constraint which encodes the "dynamics" of the theory and is the most difficult to impose.

0.1.2.1. Achilles' Heel of LQG: Dynamics

Following Dirac's quantisation procedure, the dynamics should be assigned to the quantum theory by 1) promoting the classical Hamiltonian constraint into an operator, 2) seeking the states belonging to its kernel, which are called physical states, and 3) finding a suitable inner product between the physical states. Then, the inner product of the physical Hilbert space consisting of physical states determines the dynamics, i.e. transition amplitudes between physical states. For about ten years, not even a well-defined operator could be constructed for this intractable constraint, let alone to get the desired physical Hilbert space. Finally, Thiemann, in his trailblazing work [55, 56, 57], introduced a strategy that takes advantage of an identity between the triads and the Poisson bracket among the connection and the volume function. Using this identity, he

managed to build a well-defined Hamiltonian operator on the kinematical Hilbert space. Let's denote the resulting operator by $\hat{C}(N)$, where N is the smearing function. Although the formal solutions of $\hat{C}(N)$ are known, explicit derivation of any non-trivial solution has turned out to be extremely difficult, and hence the structure of the physical Hilbert space is still scarcely perceived.

Apart from this, despite the fact that the original formulation [55, 56, 57] is anomaly free[2], it does not mean that the precise structure functions of the classical formulation are recovered. This must indeed be a delicate issue because in contrast to the classical theory, in the quantum theory the generator of spatial diffeomorphisms does not exist due to lack of continuity of 1-parameter unitary subgroups of the spatial diffeomorphism group. This is an important issue to be addressed because Thiemann's Hamiltonian constraint operator suffers from ambiguities that are expected to be resolved by achieving a faithful representation of the constraint algebra.

The problem of dynamics has become a prolific source of inspiration for several other approaches and researches attempting to fix the quantisation ambiguities, such as the master constraint programme [58, 59], algebraic quantum gravity [60, 61, 62, 63], the spin foam models [64, 65, 66], the Hamiltonian renormalisation approach [67, 68, 69, 70], etc.
In addition, working on "toy models" sharing essential features of GR, i.e. general covariance and background independence, can provide significant insight and primary directions for future progress.

0.1.2.2. Testing Grounds of Dynamics: Toy Models

A very interesting idea to better scrutinise the dynamics is to apply the techniques of LQG to simpler models possessing similar mathematical structures to that of GR, then try to find a strategy to fix possible ambiguities that naturally arise in the quantisation procedure and finally investigate what happens if we employ that strategy in the full theory. The toy models existing in the literature are symmetry reduced gravitational systems, gravity in lower dimensions, parametrized field theory (PFT) [71, 72], Husain-Kuchar model [73] and $U(1)^3$ model of Euclidean gravity [2], each of which provides a simpler arena to probe the problems of dynamics either by reducing the degrees of freedom and the number of constraints or by allowing comparison with systems whose quantisation is known through other methods. They all intend to dissect the structure of the loop quantisation and look for new improvements which are, sometimes surprisingly, fully susceptible.

For example, loop quantum cosmology (LQC) [74, 75] makes provision for probing the problem of dynamics in a mathematically simpler yet physically appealing framework because thanks to the high degree of symmetry cosmological models admit, one can construct the physical Hilbert space and analyse it meticulously (see e.g. [76, 77, 78, 79, 80]). Accordingly, LQC has improved our understanding of the dynamics of LQG by giving hints about the correction of the Hamiltonian constraint and also the extraction of physical information. On the other hand, insights from working on the loop quantisation of PFT [81, 82, 83, 84, 85] whose Fock representation had already been available, culminated in the construction of a diffeomorphism constraint operator in full LQG [86]. Therefore, insights from toy models can have profound influences on the full theory of LQG.

To improve the original formulation [55, 56, 57] and find a faithful representation of quantum

[2]It is anomaly free in the sense that $\ell([\hat{C}(N_1), \hat{C}(N_2)]\psi) = 0$ for all test functions N_1, N_2, for each state ψ in the common dense and invariant domain D of all the $\hat{C}(N)$ and all $\ell \in L$, the space of diffeomorphism invariant algebraic distribution on D (i.e. a linear functional on D without continuity conditions, D can be identified with the finite linear span of spin network functions).

dynamics for GR, we cannot resort to PFT or the Husain-Kuchar model since both do not model the aspect that the constraint algebra is not a Lie algebra but a Lie algebroid with non-trivial structure functions. On the other hand, the $U(1)^3$ model of Euclidean gravity, in which the structure group $SU(2)$ is simply replaced by $U(1)^3$, is intriguing to focus on since the constraint algebra of this model has structure functions similar to the gravitational case. As the $U(1)^3$ model shares many of the conceptual and mathematical structures of GR, it provides a fertile ground to examine various ideas of full LQG. While Dirac quantisation of this model has been studied extensively in [87, 88], in this dissertation we aim at moving toward its quantisation through other approaches, i.e. reduced phase space and path integral quantisations, to allow comparisons enlightening the pathway to progress! In particular, a reduced phase space quantisation approach has the additional advantage that it frees us from the steps to compute 1) kernel of constraints 2) physical Hilbert space and 3) Dirac observables, as one only quantises classical gauge-invariant degrees of freedom. These steps are exactly those that are still missing in the work of Varadarajan et al [87, 88].

0.2. Outline

This thesis is substantially based on the publications [89, 90, 91] arisen from collaborations of the author with Thomas Thiemann and Hossein Shojaie. The main results are presented in chapters 8, 6 and 9 together with section 5.2, respectively. Moreover, the paper [92] was also published during the author's PhD programme.

General structure:

The content of this dissertation is divided into three parts. Since the main purpose of our research course is to comprehensively study the $U(1)^3$ model of Euclidean quantum gravity, first it is necessary for the reader to have a good knowledge of different quantisation methods. This is provided in the first part. In the second part, we introduce the classical theory of the model we intend to focus on, and in the last part, we try to apply the methods presented in the first part to the model introduced in the second part. The first section of each chapter is devoted to the preliminaries required to explain the purpose of the chapter. This section helps the reader to understand the general outline of the chapter before going into details.

Part I

The first part is concerned with different approaches to quantising a constrained system. There are three main methods to quantise a system with constraints: the Dirac quantisation and the reduced phase space quantisation as canonical approaches, in addition to the path integral method being the covariant quantisation of gauge-invariant systems. The first three chapters are devoted to reviewing these approaches and, in particular, applying them to the special case of GR.

Chapter 1

The cornerstone of the Dirac quantisation dwells in imposing quantum mechanically the constraints as operator conditions on the states for identifying the physical ones. In the second section, we present the steps one has to take in order to quantise a general constrained system following which we exhibit the structure of canonical loop quantum gravity in the next section. Indeed, we apply the Dirac quantisation procedure to GR,

which is a fully constrained system, and provide the techniques utilised to solve the kinematical constraints. Discussion on the dynamics is more subtle and deserve a complete chapter, hence it will be carried over to the second part of this dissertation where the main motivation of studying the $U(1)^3$ model is presented. The content of this chapter will be extensively used in chapter 7 deriving the Dirac quantisation of the $U(1)^3$ model.

Chapter 2

The basic idea of the reduced phase space quantisation consists in first identifying the physical degrees of freedom at the classical level by factorising the constraint surface with respect to the action of the gauge group, generated by the constraints. Then the resulting Hamiltonian system is quantised as a usual unconstrained system. In the second section, we present the steps of this kind of quantisation for a general constrained system by deriving the physical observables and defining the physical Hamiltonian governing the dynamics of observables. While this method cannot be easily applied to pure gravity due to the complexity of the constraints, it is technically easier applicable to the case of GR coupled to a suitable kind of matter (e.g. dust). This is what we will provide in the succeeding section. In the end, we compare the two approaches of canonical quantisation. The content of this chapter is essential for chapter 7.

Chapter 3

There is another approach for quantising constrained systems which tackle the problem from the covariant point of view, that is path integral quantisation. This will be briefly presented in the second section for a general constrained system. As the path integral quantisation of BF theory, which is closely related to GR, is well-known we begin the succeeding section with an introduction and display path integral of the BF theory. Then we formulate GR as a "constrained" BF theory and try to impose the constraints at the quantum level. This procedure is known as spin foam models to which half of the activities in the LQG community are devoted. The main motivation of chapter 9 is to provide an interesting test ground for these models.

Part II

In this part, by presenting the dynamics of LQG and its shortcomings, we express the motivation for our research study and then we will introduce the $U(1)^3$ model as an appealing test laboratory of dynamics. Next, we study its asymptotic structure needed to perform the reduced phase space quantisation.

Chapter 4

We express the main idea of Thiemann's construction of Hamiltonian constraint which is responsible for the dynamics. Then we will specify its virtues as well as its shortcomings. Besides the ambiguities arising in the quantisation procedure, the drawback of Thiemann's Hamiltonian constraint is that although the quantum algebra is mathematically anomaly-free, it was not possible to check explicitly that it implements the classical constraint algebra at the quantum level. Because the constraint algebra of the $U(1)^3$ model involves structure functions, similar to that of GR, this simpler model can be employed to test the ideas suggested to faithful implementation of the constraint algebra.

Chapter 5

The $U(1)^3$ model is introduced as a weak coupling limit of Euclidean GR. We describe

the model using two different canonical variables, i.e. (A, E) and (B, f). We show that the Lagrangian proposed in the original paper as the covariant origin of this theory is not correct. This forces us to look for the proper Lagrangian in chapter 9.

Chapter 6

We try to produce well-defined generators for asymptotic symmetries out of the constraints of the $U(1)^3$ model. To do this, first, we seek a boundary term spoiling the differentiability of a given constraint and then it will be subtracted from the variation of the constraint. If this boundary term happens to be an exact one-form in the field space, one concludes that the resulting expression is functionally differentiable. At that point, only the examination of its finiteness is left to make sure that it is well-defined. In the case of the $U(1)^3$ model, we show that although space-time translations are admissible asymptotic symmetries, boosts and rotations do not have well-defined generators [90].

Part III

We try to provide appropriate circumstances for the quantisation methods studied in the first part to become applicable to the $U(1)^3$ model. Albeit the Dirac quantisation has taken already major steps in [87, 88], the work on the other ones is still in progress.

Chapter 7

As proposed in [87, 88], instead of working with density weight one Hamiltonian constraint we will construct the quantum counterpart of the Hamiltonian constraint of density weight 4/3 so as to produce a non-trivial constraint algebra on a subspace of the algebraic dual of a dense subset of the kinematical Hilbert space rather than on that subset itself as it is done in [55]. We see that this idea works well but still needs to be tested. Hence, if one can establish the quantum theory through other methods, the comparison of the final results can help to gain better understanding of the idea and its applicability. This is another motivation for the main content of this thesis [89, 90, 91].

Chapter 8

First, we prove two theorems concerning the construction of the physical Hamiltonian for a system with first-class constraints that are at most linear in the momenta. They are achieved by means of a choice of gauge fixings. Then, for various choices of gauge fixings, we apply our theorems to the $U(1)^3$ model to obtain the physical Hamiltonian. The degree of spatial non-locality and non-polynomiality of the physical Hamiltonian critically depends on the choice of gauge fixings. These choices are related by a gauge transformation and hence classically equivalent, but when it comes to quantisation certain choices seem to be preferred. We obtain several physical Hamiltonians displaying a rather manageable form, discuss their properties, and find the physical equations of motion [89].

Chapter 9

To begin with, we consider the $U(1)^6$ Palatini action as a possible covariant origin of the $U(1)^3$ model. Surprisingly, we reveal that this theory is topological, there are no propagating degrees of freedom. The noticeable difference with the $SO(4)$ theory is that, in a vital step of the constraint stability analysis, the Abelian nature of the model precludes solving the equations for the Lagrange multipliers but instead additional secondary constraints emerge. Then, based on the gauge group $U(1)^3$, we introduce a one-parameter family of Lagrangians. The parameter is similar to but

different from the Immirzi-Barbero parameter and yields a "twisted self-dual" and Abelian connection. It turns out that in this case, the Hamiltonian theory is indeed equivalent to the $U(1)^3$ model of Euclidean GR. As a byproduct, we prove that the same action for SU(2) also results in the Hamiltonian formulation of Euclidean General Relativity that so far was only known for the Euclidean (anti-)self-dual theory (the parameter is equal to plus/minus unity). As a technical stride, we present the analysis using half densitised tetrads and bypassing the introduction of simplicity constraints that always produce spurious solutions. Furthermore, a pure connection formulation of the theory is also derived [91].

Chapter 10

We conclude and give an outlook to future research.

Appendices

There are also two appendices devoted to displaying the detailed calculations of solving some systems of equations, the first one is related to the constraints in the (B, f) description in chapter 8 and the second one concerns the system of constraints we faced in chapter 9.

Part I.

Review on Different Approaches for Quantisation of Constrained Systems

Dirac Quantisation

1.1. Preliminaries

1.1.1. Why Constrained Systems Are Important

It is quite often the case that in modern physics theories of interest are formulated in terms of *fields* which are specified with respect to an arbitrary *reference frame*. Only those variables can be called measurables that are independent of the choice of the reference frame. The field theories have an attribute in common that the fundamental fields are not directly measurable, however, some associated quantities, such as charges, energies, etc. can be measured. In such theories it is quite possible that different configurations of *non-observable* fundamental fields lead to an identical *observable* quantity. A transformation from one such configuration to one another is called gauge transformation and the lack of change in the observable quantities, despite the fundamental field being changed, is a feature called *gauge invariance* and a theory describing such a system is called *gauge theory* [93]. In gauge theories, because of the presence of gauge transformations that are physically irrelevant, the equations of motion with given initial data do not have a unique solution, thus one can easily see that the Legendre transformation cannot be invertible. Dirac revealed that despite this non-invertibility one can still develop the Hamiltonian formalism for these theories but with this major difference that the canonical variables are not free anymore. Rather, the allowed states are constrained to be on the so-called "constrained surface" and such systems are called "constrained Hamiltonian systems" [94, 95].
Put it briefly, in modern physics, a situation is often treated as a gauge system which is always a constrained Hamiltonian system, therefore the quantisation of systems with constraints is of prominent importance in physics.

1.1.2. How a Constrained System is Mathematically Described

The Lagrangian $L(q^i, \dot{q}^i)$ is said to be singular if the momenta $p_i := \frac{\partial L}{\partial \dot{q}^i}$ are not independent functions. It follows that there exists some equations $C_a(p, q) = 0$ expressing the dependence which is called *primary constraints*. It can be shown that time evolution of any function $f(q, p)$ defined on phase space is obtained via the equation $\dot{f} = \{f, H_T\}$ where $H_T = H_C + \lambda^a C_a$ in which H_C is the canonical Hamiltonian and λ^a are Lagrange multipliers. To have a consistent theory, the primary constraints should not change over time. Hence, the consistency conditions $\dot{C} = \{C_a, H_C\} + \lambda^b \{C_a, C_b\} \approx 0$ have to be satisfied, where $\approx$ denotes weak equality, i.e. equality holding on the constraint surface[1]. The consistency conditions may lead to specifying some of Lagrange multipliers and/or emerging some new constraints, known as *secondary constraints*. In

[1] the surface which is implicitly defined by the simultaneous vanishing of all the constraints.

other words, secondary constraints are those constraints arising in the theory using the equations of motion. The procedure should be repeated until one ensures that all the constraints are weakly preserved by time. The whole system of constraints can be divided into two sets: first class constraints C_I that Poisson commute with all the constraints, and second class constraints χ_α, otherwise[2]. Mathematically speaking, $\{C_a, C_b\} \approx 0 \approx \{C_a, \chi_\alpha\}$ and $\{\chi_\alpha, \chi_\beta\} \approx M_{\alpha\beta}$ with $\det(M_{\alpha\beta}) \neq 0$ [3]. That the first-class constraints generate gauge transformations is rather clear in the case of the first-class primary constraints, since these appear explicitly in the generator of the time evolution multiplied by *arbitrary* functions. That it also holds for the first-class secondary constraints is known as the "Dirac conjecture". This conjecture can be proved under reasonable assumptions (see, e.g. [96]). The reason that the secondary first-class constraints also correspond to gauge transformations is that they appear in the brackets of the Hamiltonian with the primary first-class constraints. Thus, different choices of arbitrary functions λ^a in the dynamical equations of motion will lead to phase-space points that differ by a canonical transformation whose generator involves the secondary first-class constraints as well. Since an (strong) observable quantity O is a gauge-invariant phase-space function, and first-class constraints generate gauge transformations, it is then concluded that $\{O, C_a\} = 0$ for all C_a. If the equality is substituted by $\approx$, then O is called a *weak observable*.

As second-class constraints do not map the constraint surface on itself, they cannot generate admissible transformations. Therefore, it is advantageous to consistently eliminate them by using the Dirac bracket instead of the Poisson bracket. By construction, the Dirac bracket $\{F, G\}_D$ of two phase-space functions F and G is given by $\{F, G\}_D = \{F, G\} - \{F, \chi_\alpha\} M^{\alpha\beta} \{\chi_\beta, F\}$ where $M^{\alpha\beta}$ is the inverse to $M_{\alpha\beta}$. It fulfils two essential properties: 1) the Dirac bracket of any function with any second-class constraint vanishes, $\{F, \chi_\alpha\} = 0$, 2) the Dirac bracket of two first-class functions equals to their Poisson bracket. Thus, one can consistently eliminate the second-class constraints and replace the Poisson bracket with the Dirac bracket.

1.1.3. Purpose of This Chapter

In this chapter, we concisely represent the steps engaged in the process of a certain approach for quantising a constrained system known as Dirac quantisation [97, 98, 99, 100] in which "quantising before constraining" is the main idea. Given a symplectic manifold $(\mathscr{P}, \Omega)$ together with a set of first-class constraints C_I[4], according to the main idea by Dirac, in this approach one first quantises the unconstrained system giving rise to a kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ in which the set of elementary variables of the full phase space are represented by self-adjoint operators. Afterwards one has to promote the constraints C_I to well-defined operators $\hat{C}_I$ in $\mathcal{H}_{\text{kin}}$ the intersection of whose kernels determines the physical states. Finally, if an inner product can be defined for the physical states, the sought-after physical Hilbert space will be achieved [5]. Whether it is also applicable to take the opposite direction of this approach, i.e. "quantising after constraining", is the subject of the next chapter where a comparison of the two approaches will also be done; the advantages and disadvantages of applying them will be given.

Section 1.2 is devoted to a review of the Dirac quantisation procedure that will be employed

[2] The terminology of primary and secondary constraints is confusingly similar to that of first and second class constraints. These divisions are independent: both first and second class constraints can be either primary or secondary, so this gives altogether four different classes of constraints.

[3] It can be prooved that if $\det(M_{\alpha\beta}) \approx 0$, there exists at least one first-class constraint among the C_a's [95].

[4] Actually, $(\mathscr{P}, \Omega)$ is the full phase space in which the constraint surface $C_I = 0$ is embedded. The classical states of the theory are restricted to living on the constraint surface.

[5] Two excellent books on the quantisation of gauge systems are [95] and [94]. For a brief and comprehensive review of the Dirac algorithm, we refer the reader to chapter 24 of [1].

in chapter 7 where we wish to derive the quantum theory of $U(1)^3$ model introduced in Part II. In section 1.3, we apply the Dirac quantisation procedure to GR and achieve the kinematical Hilbert space. This is essential to discuss ambiguities existing in the dynamics of LQG which is the main motivation to study the $U(1)^3$ model.

1.2. Dirac Quantisation Procedure

What follows in this section is an adaptation of the Dirac quantisation fully described in [1], to which the reader is referred for further information.

1.2.1. Classical *-Algebra

In order to quantise a constrained system, the initial step is to introduce an appropriate set $\mathcal{E}$ of *elementary variables* using which the phase space $\mathscr{P}$ is coordinatised, such that $\mathcal{E}$ separates the points of $\mathscr{P}$ [6]. By appropriate elementary variables for quantisation, we mean those forming a closed Poisson subalgebra of the full Poisson algebra $C^\infty(\mathscr{P})$ and closed under complex conjugation. The former is required as in canonical quantisation Poisson brackets are to be replaced by $i\hbar$ times the corresponding commutator relation, and the latter is necessary since quantising complex conjugates are supposed to result in adjoints of operators. In mathematics, these requirements are satisfied by an object called *-algebra. A *-algebra is an algebra with an involution, that is a map $* : \mathscr{A} \to \mathscr{A}$ where $x \mapsto x^*$ has the following properties $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$, $(xy)^* = y^* x^*$ and $(x^*)^* = x$ for all $x, y \in \mathscr{A}$ and $\lambda, \mu \in \mathbb{C}$. Sometimes the procedure of choosing $\mathcal{E}$ is called the choice of *polarisation*. There are some simplicity principles guiding us to choosing a suitable polarisation. For instance, since only gauge invariant functions (observables) are physically relevant, we seek elementary variables whose gauge transformation behaviours obey some relatively simple rules in order to simplify the quantisation of observables. Moreover, as one of the main challenges is usually finding a representation for the Poisson algebra, any complexity here may become an obstacle to completing the quantisation programme. On the other hand, since constraints need to be implemented to obtain the physical Hilbert space, their expressions' simplicity can be considered another guideline to finding the appropriate $\mathcal{E}$.

As explained in [1], if the phase space $\mathscr{P}$ can be viewed as a cotangent bundle T^*Q over a configuration space Q, then a natural candidate for $\mathcal{E}$ is $\mathrm{Fun}(Q) \times V(Q)$, where $\mathrm{Fun}(Q)$ is a suitable algebra of smeared functions over Q and $V(Q)$ the space of vector fields over Q preserving $\mathrm{Fun}(Q)$. The product space $\mathrm{Fun}(Q) \times V(Q)$ carries a Lie algebra structure according to $\{(f, v), (f', v')\} := (v[f'] - v'[f], [v, v'])$ where $v[f]$ is the action of vector fields on functions and $[v, v']$ is the Lie bracket of vector fields.

1.2.2. Quantum *-Algebra

Now one can promote the classical *-algebra $\mathcal{E}$ to a quantum *-algebra in such a way that the Poisson brackets are replaced by commutation relations and complex conjugation by involution. In order to construct the quantum *-algebra $\mathfrak{A}$ out of $\mathcal{E}$ we, first of all, consider the tensor algebra[7]

[6] A set of functions $\mathcal{S}$ from a set D to a set C is said to separate the points of D if for any two distinct elements $x, y \in D$, there exists a function $f \in \mathcal{S}$ so that $f(x) \neq f(y)$.

[7] Suppose V is a vector space over a field F. For any integer $k \geq 0$, the kth tensor power of V is defined to be the tensor product of V with itself k times: $T^k V = V^{\otimes k}$ and $T^0 V := F$. Then, one defines $T(V)$ as the direct sum of $T^k V$ for $k = 0, 1, 2, \cdots$, i.e. $T(V) := \bigoplus_{k=0}^{\infty} T^k V$, and the canonical isomorphism $T^k V \otimes T^\ell V \to T^{k+\ell}$ determines the multiplication in $T(V)$.

$T(\mathcal{E})$ over $\mathcal{E}$. Consider the two sided ideal[8] $\mathcal{I}$ of $T(\mathcal{E})$ generated by elements of the form

$$a \otimes b - b \otimes a - i\hbar\{a, b\}, \qquad a^* - \bar{a} \tag{1.1}$$

where $a, b \in \mathcal{E}$. Then, the desired algebra is defined as the quotient space $\mathfrak{A} := T(\mathcal{E})/\mathcal{I}$.

When it comes to introducing a physically interesting representation theory (see next subsection) of $\mathfrak{A}$, for an unbounded element $a \in \mathcal{E}$ the corresponding operator $\hat{a}$ become unbounded. Since the domain of an unbounded operator can only be a dense subset of a given Hilbert space, it is not guaranteed that two different such operators can be defined on a common and invariant domain and so their commutator might be ill-defined. Thus, domain questions naturally arise which make the situation more difficult. To circumvent this issue, instead of using $a \in \mathcal{E}$ itself, it is convenient to employ a bounded function of it. As long as it is ensured that such functions also separate the points in $\mathscr{P}$, this substitution is permissible. To define a bounded function of an unbounded element $a \in \mathcal{E}$, we consider the one parameter family of unitary operators, called *Weyl elements*, $t \mapsto W_t := \exp(ita)$ for $t \in \mathbb{R}$. These operators approximate $1 + ita$ for small t and do separate the points of $\mathscr{P}$. In this case, the ideal required to define the algebra $\mathfrak{A}$ is generated by the following elements rather than (1.1)

$$W_s(a)W_t(b)W_{-s}(a) := W_t\left(\sum_{n=0}^{\infty} \frac{(is\hbar)^n}{n!}\{a, b\}_{(n)}\right) \tag{1.2}$$

$$(W_s(a))^* := W_{-s}(a) = (W_s(a))^{-1} \tag{1.3}$$

where $\{a, b\}_{(0)} = b, \{a, b\}_{(n+1)} := \{a, \{a, b\}_{(n)}\}$ is the iterated Poisson bracket.

1.2.3. Representation of the Quantum *-Algebra

The next task is to find a representation of the quantum *-algebra $\mathfrak{A}$. By representation we mean a pair $(\mathcal{H}, \pi)$ in which $\mathcal{H}$ is a Hilbert space and $\pi : \mathfrak{A} \to L(\mathcal{H})$ is a morphism from the *-algebra $\mathfrak{A}$ into the algebra of linear operators defined on $\mathcal{H}$, with a common and invariant dense domain such that

$$\pi(\mu a + \lambda a') = \mu\pi(a) + \lambda\pi(a'), \quad \pi(aa') = \pi(a)\pi(a'), \quad \pi(a^*) = \pi(a)^\dagger \tag{1.4}$$

where $a, a' \in \mathfrak{A}$ and $\mu, \lambda \in \mathbb{C}$. If $\ker(\pi) = 0$, then the representation is said to be *faithful*. Moreover, an *irreducible representation* (ρ, V) or *irrep* of an algebraic structure $\mathfrak{A}$ is a non-zero representation that has no proper non-trivial sub-representation $(\rho|_W, W)$, with $W \subset V$ invariant under the action of $\{\rho(a) : a \in A\}$.

Generally, for a given *-algebra $\mathfrak{A}$ there will be infinitely many representations. Any of these representations can in principle define a different quantum theory unless they are unitary equivalent. Two representations $\pi_1 : \mathfrak{A} \to L(\mathcal{H}_1)$ and $\pi_2 : \mathfrak{A} \to L(\mathcal{H}_2)$ is said to be equivalent if there exists a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$ for all $a \in \mathfrak{A}$. So, what is usually done to reduce the number of possible representations is introducing further physically motivated assumptions on the representation. For example, in the context of quantum mechanics (QM), the Stone-von-Neumann uniqueness theorem says: making very mild assumptions that the representation is irreducible and weakly continuous, one concludes that the Schrödinger representation is the unique representation for QM, up to unitary equivalence. Here, a representation is said to be *weakly continuous* if

$$\lim_{t\to 0}\langle\psi, \pi(W_t(a))\psi'\rangle = \langle\psi, \psi'\rangle \tag{1.5}$$

[8] Given a subalgebra $\mathcal{I}$ of an algebra $\mathfrak{A}$, we say $\mathcal{I}$ is a right (left) ideal of $\mathfrak{A}$ iff $ba \in \mathcal{I}$ ($ab \in \mathcal{I}$) for all $a \in \mathfrak{A}, b \in \mathcal{I}$. A two sided ideal is both a left and right ideal.

for all $\psi, \psi' \in \mathcal{H}$. In quantum field theory where we have an infinite number of degrees of freedom the situation is worse, so that in general for a given physical theory many inequivalent representations are possible and it is a priori not clear which is the physical relevant one. In this case, one has to look for stronger physical assumptions which hopefully lead to a similar uniqueness result. Furthermore, we are only interested in those representations in which it is possible to implement the generators of the dynamics as operators. In the case of QM, the Hamiltonian generating dynamics is usually a simple function on the phase space. Hence, in the Schrödinger representation, it can be promoted to an operator [101]. In the case of constrained field theories, we have to find a representation for which the classical constraints can be implemented on the so-called *kinematical Hilbert space* $\mathcal{H}_{\text{kin}}$.

1.2.4. Solving the Constraints

Having obtained $\mathcal{H}_{\text{kin}}$, we must solve the constraint operators since the physical states are those annihilated by quantum constraints. However, there is a glaring obstacle in proceeding with this step. If zero lies in the continuous part of the constraint operators' spectrum, then the functions in their kernels do not lie in the kinematical Hilbert space[9]; such functions are called *generalised eigenfunctions* [102]. Two constructive procedures are available to overcome such a problem: Refined Algebraic Quantisation (RAQ) and Direct Integral Decomposition (DID) that is a generalisation of RAQ. Here, we only discuss the former as it will be used in the succeeding sections. For a detailed introduction to the latter, we refer the interested reader to [103] or to section 30.2 of [1].

1.2.4.1. Refined Algebraic Quantisation (RAQ)

Let $(\hat{C}_I)_{I\in\mathfrak{I}}$ denote the constraint operators defined on the kinematical Hilbert space, for an index set $\mathfrak{I}$ labelling them. Since $\mathcal{H}_{\text{kin}}$ is too "small" to contain all the solutions to the constraints $(\hat{C}_I)_{I\in\mathfrak{I}}$, one could seek solutions in the space $\mathcal{D}^*_{\text{kin}}$ of linear functionals on $\mathcal{D}_{\text{kin}}$ where $\mathcal{D}_{\text{kin}} \subset \mathcal{H}_{\text{kin}}$ is an invariant dense domain on which all the constraint operators and their adjoints are defined. The topology on $\mathcal{D}_{\text{kin}}$ is the topology induced by the norm topology on $\mathcal{H}_{\text{kin}}$. Moreover, $\mathcal{D}^*_{\text{kin}}$ is equipped with the weak *-topology of point-wise convergence which is coarser than the topology on $\mathcal{H}_{\text{kin}}$. Therefore, we gain the following topological inclusion[10]

$$\mathcal{D}_{\text{kin}} \hookrightarrow \mathcal{H}_{\text{kin}} \hookrightarrow \mathcal{D}^*_{\text{kin}} \tag{1.6}$$

Now one is looking for a subspace $\mathcal{D}^*_{\text{phys}} \subset \mathcal{D}^*_{\text{kin}}$ of generalised eigenfunctions $\ell \in \mathcal{D}^*_{\text{phys}}$ satisfying

$$[(\hat{C}_I)'\ell](f) := \ell(\hat{C}_I^\dagger f) = 0, \quad \forall f \in \mathcal{D}_{\text{kin}}, \ \ \forall I \tag{1.7}$$

where the prime denotes a dual, anti-linear representation of the constraints on $\mathcal{D}^*_{\text{kin}}$[11].

[9] If zero lies in the discrete spectrum of the constraint operators, the eigenvectors associated with the eigenvalue $\lambda = 0$ are the physical states that do belong to $\mathcal{H}_{\text{kin}}$.

[10] In abuse of terminology we can call it a *Gel'fand triple* or *Rigged Hilbert Space*, as these names is usually employed for the case where $\mathcal{D}_{\text{kin}}$ is equipped with a nuclear topology and that $\mathcal{D}^*_{\text{kin}}$ is its topological dual, not the algebraic one.

[11] Note that if ℓ was an element of $\mathcal{H}_{\text{kin}}$ then the equation (1.7) would be

$$[\hat{C}_I'\ell](f) := \langle \hat{C}_I\ell, f\rangle_{\text{kin}} = \langle \ell, \hat{C}_I^\dagger f\rangle_{\text{kin}} =: \ell(\hat{C}_I^\dagger f) \quad \forall f \in \mathcal{D}_{\text{kin}}, \ \ \forall I \tag{1.8}$$

where $\langle\cdot,\cdot\rangle_{\text{kin}}$ is the inner product of $\mathcal{H}_{\text{kin}}$. Thus, (1.8) is viewed as the natural extension of (1.7) [104].

The sought-after physical Hilbert space $\mathcal{H}_{\rm phys}$ has not achieved yet, since if one considered $\mathcal{D}^*_{\rm phys}$ as the desired $\mathcal{H}_{\rm phys}$ all observables would be defined on the whole of $\mathcal{D}^*_{\rm phys}$ and hence would be understood as bounded operators, while the physical interesting operators are normally unbounded. This motivates us to consider a subset of $\mathcal{D}^*_{\rm phys}$ as the physical Hilbert space $\mathcal{H}_{\rm phys}$. We choose a subspace $\mathcal{D}_{\rm phys} \subset \mathcal{H}_{\rm phys}$ on which the algebra of observables is densely defined and based on which the following rigged Hilbert space is obtained

$$\mathcal{D}_{\rm phys} \hookrightarrow \mathcal{H}_{\rm phys} \hookrightarrow \mathcal{D}^*_{\rm phys} \tag{1.9}$$

The task left to accomplish is to construct an inner product for $\mathcal{H}_{\rm phys}$. A natural requirement the physical inner product is expected to fulfil is that adjoints with respect to $\langle\cdot,\cdot\rangle_{\rm phys}$ denoted by $\ddagger$ represents the corresponding adjoints with respect to $\langle\cdot,\cdot\rangle_{\rm kin}$, i.e. $\langle(\hat{O}')^\ddagger\psi,\psi'\rangle_{\rm phys} = \langle(\hat{O}^\dagger)'\psi,\psi'\rangle_{\rm phys}$ for all observables O and all $\psi,\psi' \in \mathcal{H}_{\rm phys}$. One can readily verify that such an inner product can be provided by an anti-linear map $\eta : \mathcal{D}_{\rm kin} \to \mathcal{D}^*_{\rm phys};\ \ f \mapsto \eta(f)$, called a *rigging map*. As we require the space of solutions to be preserved by the dual action of any observable operator $\hat{O}$, the rigging map has to satisfy $\hat{O}'\eta(f) = \eta(\hat{O}f)$ for any $f \in \mathcal{D}_{\rm kin}$. Furthermore, η as a map from $\mathcal{D}_{\rm kin} \times \mathcal{D}_{\rm kin}$ into $\mathbb{C}$ with $(f,f') \mapsto [\eta(f)](f')$ needs to be a positive sesquilinear form[12]. Then, the physical inner product is simply defined as

$$\langle\eta(f),\eta(f')\rangle_{\rm phys} := [\eta(f)](f'), \quad \forall\ f,f' \in \mathcal{D}_{\rm kin} \tag{1.10}$$

As we have just seen, if such a rigging map exists, constructing the physical inner product is its direct consequence. Thus, the task is reduced to introduce an appropriate η and can be accomplished by a proposal known as *group averaging* [105, 106].

Suppose that the first-class constraints $(\hat{C}_I)_{I\in\mathfrak{I}}$ we started with are self-adjoint and form an "honest Lie algebra", i.e. the commutators of the constraints is again a combination of constraints via structure constants and not functions. Using the constraints that are generators of the Lie algebra, one defines a group of unitary operators $\hat{U}(g) := \exp\left(it^I C_I\right)$ with $t^I \in \mathbb{R}$ and $g \in G$. According to (1.7), it turns out that $\hat{U}(g)$ acts trivially on any physical state $\ell \in \mathcal{D}^*_{\rm phys}$ because $[\hat{U}(g)'\ell](f) := \ell(\hat{U}^\dagger(g)f) = \ell(f)$ for all $f \in \mathcal{D}_{\rm kin}$. If G is a finite compact Lie group, then the existence of the unique Haar measure μ_H on G that is invariant under translations and inversions empowers us to define the rigging map as

$$\eta : \mathcal{D}_{\rm kin} \to \mathcal{D}^*_{\rm phys}, \quad f \mapsto \eta(f) := \int_G d\mu_H(g)\langle\hat{U}(g)f,\cdot\rangle_{\rm kin} \tag{1.11}$$

It is worth recalling that RAQ can be employed to solve the first-class and self-adjoint constraints of a constrained system only when they form an "honest" Lie algebra and the corresponding Lie group is locally compact. As in the presence of structure functions rather than structure constants, the RAQ process cannot guide us to the physical inner product, one has to work with DID strategy constructed to deal with such cases [103, 1].

1.3. Dirac Quantisation of Loop Quantum Gravity

To see how the quantisation programme introduced in the previous section works in detail for gravity, we apply it to GR and arrive at LQG. The classical Poisson algebra that canonical LQG is based on is formulated in terms of Ashtekar-Barbero variables, which in turn require a brief overview of ADM formalism for better understanding.

[12] For a complex vector space V, a map $\eta : V \times V \to \mathbb{C}$ is sesquilinear if it is linear in one argument and antilinear in the other one.

1.3.1. ADM Formalism

In Einstein's formulation of GR, the spacetime metric field $g_{\mu\nu}(x,t)$ plays the role of Lagrangian variables. Afterwards, Arnowit, Deser and Misner [23] changed these variables to *ADM variables*

$$q_{ab} = g_{ab}, \quad N = \frac{1}{\sqrt{-g^{tt}}} \quad N^a = q^{ab} g_{bt} \tag{1.12}$$

where $a, b = x, y, z$ used for spatial indices and q^{ab} is the inverse of the three-dimensional metric q_{ab} by which the spatial indices are raised and lowered. The geometrical interpretation of ADM variables is as follows. Consider a family of spacelike hypersurfaces Σ_t defined by $t =$ constant. Then q_{ab} is the 3-metric induced on the hypersurface. Let n^a be a unit normal vector field to the hypersurface Σ_t. The *lapse function* N measures the rate of flow of proper time with respect to coordinate time as one moves normally to Σ_t along the normal vector n^a. The *shift vector* measures how much the spatial coordinate system shifts tangential to Σ_1 when moving from Σ_1 to Σ_2 along n^a (for detailed construction see e.g. [1, 107]). The invariant interval can be written in the form

$$ds^2 = -N^2 dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt) \tag{1.13}$$

The action of GR in terms of ADM variables takes the form

$$S = \int d^4x \, \sqrt{q} N \left(R + K_{ab} K^{ab} - K^2 \right) \tag{1.14}$$

where q and R are the determinant and the Ricci scalar of the metric q_{ab}, respectively,

$$K_{ab} = \frac{1}{2N} \left(\partial_t q_{ab} - D_a N_b - D_b N_a \right) \tag{1.15}$$

is the extrinsic curvature of the hypersurface, and D_a is the covariant derivative of q_{ab}. Since the action (1.14) is independent of the time derivatives of N and N^a, the corresponding conjugate momenta, denoted by π and π_a respectively, are primary constraints and the pairs (N, π) and (N^a, π_a) can be taken out of the phase space. Therefore, the 3-metric and its conjugate momentum, i.e. (q_{ab}, p^{ab}) can be considered as the canonical variables of GR. The relation between the momentum and the velocity $\partial_t q_{ab}$ is determined through

$$p^{ab} = \sqrt{q}(K^{ab} - K q^{ab}) \tag{1.16}$$

where $K = K_{ab} q^{ab}$. To have a consistent theory the primary constraints $\pi = 0, \pi_a = 0$ should be stable under the time evolution. Demanding this, the following secondary constraints arise

$$C_a = -2 q_{ac} D_b \, p^{bc} \tag{1.17}$$

$$C = \frac{1}{\sqrt{q}} \left(p_{ab} p^{ab} - \frac{1}{2} p^2 \right) - \sqrt{q} R \tag{1.18}$$

where $p = p_{ab} q^{ab}$. The constraints (1.17) and (1.18) are called the diffeomorphism and Hamiltonian constraints, respectively. It turns out that they are preserved by time, hence fortunately no tertiary constraints are generated and the canonical Hamiltonian is

$$H = \int d^3x \, (NC + N^a C_a) \tag{1.19}$$

with arbitrary functions N, N^a that now serve as Lagrange multipliers.

1.3.2. Ashtekar-Barbero Formalism

Ashtekar-Barbero formalism relies on the *tetrad variables* in which the gravitational field is represented by four covariant fields $e^I_\mu(x)$ where $I, J, \cdots = 0, 1, 2, 3$ are flat Lorentz indices moved by the metric $\eta_{IJ} = \text{diag}[-1, +1, +1, +1]$. The metric variable can be expressed in terms of tetrad variables through $g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$. In this formulation, GR has an extra $SO(3,1)$ gauge invariance. The corresponding canonical formalism is usually defined in the temporal gauge $e^i_t = 0$, where $i, j, \cdots = 1, 2, 3$ are flat three-dimensional indices moved by δ_{ij}. In this gauge, the Lorentz group is reduced to $SO(3)$, and the ADM variables are $q_{ab} = e^i_a e^i_b, N = e^0_t, e^i_t = e^i_a N^a$. Then the invariant interval (1.13) is equivalent to

$$ds^2 = -N^2 dt^2 + (e_{ai} dx^a + e_{ai} N^a dt)(e^i_b dx^b + e^i_a N^a dt) \tag{1.20}$$

Then, the triad field of the ADM hypersurface, e^i_a, and its conjugate momentum, $p^a_i := p^{ab} e_{bi}$, are the reduced canonical variables. The diffeomorphism and Hamiltonian constraints are the same as in the ADM formulation, with q_{ab} and p_{ab} expressed in terms of the triad variables. In addition, there is an extra constraint generating the internal rotations

$$G_i = \epsilon_{ijk} e^j_a p^{ak} \tag{1.21}$$

When it comes to Ashtekar-Barbero formalism, the form of the constraints are simplified and the phase space of GR is cast exactly in the form of that of Yang-Mills. For this reason, it is widely used in non-perturbative quantum gravity; in particular LQG.
Ashtekar's original variables are

$$A^i_a := \frac{1}{2} \epsilon^i_{jk} \omega^{jk}_a + i K^i_a \tag{1.22}$$

$$E^a_i := \det(e)\, e^a_i \tag{1.23}$$

where $\omega^{ij} = \omega^{ij}_a dx^a$ is the (torsion-free) spin connection of the triad 1-form field $e^i = e^i_a dx^a$, determined by the Cartan equation $de^i + \omega^j_k \wedge e^k = 0$ and

$$K^i_a := e^{ib} K_{ab} = \frac{2}{\det(e)} (p^i_a - \frac{1}{2} e^i_a p) \tag{1.24}$$

where $p = e^i_a p^a_i$. Note that while the *electric field* E in (1.23) is real, the *Ashtekar connection* A introduced in (1.22) is complex, and hence it is supposed to fulfil the *reality condition*

$$A^i_a + \bar{A}^i_a = 2\Gamma^i_a[E] \tag{1.25}$$

where $\Gamma^i_a[E]$ is the Levi-Civita connection that functionally depends on E. In terms of Ashtekar variables (1.22) and (1.23), the constraints of the theory can be re-expressed in the form

$$G_i = D_a E^a_i = 0 \tag{1.26}$$

$$C_a = F^i_{ab} E^b_i = 0 \tag{1.27}$$

$$C = \frac{1}{\sqrt{\det(q)}} \epsilon_{ijk} F^i_{ab} E^a_j E^b_k = 0 \tag{1.28}$$

that are Gauß, diffeomorphism and Hamiltonian constraints, respectively. Here, F^i_{ab} is the curvature associated with the connection A. Notice that G_i, C_a and $\sqrt{\det(q)} C$ are polynomial in the canonical variables. As explained in the introduction, the problem with these variables is

that although the constraints are very simple, in addition to them, the "non-polynomial" reality condition must be satisfied. This again makes the situation complicated.

Barbero slightly generalised Ashtekar's original variables by replacing (1.22) with

$$A_a^i := \frac{1}{2}\epsilon^i_{jk}\omega_a^{jk} + \gamma K_a^i \tag{1.29}$$

where γ is an arbitrary complex number known as *Barbero-Immirzi parameter*. We refer to the canonical variables (1.29) and (1.23) as *Ashtekar-Barbero variables*. In principle, these variables form a canonically conjugate pair whose non-vanishing Poisson bracket is

$$\{E_i^a(x), A_b^j(y)\} = \delta_b^a\delta_i^j\delta(x,y) \tag{1.30}$$

For $\gamma = 1$, the equations (1.26)-(1.28) are the constraints of the Euclidean gravity. We will repeatedly return to these constraints in parts II, III. In the Lorentzian case, if one derives the constraints in terms of the Ashtekar-Barbero variables, the Gauß and diffeomorphism constraints remain untouched but the Hamiltonian constraint (1.28) is replaced by

$$H = C - \frac{1}{\sqrt{\det(q)}}(1+\gamma^2)E_j^aE_k^bK^j_{[a}K^k_{b]} \tag{1.31}$$

that is very complicated to deal with. The non-vanishing Poisson brackets of the classical constraint algebra turn out to be

$$\{G(\Lambda), G(\lambda')\} = G([\Lambda,\Lambda']), \tag{1.32}$$
$$\{\vec{C}(\vec{N}), \vec{C}(\vec{N}')\} = \vec{C}(\mathcal{L}_{\vec{N}}\vec{N}'\}, \tag{1.33}$$
$$\{\vec{C}(\vec{N}), C(N)\} = C(\mathcal{L}_{\vec{N}}N\}, \tag{1.34}$$
$$\{C(N), C(N')\} = \vec{C}(\vec{N}(q,N,N')) \tag{1.35}$$

where $\vec{N}(q,N,N') := q^{ab}[N\partial_bN' - N'\partial_bN]$ and if we replace C by H, the algebra is the same up to signs. In the literature, this algebra is also known as *Dirac algebra* or *hypersurface deformation algebra*.

1.3.3. Holonomy-Flux Algebra as the Classical *-Algebra

Now, it is time to introduce classical algebra. Inspired from lattice gauge theory [108], we can work with $SU(2)$ holonomies and fluxes, rather than the Ashtekar-Barbero variables. Some physical and mathematical reasons behind this choice will be provided in section 1.3.3.1. We initiate with the precise definition of holonomy.

Definition 1.3.1. *The holonomy $h_e(A)$ is the SU(2)-valued parallel transport of the SU(2) connection A_a^i along a piecewise analytic curve $e : [0,1] \to \sigma;\ t \to e(t)$. In other words, if $h_t(e)$ is the unique solution of the parallel transport equation*

$$\frac{d}{dt}h_t(e) = h_t(e)A_e(t) \tag{1.36}$$

where $h_{t=0}(e) = \mathbb{I}$ and $A_e(t) := \dot{e}^a(t)A_a^i(e(t))\tau_i$, then $h_e(A) := h_t(e)|_{t=1}$. Here, τ_i denote the anti-Hermitian generators of SU(2) which is normalized according to $Tr(\tau_i\tau_j) = -\frac{1}{2}\delta_{ij}$.

Integrating both sides of (1.36) from 0 to t gives rise to $h_{e(t)}(A) = \mathbb{I} + \int_0^t dt'\ h_{e(t')}(A)A_e(t)$. By repeatedly iterating this equation and evaluating the solution at $s = 1$, we find the explicit expression

$$\begin{aligned} h_e(A) &= \mathbb{I} + \sum_{n=1}^{\infty} \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\ A_e(s_n) \cdots A_e(s_1) \\ &= \mathcal{P} \exp\left(\int_0^t dt'\ A_e(t') \right) \end{aligned} \tag{1.37}$$

One can easily check that (1.36) is invariant under any reparametrisation of e, hence the holonomy depends only on the path[13] of e and not on the parametrisation one chooses to label its points. The set of all paths $\mathcal{P}$ has a special structure as follows. It is naturally equipped with composition and inverse operations. If e and e' are two paths such that the endpoint of e coincides with the beginning point of e', then the composition of them $e \circ e'$ is defined to be the path composed of e followed by e'. The inverse of a path e is simply given by reversing its direction and denoted by e^{-1}. The composition of paths is associative, i.e. $(e_1 \circ e_2) \circ e_3 = e_1 \circ (e_2 \circ e_3)$ [14] and the path $e \circ e^{-1}$ is equal to the beginning point of the path $b(e)$ [15]. Lack of two properties prevents $\mathcal{P}$ from being a group: 1) the composition of any two arbitrary paths cannot be defined in $\mathcal{P}$, only those in which the beginning point of the one coincides with the final point of the other can be composed and 2) there is no identity element in $\mathcal{P}$. In the language of category theory, a set with these properties is called a *groupoid*.

From the differential equation (1.36), we immediately see that the holonomy satisfies the properties $h_{e^{-1}}(A) = h_e^{-1}(A)$ and $h_e(A)h_{e'}(A) = h_{eoe'}(A)$. It means that for any connection A, its holonomy $h_e(A)$ defines a groupoid homomorphism from $\mathcal{P}$ into the gauge group G. In other words, there is a map

$$F : \mathcal{A} \to \mathrm{Hom}(\mathcal{P}, G) \quad A \mapsto F(A) \quad \text{s.t } (F(A))(e) := h_e(A) \tag{1.38}$$

where $\mathcal{A}$ is the set of all smooth connections and $\mathrm{Hom}(\mathcal{P}, \mathcal{G})$ is the set of all, not necessarily continuous groupoid homomorphism from $\mathcal{P}$ into G. Thus, we conclude that $\mathcal{A}$ is a subset of $\mathrm{Hom}(\mathcal{P}, G)$ that, as we will see, play a notable role in the construction of quantum theory.

Under a gauge transformation by $g : \Sigma \to SU(2)$, the holonomy is transformed as

$$h_e^g(A) = g(b(e))h_e(A)g(f(e))^{-1} \tag{1.39}$$

where $b(e)$ is the beginning point and $f(e)$ the final point of e. To prove (1.39), one can easily show that $h^g_{e(t)}(A) = g(b(e))h_{e(t)}(A)g(e(t))^{-1}$ satisfies the parallel transport equation $\frac{d}{dt}h^g_{e(t)}(A) = h^g_{e(t)}(A)A^g_e(t)$. Then, employing the uniqueness theorem of ordinary differential equations for the initial data $h^g_{b(e)}(A) = \mathbb{I}$ simply gives the claim by putting $t = 1$.

The role of momentum variable in this formulation is played by the electric flux, which is precisely defined as follows:

[13] If one identifies all the curves with the same range but different parametrizations, then any equivalence class is called a *path*. Here, in abuse of notation, we denote a path and its representative curve both by e.

[14] This would not be true if one used curves instead of paths, because in that case $(e_1 \circ e_2) \circ e_3$ and $e_1 \circ (e_2 \circ e_3)$ differ by a reparameterisation.

[15] This again would not be true if one used curves instead of paths.

Definition 1.3.2. *Let S be an orientable embedded submanifold that is a finite union of connected, analytic 2-dimensional surfaces S_i of Σ which are distinct up to their boundaries. Suppose that (s^1, s^2) are the coordinates on S and the locus of S in Σ is given by $x^a(s^1, s^2)$. Moreover, let $n_a(s) = \epsilon_{abc}(\frac{\partial x^b}{\partial s^1})(\frac{\partial x^c}{\partial s^2})$ be the normal one-form on S. The conjugate variable to the holonomy is defined by the flux of E^a_i across a surface S in Σ*

$$E_i(S) = \int_S d^2s\; n_a(s) E^a_i(x(s)) \tag{1.40}$$

and its smeared version using an su(2)-valued function $f = f^i\tau_i$ is

$$E_f(S) = \int_S d^2s\; f^i(x(s)) n_a(s) E^a_i(x(s)) \tag{1.41}$$

All information about Ashtekar-Barbero variables is encoded in holonomy and flux, as one can recover the connection A^i_a from its holonomy and the electric field E^a_i from its flux in the following way. It follows that the holonomies and electric fluxes separate the points of $\mathcal{M}$ and thus they can be considered as the fundamental variables.
Given a curve e we define a short curve $e_\epsilon(t) := e(\epsilon t)$ for $0 < \epsilon < 1$. Expanding the exponential in the definition of the holonomy in terms of ϵ, we obtain $h_{c_\epsilon}(A) = \mathbb{I}_G + \epsilon(\tau_i/2)A^i_a(c_\epsilon(0))\dot{c}^a_\epsilon(0) + O(\epsilon^2)$ from which we infer

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(h_{c_\epsilon}(A) - \mathbb{I}_G\right) = (\tau_i/2)A^i_a(c_\epsilon(0))\dot{c}^a_\epsilon(0) = A_e(t=0) \tag{1.42}$$

Consequently, by diversifying the length and positions of the curve c, one recovers all of A's.
To show that the densitised triad E^a_i is recovered from the electric fluxes, we need to consider a very small surface S_ϵ embedded in Σ by the embedding map

$$X_S : M_\epsilon \to \Sigma; \quad (s^1, s^2) \mapsto X_S(s^1, s^2) \tag{1.43}$$

where $M = [-\epsilon/2, \epsilon/2]^2$ is very small square. The electric flux through this surface is

$$\int_{M_\epsilon} ds^2\; n^S_a(X(s^1, s^2)) E^a_i(X(s^1, s^2)) \approx \epsilon^2 n^S_a(X(0,0)) E^a_i(X(0,0)) \tag{1.44}$$

Again by varying the surfaces S, we can recover $E^a_i(x)$ for all $x \in \Sigma$.

Having (1.42) in hand, it is an excellent point to express the curvature in terms of holonomies required in promoting the Hamiltonian constraint to an operator. The appearance of the connection through its curvature F_{ab} in the Hamiltonian constraint leads us to express curvature in terms of holonomies, because only holonomy operators are well-defined. In the classical theory, we can derive different components of F^i_{ab} by first computing holonomies around suitable loops, then dividing them by the area of the surfaces surrounded by these loops, and finally taking the limit as the area approaches zero. For instance, to find F_{xy} we consider a loop based at point x and of infinitesimal coordinate area ϵ^2 which lies in the $x-y$ plane. The connection around $\mathbf{x}_0$ takes the expansion $A_a(\alpha(t)) = A_a(\mathbf{x}_0) + (\alpha(t) - \mathbf{x}_0)^b \partial_b A_a(\mathbf{x}_0) + O(\epsilon^2)$ from which the holonomy

around the loop is found

$$\begin{aligned}
h_\alpha(A) =& 1 + \int_0^1 dt\ \dot\alpha^a(t) A_a(\alpha(t)) + \int_0^1 dt\ \int_0^t dt'\ \dot\alpha^a(t) A_a(\alpha(t)) \dot\alpha^b(t') A_b(\alpha(t')) + O(\epsilon^3) \\
=& 1 + A_a(\mathbf{x}_0) \int_0^1 dt\ \dot\alpha^a(t) + \partial_b A_a(\mathbf{x}_0) \int_0^1 dt\ \dot\alpha^a(t)(\alpha^b(t) - \mathbf{x}_0^b) \\
&+ A_a(\mathbf{x}_0) A_b(\mathbf{x}_0) \int_0^1 dt\ \int_0^t dt'\ \dot\alpha^a(t) \dot\alpha^b(t') + O(\epsilon^3) \\
=& 1 + (\partial_b A_a(\mathbf{x}_0) + A_a(\mathbf{x}_0) A_b(\mathbf{x}_0)) \int_0^1 dt\ \dot\alpha^a(t)(\alpha^b(t) - \mathbf{x}_0^b) + O(\epsilon^3) \\
=& 1 + F_{xy} \int_0^1 dt\ \dot\alpha^x(t)(\alpha^y(t) - \mathbf{x}_0^y) + O(\epsilon^3) \\
=& 1 + \epsilon^2 F_{xy}(\mathbf{x}_0) + O(\epsilon^3)
\end{aligned} \tag{1.45}$$

where we have used $\alpha(0) = \alpha(1) = \mathbf{x}_0$ and the fact that $\int_0^1 dt\ \dot\alpha^x(t)(\alpha^y(t) - \mathbf{x}_0^y)$ is the quantity of area enclosed by the loop. Moreover, appearance of the curvature is rooted in antisymmetric property of $\int_0^1 dt\ \dot\alpha^a(t)(\alpha^b(t) - \mathbf{x}_0^b)$ in a, b. In general, for a loop α_{ab} lying in x^a-x^b plane, we have

$$F_{ab}(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \left(h_{\alpha_{ab}}(A) - \mathbb{I}_G \right) \tag{1.46}$$

Construction of the quantum theory necessitates computing the Poisson brackets of holonomies and fluxes. This in turn requires a regularisation procedure and a discussion of the intersection features of the curves and the surfaces on which we integrate E and A to get the involved holonomies and fluxes, respectively. We will not do this in detail here, but only display the most important Poisson bracket in the simplest non-trivial intersection. For a thorough discussion consult [1].
If the surface S intersects the curve e exactly once, then the Poisson bracket of the corresponding holonomy and flux is

$$\{E_f(S), h_e(A)\} = \frac{\sigma(S,e)}{2} h(e_1) \tau_i h(e_2) f^i(S \cap e) \tag{1.47}$$

where $\sigma(S, e) = +1$ if the tangent of e points upwards with respect to the conormal of the surface at the intersection point and -1 otherwise.

1.3.3.1. Why Holonomy-Flux Variables?!

The question arising here is why the Ashtekar-Barbero variables were not employed to introduce the classical algebra and what is the need to use the holonomy-flux algebra? The answer has several physical and mathematical aspects that are listed below:

Non-locality

In a quantum field theory with a fixed background, it is usually assumed that any arbitrary small distances can be probed by the quanta of a field. However, in the presence of a dynamical background, the field will create a horizon that acts as a shield and prevents probing arbitrary small regions. It means when quantum gravity is involved, there exists a natural cut-off. Therefore, one cannot describe the gravitational field at the quantum level using local variables (i.e. fields). Instead, it is necessary to introduce non-local ones such as the aforementioned holonomy and flux variables.

Background independence

The fact that the theory is invariant under diffeomorphisms is usually called the background independence of GR. There is a drastic difference between the diffeomorphism group and the coordinate transformations that is vastly explained in [109, 110]. In principle, a diffeomorphism (usually called active diffeomorphism) is a transformation changing the position of a point, and invariance under which means by moving smoothly the fields over the manifold, solutions of the equations of motion are mapped to themselves; whereas a coordinate transformation (or passive diffeomorphism) is merely a change in the coordinate chart, and invariance under which means by changing the coordinate the form of equations of motion does not change. Moreover, the former generates a dynamical symmetry which constrains the equations of motion, contrary to the latter which is a non dynamical symmetry [111]. With this explanation, it is clear that if a theory is invariant under the group of (active) difeomorphisms, it is independent of background. Therefore, in order to respect this fundamental symmetry, the Poisson algebra has to be built from objects which do not refer to any background, like the holonomy and flux variables.

Simple behaviour under gauge transformations

Further desirable properties of the chosen variables would be simple behaviour under spatial diffeomorphisms and internal gauge transformations. As all physical pertinent information is encoded in gauge-invariant objects, if one chooses to work with fundamental variables not behaving nicely under gauge transformations it might be very hard to even construct such objects, let alone to work with them. Although the behaviour of the connection has the complicated form $A \mapsto dgg^{-1} + gAg^{-1}$, passing to the exponential form brings up some interesting group properties that ensure a nice behaviour under gauge transformations, as already shown in (1.39).

Non-distributional Poisson bracket

Note that since the Poisson bracket (1.30) is distributional, to define a physically relevant algebra, one has to seek suitable smearing versions of the elementary variables. Notice that holonomy and flux are integrated respectively against one-dimensional curves and two-dimensional surfaces. Hence the total degree of smearing is just enough to absorb the delta function present in the Poisson bracket, making the Poisson bracket between the holonomy and the flux non-singular, see (1.47). One may ask what would have happened if one had chosen 3-dimensional smearing versions of the fundamental variables. The answer is that doing so would spoil the closure property of the Poisson algebra that has been properly achieved in (1.47). In fact, three-dimensional smearing of the densitised triad would preclude the Poisson bracket to depend on a "finite" number of the elementary variables [1].

Uniqueness theorem

The uniqueness theorem [53, 54], known also as the LOST theorem, is the next reason convincing us to work with the holonomy-flux algebra. Recalling the Stone-von Neumann theorem of QM that ensures the uniqueness of the Schrödinger representation, the LOST theorem can be considered as its counterpart in the context of diffeomorphism invariant systems. Essentially, this theorem shows that applying the GNS construction to the holonomy-flux algebra together with the requirement that the representation admits a unitary action of the diffeomorphism group singles out uniquely the representation, up to unitary equivalence. The LOST theorem reveals that the diffeomorphism group and being background-independent are highly restrictive properties at the quantum level. Fur-

thermore, the unitary implementation of the diffeomorphism group on the Hilbert space forces us to allow non-continuous scalar products. Accordingly, the loop quantisation is not equivalent to the usual Fock representation and this difference emerging exactly from background independence of GR is the cornerstone feature of quantising the gravitational field [112, 113].

1.3.3.2. Cylindrical Functions and Holonomy-Flux Algebra

As the classical configuration space $\mathcal{A}$ is the space of smooth connections, it seems reasonable to study the space of continuous functions of the connection $f(A) \in C^0(\mathcal{A})$ but as spelt out in section 1.3.3.1, the well-adapted variables to work with are not connections (as they are local quantities) but holonomies that are non-local and background-independent. Therefore, we will work only with functions of the holonomies. In other words, we are restricted to work with those functions which depend on the connection A only through its holonomies. These functions are called *cylindrical functions*. To be precise, consider a graph γ living on Σ whose edges are in a finite set of paths $E(\gamma) := \{e_1, \cdots, e_n\}$ and the vertices denoted by $V(\gamma)$ are the beginning and final points of the paths. We associate to each edge $e_i \in E(\gamma)$ the holonomy of the connection A along e_i, i.e. $h_{e_i}(A) \in G = SU(2)$ and introduce a map $H_\gamma : \mathcal{A} \to SU(2)^n$ such that $H_\gamma(A) = (h_{e_1}(A), \cdots, h_{e_n}(A))$. Therefore, a cylindrical function on a graph can be defined as follows. Given a function $f : SU(2)^n \to \mathbb{C}$, the functional

$$f_\gamma : \mathcal{A} \to \mathbb{C}; \quad f_\gamma(A) := f \circ H_\gamma(A) = f(h_{e_1}(A), \cdots, h_{e_n}(A)) \tag{1.48}$$

is called a *cylindrical function over* γ and $\mathrm{Cly}^\infty(\gamma)$ is the space of all smooth cylindrical functions on γ. A given cylindrical function is cylindrical on many graphs. Consider the example [114] of a function $f_\gamma(h_e(A))$, which is cylindrical with respect to γ with $E(\gamma) = \{e\}$. Then one can easily observe that f is also cylindrical with respect to another graph γ' with $E(\gamma') = \{e_1, e_2, e_3\}$ where $e = e_1 \circ e_2$ and e_3 independent of e. This is the case because f can be written purely in terms of holonomies along edges in γ', i.e. $f_\gamma(h_e) = f_{\gamma'}(h_{e_1} h_{e_2})$. Since we would like to identify the same cylindrical functions on different graphs, the following structure is required. Let's denote by $l(\gamma)$ the subgroupoid of $\mathcal{P}$ generated by $E(\gamma)$, i.e. the set of paths in $l(\gamma)$ consist of all members of $E(\gamma)$ together with their finite compositions and inverses. Now, we define $X_\gamma := \mathrm{Hom}(l(\gamma), G)$. Based on subgroupoid relation $\subseteq$ between all subgroupoids of $\mathcal{P}$, the set of all graphs on Σ denoted by Γ is naturally equipped with a partial order relation $\preceq$ in the way that we say $\gamma \preceq \gamma'$ if $l(\gamma) \subseteq l(\gamma')$. Having this relation in hand, for $\gamma \preceq \gamma'$ one can define a surjective projection $p_{\gamma'\gamma} : X_{\gamma'} \to X_\gamma$ satisfying $p_{\gamma'\gamma} p_{\gamma''\gamma'} = p_{\gamma''\gamma}$ for all $\gamma \preceq \gamma' \preceq \gamma''$. The family $(X_\gamma, p_{\gamma\gamma'})$ is called a *projective family*. The *projective limit* $\bar{X}$ of this projective family is the subset of the direct product $X_\infty := \prod_{\gamma\in\Gamma} X_\gamma$ defined by

$$\bar{X} := \{(x_\gamma)_{\gamma\in\Gamma}; p_{\gamma'\gamma} x_{\gamma'} = x_\gamma \ \ \forall \gamma \preceq \gamma'\} \tag{1.49}$$

Now, consider two cylindrical functions on different graphs, i.e. $f_\gamma, f'_\gamma \in \bigcup_{\gamma\in\Gamma} \mathrm{Cly}^\infty(\gamma)$ we define an equivalence relation

$$f_\gamma \sim f'_\gamma \quad \text{if} \quad \exists \gamma'' \in \Gamma \quad \text{s.t.} \quad p^*_{\gamma''\gamma} f_\gamma = p^*_{\gamma''\gamma'} f_{\gamma'} \tag{1.50}$$

The space of cylindrical functions on the projective limit $\bar{X}$ is defined to be the space of equivalence classes

$$\mathrm{Cly}^\infty := \left(\bigcup_{\gamma\in\Gamma} \mathrm{Cly}^\infty(\gamma) \right) / \sim \tag{1.51}$$

For later purposes, we want to show that one can identify $\mathrm{Hom}(\mathcal{P}, G)$ with the projective limit $\bar{X}$. Indeed, the map

$$\begin{aligned} \Phi : \mathrm{Hom}(\mathcal{P}, G) &\to \bar{X} \\ \varphi &\mapsto (\varphi|_{\ell(\gamma)})_{\gamma\in\Gamma} \end{aligned} \tag{1.52}$$

where $\varphi|_{\ell(\gamma)} \in X_\gamma$, is a bijection. The proof is as follows.

Well-definedness of Φ

It is obvious by definition that if $\varphi = \varphi'$, then $\Phi(\varphi) = \Phi(\varphi')$. In fact, when two homomorphisms φ and φ' are equal, their restrictions to subgroupoid $\ell(\gamma)$ are also the same, i.e. $\varphi|_{\ell(\gamma)} = \varphi'|_{\ell(\gamma)}$ which means $\Phi(\varphi) = \Phi(\varphi')$.

Injectivity of Φ

Suppose for two homomorphisms φ and φ' we have $\Phi(\varphi) = (\varphi|_{\ell(\gamma)})_{\gamma\in\Gamma} = (\varphi'|_{\ell(\gamma)})_{\gamma\in\Gamma} = \Phi(\varphi')$ that implies for all $\gamma \in \Gamma$, the equality $\varphi|_{\ell(\gamma)} = \varphi'|_{\ell(\gamma)}$ holds. Thus, for all $e \in E(\gamma)$ we have $\varphi(e) = \varphi'(e)$ because the homomorphisms are determined by their action on the edges. Now, since an arbitrary path p is composition of edges, we conclude $\varphi(p) = \varphi'(p)$ for any $p \in \mathcal{P}$. This means $\varphi = \varphi'$.

Surjectivity of Φ

Take $(x_\gamma)_{\gamma\in\Gamma} \in \bar{X}$. In order to show that Φ is surjective we must find one $\varphi_x \in \mathrm{Hom}(\mathcal{P}, G)$ such that $\Phi(\varphi_x) = (x_\gamma)_{\gamma\in\Gamma}$. For an arbitrary path $p \in \mathcal{P}$, we can always find a graph γ_p such that $p \in l(\gamma_p)$. Then, we define $\varphi_x(p) := x_{\gamma_p}(p) \in \mathrm{Hom}(\ell(\gamma_p), G)$. This definition is independent of the graph we choose because considering another graph γ' for which $p \in \ell(\gamma')$, one can find a third graph γ'' such that $\gamma, \gamma' \preceq \gamma''$ and hence

$$x_\gamma(p) = [p_{\gamma''\gamma}(x_{\gamma''})(p)] = (x_{\gamma''})|_{\ell(\gamma)}(p) = (x_{\gamma''})(p) = (x_{\gamma''})|_{\ell(\gamma')}(p) = x_{\gamma'}(p) \tag{1.53}$$

It follows from this definition that $\Phi(\varphi_x) = (\varphi|_{\ell(\gamma)})_{\gamma\in\Gamma} = (x_\gamma)_{\gamma\in\Gamma}$. The only task left is to show that φ_x is actually a homomorphism. Suppose two paths p, p' belong to $\ell(\gamma)$ such that $f(p) = b(p')$ which implies $p \circ p' \in \ell(\gamma)$ and also $p^{-1} \in \ell(\gamma)$. Then, since $x_\gamma \in \mathrm{Hom}(\ell(\gamma), G)$ we get

$$\varphi_x(p \circ p') = x_\gamma(p \circ p') = x_\gamma(p)x_\gamma(p') = \varphi_x(p)\varphi_x(p') \tag{1.54}$$

$$\varphi_x(p^{-1}) = x_\gamma(p^{-1}) = (x_\gamma(p))^{-1} = (\varphi_x(p))^{-1} \tag{1.55}$$

Since a cylindrical function, by definition, is a function of a finite number of holonomies, by considering the action of the flux vector field $Y_f(S) := \{E_f(S), \cdot\}$ on one single holonomy displayed in the equation (1.47), one can readily obtain the action of the flux vector field on an arbitrary smooth cylindrical function. Thus, $Y_f(S)$ can be understood as vector fields on Cyl^∞. Based on this, the precise definition of the holonomy-flux algebra is [1]

Definition 1.3.3. *The classical Poisson algebra is the Lie *-subalgebra of $Cly^\infty(\mathcal{A}) \times V^\infty(\mathcal{A})$ generated by the smooth cylindrical functions $Cly^\infty(\mathcal{A})$ and the flux vector fields $Y_f(S) \in V^\infty(\mathcal{A})$ on $Cly^\infty(\mathcal{A})$. The involution on this algebra is just complex conjugation, in particular $A(e)^* = A(e^{-1})^T$ and $Y_f(S)^* = Y_f(S)$. The algebra is called the holonomy-flux algebra.*

1.3.4. Kinematical Hilbert Space $\mathcal{H}_{\text{kin}}$

In order to reach the quantum theory, firstly, one has to find a distributional extension of the classical configuration space, i.e. the space of smooth connections $\mathcal{A}$ over Σ, and then the kinematical Hilbert space $\mathcal{H}_{kin}$ is built on the distributional configuration space denoted by $\bar{\mathcal{A}}$. The strategy is similar to what is done in quantum field theories[16] with this difference that here everything has to be constructed in a background-independent manner. This construction was achieved mathematical rigorously in a series of papers [41, 42, 115, 46, 48] by using abstract Gelfand–Naimark–Segal (GNS) theory. An essential feature of this structure is that it is not limited to quantum gravity but can be applied to any background independent quantum field theory (see for example [116, 117, 118]). As we will see, since the $U(1)^3$ model, the leading theory discussed in this dissertation, is also a background independent field theory, the same construction will be employed in chapter 7.

As explained around (1.38), $\mathcal{A}$ is a subset of $\text{Hom}(\mathcal{P}, G)$ and it can be shown that it is in fact a proper subset and $\text{Hom}(\mathcal{P}, G)$ can be understood as the distributional extension of $\mathcal{A}$, i.e. $\bar{\mathcal{A}} = \text{Hom}(\mathcal{P}, G)$. It turns out that $\mathcal{A}$ is topologically densely embedded into $\bar{\mathcal{A}}$ [119].

Now that the quantum configuration space has been recognised, it must be equipped with a measure to get a well-defined inner product based on which the kinematical Hilbert space is introduced. Let us consider two cylindrical functions defined on γ with $|E(\gamma)| = n$, i.e. $f_\gamma, f'_\gamma \in \text{Cyl}^\infty(\gamma)$. Since the arguments of f_γ, f'_γ are the holonomies along each edge of γ, they are elements of the group $G = SU(2)$. Moreover, it is well-known that the natural invariant measure on a Lie group is given by the Haar measure μ_H. Accordingly, the inner product of the cylindrical functions f_γ and f'_γ, that depend on n copies of the group $SU(2)$, is defined in the form, therefore:

$$\begin{aligned}\langle f_\gamma, f'_\gamma \rangle &= \int_{SU(2)^n} d\mu_{\text{H}}(g_1) \cdots d\mu_{\text{H}}(g_n)\ \overline{f_\gamma(g_1, \cdots, g_n)} f'_\gamma(g_1, \cdots, g_n) \\ &= \int_{SU(2)^n} (d\mu_\gamma)_{AL}\ \overline{f_\gamma(g_1, \cdots, g_n)} f'_\gamma(g_1, \cdots, g_n) \end{aligned} \tag{1.56}$$

where $(d\mu_\gamma)_{AL}$ is called the Ashtekar-Lewandowski measure [47]. In order to determine the scalar product between two functions f_γ and $f'_{\gamma'}$, cylindrical on two different graphs γ and γ', one takes any graph γ'' such that $\gamma, \gamma' \preceq \gamma''$, and views $f_{\gamma''}$ and $f'_{\gamma''}$ as cylindrical functions on γ''. The desired scalar product can then be defined as $\langle f_\gamma, f'_{\gamma'} \rangle := \langle f_{\gamma''}, f'_{\gamma''} \rangle$ that is independent of the chosen graph γ''. This independence is encoded in the measure $d\mu_{AL}$ which is said to be a *cylindrically consistent* measure. To be precise, associating a measure $d\mu_\gamma$ to every graph γ, we say the family of measures $\{d\mu_\gamma\}_{\gamma \in \Gamma}$ is consistent if for any $\gamma \preceq \gamma'$

$$\int_{SU(2)^n} d\mu_\gamma\ f_\gamma = \int_{SU(2)^n} d\mu_{\gamma'}\ (p_{\gamma'\gamma})^* f_\gamma \tag{1.57}$$

It can be shown that every consistent family of measures on X_γ defines a unique measure on $\bar{X}$ and, conversely, every measure on $\bar{X}$ can be obtained from a consistent family of measures on X_γ [43]. Choosing the Haar measure on X_γ, one gets a consistent family of measures which in turn defines a measure on $\bar{X} = \text{Hom}(\mathcal{P}, G)$ (recall the map (1.52)) known as the *Ashtekar-Lewandowski measure*.

Moreover, the laws sending the holonomy $h_e(A)$ to $g(b(e))h_e(A)g^{-1}(f(e))$ and $h_{\phi(e)}(A)$ under the gauge transformation g and the diffeomorphism ϕ, respectively, together with the left and

[16] In general, the measure that is essential to build the scalar product of the theory is supported on a distributional extension of the classical configuration space.

right invariance of the Haar measure, imply that the inner product introduced here is gauge and diffeomorphism invariant. The desired kinematical Hilbert space is then defined as the Cauchy completion of Cyl with respect to the norm following from the inner product (1.56), i.e, $\mathcal{H}_{\text{kin}} = \overline{\text{Cyl}}$. This Hilbert space is in fact isomorphic to the space of square integrable functions on $\bar{\mathcal{A}}$, i.e.

$$\mathcal{H}_{\text{kin}} = L^2(\bar{\mathcal{A}}, d\mu_{AL}) \tag{1.58}$$

Now, we want to introduce an orthonormal basis for $\mathcal{H}_{\text{kin}}$, that henceforth is called *spin network basis*. Based on the Peter-Weyl theorem[17] [120] , we can find a basis for $\mathcal{H}_e = L^2(SU(2); dg_H)$ as $\sqrt{\mathrm{d}_j}[\pi_j]_{mn}(h_e)$, where d_j is the dimension of the representation space of the unitary representation π_j of the group $SU(2)$ and the indices m, n label the matrix elements of π_j. Since the space $\mathcal{H}_\gamma := L^2(G^n, d\mu_\gamma)$ is essentially a tensor product of the spaces $\mathcal{H}_e := L^2(G, d\mu)$ over the edges of γ, it follows that an orthonormal basis on $\mathcal{H}_\gamma := L^2(G^n, d\mu_\gamma)$ is formed by products of the functions

$$T_{\gamma,\vec{\pi},\vec{m},\vec{n}} : \bar{\mathcal{A}} \to \mathbb{C}; \qquad A \mapsto \prod_{e\in E(\gamma)} \sqrt{\mathrm{d}_{j_e}}[\pi_{j_e}(h_e(A))]_{m_e n_e} \tag{1.61}$$

whereby, the vectors $\vec{m} := \{m_e\}_{e\in E(\gamma)}$ and $\vec{n} := \{n_e\}_{e\in E(\gamma)}$ denote a certain collection of matrix element labels of the representations, that is $m_e, n_e \in \{1, \cdots, 2j_e + 1\}$. We call the above spin network functions gauge variant, because they are in general not invariant under $SU(2)$ gauge transformations. In order to define also gauge invariant spin network functions (SNFs) which are the ones being physically relevant, we will rewrite the SNFs with the help of so called intertwiners.

Note that according to the Peter-Weyl theorem, spin network functions can be defined for any compact lie group G, but in what follows we focus on the group $G = SU(2)$, because this is the one required for Loop quantum gravity. In chapter 7 we will specialise our discussion to the compact group $U(1)^3$ leading to functions so-called *charge network functions*.

1.3.4.1. Some Useful Quantum Operators

The holonomy operator acts on a cylindrical function f_γ by multiplication:

$$\hat{h}_e \, |f_\gamma\rangle(A) = (h_e(A)) f_\gamma \left(h_{e_1(A)}, \cdots, h_{e_N}(A)\right) \tag{1.62}$$

where $N = |E(\gamma)|$. Note that the relative position of the edge e and the graph γ determines the character of the resulting expression:

- If $e \in \ell(\gamma)$, then (1.62) is an element of $\mathcal{H}_\gamma$.
- If $e \notin \ell(\gamma)$, then (1.62) is an element of $\mathcal{H}_{\gamma\cup e}$.

In principle, in the former case, the action of the holonomy operator modifies the spin quantum number associated to the edge e of γ, and in the latter case, it adds a new edge to the underlying

[17] The Peter-Weyl theorem says that for any compact group G, the space $L^2(G, d\mu_H)$ of square-integrable functions, with inner product $\langle k, h\rangle = \int_G k\bar{h} d\mu_H$ ($d\mu_H$ denotes the normalised Haar measure), has an orthonormal basis

$$\{\sqrt{\dim(\pi)}\pi_{ij} | \pi \in \Pi(G),\ 1 \le i, j \le \dim(\pi)\} \tag{1.59}$$

where $\Pi(G)$ is the set of all equivalence classes of irreducible representations of G and π_{ij} denotes the ij matrix element of π. Here, the orthonormality means

$$\langle \sqrt{\dim(\pi)}\pi_{ij}, \sqrt{\dim(\pi')}\pi'_{kl}\rangle = \delta_{\pi\pi'}\delta_{ik}\delta_{jl} \tag{1.60}$$

graph on which the cylindrical function is defined. Therefore, in both cases, the action of the holonomy operator changes the cylindrical function in such a way that the resulting cylindrical function $\hat{h}_e\ |f_\gamma\rangle$ is typically orthogonal to the original function $|f_\gamma\rangle$ with respect to the inner product (1.56). This crucial observation prevents the connection A to have a well-defined quantum operator. Indeed, we know from (1.42) that one should define a connection operator through the following limit

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}(\hat{h}_{e_\epsilon}-\mathbb{I})|f_\gamma\rangle \tag{1.63}$$

where e_ϵ is a path of coordinate length ϵ. Because of the aforementioned orthogonality property, in general, the limit (1.63) fails to exist.

To obtain the flux operator rigorously, one first must regularise the classical flux as $E_i^\epsilon(S)=\int_{-\epsilon}^{+\epsilon}dt\delta(t,0)\ \int_{S_t}E_i^a n_a^S$ and then the action of the flux operator is defined as the action of the regularised one when the regularisation parameter ϵ is removed, i.e. $\hat{E}_i(S):=i\hbar\lim_{\epsilon\to 0}\{E_i^\epsilon(S),\ \cdot\ \}$. In what follows, we will not go through the regularisation procedures and just refer the reader to [1], but instead, we derive the desired operator in a heuristic manner.

In order to define the flux operator, first, we need to introduce the left and right invariant vector fields on the group G

$$(L_i f)(g)=i\frac{d}{dt}|_{t=0}f(ge^{t\tau_i}),\qquad (R_i f)(g)=i\frac{d}{dt}|_{t=0}f(e^{-t\tau_i}g) \tag{1.64}$$

One easily generalises (1.64) to operators $L_i^{(e)}$, $R_i^{(e)}$ acting cylindrical functions. Let f_γ be a cylindrical function such that e is among the edges of the underlying graph γ, then

$$(L_i^{(e)}f_\gamma)(A):=i\frac{d}{dt}|_{t=0}f_\gamma\left(h_{e_1}(A),\cdots,h_e(A)\ e^{t\tau_i},\cdots,h_{e_n}(A)\right), \tag{1.65}$$

$$(R_i^{(e)}f_\gamma)(A):=i\frac{d}{dt}|_{t=0}f_\gamma\left(h_{e_1}(A),\cdots,e^{-t\tau_i}\ h_e(A),\cdots,h_{e_n}(A)\right) \tag{1.66}$$

in the case $e\notin E(\gamma)$, we define $(L_i^{(e)}f_\gamma)(A)=(R_i^{(e)}f_\gamma)(A)=0$. Using these operators, we can construct the following practical operators on $\mathcal{H}_{\text{kin}}$

$$(\mathcal{J}_i^{(e,v)}f_\gamma)(A):=\begin{cases}(L_i^{(e)}f_\gamma)(A), & \text{if } e\in E(\gamma),\ b(e)=v\\ (R_i^{(e)}f_\gamma)(A), & \text{if } e\in E(\gamma),\ f(e)=v\\ 0, & \text{if } e\notin E(\gamma)\end{cases} \tag{1.67}$$

Now, following the canonical quantisation rule $E_i^a\to -i\hbar\frac{\delta}{\delta A_a^i}$, we can derive the action of the quantum counterpart of the classical flux (1.40) on a single holonomy

$$\begin{aligned}\hat{E}_i(S)|h_e\rangle(A)&=-i\hbar\int_S d^2s\ n_a(s)\frac{\delta}{\delta A_a^i(x(s))}h_e(A)\\ &=\begin{cases}+\ i\hbar\dfrac{\sigma(S,e)}{2}h_e(A)\tau_i, & S\cap e=b(e)\\ -\ i\hbar\dfrac{\sigma(S,e)}{2}\tau_i h_e(A), & S\cap e=f(e)\end{cases}\\ &=\frac{\hbar}{2}\ \sigma(S,e)\ \mathcal{J}_i^{(e,v)}h_e(A)\end{aligned} \tag{1.68}$$

where in the second equality we have used (1.47) and in the last step we simply used the definition of the operators $\mathcal{J}_i^{(e,v)}$. If the edge e does not intersect the surface S, or intersects it tangentially, the result vanishes. Since any $f_\gamma \in \mathcal{H}_{\rm kin}$ is a function of finite holonomies, the action of the flux operator $\hat{E}_i(S)$ can be readily defined on whole $\mathcal{H}_{\rm kin}$

$$\hat{E}_i(S)|f_\gamma\rangle(A) = \frac{\hbar}{2} \sum_{v\in S} \sum_{v\cap e\neq\emptyset} \sigma(S,e)\, \mathcal{J}_i^{(e,v)} f_\gamma(A) \tag{1.69}$$

We end this section by introducing the volume operator. This is important for our purposes in the succeeding chapters because this operator enters crucially into the construction of the dynamics of LQG and also the $U(1)^3$ model through Thiemann's identities (see chapter 4). In order to quantise the three-dimensional volume $V(R) = \int_R d^3x \sqrt{|\det(E_i^a)|}$ for a given region R, first one needs to regularise E_i^a suitably and then express it in terms of the flux operator. In the literature there exist two different volume operators: Rovelli-Smolin (RS) [121] and Ashtekar-Lewandowski (AL) [122] volume operators developed using different regularisation techniques. Both volume operators act non-trivially only on vertices where at least three edges intersect. At a given vertex the operators have the following form

$$\hat{V}_{\rm RS}(R)|f_\gamma\rangle = C_{\rm RS} \sum_{v\in V(\gamma)\cap R} \sum_{e_I\cap e_j\cap e_K=v} \sqrt{|\hat{Q}^{(v)}_{IJK}|}|f_\gamma\rangle \tag{1.70}$$

$$\hat{V}_{\rm AL}(R)|f_\gamma\rangle = C_{\rm AL} \sum_{v\in V(\gamma)\cap R} \sqrt{|\sum_{e_I\cap e_j\cap e_K=v} \epsilon(e_I,e_J,e_K)\hat{Q}^{(v)}_{IJK}|}|f_\gamma\rangle \tag{1.71}$$

where

$$\hat{Q}^{(v)}_{IJK} := \epsilon^{ijk}\, \mathcal{J}_i^{(e_I,v)}\, \mathcal{J}_j^{(e_J,v)}\, \mathcal{J}_k^{(e_K,v)} \tag{1.72}$$

and $C_{\rm RS}, C_{\rm AL}$ are regularisation constants. Moreover, the second sum in both volume operators runs over all ordered triples of edges incident at v and the factor $\epsilon(e_I,e_J,e_K)$ is defined to equal $+1$ if the ordered tangents to the edges are positively oriented, -1 if it is negatively oriented, and 0 if the tangents are not independent. In contrast to (1.70), the presence of the orientation factor tells us that if one works with (1.71), planar triples of edges will not contribute and that is the main difference between the two operators. It is also worth mentioning that, in a gauge-invariant spin network function, it is a "four-valent vertex" (not a three-valent one) that is the simplest one on which the AL-volume operator acts non-trivially. This is again a direct consequence of the presence of the factor orientation.

1.3.5. Solving the Constraints

Implementation of the constraints is the next task one should accomplish after constructing the kinematical Hilbert space $\mathcal{H}_{\rm kin}$. Recalling the Poisson algebra (1.32)-(1.35) of the constraints, one sees that the subalgebra generated by the Gauß constraints forms a 2-sided ideal. Thus, we can solve the Gauß constraints independently of the other constraints and arrive at a Hilbert space whose constituents are gauge-invariant quantum states. Then, although the subalgebra generated by the diffeomorphism constraints is not an ideal, for the technical convenience, we still solve them independently of the Hamiltonian constraints. The incorporation of the Hamiltonian constraints will be postponed to chapter 4.

1.3.5.1. Gauß Constraint

In this section, we want to derive the solutions of the quantum Gauß constraint. They are characterised by $SU(2)$-gauge-invariant states in $\mathcal{H}_{\text{kin}}$. These states form a Hilbert space spanned by gauge-invariant spin network functions that are a very important tool in LQG. Having these few words as an introduction, let's dive in.

Paying attention to the Dirac algebra (1.32)-(1.35), one observes that the Gauß constraints form an honest Lie algebra. Thus, one can solve them using the procedure of RAQ discussed in section 1.2.4.1. In this method, we construct solutions of the Gauß constraint by averaging arbitrary cylindrical functions with respect to the action of the $SU(2)$ gauge transformations. Consider the formal projection operator $\hat{P}_G := \int_G d\mu(g)\hat{U}(g)$ where $\hat{U}(g)$ is the unitary operator corresponding to the $SU(2)$ gauge transformations on cylindrical functions, and $d\mu(g)$ is an appropriate measure related to the $SU(2)$ Haar measure. Using group properties and the features of the Haar measure, we show that the operator $\hat{P}_G$ satisfies $\hat{U}(g)\hat{P}_G = \hat{P}_G$, in fact

$$\begin{aligned}\hat{U}(g)\hat{P}_G f_\gamma &= \hat{U}(g)\int_G d\mu_H(g')\hat{U}(g')f_\gamma = \int_G d\mu_H(g')\hat{U}(g)\hat{U}(g')f_\gamma = \int_G d\mu_H(g')\hat{U}(gg')f_\gamma \\ &= \int_G d\mu_H(g^{-1}g'')\hat{U}(g'')f_\gamma = \int_G d\mu_H(g'')\hat{U}(g'')f_\gamma = \hat{P}_G f_\gamma \end{aligned} \tag{1.73}$$

where $g'' := gg'$ and we have used that $\hat{U}$ is a representation of G and the left-invariance feature of the Haar measure. Therefore, starting with an arbitrary cylindrical function f_γ, we expect the function $\hat{P}_G f_\gamma$ to be $SU(2)$-gauge invariant.

The gauge transformation of a cylindrical function on a graph γ is completely determined by the values of the gauge function $g(x)$ at the vertices of γ. Therefore, we can explicitly define the projection operator $\hat{P}_G$ on $\mathcal{H}_\gamma$

$$\hat{P}_G|_\gamma = \int_{SU(2)^m}\left(\prod_{i=1}^m d\mu_H(g_i)\right)\hat{U}(g) \tag{1.74}$$

where μ_H is the unique Haar measure on $SU(2)$. Using the gauge transformation (1.39), one can compute $\hat{U}(g)f_\gamma(A)$ and conclude that

$$\hat{P}_G f_\gamma(A) = \int_{SU(2)^M}\left(\prod_{i=1}^M d\mu_H(g_i)\right) f_\gamma\left(g(b(e_1))h_{e_1}(A)g(f(e_1))^{-1},\cdots,g(b(e_N))h_{e_N}(A)g(f(e_N))^{-1}\right) \tag{1.75}$$

is a gauge-invariant function. To derive the form of this function, let us expand the function $f_\gamma(A)$ in the basis (1.61) as

$$f_\gamma(A) = \sum_{\vec{j},\vec{m},\vec{n}} c^{\vec{j},\vec{m},\vec{n}}\, T_{\gamma,\vec{j},\vec{m},\vec{n}}(A) \tag{1.76}$$

Plugging (1.76) into (1.75), one finds

$$\hat{P}_G f_\gamma(A) = \sum_{\vec{j},\vec{I}} c^{\vec{j},\vec{I}}\left(\prod_{v\in V(\gamma)} I_v\right)\cdot\left(\prod_{e\in E(\gamma)}\sqrt{d_{j_e}}\,\pi^{(j_e)m_e}{}_{n_e}h_e(A)\right) \tag{1.77}$$

where we have used the following identity

$$\int_G d\mu_H(g)\left(\prod_{i=1}^N \pi^{(j_i)m_i}{}_{n_i}(g)\right)\left(\prod_{i=1}^N \pi^{(j_i')m_i'}{}_{n_i'}(g^{-1})\right) = \sum_I I^{m_1\cdots m_N}_{m_1'\cdots m_N'} I^{n_1\cdots n_N}_{n_1'\cdots n_N'} \tag{1.78}$$

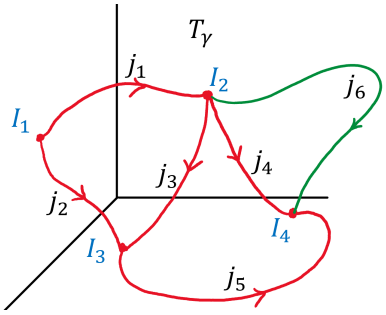

Figure 1.1.: A Gauge-invariant spin network

in which the sum over I runs through an orthonormal basis of the space $\mathrm{Inv}\left(\bigotimes_{i=1}^N \mathcal{H}_{j_i} \otimes \bigotimes_{i=1}^N \mathcal{H}_{j_i}^*\right)$. According to (1.77), the following states

$$T_{\gamma,\vec{j},\vec{I}}(A) = \left(\prod_{v\in V(\gamma)} I_v\right) \cdot \left(\prod_{e\in E(\gamma)} \sqrt{d_{j_e}}\pi^{(j_e)m_e}{}_{n_e} h_e(A)\right) \tag{1.79}$$

span the gauge-invariant Hilbert space $\mathcal{H}_{\mathrm{kin}}^G$ and are called *gauge-invariant spin network functions*. Notice that as $\mathcal{H}_{\mathrm{kin}}^G$ is a proper subspace of $\mathcal{H}_{\mathrm{kin}}$, the inner product on $\mathcal{H}_{\mathrm{kin}}^G$ is immediately inherited from $\langle\cdot,\cdot\rangle_{\mathrm{kin}}$. With respect to this inner product, gauge-invariant spin network functions form an orthonormal basis for $\mathcal{H}_{\mathrm{kin}}^G$, provided that appropriate considerations are taken into account. Given a graph γ, we identify γ with any graph differing from it only by 1) different orientations of some of their edges, 2) a number of trivial bivalent vertices and 3) edges carrying zero spin. Denoting the equivalence class by $[\gamma]$, we obtain the orthogonal decomposition

$$\mathcal{H}_{\mathrm{kin}}^G = \bigoplus_{[\gamma]} \mathcal{H}_\gamma \tag{1.80}$$

1.3.5.2. Diffeomorphism Constraint

We now turn to solve the diffeomorphism constraint. The action of the finite diffeomorphism is defined on the spin network function T_s based on a multi-label $s = (\gamma, \vec{j}, \vec{m}, \vec{n})$ as

$$\hat{U}(\varphi)T_s := T_{\varphi(s)}, \quad \text{with} \quad \varphi(s) := (\varphi(\gamma), \vec{j}, \vec{m}, \vec{n}) \tag{1.81}$$

for all $\varphi \in \mathrm{Diff}_{sa}^\omega(\Sigma)$[18], meaning that $\hat{U}(\varphi)$ maps the graph simply to its image under the diffeomorphism without changing the colours of its edges and vertices. Let φ_t be a one-parameter family of diffeomorphisms generated by the non-zero vector field V. We choose γ in the support of V , then there exists $\epsilon > 0$ such that $\varphi_t^V(\gamma) \neq \gamma$ for all $0 < t < \epsilon$. Then we have

$$\lim_{t\to 0}\langle T_s, \hat{U}(\varphi_t)T_s\rangle_{\mathrm{kin}} = \lim_{t\to 0}\langle T_s, T_{\varphi_t(s)}\rangle_{\mathrm{kin}} = 0 \neq 1 = \langle T_s, T_s\rangle_{\mathrm{kin}} \tag{1.82}$$

where the second equality comes from the fact that $\varphi_t(\gamma) \neq \gamma$. (1.82) shows that the action of $\hat{U}(\varphi)$ is not weakly continuous. Thus, according to the Stone's theorem [123] the infinitesimal generators of $\mathrm{Diff}_{sa}^\omega(\Sigma)$ cannot be defined on $\mathcal{H}_{kin}$. Recalling that only those representations of the

[18] $\mathrm{Diff}_{sa}^\omega(\Sigma)$ denotes the group of semianalytic diffeomorphisms of Σ.

quantum algebra $\mathfrak{A}$ are interesting on which the constraints can be implemented as operators, one might be worried that no operator corresponds to the diffeomorphism constraint H_a. However, since the finite diffeomorphism has a well-defined action, there are no problems in applying Dirac's quantisation procedure. Now, one is supposed to find solutions of the finite diffeomorphism constraint using the RAQ method discussed in section (1.2.4.1). A remarkable virtue of this method is that group averaging may be employed even if a well-defined constraint operator is not available. We are looking for generalised eigenfunctions $\ell \in \mathcal{D}^*$ with $\mathcal{D} = C^\infty(\bar{\mathcal{A}})$ satisfying

$$\ell(\hat{U}^\dagger(\varphi)T_s) = \ell(\hat{U}(\varphi^{-1})T_s) = \ell(T_s) \tag{1.83}$$

for all labels s and all $\varphi \in \mathrm{Diff}^\omega_{sa}(\Sigma)$. Note that the validity of (1.83) for T_s, instead of an arbitrary $f \in \mathcal{D}$, is sufficient because the spin network functions lie dense in $\mathcal{D}$. Since every $\ell \in \mathcal{D}^*$ is completely specified by its action on the spin network functions, we can express it as $\ell = \sum_s \ell_s \langle T_s, \cdot\rangle_{\mathrm{kin}}$ where $\ell_s := \ell(T_s) \in \mathbb{C}$. Using this expression in (1.83) says that we are actually looking for ℓ satisfying $\ell_s = \ell_{\varphi(s)}$. Defining the orbit of s by $[s] := \{\varphi(s), \varphi \in \mathrm{Diff}^\omega_{sa}(\Sigma)\}$ and $\ell_{[s]} := \sum_{s'\in[s]} \langle T_{s'}, \cdot\rangle_{\mathrm{kin}}$, one can verify that $\ell = \sum_{[s]} c_{[s]}\ell_{[s]}$ is a general solution of the diffeomorphism constraint, because

$$\begin{aligned}
\ell\left(\hat{U}(\varphi^{-1})T_{s'}\right) &= \sum_{[s]} c_{[s]}\ell_{[s]}\hat{U}(\varphi^{-1})T_{s'} = \sum_{[s]} c_{[s]} \sum_{s''\in[s]} \langle T_{s''}, \hat{U}(\varphi^{-1})T_{s'}\rangle \\
&= \sum_{[s]} c_{[s]} \sum_{s''\in[s]} \langle \hat{U}(\varphi)T_{s''}, T_{s'}\rangle = \sum_{[s]} c_{[s]} \sum_{s''\in[s]} \langle T_{\varphi(s'')}, T_{s'}\rangle \\
&= \ell(T_{s'})
\end{aligned} \tag{1.84}$$

The space of solutions to the diffeomorphism constraint will be denoted by $\mathcal{H}_{\mathrm{Diff}}$ on which we need to construct an inner product to get a Hilbert space. A suitable rigging map through which the inner product of $\mathcal{H}_{\mathrm{Diff}}$ can be constructed is $\eta(T_s) := \eta_{[s]}\ell_{[s]} = \eta_{[s]} \sum_{s'\in[s]} \langle T_{s'}, \cdot\rangle$ with $0 < \eta_{[s]} \in \mathbb{R}$. Consequently, we define the desired inner product as

$$\langle \eta(T_s), \eta(T_{s'})\rangle_{\mathrm{Diff}} := \eta(T_{s'})[T_s] = \eta_{[s']} \sum_{s''\in[s']} \langle T_{s''}, T_s\rangle = \eta_{[s']} \sum_{s''\in[s']} \delta_{s''s} = \eta_{[s']}\chi_{[s']}(s) \tag{1.85}$$

where χ denotes the characteristic function that is either one or zero. It is easy to check that the inner product (1.85) is well-defined.

Summarising, we identified all graphs that can be mapped to the graph γ by applying a diffeomorphism and defined the equivalence class $[\gamma]$. The graphs in another equivalence class can not be related to γ through a diffeomorphism. These diffeomorphism equivalence classes are called *knots*. Asking for diffeomorphism invariant quantum states, one obtains a gauge-invariant spin network based on a knot. Such a diff- and SU(2)- invariant state is called an *s-knot* and two s-knots are orthogonal with respect to the inner product (1.85) if their knots are not the same, i.e. their underlying graphs do not belong to the same diffeomorphism equivalence class.

There is still a delicacy about the $\mathcal{H}_{\mathrm{Diff}}$'s structure arising from the question of the separability of the space. Since the diffeomorphism invariant states keep in memory the differential structure at the vertices of the graph, those states in $\mathcal{H}_{\mathrm{Diff}}$ having vertices of sufficiently high valence carry continuous, diffeomorphism invariant information that is called *moduli*, leading to an uncountable basis for $\mathcal{H}_{\mathrm{Diff}}$, i.e. $\mathcal{H}_{\mathrm{Diff}}$ is non-separable (see e.g. [124]). However, as is pointed out in [125, 57], the known operators in LQG are insensitive to the values of the moduli. As a result, in practice, one can arbitrarily fix the values of the moduli and work with a separable subspace of $\mathcal{H}_{\mathrm{Diff}}$. It means that despite the non-separability of the entire space $\mathcal{H}_{\mathrm{Diff}}$, any set of practically pertinent operators always select a separable subspace called the *superselection sector* of $\mathcal{H}_{\mathrm{Diff}}$.

1.3.5.3. Hamiltonian Constraint

We have defined the kinematical Hilbert space and implemented the Gauß and diffeomorphism constraints and eventually reach the last step of the quantisation. It is left to implement the Hamiltonian constraint and then to build up the physical Hilbert space. Since the Gauß and spatial diffeomorphism constraints generate gauge transformations at a fixed time, they cannot be considered as a generator of dynamics and rather they are often referred to as the kinematical constraints. On the other hand, from the spacetime point of view, the Hamiltonian constraint generates the time evolution and so any question about the quantum dynamics dwells in proper incorporation of the Hamiltonian constraint and finding its kernel to construct the physical Hilbert space. As the Hamiltonian constraints do not form an honest Lie algebra, according to the Dirac algebra (1.35), we are precluded from employing RAQ techniques to implement and solve them. Therefore, it is thrust upon one to derive a well–defined operator from the classical form of the constraint by using a precise regularisation process, then trying to find its kernel and finally promoting the solution space to Hilbert space.

The first concrete implementation of the Hamiltonian constraint was proposed by Thiemann in [55, 56, 57]. Although it was great progress toward completing the quantisation, some ambiguities [126] in the construction prevents us to call it the final result. Up to now, incorporation of the dynamics remains a difficult task that has generated a lot of work both in the canonical (e.g. [127, 128]) and the covariant (see e.g. [65] and references therein) approach. In chapter 4, we will elaborately discuss this issue which is our main motivation to work on the $U(1)^3$ model.

Reduced Phase Space Quantisation

2.1. Preliminaries

2.1.1. What the Reduced Phase Space Is

The physical relevant quantities are those which are invariant under gauge transformation. As explained in section 1.1.2, generators of the gauge transformations are first-class constraints whose orbits on the constraint surface is called the *gauge orbits*. In order to obtain a space free of any gauge variable one must identify the phase-space points in the same gauge orbit. The resulting space, which is also equipped with a symplectic structure, is called *reduced phase space*.

We can remove the first-class constraints utilising gauge conditions $\mathcal{G}_a = 0$, which are some conditions on the phase-space variables, intersecting each gauge orbit "once and only once". This requirement is (locally) equivalent to

$$\{\mathcal{G}_a, C_b\}\lambda^b \approx 0 \Rightarrow \ \lambda^b \approx 0 \tag{2.1}$$

because the gauge transformations are generated by the first-class constraints C_b. Note that (2.1) is equivalent to $\det\left(\{\mathcal{G}_a, C_b\}\right) \neq 0$, hence according to the third footnote of section 1.1.2, the constraints $\mathcal{G}_a, C_b$ together produce a second-class system. In other words, when the gauge conditions $\mathcal{G}_a$ are included, no first-class constraint is left. Then, by introducing the corresponding Dirac bracket, we can remove all the constraints and gauge conditions.

For gauge-invariant functions, the resulting Dirac bracket agrees with the original Poisson bracket. Therefore, the reduced phase space is the space achieved after this "reduction". It is free of any constraint and is of dimension $2n - 2n_1 - n_2$, where n is the dimension of the original configuration space, n_1 is the number of first-class constraints, and n_2 is the number of second-class constraints. The first-class constraints strike twice because they require gauge conditions. From this perspective, the observables can be considered as the reduced phase-space functions that form a Poisson algebra.

2.1.2. Problem of Time

The formulation of quantum theory is imbued with the idea of measuring quantities at a particular instant of time. According to the conventional Copenhagen interpretation, time is considered an external parameter to the system relating to the classical world, and the lack of a time operator in the quantum theory reflects the fact that time is not a physical observable [129]. This particular characteristic of time pertains to non-relativistic quantum theory, relativistic particle dynamics, and quantum field theory. On the other hand, when GR is taken into account, the invariance of the theory under the action of active diffeomorphisms intimates that one cannot assign intrinsic

physical significance to the individual points of both space and time. In other words, the action of spacetime diffeomorphisms is comprehended as a transformation between two different reference frames. The invariance under such a transformation implies that physics is the same for all observers, independent of their chosen reference frame. Consequently, two quantities that relate only by a spacetime diffeomorphism are, indeed, physically identified, and in this sense, GR can be understood as a gauge theory with the gauge group of all spacetime diffeomorphisms. In general, observables of any constrained theory must be unchanged along the gauge orbits and hence Poisson commute with all the constraints. For a fully constrained theory, such as GR, the canonical Hamiltonian $\mathbf{H}_{\text{can}}$ is a linear combination of the constraints and therefore vanishes on the constraint surface. Thus, if we interpret $\mathbf{H}_{\text{can}}$ as the generator of physical evolution, then we immediately run into the trouble that observables do not evolve dynamically and we get a frozen picture which is sometimes referred to as *the problem of time* [129, 130, 131]. This is of course in conflict with reality where for instance in cosmology we can observe the gravitational interaction as a dynamical process.

As discussed in the previous chapter, the quantum theory of GR can be thought of as an impeccable theory, only when one can specify the dynamics of the quantum states. Apart from technical difficulties one may encounter in order to define the dynamics, the problem of time makes the situation also conceptually intricate. The resolution to this problem, which occurs in any generally covariant theory, stems from the fact that $\mathbf{H}_{\text{can}}$ describes evolution with respect to the "coordinate" time, which is meaningless because the invariance under general coordinate transformations deprives the coordinates of any physical meaning. What we observe in reality is, in fact, evolution with respect to other fields and not the coordinate time. This idea first discussed by Bergmann [130, 132], Komar [133] and Kuchar [131] and further refined by Anishetty and Vytheeswaran [134], Mitra and Rajaraman [135], and Rovelli [136, 137, 138, 139, 140]. This idea is known as *relational formalism* whose mathematical description developed by Dittrich [141, 142] and Thiemann [143] will be presented in the following.

2.1.3. Purpose of This Chapter

Dirac's main impetus in cultivating the Hamiltonian theory of a constrained system and its quantisation was to find a technique for quantising gravity. However, the application of his scheme turned out to be problematic on both technical and conceptual levels, as, for instance, the Hamiltonian constraint is perplexing, and the above-mentioned problem of time arises.

As we have already seen in chapter 1, the underlying idea of the Dirac quantisation method is comprised of imposing the first-class constraints as quantum operator conditions on states for picking out the physical ones. It means that in the Dirac method, the reduction is imposed after the quantisation. Hoping to overcome Dirac's quantisation problems, one may consider a method working the other way round, called the *reduced phase space quantisation*: factoring out the constraint surface with respect to the action of the gauge group, generated by the first-class constraints, one first identifies the physical degrees of freedom at the classical level and then quantises the resulting reduced phase space as a usual unconstrained system (see figure (2.1)).

In this chapter, after introducing the reduced phase space quantisation, we study its application in the context of loop quantum gravity and finally compare it with Dirac quantisation [144, 145]. The contents of this chapter will be used extensively in chapter 8.

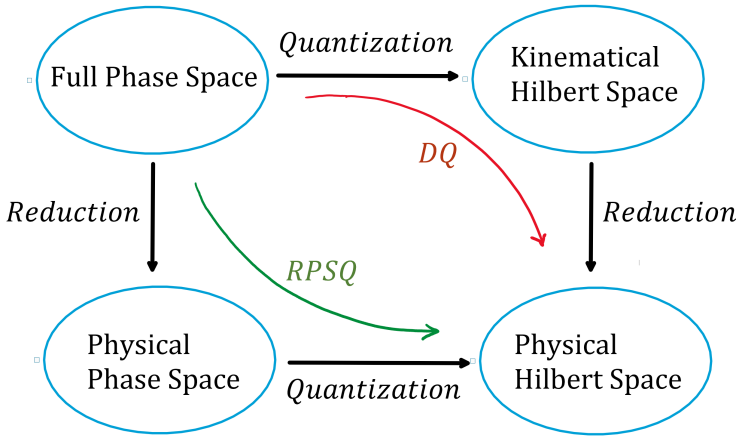

Figure 2.1.: Dirac quantisation (DG) versus reduced phase space quantisation (RPSQ)

2.2. Reduced Phase Space Quantisation Procedure

2.2.1. Relational Formalism and Constructing Observables

Giving serious consideration to the fact that the Einstein equations cannot be described as physical evolution equations, Rovelli introduced relational formalism to bypass the above-mentioned problem of time. In fact, one should interpret the Einstein equations as the evolution of quantities under a gauge flow generated by the diffeomorphism and the Hamiltonian constraint. The underlying idea of relational formalism is to establish so-called clocks using which a physical meaning can be attributed to spacetime points and with respect to which one describes the evolution of observables.

Take two arbitrary gauge variant functions f and T defined on the phase space. Under the action of the constraints, these two functions just move along the gauge orbits parametrised by t. Being equivalent up to a gauge transformation, all points on a gauge orbit are physically identified. We choose one of the functions, say T, as a clock. Now, instead of stating the evolution of f with respect to t, we can construct a gauge-invariant expression for f, denoted by $\mathcal{O}_{f,T}$, with respect to the values that T takes while evolving along the gauge orbit. In other words, the physically important quantity $\mathcal{O}_{f,T}$ is actually defined as the value of f at those values of t where T takes a certain value τ. In this way, the function $\mathcal{O}_{f,T}$ can be expressed as a function of τ. For instance, a scalar field φ coupled to gravity can be considered as a clock and thus the values of the spatial metric q_{ab} at those values of t where $\varphi(t) = \tau$ construct the gauge invariant quantity $\mathcal{O}_{q_{ab},\varphi}$.

The philosophy behind this construction of Dirac observables is as follows. In our daily life, we parametrise the motion of any object by time. But time itself is parametrised by the arm of a clock, meaning that we actually parametrise the motion of any object by the arm of a clock. Thus, the motion of an object is always parametrised using the motion of another object. For example, as we will see in the next section, we can parametrise the motion of the gravitational field by using the motion of a dust field. In fact, there, the dust field behaves like a reference field.

Mathematically speaking, one can explicitly express $\mathcal{O}_{f,T}$ as a power series in the clock variables with coefficients including iterated Poisson brackets between f and the constraints. As a simple case, consider a system with only one constraint C in which an arbitrary phase space function T

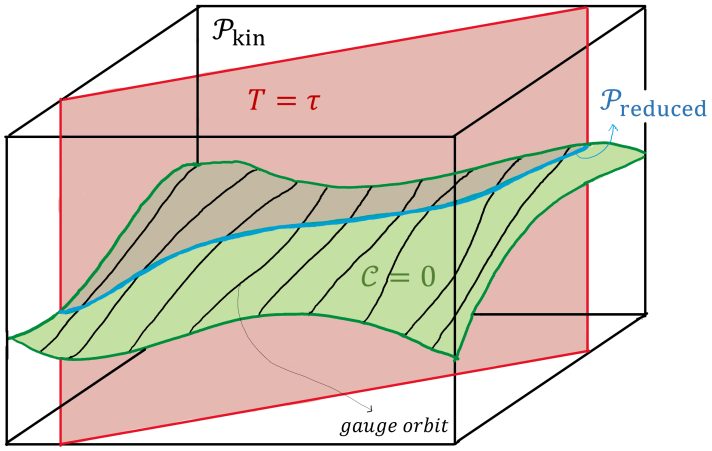

Figure 2.2.: Reduced phase space

is chosen as a clock. Under the flow generated by C the clock T is transformed to

$$\alpha_t(T) := \sum_{n=0}^{\infty} \frac{t^n}{n!} \{C, T\}_{(n)} \tag{2.2}$$

where we define the iterated Poisson bracket inductively by $\{C,T\}_{(n+1)} = \{C, \{C,T\}_{(n)}\}$ with the base case $\{C,T\}_{(0)} = T$. Now, in order to delineate the evolution of an arbitrary function f with respect to the clock T, first we have to solve the equation $\alpha_t(T) = \tau$ for the gauge parameter t. One can find such a solution, denoted by $t_T(\tau)$, if the map $t \mapsto \alpha_t(T)$ is assumed to be locally invertible. Therefore the gauge invariant extension of f with respect to T is simply

$$\mathcal{O}_{f,T}(\tau) := [\alpha_t(f)]_{t=t_T(\tau)} \tag{2.3}$$

It is straightforward to see that (2.3) is a Dirac observable

$$\begin{aligned}
\{C, \mathcal{O}_{f,T}(\tau)\} &= \sum_{n=0}^{\infty} \frac{1}{n!} \{C, [t^n]_{t=t_T(\tau)} \{C, f\}_{(n)}\} \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \{C, [t^n]_{t=t_T(\tau)}\} \{C, f\}_{(n)} + \sum_{n=0}^{\infty} \frac{[t^n]_{t=t_T(\tau)}}{n!} \{C, \{C, f\}_{(n)}\} \\
&= \sum_{n=1}^{\infty} \frac{[t^{n-1}]_{t=t_T(\tau)}}{(n-1)!} \{C, t_T(\tau)\} \{C, f\}_{(n)} + \sum_{n=1}^{\infty} \frac{[t^{n-1}]_{t=t_T(\tau)}}{(n-1)!} \{C, f\}_{(n)} \\
&= \sum_{n=1}^{\infty} \frac{[t^{n-1}]_{t=t_T(\tau)}}{(n-1)!} \{C, f\}_{(n)} \left(\{C, t_T(\tau)\} + 1\right) \\
&= \left[\frac{d}{dt} \alpha_t(f)\right]_{t=t_T(\tau)} \left(\{C, t_T(\tau)\} + 1\right)
\end{aligned} \tag{2.4}$$

As f is assumed to be gauge variant, one concludes $\left[\frac{d}{dt}\alpha_t(f)\right]_{t=t_T(\tau)} \neq 0$ and thus $\mathcal{O}_{f,T}$ is an observable if and only if $\{C, t_T(\tau)\} + 1 = 0$ that holds because τ is a constant, i.e.

$$0 = \{C, \tau\} = \{C, [\alpha_t(T)]_{t=t_T(\tau)}\} = \left[\frac{d}{dt} \alpha_t(T)\right]_{t=t_T(\tau)} \left(\{C, t_T(\tau)\} + 1\right) \tag{2.5}$$

Again, since the clock T is gauge variant, $\left[\frac{d}{dt}\alpha_t(T)\right]_{t=t_T(\tau)} \neq 0$ that immediately results in $\{C, t_T(\tau)\} + 1 = 0$. So $\mathcal{O}_{f,T}$ is a Dirac observable, that is $\{C, \mathcal{O}_{f,T}\} = 0$.

Having found the observables of a system with a single constraint, we are ready to examine a system possessing several constraints C_I with $\{C_I, C_J\} = f_{IJ}^K C_K$ where I, J, K are labels of some index set $\mathfrak{I}$. It is required to choose as many clocks T_I as there are constraints. If $\alpha_{t_I}^{C_I}$ denotes the gauge flow generated by C_I, the following Dirac observable can be constructed from a gauge variant phase-space function f

$$\mathcal{O}_{f,\{T\}}(\{\tau\}) := \left[\left(\circ_{I\in\mathfrak{I}} \ \alpha_{t_I}^{C_I}\right)(f)\right]_{\alpha_{t_I=t_{T_I}(\tau_I)}} \tag{2.6}$$

where $t_{T_I}(\tau_I)$ is the solution of the equation $\alpha_{t_I}^{C_I}(T_I) = \tau_I$ for t_I. Now, if the constraints mutually commute, i.e. $\{C_I, C_J\} = 0$ for all $I, J \in \mathfrak{I}$, the actual order in applying the gauge flows $\alpha_{t_I}^{C_I}(T_I)$ on the function f is beside the point. In this case, the observable (2.6) is simply the gauge invariant extension of f giving the value of f when the clocks T_I take the values τ_I.

If the system under consideration has several non-commuting constraints C_I, the order of the gauge flows in (2.6) gains importance and thus constructing observables will be more complicated. This is what we are dealing with in GR where the algebra of the constraints, that is (1.32)-(1.35), is not Abelian. However, it can be shown that for a general system with only first-class constraints, we are able to replace non-commuting constraints C_I by a set of equivalent ones C'_I whose corresponding Hamiltonian vector fields $X_I := \{C_I, \cdot\}$ weakly mutually commute. By this method that is known as *weak Abelianisation*, any general system can be reduced back to the case of Abelian constraints we have just discussed [141, 142, 143]. To see how it is possible, consider the constraint surface $\mathcal{C} := \{p \in \mathscr{P} : C_I(p) = 0 \ \ \forall I \in \mathfrak{I}\}$ embedded in the phase space $\mathscr{P}$. The gauge flow of an arbitrary phase space function f is given by $\alpha_\beta(f) := \exp(\chi_\beta) \cdot f$ where $C_\beta := \beta^I C_I$ and $\chi_\beta := \{C_\beta, \cdot\}$. We choose as many clocks T_I as there are constraints C_I. Moving along their gauge orbits, these clocks T_I take the values τ_I when the following equation holds

$$[\alpha_\beta(T_I)](p) = T_I(\alpha_\beta(p)) = \tau_I \tag{2.7}$$

We can invert (2.7) for β and get the solution $\beta_{\{T\}}^{\{\tau\}}(p)$ depending on the phase space point p. Such a solution can be found at least locally provided that the matrix $A_{IJ} := \{C_I, T_J\}$ is locally invertible. The gauge invariant extension of f denoted by $\mathcal{O}_{f,T}^\tau$ is then defined as

$$\mathcal{O}_{f,T}^\tau(p) := [\alpha_\beta(f)](p)_{[\alpha_\beta(T_I)](p)=\tau_I} = [f(\alpha_\beta(p))]_{\beta=\beta_{\{T\}}^{\{\tau\}}(p)} \tag{2.8}$$

We introduce the gauge fixing surface $\mathcal{C}_\tau := \{p \in \mathcal{C} : T_I(p) = \tau_I\}$ on which $\beta_{\{T\}}^{\{\tau\}}(p) = 0$ and hence $\mathcal{O}_{f,T}^\tau(p) = f(p)$. Away from $\mathcal{C}_\tau$ we can expand $\mathcal{O}_{f,T}^\tau$ as a Taylor series in powers of the clock variables. We make the following Ansatz

$$\mathcal{O}_{f,T}^\tau = \sum_{\{k_I\}=0}^{\infty} \prod_{I\in\mathfrak{I}} \frac{(\tau_I - T_I)^{k_I}}{k_I!} f_{\{k_I\}_{I\in\mathfrak{I}}} \tag{2.9}$$

with $f_{\{k_I\}=\{0\}} = f$. By requiring that $\mathcal{O}_{f,T}^\tau$ has to weakly commute with the constraints, that is $\{C_I, \mathcal{O}_{f,T}^\tau\} \approx 0$, it is possible to derive a formal solution for $f_{\{k_I\}}$ given by

$$f_{\{k_I\}} = \prod_{I\in\mathfrak{I}} (X'_I)^{k_I} \cdot f \tag{2.10}$$

where $X'_I := \sum_{J\in\mathfrak{I}}(A^{-1})_{IJ}\{C_J,\cdot\}$ and $(A^{-1})_{IJ}$ is the inverse of the matrix A_{IJ}. Note that we have not specified the order of applying the vector fields in (2.10) yet. As already mentioned, this is important because in general the vector fields X'_I do not mutually commute. The weak Abelianisation method introduces a set of equivalent constraints defined by

$$C'_I := \sum_{J\in\mathfrak{I}}(A^{-1})_{IJ}C_J \tag{2.11}$$

whose corresponding Hamiltonian vector fields $X_I := \{C'_I,\cdot\}$ are weakly equivalent to the vector fields X'_I, because for an arbitrary phase space function f we have

$$\begin{aligned} X_I f &= \sum_{J\in\mathfrak{I}}\{(A^{-1})_{IJ}C_J, f\} = \sum_{J\in\mathfrak{I}}\Big((A^{-1})_{IJ}\{C_J,f\} + \{(A^{-1})_{IJ},f\}C_J\Big) \\ &\approx \sum_{J\in\mathfrak{I}}(A^{-1})_{IJ}\{C_J,f\} = X'_I f \end{aligned} \tag{2.12}$$

Furthuemore,

$$\begin{aligned} X_J X_I f =& \{\sum_{L\in\mathfrak{I}}(A^{-1})_{JL}C_L, \{\sum_{K\in\mathfrak{I}}(A^{-1})_{IK}C_K, f\}\} \\ =& \sum_{K,L\in\mathfrak{I}}\Big(C_L\{(A^{-1})_{JL},(A^{-1})_{IK}\}\{C_K,f\} + C_L\{(A^{-1})_{JL},\{C_K,f\}(A^{-1})_{IK} \\ &+ (A^{-1})_{JL}\{C_L,(A^{-1})_{IK}\}\{C_K,f\} + (A^{-1})_{JL}(A^{-1})_{IK}\{C_L,\{C_K,f\}\} \\ &+ C_L\{(A^{-1})_{JL},C_K\}\{(A^{-1})_{IK},f\} + C_L\{(A^{-1})_{JL},\{(A^{-1})_{IK},f\}\} \\ &+ (A^{-1})_{JL}\{C_L,\{(A^{-1})_{IK},f\}\}C_K + (A^{-1})_{JL}\{C_L,C_K\}\{(A^{-1})_{IK},f\}\Big) \\ \approx& \sum_{K,L\in\mathfrak{I}}\Big((A^{-1})_{JL}\{C_L,(A^{-1})_{IK}\}\{C_K,f\} + (A^{-1})_{JL}(A^{-1})_{IK}\{C_L,\{C_K,f\}\}\Big) \\ =& X'_J X'_I f \end{aligned} \tag{2.13}$$

where we took advantage of the assumption that the constraints are first-class to conclude $\{C_L,C_K\}\approx 0$. The conclusion of (2.13) can be inductively generalised for more than two vector fields, i.e. $\prod_{i=1}^n X_{I_i} \approx \prod_{i=1}^n X'_{I_i}$ for $I_i \in \mathfrak{I}$. The advantage of using X_I over X'_I is this crucial

observation that the former are mutually commuting.

$$\begin{aligned}
[X_J, X_I]f &= \sum_{K,L\in\mathfrak{I}} \Big(\{(A^{-1})_{JL}C_L, \{(A^{-1})_{IK}C_K, f\}\} - \{(A^{-1})_{IL}C_L, \{(A^{-1})_{JK}C_K, f\}\}\Big) \\
&\approx \sum_{K,L\in\mathfrak{I}} (A^{-1})_{JL}\Big(\{C_L, (A^{-1})_{IK}\}\{C_K, f\} + (A^{-1})_{IK}\{C_L, \{C_K, f\}\}\Big) \\
&\quad + \sum_{K,L\in\mathfrak{I}} (A^{-1})_{IL}\Big(\{C_L, (A^{-1})_{JK}\}\{C_K, f\} + (A^{-1})_{JK}\{C_L, \{C_K, f\}\}\Big) \\
&= 2\sum_{K,L\in\mathfrak{I}} (A^{-1})_{JL}\left((A^{-1})_{JK}\{C_{[L}, \{C_{K]}, f\}\} - \sum_{M,N}(A^{-1})_{IM}(A^{-1})_{NK}\{C_K, f\}\{C_{[L}, \{C_{M]}, T_N\}\}\right) \\
&= \sum_{K,L\in\mathfrak{I}} (A^{-1})_{JL}\left(\sum_{M,N}(A^{-1})_{IM}(A^{-1})_{NK}\{C_K, f\}\{T_N, \{C_L, C_M\}\} - (A^{-1})_{JK}\{f, \{C_L, C_K\}\}\right) \\
&= \sum_{K,L\in\mathfrak{I}} (A^{-1})_{JL}\left(-\sum_{M,N,P}(A^{-1})_{IM}(A^{-1})_{NK}\{C_K, f\}f^P_{LM}A_{PN} + \sum_P (A^{-1})_{JK}f^P_{LK}\{C_P, f\}\right) \\
&= \sum_{K,L\in\mathfrak{I}} (A^{-1})_{JL}\left(-\sum_{M,P}(A^{-1})_{IM}\{C_P, f\}f^P_{LM} + \sum_P (A^{-1})_{JK}f^P_{LK}\{C_P, f\}\right) \\
&= 0
\end{aligned} \tag{2.14}$$

Consequently, as far as weak equalities are concerned, the order of applying the vector fields in (2.10) is irrelevant and hence the gauge-invariant extension of f can be expressed as

$$\mathcal{O}^\tau_{f,T} = \sum_{\{k_I\}=0}^{\infty}\prod_{I\in\mathfrak{I}}\frac{(\tau_I - T_I)^{k_I}}{k_I!}\prod_{I\in\mathfrak{I}}(X_I)^{k_I}\cdot f \tag{2.15}$$

It is worth mentioning that although (2.15) is greatly simple, its exact expression, and even the inverse A^{-1}, will be hard to calculate especially in quantum field theory.

To summarise the above observation, constructing observables is founded on the fact that they are constant along gauge orbits. Accordingly, if we know the values of a function f in a specific gauge determined by a clock T, we can extend f off that gauge in a gauge-invariant way, yielding the function $\mathcal{O}^\tau_{f,T}$ defined in (2.15). In fact, the *observable map*

$$\mathcal{O}^\tau_T : C^\infty(\mathscr{P}) \to C^\infty(\mathscr{P}); \quad f \mapsto \mathcal{O}^\tau_{f,T} \tag{2.16}$$

sends an arbitrary function to an observable. Note that the relation between f and $\mathcal{O}^\tau_{f,T}$ depends on the gauge-fixing conditions unless f happens to be gauge invariant, in which case $f \approx \mathcal{O}^\tau_{f,T}$. A significant property of the observable map is that it can be sighted as homomorphisms for two different mathematical structures.

Ring Homomorphism [141]

The map $\mathcal{O}^\tau_T$ is a ring[1] homomorphism[2] from the ring of smooth functions $(C^\infty(\mathscr{P}), +, \cdot)$

[1] A *ring* is a set R equipped with two binary operations $+$ (addition) and $\cdot$ (multiplication) such that $(R, +)$ is an Abelian group, $(R, \cdot)$ is a semigroup and the multiplication $\cdot$ is distributive with respect to the addition $+$.

[2] A *ring homomorphism* from a ring $(R, +, \cdot)$ to a ring $(S, \ddagger, *)$ is a function $\varphi : R \to S$ preserving the ring operations, i.e. for all $r, r' \in R$

$$\varphi(r + r') = \varphi(r) \ddagger \varphi(r'), \quad \varphi(r \cdot r') = \varphi(r) * \varphi(r')$$

to the ring of observables with addition and multiplication as its binary operations, i.e, $\mathcal{O}^\tau_{f+f',T} = \mathcal{O}^\tau_{f,T} + \mathcal{O}^\tau_{f',T}$ and $\mathcal{O}^\tau_{ff',T} = \mathcal{O}^\tau_{f,T} \cdot \mathcal{O}^\tau_{f',T}$. Consequently, for an arbitrary analytic function g of the phase-space functions $f_1, \cdots, f_n$, we have

$$\mathcal{O}^\tau_{g(f_1,\cdots,f_n),T} = g(\mathcal{O}^\tau_{f_1,T}, \cdots, \mathcal{O}^\tau_{f_n,T}) \tag{2.17}$$

Symplectic Homomorphism [143]

The map $\mathcal{O}^\tau_T$ is a symplectic homomorphism[3] from the symplectic manifold of smooth functions $(C^\infty(\mathscr{P}), \{\cdot,\cdot\}^*)$ to the symplectic manifold of observables with the Poisson bracket $\{\cdot,\cdot\}$. More precisely, the Poisson bracket of two observables coincides weakly with the observable of the Dirac bracket of the two phase-space functions, i.e. for two phase-space functions f and f' it holds

$$\{\mathcal{O}^\tau_{f,T}, \mathcal{O}^\tau_{f',T}\} \approx \mathcal{O}^\tau_{\{f,f'\}^*,T} \tag{2.18}$$

The above Dirac bracket is defined by introducing the gauge conditions $\mathcal{G}_I := T_I - \tau_I$ and considering the system of second-class constraints C_μ containing $\{C_I\}_{I\in\mathfrak{I}}$ and $\{\mathcal{G}\}_{I\in\mathfrak{I}}$ as

$$\{f,f'\}^* := \{f,f'\} - \{f,C_\mu\}K^{\mu\nu}\{C_\nu,f'\} \tag{2.19}$$

where $K^{\mu\nu}$ is the inverse of $K_{\mu\nu} := \{C_\mu, C_\nu\}$, i.e. $K^{\mu\rho}K_{\rho\nu} = \delta^\mu_\nu$.

2.2.2. Time Evolution of Observables: Physical Hamiltonian

Suppose α'_τ is the flow generated by the Hamiltonian vector field $\tilde{X} := \{\sum_I \tau_I C'_I, \cdot\}$, where C'_I are the abelianised constraints (2.11). Using the multinomial theorem, we have

$$\alpha'_\tau(f) = \sum_{n=0}^\infty \frac{1}{n!}\left(\sum_I \tau_I X_I\right)^n f = \sum_{n=0}^\infty \frac{1}{n!}\sum_{\{k_I\}} \frac{n!}{\prod_I k_I!}\prod_I \tau_I^{k_I}\prod_I X_I^{k_I} f = \sum_{\{k_I\}}\prod_I \frac{\tau_I^{k_I}}{k_I!}\prod_I X_I^{k_I} f \tag{2.20}$$

where in the second equality $\sum_I k_I = n$. We are interested in seeing how a given observable evolves while the hands of the clock T_I move from $\bar{\tau}_I$ to $\bar{\tau}_I + \tau_I$.

$$\begin{aligned}
\mathcal{O}^{\bar{\tau}+\tau}_{f,T} &= \sum_{\{k_I\}=0}^\infty \prod_I \frac{(\tau_I + \bar{\tau}_I - T_I)^{k_I}}{k_I!}\prod_I (X_I)^{k_I}\cdot f \\
&\approx \sum_{\{k_I\}}\prod_I \frac{(\bar{\tau}_I - T_I)}{k_I!}\prod_I X_I^{k_I}\cdot \underbrace{\left(\sum_{\{\ell_I\}}\prod_I \frac{\tau_I^{\ell_I}}{\ell_I!}\prod_I X_I^{\ell_I}\right)\cdot f}_{\alpha'_\tau(f)} \\
&= \mathcal{O}^{\bar{\tau}}_{\alpha'_\tau(f),T}
\end{aligned} \tag{2.21}$$

Thus, it is a gauge transformation on f that induces the time evolution on the observables [143]. In other words, there is a map α^τ describing the physical evolution on the space of observables by $\alpha^\tau(\mathcal{O}^{\bar{\tau}}_{f,T}) = \mathcal{O}^{\bar{\tau}+\tau}_{f,T}$. Using (2.18), one can easily see that $\{\alpha^\tau(\mathcal{O}^{\bar{\tau}}_{f,T}), \alpha^\tau(\mathcal{O}^{\bar{\tau}}_{f',T})\} \approx$

[3] A *symplectic homomorphism* from a symplectic manifold $(P, \{\cdot,\cdot\}^*)$ to a ring $(Q, \{\cdot,\cdot\})$ is a function $\varphi : P \to Q$ preserving the Poisson bracket, i.e. for all $f, f' \in P$

$$\varphi(\{f,f'\}^*) = \{\varphi(f), \varphi(f')\}$$

$\alpha^\tau(\{\mathcal{O}^{\bar{\tau}}_{f,T}, \mathcal{O}^{\bar{\tau}}_{f',T}\})$. Here, τ is called the *multi-fingered time* and the evolution with respect to it is known as *multi-fingered time evolution* [143]. Now, we choose the value τ_0 of the clock T_0 as *physical time* and if there exists a function h_{phys} generating evolution with respect to the physical time on the space of observables, we call it the *physical Hamiltonian*, i.e,

$$\frac{d}{d\tau_0}\mathcal{O}^\tau_{f,T} \approx \{h_{\text{phys}}, \mathcal{O}^\tau_{f,T}\} \tag{2.22}$$

Implementation of the multi-fingered time evolution can be done by complying with the gauge-fixing formalism steps:

- Take as many clocks T^I as there are constraints C_I and consider gauge fixing conditions $\mathcal{G}^I = T^I - \tau^I$, where τ^I is assumed to be phase-space independent.
- Solve the constraints for the momenta P_I conjugate to T^I.
- Consider the equivalent constraints

  $$\tilde{C}_I = P_I + h_I(q^a, p_a, T^J) \tag{2.23}$$

  where (q^a, p_a) are the "true" degrees of freedom. The constraints $\tilde{C}_I$ are strongly Abelian. To see this, first note that the first-class property of the original constraints tells us that $\tilde{C}_I$ also must satisfy this property for some new structure functions, i.e., $\{\tilde{C}_J, \tilde{C}_K\} = \tilde{f}_{JK}{}^I\, \tilde{C}_I$. Because the left-hand side is independent of P_I while the right-hand side is not, one concluded that $\tilde{f}_{JK}{}^I = 0$.
- Fix some of the Lagrange multipliers so that the gauge fixing conditions are conserved under the action of the canonical Hamiltonian $\mathbf{H}_{\text{can}} = \lambda^I C_I$, i.e., one has to solve the equations

  $$0 = \frac{d}{dt}\mathcal{G}^J = \frac{\partial}{\partial t}\mathcal{G}^J + \{\mathbf{H}_{\text{can}}, \mathcal{G}^J\} = \frac{\partial}{\partial t}\mathcal{G}^J + \lambda^I\{C_I, \mathcal{G}^J\} =: \frac{\partial}{\partial t}\mathcal{G}^J + \lambda^I A^J_I \tag{2.24}$$

  for λ^I. If A_{IJ} is invertible, the fixed Lagrange multipliers are $\lambda^I_0 = -\frac{\partial \mathcal{G}^J}{\partial t}(A^{-1})^I_J$.
- Try to find the Physical Hamiltonian H_{phys} satisfying

  $$\{H_{\text{phys}}, f(q,p)\} = \{\mathbf{H}_{\text{can}}, f(q,p)\}_{C=\mathcal{G}=\lambda-\lambda_0=0} \tag{2.25}$$

  for any function f depending on the true degrees of freedom. Note that (2.25) must be fulfilled because, when the constraints and the gauge fixing conditions are satisfied and the Lagrange multipliers assumed to be their fixed values, H_{phys} is supposed to generate the same equations of motion for (q,p) as the canonical Hamiltonian does.

The question arises whether different gauge-fixing conditions can give rise to inequivalent quantum theories from the same classical theory. To answer it first note that concerning the relation between gauge-fixing and gauge-invariant formalism, there is a one-to-one correspondence between a choice of gauge fixing and a preferred set of gauge invariant functions which generate the full algebra of gauge-invariant functions [146]. The two formalisms are therefore equivalent at generic points of the reduced phase space at which the Dirac matrix (which is a non-trivial function on phase space in every interacting theory) is non-singular. In the same sense, different gauge fixing conditions are generically (i.e., locally in phase space) equivalent. As usual, global differences may have an effect on the quantisation in different gauge choices. However, our attitude is that in quantum gravity global non-equivalence of gauge fixed theories is a second-order concern, one would be happy to have at least one working quantisation at one's proposal to start with which then can be further improved.

2.2.2.1. Deparametrisation

The construction of the observables and derivation of the physical Hamiltonian will be simplified if the constraints can be expressed in the following *deparametrised* form. In this special case, we can always find canonical coordinates consisting of two sets (T^I, P_I) and (q^a, p_a) such that all constraints C_I can be expressed as

$$C_I = P_I + h_I(q^a, p_a) \tag{2.26}$$

In practice, in most constrained systems only part of the constraints, if at all, can be deparametrised. However, in this section, we consider a fully deparametrised system. As in this case $A^J_I = \{C_I, T^J\} = \delta^J_I$, the task of inverting the complicated matrix A^I_J appearing in (2.8) and (2.24) for construction of observables and derivation of the physical Hamiltonian, respectively, is not a challenge anymore. Comparing (2.23) and (2.26), one finds that the virtue of *deparametrisation* is that the constraints do not depend on T^I. Putting this property next to the above-proven fact that if all constraints are linear in the momenta P_I, then the constraint algebra is Abelian (see the argument after (2.23)), we find that

$$0 = \{C_I, C_J\} = \{P_I + h_I(q^a, p_a), P_J + h_J(q^a, p_a)\} = \{h_I, h_J\} \tag{2.27}$$

from which one immediately observes that h_I are already Dirac observables, as they commute with all the constraints

$$\{C_I, h_J\} = \{P_I + h_I(q^a, p_a), h_J(q^a, p_a)\} = \{h_I, h_J\} = 0 \tag{2.28}$$

Following the steps presented in (2.2.1), we can construct an observable from any function f depending only on (q^a, p_a). Note that

1. Since q^a and p_a both commute with all momenta P_I, the action of the Hamiltonian vector field associated with C_I, that is $X'_I = \{C_I, \cdot\}$, on $f(q^a, p_a)$ equals to the action of the Hamiltonian vector field of h_I, i.e. $X_I = \{h_I, \cdot\}$, on it. Thus, $X'_I \cdot f = X_I \cdot f$.

2. Since q^a and p_a both commute with all reference fields T^I, when the action of $X_\beta = \{\beta^I C_I, \cdot\}$ on $f(q^a, p_a)$ is concerned, one can replace β^I by the corresponding gauge $T^I - \tau^I$ and work with $X_\tau := \{(T^I - \tau^I)h_I, \cdot\}$.

Hence, the observable corresponding to the function $f(q^a, p_a)$ is

$$\mathcal{O}_f(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n_\tau \cdot f \tag{2.29}$$

The observables associated with the elementary variables q^a and p_a are denoted by $Q^a := \mathcal{O}_{q^a}(\tau)$ and $P_a = \mathcal{O}_{p_a}(\tau)$, respectively. Note that since h_I are already observables $\mathcal{O}_{h_I(q^a,p_a)}(\tau) = h_I(Q^a, P_a) = h_I$. Now, we are seeking a function driving the evolution of the observables (2.29). Recall that the dynamics cannot be generated by the canonical Hamiltonian since $\mathcal{O}_f(\tau)$ Poisson commutes with all the constraints. Because $\mathcal{O}_f(\tau)$ gives us the value of f when the clocks T^I take the values τ^I, one of the chosen clocks, say T^0, can be associated with the physical time denoted by τ^0. How $\mathcal{O}_f(\tau)$ changes with respect to τ^0 is then interpreted as its time evolution. Considering the constraint $C_0 := P_0 + h_0$ associated with the reference field T^0, one immediately observes that $H_{\text{phys}} = h_0$ [143], i.e.

$$\frac{\partial}{\partial \tau^0} \mathcal{O}_{f,T}(\tau) = \{h_0, \mathcal{O}_{f,T}(\tau)\} \tag{2.30}$$

Due to the complicated form of the constraints in General Relativity with/without standard matter, its reduced phase space is hard to construct explicitly. However, in section (2.3) we will see that when general relativity is coupled to very special matter fields, it falls into the class of deparameterizable constrained systems and one can employ the findings of this section to build the reduced phase space. For more information, we refer the reader to [147] where a general discussion about matter fields leading to deparameterised constrained systems is provided.

2.2.3. Poisson Algebra of the Observables and its Quantisation

The next step is to quantise the classical algebra of the constructed observables completely determined in (2.18). It is assumed that we choose a canonical coordinate system consisting of canonical pairs (q^a, p_a) and (T_I, P^I) so that the only non-vanishing brackets are $\{p_a, q^b\} = \delta_a^b$, $\{P^J, T_I\} = \delta_I^J$. Now, if we restrict to functions depending only on (q^a, p_a) the Dirac bracket (2.19) reduces to the Poisson bracket because according to the above assumption on the canonical pairs $f(q^a, p_a)$ commute with all reference fields T^I. In particular, for the algebra of the observables of elementary variables $Q^a := \mathcal{O}_{q^a,T}(\tau)$ and $P_a := \mathcal{O}_{p_a,T}(\tau)$, we have

$$\{P_a(\tau), Q^b(\tau)\} = \{\mathcal{O}_{p_a,T}(\tau), \mathcal{O}_{q^b,T}(\tau)\} = \mathcal{O}_{\{p_a,q^b\}^*,T}(\tau) = \mathcal{O}_{\{p_a,q^b\},T}(\tau) = \mathcal{O}_{\delta_a^b,T}(\tau) = \delta_a^b \quad (2.31)$$

where we have used (2.18). As is transparent from (2.31), the symplectic structure of the reduced phase space in terms of the coordinates (Q^a, P_a) is very simple and manageable for quantisation. This a direct consequence of the assumption on the chosen canonical variables, because if $\{p_a, q^b\}^* \neq \{p_a, q^b\}$ the calculations (2.31) would stop after the second equality and we would only have $\{P_a(\tau), Q^b(\tau)\} = \mathcal{O}_{\{p_a,q^b\}^*,T}(\tau)$. Hence, due to the presence of Dirac bracket on the right-hand side, the algebra among the Q^a, P_a would be too complicated to have hope for its quantisation.

Let $\mathcal{D}$ be the Poisson algebra generated by Q^a, P_a. Now, we should seek a representation $\pi : \mathcal{D} \to L(\mathcal{H})$ where $L(\mathcal{H})$ denotes the linear operators defined on a Hilbert space $\mathcal{H}$ such that $[\pi(P_a), \pi(Q^b)] = i\hbar\delta_a^b$. It seems that the reduced phase space quantisation is a trivial task since there are no constraints to be implemented anymore and therefore one can choose any of the standard kinematical representations. However, the reduced phase space quantisation is not that simple because the crucial remaining question is whether the chosen representation allows us to define the quantised version of the physical Hamiltonian. As the physical Hamiltonian usually have a complicated form, finding a representation supporting its quantum counterpart is difficult.

2.3. Reduced Phase Space Quantisation of Loop Quantum Gravity

Although it is difficult to construct the reduced phase space of GR explicitly, it is shown in [147] that adding a scalar field to the theory provides suitable circumstances for building the algebra of classical observables. In [146, 148], the pressure-free dust of Brown and Kuchar [149] was employed as a specific scalar field to improve further the framework introduced in [147] and to define a physical Hamiltonian generating physical time evolution. While the obtained physical Hamiltonian density is rather complicated, the authors of [63] could find a representation of the algebra of observables that supports the quantised version of the physical Hamiltonian. In these works, one considers the dust as a material "clock" coupled dynamically as fields rather than test observers. One can examine various matter fields in this regard. In [150], different reference fields were compared and it turned out that the physical Hamiltonian corresponding to the Gaussian dust [151] has a very simple expression and the physical Hilbert space is accessible [150]. On the

other hand, it is not always the case that one has to define all the clocks as additional matter fields. Suppose the geometric constraints are simple enough to solve for some momenta. In that case, we can choose all the clocks from the geometric degrees of freedom without introducing an additional matter field. This is exactly what we will do in the case of $U(1)^3$ model in chapter 8. Another option is to define the clocks as a combination of geometric and matter degrees of freedom. This is exemplified in the work undertaken in [152, 153, 154] in which matter and inflaton perturbations are used as clocks to construct observables in cosmology on the non-linear level. Therefore, it is obvious that the choice of clocks in relational formalism is entirely arbitrary. However, this choice has a great influence on the form of the observable algebra and also on the form of the physical Hamiltonian. Since our ultimate goal is the quantisation of the reduced theory, we wish to choose the reference fields leading to a simple observable algebra and also to a physical Hamiltonian which can be promoted to an operator in the physical Hilbert space. In this section, we will present the principal findings of [146, 148, 63] as an excellent example of the reduced phase space quantisation.

2.3.1. Brown-Kuchar Model: Classical Part

The main idea of the Brown-Kuchar Model [149] is to consider additional matter dust fields as a reference system for GR. In principle, we require one reference field for each constraint that is going to be reduced at the classical level. The dust action that has to be added to the Einstein-Hilbert action is

$$S_{\text{dust}} = -\frac{1}{2}\int_M d^4x\, \sqrt{|\det(g)|}\rho(g^{\mu\nu}U_\mu U_\nu + 1) \tag{2.32}$$

where g is the metric of the spacetime M, ρ is the dust energy density and $U_\mu = -T_{,\mu} + W_j S^j_{,\mu}$ is the dust four velocity that in turn consists of the scalar fields T, S^j, W_j where $j \in \{1,2,3\}$. Therefore, the action (2.32) depends on the tensor field g and also the scalar field ρ, T, S^j, W_j. Looking at the Euler-Lagrange equations for the scalar fields, one concludes that $U_\mu = g^{\mu\nu}U_\nu$ is a geodesic congruence with proper time T that the fields W_j, S^j are constant along. The canonical analysis of this system reveals that second-class constraints arise. Thus, one is supposed to define the appropriate Dirac bracket and solve the second-class constraints. It turns out that the scalar fields ρ and W_j are not independent phase space variables; instead, they can be expressed in terms of the other degrees of freedom. Writing all geometric quantities in terms of Ashtekar variables (A, E), we see that for the variables T, S^j, A and their associated conjugate momenta P, P_j, E the Dirac bracket reduces to the Poisson bracket. A natural choice for the reference fields associated with time and space are T and S^j, respectively. This choice justifies since the former is the proper time and the latter are constant along the congruence of geodesics. In other words, the field T serves as a clock to construct observables with respect to the Hamiltonian constraint and the three fields S^j serve as rulers (in the language of the previous section clocks) for the diffeomorphism constraints. As mentioned before, these are not the only choices for the clocks but, as we will see, they are appropriate choices to complete the quantisation program. The first-class constraints of this system consisting of gravity plus dust are of the following form

$$C^{\text{tot}} = C + C^{\text{dust}}, \quad C^{\text{dust}} = -\sqrt{P^2 + q^{ab}(PT_{,a} + P_i S^i_{,a})(PT_{,b} + P_j S^j_{,b})} \tag{2.33}$$

$$C^{\text{tot}}_a = C_a + C^{\text{dust}}_a, \quad C^{\text{dust}}_a = PT_{,a} + P_j S^j_a \tag{2.34}$$

where C, C_a denote the gravitational parts of the Hamiltonian and diffeomorphism constraints and the Gauß constraint remains unchanged. Two essential properties of the Brown-Kuchar model are that

1. One can solve the constraints $C^{\rm tot}$ and $C^{\rm tot}_a$ for P and P_j respectively,
2. Exactly the same combination of P and T that makes up $C^{\rm dust}_a$ appears in $C^{\rm tot}$.

Consequently, the equivalent abelianised constraints are

$$\tilde{C}^{\rm tot} = P + h, \quad h(A,E) := \sqrt{C^2 - q^{ab}C_aC_b} \tag{2.35}$$

$$\tilde{C}^{\rm tot}_j = P_j + h_j, \quad h_j(T,S^j,A,E) := S^a_j(C_a - hT_{,a}) \tag{2.36}$$

where in (2.35) we have used the constraint $C^{\rm dust}_a = -C_a$ together with the second property listed above and to solve the diffeomorphism constraint as (2.36) we assumed that the inverse of $S^j_{,a}$ exists and is denoted by S^a_j. Invertibility assumption on $S^j_{,a}$ says implicitly that $S^j : \Sigma \to \mathcal{S}$ defines a diffeomorphism, where $\mathcal{S} := \{S^j(x);\ \ x \in \Sigma\}$ is the *dust space*. Note that the function h in (2.35) is independent of the dust degrees of freedom, hence according to the section 2.2.2.1 the abelianised constraint $\tilde{C}^{\rm tot}$ is deparametrised. In contrast, this is not the case for $\tilde{C}^{\rm tot}_j$ because h_j still depends on T, S^j. Deparametrisation of the Hamiltonian constraint not only technically simplifies the construction of observables but also ensures that the physical Hamiltonian will be independent of time.

Following the general strategy discussed in section 2.2.1 and introducing the fields $T^0 := T$ and $T^j := S^j$ as the clocks, we can construct observables $\mathcal{O}^\tau_{f,T} = [\alpha_\beta(f)]_{\alpha_\beta(T)=\tau}$, where α_β is the flow corresponding to the Hamiltonian vector field of $C_\beta = \int_\Sigma d^3x\ (\beta^0\tilde{C}^{\rm tot} + \beta^j\tilde{C}^{\rm tot}_j)$. Since S^j Poisson commutes with $\tilde{C}^{\rm tot}$, we can first construct spatial diffeomorphism invariant functions and then reduce them with respect to the Hamiltonian constraint, i.e. we can write $\mathcal{O}^\tau_{f,T} = [\alpha_{\beta^0}([\alpha_{\vec{\beta}}(f)]_{\alpha_{\vec{\beta}}(S^j)=\sigma^j})]_{\alpha_{\beta^0}(T)=\tau}$.

We start constructing observables first by reducing with respect to the diffeomorphism constraint. The diffeomorphism-invariant functions corresponding to the main canonical variables A, E, T, giving the value of the fields while the reference fields S^j take the values σ^j, can be derived as [146]

$$[\alpha_{\vec{\beta}}(A^I_a)]_{\alpha_{\vec{\beta}}(\vec{S})=\vec{\sigma}} = [A^I_a(x)]_{\vec{S}(x)=\vec{\sigma}} = A^I_a S^a_i =: A^I_i(\sigma) \tag{2.37}$$

$$[\alpha_{\vec{\beta}}(E^a_I)]_{\alpha_{\vec{\beta}}(\vec{S})=\vec{\sigma}} = [E^a_I(x)]_{\vec{S}(x)=\vec{\sigma}} = \frac{1}{J}E^a_I S^i_{,a} =: E^i_I(\sigma) \tag{2.38}$$

$$[\alpha_{\vec{\beta}}(T)]_{\alpha_{\vec{\beta}}(\vec{S})=\vec{\sigma}} = [T(x)]_{\vec{S}(x)=\vec{\sigma}} =: T(\sigma) \tag{2.39}$$

respectively, where $J := |\det(S^j_{,a})|$ and I is used for the $\mathfrak{su}(2)$ index. As the clocks used here are chosen to be S^j, the abstract points $x \in \Sigma$ are labelled by the dust fields S^j. Although in [149] analogous diffeomorphism-invariant functions for the ADM variables have been constructed, instead of reducing with respect to $C^{\rm tot}$ a "formal" Dirac quantisation was performed in the sense that no representation has been found for GR written in terms of ADM variables in which the constraints can be implemented. Inserting (2.37) and (2.38) into the standard formula for observables using the clock T, we explicitly obtain [63]

$$\mathcal{O}_{A^I_i,T}(\vec{\sigma},\tau) = \sum_{n=0}^\infty \frac{1}{n!}\{\mathtt{h}(\tau), A^I_i(\vec{\sigma})\}_{(n)} =: \mathtt{A}^I_i(\vec{\sigma},\tau) \tag{2.40}$$

$$\mathcal{O}_{E^i_I,T}(\vec{\sigma},\tau) = \sum_{n=0}^\infty \frac{1}{n!}\{\mathtt{h}(\tau), E^i_I(\vec{\sigma})\}_{(n)} =: \mathtt{E}^i_I(\vec{\sigma},\tau) \tag{2.41}$$

with

$$\mathtt{h}(\tau) := \int_{\mathcal{S}} d^3\sigma\ (\tau - T(\sigma))h(\sigma) \tag{2.42}$$

One can explicitly verify that $\mathtt{A}^I_i(\vec{\sigma},\tau)$ and $\mathtt{E}^i_I(\vec{\sigma},\tau)$ indeed Poisson commute with $C^{\rm tot}(\vec{\sigma})$ and $C^{\rm tot}_j(\vec{\sigma})$. The quantities (2.40) and (2.41) are interpreted in the way that they give the values of A^I_a and E^a_I respectively when the reference fields T and S^j takes the values τ and σ^j, respectively[4].

The remaining Gauß constraint will be solved in the quantum theory through Dirac quantisation. Now, we are interested in finding the physical Hamiltonian generating the dynamics of the observables constructed above. It turns out that the observable constructed out of the function h in (2.35) is the Hamiltonian density generating the physical dynamics. We can derive this observable by simply substituting A and E with their associated observables, according to the homomorphism property (2.17) of the observable map. Then, the result is[5]

$$h_{\rm phys}(\sigma) = \sqrt{\mathtt{C}(\sigma)^2 - \mathtt{q}^{ij}(\sigma)\mathtt{C}_i(\sigma)\mathtt{C}_j(\sigma)} \tag{2.45}$$

where $\mathtt{C}(\sigma) := C/J$, $\mathtt{C}_j(\sigma) := C_a S^a_j/J$, $\mathtt{q}^{ij} = \delta^{IJ}\mathtt{E}^i_I\mathtt{E}^j_J/\det(\mathtt{q})$ and the physical Hamiltonian is then given by

$$H_{\rm phys} = \int_{\mathcal{S}} d^3\sigma\ h_{\rm phys} \tag{2.46}$$

from which the equation of motion for the observables follows

$$\frac{d}{d\tau}\mathcal{O}_{f,T}(\sigma,\tau) = \{H_{\rm phys}, \mathcal{O}_{f,T}(\sigma,\tau)\} \tag{2.47}$$

As all constituents of (2.47) are manifestly gauge invariant, it can be thought of as the gauge-invariant counterpart of Einstein's equations. It is also worth emphasising that the dynamics defined here is free of the Hamiltonian and diffeomorphism constraints, in contrast to what we had while working with the canonical Hamiltonian.

Furthermore, the group Diff($\mathcal{S}$) is a symmetry of $H_{\rm phys}$, because a density of weight one is integrated over $\mathcal{S}$. Note that since $\mathcal{S}$ is a label space for geodesics, not a coordinate manifold, Diff($\mathcal{S}$) is the group of "active" diffeomorphisms [63].

2.3.2. Brown-Kuchar Model: Quantum Part

After constructing observables, one is to quantise their algebra. The algebra of the elementary observables, following (2.18), is of the form

$$\{\mathcal{O}_{A,T}, \mathcal{O}_{E,T}\} \approx \mathcal{O}_{\{A,E\}^*,T} \tag{2.48}$$

[4] τ and σ^j can be understood as the physical time and space parameter, respectively.

[5] If one employs Gaußian dust [150] as reference fields, instead of Brown-Kuchar dust, the physical Hamiltonian turns out to be

$$h_{\rm phys} = \mathtt{C} \tag{2.43}$$

and using four Klein-Gordon fields [155] as reference fields, leads to the following physical Hamiltonian

$$h_{\rm phys} = \sqrt{-2\sqrt{\mathtt{q}}\,\mathtt{C} + 2\sqrt{q}\sum_i \sqrt{\mathtt{q}^{ii}\mathtt{C}_i\mathtt{C}_i}} \tag{2.44}$$

that, in general, can be complicated because of the Dirac bracket appearing on the right hand side. However, a significant property of parametrised models is that the algebra of the elementary observables is isomorphic to the kinematical one, i.e.

$$\{\mathtt{A}_i^I(\sigma), \mathtt{E}_J^j(\sigma')\} = \delta_i^j \delta_J^I \delta(\sigma, \sigma') \tag{2.49}$$

Note that in the unconstrained phase space the canonical conjugate pairs are (A_a^I, E_I^a), (T, P) and (S^j, P_j), where (A_a^I, E_I^a) are for the gravitational field and the others for the dust field. On the constraint surface again we have (A_a^I, E_I^a) among our degrees of freedom but from dust degrees of freedom, we should count only T, S^j, because we get rid of P, P_j by solving the constraints for them. When it comes to the reduced phase space, we can get rid of another four degrees of freedom which are gauge freedoms by fixing the values of T and S^j. Hence, we get $(\mathtt{A}_i^I, \mathtt{E}_I^i)$ parametrised by τ and σ as our degrees of freedom. So, the reduced phase space is just coordinated by the gravitational field satisfying the same Poisson bracket as before (see (2.49)). One can easily see that it is important to couple "four" scalar fields to gravity, one corresponding to time and the others to space so that at the end we are only left with the gravitational fields on the reduced phase space and the dust fields become parameters to parametrise the gravitational fields. Note that on the reduced phase space there is no constraint anymore and its structure is the same as the phase space of unconstrained pure gravity.

Since there is no constraint left, the quantisation of the reduced phase space seems to be a trivial task. Looking at the algebra, one might even think that a Fock quantisation would be viable. However, this is not the case because only those representations in which the physical Hamiltonian H_{phys} can be promoted to a well-defined operator are desirable. On the other hand, the expression of H_{phys} involves the gravitational contribution of the constraints, hence Fock quantisation is excluded.

The kinematical Hilbert space in LQG, i.e., $\mathcal{H}_{\text{phys}} = L_2(\bar{\mathcal{A}}, d\mu_{AL})$, when it is restricted to its gauge-invariant subspace, would be a possible representation. Note that here the Hilbert space $\mathcal{H}_{\text{phys}}$ is physical since we are quantising the reduced phase space. Exactly similar to what we did in the previous chapter, we introduce a graph γ in the dust space $\mathcal{S}$. Then, the connection $\mathtt{A}_i^I$ is discretised on the graph and gives us the holonomy $h_e(\mathtt{A})$ like before and when we quantise we get the holonomy operator $\hat{h}_e(\mathtt{A})$.

Now, the next task is to quantise the physical Hamiltonian H_{phys}. First, the classical expression needs to be regularised: we partitionise the spatial dust manifold $\mathcal{S}$ by means of a triangulation $\mathfrak{T}$ consisting of 3-dimensional cells Δ. Hence, H_{phys} can be written as

$$H_{\text{phys}} = \sum_{\Delta} \int_{\Delta} d^3\sigma \sqrt{|\mathtt{C}^2 - \mathtt{q}^{ij}\mathtt{C}_i\mathtt{C}_j|} \tag{2.50}$$

If we denote the volume of the cells by $V(\Delta)$ and a point inside Δ by σ_Δ denote, then using a Riemann sum we can write H_{phys} in the continuum limit $\mathfrak{T} \to \mathcal{S}$ as

$$H_{\text{phys}} = \lim_{\mathfrak{T}\to\mathcal{S}} \sum_{\Delta} V(\Delta) \sqrt{|\mathtt{C}^2 - \mathtt{q}^{ij}\mathtt{C}_i\mathtt{C}_j|}(\sigma_\Delta) \tag{2.51}$$

Now, we should reformulate the expression under the square root so that it can be manageable to quantise. Here we want to use the techniques employed in [55] successfully to promote the Hamiltonian constraint to a well-defined operator (See also section 4.2). Here we quantise only the Euclidean part of $\mathtt{C}$ since afterwards quantisation of the remaining part would be straightforward. Let us work with magnetic field given by $\mathtt{B}_I^i := \frac{1}{2}\epsilon_{ijk}\mathtt{F}_{jk}^I$ and its contraction with a co-triad, where

$\mathtt{F}^I_{jk}$ is the curvature associated to the connection $\mathtt{A}^I_i$. Using $\mathrm{Tr}(\tau_J\tau_K) = -\frac{1}{2}\delta_{JK}$, we get

$$\begin{aligned}\mathrm{Tr}(\mathtt{B}) &:= \mathrm{Tr}\left(\sqrt{\det(\mathtt{q})}\,\mathtt{B}^i_I\tau_I\mathtt{E}^J_i\tau_J\right) = \sqrt{\det(\mathtt{q})}\,\mathtt{B}^i_I\mathtt{E}^J_i\mathrm{Tr}(\tau_I\tau_J) = \sqrt{\det(\mathtt{q})}\,\mathtt{B}^i_I\mathtt{E}^J_i(-2\delta_{IJ})\\ &= -2\sqrt{\det(\mathtt{q})}\,\mathtt{B}^i_I\mathtt{E}^I_i = -\sqrt{\det(\mathtt{q})}\,\epsilon^{ijk}\mathtt{F}^I_{jk}\mathtt{E}^I_i = -\sqrt{\det(\mathtt{q})}\,\mathtt{F}^I_{jk}\frac{\epsilon^{IJK}\mathtt{E}^j_J\mathtt{E}^k_K}{\det(\mathtt{E})}\\ &= -\mathrm{sgn}(\det(\mathtt{E}))\,\frac{\mathtt{F}^I_{jk}\epsilon^{IJK}\mathtt{E}^j_J\mathtt{E}^k_K}{\sqrt{\det(\mathtt{q})}} = -\mathrm{sgn}(\det(\mathtt{E}))\,\mathtt{C}\end{aligned} \tag{2.52}$$

Consequently, we have $\mathrm{Tr}(\mathtt{B})^2 = \mathtt{C}^2$ that is the first term under the square root in (2.51).

To find a similar expression for the second term under the square root appearing in the physical Hamiltonian (2.51), we have

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}(\mathtt{B}\tau_K) &= \frac{1}{2}\mathrm{Tr}\left(\sqrt{\det(\mathtt{q})}\,\mathtt{B}^i_I\tau_I\mathtt{E}^J_i\tau_J\tau_K\right) = \frac{1}{2}\sqrt{\det(\mathtt{q})}\,\mathtt{B}^i_I\mathtt{E}^J_i\mathrm{Tr}\left(\tau_I\tau_J\tau_K\right)\\ &= \frac{1}{2}\sqrt{\det(\mathtt{q})}\,\mathtt{B}^i_I\mathtt{E}^J_i\,(-2\epsilon_{IJK}) = -\frac{1}{2}\sqrt{\det(\mathtt{q})}\,\epsilon_{IJK}\epsilon^{ijk}\mathtt{F}^I_{jk}\mathtt{E}^J_i\\ &= -\frac{1}{2}\sqrt{\det(\mathtt{q})}\,\epsilon_{IJK}\mathtt{F}^I_{jk}\frac{\epsilon^{JLM}\mathtt{E}^j_L\mathtt{E}^k_M}{\det(\mathtt{E})} = -\frac{1}{2}\sqrt{\det(\mathtt{q})}\,\left(\delta^M_I\delta^L_K - \delta^L_I\delta^M_K\right)\frac{\mathtt{F}^I_{jk}\mathtt{E}^j_L\mathtt{E}^k_M}{\det(\mathtt{E})}\\ &= -\mathrm{sgn}(\det(\mathtt{E}))\,\frac{\mathtt{F}^I_{jk}\mathtt{E}^k_I\mathtt{E}^j_K}{\sqrt{\det(\mathtt{q})}} = -\mathrm{sgn}(\det(\mathtt{E}))\,\frac{\mathtt{C}_j\mathtt{E}^j_K}{\sqrt{\det(\mathtt{q})}}\\ &=: -\mathrm{sgn}(\det(\mathtt{E}))\,\mathtt{C}_K\end{aligned} \tag{2.53}$$

that immediately leads to

$$\delta^{JK}\mathtt{C}_J\mathtt{C}_K = \delta^{JK}\frac{\mathtt{C}_j\mathtt{E}^j_J\mathtt{C}_k\mathtt{E}^k_K}{\det(\mathtt{q})} = \mathtt{q}^{jk}\mathtt{C}_j\mathtt{C}_k \tag{2.54}$$

As a consequence, by defining $\mathtt{C}_\mu(\Delta) := \int_\Delta d^3\sigma\ \mathtt{C}_\mu(\sigma)$ in which $\mu = 0, \cdots, 3$ and $\mathtt{C}_0 := \mathtt{C}$, in the refinement limit, one can re-express the physical Hamiltonian as

$$H_{\mathrm{phys}} = \lim_{\mathfrak{T}\to\mathcal{S}}\sum_\Delta\sqrt{|\mathtt{C}^2(\Delta) - \delta^{IJ}\mathtt{C}_I(\Delta)\mathtt{C}_J(\Delta)|} \tag{2.55}$$

Hence, if we can quantise $C_\mu(\Delta)$, then the quantum physical Hamiltonian will define as

$$\hat{H}_{\mathrm{phys}} = \lim_{\mathfrak{T}\to\mathcal{S}}\sum_\Delta\sqrt{|\hat{\mathtt{C}}^\dagger(\Delta)\hat{\mathtt{C}}(\Delta) - \delta^{IJ}\hat{\mathtt{C}}^\dagger_I\hat{\mathtt{C}}_J(\Delta)|} \tag{2.56}$$

provided that the limit exists. To find the quantum correspondence of $\mathtt{C}_\mu(\Delta)$ first note that

$$\begin{aligned}\mathtt{C}_\mu(\Delta) &:= \int_\Delta d^3\sigma\ \mathtt{C}_\mu(\sigma) = \frac{-\mathrm{sgn}(\det(\mathtt{E}))}{2}\int_\Delta d^3\sigma\ \mathrm{Tr}(\mathtt{B}\bar{\tau}_\mu)\\ &= -\,\mathrm{sgn}(\det(\mathtt{E}))\int_\Delta d^3\sigma\ \epsilon^{ijk}\mathrm{Tr}\left(\mathtt{F}_{jk}\left(\frac{\sqrt{\det(\mathtt{q})}}{2}\,\mathtt{E}_i\right)\bar{\tau}_\mu\right)\\ &= -\,\mathrm{sgn}(\det(\mathtt{E}))\int_\Delta d^3\sigma\ \epsilon^{ijk}\mathrm{Tr}\left(\mathtt{F}_{jk}\{A_i, \mathtt{V}(\Delta)\}\bar{\tau}_\mu\right)\end{aligned} \tag{2.57}$$

where $\mathtt{V}(\Delta) := \int_\Delta d^3\sigma\ \sqrt{\det(\mathtt{q})}$ is the physical volume of Δ and in the last step we have used the following relation

$$\begin{aligned}
\{\mathtt{A}_l^L(\sigma), \mathtt{V}(\Delta)\} &= \int d^3\sigma'\ \{\mathtt{A}_l^L(\sigma), \sqrt{|\det(\mathtt{E})|}(\sigma')\} \\
&= \int d^3\sigma'\ \frac{1}{2\sqrt{|\det(\mathtt{E})|}}\{\mathtt{A}_l^L(\sigma), |\det(\mathtt{E})|(\sigma')\} \\
&= \int d^3\sigma'\ \frac{\mathrm{sgn}(\det(\mathtt{E}))}{4\sqrt{|\det(\mathtt{E})|}}\epsilon^{IJK}\mathtt{E}^i_I\mathtt{E}^j_J\{\mathtt{A}_l^L(\sigma), \mathtt{E}^k_K(\sigma')\}\epsilon_{ijk} \\
&= \int d^3\sigma'\ \frac{\mathrm{sgn}(\det(\mathtt{E}))}{4\sqrt{|\det(\mathtt{E})|}}\epsilon^{IJK}\mathtt{E}^i_I\mathtt{E}^j_J\epsilon_{ijk}\delta^L_K\delta^k_l\delta(\sigma,\sigma') \\
&= \frac{\mathrm{sgn}(\det(\mathtt{E}))}{4\sqrt{|\det(\mathtt{E})|}}\epsilon^{IJL}\mathtt{E}^i_I\mathtt{E}^j_J\epsilon_{ijl} \\
&= \frac{1}{2}\sqrt{|\det(\mathtt{E})|}\ \mathtt{E}^L_l(\sigma) \\
&= \frac{1}{2}\sqrt{\det(\mathtt{q})}\ \mathtt{E}^L_l(\sigma)
\end{aligned} \tag{2.58}$$

The sign of $\det(\mathtt{E})$ involved in (2.57) is not of any importance because it will be cancelled in (2.55). Since the physical volume $\mathtt{V}(\Delta)$ can be quantised on the LQG Hilbert space and the Poisson bracket is replaced by the commutator divided by $i\hbar$, hence only quantisation of the connection $\mathtt{A}$ and its curvature $\mathtt{F}$ remain on the agenda. But, as discussed in the previous chapter, $\mathtt{A}, \mathtt{F}$ cannot be promoted to well-defined operators (see (1.63) and its following explanation) and one only can approximate them by holonomies along paths and loops respectively. This in turn causes a lot of ambiguities because these approximations can be made in infinitely many ways and while classically all of them yield to the same continuum limit, in the quantum theory they lead to different regularised operators [55]. This will be discussed in more detail in chapter 4. As previously mentioned, the classical physical Hamiltonian (2.51) enjoys from the symmetry of being invariant under (active) diffeomorphisms on $\mathcal{S}$ that we wish to be preserved also in the quantum theory. Consequently, $\hat{H}_{\mathrm{phys}}$ requires to be a spatially diffeomorphism invariant operator. Note that in Dirac Quantisation of LQG, we also expected the Hamiltonian constraint to be invariant under the action of this group, but the discrepancy manifests in the fact that in the unreduced LQG spatial diffeomorphism group is a gauge group, but in reduced LQG it is a symmetry group of the dynamics. It was revealed in [48] that for the representation $L_2(\bar{\mathcal{A}}, \mu_{AL})$, spatially diffeomorphism invariant functions has to be quantised in a graph preserving way which means that the underlying graph of a spin network function must not be modified by the action of the corresponding quantum operators. However, in the usual quantisation proposed in [55], the function $\mathtt{C}_0$ is quantised in a graph changing way where for approximating the curvature at a vertex v, one adds an additional edge to the underlying graph connecting two edges starting at v. Therefore, one needs to generalise the strategy and call for AQG [60, 61, 62] where the idea is to use already existing edges of the graph instead of adding new edges.

It is required to introduce minimal loops as follows. Given a graph γ, consider a vertex $v \in V(\gamma)$ and a pair of edges $e_1, e_2 \in E(\gamma)$ of edges starting at the vertex v. A loop $\alpha_{\gamma,e_1,e_2} = e_1 \circ \cdots \circ e_2^{-1}$ in γ is said to be *minimal* provided that there exist no other loop starting with e_1 and ending with e_2^{-1} in γ along which fewer edges traversed. The set of such minimal loops is denoted by L_{γ,v,e_1,e_2}. For instance, in the figure (2.3), the green loops $e_1 \circ e_3 \circ e_2^{-1}, e_1 \circ e_7 \circ e_2^{-1}$ belong to L_{γ,v,e_1,e_2}, but the loop $e_1 \circ e_4 \circ e_5^{-1} \circ e_2^{-1}$ is not minimal. Furthermore, let $T_v(\gamma)$ be the set of ordered triples of distinct edges at v. Then, for each graph, one can define an operator at

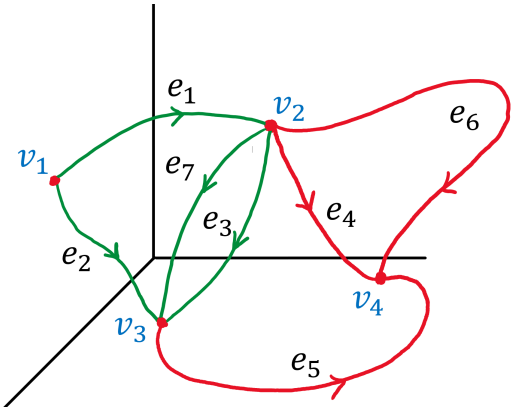

Figure 2.3.: Minimal loop

$v \in V(\gamma)$

$$\hat{\mathtt{C}}_{\mu,\gamma,v} = \frac{1}{\ell_p^2 |T_v(\gamma)|} \sum_{(e_1,e_2,e_3)\in T_v(\gamma)} \epsilon^{IJK} \frac{1}{|L_{\gamma,v,e_I,e_J}|} \sum_{\alpha \in L_{\gamma,v,e_I,e_J}} \mathrm{Tr}\left(\bar{\tau}_\mu \hat{h}_\alpha \hat{h}_{e_K} [\hat{h}_{e_K}^{-1}, \hat{V}_{\gamma,v}]\right) \tag{2.59}$$

Here, $\hat{V}_{\gamma,v}$ is the Ashtekar-Lewandowski volume operator (1.71). Now, we can correspond to the physical Hamiltonian of each graph γ a well-defined operator. The only subtlety remains to be care of is to ensure that the operator is graph-preserving for which we need to utilise the orthogonal projection operator $P_\gamma : \mathcal{H} \to \mathcal{H}_\gamma$ similar to the orthogonal decomposition (1.80). Accordingly, we get

$$\hat{H}_{\mathrm{phys},\gamma} := \sum_{v\in V(\gamma)} \sqrt{|P_\gamma \left(\hat{\mathtt{C}}^\dagger_{\gamma,v} \hat{\mathtt{C}}_{\gamma,v} - \delta^{IJ} \hat{\mathtt{C}}^\dagger_{I,\gamma,v} \hat{\mathtt{C}}_{J,\gamma,v}\right) P_\gamma|} \tag{2.60}$$

and the conclusive physical Hamiltonian operator is introduced as

$$\hat{H}_{\mathrm{phys}} = \bigoplus_\gamma \hat{H}_{\mathrm{phys},\gamma} \tag{2.61}$$

This ends the procedure of reduced phase space quantisation. Let us state some features of this approach:

- As there is no constraint to be solved, the difficulties and ambiguities one usually confronts to solve the constraints operators and to obtain the physical Hilbert space are absent.
- The problem of time is resolved.
- The physical Hamiltonian $\hat{H}_{\mathrm{phys}}$ is self-adjoint and thus the dynamics is manifestly unitary.
- We obtained an unconstrained physical Hilbert space and a Hamiltonian operator. Thus, the situation of the reduced phase space of LQG is similar to lattice gauge theory [156, 157, 158, 159], despite the fact that here we have a more complicated Hamiltonian. Still the quantum dynamics can be studied (see e.g. [160, 161]).
- The physical Hamiltonian operator is defined in the continuum, i.e. no continuum limit has to be taken in contrast to lattice gauge theory.

2.4. Comparison Between Dirac and Reduced Phase Space Quantisation

The difference between the Dirac quantisation and the reduced phase space quantisation is rooted in how they manage the constraints in the path from the classical to the quantum theory. Let us summarise the steps one has to take for both quantisations. In the Dirac quantisation, the entire kinematical phase space is quantised and the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ is obtained that in addition to the physical degrees of freedom involves also the gauge degrees of freedom. The reduction of the system to the physical degrees of freedom is transferred to the quantum level where one implements the first-class constraints as well-defined operators on $\mathcal{H}_{\text{kin}}$. Then physical states are those annihilated by all constraint operators. If one can define an inner product on the set of all physical states, the desired physical Hilbert space $\mathcal{H}_{\text{phys}}$ will be attained. On the other hand, in the reduced phase space quantisation, the reduction of the constraints is performed at the classical level. By this, the algebra of observables is obtained that encodes exclusively the physical degrees of freedom. Then, finding a representation for the observable algebra provides direct access to the physical Hilbert space $\mathcal{H}_{\text{phys}}$. The dynamics of the observables on the reduced phase space is driven by the physical Hamiltonian, that in contrast to the canonical Hamiltonian, does not vanish on the physical sector of the theory. Both procedures have their advantages and disadvantages. For instance, the advantage of the Dirac approach is that the algebra of canonical variables that coordinate the full phase space is sufficiently simple and representations thereof are easy to construct, while in the reduced phase space quantisation the induced algebra of observables is so difficult that representations thereof are hard to find. The disadvantage of Dirac quantisation is that one has to deal with spurious degrees of freedom which is the possible source of ambiguities and anomalies in the gauge symmetry algebra. However, in the reduced phase space approach, one never has to care about kinematical Hilbert space representations.

An important question arises whether two quantum theories achieved from an identical classical constrained system by applying Dirac quantisation and reduced phase space quantisation coincide. In other words, one is interested to investigate whether the quantisation commutes with the reduction (see figure (2.1)). For certain classes of constrained systems, it can be proved [162, 163, 164, 165] that the Dirac route is equivalent to the reduced phase space method, but in general, this is not the case. For instance in the systems studied in [166, 167, 168, 169] there exist substantial differences between the two quantum theories, as the spectra of physical operators do not match. In [143], the origin of this conflict is stated as follows:

By reviewing the quantisation steps, we find that in the reduced phase space quantisation the clock variables T^I are replaced by real numbers τ^I through the gauge fixing conditions $\mathcal{G}^I = T^I - \tau^I$ and the momenta P_I by the functions $-h_I(Q^a, P_a, \tau^I)$ through (2.23), while in the Dirac quantisation all variables are treated in the same way, meaning that even the clock variables and their momenta (T^I, P_I) have to be quantised. Based on this discrepancy, quantum fluctuations of T^I are suppressed in reduced phase space quantisation while they are involved in the Dirac quantisation.

Hence one might think that:

Although, as explained at the end of the previous section, the reduced phase space quantisation bypasses various difficulties of the Dirac quantisation, the latter should be regarded as more fundamental than the former. Therefore, the reduced phase space quantisation is not suitable in extreme regimes such as the big bang, where the clocks T^I cannot be assumed to behave classically, thus in those situations, access to the Dirac quantisation of the system is essential. Subsequently, as in the reduced phase space route the clock variables are basically considered classical all the way

down to the deep quantum regime, one has to be careful about the predictions for extreme regions, obtained using the techniques of the reduced phase space quantisation. Moreover, when the reduced phase space quantisation is performed with respect to different reference fields $T \neq T'$, different quantum fluctuations will be suppressed and hence, in extreme regimes, we will in general get different predictions from the two quantum theories obtained. Keeping these problems in mind, one might doubt the fundamental validity of the models quantised by this technique.

However, one should think of this more carefully. In fact, Dirac quantisation has the advantage of bypassing gauge fixing conditions and thus global problems such as Gribov copies and in that sense is superior to reduced phase space quantisation. But, it is not at all clear whether the existence of fluctuations in the clock variables in the Dirac quantisation method have any physical significance because the clock field is not an observable and the physical Hilbert space therefore simply does not carry a representation of it, hence also not its fluctuations. This is easy to see in the deparametrised case but needs independent study in the non-deparametrised case. Even if it turns out that clock fluctuations are physically important, note that if quantum gravity can be experimentally tested in the future, it will be the semi-classical regime that is of first interest. Since by applying this method of quantisation, valuable insights can be gained about the quantum dynamics of a system in the semi-classical regime, reduced phase space quantisation always remains an interesting ground to study.

Path Integral Quantisation: Spin Foam Models

3.1. Preliminaries

3.1.1. Feynman's Path Integral for Unconstrained Systems

According to Feynman's original exposition [170] of path integral, the quantum transition amplitude for a particle propagating from the spacetime point (q_i, t_i) to another one (q_f, t_f) can be reckoned as a generalisation of double-slit experimental configuration by envisaging an infinite number of screens each of which has an infinite number of slits. The sum of all the probability amplitudes is now tantamount to a sum over all possible paths connecting (q_i, t_i) and (q_f, t_f). If $U(t_f - t_i)$ denotes the evolution operator generated by the Hamiltonian H, then the amplitude is

$$K(q_f, t_f; q_i, t_i) := \langle q_f | U(t_f - t_i) | q_i \rangle, \quad U(t_f - t_i) = \exp\left(-iH(t_f - t_i)\right) \tag{3.1}$$

Hence, the wave function $\psi(q_f, t_f)$ can be computed from $\psi(q_i, t_i)$ as

$$\psi(q_f, t_f) = \int dq_i \; K(q_f, t_f; q_i, t_i) \; \psi(q_i, t_i) \tag{3.2}$$

By decomposing the identity with $t_n = t_i + \frac{n}{N+1}(t_f - t_i)$ for $n \in \{0, \cdots, N\}$, the evolution operator turns to $U(t_f, t_i) = \prod_{n=1}^{N} \exp\left(-iH(t_n - t_{n-1})\right)$ by means of which we get

$$K(q_f, t_f; q_i, t_i) = \int \prod_{n=1}^{N} dq_n \prod_{m=1}^{N} K(q_{m+1}, t_{m+1}; q_m, t_m) \tag{3.3}$$

In the momentum representation K is of the form

$$\begin{aligned} K(q_{m+1}, t_{m+1}; q_m, t_m) &= \int \frac{dp_n}{(2\pi)^{\alpha/2}} e^{ip_n q_{n+1}} \langle p_n | U(\frac{t_f - t_i}{N+1}) | q_n \rangle \\ &\approx \int \frac{dp_n}{(2\pi)^{\alpha/2}} \; e^{ip_n q_{n+1}} \langle p_n | \mathbb{I} - i\frac{t_f - t_i}{N+1} H | q_n \rangle \\ &= \int \frac{dp_n}{(2\pi)^{\alpha}} \; e^{ip_n q_{n+1}} e^{i\left[p_n q_n - \frac{t_f - t_i}{N+1} H(p_n, q_n)\right]} \\ &= \int \frac{dp_n}{(2\pi)^{\alpha}} \; e^{i\left[p_n (q_{n+1} - q_n) - \frac{t_f - t_i}{N+1} H(p_n, q_n)\right]} \end{aligned} \tag{3.4}$$

where α is the number of degrees of freedom. Plugging (3.4) into (3.3), in the limit $N \to \infty$, we obtain

$$K(q_f, t_f; q_i, t_i) = \lim_{N \to \infty} \int \left(\prod_{n=1}^{N} \frac{dq^n dp_n}{(2\pi)^{\alpha}} \right) \frac{dp_0}{2\pi} \; e^{i \sum_{n=0}^{N} \left[p_n (q_{n+1} - q_n) - \frac{t_f - t_i}{N+1} H(p_n, q_n)\right]} \tag{3.5}$$

Now, one can define an skeletonised path $(q(t), p(t))$ in phase space such that $q(t_n) := q_n$ and $p(\frac{t_{n+1}-t_n}{2}) := p_n$. Let us consider the quantities $\int_{t_n}^{t_{n+1}} dt p(t)q(t)$ and $\int_{t_n}^{t_{n+1}} dt H(p(t), q(t))$. The former can be approximated by $p_n(q_{n+1} - q_n)$ and the latter by $\frac{t_f - t_i}{N+1} H(p_n, q_n)$. Subsequently, (3.5) can be rewritten symbolically as a sum over phase space paths

$$K(q_f, t_f; q_i, t_i) = \int \mathcal{D}q\mathcal{D}p \; e^{iS(p,q)}, \qquad S(p,q) = \int_{t_i}^{t_f} (p\dot{q} - H)dt \tag{3.6}$$

where the measure in a sympolical notation is $\mathcal{D}q\mathcal{D}p = \prod_t \frac{dq(t)dp(t)}{2\pi}$ and S is the action of the theory, that is extremized when $q(t)$ is a solution to the classical equation. This observation can be easily generalised to field theories. It is worth noting that the expression (3.6) that provides the transition amplitude from (q_i, t_i) to (q_f, t_f) involves only a sum over classical paths $q(t)$ starting at q_i and ending at q_f, and the classical action S depending on the path [171]. It seems that the classical theory is the only data that enters in (3.6), but the quantum theory is also involved in only one way: the spectrum of $\hat{q}$ is also determined by (3.6). At this point, one should be careful that it is the quantum theory that determines the spectrum of the position operator $\hat{q}$, and hence the allowed positions over which one sums. This observation is not relevant in the case of the free particle we discussed above because the position spectrum includes all real numbers and hence the sum is equivalent to a sum over all classical histories. However, in other theories, this is not necessarily the case [171].
It should be noted that the derivation of path integral is, in fact, "mathematical non-sence", because 1) there exists no infinite dimensional Lebesgue measure $\mathcal{D}q\mathcal{D}q$, 2) the phase space integral is not only on classical paths but arbitrarily discontinuous ones such that the action is not a measurable function. However, in this section we just wanted to review the heuristic idea and not the concrete mathematical structure involving derivation of Wightman functions from Schwinger functions using analytic continuation in time.

3.1.2. Palatini and Holst Actions of Gravity

Although working with Einstein-Hilbert action is the standard approach to GR, there exist many alternatives but equivalent formulations of gravity, that differ in the choice of the relevant mathematical objects describing the world. In principle, it is not in vain to find different formulations of a theory because equivalent classical theories may lead to inequivalent quantum theories. For instance, the action

$$S_P[e, A] = \frac{1}{2} \int_M d^4x \; \det(e) e^\mu_I e^\nu_J F^{IJ}_{\mu\nu} \tag{3.7}$$

known as *Palatini action* is the "first order" formulation of gravity[1]. Here, e^μ_I denotes the tetrad introduced in section 1.3.2, $\det(e)$ is the determinant of its inverse, and

$$F^{IJ}_{\mu\nu} := 2\partial_{[\mu} A^{IJ}_{\nu]} + 2A^I_{[\mu|K|} A^{KJ}_{\nu]} \tag{3.8}$$

is the curvature associated to the $SO(1;3)$ or $SO(4)$ connection A^{IJ}_μ for Lorentzian and Euclidean signature respectively. In (3.7), the frame fields e and the connection A are considered as independent variables[2]. Varying the action with respect to e and A, one readily observes that Einstein's vacuum field equations are reproduced. From the Hamiltonian point of view,

[1] In the first and second order formulations of gravity the actions are of first and second order in the derivatives of the fields, respectively.

[2] Although the action (3.7) is called Palatini action in the literature, according to [172], in Palatini's paper [173] the connection was not considered independently and Einstein was the first one who did so in [174].

the emergence of second class constraints makes the Hamiltonian analysis of (3.7) very intricate. However, it was revealed in [29] that after solving all second class constraints, one reverts back to the ADM formulation for arbitrary spacetime dimensions > 2 [3].

Adding a term to the action (3.7), one arrives at the *Holst action*

$$S_H[e,A] = \frac{1}{2}\int_M d^4x \ \det(e) e^\mu_I e^\nu_J \left(F^{IJ}_{\mu\nu} + \frac{\gamma}{2}\epsilon^{IJ}{}_{KL}F^{KL}_{\mu\nu}\right) \tag{3.9}$$

where γ is the Barbero-Immirzi parameter. As the second term, the *Holst term*, does not affect the equations of motion at the classical level the actions (3.7) and (3.9) are equivalent. In contrast to the Palatini action that gives rise to GR in any arbitrary dimension, the Holst term only exists for 4 dimensional spacetime. It was shown in [177] (using time gauge) and in [178] (without any gauge fixing) that by passing to the canonical theory, the Holst action yields Ashtekar-Barbero variables [35, 36, 38, 40]. More precisely, similar to the Palatini case, the conjugate momentum corresponding to the connection A^{IJ}_a, i.e. $\pi^a_{IJ} = \epsilon^{abc}_{IJKL} e^K_b e^L_c$, are not independent and are subject to fulfil the *simplicity constraints*

$$\epsilon^{IJKL}\pi^a_{IJ}\pi^b_{KL} = 0 \tag{3.10}$$

whose solutions fall into the following sectors [179]

$$(\mathtt{I}\pm)\ \pi^a_{IJ} = \pm\epsilon^{abc}e^I_b e^J_c; \qquad (\mathtt{II}\pm)\ \pi^a_{IJ} = \pm\frac{1}{2}\epsilon^{abc}\epsilon_{IJKL}e^K_b e^L_c; \qquad (\mathtt{deg})\ \pi^a_{IJ} = 0 \tag{3.11}$$

The non-trivial part of the canonical analysis stems from the presence of the constraints (3.10) whose stabilisation lead to new secondary constraints that are second class. The analysis considerably simplifies in the time gauge $e^I_\mu n^\mu = \delta^I_0$, because the simplicity constraints (3.10) do not appear, and one can easily see that the Ashtekar-Barbero variables are recovered.

3.1.3. Purpose of This Chapter

In the previous two chapters, we mentioned the important issue that, what prevents LQG from being considered as a complete theory is the problem of dynamics and its pertinent ambiguities. One approach to attack this problem is spin foam models that avoid defining the Hamiltonian constraint operator by working within the path integral approach to quantization. To apply the path integral approach to GR, two considerations must be taken into account: 1) GR is a constrained system, 2) path integral must be a sum over histories of geometry. For the former, one has to find a way to deal with the constraints to avoid getting divergent path integral. Regarding the latter, as one of the seminal results of LQG is that geometry is quantised one should not sum over all histories of classical geometries, but rather over histories of the permissible quantum geometries predicted by LQG [171]. The spin foam program is based on these two considerations. In this chapter, after discussing path integral for constrained systems, we will show that GR can be viewed as a constrained topological theory. Since the path integral of the unconstrained topological theory (i.e. BF theory) is well-understood, the task is confined to implement the constraint that, of course, is not an easy task!

Since the main goal of delving into the $U(1)^3$ model is to improve our insights about the dynamics of LQG, it is quite natural to study also the path integral approach of this model. As shown in section 3.1.1, one of the prime ingredients of the path integral is the action, we are first supposed to find the covariant origin of the Hamiltonian $U(1)^3$ model. This is what we are going to address in chapter 9, based on the information in this chapter.

[3] For 3, 4-dimensional spacetimes the Hamiltonian analysis was derived in [175]. Moreover, using a gauge fixing that simplifies the analysis, the authors of [176] completed the Hamiltonian analysis for arbitrary dimensions > 2, while in [29] no gauge fixing has been used.

3.2. Path Integral Quantisation of Constrained Systems

Recall the relation between gauge theories and constrained systems explained in section 1.1.1. If one naively uses (3.6) to obtain the transition amplitude for a constrained system, one confronts the problem of divergence because of the integration over the pure gauge degrees of freedom [95]. In this section, we make a brief review of two ideas always on the table to study constrained systems, i.e. reducing before quantising and vice versa.

3.2.1. Reduction Before Quantisation

In [180, 181] an approach was formulated to deal with constraints in the path integral formalism of a theory with first-class constraints and a generalization to systems with second-class constraints was given in [182, 183]. In this approach the constraints are imposed at the classical level, i.e. the idea is to make reduction before quantisation.

The formal path integral for a system with first-class constraints C_I is of the form

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}p\, \mathcal{D}q\, \mathcal{D}\lambda\, e^{i\int_{t_i}^{t_f} dt\, (p_a\dot{q}^a - H(p,q) + \lambda^I C_I)} \\ &= \int \mathcal{D}p\, \mathcal{D}q\, \delta(C,0) e^{i\int_{t_i}^{t_f} dt\, (p_a\dot{q}^a - H(p,q))}\end{aligned} \tag{3.12}$$

where $\delta(C,0) := \prod_I \delta(C_I,0)$. Still the integral (3.12) may encounter divergences. In order to cure this issue, subsidiary conditions $\mathcal{G}_I(q,p) = 0$ should be imposed to single out (ideally) one point per gauge orbit. Furthermore, to formally preserve canonical covariance, one adds a factor of a determinant form to the resulting expression (see below). Recall from section 2.1.1 that, in addition to the fact that we should impose as many gauge-fixing conditions as there are first-class constraints, they all together form a second-class system, that is

$$\det(\{C_I, \mathcal{G}_J\}) \neq 0 \tag{3.13}$$

Taking into account these considerations, we obtain the ensuing path integral in the form

$$\mathcal{Z} = \int \mathcal{D}p\mathcal{D}q\, \underbrace{\delta(C,0)\det(\{\mathcal{G}_J, C_I\})\delta(\mathcal{G},0)}_{*} e^{i\int_{t_i}^{t_f} dt\, (p_a\dot{q}^a - H(p,q))} \tag{3.14}$$

where $\delta(\mathcal{G},0) := \prod_J \delta(\mathcal{G}_J,0)$ and we wish to simplify the starred term in subsequent. Let us denote the physical canonical variables of the theory by $\mathfrak{q}^{\mathfrak{a}}, \mathfrak{p}_{\mathfrak{a}}$ using which the reduced phase space $\mathcal{G}_I = C_I = 0$ is parametrised. Then, the complete set of canonical (both physical and non-physical) variables can be appointed as

$$q = (\mathcal{G}^I, \mathfrak{q}^{\mathfrak{a}}), \qquad p = (P_I, \mathfrak{p}_{\mathfrak{a}}) \tag{3.15}$$

where P_I are the momenta canonically conjugated to $\mathcal{G}^I$. The condition (3.13) is now equivalent to

$$\det\left(\frac{\partial C_I}{\partial P_J}\right) \neq 0 \tag{3.16}$$

so that one is allowed to express the momenta P_I in terms of other variables via the constraints $C_I = 0$. Thus, the reduced phase space can be characterised by the equations

$$Q^I := \mathcal{G}^I = 0, \qquad P_I = P_I(\mathfrak{q}^{\mathfrak{a}}, \mathfrak{p}_{\mathfrak{a}}) \tag{3.17}$$

The starred term in (3.14) is then simplified as

$$\delta(C,0)\det(\{\mathcal{G}_J,C_I\})\delta(\mathcal{G},0)=\delta(C,0)\det\left(\frac{\partial C_I}{\partial P_J}\right)\delta(\mathcal{G},0)=\delta(Q,0)\;\delta(P-P(\mathtt{q}^{\mathfrak{a}},\mathtt{p}_{\mathfrak{a}}),0) \tag{3.18}$$

Finally, plugging (3.18) into (3.14) yields

$$\mathcal{Z}=\int\prod_{\mathfrak{a}}\mathcal{D}\mathtt{q}^{\mathfrak{a}}\mathcal{D}\mathtt{p}_{\mathfrak{a}}\exp\left(i\int d\tau\;[\dot{\mathtt{q}}^{\mathfrak{b}}\mathtt{p}_{\mathfrak{b}}-H_{\text{phys}}]\right) \tag{3.19}$$

Notice that the resultant (3.19) is an unconstrained path integral (3.19) for the reduced phase space. Indeed, the δ-functionals in the starred terms have been used to eliminate $2n_1$ integration variables, for n_1 = the number of first-class constraints. It should be pointed out that since the path integral (3.19) generally includes curvilinear coordinates, it is typically ill-defined. Moreover, the determinant factor in the starred term often suffers from ambiguities related to impermissible gauge-fixing conditions [184, 185, 186]. Subsequently, this extensively used prescription for path integral quantisation of a constrained system is often associated with difficulties and obstacles. In our discussion, second-class constraints were not involved but including them is quite straightforward. For more information, we refer the reader to [187, 188, 189, 190, 191].

3.2.2. Quantisation Before Reduction

Due to the difficulties one usually faces to obtain the reduced phase space and the physical Hamiltonian (see chapter 2), one may prefer to write the path integral in terms of the original canonical variables (q^i,p_i) and carry over the imposition of the constraints to the quantum level, i.e. the idea is to make reduction after quantisation.

This strategy can be sketched in the following steps:

1. Writing the path integral for the unconstrained system;
2. Discretising the classical theory by a cellular decomposition;
3. Quantising the unconstrained part of the discretised theory;
4. Trying to discretise the constraints;
5. Imposing the constraints at the quantum level.

Since this prescription is used in spin foam models, we spend the rest of this chapter explaining it in sufficient detail.

3.3. Path Integral Quantisation of Loop Quantum Gravity: Spin Foam Models

Here, we provide a structural overview of spin foam models by deliberating the main ideas and applying the steps listed in section 3.2.2 to GR. In practice, we first quantise a general BF theory and then discuss that GR can be written as a "constrained" BF theory to which one can apply the prescription of section 3.2.2.

3.3.1. BF Theory

BF theory is a field theory whose action is of the form

$$S_{BF}[A,B] = \int_M \mathrm{Tr}(B \wedge F) \tag{3.20}$$

To understand the constituents of the action (3.20), recall that if P is a principal G-bundle[4] over a smooth n dimensional manifold M, a connection A is a $\mathfrak{g}$-valued[5] one-form on a local trivialisation of P. Now, F appearing in (3.20) is the curvature two-form associated with the connection A, i.e., $F = d_A A = dA + [A \wedge A]$ where $[\cdot,\cdot]$ is the Lie bracket and d_A is the exterior covariant derivative, and when $F = 0$ the connection is said to be flat. A gauge transformation by $g \in G$ changes these quantities as $A \mapsto g^{-1}Ag - g^{-1}dg$ and $F \mapsto g^{-1}Fg$. Another constituent of (3.20) is B that refers to an additional $\mathfrak{g}$-valued $(n-2)$-form field. Under a gauge transformation $B \mapsto g^{-1}Bg$ in a local trivialisation. Notice that the semblance of the action (3.20) justifies the name of the "BF" theory.

This theory, in addition to the gauge transformations just mentioned, has another sort of symmetry. Suppose we define a transformation of the A and B fields by $A \mapsto A$, $B \mapsto B + d_A\eta$ for some $(n-3)$-form η. The action (3.20) remains unchanged under this transformation:

$$\int_M \mathrm{Tr}([B + d_A\eta] \wedge F) = \int_M \mathrm{Tr}(B \wedge F + (-1)^n \eta \wedge d_A F) = \int_M \mathrm{Tr}(B \wedge F) \tag{3.21}$$

in which integration by parts and the Bianchi identity $d_A F = 0$ have been used. In other words, this is a gauge symmetry and any two solutions differing by it are indeed physically equivalent.

Varying the action, we can readily obtain the field equations. Putting the variation equal to zero is equivalent to

$$0 = \delta S_{BF}[A,B] = \int_M \mathrm{Tr}(\delta B \wedge F + B \wedge d_A \delta A) = \int_M \mathrm{Tr}(\delta B \wedge F + (-1)^n d_A B \wedge \delta A) \tag{3.22}$$

where again we have used an integration by parts. The equation (3.22) holds for all δB and δA if and only if the field equations

$$F = 0, \qquad d_A B = 0 \tag{3.23}$$

are satisfied. The property of being "topological field theory" is propounded by the field equations (3.23) of the BF theory, because they don't admit "locally" different solutions and hence there are no "local" degrees of freedom. To check this claim, first, note that according to the former of 3.23 the connection A is flat and, up to gauge transformations, all flat connections are locally equal. Furthermore, when A is flat, any B satisfying $d_A B = 0$ can be expressed locally as $d_A\eta$ for some η, because all closed forms are locally exact. Therefore, all solutions of (3.23) are locally equivalent modulo gauge transformations.

For $n = 3$, GR coincides with the 3-dimensional BF theory [192, 193, 194]. This enriches its quantisation procedure with a variety of methods from topological quantum field theory (TQFT) that ultimately leads to the final quantised theory [195]. Considering the Hamiltonian formulation of 3-dimensional (3d) GR gives insight into the nature of this theory [196]. As usual, the Hamiltonian formulation starts with splitting the spacetime into a foliation of spacelike hypersurfaces, e.g. the manifold has the form $\Sigma \times \mathbb{R}$. The configuration space is then the space of all spatial metrics, say h_{ab} defined on Σ. Note that h_{ab}, where $a, b = 1, 2$, has only 3

[4] A principal G bundle is a fibre bundle whose typical fibre and structure group coincide with G.
[5] We denote the Lie algebra of the gauge group G by $\mathfrak{g}$.

independent components. Just the same as what we had in section 1.3.1, n^a denotes the unit normal to the hypersurface and the lapse N and the shift N^a account the steps between abutting hypersurfaces. The covariant derivative of n^a gives us the extrinsic curvature, i.e. $K_{ab} = h_a^c \nabla_c n^b$ by means of which one can write the momentum conjugate to the elementary variable h_{ab} as $p^{ab} = \det(h)^{1/2}(Kh_{ab} - K_{ab})$ where $K = K^{ab}h_{ab}$. The Hamiltonian can be expressed in terms of these canonical variables and one figures out that N and N^a play the role of Lagrange multipliers (their corresponding conjugate momenta vanish) for the 3 constraints $H = 0$ and $H_a = 0$, respectively[6]. We can now describe the unreduced phase space of 3d GR with 6 variables h_{ab}, p^{ab} and it turns out that the constraints are first class. Recall from section 2.1.1 that in this case that the reduced phase space is of dimension $2n - 2n_1 = 6 - 6 = 0$, i.e. we are left with zero local degrees of freedom. Even once one adds matter to the theory, spacetime outside the matter is still flat and hence there are no gravitational waves in 3d GR. Even though matter cannot have any local effect, it can still cause global ones, and this is why one says that 3d GR is a "topological" theory. In contrast, the spatial metric in 4-dimensional GR has 6 components, and the unreduced phase space is 12 dimensional. As we have 4 constraints H, H_a (since now $a = 1, 2, 3$), in this case the reduced phase space is of dimension $2n - 2n_1 = 12 - 8 = 4$, i.e. 4-dimensional GR has 2 local degrees of freedom. In order to construct the spin foam sum for 3d GR, there are basically two different approaches. The first one comes from the canonical quantisation: it is imposition of the Hamiltonian constraints that leads to the sum over spin foams [197]. The second one departs from the covariant quantisation: one starts from the $n = 3$ BF-formulation and arrives at the spin foam as we will discuss in the succeeding section.

Let us return to the action (3.20) where for $n = 3$ a "two-form" field B acts as one of the basic objects. Plebanski [198] was the first who considered two-forms as fundamental variables in 4 dimensional GR, rather than the metric or tetrad. He found conditions, known as *simplicity constraints*, required to recover the tetrad from the two-forms. Remarkably, he revealed that 4-dimensional GR is "almost" a BF theory, precisely speaking, GR can be formulated as a BF theory subject to the simplicity constraints (see section 3.3.3).

3.3.2. Path Integral Quantisation of BF Theory

The path integral associated with the action (3.20) is given by

$$\mathcal{Z}_{BF} = \int_M \mathcal{D}A\mathcal{D}B\, e^{i\int_M \mathrm{Tr}(B\wedge F)} = \int_M \mathcal{D}A\, \delta(F, 0) \tag{3.24}$$

where $\delta(X, Y)$ is Dirac delta and in the second equality the integral over the B field has been formally performed. The delta distribution $\delta(F, 0)$ says only the flat connections contribute. Now, we wish to give a physical meaning to the above formal expression.

Note that since BF theory is topological and without any local degrees of freedom, we do not lose any information if we replace M with a simplicial manifold $\mathfrak{T}$, with identical topology. To be more precise, we can think of $\mathfrak{T}$ as a triangulation fabricated from glueing n-simplexes together (e.g. 3-simplex=tetrahedron) along their respective triangles and edges. We use the following notations for referring to different elements of $\mathfrak{T}$:

[6] The constraints are of the form

$$H = 2\det(h)^{-1/2}\left(p^{ab}p_{ab} - p^2\right) - \frac{1}{2}h^{-1/2}\,{}^{(2)}R = 0,$$
$$H_a = {}^{(2)}\nabla_b p_a^b = 0$$

where ${}^{(2)}R$ is the Ricci scalar for the spatial metric and ${}^{(2)}\nabla_b$ is the covariant derivative compatible with the spatial metric.

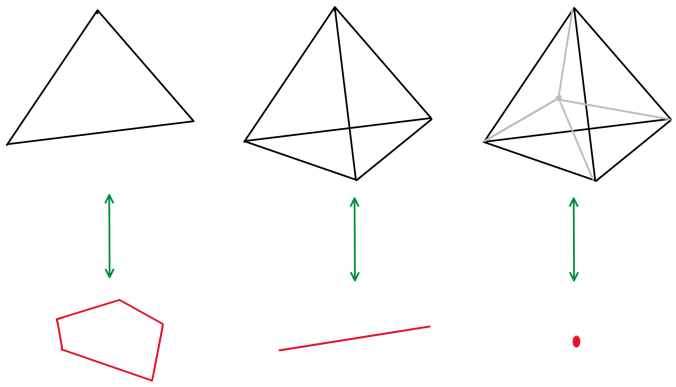

Figure 3.1.: Elements of a 4-dimensional triangulation (above) and the corresponding elements of its complex dual (below)

$$n\text{-simplex: } \Delta_n, \quad \text{tetrahedron: } \Delta_3, \quad \text{triangle: } \Delta, \quad \text{edge: } \Delta_1, \quad \text{vertex: } \Delta_0$$

We also need to construct a 2-skeleton[7] of a complex dual to $\mathfrak{T}$ and equip it with an orientation. The resulting object is called a *foam* and denoted by $\mathfrak{F}$. In fact, this is a combinatorial object consisting of a set of vertices $v \in \mathfrak{F}$ (dual to Δ_n), edges $e \in \mathfrak{F}$ (dual to Δ_{n-1}) and faces $f \in \mathfrak{F}$ (dual to Δ_{n-2}). We denote the set of all vertices, edges and faces of $\mathfrak{F}$ by $C_0(\mathfrak{F})$, $C_1(\mathfrak{F})$ and $C_2(\mathfrak{F})$, respectively. In figure (3.1), elements of a 4 dimensional triangulation and their corresponding duals have been illustrated. To gain more information and see the precise definitions of triangulation and its dual complex consult [199, 200].

As usual, we discretise the connection by attributing a group element $g_e \in G$ to each edge e of the foam. Accordingly, a connection is flat if the holonomy around each (oriented) face is trivial, i.e, $\prod_{i=1}^{N_f} g^f_{e_i} = \mathbb{I}_G$ where $g^f_e = g_e$ if the orientation of e agrees with the orientation of f and $g^f_e = g_e^{-1}$ if it does not. The discrete version of (3.24) is

$$\mathcal{Z}(\mathfrak{F}) = \int \prod_{e\in C_1(\mathfrak{F})} d\mu_H(g_e) \prod_{f\in C_2(\mathfrak{F})} \delta(\prod_{i=1}^{N_f} g^f_{e_i}, \mathbb{I}_G) \tag{3.25}$$

where μ_H denotes the Haar measure. We want to find an alternative for the delta distribution involved in (3.25). Using the Peter-Weyl theorem, the δ distribution can be expressed as $\delta(g, \mathbb{I}_G) = \sum_{\pi\in\Pi(G)} \mathrm{d}_\pi \,\mathrm{Tr}(\pi(g))$[8], where $\Pi(G)$ is the set of unitary irreducible representations of G and d_π denotes the dimension of the representation space of π. Utilising this formula in (3.25)

[7] The n-skeleton of a topological space X refers to the the union of the simplices of X of dimensions $m \leq n$.

[8] Recall the Peter-Weyl theorem stated in footnote 17 in the first chapter, according to which any $f \in L^2(G, d\mu_H)$ can be decomposed as

$$f(h) = \sum_{\pi\in\Pi(G)} \sum_{i,j=1}^{\dim(\pi)} \langle f, \sqrt{\dim(\pi)}\pi_{ij}\rangle \sqrt{\dim(\pi)}\pi_{ij}(h) = \sum_{\pi\in\Pi(G)} \dim(\pi) \sum_{i,j=1}^{\dim(\pi)} \left(\int_G f(g)\overline{\pi_{ij}(g)} d\mu_H(g)\right) \pi_{ij}(h)$$
$$= \sum_{\pi\in\Pi(G)} \dim(\pi) \sum_{i=1}^{\dim(\pi)} \left(\int_G f(g)\pi_{ii}(hg^{-1}) d\mu_H(g)\right) = \sum_{\pi\in\Pi(G)} \dim(\pi) \sum_{i=1}^{\dim(\pi)} \left(\int_G f(g^{-1}h)\pi_{ii}(g) d\mu_H(g)\right)$$

This expansion can be put formally in the form $\delta(g, \mathbb{I}_G) = \sum_{\pi\in\Pi(G)} \dim(\pi)\,\mathrm{Tr}(\pi(g))$.

yields

$$\mathcal{Z}(\mathfrak{F}) = \int \prod_{e\in C_1(\mathfrak{F})} d\mu_H(g_e) \prod_{f\in C_2(\mathfrak{F})} \sum_{\pi_f\in\Pi(G)} \mathrm{d}_{\pi_f}\, \mathrm{Tr}\left(\pi_f(\prod_{i=1}^{N_f} g^f_{e_i})\right) \tag{3.26}$$

We divide the faces including an edge e into the faces whose orientations coincide with the orientation of e, denoted by $\overrightarrow{\mathcal{F}}(e)$ and the faces having opposite orientations, denoted by $\overleftarrow{\mathcal{F}}(e)$. In order to carry out the integral over g_e in (3.26), we should simplify the expressions

$$\mathbb{P}_e(\{\pi_f\}) := \int d\mu_H(g_e) \bigotimes_{f\in\overrightarrow{\mathcal{F}}(e)} \pi_f(g_e) \otimes \bigotimes_{f\in\overleftarrow{\mathcal{F}}(e)} \pi_f(g_e^{-1}) \tag{3.27}$$

which are projectors on the G-invariant subspace of the tensor product of representation due to the group averaging. in other words,

$$\mathbb{P}_e(\{\pi_f\}) : \bigotimes_{f\in\overrightarrow{\mathcal{F}}(e)} \mathcal{H}_f \otimes \bigotimes_{f\in\overleftarrow{\mathcal{F}}(e)} \mathcal{H}_f^* \to \bigotimes_{f\in\overrightarrow{\mathcal{F}}(e)} \mathcal{H}_f \otimes \bigotimes_{f\in\overleftarrow{\mathcal{F}}(e)} \mathcal{H}_f^* \tag{3.28}$$

is orthogonal projection onto the space of invariants of the representation $\bigotimes_{f\in\overrightarrow{\mathcal{F}}(e)} \pi_f \otimes \bigotimes_{f'\in\overleftarrow{\mathcal{F}}(e)} \pi^*_{f'}$ denoted by $\mathrm{Inv}\left(\bigotimes_{f\in\overrightarrow{\mathcal{F}}(e)} \mathcal{H}_f \otimes \bigotimes_{f\in\overleftarrow{\mathcal{F}}(e)} \mathcal{H}_f^*\right)$ whose orthogonal basis makes up of the intertwiners $\imath_e \otimes \imath_e^\dagger$.

Using (3.28) one can simplify (3.26) to

$$\mathcal{Z}(\mathfrak{F}) = \sum_{\pi_f\in\Pi(G)} \prod_{f\in C_2(\mathfrak{F})} \mathrm{d}_{\pi_f}\, \mathrm{Tr}_{\pi_f} \prod_{e\in C_1(\mathfrak{F})} \mathbb{P}_e(\{\pi_f\}) \tag{3.29}$$

Now, the expansion of the projectors in the intertwiner basis results in $\mathbb{P}_e = \sum_{\imath_e} \imath_e \otimes \imath_e^\dagger$. Notice that to each vertex v a tensor $T_v := \bigotimes_{e\in\overrightarrow{v}} \imath_e \otimes \bigotimes_{e\in\overleftarrow{v}} \imath_e^\dagger$ is assigned, where $\overrightarrow{v}$ and $\overleftarrow{v}$ denote the edges ingoing and outgoing at v, respectively. For every face f intersecting v, contains exactly two edges incident at v. Therefore, in T_v there are two indices corresponding to the representation π_f and π_f^*, respectively, that should be contracted in order to get an equivalent expression for (3.29). The result of this contraction procedure is symbolically denoted by $\mathrm{Tr}(T_v)$. Applying this observation to (3.29), we arrive at

$$\mathcal{Z}(\mathfrak{F}) = \sum_{\pi_f\in\Pi(G)} \sum_{\imath_e} \prod_{f\in C_2(\mathfrak{F})} \mathrm{d}_{\pi_f} \prod_{v\in\bar{C}_0(\mathfrak{F})} \mathrm{Tr}(T_v) \tag{3.30}$$

The numbers d_{π_f} and $\mathrm{Tr}(T_v)$ are called *face and vertex amplitudes*, respectively.

For instance, for 3 dimensional BF theory which is equivalent to 3d GR, the vertex amplitude turns out to be the Wigner 6j-symbol. Hence, in this case, the expression (3.30) becomes

$$\mathcal{Z}_{SU(2)}(\mathfrak{F}) = \sum_{j_f} \prod_f (-1)^{2j_f}(2j_f+1) \prod_v \{6j\}_v \tag{3.31}$$

that is exactly the partition function of the Ponzano-Regge model [201].

Before ending this section, it is worth noting that while the partition function (3.30) depends on the global topology, it is independent of the way of triangulating M. Hence, a refinement of the triangulation does not affect the final expression (3.30) of BF theories, e.g. that of 3d GR. In contrast, there is no triangulation invariance in the context of 4 dimensional GR having local degrees of freedom.

3.3.3. GR as a Constrained BF Theory

In this section, we briefly introduce the *Plebanski action* that formulates GR as a BF theory with constraints. Although there are attempts to derive spin-foam models without reference to BF theory [202], the Plebanski formulation serves as the starting point of most spin foam models. The action is given by

$$S_{Pl}[A,B] = \int_M B^{IJ} \wedge F_{IJ} + \lambda_{IJKL} B^{IJ} \wedge B^{KL} \tag{3.32}$$

In his paper [198], Plebanski worked in 4 dimensional spacetime, but it was shown in [203] that the formulation exists for all dimensions > 2. The Hamiltonian analysis of this theory has been studied thoroughly in [204]. Here, just like BF theory, the relevant fields are the connection 1-form A inducing the curvature form $F(A)$, and the 2-form B. Additionally, λ_{IJKL} serves as a Lagrange multiplier satisfying the properties $\lambda_{IJKL} = \lambda_{KLIJ}$ and $\epsilon^{IJKL}\lambda_{IJKL} = 0$. The Lagrange multiplier enforces the constraints $C_{IJKL} := B^{IJ} \wedge B^{KL}$ restricting B in such a way that one recovers GR described by (3.7). The equations of motion for A and B lead to $DB^{IJ} = dB^{IJ} + [A,B]^{IJ} = 0$ and $F_{IJ} = -\lambda_{IJKL} B^{KL}$, respectively. Varying the action with respect to λ, one finds

$$B^{IJ} \wedge B^{KL} = \epsilon^{IJKL} \left(\frac{1}{4!} \epsilon_{MNPQ} B^{MN} \wedge B^{PQ} \right) =: \epsilon^{IJKL} e \tag{3.33}$$

where $e \in \Lambda^4 T^*M$. The equations (3.33) are called the simplicity constraints whose solutions fall into the following sectors [205]:

$$(\texttt{I}\pm)\ B^{IJ} = \pm e^I \wedge e^J; \qquad (\texttt{II}\pm)\ B^{IJ} = \pm \star e^I \wedge e^J; \qquad (\texttt{deg})\ B \text{ is degenerate} \tag{3.34}$$

for some real tetrad field $e^I = e^I_\mu dx^\mu$. Here $\star$ is the Hodge operator. Only the action corresponding to the sector $(\texttt{II}\pm)$ turns out to be the Palatini action (3.7). If one wanted to retrieve the Ashtekar-Barbero variables from the resulting action, the Barbero-Immirzi parameter would have to be $\gamma = \infty$. As γ is an important parameter appearing in the canonical quantisation introduced in chapter 1, one is interested to introduce it also in the covariant approach. In order to include a finite parameter γ we have to work with a slightly generalised action instead of starting from (3.32) [206]

$$S_{Pl}[A,B] = \int_M \left(B^{IJ} + \frac{1}{\gamma} \star B^{IJ} \right) \wedge F_{IJ} + \lambda_{IJKL} B^{IJ} \wedge B^{KL} \tag{3.35}$$

If one takes the boundary into account, it turns the unconstrained part of (3.35) would be of the form

$$S[A,B] = \int_M \left(B^{IJ} + \frac{1}{\gamma} \star B^{IJ} \right) \wedge F_{IJ} + \int_{\partial M} \left(B^{IJ} + \frac{1}{\gamma} \star B^{IJ} \right) \wedge F_{IJ} \tag{3.36}$$

It is transparent that again the constraint is (3.33) whose solutions are displayed in (3.34). To put it succinctly, the BF action with constraints (3.35) reduces to the Holst action (3.9), meaning that GR is recovered as a constrained BF theory. The essence of the above observation is that the simplicity constraints break the topological invariance of BF theory and give rise to the local degrees of freedom of gravity.

3.3.4. Path Integral Quantisation of GR

3.3.4.1. Discretising the Constraints

Now, the question to answer is how to restrict the quantum states of the BF model such that they satisfy the constraints (3.33). To do so, we derive from the constraints (3.33) some conditions

on the Lie algebra elements and then apply them in the context of a discretisation of the action. First, we should paraphrase the constraint into a discrete setting. The 2-forms B naturally correspond to triangles Δ of each 4-simplex of the chosen triangulation $\mathfrak{T}$, i.e.

$$B^{IJ}_{\Delta} = \int_{\Delta} B^{IJ}_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \tag{3.37}$$

Hence, one can smear the constraint $\epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} = e\epsilon_{\mu\nu\rho\sigma}$ over two triangles of the discretisation belonging to the same 4-simplex Δ_4 as

$$\epsilon_{IJKL} B^{IJ}_{\Delta} B^{IJ}_{\Delta'} = \int_{\Delta,\Delta'} d^2\sigma \wedge d^2\sigma' = V(\Delta,\Delta'); \qquad \forall \Delta, \Delta' \subset \Delta_4 \tag{3.38}$$

where $V(\Delta,\Delta')$ is the 4-volume spanned by the two triangles. Depending on the relative position of Δ and Δ' we get the following three types of constraints

Diagonal simplicity: If $\Delta = \Delta'$, then $\epsilon_{IJKL} B^{IJ}_{\Delta} B^{KL}_{\Delta} = 0$, (3.39)

Cross simplicity: If $\Delta \neq \Delta'$, $\Delta \cap \Delta'$ is an edge, then $\epsilon_{IJKL} B^{IJ}_{\Delta} B^{KL}_{\Delta'} = 0$, (3.40)

Volume simplicity: If $\Delta \neq \Delta'$, $\Delta \cap \Delta'$ is a vertex, then $\epsilon_{IJKL} B^{IJ}_{\Delta} B^{KL}_{\Delta'} = V(\Delta,\Delta')$ (3.41)

In fact, in the first two cases Δ, Δ' span a lower dimension than the volume form to be integrated, so the result vanishes. Moreover, B^{IJ} must also fulfil the so called *closure constraint*

$$0 = \int_{\Delta_3} dB^{IJ}(x) = \int_{\partial\Delta_3} B^{IJ}(x) = \sum_{\Delta \subset \partial\Delta_3} \int_{\Delta} B^{IJ}(x) = \sum_{\Delta \subset \partial\Delta_3} B^{IJ}_{\Delta} \tag{3.42}$$

that ensures the set of 4 triangles really builds a tetrahedron. The volume constraint is not to be implemented since (3.39), (3.40) and (3.42) already imply (3.41). The implementation of the constraints (3.39) and (3.40) can be simplified by the geometrical interpretations provided in [207]

The diagonal simplicity constraint (3.39) says that the bivector B^{IJ}_{Δ} is simple[9], i.e. there exist e^I such that each bivector can be expressed as exterior product of these vectors $B^{IJ}_{\Delta} = e^I \wedge e^J$ or its Hodge dual $B^{IJ}_{\Delta} = \star(e^I \wedge e^J)$. Besides that a bivector B^{IJ}_{Δ} in $\mathbb{R}^4$ is simple if and only if there exists a vector n^I such that $B^{IJ}_{\Delta} n_J = 0$ which holds if and only if $(\star B_{\Delta})^{IJ} n_J = 0$ [207].

The cross simplicity constraint (3.40) implies that the planes defined by the bivectors B_{Δ} and $B_{\Delta'}$ with a common edge span a 3-dimensional space. Two simple bivectors B_{Δ} and $B_{\Delta'}$ span a 3-dimensional subspace of $\mathbb{R}^4$ if and only if there exists a vector n^J such that $B^{IJ}_{\Delta} n_J = 0$ and $B^{IJ}_{\Delta'} n_J = 0$ that holds if and only if $(\star B_{\Delta})^{IJ} n_J = 0$ and $(\star B_{\Delta'})^{IJ} n_J = 0$ [207].

These conditions together with (3.42) are sufficient to construct a non-singular tetrahedron. Accordingly, from the geometrical point of view, if we consider n^I as the 4-vector normal to a hypersurface, the simplicity constraints say that all the faces of a given Δ_3 lay in the same hypersurface. The above observations lead to very simple constraints on B^{IJ}_{Δ} that are called the *linear simplicity constraints*

$$(\mathtt{I}\pm) \;\; B^{IJ}_{\Delta} n_J = 0 \;\; \forall \Delta \subset \partial\Delta_4, \qquad (\mathtt{II}\pm) \;\; (\star B_{\Delta})^{IJ} n_J = 0 \;\; \forall \Delta \subset \partial\Delta_4 \tag{3.43}$$

[9] A bivector that can be expressed as the exterior product of two vectors is simple.

Many spin foam models [208, 207, 209, 210, 211, 212, 213] are constructed based on the linear simplicity constraints, while [214, 215] used the original quadratic simplicity constraints. For example, in the EPRL model [208] instead of the quadratic form of the cross simplicity constraint (3.40), its linearised version (3.43) is to be implemented[10], and in the FK model [207] both diagonal and cross simplicity constraints are linearised. In what follows, we will briefly review the EPRL approach.

3.3.4.2. The EPRL Model

The unconstrained part of the action (3.35) is discretised in the same manner as BF-theory. Taking a triangulation $\mathfrak{T}$, one smears the B fields on the triangles Δ and approximates the curvature by a product of group elements assigned to edges of the loop enclosing the dual faces f. If we associate an inertial frame to each tetrahedron Δ_3, then two bivectors $B_\Delta(\Delta_3)$, $B_\Delta(\Delta_3')$, smearing on the same triangle Δ represented in two different frames of two distinct tetrahedra Δ_3 and Δ_3', can be related using an element of G denoted by $U_\Delta(\Delta_3,\Delta_3')\in G$, i.e.

$$B_\Delta(\Delta_3') = U_\Delta(\Delta_3,\Delta_3')^{-1} B_\Delta(\Delta_3') U_\Delta(\Delta_3,\Delta_3') \tag{3.44}$$

According to [210, 216], this yields the following discretised action corresponding to (3.36)

$$\begin{aligned} S = &- \sum_{\Delta\subset\Delta_4} \mathrm{Tr}\left[B_\Delta(\Delta_3)U_\Delta(\Delta_3) + \frac{1}{\gamma}\star B_\Delta(\Delta_3)U_\Delta(\Delta_3)\right] \\ &- \sum_{\Delta\subset\partial\Delta_4} \mathrm{Tr}\left[B_\Delta(\Delta_3)U_\Delta(\Delta_3,\Delta_3') + \frac{1}{\gamma}\star B_\Delta(\Delta_3)U_\Delta(\Delta_3,\Delta_3')\right] \end{aligned} \tag{3.45}$$

where $U_\Delta(\Delta_3) := U_\Delta(\Delta_3,\Delta_3)$. This action together with the simplicity constraints (3.43) and closure constraint (3.42) defines a discretisation of GR. The next task is to translate the constraints into restrictions on the states of the theory. Note that the variable conjugate to A in (3.35) is associated to the bivector $J^{IJ}_\Delta := \left(1+\frac{1}{\gamma}\star\right)B^{IJ}_\Delta$. This can be inverted as

$$B^{IJ}_\Delta = \frac{\gamma^2}{\gamma^2-s}\left(1-\frac{1}{\gamma}\star\right)J^{IJ}_\Delta \tag{3.46}$$

where for the Euclidean case $s=1$ and $G=Spin(4)$ whereas for the Lorentzian case $s=-1$ and $G=SL(2;\mathbb{C})$. Moreover, we have assumed $\gamma\neq 0,1,\infty$. As we will see in what follows, the generators of $\mathfrak{g}$ are considered as the quantum counterparts of J^{IJ}_Δ, by means of which the constraints are imposed at the quantum level.

Let us begin this procedure by expressing the linear simplicity constraints (3.43) in terms of the bivectors J^{IJ}_Δ

$$\begin{aligned} 0 = C^j_\Delta &= n_I \star \left(1-\frac{1}{\gamma}\star\right)J^{IJ}_\Delta = \delta^0_I\left(\epsilon^{IJ}{}_{KL}J^{KL}_\Delta - \frac{s}{\gamma}J^{IJ}_\Delta\right) \\ &= \frac{1}{2}\epsilon_{jkl}J^{kl}_\Delta - \frac{s}{\gamma}J^{0j}_\Delta =: L^j_\Delta - \frac{s}{\gamma}K^j_\Delta \end{aligned} \tag{3.47}$$

where we employed the gauge $n_I=\delta^0_I$. Utilising the commutators between the generators of $\mathfrak{g}$, it turns out that the commutator of the constraints (3.47) is [65]

$$\left[C^i_\Delta, C^j_{\Delta'}\right] = \delta_{\Delta\Delta'}\epsilon^{ij}{}_k\left(C^k_\Delta - \frac{1-\gamma^2}{\gamma^2}L^k_\Delta\right) \tag{3.48}$$

[10] As discussed above, the constraints (3.43) encode both diagonal and cross simplicity constraints but, as we will see, in the EPRL model the former is treated on its own.

from which one concludes that the algebra of the constraints (3.47) does not close. Therefore, as is well known for any second class constraints, one cannot impose them in a strong sense, because otherwise, it would lead to extra conditions that are not present in the classical theory. In other words, instead of looking for a Hilbert space $\mathcal{H}_s$ on which the equations $C^j_\Delta|\psi\rangle = 0$ (strong imposition of the constraints) hold simultaneously, we only require that the conditions $\langle\psi'|C^j_\Delta|\psi\rangle = 0$ (weak imposition of the constraints) are fulfilled on $\mathcal{H}_s$ [217].

For the diagonal simplicity constraint (3.39), we obtain

$$0 = \epsilon_{IJKL}\left(1-\frac{1}{\gamma}\star\right)J^{IJ}_\Delta\left(1-\frac{1}{\gamma}\star\right)J^{KL}_\Delta = \left(1+\frac{s}{\gamma^2}\right)J^{IJ}_\Delta(\star J_\Delta)_{IJ} - \frac{2s}{\gamma}J^{IJ}_\Delta(J_\Delta)_{IJ} \tag{3.49}$$

This construction associates a Hilbert space $\mathcal{H}_{\Delta_3}$ to each tetrahedron Δ_3 of the boundary triangulation $\partial\mathfrak{T}$ in such way that we attach an irreducible representation, $I_\alpha(\Delta_3)$, of G to each triangle of the tetrahedron and denote the carrier space of this representation by $\mathcal{H}_{I_\alpha}$. Since the tetrahedron contains 4 triangles, the index α runs from 1 to 4 and we get $\mathcal{H}_{\Delta_3} = \otimes_{\alpha=1}^4\mathcal{H}_{I_\alpha}$. Now, for the quantisation step we replace the bivectors J^{IJ}_Δ by the generators of the algebra $\mathfrak{g}$ denoted by $\hat{J}_{I_\alpha}$ in the representation I_α. Therefore, the quantum version of the constraint equation (3.49) turns out to be

$$0 = \frac{1}{2}\left(1+\frac{s}{\gamma^2}\right)\epsilon_{IJKL}\hat{J}^{IJ}_{I_\alpha}\hat{J}^{KL}_{I_\alpha} - \frac{2s}{\gamma}\hat{J}^{IJ}_{I_\alpha}(\hat{J}_{I_\alpha})_{IJ} = 4\left(1+\frac{s}{\gamma^2}\right)L^jK_j - \frac{4s}{\gamma}(L^2+sK^2) \tag{3.50}$$

from which it follows that

$$\frac{\gamma^2+s}{s\gamma}L_\Delta\cdot K_\Delta = L^2_\Delta + sK^2_\Delta \tag{3.51}$$

Let $s = 1$ and recall that since $Spin(4) = SU(2)\times SU(2)$, the irreducible representations of $Spin(4)$ are labelled by a couple of half-integers (j^ℓ, j^r). In [206], it was noticed that the constraint (3.51) does not admit any non-trivial solution. However, in [218] by choosing a different ordering than the one usually employed to define Casimirs operators, one observes (up to ordering ambiguity) that a solution arises and the resulting restriction on representations is

$$j^\ell_\Delta = \left|\frac{\gamma+1}{\gamma-1}\right| j^r_\Delta \tag{3.52}$$

which implies that when $\gamma > 0$ ($\gamma < 0$) the relation $j^\ell_\Delta > j^r_\Delta$ ($j^\ell_\Delta < j^r_\Delta$) holds. The equation (3.52) yields this noticeable result that the Immirzi parameter γ has to be rational, i.e. γ is also quantised in this approach! [219]

In order to implement the quantum version of (3.47), we need to find Hilbert space $\mathcal{H}_s \subset \mathcal{H}_{\Delta_3}$ such that all matrix elements of the constraint (3.47) on $\mathcal{H}_s$ equals zero. Inspired by the master constraint programme proposed in [58, 59], the idea of [210] is to replace the constraints (3.47) by the corresponding master constraint

$$C_\Delta := \left(L^j_\Delta - \frac{s}{\gamma}K^j_\Delta\right)\left(L^\Delta_j - \frac{s}{\gamma}K^\Delta_j\right) = L^2_\Delta - \frac{2s}{\gamma}L_\Delta\cdot K_\Delta + \frac{1}{\gamma^2}K^2_\Delta = 0 \tag{3.53}$$

Using (3.51) in (3.53), we get

$$L_\Delta\cdot K_\Delta = \gamma L^2_\Delta \tag{3.54}$$

Now for the case $\gamma > 0$, inserting (3.52) into (3.54) results in a restriction on the quantum number j corresponding to the $SU(2)$ Casimir L^2 as

$$j^2 = \left(\frac{2j^\ell}{1+\gamma}\right)^2 = \left(\frac{2j^r}{1-\gamma}\right)^2 \tag{3.55}$$

that specifies the desired Hilbert space $\mathcal{H}_s$

$$j := \begin{cases} j^\ell + j^r, & 0 < \gamma < 1, \\ j^\ell - j^r, & 1 < \gamma \end{cases} \implies \mathcal{H}_s := \begin{cases} \bigotimes_{\alpha=1}^{4} \mathcal{H}_{j^\ell + j^r}, & 0 < \gamma < 1, \\ \bigotimes_{\alpha=1}^{4} \mathcal{H}_{j^\ell - j^r}, & 1 < \gamma \end{cases} \tag{3.56}$$

in which the cross simplicity constraint (3.47) is satisfied weakly [220]. Finally, implementation of the closure condition (3.42) gives us the intertwiner space as $\mathcal{K}_s = \mathrm{Inv}_{SU(2)}(\mathcal{H}_s)$. We do not display the ultimate spin foam amplitude here and refer the reader to the original paper [210].

In summary, first, we saw that the discrete quantum BF models are topological. Then we recovered GR using constraints on the B field at the continuum level. We went on to impose the simplicity constraints at the quantum level as a selection of the unitary irreducible representations that are admissible. Finally, the amplitude is constructed from the topological $Spin(4)$ BF spin foam vertex amplitude by weakly imposing the simplicity constraints resulting in a restriction of representation labels. In this way, the space of histories of the BF theory path integral is constrained to that of (Euclidean) gravity. For the more involved Lorentzian case, similar calculations lead to amplitude describing the covariant dynamics of LQG. For more detail on this matter, we refer to [65, 221]. The semi-classical limit has been discussed in [222]. The phenomenological applications of spin foam models to quantum cosmology are investigated in [223, 224, 225, 226, 227, 228, 229, 230]. Furthermore, interesting studies on the application of these models to quantum black holes are provided in [231, 232, 233, 234]. Moreover, in an interesting work[235] the relation between Dirac, reduced phase space and path integral quantisation is studied.

Although this type of quantisation for LQG has gained structural and physical successes to describe the dynamics of LQG, it still suffers from some kind of shortcomings [236] that we will discuss in chapter 4.

Part II.

The U(1)^3 Model and Why One Needs to Study It

Dynamics of LQG

4.1. Preliminaries

4.1.1. Kinematics of LQG

In section 1.3, we discussed how the kinematics of LQG has been derived. Briefly, this construction consists of the following steps:

1. The space of all smooth connections $\mathcal{A}$ was chosen as the classical configuration space. These variables together with their conjugate momenta, i.e. the electric fields, formed the classical phase space.

2. We considered the space of all cylindrical functions on $\mathcal{A}$, i.e. those functions that depend on the connection only through a finite number of holonomies, as the functions on the configuration space we are interested in working with.

3. The space of all generalised connections $\bar{\mathcal{A}} = \mathrm{Hom}(\mathcal{P}, G)$ was introduced as the quantum configuration space. The cylindrical functions defined on $\bar{\mathcal{A}}$ was equipped naturally with a measure induced by the Haar measure on $SU(2)$ and consequently, a natural inner product was defined on Cyl. The kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}$ was then defined as the completion of Cyl with respect to this inner product.

4. The holonomies and fluxes could be represented as multiplicative and derivative operators, respectively, on $\mathcal{H}_{\mathrm{kin}}$.

5. The Gauß and diffeomorphism constraints were solved using the group averaging method and we arrived at a Hilbert space free of the kinematical constraints.

In this construction, the dynamics of LQG generated by the Hamiltonian constraint was not imposed. The first step in this regard is to promote the Hamiltonian constraint to a well-defined operator on $\mathcal{H}_{\mathrm{kin}}$: it was a task that due to the intricate expression of the Hamiltonian constraint remained unsolved for many years.

4.1.2. Purpose of This Chapter

Using the key classical identities involving Poisson brackets and with appropriate choices of intermediate regularisations, Thomas Thiemann, in his remarkable series of papers [55, 56, 57, 237, 238, 239], made significant progress in quantising the Hamiltonian constraint by implementing it as an operator well defined on $\mathcal{H}_{\mathrm{kin}}$. Then, it turned out that the quantisation of the Hamiltonian constraint proposed generates a quantum Dirac algebra without anomaly in the sense

that the commutator of two Hamiltonian constraints when acting on spatially diffeomorphism-invariant distributions, is annihilated as it should be according to the classical algebra. However, the scheme of construction exhibits a great deal of freedom in choosing the regulators, thus the Hamiltonian constraint operators suffer from a considerable amount of ambiguities. In this section, we take a brief look at Thiemann's construction of quantum Hamiltonian constraint [55, 56]. Focusing on the underlying idea and avoiding the details, we articulate its shortcomings that highlight the need to examine toy models.

4.2. Thiemann's Construction of Quantum Hamiltonian Constraint

As already mentioned in the section 1.3.2, the classical Hamiltonian constraint of GR is given by

$$H = C - \frac{(1+\gamma^2)}{\sqrt{\det(q)}} E^a_j E^b_k K^j_{[a} K^k_{b]} \tag{4.1}$$

where

$$C := \frac{1}{\sqrt{\det(q)}} F^j_{ab} \epsilon_{jkl} E^a_k E^b_l \tag{4.2}$$

is the classical Euclidean Hamiltonian constraint. Evidently, the tremendous non-linearity of H (and C) as a function of the phase space variables is a difficult barrier to quantisation, since ordering issues arise. The next difficulty presents itself as the volume element appears at the denominator. Quantising the volume element, one finds that it has a large kernel which makes its inverse ill-defined. On the other hand, for F^j_{ab} (the curvature of A^j_a) and K^j_a (the extrinsic curvature) appearing in the Hamiltonian constraint, there are no simple operators in quantum theory. An interesting question is now whether one can get rid of the problematic term $\det(q)^{-1/2}$ by rescaling the Hamiltonian constraint such that one is left with a polynomial expression. It can be shown intuitively and technically that this assumption that a function should be polynomial to be quantised is not only a false premise but the non-polynomiality feature of the Hamiltonian constraint is vital, since only integrals of density weight one scalars can be promoted into a well-defined operator without creating UV singularities. In fact, [240] revealed that UV divergences won't be displayed in any background independent quantum field theory provided that the Hamiltonian has density weight one[1] which means that $\det(q)^{-1/2}$ is a blessing in disguise! Therefore, $\det(q)^{-1/2}$ cannot be swept under the carpet, rather one has to find a way to deal with it. Considering all these difficulties, one can see how non-trivial and challenging it is to propose an explicitly formulated and mathematically well-defined Hamiltonian constraint operator.

4.2.1. The Main Idea and Regularised Hamiltonian Constraint

Thiemann's construction of the Hamiltonian operator is based on two main ideas that should be accomplished before quantization
1) parts of the Hamiltonian constraint are written in terms of Poisson brackets in order to eliminate the troublesome inverse volume element and deal with the extrinsic curvature,
2) the connection A^j_a and the curvature F^j_{ab} are expressed in terms of holonomies whose quantum operator can be used to deal with the curvature.

[1] In the standard model in flat spacetime, the Hamiltonian is of density weight 2.

For the former, one discovers the following two crucial identities [1]

$$\epsilon_{abc}\frac{\epsilon^{jkl}E^a_j E^b_k}{\sqrt{q}}(x) = \{V(R), A^l_c(x)\} \tag{4.3}$$

$$K^j_a = \{K, A^j_a\} \tag{4.4}$$

which can be used to write

$$C[N] = \frac{8}{\kappa}\int d^3x\ N\epsilon^{abc}\mathrm{tr}\left(F_{ab}\{A_c, V(R)\}\right) \tag{4.5}$$

$$H[N] - C[N] = \frac{16}{\kappa^3}\int d^3x\ N\epsilon^{abc}\mathrm{tr}\left(\{A_a, K\}\{A_b, K\}\{A_c, V(R)\}\right) \tag{4.6}$$

where

$$V(R) := \int_R d^3x\ \sqrt{\det q} \tag{4.7}$$

$$K := \int_\Sigma d^3x\ K^j_a E^a_j \tag{4.8}$$

are the volume of an open region R of Σ and The integrated trace of the extrinsic curvature, respectively. The idea of expressing parts of the Hamiltonian constraint in terms of the Poisson brackets untangles the process because in the canonical quantisation, the Poisson bracket $\{\cdot,\cdot\}$ is replaced by $\frac{1}{i\hbar}[\cdot,\cdot]$ and hence the quantisation is greatly simplified, if there exist quantum counterpart operators corresponding to $V(R)$ and K. As discussed in section 1.3.4.1, there are two volume operators: AL volume operator (1.71) and RS volume operator (1.70). In Thiemann's regularisation scheme, we are restricted to work with AL volume operator because it is only this operator that leads to a densely defined Hamiltonian constraint operator [1]. One can easily verify that K can be expressed by a Poisson bracket as

$$K = \{C[1], V_\sigma\} \tag{4.9}$$

Thus, in order to define the operator for K, we are supposed to first quantise C and then compute the commutator with the volume operator.

For the latter, as the canonical variables in the theory are holonomies along edges and fluxes along surfaces, it is required to employ the information encoded in the holonomies about A^j_a and F^j_{ab} and write the Hamiltonian constraint in terms of holonomies. In other words, the local fields have to be expressed in terms of non-local holonomies. In order to do so, one can expand the holonomy for an edge e of coordinate length ϵ as $h_{e_\epsilon} = 1 + \epsilon\dot{e}^a(0)A^j_a(e(0))\tau_j/2 + O(\epsilon^2)$ and approximate $\{\int_e \dot{e}^a A_a, V(R)\} \approx h_e^{-1}\{h_e, V(R)\}$ and $\{\int_e \dot{e}^a A_a, K\} \approx h_e^{-1}\{h_e, K\}$, where $\dot{e}^a$ is the tangent to e in a chosen parametrisation $e(t)$. Similarly, the holonomy expansion for a loop α_{ab} of coordinate area size ϵ^2 is $h_{\alpha_{ab}} = 1 + \epsilon^2 F^j_{ab}\tau_j + O(\epsilon^3)$ and therefore the approximation $\int_S F_{ab} \approx \frac{1}{2}(h_{\alpha_{ab}} - h^{-1}_{\alpha_{ab}})$ holds for an oriented surface S whose boundary is α_{ab} (see figure (4.1)).

Now, consider a triangulation $\mathfrak{T}(\epsilon)$ of the spatial manifold Σ by tetraherda, where the epsilon is a parameter showing how fine $\mathfrak{T}(\epsilon)$ is and $\epsilon \to 0$ describes a situation in which the triangulation fills out the entire space. In fact, $\epsilon \to 0$ is the signature of "continuum limit". For each tetrahedron $\Delta \in \mathfrak{T}(\epsilon)$, we fix a vertex $v(\Delta)$ and denote by $s_i(\Delta)_{i=1,2,3}$ the three outgoing edges in Δ whose beginning points are $v(\Delta)$ and whose tangents span the tangent space at $v(\Delta)$. We also denote by $a_{ij}(\Delta)$ the arc connecting the end points of $s_i(\Delta)$ and $s_j(\Delta)$. Then, one can form several

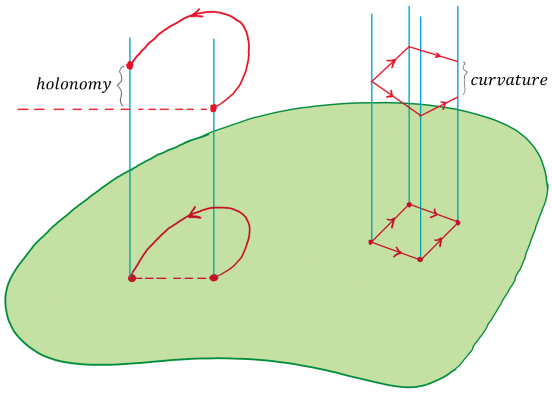

Figure 4.1.: Intuition of holonomy and curvature when one thinks of a connection as a horizontal lift in a fibre bundle.

loops by $\alpha_{ij}(\Delta) := s_i(\Delta) \circ a_{ij}(\Delta) \circ s_j(\Delta)^{-1}$. Each choice of $\{\mathfrak{T}(\epsilon), v(\Delta), \{s_i(\Delta)\}, \{\alpha_{ij}(\Delta)\}\}$ is called a *regulator*.

Employing all the information above and using a regulator, one obtains a Riemann sum approximation of (4.5) as [55]

$$C^\epsilon[N] = \frac{2}{3\kappa^2} \sum_{\Delta \in \mathfrak{T}(\epsilon)} N(v(\Delta)) \epsilon^{ijk} \mathrm{tr}\left(h^{-1}_{\alpha_{ij}(\Delta)} h^{-1}_{s_k(\Delta)} \{h_{s_k(\Delta)}, V_{R_{v(\Delta)}}\} \right) \tag{4.10}$$

such that $\lim_{\epsilon \to 0} C^\epsilon[N] = C[N]$ for any choice of regulator.

4.2.2. Continuum Limit and the Difficulties in Achieving It

Now, the regulated Hamiltonian constraint (4.10) is made up of constituents all of which have quantum counterparts, thus by replacing them by the corresponding operators, we would first arrive at $\hat{C}^\epsilon[N]$ and then it is expected to obtain the quantum counterpart of the Hamiltonian constraint simply by removing the regulator, i.e. taking the refinement limit $\epsilon \to 0$, with respect to a suitable operator topology. But the situation is more complicated than it seems at first glance as, in practice, the program faces difficulties that require individual delicacy and mathematical precision. We discuss these difficulties in the succeeding paragraphs.

- One always needs to ensure that the operator to be defined is cylindrically consistent.

Since any operator directly defined on a basis is automatically consistent, a solution to surmount this problem is to explicitly define the operator on the spin networks.

- As the volume operator existing in the quantum counterpart of (4.10) acts only on the vertices of the graph γ underlying T_s, one is in danger of getting a trivial continuum limit.

To overcome this problem, one is to work with a triangulation which is adapted to the graph γ. Obviously, this is permissible because classically any choice of refinements is as good as the other and all give rise to the same limit. A triangulation adapted to a given graph γ is the one in which each cell Δ contains at most one vertex v of γ and the outgoing edges $s_i(\Delta)$ are proper segments of the edges of γ incident at v. The loops α_{ij} are the triangular loops spanned by s_i, s_j and the connecting arc a_{ij} whose orientation is defined to be from s_i to s_j. There is no other

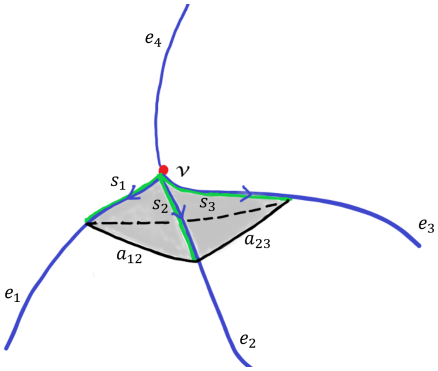

Figure 4.2.: The construction of a triangulation used in the regularisation of the Hamiltonian constraint acting on a vertex v.

points of γ contained in the loops except the edges s_i and s_j (see figure 4.2). One constructs a tetrahedron spanned by a triple (s_i, s_j, s_k) and seven additional "virtual" tetrahedra so that a neighbourhood of the vertex can be triangulated by the eight tetrahedra together. The detailed description of this triangulation can be found in [55]. With this regulator, the sum $\sum_\Delta$ in (4.10) effectively converts to the sum $\sum_{v\in V(\gamma)}$, and the action of the regularised Hamiltonian constraint on a spin network T_s based on a graph γ is given by

$$\hat{C}^\epsilon[N]|T_s\rangle = \frac{16i}{3\hbar\kappa^2}\sum_{v\in\gamma}\frac{N(v)}{E(v)}\sum_{\Delta(v)}\epsilon^{ijk}\mathrm{Tr}\left(\hat{h}^{-1}_{\alpha_{ij}(\Delta)}\hat{h}^{-1}_{s_k(\Delta)}[\hat{h}_{s_k(\Delta)}, V_v]\right)|T_s\rangle \tag{4.11}$$

where the appearance of $E(v) := \binom{n(v)}{3}$ in the denominator stems from the fact that for each triple of edges an independent triangulation is constructed and this has to be compensated.

- Since holonomies and the volume element do not commute, ordering ambiguities arise. On the other hand, one troubling consequence of this quandary can be seen in the refinement limit where the operator may create infinitely many edges and loops that may preclude the limit from being well-defined.

This quandary can be sorted out by choosing a suitable ordering. Since the volume operator acts only on the vertices, by ordering the commutator to the rightmost, the final operator of the Hamiltonian constraint acts only on spin network vertices, as well. Note that this forces the "exceptional"[2] edges that are newly created to be appended to the vertices of the spin network only. Consequently, in this case, only finitely many new edges are created.
Having been ordered in this way, the Hamiltonian constraint operator changes the graph of the spin network by two operators h_{s_k} and $h_{\alpha_{ij}}$. The former overlay segments of coordinate size ϵ on edges of γ, and the latter adds the triangular loops at the vertices, as is depicted in figure (4.3), i.e. its action is "graph-changing". It is worth noting that it creates also new vertices but they are planar[3] and hence carry no volume thanks to using the AL volume operator in (4.11)[4]. It

[2]Exceptional edges are those edges created by the quantum constraint.

[3]a vertex v is called planar if the tangent space spanned by the tangents of the edges incident at v is at most two-dimensional.

[4]Recall that, unlike the RS volume operator, planar vertices are annihilated by the AL volume operator [241]. Moreover, a mathematical consistency analysis [242, 243] shows that the RS operator [121] could be ruled out. It is noteworthy that if one wants to define the continuum limit on some subspace of Cyl^*, instead of $\mathcal{H}_{\mathrm{kin}}$, then RS volume operator is a feasible option [244].

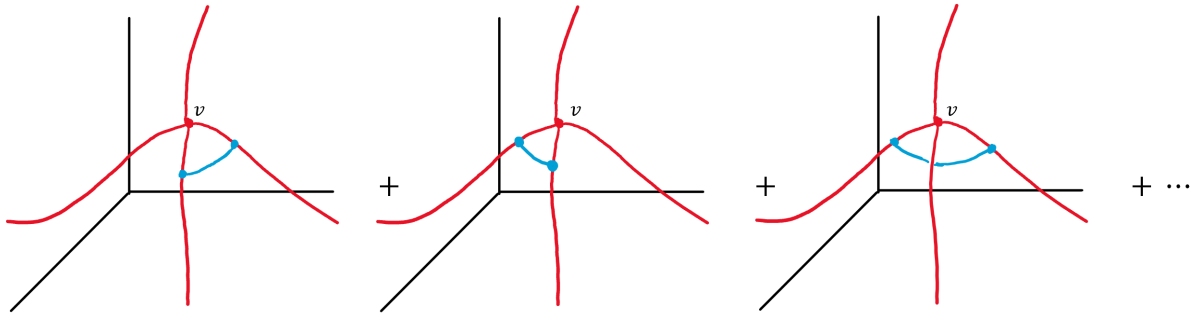

Figure 4.3.: Action of the Hamiltonian constraint adds an extra edge (in blue) to the original graph (in red) and sums up the contributions.

can be shown that the factor ordering just described is unique and any other ordering leads to an operator which is not densely defined [55], thus there is no factor ordering ambiguity left.

- So far, in the choice of regulators, there is a huge amount of ambiguities. This itself is not an issue if various regularisations lead to the same continuum limit. However, there is no guarantee that this happens, if different families of regulators converge at all.

Diffeomorphism covariance of the classical constraint (4.2) motivates Thiemann to confine the choice of triangulations. In fact, the need to make sure that the final quantum operator is also covariant under diffeomorphisms will impose restriction on how a regularisation should be implemented in the following way. Given two diffeomorphic graphs γ and γ' together with their adapted regulators $\{\mathfrak{T}(\epsilon), v(\Delta), \{s_i(\Delta)\}, \{\alpha_{ij}(\Delta)\}\}$ and $\left\{\mathfrak{T}(\epsilon'), v'(\Delta), \{s'_i(\Delta)\}, \{\alpha'_{ij}(\Delta)\}\right\}$ respectively, then one requires that for any choice of ϵ and ϵ' the regulators to be related by a diffeomorphism. This property is called "regulator covariance" [245]. If $C^\epsilon[N]|T_s\rangle = \sum_{v\in V(\gamma)} N(v)\hat{C}^\epsilon(v)|T_s\rangle$, then

$$\hat{C}^\epsilon(v)|T_s\rangle = \hat{U}(\phi)\hat{C}^{\epsilon'}(v)|T_s\rangle \tag{4.12}$$

The existence of such a family of well-defined covariant regulators in the kinematical Hilbert space has been proven in [55, 57]. Note that this is the only "physical" requirement imposed on regularisations. Although this requirement reduces many of the ambiguities in the attachment of the loops and edges, there are still infinitely many of them left in the construction, as explained in [246].

- As the operators (4.11) are graph-changing, one is confronted with the problem that $\hat{C}^\epsilon[N]|T_s\rangle$ is typically orthogonal to $\hat{C}^{\epsilon'}[N]|T_s\rangle$ for $\epsilon \neq \epsilon'$ and so generically $\hat{C}^\epsilon[N]$ fails to converge in $\mathcal{H}_{\text{kin}}$ with respect to the weak operator topology[5].

How small ϵ should be in order for the action of $\hat{C}^\epsilon[N]$ to be defined seems to depend on the underlying graph of the spin network. But at the diffeomorphism invariant level, the background-independent feature of the theory makes the coordinate size ϵ lose its physical importance. Hence, it can be verified that there exists $\tilde{\epsilon} > 0$ such that for all $\epsilon < \tilde{\epsilon}$ the action of $\hat{C}^\epsilon[N]$ on any spin network is well defined [55]. This property is called "uniform regulator covariance" [245] and

[5] We say that a net of operators O_i defined on $\mathcal{H}$ converges to an operator O in *weak operator topology* if for all $y \in \mathcal{H}^*$ and $x \in \mathcal{H}$, the net $y(O_i x)$ converges to $y(Ox)$.

was pointed out in [247] for the first time. Consequently, if $\psi \in \mathcal{H}_{\text{diff}}$ is any diffeomorphism invariant state, the value of $\psi(\hat{C}^{\epsilon}[N]|f_\gamma\rangle)$ is actually independent of ϵ. Thus, $\hat{C}^{\epsilon_0}[N]$ can simply be defined as the limit operator for an arbitrary but fixed value of ϵ_0, since the sequence of numbers $\psi(\hat{C}^{\epsilon}[N]|f_\gamma\rangle)$ obviously converge. This convergence holds with respect to the so-called URS topology[6] in which a sequence of operators $\hat{O}^{\epsilon}$ is said to converge to $\hat{O}$ if for a given $\delta > 0$, there exist an ϵ' such that $\left|\psi\left((\hat{O}^{\epsilon'} - \hat{O})|f_\gamma\rangle\right)\right| < \delta$ for all $f_\gamma \in \mathcal{H}_{\text{kin}}$ and $\psi \in \mathcal{H}_{\text{diff}}$. It has to be noted here that the use of $\mathcal{H}_{\text{diff}}$ is merely a means of setting up a suitable topology for the operators defined on $\mathcal{H}_{\text{kin}}$ [1], and it should not be mistaken to assume that the limit operator is dually defined on $\mathcal{H}_{\text{diff}}$, as is misunderstood in [244, 248, 249].

4.2.3. The Quantum Constraint Algebra

In the quantisation of constrained theories, the closure of the quantum constraint algebra is an essential consistency requirement that must be met. Preferably one desires the quantum constraint algebra to mirror that of the classical constraints but, if not feasible, at least it is expected that the commutator between any two constraints is again a linear combination of them. Even if this also is not the case, at the lowest level of expectation, it is required for the consistency that the commutator annihilates physical states. In fact, if the commutator produces an independent constraint, then the physical Hilbert space can not be consistently determined by the solutions of the original constraints. It follows that the quantum theory loses some degrees of freedom whence it does not have the proper limit. The new quantum constraint that might be produced in the process of quantisation is called "anomaly" and a consistent quantum theory is supposed to be "anomaly-free".

In what follows, we discuss the quantum algebra between the diffeomorphism and Hamiltonian constraints and we do not bother with the commutators involving the Gauss constraint as they are straightforward to check.

4.2.3.1. The Commutators Involving Diffeomorphism Constraints

Because the infinitesimal generator of diffeomorphisms does not exist on $\mathcal{H}_{\text{kin}}$ (see section 1.3.5.2), only the finite and exponentiated versions of (1.33) and (1.34) can be tested in LQG. Using the unitary representation of ϕ on the Hilbert space, that is (1.81), it is easy to verify that the formal exponentiated forms of (1.33) and (1.34) are

$$\hat{U}(\phi_1)\hat{U}(\phi_2)\hat{U}(\phi_1^{-1})\hat{U}(\phi_2^{-1}) = \hat{U}(\phi_1 \circ \phi_2 \circ \phi_1^{-1} \circ \phi_2^{-1}) \tag{4.13}$$

$$\hat{U}(\phi)^{-1}\hat{C}[N]\hat{U}(\phi) = \hat{C}[\phi^* N] \tag{4.14}$$

respectively. Needless to say, (4.13) is simply satisfied by construction, and hence the quantum diffeomorphism constraint subalgebra is anomaly-free. To verify the validity of (4.14), following [1], we calculate its both sides separately and compare them to one another. On one hand, the l.h.s. gives us

$$\begin{aligned}\left[\hat{U}(\phi)^{-1}\hat{C}[N]\hat{U}(\phi)\right] f_{\phi^{-1}(\gamma)} = \hat{U}(\phi)^{-1}\hat{C}[N]f_\gamma &= \sum_{v\in V(\gamma)} N(v)\hat{C}^{\phi^{-1}(\epsilon(\gamma))}(\phi^{-1}(v))f_{\phi^{-1}(\gamma)} \\ &= \sum_{v\in V(\gamma)} (\phi^* N)(\phi^{-1}(v))\hat{C}^{\phi^{-1}(\epsilon(\gamma))}(\phi^{-1}(v))f_{\phi^{-1}(\gamma)}\end{aligned} \tag{4.15}$$

[6] It stands for Uniform Rovelli-Smolin topology [1].

and on the other hand, the r.h.s. results in

$$\begin{aligned}\hat{C}[\phi^* N] f_{\phi^{-1}(\gamma)} &= \sum_{v\in V(\phi^{-1}(\gamma))} (\phi^* N)(v) \hat{C}^{\epsilon(\phi^{-1}(\gamma))}(v) f_{\phi^{-1}(\gamma)} \\ &= \sum_{v\in V(\gamma)} (\phi^* N)(\phi^{-1}(v)) \hat{C}^{\epsilon(\phi^{-1}(\gamma))}(\phi^{-1}(v)) f_{\phi^{-1}(\gamma)} \end{aligned} \tag{4.16}$$

The discrepancy between (4.15) and (4.16) is rooted in the fact that action of $C(N)$ on f_γ is defined through a triangulation denoted by $\epsilon(\gamma)$ which is adapted to the graph γ, and in general this triangulation is not mapped under ϕ^{-1} into the adapted triangulation $\epsilon(\phi^{-1}(\gamma))$ by which $C(N)$ is supposed to act on a cylindrical function whose underlying graph is $\phi^{-1}(\gamma)$. Succinctly, $\phi^{-1}(\epsilon(\gamma)) = \epsilon(\phi^{-1}(\gamma))$ is not necessarily the case. However, one undoubtedly can map $\phi^{-1}(\epsilon(\gamma))$ into $\epsilon(\phi^{-1}(\gamma))$ by applying another diffeomorphism denoted by $\phi'_{\phi^{-1}(\gamma)}$ and get

$$\left[\hat{U}(\phi)^{-1}\hat{C}[N]\hat{U}(\phi)\right] f_{\phi^{-1}(\gamma)} = \left(\hat{U}(\phi'_{\phi^{-1}(\gamma)})\hat{C}[\phi^* N]\hat{U}(\phi'_{\phi^{-1}(\gamma)})^{-1}\right) f_{\phi^{-1}(\gamma)} \tag{4.17}$$

Thus, although (4.14) is not fulfilled, there is no anomaly because the r.h.s. of (4.17) is a constraint operator again. Put another way, as it is shown in [57], when the dual of the relation (4.14) is realised to act on $\psi \in \mathcal{H}_{\text{diff}}$, one can rewrite it as $[\hat{U}(\phi), \hat{C}[N]] = \hat{C}[\phi^* N - N]$, which is obviously free of anomaly.

4.2.3.2. The Commutator Between Two Hamiltonian Constraints

The remaining and crucial commutator between two Hamiltonian constraints whose classical version reads (1.35) is readily computed

$$\begin{aligned}&[\hat{C}(N),\hat{C}(M)] f_\gamma \\ &= \sum_{v\in V(\gamma)} \left(M(v)\hat{C}(N) - N(v)\hat{C}(M)\right) \hat{C}^{\epsilon(\gamma)}(v) f_\gamma \\ &= \sum_{v\in V(\gamma)} \sum_{v'\in V(\gamma')} \left(M(v)N(v') - N(v)M(v')\right) \hat{C}^{\epsilon'(\gamma')}(v')\hat{C}^{\epsilon(\gamma)}(v) f_\gamma \\ &= \frac{1}{2} \sum_{v,v'\in V(\gamma)} \left(M(v)N(v') - N(v)M(v')\right) \left(\hat{C}^{\epsilon'(\gamma')}(v')\hat{C}^{\epsilon(\gamma)}(v) - \hat{C}^{\epsilon'(\gamma')}(v)\hat{C}^{\epsilon(\gamma)}(v')\right) f_\gamma \\ &= \frac{1}{2} \sum_{v,v'\in V(\gamma)} \left(M(v)N(v') - N(v)M(v')\right) \left(\hat{U}(\phi_{v',v}) - \hat{U}(\phi_{v,v'})\right) \hat{C}^{\epsilon(\gamma)}(v)\hat{C}^{\epsilon(\gamma)}(v') f_\gamma \end{aligned} \tag{4.18}$$

where $\epsilon(\gamma)$ is the triangulation adapted to γ and $\epsilon'(\gamma')$ is the triangulation adapted to the graph γ' which is created by the action of $\hat{C}[N]$ or $\hat{C}[N]$ on a state f_γ based on γ and we have used the fact that, due to (4.12), there exists a diffeomorphism $\phi_{v',v}$ such that $\hat{C}^{\epsilon'(\gamma')}(v')\hat{C}^{\epsilon(\gamma)}(v) = \hat{U}(\phi_{v',v})\hat{C}^{\epsilon(\gamma)}(v')\hat{C}^{\epsilon(\gamma)}(v)$. Note that the vertices added by the successive actions of $\hat{C}(v)$ will remain untouched, since they are planar and so annihilated by AL volume operator in the heart of $\hat{C}(v)$.

Thanks to the presence of $\left(\hat{U}(\phi_{v',v}) - \hat{U}(\phi_{v,v'})\right)$ in (4.18), this pivotal result emerges that $\mathcal{H}_{\text{diff}}$ is annihilated by the dual of the commutator and so Thiemann's quantisation of the Hamiltonian constraint is mathematically consistent and anomaly-free [7].

[7] It can be seen that the operator would be anomalous if it acted at the vertices newly created by its action.

4.2.4. Perfections and Imperfections

Having overviewed the construction of Thiemann's quantum Hamiltonian constraint, we are ready to list its perfections and imperfections. This is important for a better understanding of the need to work on the $U(1)^3$ model because it is finding a way to address the worrisome aspects of Thiemann's construction that actuates us to study toy models in order to get insights for improvements in the full theory.

Perfections

1. *There is no factor ordering ambiguity.*
   It turns out that any other ordering rather than what is chosen in (4.11) prevents $\hat{C}[N]$ from being densely defined [55, 250].

2. *The Hamiltonian constraint operator is free of UV singularities.*
   As discussed in [240], if one leaves the Hamiltonian constraint as it appears in the classical analysis, i.e. with its density weight one character, it is naturally devoid of any UV singularities. This is a direct consequence of the background independence property, which holds not only for GR but also for any background-independent field theory.

3. *$\hat{C}(N)$ has a non-trivial kernel.*
   The solutions of the Hamiltonian constraint are the states $\psi \in \mathcal{H}_{\text{diff}}$ satisfying $\psi(\bar{C}(N)f) = 0$ for all $f \in \mathcal{H}_{\text{kin}}$ and all N. The simplest solution is the LQG vacuum which is a state in $\mathcal{H}_{\text{diff}}$ and more complicated solutions have been constructed in [55, 56, 57]. Furthermore, its matrix elements have widely been studied in [251, 252, 253, 254].

4. *The quantum algebra of the constraints is anomaly-free*
   In the sense that the commutator of two constraints vanishes on $\psi \in \mathcal{H}_{\text{diff}}$. The final Hamiltonian constraint is well defined on the kinematical Hilbert space and generates a quantum Dirac algebra that is anomaly-free, in the sense that the action of the commutator of two Hamiltonian constraints on an arbitrary spin network state results in a null state, by averaging over spatial diffeomorphisms.

5. *$\hat{C}[N]$ is diffeomorphism covariant.*
   Since the classical constraint is diffeomorphism covariant, one expects that this feature will be retained in its quantum version as well. The operator $\hat{C}(N)$ fulfils this requirement decreasing a large amount of ambiguities.

Perfections in the disguise of imperfections

1. *Habitat ambiguities*
   It was shown in [244, 248] that the commutator of two Hamiltonian constraint operator annihilates elements of a space larger than $\mathcal{H}_{\text{diff}}$, called the *habitat space* (see section 7.1.1). These papers are based on the idea of taking the refinement limit $\epsilon \to 0$ by means of the elements of such a habitat. However, there may be a huge number of habitats on which the limit can be carried out. This issue is called the *habitat ambiguity* [250] and have extensively been discussed and addressed in [250, 1] in this way: since the habitats are not in the kernel of the diffeomorphism constraint, they are unphysical and the habitat ambiguity is absent. For more information consult [1].

2. *Ultra-locality*
   We have seen that the Hamiltonian constraint acts only on the vertices of a spin network and its action on a given vertex is rather independent of its action on another one. Ultimately, it is the individual contributions of each vertex that add up to obtain the final result. Based on this observation, criticism was made [255] claiming dynamical correlations between distinct vertices would be missing by considering such an *ultra-local* action for the Hamiltonian constraint. In contrast to LQG, this issue does not occur in lattice gauge theory. The answer lies in the fundamental difference between the former and the latter: Background independence feature because locality in the former strongly depends on the fixed background. Furthermore, as explained in [250, 1], the contributions from two different vertices do not commute which means they do influence each other.

3. *On-shell closure*
   The constraint algebra, with Thiemann's Hamiltonian constraint, closes off-shell, i.e. it closes on the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$. It is a misnomer to say the algebra closes on-shell just because URS topology was used to take the regulator limit (This has incorrectly been used in many papers e.g. [256, 244, 245, 257]). In fact, the Hamiltonian constraint densely defined on $\mathcal{H}_{\text{kin}}$ and not on the diffeomorphism invariant Hilbert space $\mathcal{H}_{\text{diff}}$. In particular, the operator cannot be defined on the latter, as it does not even preserve $\mathcal{H}_{\text{diff}}$.

Imperfections

1. *Loop attachment ambiguities*
   As we have a large amount of freedom in choosing the loops that have to be attached to the graph under consideration, there are an infinite number of different allowed regulators to construct the Hamiltonian constraint operator. Every two distinct regulators which are not diffeomorphically related to each other leads to different regulated operators whose continuum limits are not the same in the URS topology.

2. *Loop representation ambiguities*
   The representation that is usually assigned to the exceptional edges is $j = 1/2$. In [1] it is argued that spin $1/2$ is a rather natural choice, but in fact, a complete semi-classical regime is needed to provide a selection principle of the proper spin on the edges.

3. *Faithful representation of the constraint algebra*
   Although the quantum Dirac algebra is free of anomaly, i.e. the algebra closes, one expects a more satisfactory situation in which the quantum structure functions are precisely quantisations of the classical ones provided in [57]. The main reason for this is the lack of quantum infinitesimal generators for spatial diffeomorphisms.

4. *Semi-classical limit*
   Because of the intricate form of the Hamiltonian constraint operator (4.11), it is technically difficult to understand the connection between classical and quantum dynamics. In other words, there is no sufficient control on the classical limit of this theory, but in the context of effective Hamiltonian constraint, the semi-classical analysis has been performed in [258, 259].

5. *Physical Hilbert space*
   As already mentioned, although there exist systematical procedures to generate solutions to the quantum constraint, a sufficient control on the general form of the solutions could

not be provided. Note that since the Hamiltonian constraints do not generate an "honest" Lie algebra, one cannot use RAQ techniques to find the solutions. As a result, explicit expressions of the physical states are still in uncharted waters and construction of the physical Hilbert space is out of reach.

4.2.4.1. An Attempt to Reduce Imperfections: Master Constraint Programme

Some difficulties in dealing with the aforementioned imperfections are due to the peculiarities of the Dirac algebra:

1. The constraint algebra is not a true Lie algebra;
2. The Hamiltonian constraint cannot be defined directly on $\mathcal{H}_{\text{diff}}$, because the algebra does not preserve $\mathcal{H}_{\text{diff}}$ (Recall (1.34));
3. the implementation of (1.35) is not straightforward since there is no infinitesimal generator for spatial diffeomorphisms.

These observations motivate the so-called Master constraint programme introduced in [58, 59] and tested in [103, 260, 261, 262, 263]. The main idea is to replace the Dirac algebra by an alternative classical constraint algebra which has the same constraint surface as before but is a true Lie algebra so that the procedure of finding the solutions of the constraints becomes more manageable.

The so-called master constraint proposed in [58, 59] is

$$\mathbf{M} := \int_{\Sigma} d^3x \, \frac{C(x)^2}{\sqrt{\det(q)}} \tag{4.19}$$

It turns out that the master constraint $\mathbf{M} = 0$ is equivalent to the Hamiltonian constraints $C(x) = 0$. In addition, the Gauß, diffeomorphism and master constraints together form the following constraint algebra

$$\begin{aligned} \{\mathbf{M}, G(\Lambda)\} &= 0 \\ \{\mathbf{M}, \vec{C}(\vec{N})\} &= 0 \\ \{\mathbf{M}, \mathbf{M}\} &= 0 \end{aligned} \tag{4.20}$$

that obviously is an honest Lie algebra. The master constraint can be quantised following the same prescription employed in quantising the Hamiltonian constraint C. In the context of the master constraint programme, the quantisation is without anomaly and the faithfulness of the constraint algebra representation cancels because the operator $\mathbf{M}$ is directly defined on the space $\mathcal{H}_{\text{diff}}$. Importantly, in the context of this programme the "existence" of the physical Hilbert space of LQG was proven in [58]. However, the other issues stated in section 4.2.4 could not be addressed thoroughly and hence the programme has remained formal. For an interesting review see e.g. [264, 265].

The difficulties one faces to define a quantum counterpart for the Hamiltonian constraint led to the development of the spin foam models discussed in chapter 3. These models [214, 207, 208] give rise to a concrete realisation of the sum over histories representing the path integral version of LQG. However, they suffer from difficulties and ambiguities that prevents them from being considered as a final answer to the dynamics in LQG. In the next section, we briefly discuss these issues.

4.3. Shortcomings of the Dynamics From Covariant Approach

One may wonder why one does not use the dynamics of the covariant approach when the dynamics of the canonical approach has ambiguities that are difficult to resolve. Indeed, the dynamics in the former is also a thorny problem and suffers from many ambiguities:

1. *Imposing second-class constraints at the quantum level*
   In the quantisation procedure studied in section 3.3, we followed the strategy of "constraining before quantising" (see section 3.2.2) that has been seriously criticised in [219]. In fact, the construction of the spin foam models in 4 dimensions is not based on systemic quantisation procedures, rather is established on new methods in which "second-class" constraints are imposed at the quantum level. These methods still need to be tested adequately in known physical systems. It was shown in [266] that application of these new methods to a simple system leads to inconsistency. Thus, the question arising is whether the spin foam models can be used beyond purely topological systems. One of our motivations to derive the spin foam for the $U(1)^3$ model will be to check these strategies.

2. *Imposing the constraints at the classical level*
   An alternative approach would be the other way round, i.e. "quantising before constraining" (see section 3.2.1) in which one discretises the simplicity constraints and include them at the classical level in a discrete BF action (see for example [267]). Although the second method seems more comprehensive, it gives rise to much more complicated and even non-local spin foam amplitudes.

3. *choice of cellular decomposition*
   Spin foam models suffer from some ambiguities. For instance, as 4 dimensional GR does have local degrees of freedom, it depends on the choice of cellular decomposition of the manifold. Indeed, the parameter space underlying allowed choices of discretisations includes a very large number of decompositions among which we have to pick up one. In order to achieve a quantum theory of continuous GR, this dependence must be eliminated somehow. A candidate for recovering continuous GR is suggested by group field theory (GFT) [11, 268] where the sum over the relevant complexes is automatically provided.

4. *Using finite cell complex*
   Spin foam models use a finite cell complex and thus are not defined in the continuum in contrast to the canonical theory. GFT brings in another huge ambiguity because they sum over more than cell complexes.

5. *Lack an honest derivation from the full theory*
   Spin foam models lack an honest derivation from the full theory: As is well known (see e.g. [95]), a rigorous derivation of a path integral starts from the Hamiltonian formulation using the Dirac matrix corresponding to a choice of gauge fixing conditions (see e.g. [235] for a review and a concrete implementation for LQG). However, this has never been done.

6. *Connection with the canonical approach*
   There is no connection between the canonical and covariant framework yet, although they should be equivalent as the path integral should define the rigging map.

7. *Representation cut-off*
   Spin foam models are divergent unless one cuts the representations by hand (e.g. using quantum groups) even in Euclidean and even worse in Lorentzian signature.

All these ambiguities in the dynamics of LQG, arising in both canonical and covariant approaches, encourage us to comprehensively analyse a theory that is simpler than gravity but yet includes important and troublesome features of GR in the hope that there are lessons, hidden in studying the simpler model, that are applicable to GR and can make progress toward solving the problem of dynamics. Such a theory is the $U(1)^3$ model that will be introduced in the next chapter.

The Weak Coupling Limit of Euclidean Gravity (The U(1)^3 Model)

5.1. Preliminaries

5.1.1. An Example of a Toy Model: Parametrised Field Theory

Parametrized Field Theories (PFT) [71, 72] are just free field theories on flat spacetime written in generally covariant way by adding pure gauge degrees of freedom. The simplest PFT is the two dimensional one with a massless scalar field on a cylinder that is briefly described here. Consider the following canonical field theory on a Lorentzian Cylinder $(M = S^1 \times \mathbb{R}, \eta)$. The spatial hypersurfaces are circles coordinated by the angular coordinate $x \in [0, 2\pi]$. In this theory two fields, denoted by T, X determining the way of embedding the spatial slices in the Cylinder in addition to a matter scalar field ϕ play the role of configuration variables that together with their conjugate momenta P_T, P_X and Π, respectively, form the unconstrained phase space. One immediately concludes that T, X are gauge degrees of freedom and have to satisfy the periodic conditions. It is quite straightforward to derive the canonical formulation and see that the theory is subject to the following constraints [85, 256]

$$D := P_T T' + P_X X' + \Pi\phi' \tag{5.1}$$

$$C := P_T X' + P_X T' + \frac{1}{2}\left(\Pi^2 + [\phi']^2\right) \tag{5.2}$$

where by prime sign we mean a derivation with respect to x. The constraints (5.1) and (5.2) are the diffeomorphism and Hamiltonian constraints, respectively. Then, the Hypersurface Deformation Algebra can be easily computed

$$\{D(f), D(g)\} = D([f,g]), \qquad \{D(f), D(g)\} = C([f,g]), \qquad \{C(f), C(g)\} = D([f,g]) \tag{5.3}$$

in which f, g are test functions we used for smearing the constraints and $[f,g] := f'g - fg'$. The canonical transformation $X_\pm := T \pm X$, $P_\pm := \frac{1}{2}(P_T \pm P_X)$ and $Y_\pm := \Pi \pm \phi'$ leads to the constraints $D_\pm := \frac{1}{2}(D + C)$ whose corresponding constraint algebra is much simpler than (5.3), i.e.

$$\{D_\pm(f), D_\pm(g)\} = D_\pm([f,g]), \qquad \{D_\pm(f), D_\mp(g)\} = 0 \tag{5.4}$$

that is the well-known Witt algebra $(\mathrm{diff}(S^1) \oplus \mathrm{diff}(S^1))$. Hence, the gauge group is $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$ that is a very appealing result because in LQG it is known how to deal with the spatial diffeomorphisms using the RAQ method outlined in section 1.2.4.1. Therefore, on one hand we have the Dirac algebra (5.3) consisting of the Hamiltonian and diffeomorphism constraints

similar to that of canonical GR, and on the other hand we could complete the entire quantisation program through the Witt algebra (5.4) and obtain physical Hilbert space $\mathcal{H}_{\text{phys}}$ of the theory. It means that one could quantise the Hamiltonian and diffeomorphism constraints by means of LQG techniques and see if there is a choice of their regularisations leading to $\mathcal{H}_{\text{phys}}$ obtained through group averaging techniques. The complete analysis of this idea has been performed in [81, 82, 83, 84, 85].

5.1.2. An Example of a Toy Model: Husain-Kuchar Model

The Husain-Kuchar theory [73] is a background independent theory sharing relevant symmetries used in the quantum regime of GR as for instance the diffeomorphisms covariance, however, it lacks the Hamiltonian constraint. The action depending on tetrad fields e and connections A is of the form

$$S[e,A] = \frac{1}{4}\int_M d^4x\ \epsilon^{\mu\nu\rho\sigma} e^i_\mu e^j_\nu F^k_{\rho\sigma}\epsilon_{ijk} \tag{5.5}$$

where $\mu,\nu\cdots = 0,1,2,3$ are spacetime indices and $i,j,\cdots = 1,2,3$ are $\mathfrak{su}(2)$ indices. The action (5.5) is invariant under spacetime diffeomorphisms and the gauge group $SU(2)$. Defining $\tilde{e}^a_i := \det(e^i_a)e^a_i$ and performing Legandre transformation, one can derive the constraints

$$C_a = F^i_{ab}\tilde{e}^b_i - A^i_a G_i, \qquad G_i = -D_a\tilde{e}^a_i = -\partial_a\tilde{e}^a_i - \epsilon_{ijk}A^j_a\tilde{e}^a_k \tag{5.6}$$

that are diffeomorphism and Gauß constraint, respectively. The important feature of this theory is the absence of the Hamiltonian constraint. It turns out that the constraints are first-class saying that the theory has 3 local degrees of freedom. Notice that in this model, apart from the Gauß constraint, the only constraint is the spatial diffeomorphism one whose algebra is isomorphic to the Lie algebra (1.33). In other words, the Husain-Kuchar model is gravity without the Hamiltonian constraint and this is exactly the system whose quantisation was completely analysed in [86].

5.1.3. Purpose of This Chapter

Recall from section 4.2.4, in order to implement the dynamics of LQG, it is important to reduce the ambiguities arising in the quantisation procedure. This should be done not by imposing some criteria, but by specific physical conditions. The only physical condition used in construction of Thiemann's Hamiltonian constraint was general covariance that reduced a great deal of ambiguities. Another essential physical condition would be to demand a faithful representation for Dirac algebra of GR. While studying the aforementioned toy models turned out to be very useful, note that the constraint algebra of either of them is a true Lie algebra and this is a major simplification! Thus, they are not appropriate to gain insight for checking the correctness of the quantisation of the structure functions appearing in the Dirac algebra within the LQG framework.

In this thesis, we treat another toy model: Weak coupling limit of Euclidean gravity introduced by Smolin [2]. The gauge group of this model is the Abelian group $U(1)^3$ that simplifies the situation while it does not change the structure of the Dirac algebra; i.e. the constraint algebra of the $U(1)^3$ model is not a true Lie algebra and involves structure functions. This observation makes the $U(1)^3$ model an appealing test ground for LQG. The main aim of this thesis is to gain additional insights and experience from this model, that will be introduced in this chapter, to tackle the dynamics issue of the full LQG. The succeeding sections of this chapter are written from parts of [90] and [91], respectively.

5.2. Introduction of the $U(1)^3$ Model

In [2] Smolin introduced the weak coupling limit of Euclidean gravity, that is $G_N \to 0$ where G_N is Newton's gravitational constant, by expanding the canonical variables (A, E) as

$$\begin{aligned} E &= E_0 + G_N E_1 + G_N^2 E_2 + ... \\ A &= A_0 + G_N A_1 + G_N^2 A_2 + ..., \end{aligned} \tag{5.7}$$

at the level of action. The consequential theory should not be confused with the usual perturbation theory. To be precise, consider the canonical Hamiltonian of the Euclidean GR

$$\mathbf{H}_{\text{can}}[E, A] = \frac{1}{G_N} \int d^3x \left(N^a H_a + NH + \Lambda^i G_i \right) \tag{5.8}$$

If one rescales the dimensionful quantities in (5.8) by G_N, i.e. $A_a^i \to G_N A_a^i$ and $\Lambda^i \to G_N \Lambda^i$, the Gauß constraint (1.26) and the curvature F_{ab}^i change to

$$G_i = \mathcal{D}_a E_i^a = \partial_a E_i^a + \epsilon_{ij}{}^k G_N A_a^j E_k^a \tag{5.9}$$

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon^i{}_{jk} G_N A_a^j A_b^k \tag{5.10}$$

respectively. Note that $SU(2)$ is still the internal gauge symmetry. However, in the limit $G_N \to 0$, the ϵ terms in (5.9) and (5.10) which bring in the self-interaction are eliminated. Accordingly, in this limit, a pair of Gauß constraints are Poisson commuting because

$$\{G(\Lambda), G(\Lambda')\} \propto G_N, \tag{5.11}$$

and the gauge group $SU(2)$ contracts to three independent copy of $U(1)$, namely $U(1)^3$, each of which is associated to one of the gauge fields A^i $(i = 1, 2, 3)$. Consequently, the constraints that are still first-class take the simpler forms

$$\begin{aligned} C_j[\lambda^j] &= \int d^3x \; \Lambda^j \partial_a E_j^a \\ C_a[N^a] &= \int d^3x \; N^a \left(F_{ab}^j E_j^b - A_a^j \partial_b E_j^b \right) \\ C[N] &= \int d^3x \; N F_{ab}^j E_k^a E_l^b \epsilon_{jkl} \end{aligned} \tag{5.12}$$

where C_j, C_a and C are the Gauß, diffeomorphism and Hamiltonian constraints for the $U(1)^3$ model respectively and $F_{ab}^j = \partial_a A_b^j - \partial_b A_a^j$ is the Abelian curvature. The canonical Hamiltonian then becomes

$$\mathbf{H}_{\text{can}}[E, A] = \frac{1}{G_N} \int d^3x \left(N^a C_a + NC + \Lambda^i C_i \right) \tag{5.13}$$

and the only non-vanishing elements of the constraint algebra are

$$\{C_a[N^a], C_b[M^b]\} = C_a[\mathcal{L}_{\overrightarrow{N}} M^a] \tag{5.14}$$

$$\{C_a[N^a], C[N]\} = C[\mathcal{L}_{\overrightarrow{N}} N] \tag{5.15}$$

$$\{C[N], C[M]\} = C_a[E^{ia} E_i^b (N \partial_b M - M \partial_b N)]. \tag{5.16}$$

that is isomorphic to the Dirac algebra of GR, except for the vanishing Poisson bracket of a pair of Gauß constraints C_i's, as it can be readily compared with (1.32)-(1.35).

As we will see for our analysis in chapter 8, it is beneficial to work with the density one valued quadratic combinations

$$H_a^j = \frac{1}{2}\epsilon_{abc}\,\epsilon^{jkl}E_k^b E_l^c = \det(E)\,(E^{-1})_a^j \tag{5.17}$$

in terms of which the vector and scalar constraint can be written in an equivalent but unified density two valued form

$$\tilde{C}_j = \epsilon_{jkl}\delta^{km}B_m^a H_a^l,\ \tilde{C}_0 = B_j^a H_a^j \tag{5.18}$$

where

$$B_j^a = \epsilon^{abc}\partial_b A_c^k \delta_{kj} \tag{5.19}$$

defines the magnetic field[1] of A. As usual, we have implicitly assumed that the triad E is nowhere degenerate. Following [269, 270], we can now solve the constraints algebraically: Let

$$B_j^a = c^{jk}\delta_{kl}E_l^a \tag{5.20}$$

then the spacetime diffeomorphism constraints are simply equivalent to $c = c^T$, $\mathrm{Tr}(c) = 0$. However, this is not the case because the Bianchi identity and the Gauß constraint additionally implies

$$E_j^a \partial_a c^{kj} = 0 \tag{5.21}$$

which exhibits a system of three coupled first order PDE's which is not solvable algebraically anymore. Therefore, in order to proceed the gauge fixing analysis of chapter 8 is inevitable.

The virtue of the linearity property of the constraints in A is that the gauge transformations generated by the constraints on E_j^a have a genuinely geometric interpretation: Evidently, E_j^a is Gauß invariant and transforms as a vector density under spatial diffeomorphisms. A gauge transformation generated by the scalar constraint has the effect (using the density one version)

$$\delta_f E_j^a := \{C(f), E_j^a\} = -\epsilon^{abc}\partial_b[f e_c^j] \tag{5.22}$$

where e_c^j is the density zero co-triad. It means that the scalar constraint shifts the divergence-free part of the electric field which is in turn consistent with the Gauß constraint.

This guides us to perform the canonical transformation $(A, E) \mapsto (E, -A)$ and to view $-A$ as the momentum conjugate to E. We call this the (A, E) formulation. We can also consider a dual (f, B) formulation as follows: It is consistent with the asymptotically flat boundary conditions (see chapter 6) to solve the Gauß constraint in the form $E_j^a = \delta_j^a + \epsilon^{abc}\partial_b f_c^j$. When plugging this into the symplectic potential and employing integration by parts, we obtain

$$\theta = \int d^3x\ E_j^a \delta A_a^j = \delta[\int d^3x\ E_j^a A_a^j] - \int d^3x\ A_a^j \delta[E_j^a - \delta_j^a] = \delta[\int d^3x\ E_j^a A_a^j] - \int d^3x\ B_j^a \delta f_a^j \tag{5.23}$$

which presents $-B_j^a$ as conjugate to f_a^j. Notice that the roles of $(f, -B)$ have changed compared to (A, E): While the Gauß constraint depends only on E and hence transforms only A, we do not have a Gauß constraint anymore but instead the Bianchi identity $\partial_a B_j^a = 0$ which, when considering B as a fundamental rather than a derived field, plays the role of a "dual Gauß constraint". Besides, the Bianchi identity depends only on B and just transforms f_a^j and shifts its curl-free part. Thus, the (B, f) formulation has the advantage that all constraints, i.e. spacetime constraints together with the dual Gauß constraint, are linear in B and the former depend on the momentum B conjugate to f only algebraically. This characteristic simplifies the study of the reduced phase space of the $U(1)^3$ model, as we will see in chapter 8.

[1] Throughout this thesis, the magnetic field is defined as the dual of the derivative of the spin connection.

5.3. Hamiltonian Analysis of the Action of [2]

In [2], it is claimed that the action

$$S = \frac{1}{2}\int d^4x\, \epsilon^{\mu\nu\alpha\beta} e^i_\mu e^j_\nu F^{ij}_{\alpha\beta} \tag{5.24}$$

would be the covariant origin of the $U(1)^3$ model, without any proof. Here, $\mu, \nu, \cdots \in \{0,1,2,3\}$ are spacetime indices and $i, j, \cdots \in 1,2,3$ are Lie algebra indices. In this section, we discuss the Hamiltonian analysis of this action and show the assertion is indeed invalid!

We begin with the 3+1 decomposition of (5.24), that is

$$S = \int dt \int d^3x\, \epsilon^{abc}\left(e^i_t e^j_a F^{ij}_{bc} + e^i_a e^j_b F^{ij}_{tc}\right) \tag{5.25}$$

Then, one can easily compute the conjugate momenta corresponding to the configuration variables as

$$P_i := \frac{\delta S}{\delta \dot{e}^i_t} = 0, \quad P^a_i := \frac{\delta S}{\delta \dot{e}^i_a} = 0, \quad \pi_{ij} := \frac{\delta S}{\delta \dot{A}^{ij}_t} = 0, \quad \pi^a_{ij} := \frac{\delta S}{\delta \dot{A}^{ij}_a} = \epsilon^{abc} e^i_b e^j_c \tag{5.26}$$

leading to the primary constraints

$$P_i = 0, \quad P^a_i = 0, \quad \pi_{ij} = 0, \quad T^a_{ij} := \pi^a_{ij} - \epsilon^{abc} e^i_b e^j_c = 0 \tag{5.27}$$

The Legendre transform of (5.27) gives rise to

$$H = \int d^3x\, \{v^i P_i + v^i_a P^a_i + v^{ij}\pi_{ij} + v^{ij}_a \pi^a_{ij} - L\} \tag{5.28}$$

$$= \int d^3x\, \{v^i P_i + v^i_a P^a_i + v^{ij}\pi_{ij} + v^{ij}_a T^a_{ij} - \epsilon^{abc} e^i_t e^j_a F^{ij}_{bc} + \epsilon^{abc} e^i_a e^j_b A^{ij}_{t,c}\} \tag{5.29}$$

where $v^i, v^i_a, v^{ij}, v^{ij}_a$ are the Lagrange multipliers and L is the Lagrangian of the action (5.25). Stability of $\pi_{ij} = 0$ results in the Gauß secondary constraint

$$G_{ij} = \epsilon^{abc}\partial_c(e^i_a e^j_b) \tag{5.30}$$

Stability of $P_i = 0$ leads to three secondary constraints

$$C_i = \epsilon^{abc} e^j_a F^{ij}_{bc} \tag{5.31}$$

Stability of $P^a_i = 0$ is acquired if and only if the following 9 equations are fulfilled

$$\begin{aligned} 0 &= 2(A^{ij}_{t,c} - v^{ij}_c)\epsilon^{cab} e^j_b - \epsilon^{abc} e^j_t F^{ji}_{bc} \\ &:= f^{ij}_c \epsilon^{cab} e^j_b - \epsilon^{abc} e^j_t F^{ji}_{bc} \\ &= f^{ij}_k e^k_c \epsilon^{cab} e^j_b - \epsilon^{abc} e^j_t F^{ji}_{bc} \\ &= f_{ijk}\epsilon^{klj} e^a_l \det(e) - \epsilon^{abc} e^j_t F^{ji}_{bc} \end{aligned} \tag{5.32}$$

that are equivalent to

$$f_{ijk}\epsilon^{klj} = \epsilon^{abc} e^m_t F^{mi}_{bc} e^l_a \det(e)^{-1} =: M^l_i \tag{5.33}$$

Noticing that $f_{ijk} = 2(A^{ij}_{t,c} - v^{ij}_c)e^c_k$ is antisymmetric in i, j, one may express it as $f_{ijk} = \epsilon_{ijm} g^m_k$ for some matrix g^m_k. Therefore, (5.33) can be rewritten as

$$g^m_m \delta^l_i - g^l_i = M^l_i \tag{5.34}$$

whose trace is given by $2g^m_m = M^m_m$. Plugging this into (5.34), we obtain

$$g^l_i = \frac{1}{2}M^m_m\delta^l_i - M^l_i \tag{5.35}$$

showing that all f_{ijk}'s are fixed such that $P^a_i = 0$ is stabilised.
Stability of $T^a_{ij} = 0$ is achieved if and only of the following equation is satisfied

$$\epsilon^{cba}\partial_b\left(e^{[i}_t e^{j]}_c\right) + \epsilon^{abc}v^{[i}_b e^{j]}_c = 0 \tag{5.36}$$

Introducing $v^b_a := v^i_a e^b_i$, one observes

$$\epsilon^{abc}v^d_b e^{[i}_d e^{j]}_c = -\epsilon^{cba}\partial_b\left(e^{[i}_t e^{j]}_c\right) =: M^a_{ij} \tag{5.37}$$

We mow multiply both sides in $e^e_{[i}e^f_{j]}$ and get

$$\epsilon^{ab[f}v^{e]}_b = M^a_{ij}e^e_{[i}e^f_{j]} \tag{5.38}$$

and again by multiplying it in ϵ_{acd}, we arrive at

$$2\delta^{[f}_d v^{e]}_c = \epsilon_{acd}M^a_{ij}e^e_{[i}e^f_{j]} \tag{5.39}$$

from which the solution follows as

$$v^d_b = \frac{1}{2}M^a_{ij}\epsilon_{abc}e^d_{[i}e^c_{j]} \tag{5.40}$$

Hence, the stability of $T^a_{ij} = 0$ is obtained by fixing all the v^i_a.
Therefore, all the primary constraints are stabilised. Now, it is time to examine the stability of the secondary ones.

Stabilisation of G_{ij} is already obtained, since T^a_{ij} and $\pi^c_{ij,c}$ are stable. Modulo $T^a_{ij} = 0$ the Gauß constraint is equivalent to $\hat{G}_{ij} = \pi^c_{ij,c}$ generating U(1)^3 gauge transformations.

To verify the stability of C_i, we note that it is equivalent to $C_a := e^i_a C_i = \epsilon^{dbc}e^j_d e^i_a F^{ij}_{bc} = \epsilon^{dbc}e^j_{[d}e^i_{a]}F^{ij}_{bc}$. According to (5.27), it is immediately concluded that, modulo $T^a_{ij} = 0$, the equation $\epsilon_{abc}\pi^a_{ij} = 2e^i_{[b}e^j_{c]}$ holds. Thus, modulo $T^a_{ij} = 0$, we have

$$C_a = \epsilon^{dbc}e^j_{[d}e^i_{a]}F^{ij}_{bc} = \frac{1}{2}\epsilon^{dbc}\epsilon_{eda}\pi^e_{ij}F^{ij}_{bc} = -\frac{1}{2}(\delta^b_e\delta^c_a - \delta^b_a\delta^c_e)\pi^e_{ij}F^{ij}_{bc} = F^{ij}_{ab}\pi^b_{ij} \tag{5.41}$$

In addition, modulo $T^a_{ij} = G_{ij} = 0$ we have $C_a = F^{ij}_{ab}\pi^b_{ij} - A^{ij}_a\hat{G}_{ij}$ that generates spatial diffeomorphisms on (A^{ij}_a, π^a_{ij}). Since $\hat{P}^i_a, P_i$ is already stabilised, some terms linear in $\hat{P}^a_i, P_i$ can be added to C_a such that the resulting constraint $\hat{C}_a$ generates spatial diffeomorphisms also on the variables e^i_a, P^a_i, e^i, P_i. As all the constraints are tensor densities, the constraint $\hat{C}_a$ and thus C_a is already stabilised.

Classification of the constraints:

Note that all constraints are independent of A^{ij}_t, e^i_t, hence both π_{ij}, P_i are first-class constraints.

On the other hand, since all constraints either are invariant or covariant under Gauß transformations and spatial diffeomorphisms, both $\hat{G}_{ij}$ and $\hat{C}_a$ turn out to be first-class.
The second-class constraints are P_i^a, T_{ij}^a. To see this, it is required to show that the matrix

$$\Delta_{ij}^{ab} := \{P_i^a, T_j^b\} = -\epsilon^{abc}\epsilon_{ijk}e_c^k \tag{5.42}$$

is invertible as a symmetric (under $(a,i) \leftrightarrow (b,j)$) 9×9 matrix, where $T_j^b := T_{kl}^b \epsilon_{jkl}/2$. One can easily check that

$$\Delta_{ab}^{ij} = -\frac{3}{4}\epsilon^{ijk}\epsilon_{abc}e_k^c + \frac{1}{2}\det(e)^{-1}e_a^{(i}e_b^{j)} \tag{5.43}$$

is the inverse of (5.42), i.e., $\Delta_{ij}^{ab}\Delta_{bc}^{jk} = \delta_c^a\delta_i^k$ and thus P_i^a, T_{ij}^a form a second-class pair. Therefore, since we have 18 second-class and 12 first-class constraints, the number of our degrees of freedom is $48 - (2 \cdot 12 + 18) = 6$. This observation proves that the action of [2] cannot be a Lagrangian origin of the Hamiltonian $U(1)^3$ theory that has only four propagating degrees of freedom. Note that in the above analysis, the Hamiltonian constraint $C = F_{ab}^j\epsilon_{jkl}E_k^aE_l^b$ does not emerge as a secondary constraint, just as in the Husain-Kuchar model [73] (see section 5.1.2).

Asymptotically Flat Boundary Conditions

6.1. Preliminaries

6.1.1. Review of Asymptotically Flat boundary Conditions for the $SU(2)$ Case

In general, consistent boundary conditions are supposed to render finite integrable charges associated with the asymptotic symmetries and simultaneously preclude any divergence in symplectic structure. Here, by integrability, we mean the variation of the surface charge is an exact one-form. In the ADM formulation of asymptotically flat spacetimes, asymptotic spheres are described by the asymptotically cartesian coordinates x^a at spatial infinity; i.e. $r \to \infty$ where $r^2 = x^a x_a$. Using these coordinates, one seek appropriate boundary conditions and observes that on any hypersurface, the decay behaviours of the spatial metric q_{ab} and its conjugate momentum π^{ab} have to be

$$\begin{aligned} q_{ab} =& \delta_{ab} + \frac{h_{ab}}{r} + \mathcal{O}(r^{-2}) \\ \pi^{ab} =& \frac{p^{ab}}{r^2} + \mathcal{O}(r^{-3}) \end{aligned} \tag{6.1}$$

where h_{ab} and p^{ab} are smooth tensor fields on the asymptotic 2-sphere. In (6.1), the former is directly inherited from the form of the spacetime metric of the asymptotically flat case. On the other hand, the latter is a outcome of demanding a non-vanishing ADM momentum. In order to remove the logarithmic singularity existing in the symplectic structure, the leading terms in (6.1) require to be restricted by extra certain parity conditions as

$$h_{ab}\left(-\frac{x}{r}\right) = h_{ab}\left(\frac{x}{r}\right), \quad p^{ab}\left(-\frac{x}{r}\right) = -p^{ab}\left(\frac{x}{r}\right) \tag{6.2}$$

Indeed, since the coefficient of the singularity is the integral of $p^{ab}\dot{h}_{ab}$ over the sphere, (6.2) make it vanish and eliminate the singularity. The parity conditions also yield the finite and integrable Poincaré or ISO(4) charges.

Moreover, intending to preserve the boundary conditions (6.1) invariant under the hypersurface deformations,

$$\begin{aligned} \delta q_{ab} =& \frac{-2sN}{\sqrt{q}}(\pi_{ab} - \frac{1}{2}\pi q_{ab}) + \mathcal{L}_{\vec{N}} q_{ab} \\ \delta\pi^{ab} =& -N\sqrt{q}({}^{(3)}R^{ab} - \frac{1}{2}q^{ab\,(3)}R) - \frac{sN}{2\sqrt{q}}(\pi_{cd}\pi^{cd} - \frac{1}{2}\pi^2)q^{ab} \\ &+ \frac{2sN}{\sqrt{q}}(\pi^{ac}\pi_c{}^b - \frac{1}{2}\pi^{ab}\pi) + \sqrt{q}(D^a D^b N - q^{ab} D_a D^b N) + \mathcal{L}_{\vec{N}}\pi^{ab}, \end{aligned} \tag{6.3}$$

one also needs to restrict the dacay behaviours of the lapse function, N, and the shift vector, N^a. In (6.3), $q := \det(q_{ab})$, ${}^{(3)}R_{ab}$ is the Ricci tensor of the spatial hypersurface, D_a is the torsion free, metric compatible connection with respect to q_{ab} and s denotes the signature of spacetime mertic, i.e. $s = +1$ and $s = -1$ for Euclidean and Lorentzian spacetimes, respectively. It transpires that the most general behaviour of them including the generators of the asymptotic Poincaré and ISO(4) groups are

$$\begin{aligned} N &= \beta_a x^a + \alpha + S + \mathcal{O}(r^{-1}) \\ N^a &= \beta^a{}_b x^b + \alpha^a + S^a + \mathcal{O}(r^{-1}), \end{aligned} \tag{6.4}$$

where β_a and $\beta_{ab}(= -\beta_{ba})$ are arbitrary constants displaying boosts and rotations. Here, if v^a denotes the velocity, then the boost parameter is $\beta^a = \frac{v^a}{\sqrt{1+sv^2}} =: \gamma v^a$ satisfying the identity $\gamma^2 + s\gamma^2 v^2 = 1$. This identity shows that for a Euclidean boost, when $s = 1$, the sine and cosine appear in the transformation matrix, indicating that the Euclidean boost is nothing but a rotation in the $x^0, \vec{x}$ plane. Furthermore, the arbitrary function α, and arbitrary vector α^a represent temporal and spatial translations respectively and S, S^a which are odd functions on the asymptotic S^2 are associated with supertranslations.

On the other hand, since in vacuum GR, the smeared versions of the constraints (1.17) and (1.18), i.e.

$$\begin{aligned} H_a[N^a] &:= -2\int d^3x\; N^a D_b \pi^b_a, \\ H[N] &:= \int d^3x\; N\left(\frac{-s}{\sqrt{q}}\left[(q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd})\pi^{ab}\pi^{cd}\right] - \sqrt{q}\;{}^{(3)}R\right) \end{aligned} \tag{6.5}$$

respectively, are the generators of gauge transformations, they need to be finite and functionally differentiable so that their Poisson bracket with any phase-space function can be derived. According to (6.1) and (6.4), we can easily see that the constraints (6.5) are neither finite nor differentiable. To circumvent this problem, a surface integral destroying differentiability is supposed to be subtracted from the variation of the constraint functionals. As mentioned before, appropriate boundary conditions give rise to an exact surface term and, thus we can introduce new expressions for the constraints that now are functionally differentiable. The last step is to check the convergence of the resulting expressions. Having done this procedure, the authors of [271] derived the following well-defined generators

$$\begin{aligned} J[N] &:= H[N] + 2\oint dS_d\; \sqrt{q}q^{a[b}q^{c]d}[N\partial_b q_{ca} - \partial_b N(q_{ca} - \delta_{ca})] \\ J_a[N^a] &:= H_a[N^a] + 2\oint dS_a\; N_b \pi^{ab} \end{aligned} \tag{6.6}$$

where $\oint$ is the integration over the asymptotic 2-sphere.

The above analysis in terms of ADM variables language can be translated to the Ashtekar-Barbero variables (A^i_a, E^a_i). Remember from section 1.3.2 that the connection, A^i_a, is an $\mathfrak{su}(2)$-valued one-form and its momentum conjugate, E^a_i, is a densitised triad. However, it is challenging because in Ashtekar-Barbero formalism there is an additional internal $\mathfrak{su}(2)$ frame whose asymptotic behaviour has to be determined.

Subsequently, the boundary conditions (6.1) and (6.2) in terms of the Ashtekar-Barbero variables can be expressed as

$$\begin{aligned} E^a_i &= \delta^a_i + \frac{f^a_i}{r} + \mathcal{O}(r^{-2}) \\ A^i_a &= \frac{g^i_a}{r^2} + \mathcal{O}(r^{-3}) \end{aligned} \tag{6.7}$$

where

$$\delta^a_i = \begin{cases} 1 & \text{if } (a,i) = (x,1),(y,2),(z,3), \\ 0 & \text{otherwise.} \end{cases}$$

and f^a_i and g^i_a are tensor fields defined on the asymptotic 2-sphere admitting the definite parity conditions

$$f^a_i\left(-\frac{x}{r}\right) = f^a_i\left(\frac{x}{r}\right), \quad g^i_a\left(-\frac{x}{r}\right) = -g^i_a\left(\frac{x}{r}\right). \tag{6.8}$$

The fall-off conditions (6.7) and (6.8) ensure that the symplectic structure is well-defined.

Recall from section 1.3.2 that the smeared versions of the constraints of Euclidean GR in the (A,E)-phase space are given by

$$\begin{aligned} G_i[\Lambda^i] &= \int d^3x\ \Lambda^i \left(\partial_a E^a_i + \epsilon_{ijk} A^j_a E^a_k\right) \\ H_a[N^a] &= \int d^3x\ N^a \left(F^j_{ab} E^b_j - A^j_a G_j\right) \\ H[N] &= \int d^3x\ N \epsilon_{ijk} F^i_{ab} E^a_j E^b_k \end{aligned} \tag{6.9}$$

that are Gauß, diffeomorphism and Hamiltonian constraints, respectively.
Here,

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b, \tag{6.10}$$

and Λ^i is the Lagrange multiplier corresponding to G_i and as usual, N^a is the shift vector and N is the densitised lapse function with weight -1. We want to obtain well-defined forms of these functionals when smearing functions include the ISO(4) generators (6.4). For this goal, first, one has to determine an appropriate fall-off behaviour for Λ^i. Since the leading term of G_i is $\mathcal{O}(r^{-2})$ odd, convergence of $G_i[\Lambda^i]$ requires the decay condition

$$\Lambda^i = \frac{\lambda^i}{r} + \mathcal{O}(r^{-2}) \tag{6.11}$$

where λ^i are even functions defined on the asymptotic S^2. It is straightforward to examine that the differentiability of $G_i[\Lambda^i]$ is also assured by (6.11). Concerning the diffeomorphism and Hamiltonian constraints in (6.9), one finds that even after subtracting the surface integral spoiling differentiability, the constraints are convergent only for translations and not for boosts and rotations. This situation should be modified in such a way that 1) the generators remain functionally differentiable and 2) the well-defined generator for translations that is already available stay untouched up to a pure gauge. It is revealed in [272, 273] that the final well-defined symmetry generators are

$$\begin{aligned} J[N] &= H[N] - \oint dS_a\ N \epsilon_{ijk} A^i_b E^a_j E^b_k - G_i[\Lambda^i_B] + \oint dS_a\ E^a_i \bar{\Lambda}^i_B \\ J_a[N^a] &= H_a[N^a] - \oint dS_a\ N^a A^i_b E^b_i - G_i[\Lambda^i_R] + \oint dS_a\ E^a_i \bar{\Lambda}^i_R \end{aligned} \tag{6.12}$$

where $\Lambda^i_R = \Lambda^i + \bar{\Lambda}^i_R = \Lambda^i - \frac{1}{2}\epsilon_{ijk}\delta^j_a\delta^b_k\beta^a_b$ and $\Lambda^i_B = \Lambda^i + \bar{\Lambda}^i_B = \Lambda^i + \delta^a_i\beta_a$. The second term existing in either expressions in (6.12) is the surface term subtracted to make the original functionals (6.9) differentiable. Moreover, we have subtracted the third term to get rid of the source of divergence for boosts and rotations. But this term puts the functionals again in the status of non-differentiability which is cured by adding the last term. Note that as we expected, the volume terms added to the constraints are proportional to the Gauß constraint and thus the translation generator does not change on the constraint surface of the Gauß constraint.

6.1.2. Purpose of This Chapter

As explained before, the canonical quantisation bifurcates into two different approaches Dirac quantisation (see chapter 1) and reduced phase space quantisation (see chapter 2). In the case of the $U(1)^3$ model, much recent development has been made in the former [87, 88, 274, 275] and working on the latter has started in [89] where the study is confined to the spatial topology of $\mathbb{R}^3$ together with asymptotically flat boundary conditions. As asymptotically flat spacetimes are of prominent importance in GR, this chapter whose content is mainly from [90, 89] is devoted to investigating their properties in the $U(1)^3$ model. The results of this chapter have extensively been used in [89] which was in fact the main motivation for the present study.

To reach asymptotic symmetry generators[1], we find boundary terms for the constraints that produce well-defined phase-space functionals and Poisson brackets, while lapse function and shift vector obey fall-off conditions associated with asymptotic symmetry transformations. In the case of Lorentzian or Euclidean GR, we expect these well-defined functions to generate the Poincaré or ISO(4) group respectively depending on the signature. Regarding the $U(1)^3$ theory, one examines to what extent ISO(4) transformations can be recovered. In fact, the question is whether there are well-defined generators for all transformations of ISO(4) in this model or not, and where the possible discrepancies with the full GR come from.

6.2. Generators of Asymptotic Symmetries for the $U(1)^3$ Model in the (A, E) Description

As the $U(1)^3$ model is a testing ground for GR, we are interested to know whether the boundary conditions and asymptotic symmetries of these two theories are identical. In particular, we wish to answer the question of whether the ISO(4) group can be regarded as the asymptotic symmetries of the model. In other words, is it possible to build well-defined functionals out of the constraints (5.12) while the lapse and shift include the ISO(4) generators? Since the model being pursued is not Euclidean GR, we don't presume to have the whole ISO(4) group as its asymptotic symmetry. However, it is intriguing to study to what extent the model admits a subgroup of the ISO(4) group. In the following, we analyse the constraints (5.12) and try to make them well-defined in the presence of boundary terms.

6.2.1. Gauß Constraint

The Gauß constraint acts on the phase-space variables as

$$\begin{aligned}\delta_\Lambda A^j_a &= \{C_i[\Lambda^i], A^j_a\} = -\partial_a\Lambda^j \\ \delta_\Lambda E^a_j &= \{C_i[\Lambda^i], E^a_j\} = 0\end{aligned} \tag{6.13}$$

[1] There are lots of papers working on asymptotic symmetries, to get more information see e.g. [276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 92]

Thus we see that

$$\begin{aligned}\delta C_j[\Lambda^j] &= \int d^3x\ \Lambda^j \partial_a \delta E^a_j = \oint dS_a\ \Lambda^j \delta E^a_j - \int d^3x\ (\partial_a \Lambda^j)\delta E^a_j = -\int d^3x\ (\partial_a \Lambda^j)\delta E^a_j \\ &= \int d^3x\ \left[(\delta_\Lambda A^j_a)\delta E^a_j - (\delta_\Lambda E^a_j)\delta A^j_a\right]\end{aligned} \tag{6.14}$$

is functionally differentiable. We have dropped the surface term since $\delta E^a_j = \mathcal{O}(r^{-1})$ even and $\Lambda^j = \mathcal{O}(r^{-1})$ even. Furthermore, the constraint is also finite because $\partial_a E^a_j = \mathcal{O}(r^{-2})$ odd and consequently the integrand of $C_j[\Lambda^j]$ is $\mathcal{O}(r^{-3})$ odd.

6.2.2. Vector Constraint

The action of the vector constraint on the canonical variables is

$$\begin{aligned}\delta_{\vec{N}} A^i_c(x) &= \{C_a[N^a], A^i_c(x)\} = -\mathcal{L}_{\vec{N}} A^i_c \\ \delta_{\vec{N}} E^c_i(x) &= \{C_a[N^a], E^c_i(x)\} = -\mathcal{L}_{\vec{N}} E^c_i\end{aligned} \tag{6.15}$$

Hence, its variation is derived as

$$\begin{aligned}\delta C_a[N^a] &= \int d^3x\ N^a \left(\delta F^j_{ab} E^b_j + F^j_{ab}\delta E^b_j - \delta A^j_a \partial_b E^b_j - A^j_a \partial_b \delta E^b_j\right) \\ &= \int d^3x\ N^a \left(\partial_a \delta A^j_b E^b_j - \partial_b \delta A^j_a E^b_j + \partial_a A^j_b \delta E^b_j - \partial_b A^j_a \delta E^b_j - \delta A^j_a \partial_b E^b_j - A^j_a \partial_b \delta E^b_j\right) \\ &= \int d^3x\ \Big(\delta A^j_a \partial_b (N^a E^b_j) - \delta A^j_b \partial_a (N^a E^b_j) + N^a \partial_a A^j_b \delta E^b_j - N^a \partial_b A^j_a \delta E^b_j \\ &\qquad -N^a \delta A^j_a \partial_b E^b_j + \partial_b (N^a A^j_a)\delta E^b_j\Big) + \oint dS_a\ (N^a E^b_j \delta A^j_b - N^b E^a_j \delta A^j_b - N^b A^j_b \delta E^a_j) \\ &= \int d^3x\ \left(\delta A^j_a \left[E^b_j \partial_b N^a - \partial_b (N^b E^a_j)\right] + \delta E^b_j \left[N^a \partial_a A^j_b + A^j_a \partial_b N^a\right]\right) \\ &\quad + \oint dS_a\ (N^a E^b_j \delta A^j_b - N^b E^a_j \delta A^j_b) \\ &= \int d^3x\ \left(\delta A^j_a \left[-\mathcal{L}_{\vec{N}} E^a_j\right] + \delta E^b_j \left[\mathcal{L}_{\vec{N}} A^j_b\right]\right) + \oint dS_a\ (N^a E^b_j \delta A^b_j - N^b E^a_j \delta A^j_b) \\ &= \int d^3x\ \left(\delta A^j_a \left[\delta_{\vec{N}} E^a_j\right] - \delta E^b_j \left[\delta_{\vec{N}} A^j_b\right]\right) + \delta \oint dS_a\ (N^a E^b_j A^j_b - N^b E^a_j A^j_b)\end{aligned} \tag{6.16}$$

Here the third term of the surface integral in the fourth line can be put away because it is $\mathcal{O}(1)$ odd for a rotation and $O(r^{-1})$ even for a translation. Furthermore, in the last step we have pulled the variation out of the surface integral as the correction terms are $\mathcal{O}(r^{-1})$ even for a translation and $\mathcal{O}(1)$ odd for a rotation.

Based on this observation, the new generator should be defined as

$$C'_a[N^a] := C_a[N^a] - \oint dS_a \left(N^a E^b_j - N^b E^a_j\right) A^j_b \tag{6.17}$$

that is functionally differentiable. To check its finiteness, we re-express (6.17) as a volume integral

$$
\begin{aligned}
C'_a[N^a] &= \int d^3x \ \left[N^a F^j_{ab} E^b_j - N^a A^j_a \partial_b E^b_j - \partial_a (N^a E^b_j A^j_b - N^b E^a_j A^j_b)\right] \\
&= \int d^3x \ \left[N^a F^j_{ab} E^b_j - N^a A^j_a \partial_b E^b_j - N^a E^b_j F^j_{ab} - \partial_a (N^a E^b_j - N^b E^a_j) A^j_b\right] \\
&= -\int d^3x \ \left[N^a A^j_a \partial_b E^b_j + \partial_a (N^a E^b_j - N^b E^a_j) A^j_b\right] \\
&= -\int d^3x \ A^j_b \left[E^b_j \partial_a N^a + N^a \partial_a E^b_j - E^a_j \partial_a N^b\right] = -\int d^3x \ A^j_b \mathcal{L}_{\vec{N}} E^b_j
\end{aligned}
\tag{6.18}
$$

In the last line of (6.18), the term $AN\partial E$ is convergent, because it is $\mathcal{O}(r^{-4})$ even for a translation and $\mathcal{O}(r^{-3})$ odd for a rotation. Since α^a is a constant and β^a_b is antisymmetric, the term $A^j_b E^b_j \partial_a N^a$ vanishes for both translation and rotation. Moreover, the other term of the form $AE\partial N$ is $\mathcal{O}(r^{-2})$ odd for a rotation and vanishes for a translation. Putting all information together, one concludes $C'_a[N^a]$ is well-defined for a translation and the source of its divergence for a rotation is

$$
\begin{aligned}
\int d^3x \ A^j_b E^a_j \beta^b_a &= \int d^3x \ \beta^b_a A^j_b \left(\delta^a_j + \frac{f^a_j}{r} + \dots\right) \\
&= \int d^3x \ \beta^b_a A^j_b \delta^a_j + \int d^3x \ \beta^b_a A^j_b \frac{f^a_j}{r} + \text{finite} \\
&= \int d^3x \ \beta^b_a A^j_b \delta^a_j + \text{finite}
\end{aligned}
\tag{6.19}
$$

where the integrand of the second integral in the second line is $\mathcal{O}(r^{-3})$ odd and so the convergence is ensured. Again, since (6.19) cannot be expressed in terms of the constraints, it is not admissible to subtract the volume integral from (6.17). Consequently, while spatial translations have a well-defined generator (6.17), rotations do not!

6.2.3. Scalar Constraint

It is straightforward to obtain the action of the scalar constraint in Ashtekar-Barbero variables as

$$
\begin{aligned}
\delta_N A^i_c(x) &= \{C[N], A^i_c(x)\} = -2N\epsilon_{ijk} F^j_{ac} E^a_k \\
\delta_N E^c_i(x) &= \{C[N], E^c_i(x)\} = 2\epsilon_{ikl} \partial_b (N E^c_k E^b_l)
\end{aligned}
\tag{6.20}
$$

Immediately it follows that the variation of this constraint is

$$
\begin{aligned}
\delta C[N] &= \int d^3x \ \epsilon_{jkl} N \left(\delta F^j_{ab} E^a_k E^b_l + 2F^j_{ab} E^b_l \delta E^a_k\right) \\
&= \int d^3x \ \epsilon_{jkl} N \left((\partial_a \delta A^j_b - \partial_b \delta A^j_a) E^a_k E^b_l + 2F^j_{ab} E^b_l \delta E^a_k\right) \\
&= 2\int d^3x \ \epsilon_{jkl} \left(\partial_a (N E^a_k E^b_l \delta A^j_b) - \delta A^j_b \partial_a (N E^a_k E^b_l) + N F^j_{ab} E^b_l \delta E^a_k\right) \\
&= \int d^3x \ 2\epsilon_{jkl} \left(N F^j_{ab} E^b_l \delta E^a_k - \delta A^j_b \partial_a (N E^a_k E^b_l)\right) + 2\int dS_a (N E^a_k E^b_l \delta A^j_b) \\
&= \int d^3x \ \left(\delta A^j_b (\delta_N E^b_j) - (\delta_N A^k_a) \delta E^a_k\right) + 2\delta \oint dS_a \ \epsilon_{jkl} (N E^a_k E^b_l A^j_b)
\end{aligned}
\tag{6.21}
$$

The variation have been pulled out of the surface integral in (6.21) as the correction terms are $\mathcal{O}(r^{-1})$ even for a translation and $\mathcal{O}(1)$ odd for a boost. Now we introduce the new generator as

$$C'[N] := C[N] - 2\oint dS_a\ N\epsilon_{jkl}A^j_b E^a_k E^b_l \tag{6.22}$$

which is functionally differentiable. At this step, the finiteness has to be checked.

$$\begin{aligned}
C'[N] =& \int d^3x\ \epsilon_{jkl}\left(F^j_{ab}E^a_k E^b_l N - 2\partial_a(A^j_b E^a_k E^b_l N)\right)\\
=&2\int d^3x\ \epsilon_{jkl}\left(\frac{1}{2}F^j_{ab}E^a_k E^b_l N - E^a_k E^b_l N\partial_a A^j_b - A^j_b E^b_l N\partial_a E^a_k - A^j_b E^a_k N\partial_a E^b_l - A^j_b E^a_k E^b_l\partial_a N\right)\\
=&-2\int d^3x\ \epsilon_{jkl}\left(A^j_b E^b_l N\partial_a E^a_k + A^j_b E^a_k N\partial_a E^b_l + A^j_b E^a_k E^b_l\partial_a N\right)
\end{aligned} \tag{6.23}$$

Here terms of the form $AEN\partial E$ are convergent because they are $\mathcal{O}(r^{-4})$ even for a translation and $\mathcal{O}(r^{-3})$ odd for a boost. The last term which is of the form $AEE\partial N$ vanishes for a translation but is divergent for a boost. Therefore, $C'[N]$ is well-defined for a translation and the source of its divergence for a boost is

$$\begin{aligned}
-2\int d^3x\ \epsilon_{jkl}(A^j_b E^a_k E^b_l\beta_a) =& -\int d^3x\ \frac{1}{r^2}(\beta_a\epsilon_{jkl}g^j_b\delta^a_k\delta^b_l) - \int d^3x\ \beta_a\epsilon_{jkl}A^j_b\delta^a_k E^b_l + \text{finite}\\
=& -\int d^3x\ \beta_a\epsilon_{jkl}A^j_b\delta^a_k E^b_l + \text{finite}
\end{aligned} \tag{6.24}$$

where we have used the parity of g^j_b to drop the linear singularity in going from the first line to the second one and so we are left with the logarithmic singularity. To get rid of the divergence, we have to subtract (6.24) from (6.22) but this would not be done as it has an undesirable effect on the constraint surface since (6.24) is proportional neither to the constraints nor to a part of them. Consequently, time translations admit a well-defined generator (6.22), but boosts do not! A more detailed argument is given in section 6.3.

6.3. Comparison with the $SU(2)$ Case

In this section, we aim to put the setting under scrutiny and see what causes the $SU(2)$ and $U(1)^3$ cases so different that the latter does admit generators for boosts and rotations, but the former does not. First, we split F^i_{ab} and G_i into its Abelian and non-Abelian pieces, i.e. $F^i_{ab} = F^{i+}_{ab} + F^{i-}_{ab}$ and $G_i = G^+_i + G^-_i$ where $F^{i+}_{ab} = \partial_a A^i_b - \partial_b A^i_a$, $F^{i-}_{ab} = \epsilon_{ijk}A^j_a A^k_b$, $G^+_i = \partial_a E^a_i$ and $G^-_i = \epsilon_{ijk}A^j_a E^a_k$; accordingly the Hamiltonian and diffeomorphism constraints have also two pieces associated with the plus and minus parts of F^i_{ab} and G_i, namely $H[N] = H^+[N] + H^-[N]$ and $H_a[N^a] = H^+_a[N^a] + H^-_a[N^a]$, where

$$H^+[N] = \int d^3x\ N\epsilon_{jkl}F^{j+}_{ab}E^a_k E^b_l, \quad H^-[N] = \int d^3x\ N\epsilon_{jkl}F^{j-}_{ab}E^a_k E^b_l \tag{6.25}$$

$$H^+_a[N] = \int d^3x\ N^a(F^{j+}_{ab}E^b_j - A^j_a G^+_j), \quad H^-_a[N] = \int d^3x\ N^a(F^{j-}_{ab}E^b_j - A^j_a G^-_j) = 0. \tag{6.26}$$

Due to the boundary conditions, $F^{i-}_{ab} = \mathcal{O}(r^{-4})$ even and $G^-_i = \mathcal{O}(r^{-2})$ odd. Thus, the integrand of $H^-[N]$ is $\mathcal{O}(r^{-4})$ even for a translation and $\mathcal{O}(r^{-3})$ odd for a boost. It means that

the minus pieces of these constraints are already finite. Moreover, $H^-[N]$ is also functionally differentiable since its action on the canonical variables is

$$\begin{aligned}\delta_{N^-} A^l_c &:= \{H^-[N], A^l_c(x)\} = -2N\epsilon_{ilk}\epsilon_{imn}A^m_c A^n_b E^b_k \\ \delta_{N^-} E^c_l &:= \{H^-[N], E^c_l(x)\} = 2N\epsilon_{ijk}\epsilon_{iln}E^c_j E^b_k A^n_b\end{aligned} \tag{6.27}$$

using which one observes

$$\begin{aligned}\delta H^-[N] &= \int d^3x\ N\epsilon_{jkl}(\delta F^{j-}_{ab}E^a_k E^b_l + F^{j-}_{ab}\delta E^a_k E^b_l + F^{j-}_{ab}E^a_k \delta E^b_l) \\ &= \int d^3x\ (\delta A^l_c(2N\epsilon_{ijk}\epsilon_{iln}E^c_j E^b_k A^n_b) + \delta E^c_l(2N\epsilon_{ilk}\epsilon_{imn}A^m_c A^n_b E^b_k)) \\ &= \int d^3x\ (\delta A^l_c(\delta_{N^-}E^c_l) - \delta E^c_l(\delta_{N^-}A^l_c))\end{aligned} \tag{6.28}$$

showing that $H^-[N]$ is differentiable. Consequently, what need to be modified are $H^+[N] = C[N]$ and $H^+_a[N^a] = C_a[N^a]$, thus all failure to be well-defined is rooted in the $U(1)^3$ part of the Hamiltonian and diffeomorphism constraints existing in the $SU(2)$ case. So, as long as finding the source of divergence and non-differentiability is concerned calculations are the same in both cases. This brings us back to (6.24) and (6.19) for boosts and rotations respectively.

The source of divergence for boosts is

$$\begin{aligned}-\int d^3x\ \beta_a\epsilon_{jkl}A^j_b\delta^a_k E^b_l &= \int d^3x\ (\beta_a\delta^a_k)G^-_k = \int d^3x\ \bar{\Lambda}^k_B(G_k - G^+_k) \\ &= G_k[\bar{\Lambda}^k_B] - \int d^3x\ \partial_b(\bar{\Lambda}^k_B E^b_k) \\ &= G_k[\bar{\Lambda}^k_B] - \oint dS_b\bar{\Lambda}^k_B E^b_k\end{aligned} \tag{6.29}$$

where we used $\partial_b\bar{\Lambda}^k_B = 0$. As expected, the volume term is proportional to G_k and does not affect the constraint surface. One can check that $J[N]$ in (6.12) is the final well-defined generator. This is precisely the crux of the matter: in the $U(1)^3$ model, the absence of G^-_k is responsible for the exclusion of a well-defined boost generator.

Let's move on to (6.19) and try to get rid of it. We see

$$\begin{aligned}\int d^3x\ A^j_b\beta^b_a\delta^a_j &= \int d^3x\ A^j_b(\epsilon_{ijk}\delta^b_k\bar{\Lambda}^i_R) \\ &= \int d^3x\ A^j_b(\epsilon_{ijk}E^b_k\bar{\Lambda}^i_R) - \int d^3x\ A^j_b(\epsilon_{ijk}\frac{f^b_k}{r}\bar{\Lambda}^i_R) + \text{finite} \\ &= \int d^3x\ \bar{\Lambda}^i_R G^-_i + \text{finite} \\ &= \int d^3x\ \bar{\Lambda}^i_R(G_i - G^+_i) + \text{finite} \\ &= G_i[\bar{\Lambda}^i_R] - \oint dS_a\ \bar{\Lambda}^i_R E^a_i + \text{finite}\end{aligned} \tag{6.30}$$

where we used $\partial_a\bar{\Lambda}^i_R = 0$ and omitted the second integral in the second line because it is $\mathcal{O}(r^{-3})$ odd. As desired, again, the volume term is proportional to the Gauß constraint. It is straightforward to see that $J_a[N^a]$ is the well-defined generator for the spatial translations and rotations. As in (6.29), the presence of G^-_i (which is absent in the case of $U(1)^3$) plays a crucial role in obtaining the generator.

To the technical argument we may add another intuitive one:
The $SU(2)$ Gauß constraint generates rotations on the internal tangent space corresponding to the internal indices $j, k, l, \ldots$, while asymptotic rotations act on the spatial tangent space associated with the indices $a, b, c, \ldots$. Due to the boundary conditions $E^a_j \propto \delta^a_j$, these tangent spaces are identified in leading order, so it is not surprising that one can "undo" an unwanted asymptotic rotation by an internal one. In the case of $U(1)^3$, this cannot work because the Gauß constraint does not generate internal rotations.

6.4. Generators of Asymptotic Symmetries for the $U(1)^3$ Model in the (B, f) Description

In this section, we consider two boundary conditions. The first is simply a transcription of the boundary conditions (6.7) and (6.8), while for the other we use the reverse parity conditions. We will see in chapter 8 that the latter is beneficial to achieve a simple physical Hamiltonian in the process of reduced phase space quantisation in the (B, f) description [89].

6.4.1. Standard Parity Conditions

Since $E^a_i = \delta^a_i + \epsilon^{abc}\partial_b f^i_c$ and $B^a_i = \epsilon^{abc}\partial_b A^i_c$, transcription of the boundary conditions (6.7) and (6.8) applicable to (A, E) to the (B, f) variables yields

$$\begin{aligned} f^i_a &= c^i_a + \bar{F}^i_a + \mathcal{O}(r^{-1}) \\ B^a_i &= \frac{\bar{G}^a_i}{r^3} + \mathcal{O}(r^{-4}) \end{aligned} \tag{6.31}$$

where c^i_a are constants and $\bar{F}^i_a$ and $\bar{G}^a_i$ are tensor fields defined on the asymptotic 2-sphere with the following certain parity conditions.

$$\bar{F}^i_a\left(-\frac{x}{r}\right) = -\bar{F}^i_a\left(\frac{x}{r}\right), \quad \bar{G}^a_i\left(-\frac{x}{r}\right) = \bar{G}^a_i\left(\frac{x}{r}\right). \tag{6.32}$$

which are a direct consequence of the definite parities of the leading terms of A and E (recall that the former is odd and the latter is even). By the fall-off conditions (6.31) and (6.32), we ensure that the symplectic structure is well-defined.

6.4.1.1. Bianchi Constraint

Recall that the smeared version of the Bianchi constraint reads

$$C_i[\Lambda^i] = \int d^3x\, \Lambda^i \partial_a B^a_i \tag{6.33}$$

Since $\partial_a B^a_i$ decays as $\mathcal{O}(r^{-4})$ odd, the minimal condition for the multiplier Λ^i that ensures the convergence of the integral is

$$\Lambda^i = \lambda^i r + \mathcal{O}(1) \tag{6.34}$$

where λ^i are even functions defined on the asymptotic S^2, i.e.

$$\lambda^i\left(-\frac{x}{r}\right) = \lambda^i\left(\frac{x}{r}\right) \tag{6.35}$$

The Bianchi constraint acts on the phase-space variables as

$$\begin{aligned}\delta_\Lambda f_a^j &= \{C_i[\Lambda^i], f_a^j\} = -\partial_a \Lambda^j \\ \delta_\Lambda B_j^a &= \{C_i[\Lambda^i], B_j^a\} = 0\end{aligned} \tag{6.36}$$

Thus, the Bianchi constraint is functionally differentiable, because

$$\begin{aligned}\delta C_j[\Lambda^j] &= \int d^3x\ \Lambda^j \partial_a \delta B_j^a = \oint dS_a\ \Lambda^j \delta B_j^a - \int d^3x\ (\partial_a \Lambda^j)\delta B_j^a = -\int d^3x\ (\partial_a \Lambda^j)\delta B_j^a \\ &= \int d^3x\ \left[(\delta_\Lambda f_a^j)\delta B_j^a - (\delta_\Lambda B_j^a)\delta f_a^j\right]\end{aligned} \tag{6.37}$$

Here we dropped the surface term since $\delta B_j^a = \mathcal{O}(r^{-3})$ even and $\Lambda^j = \mathcal{O}(r)$ even.

6.4.1.2. Vector Constraint

The action of the vector constraint the canonical variables is

$$\begin{aligned}\delta_{\vec{N}} f_c^i(x) =& \{C_a[N^a], f_c^i(x)\} = \epsilon_{abc} N^a (\delta_i^b + \epsilon^{bde}\partial_d f_e^i) \\ \delta_{\vec{N}} B_i^c(x) =& \{C_a[N^a], B_i^c(x)\} = \epsilon_{abc}\epsilon^{bde}\partial_d (N^a B_i^c)\end{aligned} \tag{6.38}$$

So, its variation reads

$$\begin{aligned}\delta C_a[N^a] &= \int d^3x\ N^a \left(\epsilon_{abc}\delta B_j^c(\delta_j^b + \epsilon^{bde}\partial_d f_e^j) + \epsilon_{abc}\epsilon^{bde} B_j^c \partial_d \delta f_e^j\right) \\ &= \int d^3x\ \left(\epsilon_{abc} N^a \delta B_j^c(\delta_j^b + \epsilon^{bde}\partial_d f_e^j) - \epsilon_{abc}\epsilon^{bde}\partial_d (N^a B_j^c)\delta f_e^j\right) + \oint dS_d\ \epsilon_{abc}\epsilon^{bde} N^a B_j^c \delta f_e^j \\ &= \int d^3x\ \left(\delta B_j^a \left[\delta_{\vec{N}} f_a^j\right] - \delta f_a^j \left[\delta_{\vec{N}} B_j^a\right]\right)\end{aligned} \tag{6.39}$$

Here we put away the surface integral since it is $\mathcal{O}(1)$ odd for a rotation and $\mathcal{O}(r^{-1})$ even for a translation. Therefore, the vector constraint

$$C_a[N^a] = \int d^3x\ \epsilon_{abc} N^a B_j^c (\delta_j^b + \epsilon^{bde}\partial_d f_e^j) \tag{6.40}$$

is functionally differentiable but not convergent because it is $\mathcal{O}(r^{-1})$ even for a translation and $\mathcal{O}(1)$ odd for a rotation. Consequently, $C_a[N^a]$ is not a well-defined generator for translations, nor for rotations!

6.4.1.3. Scalar Constraint

It is easy to see that

$$\begin{aligned}\delta_N f_c^i(x) &= \{C[N], f_c^i(x)\} = \epsilon_{ikl}\epsilon_{abc} N (\delta_k^a + \epsilon^{ade}\partial_d f_e^k)(\delta_l^b + \epsilon^{bfg}\partial_f f_g^l) \\ \delta_N B_i^c(x) &= \{C[N], B_i^c(x)\} = 2\epsilon_{jil}\epsilon_{abe}\epsilon^{adc}\delta f_c^i \partial_d \left(N B_j^e[\delta_l^b + \epsilon^{bfg}\partial_f f_g^l]\right)\end{aligned} \tag{6.41}$$

Thus the variation of this constraint is

$$\begin{aligned}\delta C[N] &= \int d^3x\ \epsilon_{jkl} N \left(\epsilon_{abc}\delta B^c_j(\delta^a_k + \epsilon^{ade}\partial_d f^k_e)(\delta^b_l + \epsilon^{bfg}\partial_f f^l_g) + 2\epsilon_{abc}B^c_j(\delta^b_l + \epsilon^{bfg}\partial_f f^l_g)(\epsilon^{ade}\partial_d \delta f^k_e)\right) \\ &= \int d^3x\ \epsilon_{jkl}\left(\epsilon_{abc}N\delta B^c_j(\delta^a_k + \epsilon^{ade}\partial_d f^k_e)(\delta^b_l + \epsilon^{bfg}\partial_f f^l_g) - 2\epsilon_{abc}\epsilon^{ade}\delta f^k_e \partial_d\left(N B^c_j[\delta^b_l + \epsilon^{bfg}\partial_f f^l_g]\right)\right) \\ &\quad + 2\oint dS_d\ \epsilon_{abc}\epsilon^{ade} N B^c_j(\delta^b_l + \epsilon^{bfg}\partial_f f^l_g)\delta f^k_e \\ &= \int d^3x\ \left(\delta B^a_j(\delta_N f^j_a) - (\delta_N B^a_j)\delta f^j_a\right) \end{aligned} \tag{6.42}$$

where the surface integral vanishes because it is $\mathcal{O}(r^{-1})$ even for a translation and $\mathcal{O}(1)$ odd for a boost. Thus, the scalar constraint is functionally differentiable without requiring any modification, and now we need to check its finiteness. The Hamiltonian constraint in terms of (B, f) is

$$C[N] = \int d^3x\ \epsilon_{jkl} N \left(\epsilon_{abc}B^c_j(\delta^a_k + \epsilon^{ade}\partial_d f^k_e)(\delta^b_l + \epsilon^{bfg}\partial_f f^l_g)\right) \tag{6.43}$$

that is $\mathcal{O}(r^{-1})$ even for a translation and $\mathcal{O}(1)$ odd for a boost, both of which are divergent. Therefore, neither boosts nor translations have well-defined generators!

It is worth noting that one cannot express the source of the divergences in (6.40) and (6.43) in terms of constraints, so the problem is not cured.

6.4.2. Boundary Conditions with Opposite Parities

Now consider the opposite parity conditions

$$\bar{F}^i_a\left(-\frac{x}{r}\right) = \bar{F}^i_a\left(\frac{x}{r}\right), \quad \bar{G}^a_i\left(-\frac{x}{r}\right) = -\bar{G}^a_i\left(\frac{x}{r}\right). \tag{6.44}$$

It is obvious that with these new parity conditions, the symplectic structure is still well-defined. The Bianchi constraint is

$$C_i[\Lambda^i] = \int d^3x\ \Lambda^i \partial_a B^a_i \tag{6.45}$$

According to the new boundary conditions $\partial_a B^a_i$ fall off as $\mathcal{O}(r^{-4})$ even, thus the minimal condition on the multiplier Λ^i that ensures the convergence of the Bianchi constraint is the same as (8.31), except that λ^i must be an odd function, i.e.

$$\lambda^i\left(-\frac{x}{r}\right) = -\lambda^i\left(\frac{x}{r}\right) \tag{6.46}$$

Also in this case $C_j[\Lambda]$ is functionally differentiable because the surface term in (6.37) vanishes (recall $\delta B^a_j = \mathcal{O}(r^{-3})$ odd and $\Lambda^j = \mathcal{O}(r)$ odd).

In this case $C_a[N^a]$ is functionally differentiable since the surface term in (6.39) is $\mathcal{O}(1)$ odd for a rotation and $\mathcal{O}(r^{-1})$ even for a translation and thus vanishes. It is also convergent for a translation but not for a rotation, since (6.40) is $\mathcal{O}(r^{-1})$ odd for a translation and $\mathcal{O}(1)$ even for a rotation. Therefore $C_a[N^a]$ is a well-defined generator only for translations, but not for rotations.

Again, $C[N]$ is functionally differentiable since the surface integral appearing in (6.42) simply

vanishes because it is again $\mathcal{O}(r^{-1})$ even for a translation and $\mathcal{O}(1)$ odd for a boost. But in contrast to what we observed in the previous case, by looking at (6.43) we find that it is $\mathcal{O}(r^{-1})$ odd for a translation and $\mathcal{O}(1)$ even for a boost. Consequently, translations have a well-defined generator $C[N]$, but boosts do not!

The question that arises here is why, even though the boundary conditions carry over exactly into the (B, f) description, the use of standard parity conditions leads to well-defined generators in the (A, E) description but not in the (B, f) description. To identify the source of this discrepancy, we first consider a general situation where we have canonical variables (A, E) and a functional $\mathcal{F}(A, E)$ that depends on A only via $\partial_a A$. For simplicity, we have omitted all indices of the fields.
The variation of $\mathcal{F}[\lambda]$, where λ is a test function, would have the form

$$\delta\mathcal{F}[\lambda] = \int d^3x \ (\mathcal{A}\partial_a\delta A + \mathcal{B}\delta E) = \int d^3x \ (-(\partial_a\mathcal{A})\delta A + \mathcal{B}\delta E) + \oint dS_a \mathcal{A}\delta A \tag{6.47}$$

where $\mathcal{A}$ and $\mathcal{B}$ are simply understood as the coefficients of $\partial_a\delta A$ and δE in the variation of $\mathcal{F}[\lambda]$, respectively. If one wants to change the canonical variables (A, E) to (B, f), where $B = \partial_a A$ and $E = c + \partial_a f$ (c is constant), then the variation of $\mathcal{F}$ would become

$$\delta\mathcal{F}[\lambda] = \int d^3x \ (\mathcal{A}\delta B + \mathcal{B}\partial_a\delta f) = \int d^3x \ (\mathcal{A}\delta B - (\partial_a\mathcal{B})\delta f) + \oint dS_a \mathcal{B}\delta f \tag{6.48}$$

here $\mathcal{A}, \mathcal{B}$ are written in terms of (B, f).
From (6.47) and (6.48), it is evident that $\mathcal{F}$ would be functionally differentiable in terms of both canonical variables if we were working with compact space. But if we want to consider boundary conditions, things look quite different! Looking at the surface terms in (6.47) and (6.48), we find that they have nothing in common! So it is very likely that taking into account boundary conditions that make (6.47) functionally differentiable will cause (6.48) to stay ill-defined and vice versa.

This is exactly what occurs during the analysis of the differentiability of the Hamiltonian and diffeomorphism constraints in the $U(1)^3$ theory. Note that the Hamiltonian constraint, i.e. $C = 2\epsilon_{jkl}(\partial_a A^j_b)E^a_k E^b_l$, depends on A only via $\partial_a A$. Hence,

$$\begin{aligned}
\delta C[N] &= \int d^3x \ \left(2N\epsilon_{jkl}(\partial_a\delta A^j_b)E^a_k E^b_l + 4N\epsilon_{jkl}(\partial_a A^j_b)E^a_k\delta E^b_l\right) \\
&=: \int d^3x \ \left(\mathcal{A}^{ab}_j(\partial_a\delta A^j_b) + \mathcal{B}^l_b\delta E^b_l\right) &(6.49)\\
&= \int d^3x \ \left(-(\partial_a\mathcal{A}^{ab}_j)\delta A^j_b + \mathcal{B}^l_b\delta E^b_l\right) + \oint dS_a\mathcal{A}^{ab}_j\delta A^j_b &(6.50)
\end{aligned}$$

where $\mathcal{A}^{ab}_j := 2N\epsilon_{jkl}E^a_k E^b_l$ and $\mathcal{B}^l_b := 4N\epsilon_{jkl}(\partial_a A^j_b)E^a_k$.
Now let us express $\delta C[N]$ in terms of (B, f) where $B^a_i = \epsilon^{abc}\partial_b A^i_c$ and $E^a_i = \delta^a_i + \epsilon^{abc}\partial_b f^i_c$. We initiate from (6.49) and write everything in terms of (B, f)

$$\begin{aligned}
\delta C[N] &= \int d^3x \ \left(\mathcal{A}^{ab}_j(\partial_a\delta A^j_b) + \mathcal{B}^l_b\delta E^b_l\right) \\
&= \int d^3x \ \left(\frac{1}{2}\mathcal{A}^{ab}_j\epsilon_{cab}\delta B^c_j + \mathcal{B}^l_b\epsilon^{bac}(\partial_a\delta f^l_c)\right) \\
&= \int d^3x \ \left(\frac{1}{2}\mathcal{A}^{ab}_j\epsilon_{cab}\delta B^c_j - (\partial_a\mathcal{B}^l_b)\epsilon^{bac}\delta f^l_c\right) + \oint dS_a \ \mathcal{B}^l_b\epsilon^{bac}\delta f^l_c &(6.51)
\end{aligned}$$

Using the standard boundary conditions where $E^a_i - \delta^a_i = \mathcal{O}(r^{-1})$even, $A^i_a = \mathcal{O}(r^{-2})$odd and consequently $B^a_i = \mathcal{O}(r^{-3})$even, $f^i_a = \mathcal{O}(1)$odd, one concludes that for translations $\mathcal{A}^{ab}_j =$ constant $+ \mathcal{O}(r^{-1})$even and $\mathcal{B}^l_b = \mathcal{O}(r^{-3})$even.
Therefore, the surface terms in (6.50) and (6.51) are

$$\oint dS_a \mathcal{A}^{ab}_j \delta A^j_b = \oint dS_a \, \mathcal{O}(r^{-2})\text{odd} = \text{divergent} \tag{6.52}$$

$$\oint dS_a \, \mathcal{B}^l_b \epsilon^{bac} \delta f^l_c = \oint dS_a \, \mathcal{O}(r^{-3})\text{odd} = 0 \tag{6.53}$$

Thus, $C[N]$ is functionally differentiable in the (B, f) description but not in the (A, E) description!
Thus, to obtain a well-defined generator for the asymptotic temporal translations, we need to add to the Hamiltonian constraint a term that eliminates the divergence that occurs in its variation. In this way, we obtain a functionally differentiable expression which, if we are lucky, is already finite. In the (B, f) description, on the other hand, the variation of $C[N]$ is already functionally differentiable, since there is no such divergence, and the only factor that makes the Hamiltonian constraint ill-defined is its own divergence. Because the source of the divergence cannot be expressed in terms of the constraints, we are not allowed to subtract it from the Hamiltonian constraint, therefore $C[N]$ remains ill-defined in the (B, f) description. The same happens for the diffeomorphism constraint.

Part III.

Different Approaches to Quantise the $U(1)^3$ Model

Dirac Quantisation of the U(1)^3 model

7.1. Preliminaries

7.1.1. Lewandowski-Marolf Habitat

As we have seen in section 4.2.3, employing the URS operator topology defined by the space of diffeomorphism-invariant distributions, Thiemann's Hamiltonian constraint that is densely defined on the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$, closes off-shell but with the wrong structure functions. The source of this lies in the fact that the diffeomorphism constraint on the r.h.s. of the quantum version of (1.35) annihilates $\mathcal{H}_{\text{diff}}$, and hence as soon as the Hamiltonian constraints commute among each other, the constraints algebra is satisfied. Consequently, verification of the precise relation (1.35) is not really feasible since the r.h.s. always vanishes on the diffeomorphism-invariant distributions. Hoping to overcome this problem, in [248, 244], a totally different construction was suggested. There the Hamiltonian operator is defined on a subspace, known as a *habitat*, of the algebraic dual of $\mathcal{H}_{\text{kin}}$ including $\mathcal{H}_{\text{diff}}$ and the limit is taken differently. However, this yields vanishing (and thus also wrong) structure functions.
To see how it is defined, first recall from section 1.3.5.2 that the construction of $\mathcal{H}_{\text{diff}}$ begins with applying the group averaging method on a (gauge-invariant) spin network to obtain a diffeomorphism-invariant state. From (1.84), it turned out that a general diffeomorphism invariant state is a linear combination of the states $\ell_{[s]} = \sum_{s'\in[s]}\langle T_{s'}, \cdot\rangle_{\text{kin}}$. Now, lets dive into the generalisation of this construction. Consider any complex valued function f on n copies of the spatial hypersurface Σ i.e. $f : \Sigma^n \to \mathbb{C}$. Let $\{v_1(s), \cdots, v_n(s)\}$ which is a point belonging to Σ^n be the vertices of $[s]$. Then, the *habitat state* $\ell_{[s],f}$ is defined as

$$\ell_{[s],f} := \sum_{s'\in[s]} \langle T_{s'}| f(v_1(s'), \cdots, v_n(s'))| \cdot \rangle \tag{7.1}$$

If the weight function f, which is called a *vertex function*, is constant we recover (up to a factor) the previous construction of a diffeomorphism-invariant state. But, since a non-constant f is sensitive to diffeomorphisms, with a suitable set of functions, we enlarge $\mathcal{H}_{\text{diff}}$ into a habitat that can yield non-trivial things. Otherwise stated, habitat states are deformations of diffeomorphism invariant ones by the vertex function f [245]. The finite span of all states $\ell_{[s],f}$ is called the *Lewandowski-Marolf habitat*.

Let $\varphi_t : \sigma \to \sigma$, $t \in \mathbb{R}$ be a one-parameter family of diffeomorphisms generated by a vector field ξ^a. Using the dual action of $\hat{U}(\varphi_t)$, we have

$$
\begin{aligned}
&\left[\lim_{t\to 0}\frac{1}{t}\left(\hat{U}(\varphi_t)-1\right)\ell_{[s],f}\right]T_{\bar{s}}\\
&\qquad=\lim_{t\to 0}\frac{1}{t}\left[\ell_{[s],f}\left(\hat{U}(\varphi_{-t})T_{\bar{s}}\right)-\ell_{[s],f}(T_{\bar{s}})\right]\\
&\qquad=\lim_{t\to 0}\frac{1}{t}\sum_{s'\in[s]}\left(\langle T_{\varphi_t(s')}|f(\varphi_t(v_1),\cdots,\varphi_t(v_n))|T_{\bar{s}}\rangle-\langle T_{s'}|f(v_1,\cdots,v_n)|T_{\bar{s}}\rangle\right)\\
&\qquad=\lim_{t\to 0}\frac{1}{t}\sum_{s'\in[s]}\left(\langle T_{s'}|f(\varphi_t(v_1),\cdots,\varphi_t(v_n))|T_{\bar{s}}\rangle-\langle T_{s'}|f(v_1,\cdots,v_n)|T_{\bar{s}}\rangle\right)\\
&\qquad=\sum_{s'\in[s]}\langle T_{s'}|\left(\lim_{t\to 0}\frac{1}{t}\left[f(\varphi_t(v_1),\cdots,\varphi_t(v_n))-f(v_1,\cdots,v_n)\right]\right)|T_{\bar{s}}\rangle\\
&\qquad=\sum_{s'\in[s]}\langle T_{s'}|\mathcal{L}_\xi f(v_1,\cdots,v_n)|T_{\bar{s}}\rangle\\
&\qquad=\ell_{[s],\mathcal{L}_\xi f}(T_{\bar{s}})
\end{aligned}
\tag{7.2}
$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ. As $\ell_{[s],\mathcal{L}_\xi f}$ is still a habitat state with the new vertex function $\mathcal{L}_\xi f$, (7.2) reveals that the generator of spatial diffeomorphisms, i.e. $D_a[\xi^a] := \lim_{t\to 0}\frac{1}{t}\left(\hat{U}(\varphi_t)-1\right)$, does exist in the habitat. It is obvious from (7.2) that the kernel of $D_a[\xi^a]$, as expected, consists of those habitat states whose vertex functions are constant, that are the diffeomorphism invariant states.

7.1.2. Changing the Density Weight

In [57] the r.h.s. of (1.35) with density weight one Hamiltonian constraints was quantised. This was done by replacing partial derivatives by discrete differences and infinitesimal diffeomorphisms by finite ones and taking the limit by the URS topology. The same technique was followed in [85] where even the structure constants came out correctly, also with density weight one. In fact, density weight one is the whole beauty of gravity as it regulates infinities and makes all constructions "automatically" diffeomorphism covariant.

However, there is another proposal, working with different density weights, that has been introduced with the aim of achieving the precise realisation of (1.35). Its main idea can be readily explained as follows. The smeared Hamiltonian constraint, that is $C[N]=\int_\Sigma d^3x\, N^{[1-k]}(x)C^{[k]}(x)$, consists of the smearing function N that is a scalar density of weight $(1-k)$ and the local Hamiltonian constraint $C^{[k]}(x)=q^{(k-2)/2}FEE(x)$ with density weight k, where $q:=\det(q)$. Here, density weights are denoted by superscripts in square brackets. Recall from section 4.2 that the "fineness" of the triangulation is determined by a parameter ϵ and the quantisation of Hamiltonian constraint $H[N]$ proceeds by first approximating the integral by a Riemann sum and then approximating local fields involved in it by holonomies and fluxes, see (4.10). Recalling chapter 4, one immediately convinces oneself that $F\sim O(\epsilon^{-2})$, $E\sim O(\epsilon^{-2})$, $A\sim O(\epsilon^{-1})$, $\sqrt{q}\sim O(\epsilon^{-3})$ and $d^3x\sim O(\epsilon^3)$ by means of which the expansion of $C[N]$ in powers of the regulating parameter

ϵ can be derived as [287]

$$\begin{aligned} C[N] &= \int_\Sigma d^3x\ N^{[1-k]}(x) C^{[k]}(x) \\ &= \int_\Sigma d^3x\ N^{[1-k]}(x) \sqrt{q}^{(k-2)} FEE(x)(x) \\ &\sim \sum_{\Delta\in\mathfrak{T}(\epsilon)} (\epsilon^{-1})^{3(k-1)}\ N^{[1-k]}(v(\Delta)) \bar{C}_\Delta(v(\Delta)) \end{aligned} \tag{7.3}$$

The equation (7.3) says that for density weight one Hamiltonian constraint, i.e. when $k = 1$, there is no dependence on ϵ in the expression of $\hat{C}_\epsilon[N]$. Therefore, the commutator $[\hat{C}_\epsilon[N], \hat{C}_\epsilon[M]]$ does not depend on ϵ either, meaning that the quantised version of the r.h.s of (1.35) can never be obtained because $\int d^3x\ F^i_{ab} E^b_i [q^{ac}(N\partial_c M - M\partial_c N)] \sim O(\epsilon)$. In fact, in the continuum limit $\epsilon \to 0$, the r.h.s of (1.35) vanishes irrespective of its finer structure. As the vanishing of the r.h.s of (1.35) is a direct consequence of setting $k = 1$, this observation suggests that working with higher density constraints might provide an appropriate ground to get the precise quantum version of (1.35).

7.1.3. Purpose of This Chapter

In [57], it was shown that the quantum operator corresponding to the r.h.s of (1.35) is of the form $[\hat{U}(\varphi_\epsilon) - 1]\hat{O}$, where $\hat{O}$ is a finite operator. The presence of $\hat{U}(\varphi_\epsilon) - 1$ in this combination guarantees that the quantum version of the r.h.s of (1.35) annihilates all diffeomorphism-invariant states, just like as its l.h.s. Although this is the essential property of anomaly freeness, one still desires to be able to check the correctness of the exact algebraic form of (1.35). If this can be done, it may lead to some restrictions reducing quantisation ambiguities discussed in section 4.2.4.

The purpose of this chapter is to present the strategy of changing the density weight in the context of the $U(1)^3$ theory and discuss its shortcomings. The main idea of this strategy is as follows. Comparing (7.2) with $[\hat{U}(\varphi_\epsilon) - 1]\hat{O}$ and noting that the latter is of $O(\epsilon)$ brings to mind the idea that if the r.h.s of (1.35) is rescaled by the factor ϵ^{-2}, then the r.h.s can be saved from trivially being zero in the continuum limit $\epsilon \to 0$. It turns out that working with a density weight $\frac{4}{3}$ Hamiltonian constraint results in the desired additional factor ϵ^{-2}. A quick calculation shows that by setting $k = 4/3$ the Hamiltonian constraint $C^{[4/3]}[N] = \int d^3x\ N^{[-1/3]} q^{-1/3} FEE(x)$ is of $O(\epsilon^{-1})$ that blows up in the refinement limit. This seems to be a new trouble one has to deal with, but again by considering (7.2) one can envisage that if the action of the Hamiltonian constraint can be written as an expression proportional to $\hat{U}(\varphi_\epsilon) - 1$, then, on an appropriate habitat, the "seeming divergence" of the Hamiltonian constraint will be disappeared by changing the vertex function.

In order to test this strategy, one employs toy models. In this chapter, we mainly review the work [87] on the closure of the $U(1)^3$ constraint algebra, which was just meant as a "first" attack on the problem. In fact, the construction of the Hamiltonian constraint in [87] has some shortcomings such as lack of manifest gauge invariance. Those shortcomings were partially overcome in [88] where the construction is improved but is much more complex. For reasons of space, here we just review the simpler pioneering paper, i.e. we construct the quantum Hamiltonian constraint of density weight 4/3 for the $U(1)^3$ model and verify the precise realisation of (1.35). For more information on this perspective, we refer the interested reader to [287, 288, 289, 290].

7.2. Constraints in the Classical Theory

Recall that the constraints of the $U(1)^3$ model with $C^{[4/3]}$ are

$$G[\Lambda] = \int d^3x\ \Lambda^i \partial_a E^a_i, \quad C_a[N^a] = \int d^3x\ E^a_i \mathcal{L}_{\vec{N}} A^i_a, \quad C[N] = \frac{1}{2}\int d^3x\ \epsilon^{ijk} F^k_{ab} E^b_j N^a_i \tag{7.4}$$

where $N^a_i := \det(q)^{-1/3} N E^a_i$, for each i, is a vector field depending on the electric fields and we refer to it as the *electric shift vector field*. The structure function involved in the Poisson bracket between two Hamiltonian constraints turns out to be rescaled by $q^{-2/3}$, i.e.

$$\{C[N], C[M]\} = C_a[N^a(q,N,M)], \quad N^a(q,N,M) := q^{1/3} q^{ab}(M\partial_b N - N\partial_b M) \tag{7.5}$$

$$\begin{aligned}
C[N] &= \frac{1}{2}\int d^3x\ \epsilon^{ijk} F^k_{ab} E^b_j N^a_i \\
&= \frac{1}{2}\int d^3x\ \epsilon^{ijk}(\partial_a A^k_b - \partial_b A^k_a) E^b_j N^a_i \\
&= \frac{1}{2}\int d^3x\ \epsilon^{ijk}[N^a_i(\partial_a A^k_b) + A^k_a(\partial_b N^a_i) - (\partial_b N^a_i A^k_a)] E^b_j \\
&= \frac{1}{2}\int d^3x\ \epsilon^{ijk}[E^b_j(\mathcal{L}_{\vec{N}_i} A^k_b) + N^a_i A^k_a G_j - \partial_b(N^a_i A^k_a E^b_j)] \\
&\approx \frac{1}{2}\int d^3x\ \epsilon^{ijk}(\mathcal{L}_{\vec{N}_i} A^k_b) E^b_j
\end{aligned} \tag{7.6}$$

where we have used integration by parts and dropped the boundary term. As we will see later, the presence of the Lie derivative in the above relation causes a slight replacement in the direction of the flow generated by N^a_i on the vertices of the underlying graph of a quantum state , when $\hat{C}[N]$ is acting on it. As we want the second Hamiltonian acts on the replaced vertices, it will turn out that we should add a vanishing term to (7.6) as follows. Noting that F^i_{ab} is antisymmetric while $N^a_i E^b_i = \det(q)^{-1/3} N E^a_i E^b_i$ is symmetric in a, b, one immediately observes that $\sum_i N^a_i F^i_{ab} E^b_i = 0$. Therefore,

$$\begin{aligned}
0 &= \int d^3x\ \sum_i N^a_i F^i_{ab} E^b_i = \int d^3x\ \sum_i N^a_i(\partial_a A^i_b - \partial_b A^i_a) E^b_i \\
&= \int d^3x\ \sum_i \left(E^b_i N^a_i \partial_a A^i_b + A^i_a \partial_b(N^a_i E^b_i)\right) \\
&= \int d^3x\ \sum_i \left(E^b_i N^a_i \partial_a A^i_b + E^b_i A^i_a \partial_b(N^a_i) + A^i_a N^a_i G_i\right) \\
&\approx \int d^3x\ \sum_i E^b_i \mathcal{L}_{\vec{N}_i} A^i_b
\end{aligned} \tag{7.7}$$

where again we integrated by parts and dropped the boundary term. What we are going to construct its quantum counterpart in section 7.4 is

$$C'[N] := \frac{1}{2}\int d^3x\ \left(\epsilon^{ijk}(\mathcal{L}_{\vec{N}_i} A^k_b) E^b_j + \sum_i E^b_i \mathcal{L}_{\vec{N}_i} A^i_b\right) \tag{7.8}$$

that is classically equivalent to $C[N]$ but in quantum theory the classically vanishing term (7.7) certifies that, when the commutator between two Hamiltonian constraint operators is being computed, the second Hamiltonian constraint acts on a vertex moved by the first one [87]. It should

be noted that one of the criticisms of the approach used in [87] emanates from this point. In fact the triple sum over i in (7.7) would have no gauge-invariant generalisation to non-Abelian groups, and adding such zeros to the constraints introduces another infinite source of ambiguity. Smearing the diffeomorphism constraint with the electric shift vector fields results in

$$C_a[N_j^a] = \int d^3x\ E_i^a \mathcal{L}_{\vec{N}_j} A_a^i \tag{7.9}$$

that helps us to manage the quantum version of the r.h.s. of (7.5). In fact, a remarkable observation of [87] indicates that the r.h.s. of (7.5) is proportional to the trace of the 3×3 matrix formed by $C_a[N_i^a]$, i.e., $M_{ij} := \{C_a[N_i^a], C_b[M_j^b]\}$. The verification of this proposition is quite straightforward: (notice that in $\{C_a[N_i^a], C_b[M_i^b]\}$ there is a sum over i due to the Einstein summation convention)

$$\begin{aligned}
\{C_a[N_i^a], C_b[M_i^b]\} &= \int d^3x \int d^3y\ N(x)M(y)\{q^{-1/3}E_i^a E_j^c F_{ac}^j(x), q^{-1/3}E_i^b E_k^d F_{bd}^k(y)\} \\
&= \int d^3x \int d^3y\ \ (N(x)M(y) - N(y)M(x)) \times \\
&\qquad \Big[(E_i^a E_j^c F_{ac}^j)(x)\{q^{-1/3}(x), F_{bd}^k(y)\}(q^{-1/3}E_i^b E_k^d)(y) \\
&\qquad + (q^{-1/3}E_j^c F_{ac}^j)(x)\{E_i^a(x), F_{bd}^k(y)\}(q^{-1/3}E_i^b E_k^d)(y) \\
&\qquad + (q^{-1/3}E_i^a F_{ac}^j)(x)\{E_j^c(x), F_{bd}^k(y)\}(q^{-1/3}E_i^b E_k^d)(y)\Big] \\
&= \int d^3x \int d^3y\ \ (N(x)M(y) - N(y)M(x)) \times \\
&\qquad \Big[(E_i^a E_j^c F_{ac}^j)(x)\left(\frac{2}{3}q^{-1/3}E_{[d}^k \partial_{b]}\delta(x,y)\right)(q^{-1/3}E_i^b E_k^d)(y) \\
&\qquad + (q^{-1/3}E_j^c F_{c[d}^j)(x)\left(2\partial_{b]}\delta(x,y)\right)(q^{-1/3}E_i^b E_i^d)(y) \\
&\qquad + (q^{-1/3}E_i^a F_{a[d}^j)(x)\left(-2\partial_{b]}\delta(x,y)\right)(q^{-1/3}E_i^b E_j^d)(y)\Big] \\
&= \int d^3x\ [\ -\frac{2}{3}qq^{ab}q^{-2/3}E_j^c F_{ac}^j E_k^d E_{[d}^k (N\partial_{b]}M - M\partial_{b]}N) \\
&\qquad\qquad + 2qq^{ab}q^{-2/3}E_j^d F_{a[d}^j (N\partial_{b]}M - M\partial_{b]}N)\ ] \\
&= \int d^3x\ \left[\frac{1}{3}q^{1/3}q^{ab}E_j^c F_{ac}^j (N\partial_b M - M\partial_b N)\right] \\
&= -\frac{1}{3}C_a[N^a(q,N,M)]
\end{aligned} \tag{7.10}$$

It means that the Poisson bracket between two Hamiltonian constraints (7.5) can equivalently be written as

$$\{C[N], C[M]\} = -3\{C_a[N_i^a], C_b[M_i^b]\} \tag{7.11}$$

A quick review on the steps of the calculation (7.10) shows that if we worked with $N_i^a = Nq^{-1/2}E_i^a$ (i.e., if we worked with density weight 1 Hamiltonian constraint as usual) the commutator $\{C_a[N_i^a], C_b[M_i^b]\}$ would vanish and so it would not be equal to the r.h.s. of (7.5). As trivialisation of this commutator only occurs for density weight one Hamiltonian constraint, working with higher densities is backed up again.

7.3. Kinematical Hilbert Space

7.3.1. The Holonomy-Flux Algebra

Here we intend to introduce the kinematical Hilbert space of the $U(1)^3$ model. It is in complete analogy with the $SU(2)$ case whose detailed Dirac quantisation was presented in section 1.3. Hence, we do not repeat the details of similar structures and proofs, but merely refer to the same structure in section 1.3.

To begin with, for the same reasons mentioned in section 1.3.3.1, here again $U(1)^3$-holonomies along edges and electric fluxes through surfaces, i.e.

$$h_e(A) = \exp\left(i\sum_{j=1}^{3}\int_e A_a^j dx^a\right), \qquad E_i(S) = \int_S \epsilon_{abc}E_i^a dx^b \wedge dx^c \tag{7.12}$$

respectively, are the basic variables of our interest and the only non-trivial Poisson bracket among them is

$$\{h_e(A), E_i(S)\} = \frac{\sigma(S,e)}{2}h_e(A^i) \tag{7.13}$$

where $\sigma(S,e) = \pm 1, 0$ is the relative position of S and e like what we had in (1.47). The notions of a graph and cylindrical functions already presented in section 1.3 are used in the same way here. Going forward and employing the Haar measure of $U(1)^3$, we construct the inner product between the cylindrical functions as (1.56). As is obvious from (1.56), the Ashtekar-Lewandowski measure μ_{AL} is built from the $U(1)^3$ Haar measure, with respect to which the desired kinematical Hilbert space $\mathcal{H}_{\text{kin}} = L^2(\bar{\mathcal{A}}, d\mu_{AL})$ is set up.

7.3.2. Charge Network Functions

Following the Peter-Weyl theorem[1], in order to introduce an orthonormal basis for $\mathcal{H}_{\text{kin}}$, we need to delve into the representation theory of the group $U(1)^3$. Recall that as $U(1)$ is a commutative group, all irreducible representations are one dimensional[2] and they are given by

$$\begin{aligned} \pi_k :& U(1) \to GL(1,\mathbb{C}) \simeq \mathbb{C}^* \\ & e^{i\theta} \mapsto \pi_k(e^{i\theta}) := e^{ik\theta} \end{aligned} \tag{7.14}$$

for $k \in \mathbb{Z}$. Therefore, representations of the $U(1)$-holonomy along the edge e are simply given by

$$h_{e,q}(A) := \pi_q(h_e(A)) = e^{iq\int_e A_a dx^a} \tag{7.15}$$

and the product of three copies of them is considered as a representation of the $U(1)^3$-holonomy along the edge, i.e.

$$h_{e,\vec{q}}(A) := \prod_{j=1}^{3} h_{e,q_j}(A^j) = e^{iq_j\int_e A_a^j dx^a} \tag{7.16}$$

In principle, the holonomy in (7.12) has been stated in the fundamental representation. Since here the gauge group is $U(1)^3$ and not $SU(2)$, instead of spin networks we have *charge networks*

[1] See the footnote (17) in Chapter 1.

[2] For G commutative, $g \in G$, any representation will satisfy $\pi(h)\pi(g) = \pi(g)\pi(h)$ for all $h \in G$. If π is irreducible, Schur's lemma implies that, since they commute with all the $\pi(g)$, the matrices $\pi(h)$ are all scalar matrices, i.e., $\pi(h) = \lambda_h \mathbb{I}_G$ for some $\lambda_h \in \mathbb{C}$. π is then irreducible when it is the one-dimensional representation given by $\pi(h) = \lambda_h$.

that are defined as colored graphs labelled by representations of $U(1)^3$, i.e. by a triplet of integer valued charges $\vec{q}_I := (q_I^1, q_I^2, q_I^3)$ for the edge e_I.

Given an oriented graph γ, a *charge network function* (CNF) that form an orthogonal basis for the kinematical Hilbert space is defined, in the same way we had already introduced in (1.61), as

$$\begin{aligned} T_{\gamma,\{\vec{q}\}} : \bar{\mathcal{A}} \to \mathbb{C}; \qquad A \mapsto \prod_{e\in E(\gamma)} [\pi_{\vec{q}}(h_e(A))] &= e^{i\sum_{I=1}^{|E(\gamma)|} q_j^I \int_{e_I} A_a^j dx^a} \\ &= e^{\int d^3x \left(\sum_{I=1}^{|E(\gamma)|} i q_j^I \int_0^1 dt_I \delta^{(3)}(e_I(t_I),x)\dot{e}_I^a(t_I)\right) A_a^j(x)} \\ &=: e^{\int d^3x \left(\sum_{I=1}^{|E(\gamma)|} c_j^{aI}(x)\right) A_a^j(x)} \\ &=: e^{\int d^3x\, c_j^a(x) A_a^j(x)} \end{aligned} \tag{7.17}$$

where e_I is parametrised by t_I and as is obvious from its definition, c_j^a depends on the edges $\{e_I\}$ and the charges $\{q_j^I\}$. To find charge network functions which are invariant under $U(1)^3$ gauge transformations, note that $T_{\gamma,\vec{q}}$ will be $U(1)^3$-gauge-invariant if only if it is $U(1)_i$-gauge-invariant for each i separately. We recall that under a transformation specified by the gauge function $g(x) \in U(1)_i$, the holonomy transforms as $h_e(A^{i(g)}) = g(f(e))h_e(A^i)g(b(e))^{-1} =^3 g(f(e))g(b(e))^{-1}h_e(A^i)$ using which we can consider how a gauge transformation affects a generic CNF (7.17). Let us focus on a specific vertex v of γ. From the gauge transformation law of a holonomy, we observe that each edge ending at v contributes a $\pi_{q^j}(g(v)) = \pi_{q^j}(e^{i\theta_v}) = e^{iq^j\theta_v}$, while each edge starting from v contributes a $\pi_{q_j}(g(v))^{-1} = \pi_{q^j}(e^{i\theta_v})^{-1} = e^{-iq^j\theta_v}$, where $g(v) := e^{i\theta_v}$ is the value of the gauge function at the vertex. Taking this observation into account, one concludes that in the gauge transformation of T_{γ,q^j} the contribution coming from the vertex v is $(T_{\gamma,q^j})_v^{(g)} := e^{i\sum_{e_{I_v}} \tau(e_{I_v}) q^j_{e_{I_v}}}$, where e_{I_v} denotes the edges incident at v and $\tau(e_{I_v})$ is $+1$ for outgoing edges and -1 ingoing ones. Thus, $(T_{\gamma,q^j})_v^{(g)} = 1$ is equivalent to being invariant under the $U(1)_j$-gauge transformation. Consequently, the CNF (7.17) is invariant under $U(1)^3$ gauge transformations if and only if

$$\sum_{I_v} \tau(e_{I_v}) q_j^{I_v} = 0, \qquad \forall v \in V(\gamma), \quad \forall j \in \{1,2,3\} \tag{7.18}$$

From now on, we denote a gauge invariant CNF by T_c, similar to T_s for SNFs in chapter 1, in which c is a *charge network label* consisting of the graph γ and the triplets $(q_j^I)_{j=1,2,3}$ assigned to each $e_I \in E(\gamma)$. c is sometimes called a *colored graph*, meaning a graph colored by $(q_j^I)_{j=1,2,3}$ (see Figure 7.1).

Two CNFs T_c and $T_{c'}$ are orthogonal with respect to the inner product of $\mathcal{H}_{\text{kin}}$ if their charge network labels differ, i.e. either their underlying graphs or the labels on their edges differ. The inner product between these CNFs is denoted by $\langle T_c, T_{c'}\rangle = \delta_{cc'}$ where $\delta_{cc'}$ is 1 only when c is identical to c', otherwise it vanishes. In fact, $\mathcal{H}_{\text{kin}}$ is the Cauchy completion of the finite span of the CNFs with respect to the just mentioned inner product $\langle\cdot,\cdot\rangle$.

7.3.3. Quantum Operators

The quantum operator corresponding to the holonomy $h_{e,\vec{q}}(A)$ acts on a CNF, say T_c, as

$$\hat{h}_{e_J,\vec{q}_0}(A)|T_c\rangle = |T_{\tilde{c}}\rangle \tag{7.19}$$

[3] Note that the gauge group is Abelian

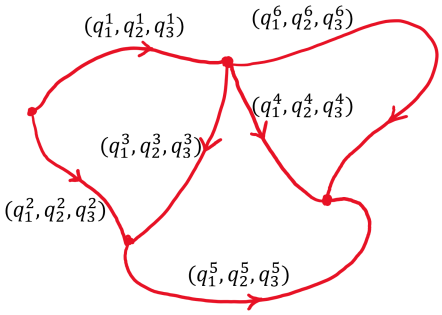

Figure 7.1.: An example of charge network label; All q_j^I's are integer, in which $I \in \{1, \cdots, |E(\gamma)|\}$ and $j \in \{1, 2, 3\}$.

where $T_{\tilde{c}}$ is defined as:
If $e_J \in E(\gamma)$, where γ is the underlying graph of $c = \left\{\gamma, \left((q_I^j)_{j\in\{1,2,3\}}\right)_{I\in\{1,\cdots,|E(\gamma)|\}}\right\}$, then

$$\tilde{c} := \left\{\gamma, \left((q_1^j)_{j\in\{1,2,3\}}, \cdots, (q_J^j + q_0^j)_{j\in\{1,2,3\}}, \cdots, (q_{|E(\gamma)|}^j)_{j\in\{1,2,3\}}\right)\right\} \tag{7.20}$$

If e_J is not contained in γ, then the action of the holonomy operator adds the new edge e_J to γ, i.e. the underlying graph of $T_{\tilde{c}}$ is $\gamma \cup \{e_J\}$ (recall section 1.3.4.1).

Considering (7.13) and using the Leibniz rule, one finds that the action of the quantum operator corresponding to the electric flux $E_i(S)$ on T_c is

$$\hat{E}_i(S)|T_c\rangle = \sum_{e_I\in E(\gamma)} \frac{\sigma(S, e_I)}{2} q_I^i |T_c\rangle \tag{7.21}$$

showing also that the flux operator $\hat{E}_i(S)$ is diagonal in the charge network basis T_c. In fact, the operators (7.19) and (7.21) equip $\mathcal{H}_{\text{kin}}$ with a representation of the holonomy-flux Poisson bracket algebra. As expected, it is obvious from their forms that the holonomy operator acts on a CNF by multiplication and electric field by differentiation, $\hat{E}_i^a \sim -i\hbar\frac{\delta}{\delta A_a^i}$.

$$\hat{E}_i^a(x)|T_c\rangle = -i\hbar\frac{\delta}{\delta A_a^i(x)} e^{\int d^3y\, c_j^b(y) A_b^j(y)} = -i\hbar c_i^a(x)\ |T_c\rangle \tag{7.22}$$

Since the action of $\hat{E}_i^a$ on a CNF is distributional (see the definition of c_a^i in (7.1)), one is supposed to regularise it. Once and for all, we need to fix an ϵ-neighbourhood $U_\epsilon(\gamma, v)$ at each vertex $v \in V(\gamma)$ which is coordinated by a coordinate chart $\{x_v\}$ in which the subscript v means that the origin is located at v. We denote the coordinate ball of radius ϵ centered at v by $B_\epsilon(v) \subset U_\epsilon(\gamma, v)$ (see figure 7.2). If we regularise the Dirac Delta function using normalised characteristic functions of coordinate balls with radius ϵ, i.e.,

$$\delta_\epsilon(x, y) = \frac{1}{V(B_\epsilon(x))}\chi_{B_\epsilon(x)}(y) = \left(\frac{4\pi\epsilon^3}{3}\right)^{-1} \chi_{B_\epsilon(x)}(y) \tag{7.23}$$

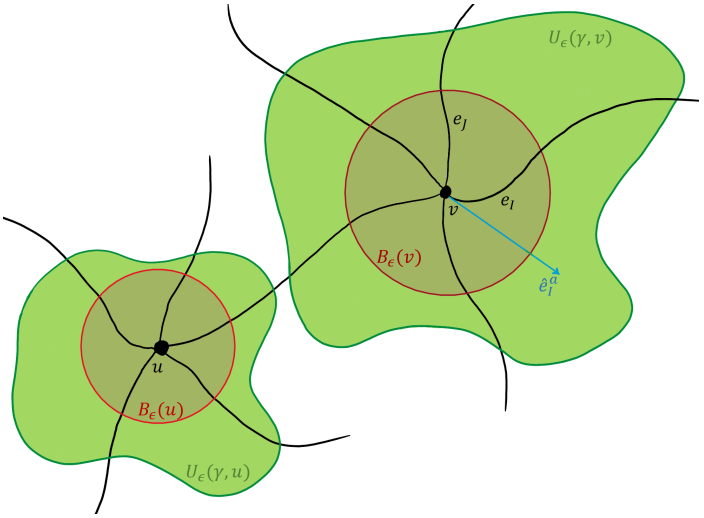

Figure 7.2.: Neighbourhoods around vertices and the unit vector $\hat{e}_I^a$ tangent to the edge e_I at v.

then the regularised version of (7.22) is

$$\begin{aligned}\hat{E}^a_{i,\epsilon}(v)|T_c\rangle = -i\hbar c^a_{i,\epsilon}(v)\,|T_c\rangle &= \frac{-3i\hbar}{4\pi\epsilon^3}\sum_{I=1}^{|E(\gamma)|} iq_i^I \int_0^1 dt_I \chi_{B_\epsilon(v)}(e_I(t_I))\dot{e}_I^a(t_I)\,|T_c\rangle \\ &= \frac{3\hbar}{4\pi\epsilon^3}\sum_{e_I\cap\{v\}\neq\emptyset} q_i^I \int_{B_\epsilon(v)\cap e_I} dt_I \dot{e}_I^a(t_I)\,|T_c\rangle \\ &= \frac{3\hbar}{4\pi\epsilon^3}\sum_{e_I\cap\{v\}\neq\emptyset} q_i^I(\epsilon\hat{e}_I^a + \mathcal{O}(\epsilon^2))\,|T_c\rangle \\ &= \frac{3\hbar}{4\pi\epsilon^2}\sum_{e_I\cap\{v\}\neq\emptyset} q_i^I\hat{e}_I^a\,|T_c\rangle + \mathcal{O}(\epsilon^{-1}) \end{aligned} \tag{7.24}$$

where $\hat{e}_I^a$ is a unit vector tangent to e_I at v (see figure 7.2). With (7.24) in hand, we can move towards acquiring the operator corresponding to the electric shift. For this purpose, we need also the regularised operator associated to $q^{-1/3}$, which has been computed in detail in [87], and the final result is $\hat{q}_\epsilon^{-1/3}|T_c\rangle = \frac{\epsilon^2}{\hbar}\nu^{-2/3}|T_c\rangle$ where ν is a number made up of the charge labels of the edges of the graph γ incident at v. We suffice with this little information about ν because its finer structure will not be required for the following discussion; for more details, we refer the interested reader to appendix A of [87]. Subsequently, the regularised electric shift turns out to be

$$\hat{N}^a_{i,\epsilon}(v)\,|T_c\rangle = N(x(v))\hat{q}_\epsilon^{-1/3}\hat{E}^a_{i,\epsilon}(v)\,|T_c\rangle = \frac{3}{4\pi}N(x(v))\nu^{-2/3}\sum_{e_I\cap\{v\}\neq\emptyset} q_i^I\hat{e}_I^a\,|T_c\rangle + \mathcal{O}(\epsilon) \tag{7.25}$$

Removing the regulator, the electric shift operator takes the form

$$\hat{N}^a_i(v)\,|T_c\rangle = \lim_{\epsilon\to 0}\hat{N}^a_{i,\epsilon}(v)\,|T_c\rangle = \underbrace{\sum_{e_I\cap\{v\}\neq\emptyset}\left(\frac{3}{4\pi}N(x(v))\nu^{-2/3}q_i^I\hat{e}_I^a\right)}_{=:N^a_i(v)}|T_c\rangle =: \sum_{e_I\cap\{v\}\neq\emptyset} N_i^{aI}(v)\,|T_c\rangle \tag{7.26}$$

The critical point to note is that because[4] the regularised operator $\hat{q}_\epsilon$ acts non-trivially only at vertices, $\hat{N}^a_i(x)|T_c\rangle$ can give us a non-vanishing quantity only when x lies on a vertex v of the charge network under consideration. We will return to the importance of this point in the next section.

7.4. Constraint Operators and Precise Realisation of the Algebra

Now that quantum operators corresponding to holonomy, electric field, and electric shift are available, it seems that by substituting them in the constraints, we can easily promote the constraints to quantum operators. However, the situation is more complicated than it seems at first glance. Four important challenges ahead on achieving the desired are

1) As we saw in the previous section, because the electric shift depends on the electric field, it becomes an operator in quantum theory. Hence the first question coming up is about the meaning of Lie derivative along an operator, in other words, how should $\mathcal{L}_{\hat{\vec{N}}_j}$ be interpreted?

The answer to this question lies in the fact that luckily the electric shift is diagonal in the charge network basis which is obvious from (7.26). Thus, one simply replaces the operator $\hat{N}^a_j$ by its eigenvalue, i.e. the quantum shift.

2) As we saw in chapter 1, the kinematics of LQG does not support infinitesimal diffeomorphism generator. On the other hand, Lie derivative along a vector field V is defined by infinitesimal diffeomorphisms, i.e. $\mathcal{L}_V f := \lim_{\epsilon\to 0}\epsilon^{-1}(\varphi^V_\epsilon - 1)f$ where φ^V_ϵ is the one parameter family of diffeomorphisms generated by V. In the absence of this infinitesimal diffeomorphisms at the quantum level, how should one define the Lie derivative?

To deal with this issue, we have to approximate the Lie derivative by small finite diffeomorphisms rather than the infinitesimal ones. Since the operator of finite diffeomorphisms (recall $\hat{U}(\varphi)$ in (1.81)) is well-known in LQG, one approximates the Lie derivative as

$$\mathcal{L}_{\vec{V}}T_c \approx \frac{1}{\epsilon}[\underbrace{\hat{U}(\varphi^V_\epsilon) - 1}_{\mathcal{O}(\epsilon)}]T_c \tag{7.27}$$

Emphasizing that the expression inside the square bracket is of $\mathcal{O}(\epsilon)$ is due to the fact that in what follows we need to approximate such expressions with its exponentiated form minus one.

3) As we saw in the previous section, the quantum shift is non-vanishing only at the vertices of a graph. Thus, it is not a smooth vector field like its classical counterpart; how does $\varphi_\epsilon^{\vec{N}_j}$ make sense?

This is precisely where we have to resort to intuition by first choosing an appropriate ϵ sized finite triangulation and then visualising the deformation generated by $\hat{\vec{e}}_I$ as pulling the vertex v and its immediate neighbourhood in the direction $\hat{\vec{e}}_I$ (see figure (7.3) or (7.4)). In what follows, the one parameter family of triangulations $\mathfrak{T}(\epsilon)$ are adapted to the charge network on which the finite triangulation approximants act. Specifically, we require that $\mathfrak{T}(\epsilon)$ (for sufficiently small ϵ) be such that every vertex v of the coarsest graph underlying the charge network is contained in

[4] Similar to what we saw in chapter 1 about the AL volume operator.

the interior of a cell $\Delta_\epsilon(v) \in \mathfrak{T}(\epsilon)$, and every cell of $\mathfrak{T}(\epsilon)$ contains at most one such vertex. The size of $\Delta_\epsilon(v)$ is restricted to be of $\mathcal{O}(\epsilon^3)$ as measured in the coordinate system $\{x\}_v$.

$$\begin{aligned}
\int d^3x\ A^i_a \mathcal{L}_{\vec{N}_j} c^a_i &= \sum_{v\in V(\gamma)} \int_{\Delta_\epsilon(v)} d^3x\ A^i_a \mathcal{L}_{\vec{N}_j} c^a_i \\
&= \frac{3}{4\pi} \sum_{v\in V(\gamma)} N(x(v))\nu^{-2/3} \sum_{I_v} q^{I_v}_j \int_{\Delta_\epsilon(v)} d^3x\ A^i_a \mathcal{L}_{\hat{e}_{I_v}} c^a_i \\
&\approx \frac{3}{4\pi\epsilon} \sum_{v\in V(\gamma)} N(x(v))\nu^{-2/3} \sum_{I_v} q^{I_v}_j \int_{\Delta_\epsilon(v)} d^3x\ A^i_a \left(\hat{U}(\varphi_\epsilon^{\hat{e}_{I_v}}) - 1\right) c^a_i \\
&\approx \frac{3}{4\pi\epsilon} \sum_{v\in V(\gamma)} N(x(v))\nu^{-2/3} \sum_{I_v} q^{I_v}_j \left(\exp\left[\int_{\Delta_\epsilon(v)} d^3x\ A^i_a \left(\tilde{c}^{aI_v}_{i\epsilon} - c^a_i\right)\right] - 1\right) \\
&= \frac{3}{4\pi\epsilon} \sum_{v\in V(\gamma)} N(x(v))\nu^{-2/3} \sum_{I_v} q^{I_v}_j \exp\left[\int d^3x\ A^i_a \left(\tilde{c}^{aI_v}_{i\epsilon} - c^a_i\right)\right] \\
&= \frac{3}{4\pi\epsilon} \sum_{v\in V(\gamma)} N(x(v))\nu^{-2/3} \sum_{I_v} q^{I_v}_j T_{\tilde{c}^{I_v}_\epsilon} T_{\bar{c}}
\end{aligned} \tag{7.28}$$

where in the first line we employed the triangulation, in the second line (7.26) has been applied, in the third line the Lie derivative has been approximated by (7.27), in the fourth line we defined $\tilde{c}^{aI_v}_{i\epsilon} := \hat{U}(\varphi_\epsilon^{\hat{e}_{I_v}}) c^a_i$ and used the order of the integrand, that is of $\mathcal{O}(\epsilon)$, to approximate it by its exponentiated version minus one, in the fifth line, first the gauge invariance condition (7.18), with this assumption that all the edges incident at v are outgoing, has been used to drop -1 and then to expand the integral on $\Delta_\epsilon(v)$ to the integral on whole space we invoked the fact that $\varphi_\epsilon^{\hat{e}_{I_v}}$ moved only a small neighbourhood of the vertex v while keeping the rest of space untouched, i.e. $\forall x \notin \Delta_\epsilon(v)$ we have $\tilde{c}^{aI_v}_{i\epsilon}(x) = c^a_i(x)$. In other words

$$\begin{aligned}
\int_{\Delta_\epsilon(v)} d^3x\ A^i_a \left(\tilde{c}^{aI_v}_{i\epsilon} - c^a_i\right) &= \int_{\Delta_\epsilon(v)} d^3x\ A^i_a \left(\tilde{c}^{aI_v}_{i\epsilon} - c^a_i\right) + \sum_{u\neq v} \int_{\Delta_\epsilon(u)} d^3x\ A^i_a \left(\tilde{c}^{aI_v}_{i\epsilon} - c^a_i\right) \\
&= \int d^3x\ A^i_a \left(\tilde{c}^{aI_v}_{i\epsilon} - c^a_i\right)
\end{aligned} \tag{7.29}$$

Finally in the last line of (7.28) we used the definition of CNF (7.1) for the following charge network labels

$$\bar{c} = \left(\gamma, (-q^{I_v}_1, -q^{I_v}_2, -q^{I_v}_3)_{v\in V(\gamma)}\right) \tag{7.30}$$

$$\tilde{c}^{I_v}_\epsilon = \left(\gamma^{I_v}_\epsilon, (q^{I_v}_1, q^{I_v}_2, q^{I_v}_3)_{v\in V(\gamma^{I_v}_\epsilon)}\right) \tag{7.31}$$

in which $\gamma^{I_v}_\epsilon$ is the graph obtained by starting with γ and moving the vertex v and its immediate neighbourhood as much as ϵ along the tangent vector of the edge e_{I_v}.

4) After fixing all above issues and obtaining regularised constraint operators, one confronts the problem of how to take the continuum limit. Which topology should be chosen to define the refinement limit? In particular, we wish to find a topology by means of which precise realisation of the commutator between the Hamiltonian constraints can be achieved.

Recall from chapter 4 that in Thiemann's construction of the Hamiltonian constraint, the USR topology was taken to define the continuum limit [55]. This was a topology on the space of

operators on $\mathcal{H}_{\text{kin}}$ which was defined using diffeomorphism-invariant distributions lying in the algebraic dual of $\mathcal{H}_{\text{kin}}$. In this chapter the strategy of providing a suitable topology is the same as that of chapter 4, except that the topology employed here is defined by a different subspace of the dual of $\mathcal{H}_{\text{kin}}$ which is in analogy with the Lewandowski-Marolf habitat introduced in section (7.1.1). The states of this subspace are called *vertex smooth algebraic states* (VSA states) and the operator topology defined by them is called the *VSA topology* [87]. Precisely speaking, a VSA state is a weighted sum over states which belong to a certain subset of the dual of $\mathcal{H}_{\text{kin}}$. This subset denoted by B_{VSA} is supposed to satisfy some particular assumptions that are essential to achieve the desired precise realisation of (7.11). These assumptions and the proof of the existence of a set B_{VSA} fulfilling them are presented in great detail in [87]. In the following, we will state the assumptions where necessary, but the proof of the existence is beyond the scope of this chapter. Suppose that all states $\langle T_c| \in B_{VSA}$ have n non-degenerate vertices and the evaluation of $f : \Sigma^n \to \mathbb{C}$ on the non-degenerate vertices of the graph underlying $\langle T_c|$ is considered as the weight of $\langle T_c|$. Moreover, we assume that $\langle T_c| \in B_{VSA}$ is based on a graph that does not have any symmetry on interchanging its non-degenerate vertices. If we denote the set of charge networks c underlying the states $\langle T_c| \in B_{VSA}$ by $\{c\}$, then a VSA state is defined as

$$\ell_{\{c\},f} := \sum_{c' \in \{c\}} \kappa_{c'} \langle T_{c'}| f(v_1, \cdots, v_n)| \cdot \rangle \tag{7.32}$$

where $\kappa_{c'}$ is a real number that depends on c'. Notice the similarity of (7.32) with (7.1).

7.4.1. Regularised Electric Diffeomorphism Constraint Operator

Now, if in (7.9) we replace the electric field E^a_i with the operator $-i\hbar\frac{\delta}{\delta A^i_a}$ and use the quantum shift instead of the electric shift, the heuristic form of the diffeomorphism constraint is obtained as

$$\hat{C}_a[N^a_j] = -i\hbar \int d^3x \, (\mathcal{L}_{\vec{N}_j} A^i_a) \frac{\delta}{\delta A^i_a} \tag{7.33}$$

Considering all the above-mentioned remarks and using a triangulation $\mathfrak{T}(\epsilon)$ adapted to T_c, one can see how the quantum operator of diffeomorphism constraint acts on the CNF T_c.

$$\begin{aligned}
\hat{C}^\epsilon_b[N^b_j]|T_c\rangle &= \hat{C}_b[N^b_j] \left(e^{\int d^3x \, c^a_i A^i_a}\right) \\
&= \left(\int d^3x \; c^a_i \; \hat{C}_b[N^b_j] A^i_a\right) \left(e^{\int d^3x \, c^c_k A^k_c}\right) \\
&= -i\hbar \left(\int d^3x \; c^a_i \; \mathcal{L}_{\vec{N}_j} A^i_a\right) T_c \\
&= i\hbar \left(\int d^3x \; A^i_a \; \mathcal{L}_{\vec{N}_j} c^a_i\right) T_c \\
&= i\hbar \left(\frac{3}{4\pi\epsilon} \sum_{v \in V(\gamma)} N(x(v)) \nu^{-2/3} \sum_{I_v} q^{I_v}_j T_{\tilde{c}^{I_v}_\epsilon} T_{\bar{c}}\right) T_c \\
&= \frac{3i\hbar}{4\pi\epsilon} \sum_{v \in V(\gamma)} N(x(v)) \nu_v^{-2/3} \sum_{I_v} q^{I_v}_j T_{\tilde{c}^{I_v}_\epsilon}
\end{aligned} \tag{7.34}$$

where in the fourth step we have dropped the boundary term, in the fifth one (7.28) has been used, and in the last one we have simplified the expression by $T_{\bar{c}} T_c = 1$. See figure (7.3) to

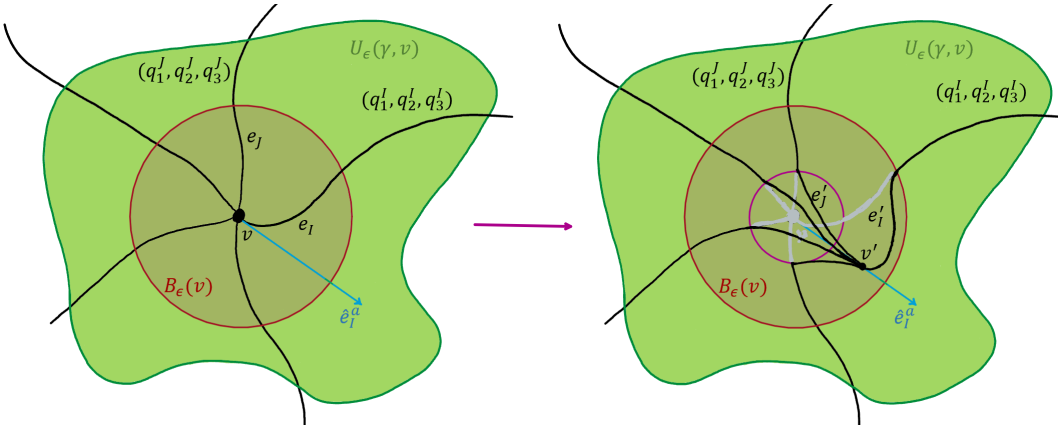

Figure 7.3.: Action of diffeomorphism constraint on a vertex of a CNF; the vertex is displaced but the labels are untouched.

visualise the action of $\hat{C}_b[N_j^b]$ on a given vertex of a CNF (In the right picture, we deliberately did not completely erase the original graph to make the comparison easier). Note that the vertex is displaced and the edges are dragged along $\hat{e}_I^a$ with this extra assumption that

$$\left(\frac{d}{dt_J}(\dot{e}_J)|_{v'}\right)^a = -\left(\frac{d}{dt_I}(\dot{e}_I)|_{v'}\right)^a = -\hat{e}_I^a, \qquad \forall J \neq I \tag{7.35}$$

where e' denotes the dragged edges. Note that the labels remain untouched. For more detail consult the original paper [87].

7.4.1.1. Continuum limit of the Commutator Between Two Regularised Electric Diffeomorphism Constraints

In the following calculations $v'_{I_v,\epsilon}$ denotes the point obtained by moving the vertex $v \in V(\gamma)$ as much as ϵ along the unit vector $\hat{e}^a_{I_v}$ tangent to the edge e_{I_v}. Likewise, if we displace $v'_{I_v,\epsilon}$ as much as ϵ' in the direction of $\hat{e}^a_{J_v}$ tangent to the edge e_{J_v} we arrive at the $v''_{(I_v,\epsilon),(J_v,\epsilon')}$. Suppose the underlying graph of the charge network c has $n+1$ vertices, i.e. $V(\gamma) = \{v_1, \cdots, v, \cdots, v_n\}$. Then the vertices of the underlying graph of $\tilde{c}^{I_v}_\epsilon$ are $V(\gamma^\epsilon_{I_v}) = \{v_1, \cdots, v'_{I_v,\epsilon}, \cdots, v_n\}$ and those of $\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}$ are $V(\gamma^{I_v,J_v}_{\epsilon,\epsilon'}) = \{v_1, \cdots, v''_{(I_v,\epsilon),(J_v,\epsilon')}, \cdots, v_n\}$. In order to compute the continuum limit of the commutator between two operators of the form (7.34), we need to specify the conditions that the set B_{VSA} has to satisfy (recall item 4 in section 7.4). Given a point $v \in \Sigma$ and a charge network c, either,

1. there exists $\epsilon_0(c) := \epsilon_0$ such that $\forall \epsilon \leq \epsilon_0$ there exists $\epsilon'_0(\epsilon)$ such that $\forall \delta' \leq \delta'_0(\delta)$ we have

$$\forall I_v, J_v \qquad \langle T_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}| \in B_{VSA}, \qquad \text{and} \quad \kappa_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}} = -\frac{1}{12}; \tag{7.36}$$

2. OR $\forall \epsilon, \epsilon'$ for which $\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}$ is defined, we have $\langle T_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}| \notin B_{VSA}, \ \ \forall I_v, J_v$.

Now, we compute the continuum limit using VSA states (7.32): (We have enumerated the following equalities because the reason for each step is mentioned afterwards.)

$$
\begin{aligned}
&\lim_{\epsilon'\to 0}\lim_{\epsilon\to 0} \ell_{\{c\},f}\left[\hat{C}_a^{\epsilon'}[M_i^a], \hat{C}_b^{\epsilon}[N_i^b]\right]|T_c\rangle \\
&\overset{1}{=} \lim_{\epsilon'\to 0}\lim_{\epsilon\to 0} \ell_{\{c\},f}\left[\frac{3i\hbar}{4\pi\epsilon}\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v} q_i^{I_v}\hat{C}_a[M_i^a]|T_{\tilde{c}_\epsilon^{I_v}}\rangle - (N\leftrightarrow M)\right] \\
&\overset{2}{=} \lim_{\epsilon'\to 0}\lim_{\epsilon\to 0} \ell_{\{c\},f}\left[\frac{3i\hbar}{4\pi\epsilon}\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v} q_i^{I_v}\frac{3i\hbar}{4\pi\epsilon'}M(x'(v'_{I_v,\epsilon}))\nu_{v'_{I_v}}^{-2/3}\sum_{J_v} q_i^{J_v}T_{\tilde{c}_{\epsilon,\epsilon'}^{I_v,J_v}} - (N\leftrightarrow M)\right] \\
&\overset{3}{=} \left(\frac{3i\hbar}{4\pi}\right)^2 \lim_{\epsilon'\to 0}\lim_{\epsilon\to 0} \\
&\quad\left[\frac{1}{\epsilon}\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v} q_i^{I_v}\frac{1}{\epsilon'}\mathcal{J}_{I_v,\epsilon}^{-\frac{1}{3}}\,[M(x(v))+\epsilon\hat{e}_{I_v}^a\partial_a M(x(v))]\nu_{v'_{I_v}}^{-2/3}\sum_{J_v} q_i^{J_v}\ell_{\{c\},f}(T_{\tilde{c}_{\epsilon,\epsilon'}^{I_v,J_v}}) - (N\leftrightarrow M)\right] \\
&\overset{4}{=} -\frac{1}{12}\left(\frac{3i\hbar}{4\pi}\right)^2 \lim_{\epsilon\to 0} \\
&\quad\left[\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v} q_i^{I_v}\mathcal{J}_{I_v,\epsilon}^{-\frac{1}{3}}\,\hat{e}_{I_v}^a\partial_a M(x(v))\nu_{v'_{I_v}}^{-2/3}\sum_{J_v} q_i^{J_v}\lim_{\epsilon'\to 0}\frac{f(v''_{(I_v,\epsilon),(J_v,\epsilon')})}{\epsilon'} - (N\leftrightarrow M)\right] \\
&\overset{5}{=} -3\left(\frac{i\hbar}{8\pi}\right)^2 \lim_{\epsilon\to 0}\left[\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v} q_i^{I_v}\mathcal{J}_{I_v,\epsilon}^{-\frac{1}{3}}\,\hat{e}_{I_v}^a\partial_a M(x(v))\nu_{v'_{I_v}}^{-2/3}\sum_{J_v} q_i^{J_v}\hat{e}'^b_{J_v}\partial_b f - (N\leftrightarrow M)\right] \\
&\overset{6}{=} -6\left(\frac{i\hbar}{8\pi}\right)^2 \sum_{v\in V(\gamma)}\sum_{I_v}(q^{I_v})^2\nu_v^{-2/3}\nu_{v_{I_v}}^{-2/3}\hat{e}_{I_v}^a\hat{e}_{I_v}^b\,[N(x(v))\partial_a M(x(v)) - M(x(v))\partial_a N(x(v))]\,\partial_b f(v)
\end{aligned}
\tag{7.37}
$$

1: We have just used (7.34) for the first electric diffeomorphism operator and the second one acts only on $|T_{\tilde{c}_\epsilon^{I_v}}\rangle$.

2: In this step, again we used (7.34) and paid attention to the fact that those terms in which the vertices are not replaced by the first constraint would cancel each other due to the antisymmetrisation of N and M.

3: Given that M is a scalar density of weight $-1/3$, its coordinate transformation obeys the rule

$$
\begin{aligned}
M(x'(v'_{I_v,\epsilon})) &= \det\left(\frac{\partial x}{\partial x'}|_{v'_{I_v,\epsilon}}\right)^{-\frac{1}{3}} M(x(v'_{I_v,\epsilon})) =: \mathcal{J}_{I_v,\epsilon}^{-\frac{1}{3}}\, M(x(v'_{I_v,\epsilon})) \\
&= \mathcal{J}_{I_v,\epsilon}^{-\frac{1}{3}}\,[M(x(v)) + \epsilon\hat{e}_{I_v}^a\partial_a M(x(v)) + \mathcal{O}(\epsilon^2)]
\end{aligned}
\tag{7.38}
$$

where in the last equality we simply used the Taylor expansion. The terms hidden in $\mathcal{O}(\epsilon^2)$ have not been displayed in (7.37) because they would vanish in the refinement limit.

4: First note that due to the antisymmetrisation of N and M the contribution from $N(x(v))M(x(v))$

vanishes. The action of the VSA state $\ell_{\{c\},f}$ on the charge network $T_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}$ results in

$$\ell_{\{c\},f}(T_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}) = -\frac{1}{12}\sum_{c'\in\{c\}}\langle T_{c'}|f(v_1,\cdots,v_n)|\, T_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}\,\rangle = -\frac{1}{12}\, f(v_1,\cdots,v''_{(I_v,\epsilon),(J_v,\epsilon')},\cdots,v_n)$$
$$=: -\frac{1}{12}\, f(v''_{(I_v,\epsilon),(J_v,\epsilon')}) \tag{7.39}$$

where the condition (7.36) has been used and for convenience we have dropped those elements of f that do not change.

5: Because of the gauge invariance condition (7.18) the vanishing term $-\sum_{J_v} q_i^{J_v} f(v'_{I_v,\epsilon})$ can be appended to $\sum_{J_v} q_i^{J_v} f(v''_{(I_v,\epsilon),(J_v,\epsilon')})$ such that the directional derivative of f arises in the refinement limit $\epsilon'\to 0$. In other words,

$$\sum_{J_v} q_i^{J_v}\lim_{\epsilon'\to 0}\frac{f(v''_{(I_v,\epsilon),(J_v,\epsilon')})}{\epsilon'} = \sum_{J_v} q_i^{J_v}\lim_{\epsilon'\to 0}\frac{f(v''_{(I_v,\epsilon),(J_v,\epsilon')})-f(v'_{I_v,\epsilon})}{\epsilon'} = \sum_{J_v} q_i^{J_v}\hat{e}'^b_{J_v}\partial_b f \tag{7.40}$$

6: The Jacobian $\mathcal{J}_{I_v,\epsilon}$ has disappeared because in the limit $\epsilon\to 0$ it is equals to identity. Using the assumption (7.35) and the gauge invariance condition (7.18), we conclude

$$\sum_{J_v} q_i^{J_v}\hat{e}'^b_{J_v} = q_i^{I_v}\hat{e}'^b_{I_v} + \sum_{J_v\neq I_v} q_i^{J_v}\hat{e}'^b_{J_v} = q_i^{I_v}\hat{e}^b_{I_v} - \hat{e}^b_{I_v}\sum_{J_v\neq I_v} q_i^{J_v} = q_i^{I_v}\hat{e}^b_{I_v} - \hat{e}^b_{I_v}(-q_i^{I_v}) = 2\, q_i^{I_v}\hat{e}^b_{I_v} \tag{7.41}$$

We have also used the notation $(q^{I_v})^2 := q_i^{I_v} q_i^{I_v}$.

7.4.2. Regularised Hamiltonian Constraint Operator

Similar to (7.33), the heuristic form of the Hamiltonian constraint operator, which corresponds to (7.8), is

$$\hat{C}'[N] := \frac{1}{2}\int d^3x\,\left(\epsilon^{ijk}(\mathcal{L}_{\vec{N}_i}A^k_b)\frac{\delta}{\delta A^j_b} + \sum_i(\mathcal{L}_{\vec{N}_i}A^i_b)\frac{\delta}{\delta A^i_b}\right) \tag{7.42}$$

To see how it acts on a generic CNF, we need to define the following charge networks

$$\begin{aligned}
{}^i\tilde{c}^{I_v}_{\epsilon} &:= \left(\gamma^{I_v}_{\epsilon}, \left({}^i\tilde{q}^{I_v}_j := (\delta^{ijk}-\epsilon^{ijk})q^{I_v}_k\right)_{v\in V(\gamma^{I_v}_{\epsilon})}\right)\\
{}^i\bar{c} &:= \left(\gamma, \left({}^i\bar{q}_j := -(\delta^{ijk}-\epsilon^{ijk})q_k\right)_{v\in V(\gamma)}\right)\\
{}^i\hat{c} &:= \left(\gamma, \left({}^i\hat{q}^{:}_j = (\epsilon^{ijk}+\hat{\delta}^{ijk})q_k\right)_{v\in V(\gamma)}\right)
\end{aligned} \tag{7.43}$$

where

$$\delta^{ijk} := \begin{cases}1, & i=j=k,\\ 0, & \text{otherwise}\end{cases}, \qquad \hat{\delta}^{ijk} := \begin{cases}1, & i\neq j=k,\\ 0, & \text{otherwise}\end{cases}$$

In (7.43), the left superscript i determines which $\vec{N}_i$ is involved to define the charge networks (see the calculation below).

Now, by employing a triangulation $\mathfrak{T}(\epsilon)$ adapted to T_c, a calculation similar to what we did in (7.28) and (7.34) reveals

$$
\begin{aligned}
&\hat{C}'_\epsilon[N]|T_c\rangle = \hat{C}'_\epsilon[N]\left(e^{\int d^3x\, c_i^a A_a^i}\right)\\
&= \left(\int d^3x\, c_i^a\, \hat{C}'_\epsilon[N] A_a^i\right)\left(e^{\int d^3x\, c_j^a A_a^j}\right)\\
&= \frac{-i\hbar}{2}\left(\int d^3x\, c_i^a\, (\epsilon^{ijk}\mathcal{L}_{\vec{N}_j} A_a^k + \mathcal{L}_{\vec{N}_i} A_a^i)\right) T_c\\
&= \frac{i\hbar}{2}\sum_{v\in V(\gamma)}\int_{\Delta_\epsilon(v)} d^3x\left(A_a^i\, (\epsilon^{ijk}\mathcal{L}_{\vec{N}_j} c_k^a + \mathcal{L}_{\vec{N}_i} c_i^a)\right) T_c\\
&= \frac{i\hbar}{2}\sum_{v\in V(\gamma)}\int_{\Delta_\epsilon(v)} d^3x\left(A_a^1\,\mathcal{L}_{\vec{N}_1} c_1^a - A_a^2\,\mathcal{L}_{\vec{N}_1} c_3^a + A_a^3\,\mathcal{L}_{\vec{N}_1} c_2^a\right.\\
&\qquad\qquad + A_a^1\,\mathcal{L}_{\vec{N}_2} c_3^a + A_a^2\,\mathcal{L}_{\vec{N}_2} c_2^a - A_a^3\,\mathcal{L}_{\vec{N}_2} c_1^a\\
&\qquad\qquad \left. -A_a^1\,\mathcal{L}_{\vec{N}_3} c_2^a + A_a^2\,\mathcal{L}_{\vec{N}_3} c_1^a + A_a^3\,\mathcal{L}_{\vec{N}_3} c_3^a\right) T_c\\
&= \frac{3i\hbar}{8\pi\epsilon}\sum_{v\in V(\gamma)} N(x(v))\nu^{-2/3}\sum_{I_v}\{q_1^{I_v}\exp\left(\int d^3x\,\left[A_a^1(\tilde{c}_{1\epsilon}^{aI_v} - c_1^a) - A_a^2(\tilde{c}_{3\epsilon}^{aI_v} - c_3^a) + A_a^3(\tilde{c}_{2\epsilon}^{aI_v} - c_2^a)\right]\right)\\
&\qquad + q_2^{I_v}\exp\left(\int d^3x\,\left[A_a^1(\tilde{c}_{3\epsilon}^{aI_v} - c_3^a) + A_a^2(\tilde{c}_{2\epsilon}^{aI_v} - c_2^a) - A_a^3(\tilde{c}_{1\epsilon}^{aI_v} - c_1^a)\right]\right)\\
&\qquad + q_3^{I_v}\exp\left(\int d^3x\,\left[-A_a^1(\tilde{c}_{2\epsilon}^{aI_v} - c_2^a) + A_a^2(\tilde{c}_{1\epsilon}^{aI_v} - c_1^a) + A_a^3(\tilde{c}_{3\epsilon}^{aI_v} - c_2^a)\right]\right)\}T_c\\
&= \frac{3i\hbar}{8\pi\epsilon}\sum_{v\in V(\gamma)} N(x(v))\nu^{-2/3}\sum_{I_v}\left[q_1^{I_v}\, e^{\int d^3x\, A_a^j\ {}^1\tilde{c}_{j\epsilon}^{aI_v}}\, e^{\int d^3x\, A_a^j\ {}^1\bar{c}_j^a}\right.\\
&\qquad + q_2^{I_v}\, e^{\int d^3x\, A_a^j\ {}^2\tilde{c}_{j\epsilon}^{aI_v}}\, e^{\int d^3x\, A_a^j\ {}^2\bar{c}_j^a}\\
&\qquad \left. + q_3^{I_v}\, e^{\int d^3x\, A_a^j\ {}^3\tilde{c}_{j\epsilon}^{aI_v}}\, e^{\int d^3x\, A_a^j\ {}^3\bar{c}_j^a}\right] T_c\\
&= \frac{3i\hbar}{8\pi\epsilon}\sum_{v\in V(\gamma)} N(x(v))\nu^{-2/3}\sum_{I_v,i} q_i^{I_v}\, T_{i\tilde{c}_\epsilon^{I_v}}\, T_{i\bar{c}}\, T_c\\
&= \frac{3i\hbar}{8\pi\epsilon}\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v,i} q_i^{I_v}\, T_{i\tilde{c}_\epsilon^{I_v}}\, T_{i\hat{c}} \qquad (7.44)
\end{aligned}
$$

Here we do not repeat the reasons we had already explained for the previous calculations, just note that in the sixth equality we rearranged the terms so that they can be expressed as holonomies in the next equality. Furthermore, in the last step we defined $T_{i\hat{c}} := T_{i\bar{c}}\, T_c$. Figure (7.4) visualises the action of the Hamiltonian constraint operator on a given vertex of a CNF. Note that in this case in addition to the replacement of the vertex, the labels are also flipped! The undeformed piece of the charge network has charges obeying ${}^i\hat{c}$ in (7.43) (In the right picture, we deliberately did not completely erase the original graph to make the comparison easier).

7.4.2.1. Commutator Between Two Hamiltonian Constraints

Here, in an abuse of notation we write $\hat{C}^\epsilon[N]$ instead of $\hat{C}'_\epsilon[N]$. We intend to compute the continuum limit of the commutator between two operators of the form (7.44), again it is required to specify the conditions that the set B_{VSA} has to satisfy (recall item 4 in section 7.4). Given a point $v \in \Sigma$ and a charge network c, either,

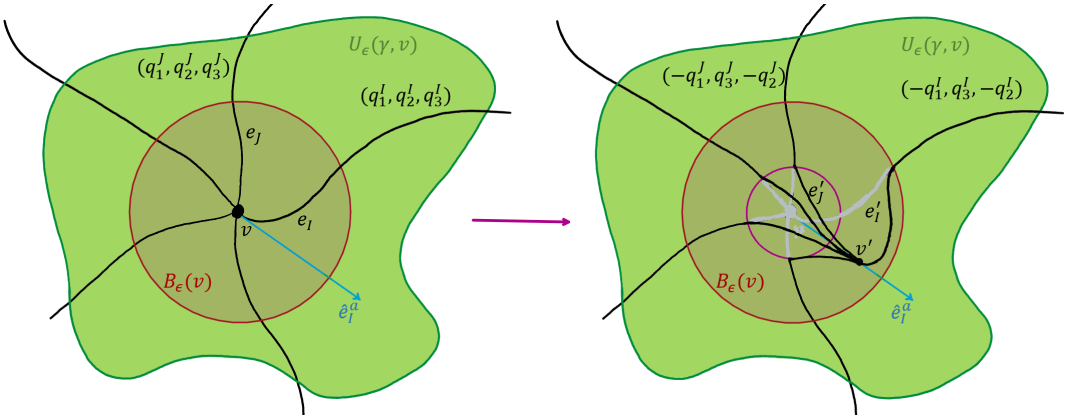

Figure 7.4.: The action of the Hamiltonian constraint on a vertex of a CNF; the vertex is replaced and the charges are flipped!

1. there exists $\epsilon_0(c) := \epsilon_0$ such that $\forall \epsilon \leq \epsilon_0$ there exists $\epsilon'_0(\epsilon)$ such that $\forall \epsilon' \leq \epsilon'_0(\epsilon)$ we have

$$\forall i, j \;\; \forall I_v, J_v \qquad \langle T_{j,i\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}| \in B_{VSA}, \qquad \text{and} \quad \kappa_{j,i\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}} = 1; \tag{7.45}$$

or

2. $\forall \epsilon, \epsilon'$ for which ${}^{j,i}\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}$ is defined, we have $\langle T_{j,i\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}| \notin B_{VSA}, \;\; \forall i, j \;\; \forall I_v, J_v$.

where $T_{j,i\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}$ is the CNF produced by acting two Hamiltonian constraint operators on T_c. Moreover, we assume that $\langle T_{j,i\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}| \in B_{VSA}$ holds if and only if $\langle T_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}| \in B_{VSA}$ holds and $\langle T_{j,i\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}| \notin B_{VSA}$ holds if and only if $\langle T_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}| \notin B_{VSA}$ holds.

Note that Now, we can compute the continuum limit using VSA states (7.32): (Again we have enumerated the following equalities because the reason for each step is mentioned afterwards.)

$$
\begin{aligned}
&\lim_{\epsilon'\to 0}\lim_{\epsilon\to 0} \ell_{\{c\},f}\left[\hat{C}^{\epsilon'}[M],\hat{C}^{\epsilon}[N]\right]|T_c\rangle \\
&\overset{1}{=} \lim_{\epsilon'\to 0}\lim_{\epsilon\to 0} \ell_{\{c\},f}\left[\frac{3i\hbar}{8\pi\epsilon}\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v,i} q_i^{I_v}\,\hat{C}^{\epsilon'}[M]|T_{i_{\tilde{c}^{I_v}_{\epsilon}}}\rangle - (N\leftrightarrow M)\right] \\
&\overset{2}{=} \lim_{\epsilon'\to 0}\lim_{\epsilon\to 0} \ell_{\{c\},f}\left[\frac{3i\hbar}{8\pi\epsilon}\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v} q_i^{I_v,i}\frac{3i\hbar}{8\pi\epsilon'}M(x'(v'_{I_v,\epsilon}))\nu_{v'_{I_v}}^{-2/3}\sum_{J_v,j} {}^i\tilde{q}_j^{J_v} T_{j,i_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}} - (N\leftrightarrow M)\right] \\
&\overset{3}{=} \left(\frac{3i\hbar}{8\pi}\right)^2 \lim_{\epsilon'\to 0}\lim_{\epsilon\to 0} \\
&\quad\left[\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v,i} q_i^{I_v}\frac{1}{\epsilon'}\mathcal{J}_{I_v,\epsilon}^{-\frac{1}{3}}\,\hat{e}^a_{I_v}\partial_a M(x(v))\nu_{v'_{I_v}}^{-2/3}\sum_{J_v,j} {}^i\tilde{q}_j^{J_v}\ell_{\{c\},f}(T_{j,i_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}}) - (N\leftrightarrow M)\right] \\
&\overset{4}{=} \left(\frac{3i\hbar}{8\pi}\right)^2 \lim_{\epsilon\to 0} \\
&\quad\left[\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v,i} q_i^{I_v}\mathcal{J}_{I_v,\epsilon}^{-\frac{1}{3}}\,\hat{e}^a_{I_v}\partial_a M(x(v))\nu_{v'_{I_v}}^{-2/3}\sum_{J_v,j} {}^i\tilde{q}_j^{J_v}\lim_{\epsilon'\to 0}\frac{f(v''_{(I_v,\epsilon),(J_v,\epsilon')})}{\epsilon'} - (N\leftrightarrow M)\right] \\
&\overset{5}{=} \left(\frac{3i\hbar}{4\pi}\right)^2 \lim_{\epsilon\to 0}\left[\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v,i} q_i^{I_v}\mathcal{J}_{I_v,\epsilon}^{-\frac{1}{3}}\,\hat{e}^a_{I_v}\partial_a M(x(v))\nu_{v'_{I_v}}^{-2/3}\sum_{J_v,j} {}^i\tilde{q}_j^{J_v}\hat{e}'^b_{J_v}\partial_b f - (N\leftrightarrow M)\right] \\
&\overset{6}{=} 2\left(\frac{3i\hbar}{8\pi}\right)^2 \lim_{\epsilon\to 0}\left[\sum_{v\in V(\gamma)} N(x(v))\nu_v^{-2/3}\sum_{I_v,i,j} q_i^{I_v}\mathcal{J}_{I_v,\epsilon}^{-\frac{1}{3}}\,\hat{e}^a_{I_v}\partial_a M(x(v))\nu_{v'_{I_v}}^{-2/3}({}^i\tilde{q}_j^{I_v}\hat{e}'^b_{I_v})\partial_b f - (N\leftrightarrow M)\right] \\
&\overset{7}{=} 18\left(\frac{i\hbar}{8\pi}\right)^2 \sum_{v\in V(\gamma)}\sum_{I_v}(q^{I_v})^2\nu_v^{-2/3}\nu_{v_{I_v}}^{-2/3}\hat{e}^a_{I_v}\hat{e}^b_{I_v}\left[N(x(v))\partial_a M(x(v)) - M(x(v))\partial_a N(x(v))\right]\partial_b f(v)
\end{aligned}
\tag{7.46}
$$

1: We have just used (7.44) for the first Hamiltonian operator and the second one acts only on $|T_{i_{\tilde{c}^{I_v}_{\epsilon}}}\rangle$. Note that since the CNF $T_{i_{\hat{c}}}$ existing (7.44) in is associated to the undeformed piece of the charge network it does not affect the final result of the commutator.

2: In this step, again we used (7.44). Note that those terms in which the vertices are not replaced by the first constraint would cancel each other due to the antisymmetrisation of N and M.

3: Again we used (7.38).

4: We used the antisymmetrisation of N and M to eliminate the contribution from $N(x(v))M(x(v))$. The action of the VSA state $\ell_{\{c\},f}$ on the charge network $T_{j,i_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}}$ results in

$$
\ell_{\{c\},f}(T_{j,i_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}}) = \sum_{c'\in\{c\}}\langle T_{c'}|f(v_1,\cdots,v_n)|\, T_{j,i_{\tilde{c}^{I_v,J_v}_{\epsilon,\epsilon'}}}\,\rangle = f(v''_{(I_v,\epsilon),(J_v,\epsilon')}) \tag{7.47}
$$

where the condition (7.45) has been used.

5: We have used the same reasoning as (7.40) to write the partial derivative.

6: The relation (7.41) has been used to eliminate the sum over J_v.

7: The Jacobian $\mathcal{J}_{I_v,\epsilon}$ has disappeared because in the limit $\epsilon \to 0$ it is equals to identity and we have also used the relation

$$\sum_{i,j} q_i^{I_v}\, {}^i\tilde{q}_j^{I_v} = \sum_{i,j} q_i^{I_v}([\delta^{ijk} - \epsilon^{ijk}]q_k^{I_v}) = (q^{I_v})^2 \tag{7.48}$$

7.4.3. Conclusion

Comparing two equations (7.37) and (7.46), one easily reads

$$[\hat{C}[N], \hat{C}[M]] = -3[\hat{C}_a[N_i^a], \hat{C}_b[M_i^b]] \tag{7.49}$$

which is the precise realisation of (7.11)! Thus, with this method we could verify the precise realisation of the equation (5.16). Still the verifications of (5.14) and (5.15) are left to perform. By a simple reasoning it turns out that the former is valid [88], but the latter is challenging because what must be hold is indeed the equation (4.14), that is diffeomorphism covariance of the Hamiltonian constraint. The problem is that since the lapse function we employed in this chapter is a scalar "density" its evaluation is coordinate dependant and hence the equation (4.14) is obviously violated. However, Varadarajan in [88] could suggest a way to deal with this problem which is behind the scope of this dissertation. We just refer the interested reader to the original paper.

A critical assessment of the electric shift perspective

Although the above structure seems to work perfectly, it is subjected to considerable criticism:

1. What is really shown in this chapter is that the classical identity (7.11), with certain density weight, can be represented in the quantum theory on some subspace of the algebraic dual. Classically both sides of (7.11) are equal to $\vec{C}(M,N) := \vec{C}(q^{-1}(MdN - NdM))$ for certain density weight. Note that by quantum hypersurface deformation algebra, we mean faithful implementation of $\{C(M), C(N)\} = \vec{C}(M,N)$. But if $\sum_j \{\vec{C}(ME_j), \vec{C}(NE_j)\}$ has nothing to do with the direct quantisation of $\vec{C}(M,N)$ (see [57]) then nothing has been gained, i.e. again we have wrong structure functions.

2. If one adds a cosmological constant term and works with density weight different from 1 then the cosmological constant term either diverges or becomes trivial which sheds considerable doubt on the whole procedure. The same is true for various matter terms while in [238] it was shown that weight 1 has a finite limit in URS topology for all matter terms universally.

3. If one considers Lorentzian signature, the additional $KKEE$ term (see (1.31)) blows up with density weight different from one.

4. It is unclear whether electric shift perspective can be generalised to non-Abelian theory because electric field operator is no longer diagonal and this does not provide an electric eigenvector field.

Toward the Reduced Phase Space Quantisation of the U(1)^3 Model

8.1. Preliminaries

8.1.1. A Sneak Peek at the Constraints in the (A, E) and (B, f) Descriptions

Recall from section 5.2 that the constraints of the $U(1)^3$ model in terms of the canonical variables (A, E) are of the form

$$C_j = \partial_a E^a_j, \quad C_a = F^j_{ab} E^b_j, \quad C = F^j_{ab} \epsilon_{jkl} E^a_k E^b_l |\det(E)|^{-1/2} \tag{8.1}$$

that are Gauß, diffeomorphism and Hamiltonian constraints, respectively. Here, $F^j_{ab} = \partial_a A^j_b - \partial_b A^j_a$. The significant property of (8.1) is that the Gauß constraint is linear in E and the other ones are linear in A.

The canonical transformation (5.23) introduced the canonical variables (B, f). The constraints of the $U(1)^3$ model in terms of this new variables are given by

$$\hat{C}_j = \partial_a E^a_j, \quad \tilde{C}_j = \epsilon_{jkl} B^a_k H^l_a, \quad \tilde{C}_0 = B^a_j H^j_a \tag{8.2}$$

that are Bianchi, diffeomorphism and Hamiltonian constraints, respectively, and H^j_a has already been defined in (5.17). The considerable feature of (8.2) is linearity of all constraints in B.

8.1.2. Purpose of This Chapter

In this chapter, we propose quantisation of the $U(1)^3$ model using the reduced phase space approach (see chapter 2), rather than operator constraint methods (see chapter 1). The advantage of the former is that one can work directly with the physical Hilbert space, the physical observables, and the physical dynamics, which in the latter must be determined after implementing the constraints on an unphysical, kinematical Hilbert space [89]. On the other hand, the disadvantage of the latter is that it generally involves implicit functions, inversions of phase space-dependent differential operators and square roots, which are difficult to quantise in practice [291]. Although this problem can be bypassed in the presence of suitable matter [149, 292, 151], it is generic in vacuum [89] (recall reduced phase space quantisation of the Brown-Kuchar model in section 2.3).

What makes the reduced phase space approach a practical feasibility for the vacuum $U(1)^3$ model is the feature that its constraints are "at most linear" in one of the canonical variables, while in the $SU(2)$ case the connection dependence is "quadratic" [89] (Compare (8.1) with (6.9) where the curvature is given in (6.10)). If one considers the connections as momentum variables after a trivial canonical transformation, then there are no square roots that need to

be taken to solve the constraints [89]. In this chapter, we follow [89], based on the procedure presented in chapter 2 (in particular, the steps outlined in section 2.2.2 for obtaining the physical Hamiltonian), to derive the reduced phase space and some rather simple physical Hamiltonians. We will defer the task of quantisation to our future work.

8.2. Physical Hamiltonian for First Class Constrained Systems with Certain Linearity Properties

In this section, we state and prove two simple results on the reduced dynamics of constrained systems that have a certain linear structure in their constraints with respect to the momenta. This linear structure is acclimatised to the appearance of the constraints of the $U(1)^3$ model presented in the previous section. Hence the succeeding results are applicable to the $U(1)^3$ model in both (A, E) and (B, f) descriptions. In what follows we distinguish, as usual, between constraint surface (constraints vanish) and reduced phase space (constraint surface modulo gauge transformation) or (locally) equivalently the gauge cut (constraints and gauge conditions vanish).

8.2.1. The (A, E) Description

In section 8.1.1, we have seen that the $U(1)^3$ model has seven first-class constraints for each spatial point x : Three Gauß constraints $C_j(x),\ j = 1, 2, 3$ which do not depend on the connection A_a^j, three density weight two diffeomorphism constraints $\tilde{C}_j(x)$ linear in the connection and one density weight two Hamiltonian constraint $\tilde{C}(x)$ also linear in the connection. With respect to a smooth orthonormal basis of test functions b^α on $L_2(d^3x,\ \Sigma)$ for α taking values in an appropriate index set, one integrates the constraints yielding $C_j^\alpha = \int\ d^3x\ C_j(x)\ b^\alpha(x)$ etc. The C_j^α is collectively denoted by C_A and the $\tilde{C}_\mu^\alpha,\ \mu = 0, .., 3$ by C_I for suitable ranges of the indices A, I. With respect to the same test functions, we integrate the canonical pairs (A_a^j, E_j^a) and subdivide them as $w = (u^A, v_A),\ z = (x^I, y_I),\ r = (q^{\mathfrak{a}}, p_{\mathfrak{a}})$. These are still conjugate pairs and in particular, all variables from w Poisson commute with all variables from z, r and all variables from z Poisson commute with the variables from r. Then the completeness relation for the basis b^α is used to express the constraints in the form

$$C_A = C_A(u, x, q), \quad C_I = M_I{}^J(u, x, q)\ y_J + N_I{}^A(u, x, q)\ v_A + h_I(u, x, q, p) \tag{8.3}$$

In our case, C_A depends linearly on u, x, q and h_I depends linearly on p but we will not require to utilise this. Furthermore, $M_I{}^J,\ N_I{}^A,\ h_I$ are homogeneous and quadratic in u, x, q but we will not need to utilise this either. What will be needed is that the subdivision of the canonical pairs into groups is performed in such a way that the "matrices" $\sigma_{AB} := \{C_A, v_B\}$ and $M_I{}^J$ are non-singular and this leads to the above subdivision of canonical pairs. This indicates that $G_A := v_A = 0$ is a suitable gauge fixing condition for C_A and $G^I := F^I(x) - \tau^I$ for C_I where τ^I are constants on the phase space but possibly functions of physical time τ (see section 2.2.2 for the definition of physical time) and $\Delta_J{}^I := \{y_J, F^I\}$ is non-singular. Indeed, recall [94] that depending on whether the spatial manifold Σ has a boundary or not, the time dependence of τ is not necessary or necessary respectively in order to get the non-trivial physical dynamics due to the different boundary conditions on the canonical variables in these two cases. By the implicit function theorem, our assumptions indicates that $C_A = 0$ can locally be solved for $u^A = g^A(x, q)$ and $G^I = 0$ for $x^I = k^I(\tau)$ for suitable functions g^A, k^I.

So we declare r as our physical degrees of freedom. Note that in field theory, the statement that infinite-dimensional matrices are non-singular must be taken with caution: for example,

if we are dealing with differential operators, then an inverse of, say $M_I{}^J$, exists only if one specifies a suitable function space, possibly with boundary conditions on them such that otherwise unspecified integration constants are uniquely determined.

The first-class Hamiltonian reads

$$H(\lambda,\Lambda)=\lambda^A\,C_A+\Lambda^I\,C_I \tag{8.4}$$

where λ^A are the Lagrange multipliers of the Gauß constraint integrated against the basis, and similarly Λ^I are the density weight minus one lapse function and shift vector integrated against that basis, in particular, they are phase space independent. They are collectively denoted by $l=(\lambda,\Lambda)$. The stability of the gauge conditions under gauge transformations determines the Lagrange multipliers

$$\dot{G}_A=\{H,G_A\}=\lambda^B\,\sigma_{BA}+\Lambda^I\{C_I,G_A\}=0;\quad \dot{G}^I=\{H,G^I\}=\Lambda^J\,M_J{}^K\,\Delta_K{}^I=\dot{\tau}^I \tag{8.5}$$

which has the explicit solution

$$\lambda_0^A=-\Lambda_0^I\,\{C_I,G_B\}\,(\sigma^{-1})^{BA}+\kappa^A,\quad \Lambda_0^J M_J{}^I=(\Delta^{-1})^I_J\,\dot{\tau}^J+\kappa^I=:\delta^I \tag{8.6}$$

where

$$\Lambda_0^I=(M^{-1})_J{}^I\,\delta^J+\hat{\kappa}^I \tag{8.7}$$

Here we have presumed that both σ,Δ have an unambiguous inverse σ^{-1},Δ^{-1} on an appropriate space of functions and we allow the space of Lagrange multiplier functions to be larger and contain the kernel of σ,Δ yielding the "integration constants" κ^A,κ^I. By construction, κ^A can only depend on u,x,q and κ^I only on x. In the $U(1)^3$ model, κ^A is indeed phase space independent. As far as M,M^{-1} are concerned, we allow for a similar kernel function $\hat{\kappa}^I$ which in general may depend on u,x,q.

The ambiguities $\kappa^A,\kappa^I,\hat{\kappa}^I$ are to be fixed by the boundary conditions (see chapter 6), and we suppose that they imply in particular $\kappa^A=0$. We emphasise that $l_0=(\lambda_0,\Lambda_0)$ are phase space dependent and defined by (8.6) and (8.7) on the whole phase space and not only on the constraint surface $C_A=C_I=0$ or the reduced phase space $C_A=G_A=C_I=G^I=0$. Note, however, that Λ_0 does not depend on the momenta v,y,p and $\Lambda_0^I M_J{}^I=\delta^I$ depends only on x.

Suppose $\mathcal{F}=\mathcal{F}(r)$ is a function on the reduced phase space. Its evolution is then the gauge motion generated by H restricted to the reduced phase space, i.e.

$$\dot{\mathcal{F}}=\{H,\mathcal{F}\}_{C=0,G=0,l=l_0} \tag{8.8}$$

Now, the physical Hamiltonian h, if it exists, is a function $h=h(r,\tau,\dot{\tau})$ satisfying $\dot{\mathcal{F}}=\{h,\mathcal{F}\}$.

Abstracting from the concrete $U(1)^3$ model, we can now assert the following general theorem [89]:

Theorem 8.2.1. *Let C_A,C_I,G_A,G^I be as above. Then*

$$h=(\Lambda_0^I h_I)_{C_A=0,G^I=0} \tag{8.9}$$

This implies that we do not have to solve $C_I=0$ for y_I, we only require to solve for Λ^I as in (8.7) and not λ^A and only at $C_A=G^I=0$ which is merely a restriction on the configuration degrees of freedom. Given the flexibility in the choice of the gauge condition G^I, it is thus conceivable that one arrives at an explicit expression for h.

Proof. Due to the imposition of $C_I = C_A = 0$, we have

$$\begin{aligned} \dot{f} &= (\lambda^A\{C_A, f\} + \Lambda^I\{C_I, f\})_{C=0,G=0,l=l_0} \\ &= \lambda_0^A\{C_A, f\}_{C=0,G=0} + \Lambda_0^I\{C_I, f\}_{C=0,G=0} \\ &= (\{\lambda_0^A C_A + \Lambda_0^I C_I, f\})_{C=0,G=0} \end{aligned} \tag{8.10}$$

Now by (8.7)

$$\Lambda_0^I C_I = \delta^I\ y_I + \Lambda_0^I\ [N_I\ {}^A v_A + h_I]] \tag{8.11}$$

The first term depends only on $z = (x, y)$, the second one only on u, x, q, v and is linear in $v_A = G_A$. The first term hence has Poisson commute with both f and v_A, the second term has a non-vanishing Poisson bracket with both f and v_A, that is linear in G_B. It hence follows from (8.6) and our assumption $\kappa^A = 0$

$$\begin{aligned} &\{\lambda_0^A C_A, f\}_{C=0,G=0} \\ &\qquad = -(\Lambda_0^I\ \{C_I, G_B\}\ (\sigma^{-1})^{BA}\ \{C_A, f\})_{C=0,G=0} \\ &\qquad = -(\{\Lambda_0^I\ C_I, G_B\}\ (\sigma^{-1})^{BA}\ \{C_A, f\})_{C=0,G=0} \\ &\qquad = -(\{\tilde{h}, G_B\}\ (\sigma^{-1})^{BA}\ \{C_A, f\})_{C=0,G=0} \end{aligned} \tag{8.12}$$

where

$$\tilde{h} = \Lambda_0^I h_I \tag{8.13}$$

For the same reason

$$\{\Lambda_0^I C_I, f\}_{C=0,G=0} = \{\tilde{h}, f\}_{C=0,G=0} \tag{8.14}$$

Therefore both terms (8.12) and (8.14) are independent of v, y so that G_A, C_I do not have to be imposed anymore, we only must impose C_A, G^I. Now it follows explicitly from (8.9)

$$h = \tilde{h}_{C_A=0,G^I=0} = \tilde{h}(u = g(x,q), x, q, p)_{x=k(\tau)} \tag{8.15}$$

so that

$$\begin{aligned} \{h, f\} = &\quad \{\tilde{h}, f\}_{C_A=0,G^I=0} - \left(\tfrac{\partial \tilde{h}}{\partial u^A}\ \tfrac{\partial g^A}{\partial q^{\mathfrak{a}}}\ \tfrac{\partial f}{\partial p^{\mathfrak{a}}}\right)_{C_A=0,G^I=0} \\ = &\quad \{\tilde{h}, f\}_{C_A=0,G^I=0} - (\{\tilde{h}, v_A\}\ \{g^A, f\})_{C_A=0,G^I=0} \end{aligned} \tag{8.16}$$

As $C_A(u = g(q,x), x, q) \equiv 0$ by taking the $q^{\mathfrak{a}}$ derivative of this identity, we have

$$(\{C_A, f\} - \{C_A, v_B\}\ \{g^B, f\})_{C_A=0,G^I=0} = (\{C_A, f\} - \sigma_{AB}\ \{g^B, f\})_{C_A=0,G^I=0} = 0 \tag{8.17}$$

Hence comparing (8.12) with the second term in (8.16), we reach the desired result. □

8.2.2. The (B, f) Description

As stated in section 8.1.1, in the (B, f) description, the $U(1)^3$ model has seven first-class constraints: Three "Bianchi" constraints $\hat{C}_j$, which are linear in B with phase space independent coefficients together with the already discussed constraints $\tilde{C}_\mu$, which are also linear in B. This feature makes the analysis even simpler. Employing the basis b^α introduced in the previous section, we now subdivide the canonical pairs (B_j^a, f_a^j) into only two groups $z = (x^I, y_I)$ and $r = (q^{\mathfrak{a}}, p_{\mathfrak{a}})$ and express the constraints in the form

$$C_I = M_I\ {}^J(x,q)\ y_J + h_I(x,q,p) \tag{8.18}$$

where again $M_I{}^J$, h_I depend on x, q at most quadratic and the h_I on p at most linear, however, this is not required in the following discussion. As discussed earlier, we presume that the subdivision is such that $M_I{}^J$ is invertible on a sufficiently large space of functions. We will impose gauge fixings of the form $G^I(x) = F^I(x) - \tau^I =$ whose stability under $H = \Lambda^I C_I$ determines Λ

$$\Lambda_0^J M_J{}^I = (\sigma^{-1})_J{}^I \dot{\tau}^J + \kappa^I =: \delta^I, \quad \Lambda_0^I = (M^{-1})_J{}^I \delta^J + \hat{\kappa}^I \tag{8.19}$$

where $\sigma_I{}^J = \{y_I, G^J\}$ depends only on x, is invertible on a suitable space of functions, and κ^I belongs to its kernel, so depends only on x. In particular, δ^I also depends only on x. Similarly, $\hat{\kappa}$ belongs to the kernel of M and can in general depend on x, q.

Abstracting from the $U(1)^3$ model, we arrive at the general theorem [89]:

Theorem 8.2.2. *Let C_I, G^I be as above. Then the physical Hamiltonian reads*

$$h = (\Lambda_0^I h_I)_{G=0} \tag{8.20}$$

This signifies that, as far as the dynamics of the physical degrees of freedom r on the reduced phase space $C = G = 0$ is concerned we can completely forget about the constraints. It is only needed to solve $G = 0$ and derive Λ_0 as in (8.19). which is also practically conceivable depending on the choice of G.

Proof. Using that

$$\Lambda_0^I C_I = \delta^I y_I + \tilde{h}, \quad \tilde{h} = \Lambda_0^I h_I \tag{8.21}$$

we have by steps familiar from the previous subsection for $f = f(r)$

$$\dot{f} = \{\Lambda_0 C_I, f\}_{C=G=0} = \{\tilde{h}, f\}_{G=0} \tag{8.22}$$

Since the solution of $G^I = 0$ is of the form $x^I = g^I(\tau)$ and does not depend on r, we can also put $G = 0$ before calculating the Poisson bracket. □

8.3. Reduced Phase Dynamics in Specific Gauges

We build the reduced phase and its physical Hamiltonian in various gauges which we propose in section 8.3.1. To keep the technicalities at a minimum, we restrict attention to linear gauges. In section 8.3.2, we then apply the theorems (8.2.1) and (8.2.2) to derive the corresponding physical Hamiltonian.

8.3.1. Gauge Fixing Choices

The discussion is divided into the (A, E) and (B, f) descriptions, respectively.

8.3.1.1. (A, E) Description

Since the Gauß constraint only depends on E, we have to use a gauge fixing condition that involves A. In the literature on Abelian gauge theories, the Coulomb gauge or axial gauge is popular. Here we work with an extension of these two to fix also spacetime diffeomorphism gauge symmetry. In what follows, we arbitrarily choose the z-coordinate as the longitudinal direction and the x, y coordinates as the transversal directions. The transversal spatial indices are now $I, J, \cdots \in \{x, y\}$ while the longitudinal index is denoted by $a = z$. We also consider "transversal" $\mathfrak{u}(1)^3$ indices $\alpha, \beta, \cdots \in \{1, 2\}$ and denote the longitudinal one by 3. The indices $\alpha, \beta, \ldots$ such as $j, k, \ldots$ are raised and lowered by the Kronecker $\delta_{\alpha\beta}$, so the index position of $\alpha, \beta, \ldots$ is also irrelevant.

A. transversally magnetic Coulomb - transversally electric anti-Coulomb - longitudinally electric axial gauge (TMC-TEaC-LEA)

The gauge conditions are

$$G^j = \delta^{IJ}\partial_I A^j_J, \quad \tilde{G}_j = \epsilon_{IJ}\delta^{IK}\partial_K E^J_j, \quad \tilde{G}_0 = E^z_3 - f \tag{8.23}$$

where f is a function that depends on coordinates and possibly also on physical time. Here $\epsilon_{IJ} = \epsilon^{IJ}$ is the fully skew symbol in two dimensions, and δ^I_J is the Kronecker symbol in two dimensions. We set

$$\partial^I := \delta^{IJ}\partial_J, \quad \hat{\partial}^I := \epsilon^{IJ}\partial_J, \quad \hat{\partial}_I := \epsilon_{IJ}\partial^J \tag{8.24}$$

to simplify the notation in what follows. It is clear that these structures break the (spatial) diffeomorphism covariance, as they should in order to be suitable gauge fixings, especially for spatial diffeomorphism gauge symmetry.

We see that the three sets of gauges (8.23) affect three different sets of canonical pairs that mutually Poisson commute among each other, specifically the transversal Coulomb pair $(\partial^I A^j_I, \Delta^{-1}\partial_I E^I_j)$, the transversal anti-Coulomb pair $(\hat{\partial}^I A^j_I, -\Delta^{-1}\hat{\partial}_I E^I_j)$ and the longitudinal axial pair (A^3_z, E^z_3). Here we have used the two-dimensional Laplacian

$$\Delta := \partial^I\partial_I \tag{8.25}$$

and Δ^{-1} is a Green function, namely $\Delta^{-1}(x_1, x_2) = (2\pi)^{-1}\ln(||x_1 - x_2||)$, where the flat two-dimensional metric has been used. Note that the transformation to these canonical coordinates is canonical, although Δ has a kernel. When inverting this transformation, one must take into account the kernel employing the boundary conditions discussed in chapter 6.

We perceive that the Gauß constraint $C_j = \partial_a E^a_j = 0$, together with the electric gauge conditions, implies that the "transversal curl" $\hat{\partial}_I E^I_j$ vanishes and that the "transversal divergence" $\partial_I E^I_j = -\partial_z E^z_j$ is determined in terms of E^z_j. In particular, if we choose the coordinate function f to be independent of z, as we will, then $\partial_I E^I_3 = \hat{\partial}_I E^I_3 = 0$. Since $\Sigma = \mathbb{R}^3$ is simply connected, $\hat{\partial}_I E^I_3 = 0$ indicates that $\delta_{IJ}E^J_3$ is an exact 1-form, that is, $E^I_3 = \delta^{IJ}\partial_J g$ for a certain 0-form g and thus $\partial_I E^I_3 = \Delta g = 0$, so the g is a harmonic function with respect to the x, y dependence.

By our boundary conditions (see section 6.2), E^I_3, which is harmonic itself, must approach zero at infinity. As is well known, in $\mathbb{R}^3$, the only smooth harmonic function decaying at infinity is the trivial function. Therefore, our gauge conditions signify $E^I_3 = 0$. Using the same argument from $\hat{\partial}_I E^I_\alpha = 0$, we conclude that $E^I_\alpha = \partial_I g_\alpha$. Plugging this into the Gauß constraint gives $\Delta g_\alpha = -\partial_z E^z_\alpha$ that is equivalent to $g_\alpha = h_\alpha - \Delta^{-1}\partial_z E^z_\alpha$ where h_α is harmonic and thus $E^I_\alpha = \partial^I[h_\alpha - \Delta^{-1}\partial_z E^z_\alpha]$. The second term vanishes at infinity because E^z_α does, hence the first term must approach δ^I_α at infinity. It follows that $\partial^I g_\alpha - \delta^I_\alpha$ is a harmonic function decaying at infinity, hence must vanish itself. Accordingly, $E^I_\alpha = \delta^I_\alpha - \partial_I\Delta^{-1}\partial_z E^z_\alpha$ is fully expressed in terms of E^z_α. This shows that the physical degrees of freedom, in this gauge, correspond to (A^α_z, E^z_α). Indeed, the condition $\partial^I A^j_I = 0$ together with the boundary conditions, implies that A^j_I is a two-dimensional curl $A^j_I = \hat{\partial}_I g^j$ for certain g^j and one would solve the constraints for both g^j and A^3_z such that A^α_z remains unconstrained.

It is essential to remark that the gauge conditions (8.23) do not contradict the requirement that the density weight two inverse spatial metric $Q^{ab} = E^a_j E^b_k \delta^{jk}$ is not degenerate.

B. Magnetic longitudinal axial - electric tranverse transverse axial (MLA-ETTA) gauge

The gauge conditions are

$$G^j := A^j_z, \quad G^I_\alpha := E^I_\alpha - f^I_\alpha \tag{8.26}$$

for certain functions f^I_α dependent of coordinate and possibly also of physical time, e.g. $f^I_\alpha = f\delta^I_\alpha$. This gauge is somewhat simpler because it does not involve derivatives. We assume that f^I_α do not depend on x, y. Hence, the Gauß constraint gives $\partial_z E^z_\alpha = 0$, which means that E^z_α is independent of z. This implies $E^z_\alpha = 0$ because if one moves to infinity along the $z-$axis at fixed finite values of x, y, E^z_α can only decay when it vanishes identically. The third Gauß constraint $\partial_z E^z_3 + \partial_I E^I_3 = 0$ is solved by $E^z_3 = k - \partial_z^{-1}\partial_I E^I_3$ where the Green function ∂_z^{-1} is chosen to be $(\partial_z^{-1})(z_1, z_2) = \frac{1}{2}\text{sgn}(z_1 - z_2)$ which makes it an antisymmetric translation-invariant kernel and k is a function independent of z. Since the second term in E^z_3 decays at infinity, k has to approach unity at infinity and $k - 1$ must vanish. However, since it is independent of z this is only possible if $k = 1$ identically. Thus, the true degrees of freedom, in this gauge, are (A^3_I, E^I_3) since one would solve the spacetime diffeomorphism constraints for A^α_I.

Note that this gauge is complementary to the previous one as it has disjoint sets of canonical pairs as physical degrees of freedom. Again it is easy to check that the gauge conditions (8.26) are not in conflict with the requirement that the density weight two inverse spatial metric $Q^{ab} = E^a_j E^b_k \delta^{jk}$ should not be degenerate.

8.3.1.2. (B, f) Description

First of all, we need to determine the asymptotic behaviours of the canonical variables (B, f). Recall from chapter 6 that by transcribing the boundary conditions of the (A, E) variables, i.e. (6.7), to (B, f), we get

$$\epsilon^{abc}\partial_b f^j_c \to F^a_j(n)/r + \mathcal{O}(1/r^2), \tag{8.27}$$

$$B^j_a \to \bar{G}^j_a(n)/r^3 + \mathcal{O}(1/r^4) \tag{8.28}$$

where $\bar{G}^j_a$ are even functions on the asymptotic 2-sphere and $\bar{G}^a_j = \epsilon^{abc}(\delta^d_b - n_b n^d)\frac{\partial G^j_c}{\partial n^d}$. Using (8.28) and also taking into account the requirement od having a well-defined symplectic structure, the asymptotic behaviour of f^a_j is determined

$$f^j_a \to c^j_a + \bar{F}^j_a(n) + \mathcal{O}(1/r) \tag{8.29}$$

where $\bar{F}^j_a(n)$ are odd functions on the asymptotic S^2 such that $F^a_j(n) = \epsilon^{abc}\left(\delta^d_b - n_b n^d\right)\frac{\partial \bar{F}^j_c}{\partial n^d}$ and c^j_a are constants. As shown in detail in section 6.4.1, utilising these boundary conditions one cannot find well-defined generators for asymptotic symmetries in the (B, f) description. As a result, the use of them is not admissible. In section 6.4.2, an alternative is considered, which ultimately leads to the conclusion that Hamiltonian and diffeomorphism constraints are well-defined generators of asymptotic temporal and spatial translations, respectively. Since here we just deal with the spacetime translations, the lack of generators for boosts and rotations does not affect the following computations. The lack of well-defined generators for boosts and rotations transpires not only in the (B, f) description but also in that of (A, E). The latter has been studied in detail in [90] (see section 6.2). Recall that since the $U(1)^3$ model is not GR, we do not expect to have full Poincaré group as its asymptotic symmetries. The appropriate boundary conditions have the decay behaviours just the same as (8.28) and (8.29) but with opossite parity conditions

$$\bar{F}^i_a(-n) = \bar{F}^i_a(n), \quad \bar{G}^a_i(-n) = -\bar{G}^a_i(n). \tag{8.30}$$

Although the chosen boundary conditions do not match those that one would choose in GR, because the $U(1)^3$ theory is just a toy model for a generally covariant theory with non-trivial

dynamics and an infinite number of degrees of freedom, we are allowed to exploit the freedom we have in defining the theory (the choice of boundary condition and choice of polarisation of the phase space is such an element of freedom) in order to learn as much as possible about the actual theory. Note that in the case of topologies without boundary, the choice of boundary conditions is irrelevant and the (A, E) description and the (B, f) description are totally equivalent (see chapter 6).

Recall from section 6.4.1.1 that the minimal condition on the Lagrange multiplier Λ^i associated with the Bianchi constraint, that ensures differentiability and convergence of the smeared constraint $C_i[\lambda^i] = \int d^3x\, \Lambda^i \partial_a B^a_i$, is

$$\Lambda^i = \lambda^i r + \mathcal{O}(1) \tag{8.31}$$

where λ^i are odd functions defined on the asymptotic S^2, i.e.

$$\lambda^i(-n) = -\lambda^i(n) \tag{8.32}$$

In what follows, the decay conditions imposed on Λ^i plays an important role. It is essential to note that the gauge fixings we discuss below are consistent with the above boundary conditions. We need also to express H^j_a in terms of f^j_a which can be easily obtained from combining (5.17) and $E^a_j = \delta^a_j + \epsilon^{abc}\partial_b f^j_c$,

$$H^j_a = \delta^j_a + \epsilon^{jkl}(\partial_a f^l_b - \partial_b f^l_a)[\delta^b_k + \frac{1}{2}\epsilon^{bde}(\partial_d f^k_e)] \tag{8.33}$$

As in the (A, E) description, here we view z-coordinate as the longitudinal direction and $I, J, \cdots \in \{x, y\}$ as the transversal ones. Likewise, in the internal indices, we consider 3 as the longitudinal and $\alpha, \beta, \cdots \in \{1, 2\}$ as the transversal directions.
In what follows, when we write an operator X as a polynomial of derivative and multiplication operators, its action on a function u is to be understood as the corresponding linear combination of consecutive applications of derivative and multiplication operators, for instance when $X := \partial_x g \partial_y^{-1} f + h$ in which f, g, h are functions, then $Xu = \partial_x(g\partial_y^{-1}(fu)) + hu$.

A. Electric transverse axial - longitudinally electric axial (ETA-LEA) gauge

The gauge conditions are

$$G^j_I = f^j_I - \sigma^j_I, \quad G = f^3_z - \sigma \tag{8.34}$$

in which σ^j_I and σ are phase space independent functions which may depend only on the physical time, τ. We assume that σ^j_I and σ are independent of the spatial coordinates, then not only will the calculations be more convenient, but also it ensures that (8.34) is consistent with the boundary conditions (8.29) and (8.30). Applying the gauge fixing (8.34) on H^j_a leads to

$$\begin{aligned}
H^j_x =& \delta^j_x + \epsilon^{jkl}(\partial_x f^l_b - \partial_b f^l_x)[\delta^b_k + \frac{1}{2}\epsilon^{bde}(\partial_d f^k_e)] = \delta^j_x + \epsilon^{j3l}(\partial_x f^l_z)\\
H^j_y =& \delta^j_y + \epsilon^{jkl}(\partial_y f^l_b - \partial_b f^l_y)[\delta^b_k + \frac{1}{2}\epsilon^{bde}(\partial_d f^k_e)] = \delta^j_y + \epsilon^{j3l}(\partial_y f^l_z)\\
H^j_z =& \delta^j_z + \epsilon^{jkl}(\partial_z f^l_b - \partial_b f^l_z)[\delta^b_k + \frac{1}{2}\epsilon^{bde}(\partial_d f^k_e)] = \delta^j_z - \epsilon^{j1l}(\partial_x f^l_z) - \epsilon^{j2l}(\partial_y f^l_z) + \epsilon^{jkl}(\partial_y f^l_z)(\partial_x f^k_z)
\end{aligned} \tag{8.35}$$

Thus, $H^3_I = H^\alpha_z = 0$ and the non-vanishing ones are

$$\begin{aligned} H^1_x &= 1 - \partial_x f^2_z, \quad H^2_x = \partial_x f^1_z, \\ H^1_y &= -\partial_y f^2_z, \quad H^2_y = 1 + \partial_y f^1_z, \\ H^3_z &= 1 - \partial_x f^2_z + \partial_y f^1_z - (\partial_x f^2_z)(\partial_y f^1_z) + (\partial_x f^1_z)(\partial_y f^2_z) \\ &= H^1_x H^2_y - H^2_x H^1_y \end{aligned} \tag{8.36}$$

With the H^a_j evaluated at the gauge cut, the constraints are much easier to solve for B^I_i and B^z_3. Indeed, the matrix representation of (8.18) in this gauge is

$$\begin{bmatrix} \partial_x & 0 & 0 & \partial_y & 0 & 0 & 0 \\ 0 & \partial_x & 0 & 0 & \partial_y & 0 & 0 \\ 0 & 0 & \partial_x & 0 & 0 & \partial_y & \partial_z \\ 0 & 0 & -H^2_x & 0 & 0 & -H^2_y & 0 \\ 0 & 0 & H^1_x & 0 & 0 & H^1_y & 0 \\ H^2_x & -H^1_x & 0 & H^2_y & -H^1_y & 0 & 0 \\ H^1_x & H^2_x & 0 & H^1_y & H^2_y & 0 & H^3_z \end{bmatrix} \begin{bmatrix} B^x_1 \\ B^x_2 \\ B^x_3 \\ B^y_1 \\ B^y_2 \\ B^y_3 \\ B^z_3 \end{bmatrix} + \begin{bmatrix} \partial_z B^z_1 \\ \partial_z B^z_2 \\ 0 \\ B^z_2 H^3_z \\ -B^z_1 H^3_z \\ 0 \\ 0 \end{bmatrix} = 0 \tag{8.37}$$

Note that the existence of the solutions for this system of equations essentially depends on the non-vanishing requirement of H^3_z which is indeed guaranteed in the MTA-MLA gauge (see Appendix A). According to (5.17) and the fact that in this gauge $H^1_z = H^2_z = 0$, vanishing of H^3_z would lead to a degenerate E, conflicting with the necessity of a non-degenerate spatial metric. Therefore, $H^3_z \neq 0$ everywhere.
As one can see in detail in Appendix A, solving the system (8.37) with the above-mentioned boundary conditions completely determines (B^I_i, B^z_3) in terms of B^z_α. Thus, the physical degrees of freedom, in this gauge, are (B^z_α, f^α_z).

In this gauge, one can easily check that

$$E^a_i = \begin{bmatrix} H^2_y & -H^1_y & 0 \\ -H^2_x & H^1_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.38}$$

Hence, the gauge conditions do not contradict the requirement that the density weight two inverse spatial metric $Q^{ab} = E^a_j E^b_k \delta^{jk}$ should be non-degenerate.

B. Electric transverse transverse axial- electric longitudinal axial (ETTA-ELA) gauge

The gauge conditions are

$$G^\alpha_I = f^\alpha_I - \sigma^\alpha_I, \quad G^j = f^j_z - \sigma^j \tag{8.39}$$

where σ^α_I and σ^j are phase space independent functions which may depend only on the physical time, τ. We assume that σ^α_I and σ^j are independent of the spatial coordinates to make the following calculations simpler. Note that (8.39) is compatible with the boundary conditions (8.29) and (8.30). A straightforward calculation attests that, in this gauge, H^j_a has a pretty

simple form, that is

$$\begin{aligned} H^1_x &= H^2_y = 1 + \partial_x f^3_y - \partial_y f^3_x \\ H^1_z &= \partial_z f^3_y - \partial_y f^3_z = \partial_z f^3_y \\ H^2_z &= \partial_x f^3_z - \partial_z f^3_x = -\partial_z f^3_x \\ H^2_x &= H^3_x = H^1_y = H^3_y = 0, \quad H^3_z = 1 \end{aligned} \tag{8.40}$$

With these H's, we can observe that the system of equations to be solved here is even simpler than that of the previous gauge, i.e. (8.37). Indeed, in this gauge, the matrix representation of (8.18) is

$$\begin{bmatrix} \partial_x & 0 & \partial_y & 0 & \partial_z & 0 & 0 \\ 0 & \partial_x & 0 & \partial_y & 0 & \partial_z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_z \\ 0 & 0 & 0 & 0 & 0 & 1 & -H^2_z \\ 0 & 0 & 0 & 0 & -1 & 0 & H^1_z \\ 0 & -H^1_x & H^1_x & 0 & H^2_z & -H^1_z & 0 \\ H^1_x & 0 & 0 & H^1_x & H^1_z & H^2_z & 1 \end{bmatrix} \begin{bmatrix} B^x_1 \\ B^x_2 \\ B^y_1 \\ B^y_2 \\ B^z_1 \\ B^z_2 \\ B^z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \partial_x B^x_3 + \partial_y B^y_3 \\ -H^1_x B^y_3 \\ H^1_x B^x_3 \\ 0 \\ 0 \end{bmatrix} = 0 \tag{8.41}$$

Note that the existence of the solutions for (8.41) essentially depends on the non-vanishing necessity of H^1_x that is assured in the ETTA-ELA gauge (see Appendix A). According to (5.17) and the fact that in this gauge $H^3_I = H^2_x = H^1_y = 0$, vanishing of H^1_x would lead to a degenerate E, which is in conflict with the requirement that the spatial metric must be non-degenerate. Thus, $H^1_x \neq 0$ everywhere.
In Appendix A, we have explicitly shown that by solving the system (8.41) using the boundary conditions, (B^I_α, B^z_j) can be completely expressed in terms of B^I_3. Consequently, the true degrees of freedom, in this gauge, are (B^I_3, f^3_I).

In this gauge, the electric field is given by

$$E^a_i = \begin{bmatrix} 1 & 0 & -H^1_z \\ 0 & 1 & -H^2_z \\ 0 & 0 & H^1_x \end{bmatrix} \tag{8.42}$$

Accordingly, again, the gauge conditions do not contradict the requirement that the density weight two inverse spatial metric $Q^{ab} = E^a_j E^b_k \delta^{jk}$ should be non-degenerate.

8.3.2. Reduced Phase Space Dynamics in Various Gauges

In this section, we discuss the dynamics of the reduced phase space in various gauges introduced in section 8.3.1 and using different pairs of canonical variables.

8.3.2.1. MLA-ETTA Gauge in (A, E) Description

We verify that the assumptions of theorem 8.2.1 apply:
Recall first that in this gauge we have $w = (E^z_j, A^j_z)$, $z = (E^I_\alpha, A^\alpha_I)$ and $r = (E^z_\alpha, A^\alpha_z)$. We consider (minus) the A^j_z as the momenta v_A and (minus) the A^α_I as the momenta y_I. Likewise, the Gauß constraints are denoted by C_A and the spacetime constraints by C_I.

We have

$$\{C(\lambda), G_j\} = \{C(\lambda), A^j_z\} = -\kappa\partial_z\lambda^j \tag{8.43}$$

which has as a kernel the space of z-independent functions. If we consider as space of functions λ^j those decaying at infinity at least as $1/r^2$ this implies that the kernel vanishes and the matrix σ_{AB} regarded as the integral kernel $\{C_j(x), G_j(y)\}$ is regular.

Next we have

$$\tilde{C}(U,F) = \int d^3x\; B^a_j\; (FH^j_a + \epsilon^{jkl}H^k_aU^l) =: \int d^3x\; B^a_j\; u^j_a =: \int d^3x\; A^j_a v^a_j \tag{8.44}$$

with

$$v^a_j = \epsilon^{abc}\partial_b u^j_c,\;\; u^j_a = FH^j_a + \epsilon^{jkl}H^k_aU^l \tag{8.45}$$

Geometrically, the relation between the density weight minus smearing functions F, U^j one and the density weight zero lapse and shift functions N, N^a is that

$$F = N|\det(E)|^{-1/2},\;\; U^jE^a_j = N^a \tag{8.46}$$

i.e. N^a is an "electric shift" [87] if U, F are assumed to be phase space independent[1].

Obviously

$$\{\tilde{C}(U,F), E^a_j\} = -\kappa v^a_j \tag{8.47}$$

and we need to confirm that, given asymptotically flat boundary conditions, $-\kappa v^\alpha_I = \dot{\tau}^\alpha_I$ signifies unique values for F, U, so that the gauge fixing be admissible. It is enough to verify this at the gauge cut $C_j = \partial_a E^j = 0,\;\; E^I_\alpha = \tau^I_\alpha$, because then by continuity and using the usual argumentation of the implicit function theorem, it holds in a neighbourhood of the gauge cut (to make this more precise, one should give the phase space a suitable (Banach) manifold topology).

Specifically, we work with $\tau^I_\alpha = f\delta^I_\alpha$, where f depends at most on the physical time. Indeed, this is consistent with asymptotic flatness if we further specify $f \equiv 1$, but for the moment, let us be more general. As mentioned earlier, it follows from the Gauß constraint that $\partial_z E^z_\alpha = 0$, which implies $E^z_\alpha = 0$ if we use the boundary conditions. Moreover, $\partial_a E^a_3 = 0$ is uniquely solved by $E^z_3 = 1 - \partial_z^{-1}\partial_I E^I_3$, so E^z_3 is completely determined by E^I_3. Evaluation of the H^j_a at the gauge cut results in

$$\begin{aligned}
H^3_z &= \det(\{E^I_\alpha\}) = f^2\\
H^3_I &= \epsilon^{\alpha\beta}\epsilon_{IJ}E^I_\beta E^z_\beta = 0\\
H^\alpha_z &= \epsilon_{IJ}\epsilon^{\alpha\beta}E^I_\beta E^J_3 = -f\delta^\alpha_I\; E^I_3\\
H^\alpha_I &= \epsilon_{IJ}\epsilon^{\alpha\beta}[E^J_\beta E^z_3 - E^J_3 E^z_\beta] = f\delta^\alpha_I E^z_3
\end{aligned} \tag{8.48}$$

and find with $\hat{U}_\alpha = \epsilon_{\alpha\beta}U^\beta,\;\hat{U}^\alpha = \epsilon^{\alpha\beta}U_\beta$

$$\begin{aligned}
u^3_z &= FH^3_z + \epsilon_{\alpha\beta}H^\alpha_z U^\beta = FH^3_z + H^\alpha_z\hat{U}_\alpha\\
u^\alpha_z &= FH^\alpha_z + \epsilon_{\alpha\beta}(H^\beta_z U^3 - H^3_z U^\beta) = [F\delta^\alpha_\beta + U^3\epsilon_{\alpha\beta}]H^\beta_z - H^3_z\hat{U}^\alpha\\
&=: \Sigma^\alpha_\beta H^\beta_z - H^3_z\hat{U}^\alpha\\
u^3_I &= FH^3_I + \epsilon^{\alpha\beta}H^\alpha_I U^\beta = H^\alpha_I\hat{U}_\alpha\\
u^\alpha_I &= FH^\alpha_I + \epsilon^{\alpha\beta}(H^\beta_I U^3 - H^3_I U^\beta) = \Sigma^\alpha_\beta H^\beta_I
\end{aligned} \tag{8.49}$$

[1] Recall from section 1.3.1 that the lapse function N and shift vector N^a can be viewed as Lagrange multipliers of the Hamiltonian and diffeomorphism constraints, respectively. Alternatively, we can consider the density weight minus one versions of them as the smearing functions of the constraints, i.e., $N|\det(E)|^{-1/2}$ and $N^aE^j_a$, respectively. This is the geometric origin of (8.46).

The stability condition emanating from (8.47) is

$$-\dot{f}/\kappa\delta^I_\alpha = v^I_\alpha = \epsilon^{IJ}[\partial_J u^\alpha_z - \partial_z u^\alpha_J] \tag{8.50}$$

This can be transformed into

$$\partial_I u^\alpha_z = \frac{\dot{f}}{\kappa}\epsilon_{IJ}\delta^J_\alpha + \partial_z u^\alpha_I \tag{8.51}$$

Let us define

$$a := fE^z_3 F,\ b := fE^z_3 U^3 \tag{8.52}$$

then employing (8.48), (8.49)

$$u^\alpha_I = a\delta^\alpha_I + b\epsilon_{\alpha\beta}\delta^\beta_I \tag{8.53}$$

and one can disentangle the four equations (8.51)

$$\partial_x u^1_z = \partial_y u^2_z = \partial_z a;\ \ \partial_y u^1_z = -\partial_x u^2_z = \partial_z b - \frac{\dot{f}}{\kappa} \tag{8.54}$$

So we note that (u^1_z, u^2_z) forms a Cauchy-Riemann pair, i.e. $w = u^1_z + iu^2_z$ is a holomorphic function in $x + iy$. On the other hand, it follows from (8.48) and (8.49) that u^α_z is a bounded function if U^j, F, E^I_3 are or that it decays at least as r^{-1} if we let N, N^a and hence F, U^j (recall (8.46)) asymptotically approach at most a constant and zero, respectively. In the former case, we conclude from Liouville's theorem that u^α_z =const. In the latter case, we immediately obtain $u^\alpha_z = 0$ since a holomorphic function decays at most in $x^2 + y^2$ but not in r^2.

In either case

$$\partial_z a = 0 = \partial_z b - \frac{\dot{f}}{\kappa} \tag{8.55}$$

that is solved by

$$a = a_0(x, y),\ b = b_0(x, y) + \frac{\dot{f}}{\kappa}z \tag{8.56}$$

Comparing with (8.52), we infer that, up to $\mathcal{O}(1/r)$ corrections, a approaches a constant asymptotically and that b falls off at least as $1/r$. Indeed, since a function of x, y cannot decay in z as r does, we must have a_0 =const. and $b_0 = \dot{f} = 0$. Moreover, by comparing the asymptotic values we find $a_0 = f$ and $f = 1$ follows from $E^a_j = \delta^a_j + \mathcal{O}(1/r)$. Accordingly

$$F = \frac{1}{E^z_3},\ U^3 = 0 \tag{8.57}$$

and thus

$$u^\alpha_z = FH^\alpha_z - H^3_z\hat{U}^\alpha = c^\alpha \tag{8.58}$$

However, because $H^3_z = 1$, a non-vanishing constant is conflicting with a decaying U^α, so that indeed $u^\alpha_z = 0$ and

$$\hat{U}^\alpha = \frac{H^\alpha_z}{E^z_3} = -\delta^\alpha_I\frac{E^I_3}{E^z_3} \tag{8.59}$$

We deduce that the gauge fixing has led to a unique solution for F, U^j and is therefore admissible.

We now employ the theorem 8.2.1 to derive the physical Hamiltonian h, which depends only on true degrees of freedom. To do this, first, we decompose

$$\tilde{C}(F, U) = \int d^3x\ [v^I_\alpha A^\alpha_I + v^z_j A^j_z + v^I_3 A^3_I] \tag{8.60}$$

and identify the first term with $\Lambda^I M_I\ {}^J y_I$, the second term with $\Lambda^I N_I\ {}^A v_A$ and the third term with $\Lambda^I h_I$, where Λ^I corresponds to (F, U). Integrating by parts, we obtain

$$\begin{aligned}
\Lambda^I h_I &= \int d^3x\ \epsilon^{IJ}(\partial_J u^3_z - \partial_z u^3_J) A^3_I \\
&= \int d^3x\ [u^3_z(\epsilon^{IJ}\partial_I A^3_J) + u^3_I(-\epsilon^{IJ}\partial_z A^3_J)] \\
&= \int d^3x\ [u^3_z B^z_3 + u^3_I B^I_3] \\
&= \int d^3x\ [F(H^3_z B^z_3 + H^3_I B^I_3) + \hat{U}^\alpha(H^\alpha_z B^z_3 + H^\alpha_I B^I_3)] \\
&= \int d^3x\ [F H^3_a B^a_3 + \hat{U}_\alpha H^\alpha_a B^a_3] \qquad (8.61)
\end{aligned}$$

where B^I_3 is to be evaluated in the magnetic longitudinal axial gauge $A^3_z = 0$. Notice that (8.61) in fact is independent of U^3 even before applying the constraints, gauge conditions and gauge-fixed Lagrange multipliers.

According to the theorem 8.2.1, we need only impose the gauge $E^I_\alpha = \delta^I_\alpha$ and use the solution $E^z_\alpha = 0,\ E^z_3 = 1 - \partial_z^{-1}\partial_I E^I_3$ of the Gauß constraint as well as the solution for $\Lambda^I_0 = (F_0, U_0)$ at these values, which can be found in (8.57), (8.59), and then plug them into (8.61). This gives the final expression

$$\begin{aligned}
h &= \int d^3x\ [B^z_3(F + H^\alpha_z \hat{U}_\alpha) + B^I_3 H^\alpha_I \hat{U}_\alpha] \\
&= \int d^3x\ [B^z_3(\frac{1}{E^z_3} + \delta_{IJ}\frac{E^I_3 E^J_3}{E^z_3}) - B^I_3 \delta_{IJ} E^J_3] \\
&= \int d^3x\ A^3_I\ \epsilon^{IJ}[\partial_J(\frac{1 + \delta_{IJ}E^I_3 E^J_3}{E^z_3}) - \partial_z \delta_{JK} E^K_3] \qquad (8.62)
\end{aligned}$$

As expected, the physical Hamiltonian (8.62) has the following features:

(i) Linearity in momentum A^3_I.

(ii) Non-polynomiality in the configuration variable E^I_3.

(iii) Spatial non-locality.

The latter two attributes are due to the appearance of $1/E^z_3$ with $E^z_3 = 1 - \partial_z^{-1}\partial_I E^I_3$. Note that the term $1 + \delta_{IJ}E^I_e E^J_3$ makes sense from a dimensional point of view since E^a_j is dimensionless and A^j_a has dimension cm^{-1}. Thus the Hamiltonian density has dimension cm^{-2} and to turn it into a quantity with dimension of energy, we have to divide h by κ (when time is multiplied by the speed of light). Actually, we should have been working with constraints rescaled by $1/\kappa$ all along, as would naturally arise from an action [91]. The fact that the three individual terms in the Hamiltonian have different density weights is due to the gauge fixing condition $E^I_\alpha = \delta^I_\alpha$, which breaks the density weight of E^I_α.

We can now acquire the physical equations of motion and show strategies for solving them despite the complexity of the Hamiltonian (8.62). Using $\{E^I_3(x), A^3_J(y)\} = \kappa\delta^I_J\delta(x, y)$, for E^I_3 we get

$$\dot{E}^I_3 = \{\frac{h}{\kappa}, E^I_3\} = -\epsilon^{IJ}[\partial_J(\frac{1 + \delta_{IJ}E^I_3 E^J_3}{E^z_3}) - \partial_z \delta_{JK} E^K_3] \qquad (8.63)$$

As expected, due to the linearity of h in A, the equation of motion for E^I_3 is first-order in time and self-closing, i.e., its time derivative no longer involves A^3_I and we can study (8.63) completely independently of the equation of motion for A^3_I. Next we have

$$\dot{A}^3_I(x) = \{\frac{h}{\kappa}, A^3_I(x)\} = (-B^I_3 + 2B^z_3 \frac{E^I_3}{E^3_z})(x) - \int d^3y \, (B^z_3 \frac{1+\delta_{JK}E^J_3 E^K_3}{(E^z_3)^2})(y) \, \frac{\delta E^z_3(y)}{\delta E^I_3(x)} \tag{8.64}$$

Employing $E^z_3 = -\partial_z^{-1}\partial_J E^J_3$, integrations by parts and the antisymmetry of the integral kernel of ∂_z^{-1} as well as $[\partial_z^{-1}, \partial_I] = 0$, for any smearing function s we obtain

$$\int d^3y \, s(y) \frac{\delta E^z_3(y)}{\delta E^I_3(x)} = \int d^3y \, (\partial_z^{-1}\partial_J s)(y) \frac{\delta E^J_3(y)}{\delta E^I_3(x)} = (\partial_z^{-1}\partial_J s)(x) \tag{8.65}$$

whence

$$\dot{A}^3_I(x) = \{\frac{h}{\kappa}, A^3_I(x)\} = -B^I_3 + 2B^z_3 \frac{E^I_3}{E^3_z} - \partial_z^{-1}\partial_I (B^z_3 \frac{1+\delta_{JK}E^J_3 E^K_3}{(E^z_3)^2}) \tag{8.66}$$

Note the abbreviations $B^I_3 := -\epsilon^{IJ}\partial_z A^3_J$, $B^z_3 := \epsilon^{IJ}\partial_I A^3_J$.

Equations (8.63), (8.66) suggest the following solution strategy: First solve (8.63), which does not depend on A. Then put this solution into (8.66), which is then a first-order linear integro-differential equation system in all derivatives and the anti-derivative ∂_z^{-1}. For the first task, we introduce the "divergence" and the "curl" of E^I_3

$$D := \partial_I E^I_3, \; C := \epsilon^{IJ}\partial_I(\delta_{JK}E^K_3) \tag{8.67}$$

and decompose

$$E^I_3 = \Delta^{-1}[\delta^{IJ}\partial_J D - \epsilon^{IJ}\partial_J C] \tag{8.68}$$

where Δ is the transversal Laplacian and we used the boundary conditions (E^I_3 must decay as $1/r$) to exclude a non-vanishing kernel of Δ. Taking the divergence and curl of (8.63), one finds

$$-\dot{D} = -\partial_z C, \;\; -\dot{C} = -\Delta[\frac{1+\delta_{IJ}E^I_3 E^J_3}{E^z_3}] + \partial_z D \tag{8.69}$$

Note that if we were to omit the nonlinear interaction term in the second equation of (8.69) and would iterate it, we would get

$$\ddot{D} + \partial_z^2 D = \ddot{C} + \partial_z^2 C = 0 \tag{8.70}$$

which is a "Euclidean" wave operator confined to the z-direction. We interpret this to display the closeness of the model to Euclidean gravity. Continuing with (8.69), we recall that $D = -\partial_z E^z_3$, so the first equation in (8.69) implies (again, possible integration constants have to vanish)

$$C = -\dot{E}^z_3 \tag{8.71}$$

This means that one can express all of E^I_3 just in terms of $F := E^z_3$

$$E^I_3 = -\Delta^{-1}[\delta^{IJ}\partial_J\partial_z - \epsilon^{IJ}\partial_J\partial_t]F \tag{8.72}$$

which allows writing the second equation in (8.69) just in terms of F

$$\ddot{F} + \partial_z^2 F = -\Delta[\frac{1+\delta_{IJ}E^I_3 E^J_3}{F}] \tag{8.73}$$

where (8.72) is to be applied to the r.h.s.. The equation (8.73) has the undesirable property of being non-polynomial (due to $1/F$) and spatially non-local (owing to Δ^{-1} in (8.72)). We can get rid of the first property by multiplying (8.73) by F^3 because the Laplacian on the r.h.s. produces factors at most of $1/F^3$. We can get rid of the second property by introducing $G = \Delta^{-1}F,\ F = \Delta G$. Consequently, we write

$$E^I_3 = -[\delta^{IJ}\partial_J\partial_z - \epsilon^{IJ}\partial_J\partial_t]G \tag{8.74}$$

and

$$(\Delta G)^3[\partial_t^2 + \partial_z^2](\Delta G) = -(\Delta G)^3[\Delta[\frac{1+\delta_{IJ}E^I_3E^J_3}{\Delta G}] \tag{8.75}$$

where we must substitute (8.74). Hence, (8.75) is a polynomial in G of degree four and involves temporal derivatives up to second-order and spatial derivatives up to fourth-order. To see this, we write with $K := \delta_{IK}E^I_3E^J_3$

$$\begin{aligned} F^3\Delta(\frac{1+K}{F}) &= F^3(\frac{\Delta K}{F} - 2\frac{\partial_I K}{F^2}(\partial^I F) + (1+K)[-\frac{\Delta F}{F^2} + 2\frac{(\partial_I F)(\partial^I F)}{F^3}]) \\ &= F^2(\Delta K) - 2F(\partial_I K)(\partial^I F) + (1+K)[-F(\Delta F) + 2(\partial_I F)(\partial^I F)] \end{aligned} \tag{8.76}$$

We leave the further analysis of (8.75) to future work. Remarkably, half of the equations of motion can be encoded with only a "single" PDE! The fact that the theory is self-interacting is expressed by the fact that this PDE is far from being linear and has a spatial degree higher than two (namely four) but still has a temporal degree of at most two. In particular, it is a quasi-linear equation in terms of the highest temporal derivatives.

As for Hilbert space representations supporting (some ordering of) h as a densely defined operator as well as the corresponding spectral problem, we consider the possibility of a representation in which E^I_3 acts as a multiplication operator, i.e. $\mathcal{H} = L_2(\widehat{\mathcal{E}}, d\mu)$, where $\widehat{\mathcal{E}}$ is a suitable distributional extension of the set $\mathcal{E}$ of the classical fields E^I_3 and μ is a probability measure on it. Then $A^3_I = i\ell_P^2\delta/\delta E^I_3 + D_I(E)$, where D_I is taken so that A^3_I is a symmetric, operator-valued distribution, and $\ell_P^2 = \hbar\kappa$ is the Planck area. Then for any symmetric ordering of h, after reordering in such a way that the functional derivatives act directly on the Hilbert space vector $\psi \in \mathcal{H}$, we see that h acts as

$$(h\psi)[E] = i\ell_P^2 \int d^3x\ V^I_3(E(x))\ \frac{\delta\psi}{\delta E^I_3(x)} + U[E]\psi[E] \tag{8.77}$$

where the potential term $U[E]$ acts as a multiplication operator and emanates from reordering h into the form presented as well as from the contribution $V^I D_I$ where

$$V^I_3(E(x)) = \epsilon^{IJ}[\partial_J(\frac{1+K}{F}) - \partial_z E^J_3] \tag{8.78}$$

Reordering h in the form exhibited generally generates singularities in the form of (derivatives of) δ-distributions evaluated at zero, which must be regularised and which guide the choice of μ and hence D_I to cancel them. Also, the form of V^I, U may bring up non-trivial domain questions.

We can now recast the spectral problem for h into the form

$$i\ell_P^2 < V^I, \frac{\delta}{\delta E^I_3} >\ \psi = (\lambda - U)\ \psi \tag{8.79}$$

where $< .,. >$ denotes the inner product on $L_2(d^3x, \mathbb{R}^3)$ or equivalently with the WKB Ansatz $\psi = \exp(-i\frac{S}{\ell_P^2})$

$$< V^I, \frac{\delta S}{\delta E^I_3} >= (\lambda - U) \tag{8.80}$$

Equation (8.80) has the form of a "first-order linear functional partial differential equation", that is, the infinite-dimensional analogue of the familiar case of a first-order linear partial differential equation. This is the simplest kind of first-order FPDE (functional PDE) imaginable. It is not even quasi-linear (which would let V^I, U depend on S) or even non-linear (which would let V^I, U depend on $S, \delta S/\delta E^I_3$). Note also that, unlike Hamiltonians with at least quadratic dependence on the momenta, the WKB Ansatz leads to a Hamilton-Jacobi equation (8.80) which is "exact" in this case.

One can then try to solve (8.80) by a functional version of the method of characteristics [293] which brings out the closeness of the spectral problem (8.80) to the solution of the classical equations of motion (8.63) in this case. For this purpose, we solve the equations of motion (8.63), i.e., we determine the integral curves of the vector field V^I. Suppose that we have found the maximal unique solution $X^I(t, x; G)$ given initial data $X^I(0, x) = G^I(x)$, where G ranges in some submanifold $\Sigma \subset \mathcal{E}$ of co-dimension one, which is everywhere transversal to the vector field V [89]. We also solve

$$\dot{s}(t) = \lambda - U[E = X(t, .; G)] \tag{8.81}$$

with initial condition $s(0) = s_0[G]$ giving rise to a unique maximal solution $s(t; G)$. The transversality of Σ to the flow lines of V implies that at least for small t i.e. close to Σ we can invert the equation $E^I_3(x) = X^I(t, x; G)$ for

$$t = \tau[E],\ G^I(x) = \sigma^I(x; E) \tag{8.82}$$

and then

$$S[E] := s(t, G)_{t=\tau, G=\sigma} \tag{8.83}$$

solves (8.80) with boundary condition $S[G] = s_0[G]$ i.e. $S_{|\Sigma} = s_0$.

The exact technical implementation of these steps may be quite involved, but it is remarkably much less intricate than one might have worried. We leave the details of this programme to future work.

8.3.2.2. TMC-TEaC-LEA Gauge in (A, E) Description

We verify that the assumptions of the theorem 8.2.1 apply:
We consider (minus) the $v^j := \partial^I A^j_I$ as the momenta v_A and (minus) the $y^j := \hat{\partial}^I A^j_I, A^3_z$ as the momenta y_I. The configuration variables corresponding to these momenta are $u_j := \Delta^{-1}\partial_I E^I_j$, $x_j := \Delta^{-1}\hat{\partial}_I E^I_j$ and E^z_3, respectively. Similarly, the Gauß constraints is denoted by C_A and the spacetime constraints by C_I. In what follows, the canonical variables are $w = (u_j, v^j), z = \{(x_j, y^j), (E^z_3, A^3_z)\}$ and $r = (E^z_\alpha, A^\alpha_z)$, where the latter plays the role of our true degrees of freedom. Thus, we need to express E^I_j and A^j_I in terms of u_j, x_j and v^j, y^j, respectively

$$E^I_j = \delta^I_j + \partial^I u_j + \hat{\partial}^I x_j \tag{8.84}$$

$$A^j_I = \Delta^{-1}\left(\partial_I v^j + \hat{\partial}_I y^j\right) \tag{8.85}$$

And the constraints written in terms of the canonical variables are

$$\begin{aligned}\tilde{C}_j &= \epsilon_{jkl}(B^I_k H^l_I + B^z_k H^l_z)\\ &= \epsilon_{jkl}\left(\epsilon^{IJ}H^l_I(\partial_J A^k_z - \partial_z(\partial_J v^k + \hat{\partial}_J y^k)) + H^l_z \Delta y^k\right)\\ &= \epsilon_{j3l}\epsilon^{IJ}H^l_I\partial_J A^3_z + \epsilon_{jkl}\left(H^l_z\Delta - \epsilon^{IJ}H^l_I\partial_z\hat{\partial}_J\right)y^k - \epsilon_{jkl}\epsilon^{IJ}H^l_I\partial_z\partial_J v^k + \epsilon_{j\alpha l}\epsilon^{IJ}H^l_I\partial_J A^\alpha_z \end{aligned} \tag{8.86}$$

and

$$\begin{aligned}\tilde{C}_0 &= B^I_j H^j_I + B^z_j H^j_z\\ &= \epsilon^{IJ}H^j_I\left(\partial_J A^j_z - \partial_z(\partial_J v^j + \hat{\partial}_J y^j)\right) + H^j_z\Delta y^j\\ &= \epsilon^{IJ}H^3_I\partial_J A^3_z + \left(H^j_z\Delta - \epsilon^{IJ}H^j_I\partial_z\hat{\partial}_J\right)y^j - \epsilon^{IJ}H^j_I\partial_z\partial_J v^j + \epsilon^{IJ}H^\alpha_I\partial_J A^\alpha_z\end{aligned} \tag{8.87}$$

where we have used $B^I_j = \epsilon^{IJ}\left(\partial_J A^j_z - \partial_z(\partial_J v^j + \hat{\partial}_J y^j)\right)$ and $B^z_j = \Delta y^j$. It can be seen from (8.86) and (8.87) that h_I introduced in (8.3) is $(\epsilon_{j\alpha l}\epsilon^{IJ}H^l_I\partial_J A^\alpha_z, \epsilon^{IJ}H^\alpha_I\partial_J A^\alpha_z)$. On the other hand, translating the gauge conditions (8.23) to these canonical variables results in

$$G^j = v^j,\ \tilde{G}_j = x_j,\ \tilde{G}_0 = E^z_3 - f \tag{8.88}$$

Recall that f has been assumed to be independent of z. The non-zero H's evaluated at the gauge cut are

$$\begin{aligned}H^1_x &= f(1+\partial_y u^2) - E^z_2(\partial_y u^3)\\ H^2_x &= E^z_1(\partial_y u^3) - f(\partial_y u^1)\\ H^3_x &= E^z_2(\partial_y u^1) - E^z_1(1+\partial_y u^2)\\ H^1_y &= E^z_2(\partial_x u^3) - f(\partial_x u^2)\\ H^2_y &= f(1+\partial_x u^1) - E^z_1(\partial_x u^3)\\ H^3_y &= E^z_1(\partial_x u^2) - E^z_2(1+\partial_x u^1)\\ H^3_z &= (1+\partial_x u^1)(1+\partial_y u^2) - (\partial_x u^2)(\partial_y u^1)\end{aligned}$$

and $H^\alpha_z = 0$. The stability conditions for $\tilde{G}_j$ and $\tilde{G}_0$ at the gauge cut are

$$0 = \{H, x_1\} = \partial^I\left[-\partial_I(\lambda^2 H^3_z) - \partial_z(\lambda^3 H^2_I - \lambda^2 H^3_I + \lambda H^1_I)\right] \tag{8.89}$$

$$0 = \{H, x_2\} = \partial^I\left[\partial_I(\lambda^1 H^3_z) - \partial_z(\lambda^1 H^3_I - \lambda^3 H^1_I + \lambda H^2_I)\right] \tag{8.90}$$

$$0 = \{H, x_3\} = \partial^I\left[\partial_I(\lambda H^3_z) - \partial_z(\lambda^2 H^1_I - \lambda^1 H^2_I + \lambda H^3_I)\right] \tag{8.91}$$

$$0 = \{H, E^z_3\} = \hat{\partial}^I\left[\lambda^2 H^1_I - \lambda^1 H^2_I + \lambda H^3_I\right] \tag{8.92}$$

respectively, which are supposed to be solved for λ^i and λ. From (8.91) and (8.92), one concludes

$$\partial_I(\lambda H^3_z) - \partial_z(\lambda^2 H^1_I - \lambda^1 H^2_I + \lambda H^3_I) = \hat{\partial}_I g_1 \tag{8.93}$$

$$\lambda^2 H^1_I - \lambda^1 H^2_I + \lambda H^3_I = \partial_I g_2 \tag{8.94}$$

respectively, where g_1 and g_2 are certain 0-forms. Substituting (8.94) into (8.93) and applying $\hat{\partial}^I$ to both sides leads to $\Delta g_1 = 0$. Since g_1 is harmonic and also decays at infinity due to the boundary conditions, we get $g_1 = 0$. Consequently, (8.93) becomes

$$\lambda^2 H^1_I - \lambda^1 H^2_I = \partial_I\partial_z^{-1}(\lambda H^3_z) + \lambda H^3_I + g_I(x,y) \tag{8.95}$$

in which g_I are functions that depend only on x, y. Going to infinity along the z-axis while x, y are finite and fixed, it turns out that $g_I = \epsilon_{IJ}\delta^J_\alpha \lambda^\alpha_0$, where λ^α_0 are the leading terms of λ^α (recall that λ^i are of $\mathcal{O}(1)$ and the leading terms are constants). Now (8.95) are two algebraic equations that can be solved for λ^α as

$$\lambda^\alpha = \frac{1}{H^3_z}\left[\epsilon^{IJ}H^\alpha_I\left(\partial_J\partial_z^{-1}(\lambda H^3_z) - H^3_J\right) + \lambda^\alpha_0\right] \tag{8.96}$$

Note that substituting (8.96) into (8.89) and (8.90) yields a system of two integro-differential equations that is too complicated to solve. Even if one could solve the system for λ^3 and λ, the resulting physical Hamiltonian would be very complicated to quantize. Therefore, we refrain from further analysis of this gauge and continue the quantisation process in the description of (A, E) with the MLA-ETTA gauge which led to a relatively simple physical Hamiltonian (8.62).

8.3.2.3. ETA-LEA Gauge in (B, f) Description

Recall first that in this gauge condition we have $z = \{(f^i_I, B^I_i), (f^3_z, B^z_3)\}$ and $r = (f^\alpha_z, B^z_\alpha)$. The equations for the stability of the gauge conditions are

$$\begin{aligned}
\dot\sigma^i_I = \{H, f^i_I(y)\} &= \int d^3x\, [\lambda^k\partial_a + \tilde\lambda^j\epsilon_{jkl}H^l_a + \bar\lambda H^k_a]\{B^a_k(x), f^i_I(y)\} \\
&= -\partial_I\lambda^i + \tilde\lambda^j\epsilon_{jil}H^l_I + \bar\lambda H^i_I \\
\dot\sigma = \{H, f^3_z(y)\} &= \int d^3x\, [\lambda^k\partial_a + \tilde\lambda^j\epsilon_{jkl}H^l_a + \bar\lambda H^k_a]\{B^a_k(x), f^3_z(y)\} \\
&= -\partial_z\lambda^3 + \tilde\lambda^j\epsilon_{j3l}H^l_z + \bar\lambda H^3_z
\end{aligned} \tag{8.97}$$

Which should be solved uniquely for the Lagrange multipliers. As already mentioned, it is sufficient to solve them at the gauge cut where the system of equations (8.97) can be represented as

$$\begin{bmatrix}
-\partial_x & 0 & 0 & 0 & 0 & H^2_x & H^1_x \\
0 & -\partial_x & 0 & 0 & 0 & -H^1_x & H^2_x \\
0 & 0 & -\partial_x & -H^2_x & H^1_x & 0 & 0 \\
-\partial_y & 0 & 0 & 0 & 0 & H^2_y & H^1_y \\
0 & -\partial_y & 0 & 0 & 0 & -H^1_y & H^2_y \\
0 & 0 & -\partial_y & -H^2_y & H^1_y & 0 & 0 \\
0 & 0 & -\partial_z & 0 & 0 & 0 & H^3_z
\end{bmatrix}
\begin{bmatrix}
\lambda^1 \\ \lambda^2 \\ \lambda^3 \\ \tilde\lambda^1 \\ \tilde\lambda^2 \\ \tilde\lambda^3 \\ \bar\lambda
\end{bmatrix}
=
\begin{bmatrix}
\dot\sigma^1_x \\ \dot\sigma^2_x \\ \dot\sigma^3_x \\ \dot\sigma^1_y \\ \dot\sigma^2_y \\ \dot\sigma^3_y \\ \dot\sigma
\end{bmatrix}
=: \dot\Sigma^I \tag{8.98}$$

First, we need to determine the space of functions such that all integral constants are fixed when solving the system of equations. For this goal, we work only with the functions that have the following form

$$\begin{aligned}
\lambda^i &= \lambda^i_0\delta_{ij}\delta^j_a x^a + \mathcal{O}(r^{-1}) \\
\tilde\lambda^i &= \tilde\lambda^i_0 + \mathcal{O}(r^{-1}) \\
\bar\lambda &= \bar\lambda_0 + \mathcal{O}(r^{-2})
\end{aligned} \tag{8.99}$$

where $\lambda^i_0, \tilde\lambda^i_0, \bar\lambda_0$ are arbitrary constants. Note that the first equation of (8.99) is completely consistent with (8.31) and (8.32) and in the second and third equations rotations and boosts are excluded, respectively, since there are no well-defined generators for them [90] (see chapter 6). The reason for the absence of the term r^{-1} in the lapse function is that the use of anti-derivatives

in the following calculations would lead to a logarithmic divergence, which prevents us from specifying some integration constants. For this very reason, we also use

$$f_a^j \to c_a^j + \mathcal{O}(1/r) \tag{8.100}$$

instead of (8.29) in what follows.

Here we want to work with $\dot{\Sigma}_I = 0$. Solving the first equation of (8.98) for λ^1 gives $\lambda^1 = \bar{\lambda}_0 x + \partial_x^{-1}\left(H_x^1\bar{\lambda} + H_x^2\tilde{\lambda}^3\right) + g_1(y,z)$, where we have used $H_x^1 = 1 + \mathcal{O}(r^{-2})$ and $\bar{\lambda} = \bar{\lambda}_0 + \mathcal{O}(r^{-2})$ and the fact that all constants lie in the kernel of ∂_a^{-1}. Considering that according to (8.99) $\lambda^1 - \bar{\lambda}_0 x \to 0$ asymptotically and moving to infinity along the x-axis at fixed finite values of y, z, one observes $g_1 = 0$. It follows that

$$\lambda^1 = \bar{\lambda}_0 x + \partial_x^{-1}\left(H_x^1\bar{\lambda} + H_x^2\tilde{\lambda}^3\right) \tag{8.101}$$

By the same argument, the fifth equation of (8.98) can be solved for λ^2 as $\lambda^2 = \bar{\lambda}_0 y + \partial_x^{-1}\left(H_y^2\bar{\lambda} - H_y^1\tilde{\lambda}^3\right) + g_2(x,z)$. Going to infinity along the y-axis at fixed finite values of x, z gives rise to $g_2 = 0$, because $\lambda^2 - \bar{\lambda}_0 y \to 0$. Therefore,

$$\lambda^2 = \bar{\lambda}_0 y + \partial_x^{-1}\left(H_y^2\bar{\lambda} - H_y^1\tilde{\lambda}^3\right) \tag{8.102}$$

The last equation can be solved for λ^3 as $\lambda^3 = \bar{\lambda}_0 z + \partial_z^{-1}\left(H_z^3\bar{\lambda}\right) + g_3(x,y)$, where we have used $H_z^3 = 1 + \mathcal{O}(r^{-2})$. If we go to infinity along the z-axis while x, y are fixed and finite, we derive $g_3 = 0$. Consequently,

$$\lambda^3 = \bar{\lambda}_0 z + \partial_z^{-1}\left(H_z^3\bar{\lambda}\right) \tag{8.103}$$

If we plug λ^1 and λ^2 into the second and fourth equations, we get a system of two integro-differential equations for $\tilde{\lambda}^3, \bar{\lambda}$

$$(-\partial_x\partial_y^{-1}H_y^2 + H_x^2)\bar{\lambda} - (-\partial_x\partial_y^{-1}H_y^1 + H_x^1)\tilde{\lambda}^3 = 0 \tag{8.104}$$

$$(-\partial_y\partial_x^{-1}H_x^1 + H_y^1)\bar{\lambda} + (-\partial_y\partial_x^{-1}H_x^2 + H_y^2)\tilde{\lambda}^3 = 0 \tag{8.105}$$

Looking at (8.36), one can easily verify that all constants belong to the kernel of the two operators $Y_2 := -\partial_x\partial_y^{-1}H_y^2 + H_x^2$ and $X_1 := -\partial_y\partial_x^{-1}H_x^1 + H_y^1$. In fact, if u is a constant

$$\begin{aligned} Y_2 u &= -\partial_x\partial_y^{-1}(H_y^2 u) + H_x^2 u \\ &= u\left(-\partial_x\partial_y^{-1}H_y^2 + H_x^2\right) \\ &= u\left(-\partial_x\partial_y^{-1}(1 + \partial_y f_z^1) + \partial_x f_z^1\right) \\ &= 0 \end{aligned}$$

and by a similar reasoning $X_1 u = 0$ as well. On the other hand, both operators $Y_1 := -\partial_x\partial_y^{-1}H_y^1 + H_x^1$ and $X_2 := -\partial_y\partial_x^{-1}H_x^2 + H_y^2$ act on constants like identity, in the sense that $Y_1 u = X_2 u = u$ and the ground would be

$$\begin{aligned} Y_1 u &= -\partial_x\partial_y^{-1}(H_y^1 u) + H_x^1 u \\ &= u\left(-\partial_x\partial_y^{-1}H_y^1 + H_x^1\right) \\ &= u\left(-\partial_x\partial_y^{-1}(-\partial_y f_z^2) + (1 - \partial_x f_z^2)\right) \\ &= u \end{aligned}$$

and by the same argument $X_2 u = u$, for all u =const. Thus, two integro-differential equations (8.104) and (8.105) are equivalent to

$$Y_2(\bar{\lambda} - \bar{\lambda}_0) - Y_1(\tilde{\lambda}^3 - \tilde{\lambda}^3_0) - \tilde{\lambda}^3_0 = 0 \tag{8.106}$$
$$X_1(\bar{\lambda} - \bar{\lambda}_0) + X_2(\tilde{\lambda}^3 - \tilde{\lambda}^3_0) + \tilde{\lambda}^3_0 = 0 \tag{8.107}$$

Since $\bar{\lambda} - \bar{\lambda}_0$ and $\tilde{\lambda}^3 - \tilde{\lambda}^3_0$ are of $\mathcal{O}(r^{-1})$, the highest order term of both (8.106) and (8.107) is $\tilde{\lambda}^3_0$, which must vanish separately. Therefore $\tilde{\lambda}^3 = \sum_{n=1}^{\infty} \tilde{\lambda}^3_n r^{-n}$ and (8.106) and (8.107) reduce to

$$Y_2(\bar{\lambda} - \bar{\lambda}_0) - Y_1\tilde{\lambda}^3 = 0 \tag{8.108}$$
$$X_1(\bar{\lambda} - \bar{\lambda}_0) + X_2\tilde{\lambda}^3 = 0 \tag{8.109}$$

From (8.108) it follows that $\tilde{\lambda}^3 = Y_1^{-1}Y_2(\bar{\lambda} - \bar{\lambda}_0) + \tilde{\kappa}$, where $\tilde{\kappa}$ is in the kernel of Y_1. Since $\tilde{\lambda}^3 = \mathcal{O}(r^{-1})$, $\tilde{\kappa}$ must be of the form $\tilde{\kappa} = \sum_{n=1}^{\infty} \tilde{\kappa}_n r^{-n}$, where $\tilde{\kappa}_n$ are functions on the asymptotic sphere. We have

$$0 = Y_1\tilde{\kappa} = H^1_x\tilde{\kappa} - \partial_y\partial_x^{-1}(H^1_y\tilde{\kappa}) \tag{8.110}$$

Since $H^1_I = \delta^1_I + \mathcal{O}(r^{-2})$ and $\tilde{\kappa} = \mathcal{O}(r^{-1})$, the first and the second terms of (8.110) are of $\mathcal{O}(r^{-1})$ and $\mathcal{O}(r^{-3})$, respectively. Thus $\tilde{\kappa}_1/r$ is the highest order term in (8.110) which must vanish separately, i.e. $\tilde{\kappa}_1 = 0$ and consequently $\tilde{\kappa} = \mathcal{O}(r^{-2})$. Now the highest order term in (8.110) is $\tilde{\kappa}_2/r^2$, which must vanish by the same reasoning. By induction, one infers $\tilde{\kappa}_n = 0$ for all $n > 0$, which implies $\tilde{\kappa} = 0$. Hence,

$$\tilde{\lambda}^3 = Y_1^{-1}Y_2(\bar{\lambda} - \bar{\lambda}_0) \tag{8.111}$$

Inserting (8.111) into (8.109), we have

$$0 = (X_2Y_1^{-1}Y_2 + X_1)(\bar{\lambda} - \bar{\lambda}_0) = -\partial_x^{-1}\partial_y[Y_1 + Y_2Y_1^{-1}Y_2](\bar{\lambda} - \bar{\lambda}_0) \tag{8.112}$$

where we used $X_\alpha = -\partial_x^{-1}\partial_y Y_\alpha$ when writing the second equation. (8.112) tells us that $\partial_y[Y_1 + Y_2Y_1^{-1}Y_2](\bar{\lambda} - \bar{\lambda}_0) = g(y,x)$, where g is an arbitrary function that depends only on y, z. Since $\bar{\lambda} - \bar{\lambda}_0 = \mathcal{O}(r^{-1})$, moving to infinity along the x-axis for fixed finite values of y, z shows that $g(y,x) = 0$. Therefore, $[Y_1 + Y_2Y_1^{-1}Y_2](\bar{\lambda} - \bar{\lambda}_0) = h(x,z)$, where h is an arbitrary function independent of y. Again, going to infinity along the y-axis while x, z have fixed finite values gives rise to $h(x,z) = 0$. Thus (8.112) is equivalent to

$$[Y_1 + Y_2Y_1^{-1}Y_2](\bar{\lambda} - \bar{\lambda}_0) = 0 \tag{8.113}$$

In general, it is straightforward to show that $(S+P)^{-1} = S^{-1} - S^{-1}P(S+P)^{-1}$ for any two operators S, P. By repeatedly inserting this relation into its r.h.s. one obtains

$$(S+P)^{-1} = S^{-1}\sum_{n=0}^{\infty}(-PS^{-1})^n \tag{8.114}$$

Based on this relation, we have

$$Y_1^{-1} = (H^1_x - \partial_y\partial_x^{-1}H^1_y)^{-1} = \frac{1}{H^1_x}\sum_{n=0}^{\infty}(\partial_y\partial_x^{-1}\frac{H^1_y}{H^1_x})^n \tag{8.115}$$

Since $H^\alpha_I = \delta^\alpha_I + \mathcal{O}(r^{-2})$, Y_1^{-1} is expanded as $Y_1^{-1} = 1 + \mathcal{O}(r^{-2})$ and $Y_2 = -\partial_x\partial_y^{-1} + \mathcal{O}(r^{-2})$. Assuming $\bar{\lambda} - \bar{\lambda}_0 = \sum_{n=1}^{\infty}\bar{\lambda}_n r^{-n}$, where $\bar{\lambda}_n = \bar{\lambda}_n(\theta,\varphi)$ in the spherical coordinates, we can

extract the highest order term of (8.113) as $\left(\partial_x^2\partial_y^{-2}+1\right)\frac{\bar{\lambda}_1}{r}=0$. Applying ∂_y^2 to both sides of this equation, we see that $\bar{\lambda}_1/r$ must satisfy the 2-dimensional Laplace equation, i.e.

$$(\partial_x^2+\partial_y^2)\frac{\tilde{\lambda}_1}{r}=0 \tag{8.116}$$

If one introduces $R=\sqrt{x^2+y^2}$, it is easy to rewrite the Laplace equation in the polar coordinate system in $x-y$ plane

$$\left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2}\right)\frac{\bar{\lambda}_1(\theta,\varphi)}{\sqrt{R^2+z^2}}=0 \tag{8.117}$$

where $z=r\cos\theta$ and $r=\sqrt{R^2+z^2}$. Carrying out the derivatives, we see that (8.117) is reduced to

$$-\bar{\lambda}_1\left(\frac{2z^2-R^2}{(R^2+z^2)^2}\right)+\frac{1}{R^2}\partial_\varphi^2\bar{\lambda}_1=0 \tag{8.118}$$

Moving to infinity along the z-axis at fixed finite values of x,y results in $\partial_\varphi^2\bar{\lambda}_1=0$. Substituting this into (8.118), we see that $\bar{\lambda}_1$ has to vanish. Thus $\bar{\lambda}-\bar{\lambda}_0=\sum_{n=2}^{\infty}\bar{\lambda}_n r^{-n}$. Now the highest order term of (8.113) is $\left(\partial_y^2\partial_x^{-2}+1\right)\frac{\bar{\lambda}_2}{r^2}=0$ and by a similar reasoning it is easily concluded that $\bar{\lambda}_2=0$ and finally by induction $\bar{\lambda}_n=0$ for all $n>0$. As a final result of this part, we have $\bar{\lambda}-\bar{\lambda}_0=0$ and consequently $\tilde{\lambda}^3=0$ (recall (8.111)). From (8.103) it follows that $\lambda^3=\bar{\lambda}_0 z+\bar{\lambda}_0\partial_z^{-1}H_z^3$. Consequently, the third and sixth equations of (8.98) are a system of two algebraic equations that simply leads to

$$\tilde{\lambda}^\alpha=\frac{\bar{\lambda}_0}{H_z^3}(H_y^\alpha\partial_x-H_x^\alpha\partial_y)\partial_z^{-1}H_z^3 \tag{8.119}$$

for $\alpha=1,2$ because $H_x^1H_y^2-H_x^2H_y^1=H_z^3\neq 0$. Finally, (8.101) and (8.102) give rise to

$$\begin{aligned}\lambda^1&=\bar{\lambda}_0x+\bar{\lambda}_0\partial_x^{-1}H_x^1=\bar{\lambda}_0x-\bar{\lambda}_0f_z^2\\ \lambda^2&=\bar{\lambda}_0y+\bar{\lambda}_0\partial_x^{-1}H_y^2=\bar{\lambda}_0y+\bar{\lambda}_0f_z^1\end{aligned} \tag{8.120}$$

respectively.
Thus it is proved that the solution of the system of PDEs (8.98) with $\dot{\Sigma}^I=0$ is of the form

$$\Lambda_0^I=\bar{\lambda}_0\left(x-f_z^2,y+f_z^1,z+\partial_z^{-1}H_z^3,\frac{1}{H_z^3}(H_y^1\partial_x-H_x^1\partial_y)\partial_z^{-1}H_z^3,\frac{1}{H_z^3}(H_y^2\partial_x-H_x^2\partial_y)\partial_z^{-1}H_z^3,0,1\right)^T \tag{8.121}$$

where $\bar{\lambda}_0$ is an arbitrary constant. In order to have a unique solution, it is necessary to specify the asymptotic behaviour of $\bar{\lambda}$. For simplicity, we consider $\bar{\lambda}_0=1$, which means that from the scratch one should work only with those lapse functions in (8.99) which are of the form $\bar{\lambda}=1+\mathcal{O}(r^{-2})$. Based on the theorem 8.2.2 it suffices to multiply Λ_0^I by $(h_I)_{G=0}=(\partial_zB_1^z,\partial_zB_2^z,0,B_2^zH_z^3,-B_1^zH_z^3,0,0)$ to obtain the corresponding physical Hamiltonian for this gauge fixing. Consequently, for this particular gauge fixing, we arrive at

$$\begin{aligned}h_{\Sigma^I=0}&=\int d^3x\,[(\delta_I^\beta x^I+\epsilon^{\alpha\beta}f_z^\alpha)(\partial_zB_\beta^z)+\epsilon^{\alpha\beta}\epsilon^{IJ}B_\alpha^zH_I^\beta\partial_J\partial_z^{-1}H_z^3]\\&=\int d^3x\,\epsilon^{\alpha\beta}[f_z^\alpha(\partial_zB_\beta^z)+\epsilon^{IJ}B_\alpha^zH_I^\beta\partial_J\partial_z^{-1}H_z^3]+\oint dS_a\,\delta_I^\beta x^I\delta_z^aB_\beta^z\\&=\int d^3x\,\epsilon^{\alpha\beta}[f_z^\alpha(\partial_zB_\beta^z)+\epsilon^{IJ}B_\alpha^zH_I^\beta\partial_J\partial_z^{-1}H_z^3]\end{aligned} \tag{8.122}$$

where the surface term has been omitted because B^z_β is $\mathcal{O}(r^{-3})$ odd.
Two other suitable choices for fixing the gauge are $\Sigma^I = (0,0,\tau,0,0,0,0)^T$ and $\Sigma^I = (0,0,0,0,0,\tau,0)^T$, which lead to

$$\Lambda^I_0 = \left(x - f^2_z, y + f^1_z, z + \partial_z^{-1} H^3_z, \frac{H^1_y}{H^3_z} + \frac{1}{H^3_z}(H^1_y\partial_x - H^1_x\partial_y)\partial_z^{-1}H^3_z, \frac{H^2_y}{H^3_z} + \frac{1}{H^3_z}(H^2_y\partial_x - H^2_x\partial_y)\partial_z^{-1}H^3_z, 0, 1\right), \tag{8.123}$$

$$\Lambda^I_0 = \left(x - f^2_z, y + f^1_z, z + \partial_z^{-1} H^3_z, \frac{1}{H^3_z}(H^1_y\partial_x - H^1_x\partial_y)\partial_z^{-1}H^3_z - \frac{H^1_x}{H^3_z}, \frac{1}{H^3_z}(H^2_y\partial_x - H^2_x\partial_y)\partial_z^{-1}H^3_z - \frac{H^2_x}{H^3_z}, 0, 1\right) \tag{8.124}$$

respectively. And the associated physical Hamiltonians are obtained as

$$h_{\Sigma^I=(0,0,\tau,0,0,0,0)^T} = \int d^3x\, \epsilon^{\alpha\beta}[f^\alpha_z(\partial_z B^z_\beta) + \epsilon^{IJ}B^z_\alpha H^\beta_I \partial_J \partial_z^{-1} H^3_z + H^\alpha_y B^z_\beta] \tag{8.125}$$

and

$$h_{\Sigma^I=(0,0,0,0,0,\tau,0)^T} = \int d^3x\, \epsilon^{\alpha\beta}[f^\alpha_z(\partial_z B^z_\beta) + \epsilon^{IJ}B^z_\alpha H^\beta_I \partial_J \partial_z^{-1} H^3_z - H^\alpha_x B^z_\beta] \tag{8.126}$$

respectively, where the expressions of H^α_I and H^3_z in terms of f^α_z have been displayed in (8.36).

The Physical Hamiltonians (8.122), (8.125) and (8.126) have the following features:

(i) Linearity in momentum B^z_α.

(ii) Polynomiality in the configuration variable f^α_z.

(iii) Spatial non-locality.

As the three physical Hamiltonians here obtained are very similar, we shall in what follows deal only with (8.122).

Before ending this section, we derive the equations of motion driven by the physical Hamiltonian (8.122) and $\{B^z_\alpha(x), f^\beta_z(y)\} = \delta^\alpha_\beta\delta(x,y)$. For f^α_z, one can effortlessly see that

$$\dot{f}^\alpha_z(x) = \{h, f^\alpha_z(x)\} = \epsilon^{\alpha\beta}[\partial_z f^\beta_z + \epsilon^{IJ}H^\beta_I\partial_J\partial_z^{-1}H^3_z] \tag{8.127}$$

where

$$H^\alpha_I = \delta^\alpha_I - \epsilon^{\alpha\beta}\partial_I f^\beta_z \tag{8.128}$$

$$H^3_z = 1 + \epsilon^{\alpha\beta}\delta^I_\beta\partial_I f^\alpha_z + \frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{IJ}(\partial_I f^\alpha_z)(\partial_J f^\beta_z) \tag{8.129}$$

Not surprisingly, the equation of motion for f^α_z closes on itself thanks to the linearity property of the physical Hamiltonian in B^z_α. Therefore, (8.127) can be solved separately without the need to refer to the equation of motion of B^a_i. To obtain the time evolution of B^a_i, one needs to know the variations of H^β_I and H^3_z with respect to f^α_z, which can be directly derived from

$\frac{\delta H_I^\beta(x)}{\delta f_z^\alpha(y)} = \epsilon^{\alpha\beta}\partial_I\delta(x,y)$ and $\frac{\delta H_z^3(x)}{\delta f_z^\alpha(y)} = \epsilon^{\alpha\beta}\left[\delta_\beta^I + \epsilon^{IJ}(\partial_J f_z^\beta)\right]\partial_I\delta(x,y)$. This gives

$$\begin{aligned}\dot{B}_\alpha^z(x) =&\{h, B_\alpha^z(x)\} = -\epsilon^{\alpha\beta}(\partial_z B_\beta^z)\\ &- \int d^3x\ \epsilon^{\gamma\beta}\left\{\epsilon^{IJ}B_\gamma^z(\partial_J\partial_z^{-1}H_z^3)\epsilon^{\alpha\beta}\partial_I\delta(x,y) + \epsilon^{KJ}B_\gamma^z H_K^\beta\partial_J\partial_z^{-1}\left(\epsilon^{\alpha\lambda}\left[\delta_\lambda^I + \epsilon^{IL}(\partial_L f_z^\lambda)\right]\partial_I\delta(x,y)\right)\right\}\\ =&\ -\epsilon^{\alpha\beta}(\partial_z B_\beta^z) + \epsilon^{IJ}(\partial_I B_\alpha^z)(\partial_J\partial_z^{-1}H_z^3) - \int d^3x\ \epsilon^{\gamma\beta}\epsilon^{KJ}\epsilon^{\alpha\lambda}\left(\delta_\lambda^I + \epsilon^{IL}(\partial_L f_z^\lambda)\right)\partial_J\partial_z^{-1}(B_\gamma^z H_K^\beta)\partial_I\delta(x,y)\\ =&\ -\epsilon^{\alpha\beta}(\partial_z B_\beta^z) + \epsilon^{IJ}(\partial_I B_\alpha^z)(\partial_J\partial_z^{-1}H_z^3) + \epsilon^{\gamma\beta}\epsilon^{KJ}\epsilon^{\alpha\lambda}\left(\delta_\lambda^I + \epsilon^{IL}(\partial_L f_z^\lambda)\right)\partial_I\partial_J\partial_z^{-1}(B_\gamma^z H_K^\beta)\end{aligned} \tag{8.130}$$

where in the last step we used $\partial_I\left[\delta_\lambda^I + \epsilon^{IL}(\partial_L f_z^\lambda)\right] = 0$. Since the equations (8.127) and (8.130) are complicated to solve, we leave further analysis of these equations to future work.

8.3.2.4. ETA-LEA Gauge in (B, f) Description

Recall first that in this gauge condition we have $z = \{(f_I^\alpha, B_\alpha^I), (f_z^j, B_j^z)\}$ and $r = (f_I^3, B_3^I)$. The equations for the stability of the gauge conditions are

$$\begin{aligned}\dot{\sigma}_I^\alpha = \{H, f_I^\alpha(y)\} &= \int d^3x\ [\lambda^k\partial_a + \tilde{\lambda}^j\epsilon_{jkl}H_a^l + \bar{\lambda}H_a^k]\{B_k^a(x), f_I^\alpha(y)\}\\ &= -\partial_I\lambda^\alpha + \tilde{\lambda}^j\epsilon_{j\alpha l}H_I^l + \bar{\lambda}H_I^\alpha\\ \dot{\sigma}^i = \{H, f_z^i(y)\} &= \int d^3x\ [\lambda^k\partial_a + \tilde{\lambda}^j\epsilon_{jkl}H_a^l + \bar{\lambda}H_a^k]\{B_k^a(x), f_z^i(y)\}\\ &= -\partial_z\lambda^i + \tilde{\lambda}^j\epsilon_{jil}H_z^l + \bar{\lambda}H_z^i\end{aligned} \tag{8.131}$$

which should be solved uniquely for the Lagrange multipliers. The system of equations (8.131) can be represented at the gauge cut as

$$\begin{bmatrix} -\partial_x & 0 & 0 & 0 & 0 & 0 & H_x^1\\ 0 & -\partial_x & 0 & 0 & 0 & -H_x^1 & 0\\ -\partial_y & 0 & 0 & 0 & 0 & H_x^1 & 0\\ 0 & -\partial_y & 0 & 0 & 0 & 0 & H_x^1\\ -\partial_z & 0 & 0 & 0 & -1 & H_z^2 & H_z^1\\ 0 & -\partial_z & 0 & 1 & 0 & -H_z^1 & H_z^2\\ 0 & 0 & -\partial_z & -H_z^2 & H_z^1 & 0 & 1\end{bmatrix}\begin{bmatrix}\lambda^1\\ \lambda^2\\ \lambda^3\\ \tilde{\lambda}^1\\ \tilde{\lambda}^2\\ \tilde{\lambda}^3\\ \bar{\lambda}\end{bmatrix} = \begin{bmatrix}\dot{\sigma}_x^1\\ \dot{\sigma}_x^2\\ \dot{\sigma}_x^3\\ \dot{\sigma}_y^1\\ \dot{\sigma}_y^2\\ \dot{\sigma}_y^3\\ \dot{\sigma}\end{bmatrix} =: \dot{\Sigma}^I \tag{8.132}$$

Again, we will work with the space of functions introduced in (8.99). And we consider (8.100), (8.28) and (6.44) as boundary conditions imposed on the canonical variables.

Now we can solve (8.132) under the assumption $\dot{\Sigma}^I = 0$. It follows immediately from the first and fourth equations of (8.132) that $\lambda^\alpha = \lambda_0\delta_I^\alpha x^I + \delta_\alpha^I\partial_I^{-1}(\bar{\lambda}H_x^1) + g_\alpha(y,z)$. Since it is assumed that $H_x^1 = 1 + \mathcal{O}(r^{-2})$, the asymptotic behaviours of (8.99) implies that $g_1 = 0$. Consequently,

$$\lambda^\alpha = \lambda_0\delta_I^\alpha x^I + \delta_\alpha^I\partial_I^{-1}(\bar{\lambda}H_x^1) \tag{8.133}$$

Plugging (8.133) into the second and third equations of (8.132) results in

$$\begin{aligned}-\partial_x\partial_y^{-1}(\bar{\lambda}H_x^1) - H_x^1\tilde{\lambda}^3 = 0\\ -\partial_y\partial_x^{-1}(\bar{\lambda}H_x^1) + H_x^1\tilde{\lambda}^3 = 0\end{aligned} \tag{8.134}$$

respectively. We can readily solve the first equation of (8.134) for $\tilde{\lambda}^3$ and get

$$\tilde{\lambda}^3 = -\frac{1}{H^1_x}\partial_x\partial_y^{-1}(\bar{\lambda}H^1_x) \tag{8.135}$$

Substituting (8.135) into the second equation of (8.134) gives $(\partial_y\partial_x^{-1} + \partial_x\partial_y^{-1})u = 0$ where $u := \bar{\lambda}H^1_x = \bar{\lambda}_0 + \mathcal{O}(r^{-2})$. Applying the operator $\partial_x\partial_y$ to both sides of this equation, we see that u has to satisfy the 2-dimensional Laplace equation $0 = (\partial_x^2 + \partial_y^2)u = (\partial_x^2 + \partial_y^2)(u - \bar{\lambda}_0)$, where $u - \bar{\lambda}_0$ is of $\mathcal{O}(r^{-2})$ and hence can be written as $u - \bar{\lambda}_0 = \sum_{n=2}^{\infty} u_n r^{-n}$. The highest order term of the Laplace equation under consideration is $(\partial_x^2 + \partial_y^2)(u_2 r^{-2})$, which can be rewritten in the polar coordinate system in the $x - y$ plane as

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2}\right)\frac{u_2(\theta,\varphi)}{R^2+z^2} = 0 \tag{8.136}$$

where $R = \sqrt{x^2+y^2}$, $z = r\cos\theta$ and $r = \sqrt{R^2+z^2}$. Carrying out the derivatives, we see that (8.117) is reduced to

$$2u_2\left(\frac{R^2-z^2}{(R^2+z^2)^2}\right) + \frac{1}{R^2}\partial_\varphi^2 u_2 = 0 \tag{8.137}$$

Moving to infinity along the z-axis at fixed finite values of x, y leads to $\partial_\varphi^2 u_2 = 0$. Substituting this into (8.137), we see that u_2 must vanish. Thus $u - \bar{\lambda}_0 = \sum_{n=3}^{\infty}\tilde{\lambda}_n r^{-n}$. Repeating the same argument for the lowest order term of Laplace equation, that is now $(\partial_x^2 + \partial_y^2)(u_3 r^{-3})$, we see $u_3 = 0$. Finally by induction one arrives at $u_n = 0$ for all $n > 1$, which implies $\bar{\lambda}H^1_x = u = \bar{\lambda}_0$. Consequently, it follows from (8.133) and (8.135) that

$$\lambda^\alpha = \lambda_0\delta^\alpha_I x^I, \quad \tilde{\lambda}^3 = 0 \tag{8.138}$$

respectively. Plugging (8.138) and $\bar{\lambda} = \frac{\bar{\lambda}_0}{H^1_x}$ into the fifth and sixth equations of (8.132), we get

$$\tilde{\lambda}^\alpha = -\frac{\bar{\lambda}_0}{H^1_x}\epsilon^{\alpha\beta}H^\beta_z \tag{8.139}$$

and ultimately putting all these results in the last equation of (8.132), one gets

$$\lambda^3 = \bar{\lambda}_0 z + \bar{\lambda}_0\partial_z^{-1}\left(\frac{1+(H^1_z)^2+(H^2_z)^2}{H^1_x}\right) \tag{8.140}$$

Note that in the expression of λ^3 an integration constant should appear which depends only on x and y, but due to the asymptotic behaviour of λ^3 introduced in (8.99), it must vanish.
This ends showing that the solutions of the PDE system (8.132) have the following form

$$\Lambda^I_0 = \bar{\lambda}_0\left(x, y, z + \partial_z^{-1}\left(\frac{1+(H^1_z)^2+(H^2_z)^2}{H^1_x}\right), -\frac{H^2_z}{H^1_x}, \frac{H^1_z}{H^1_x}, 0, \frac{1}{H^1_x}\right)^T \tag{8.141}$$

where $\bar{\lambda}_0$ is an arbitrary constant.
As explained in the paragraph following (8.121), it is necessary to specify $\bar{\lambda}_0 = 1$. Based on the theorem (8.2.2), Λ^I_0 should be multiplied by $(h_I)_{G=0} = (0, 0, \partial_I B^I_3, -H^1_x B^y_3, H^1_x B^x_3, 0, 0)$ to obtain

the corresponding physical Hamiltonian for this gauge. Consequently, for this particular gauge fixing, one obtains

$$\begin{aligned} h &= \int d^3x \left[\left(z+\partial_z^{-1}\left(\frac{1+(H_z^1)^2+(H_z^2)^2}{H_x^1}\right)\right)\partial_I B_3^I + \delta_I^\alpha H_z^\alpha B_3^I\right] \\ &= \int d^3x \left[\partial_z^{-1}\left(\frac{1+(H_z^1)^2+(H_z^2)^2}{H_x^1}\right)\partial_I B_3^I + \epsilon^{IJ}B_3^I(\partial_z f_J^3)\right] + \oint dS_a(z\delta_I^a B_3^I) \\ &= \int d^3x \left[\partial_z^{-1}\left(\frac{1+(H_z^1)^2+(H_z^2)^2}{H_x^1}\right)\partial_I B_3^I + \epsilon^{IJ}B_3^I(\partial_z f_J^3)\right] \end{aligned} \tag{8.142}$$

Here the surface term has been omitted since according to the boundary conditions (8.28) and (8.30), it is $\mathcal{O}(1)$ odd.

The Physical Hamiltonians (8.142) has the following features:

(i) Linearity in momentum B_α^z.

(ii) Non-polynomiality in the configuration variable f_z^α.

(iii) Spatial non-locality.

For the (B, f) description, the comparison of (8.142) and (8.122) shows us that the latter has simpler properties, being polynomial in the configuration variables. Therefore, we will proceed with (8.122) in further analysis in future work.

8.3.2.5. Remark on Polynomial Degree of the Physical Hamiltonian

One may wonder why, on some gauges the physical Hamiltonian is polynomial, while in others it is not. The answer lies in the choice of the different polarisations (separation between configuration and momentum degrees of freedom) of the phase space and the polynomial degree in which they enter the constraints: In the (A, E) polarisation, the Gauß constraint is indeed independent of the momentum A while in the (B, f) polarisation all constraints are linear in the momentum B. This makes it inaccessible to impose gauge fixings just in terms of configuration coordinates E in the (A, E) polarisation, while it is possible to do so for all configuration coordinates f in the (B, f) polarisation. While (B, f) is a linear canonical transformation of (A, E), it is not true that the Gauß constraint $\mathrm{div}E = 0$ in the polarisation (A, E) and the Bianchi constraint $\mathrm{div}B = 0$ in the (B, f) polarisation are simply rewritings of each other, in fact, they are not at all: since $B := *dA$ in the (A, E) polarisation is a derived quantity, the relation $\mathrm{div}B \equiv 0$ is regarded as an identity and not a constraint. Conversely, since $E := *df$ in the (B, f) polarisation is a derived quantity, the relation $\mathrm{div}E \equiv 0$ is viewed as an identity and not a constraint. On the other hand, E and B are counted as independent quantities in the (A, E) and (B, f) polarisation, respectively, before imposing the respective Gauß constraints $\mathrm{div}E = 0$ and $\mathrm{div}B = 0$. Therefore, the Gauß and Bianchi constraints act respectively on disjoint sets of canonical coordinates.

It turns out that this critical difference has a significant impact on the available gauge choices and the entries of the associated gauge fixing matrix, as far as the polynomial degree with respect to E respectively f is concerned.

Toward the Path Integral Quantisation of the $U(1)^3$ Model

9.1. Preliminaries

9.1.1. Review: $SO(4)$ Holst action

As we saw in chapter 3, the non-Abelian $SU(2)$ formulation can be obtained from the Palatini action or the Holst action for the gauge group $SO(4)$. The two actions differ by a topological term multiplied by the Immirzi-Barbero parameter [178]. The way this works is that the momentum conjugate to the $SO(4)$ connection must come from a tetrad, which we call *tetrad constraint* $T^{ab},\ a,b=1,2,3,\ T^{[ab]}\equiv 0$. From the tetrad constraint, one derives the *simplicity constraint* S^{ab} which has the advantage that it can be formulated purely in terms of the momentum conjugate to the connection. Unfortunately, the tetrad constraint and the simplicity constraint are inequivalent: the simplicity constraint admits a solution that is not equivalent to the tetrad constraint solution and is usually neglected by hand (recall section 3.3). Therefore, to avoid any dubious reasoning, we steer clear of the simplicity constraint and instead work with the proper tetrad constraint [91].

However, the requirement of dynamical stability of the primary simplicity constraint yields among others a secondary constraint which we call *dynamical constraint* D^{ab}. In the non-Abelian theory, the Dirac algorithm ends here (there are no tertiary constraints), the constraints S^{ab}, D^{ab} form a second-class pair, and the Lagrange multiplier of S^{ab} within the primary Hamiltonian is therefore completely determined by the stability condition. To arrive at the Hamiltonian formulation in terms of $SU(2)$, one uses the Lie algebra isomorphism $\mathfrak{so}(4)\cong\mathfrak{su}(2)\oplus\mathfrak{su}(2)$, and gauge fixes a copy of the decoupled $\mathfrak{su}(2)$ Gauß constraints using the time gauge $e^t_j=0$ (e^A_I is the tetrad with tensorial indices $A=t,a;\ a=1,2,3$, and Lie algebra indices $I=0,j;\ j=1,2,3$). The Hamiltonian formulation is then obtained by solving the second-class constraints, half of the Gauß constraints and the time gauge, and passing to the corresponding Dirac bracket [91].

9.1.2. Purpose of This Chapter

In the previous chapter, we concentrated on the Hamiltonian formulation of the theory, which works well as a starting point for a canonical quantisation of the reduced theory. In this chapter, we study the Lagrangian formulation of the $U(1)^3$ theory with a view to the subsequent path integral quantisation Note that the $U(1)^3$ theory is usually introduced as a manual truncation of the Hamiltonian formulation of Euclidean GR and is not derived from an action principle. So the question is whether there are such actions and how they look [91].

The answer is not a priori obvious: how should the Lagrangian gauge group be determined? For the Euclidean GR, one can start with the $SO(4)$ Palatini-Holst action [178] and after a lengthy analysis of the constraints involving second-class constraints[1] and using the time gauge to fix the "boost" part of $SO(4)$ (rather one of the two copies of $SU(2)$ into which it factorises modulo sign issues), one obtains the usual Hamiltonian $SU(2)$ formulation. One might thus suppose that a suitable covariant action starts from a proper "abelianised" version of the $SO(4)$ Palatini Holst action, perhaps $SU(2)\times U(1)^3$ or $U(1)^6$. In [2], $U(1)^3$ itself was used to propose an action that should serve as a covariant origin of the Hamiltonian $U(1)^3$ model. We could not find any proof in the literature that this action indeed serves this purpose. In section 5.3, we revealed that the action of [2] has too many degrees of freedom (it has 6 instead of 4 propagating degrees of freedom). This is perhaps not too surprising, since except for the substitution of $SU(2)$ by $U(1)^3$, the action of [2] is the same as that of [73], for which an explicit analysis also showed 6 propagating degrees of freedom (see section 5.1.2). So we need to look for another covariant action. The motivation for considering $U(1)^6$ was that it is the natural compact Abelian group of the same dimension as the compact $SO(4)$, analogous as $U(1)^3$ is the natural Abelian group of the same dimension as the compact $SO(3)$ or $SU(2)$. Then the boiling down of $U(1)^6$ to $U(1)^3$ is to occur in time gauge [91]. The contents of this section are taken from [91]. The path integral quantisation of the action proposed here will be the subject of our future work.

9.2. Covariant $U(1)^6$ Theory

In this section, we consider an Abelian analogue of the Palatini or Holst action as a potential Lagrangian formulation of the $U(1)^3$ theory. For this purpose, first, in section 9.2.1, we summon the procedure that one follows in the non-Abelian case. Then, we utilise it as a guideline to perform the corresponding analysis in the Abelian case in section 9.2.2.

9.2.1. The $U(1)^6$ Model

As we have just seen, in the non-Abelian case, one obtains the Hamiltonian $\mathfrak{su}(2)$ formulation from the Lagrangian $\mathfrak{so}(4)$ formulation by gauge fixing one of the $\mathfrak{su}(2)$ copies. In the Abelian case, this suggests working with the Lie algebra $\mathfrak{g}=\mathfrak{u}(1)\oplus\mathfrak{u}(1)\oplus\mathfrak{u}(1)\oplus\mathfrak{h}$ where $\mathfrak{h}$ is a Lie algebra such that $\mathfrak{g}$ admits a four-dimensional representation (in which the tetrad transforms while the connection transforms in the adjoint). Wishing to stay as close as possible to the non-Abelian theory, one should start from six-dimensional $\mathfrak{g}$. Since the gauge group corresponding to $\mathfrak{g}$ is supposed to be compact, the one corresponding to $\mathfrak{h}$ turns out to be also compact. Finally, the requirement that $\mathfrak{g}$ is to be Abelian determines $\mathfrak{g}$ as a direct sum of six $\mathfrak{u}(1)$ copies.

Therefore, the Lagrangian variables are a $U(1)^6$ connection A_B^{IJ}, $A_B^{(IJ)}=0$ and a tetrad e_I^A transforming into the trivial representation of $U(1)^6$. One then expects that, similar to the non-Abelian case, three copies of $U(1)$ get gauge fixed in the Dirac algorithm, either by hand as in the non-Abelian case or as a consequence of the stability conditions. In the former case, it is transparent that the time gauge cannot work as a gauge fixing condition since the tetrad is Gauß-invariant. In the latter case, the time gauge could emerge as a second-class constraint.

[1] It is worth mentioning that in [294] the second-class constraints of the Holst action are solved in a manifestly Lorentz-covariant way and in [295] the authors performed the canonical analysis of the Holst action using a suitable parametrisation of the tetrad and the connection. Finally, they reached the Hamiltonian formulation of the Holst action which, after integrating some auxiliary fields, contains only first-class constraints. Their procedure avoids the introduction of second-class constraints.

Surprisingly, the $U(1)^6$ model turns out to be topological, although it is consistent. As in the non-Abelian case, the Dirac analysis ceases at the secondary level, i.e., there are no tertiary constraints. The secondary constraints involve the Abelian analogue of the dynamical constraint D^{ab}. However, exactly due to the Abelian nature of the model, the constraint pair (T^{ab}, D^{ab}) is now first-class rather than second-class. By the customary naive counting method, this gives rise to a reduction by 24, instead of 12, degrees of freedom. Since the non-Abelian theory has only 4 propagating degrees of freedom, this proves that the resulting theory is topological.

In the following, we present the details of the analysis. We begin with the action

$$S = \frac{1}{2} \int dt\, d^3x\, F^{IJ}_{AB}\, \hat{\sigma}^{AB}_{IJ}, \qquad \hat{\sigma}^{AB}_{IJ} = \hat{\Sigma}^{AB}_{IJ} + \frac{1}{2}\gamma\epsilon_{IJ}{}^{KL}\hat{\Sigma}^{AB}_{KL}, \qquad \hat{\Sigma}^{AB}_{IJ} = \hat{e}^A_{[I}\, \hat{e}^B_{J]} \tag{9.1}$$

where Lie algebra indices are moved with the Kronecker delta. Here we work with half density valued tetrads

$$\hat{e}^A_I = \det(\{e^J_B\})^{1/2}\, e^A_I \tag{9.2}$$

in terms of which the action is polynomial. We assume that the tetrad is nowhere degenerate so that w.l.g. its determinant is everywhere positive. Here γ is the Immirzi-Barbero parameter. In this section, it is assumed that $\gamma \neq \pm 1$, since otherwise the connection is projected onto its (anti-)self-dual part. The values $\gamma = \pm 1$ will be studied as a special case in section 9.2.2. The analysis for $\gamma \neq \pm 1$ is analogous to the case $\gamma = 0$, so we set $\gamma = 0$ for the rest of this section.

The 3+1 decomposition exhibits

$$S = \int dt\, d^3x\, [F^{IJ}_{ta}\, \hat{\Sigma}^{ta}_{IJ} + \frac{1}{2}\, F^{IJ}_{ab}\, \hat{\Sigma}^{ab}_{IJ}] \tag{9.3}$$

Computing the momenta π^B_{IJ}, $\hat{P}^I_A$ conjugate to A^{IJ}_B, $\hat{e}^A_I$, we find the primary constraints

$$\pi^t_{IJ} = 0, \qquad T^a_{IJ} := \pi^a_{IJ} - \hat{\Sigma}^{ta}_{IJ} = 0, \qquad \hat{P}^I_A = 0 \tag{9.4}$$

If $v^{IJ}_a, \hat{v}^A_I$ denote the velocities that one cannot solve for, the Legendre transform of (9.3) yields the primary Hamiltonian

$$\begin{aligned} H &= \int d^3x\, \{v^{IJ}_A\, \pi^A_{IJ} + \hat{v}^A_I \hat{P}^I_A - L\} \\ &= \int d^3x\, \{v^{IJ}_a\, T^a_{IJ} + \hat{v}^A_I \hat{P}^I_A + v^{IJ}_t\, \pi^t_{IJ} + A^{IJ}_{t,a}\, \hat{\Sigma}^{ta}_{IJ} - \frac{1}{2} F^{IJ}_{ab}\, \hat{\Sigma}^{ab}_{IJ}\} \end{aligned} \tag{9.5}$$

Stability of $\pi^t_{IJ} = 0$ leads to the $U(1)^6$ Gauß secondary constraint

$$G_{IJ} = \partial_a\, \hat{\Sigma}^{ta}_{IJ} \tag{9.6}$$

With the Lagrange multiplier $f^{IJ}_a := v^{IJ}_a - \partial_a A^{IJ}_t$, the stability of $\hat{P}^I_A = 0$ leads to the 16 equations

$$f^{IJ}_a\, \hat{e}^a_J = F^{IJ}_{ab}\, \hat{e}^b_J - f^{IJ}_a\, \hat{e}^t_J = 0 \tag{9.7}$$

We solve them by decomposing $f^{IJ}_a = f^{BC}_a\, \hat{\Sigma}^{IJ}_{BC}$ where $\hat{\Sigma}^{IJ}_{AB} = \hat{e}^{[I}_A\, \hat{e}^{J]}_B$, $\hat{e}^A_I\, \hat{e}^I_B = \delta^A_B$, $\hat{e}^A_I\, \hat{e}^J_A = \delta^J_I$ and employing the relations

$$\hat{\Sigma}^{AB}_{IJ}\, \hat{\Sigma}^{IJ}_{CD} = \delta^A_{[C}\, \delta^B_{D]}, \qquad \hat{\Sigma}^{AB}_{IJ}\, \hat{\Sigma}^{KL}_{AB} = \delta^K_{[I}\, \delta^L_{J]} \tag{9.8}$$

We attain the succeeding restrictions on the Lagrange multipliers

$$f_a^{bt} = F_{ac}^{IJ}\ \hat{\Sigma}_{IJ}^{bc}, \qquad f_a^{bc} = \epsilon^{bcd} s_{ad}, \qquad s_{[ad]} = 0 \tag{9.9}$$

and the 4 secondary constraints

$$C_a := F_{ab}^{IJ}\ \hat{\Sigma}_{IJ}^{tb}, \qquad C := F_{ab}^{IJ}\ \hat{\Sigma}_{IJ}^{ab} \tag{9.10}$$

It means that we have fixed 12 out of 18 Lagrange multipliers, while the 6 degrees of freedom encoded by the symmetric tensor s_{ab} remain free. The 16 relations (9.9) and (9.10) ensure the stability of $\hat{P}_A^I$.

Stability of T_{IJ}^a yields

$$\partial_b \hat{\Sigma}_{IJ}^{ab} + \hat{v}_{[I}^t \hat{e}_{J]}^a + \hat{e}_{[I}^t \hat{v}_{J]}^a = 0 \tag{9.11}$$

We decompose $\hat{v}_I^A = \hat{v}_B^A\ \hat{e}_I^B$ and find the following restrictions on the Lagrange multipliers

$$\hat{v}_b^a = -[2\hat{\Sigma}_{tb}^{IJ}\ \hat{\Sigma}_{IJ,c}^{ca} + \hat{v}_t^t \delta_b^a], \qquad \hat{v}_a^t = -\hat{\Sigma}_{ab}^{IJ}\ \hat{\Sigma}_{IJ,c}^{cb} \tag{9.12}$$

and the 6 secondary constraints

$$D^{ab} = \epsilon^{cd(a}\ \hat{\Sigma}_{IJ,e}^{b)e}\ \hat{\Sigma}_{cd}^{IJ} \tag{9.13}$$

That is, 12 out of 16 Lagrange multipliers have been fixed while the 4 multipliers $\hat{v}_t^A$ remain free. In total, the 18 equations (9.12), (9.13) ensure stability of $T_{IJ}^a = 0$.

We must now stabilise G_{IJ}, C_a, C, D^{ab}. The easiest way to do this is as follows. Modulo $T_{IJ}^a = 0$, we have $G_{IJ} = \partial_a \pi_{IJ}^a$. As T_{IJ}^a is already stabilised, we can equivalently stabilise the constraint in the form $\hat{G}_{IJ} := \pi_{IJ,a}^a = 0$, which generates $U(1)^6$ gauge transformations. Since H depends on A_a^{IJ} only through its curvature $F_{ab}^{IJ} = 2\partial_{[a} A_{b]}^{IJ}$, the Gauß constraint $\hat{G}_{IJ}$ is already stabilised. Next, modulo $T_{IJ}^a = G_{IJ} = 0$, we have $C_a = F_{ab}^{IJ} \pi_{IJ}^b - A_a^{IJ} \hat{G}_{IJ}$, in which form it generates spatial diffeomorphisms on (A_a^{IJ}, π_{IJ}^a). Since $\hat{P}_A^I$ has already been stabilised, we add to C_a linear terms in $\hat{P}_A^I$ so that the resulting constraint $\hat{C}_a$ generates spatial diffeomorphisms also on the variables $\hat{e}_I^A, \hat{P}_A^I$ (note the density weight $\pm 1/2$). Since the primary Hamiltonian depends only on constraints that are tensor densities, the constraint $\hat{C}_a$ and hence C_a is stabilised.

Unfortunately, these abstract reasoning are not available for C, D^{ab}. We display some steps of the surprisingly lengthy calculation

$$\begin{aligned}
\{H, C(f)\} &= \int d^3x\ \{v_a^{IJ} \pi_{IJ}^a + \hat{v}_I^A\ \hat{P}_A^I, C(f)\} \\
&= 2\int d^3x\ f\ [\hat{\Sigma}_{IJ}^{bc}\ \partial_{[b} v_{c]}^{IJ} + \hat{v}_K^b\ \hat{e}_L^c\ F_{bc}^{KL}] \\
&= 2\int d^3x\ f\ [\hat{\Sigma}_{IJ}^{bc}\ \partial_{[b} f_{c]}^{IJ} + \hat{v}_K^b\ \hat{e}_L^c\ F_{bc}^{KL}] \\
&= 2\int d^3x\ f\ [(f_b^{ab})_{,a} - \hat{\Sigma}_{IJ,a}^{ab}\ f_b^{IJ} + \hat{v}_t^b\ C_b + \hat{v}_a^b\ f_b^{at}] \\
&= 2\int d^3x\ f\ [-\hat{\Sigma}_{IJ,a}^{ab}\ \hat{\Sigma}_{AB}^{IJ}\ f_b^{AB} + \hat{v}_a^b\ f_b^{at}] \\
&= 2\int d^3x\ f\ [-s_{ab}\ D^{ab}\ + f_b^{tc}\ (\hat{v}_c^b - \delta_c^b\ \hat{v}_t^t) + \hat{v}_a^b\ f_b^{at}] \\
&= 2\int d^3x\ f\ \hat{v}_t^t\ C
\end{aligned} \tag{9.14}$$

where in the first step, the contributing part of H was isolated, in the second step, we performed the Poisson brackets and used integration by parts, in the third step, we found that the r.h.s.

depends on v_a^{IJ} only via f_a^{IJ}, in the fourth step, (9.9) and (9.12) were inserted, in the fifth step we dropped C_b and exploited $f_b^{ab} = 0$ and decomposed f_b^{IJ}, in the sixth step we used (9.9) and (9.12) again, and in the last step we have cancelled terms, dropped D^{ab} and applied $C = f_a^{at}$. Therefore, C is stable, where the explicit form of $f_a^{IJ}, \hat{v}_I^A$ and the secondary constraints C_a, C, D^{ab} had to be used crucially.

In order to perform the analysis for D^{ab} it is helpful to rewrite it in the more manageable form

$$D^{ab} = \epsilon^{cd(a}\, w_c{}^{b)}{}_d, \qquad w_c{}^b{}_d = \hat{e}_c^I\, \hat{e}_{I,d}^b \tag{9.15}$$

If one also uses the abbreviation $\sigma_b^a := w_t{}^a{}_b = \hat{e}_t^I\, \hat{e}_{I,b}^a$ it is not difficult to verify that

$$\hat{v}_b^a = \sigma_b^a - [v_t^t + \sigma_c^c]\delta_b^a, \qquad \hat{v}_a^t = -w_{(a}{}^b{}_{b)} \tag{9.16}$$

With this machinery, we calculate

$$\begin{aligned}\{H, D^{ab}(g_{ab})\} &= \{\int d^3x\, \hat{v}_I^A\, \hat{P}_A^I, D^{ab}(g_{ab})\} \\ &= -\int d^3x\, g_{ab}\, \epsilon^{acd}\, [\hat{v}_c^t\, \sigma_d^b + \hat{v}_c^e\, w_e{}^b{}_d + \hat{v}_t^b\, [\hat{e}_I^t\, \hat{e}_{c,d}^I] - \hat{v}_{c,d}^b + \sigma_e^b\, w_c{}^e{}_d] \\ &= -\int d^3x\, g_{ab}\, \epsilon^{acd}\, [w_{(c}{}^e{}_{e)}\, \sigma_d^b + \sigma_c^e\, w_e{}^b{}_d + \hat{v}_t^b\, [\hat{e}_I^t\, \hat{e}_{c,d}^I] - \sigma_{c,d}^b + \sigma_e^b\, w_c{}^e{}_d]\end{aligned} \tag{9.17}$$

where, in the first step, the contributing part of H was isolated, in the second step, we performed the Poisson bracket and used integration by parts and the above abbreviations, in the last step, we found that the part $\propto \delta_b^a$ in $\hat{v}_b^a$ is omitted owing to the symmetry of g_{ab}. It is easy to check

$$\epsilon^{acd}\, \sigma_{c,d}^b = -\epsilon^{acd}[\hat{e}_t^I\, \hat{e}_{I,d}^t \sigma_c^b + \sigma_d^e\, w_e{}^b{}_c] \tag{9.18}$$

and with (9.15) and decomposing into the symmetric and antisymmetric part with respect to indices a, e

$$\epsilon^{acd} w_c{}^e{}_d = D^{ae} + \epsilon^{aed}\, w_{[c}{}^c{}_{d]} \tag{9.19}$$

If one drops the term $\propto D^{ae}$, the Poisson bracket (9.17) is simplified

$$\{H, D^{ab}(g_{ab})\} = \int d^3x\, g_{ab}\, [\sigma_c^b\, \epsilon^{acd}\, w_d - \hat{v}_t^b\, u^a] \tag{9.20}$$

where

$$w_a := w_a{}^b{}_b + \hat{e}_t^I\, \hat{e}_{I,a}^t, \qquad u^a := \epsilon^{acd} \hat{e}_I^t\, \hat{e}_{c,d}^I \tag{9.21}$$

The last step is to check

$$u^a = -\frac{1}{2}\epsilon^{abc}\, \hat{\Sigma}_{bc}^{IJ}\, G_{IJ}, \qquad w_a = 2\hat{\Sigma}_{ta}^{IJ}\, G_{IJ} \tag{9.22}$$

which indicates that D^{ab} is stable modulo D^{ab}, G_{IJ}.

Consequently, no tertiary constraints arise. Nevertheless, the reason for the absence of tertiary constraints is quite different from the non-Abelian theory. There, the absence was ensured by the specification of Lagrange multipliers in the primary Hamiltonian, whereas in the Abelian theory no Lagrange multipliers are specified, but stability is already ensured by the secondary constraints.

This difference also leads to a crucial discrepancy in the counting of the degrees of freedom[2] between the Abelian and non-Abelian theories: In the $U(1)^6$ theory we have a total of $2\cdot(4\cdot 6+4\cdot 4)=80$ degrees of freedom in phase space. We have $6+18+16=40$ primary constraints $\pi^t_{IJ}, T^a_{IJ}, \hat{P}^I_A$ and $6+4+6=16$ secondary constraints G_{IJ}, C_a, C, D^{ab}. In total, 12 of the 18 v^{IJ}_a and 12 of the 16 $\hat{v}^A_I$ were fixed during the stability analysis, while 6 of the v^{IJ}_a (coded by s_{ab}) and 4 of the $\hat{v}^I_A$ (encoded by $\hat{v}^A_t$) are still non-restricted. Performing the same analysis as above, one can verify that $\hat{C}$ is a first-class constraint, where $\hat{C}$ is the integrand of H. This implies that C is also first class. Moreover, the fact that s_{ab}, $\hat{v}^A_t$ remain free implies that the $6+4$ constraints $S^{ab}, \hat{P}_A$ multiplying them are also first class. Explicitly

$$S^{ab}=\pi^{(a}_{IJ}\,\epsilon^{b)cd}\hat{\Sigma}^{IJ}_{cd}, \qquad \hat{P}_t=\hat{P}^I_t\hat{e}^t_I-\hat{P}^I_a\hat{e}^a_I, \qquad \hat{P}_a=\hat{P}^I_a\hat{e}^t_I \tag{9.23}$$

Lastly, D^{ab} has weakly Poisson commute with all constraints except $\hat{P}^I_A$. However, one can add a term $\propto T^a_{IJ}$ to D^{ab} such that the resulting $\hat{D}^{ab}$ has exactly vanishing Poisson brackets with all constraints.

$$\begin{aligned}\hat{D}^{ab}(g_{ab})&=D^{ab}(g_{ab})+T^a_{IJ}(\hat{\Sigma}^{IJ}_{AB}h^{AB}_a(g)),\\ h^{ca}_a(g)&=-g_{ab}\epsilon^{acd}\hat{e}^I_t\hat{e}^b_{I,d}, \qquad h^{tc}_e(g)=2\epsilon^{ad[c}\,w_e{}^{b]}_d\,g_{ab}+\epsilon^{acd}g_{ae,d}\end{aligned} \tag{9.24}$$

To sum up: We have 12 pairs of second-class constraints, consisting of 12 out of 18 T^a_{IJ} and 12 of 16 $\hat{P}^A_I$ and $6+6+4+6+4+6=32$ first-class constraints $\pi^t_{IJ}, S^{ab}, \hat{P}_A, G_{IJ}, C, C_a, \hat{D}^{ab}$. In total, these are $2\cdot 12+32=56=40+16$ constraints, which reduce $24+2\cdot 32=88$ degrees of freedom. In the non-Abelian theory, the analogues of S^{ab}, D^{ab} form a second-class pair. Thus we have 56 constraints there too, but now we have 36 second-class constraints and only 20 first-class constraints, reducing only $2\cdot 20+36=76$ degrees of freedom and 4 propagating ones remain. Accordingly, the $U(1)^6$ theory is consistent in the sense that the Dirac analysis does not lead to a contradiction, but it is topological in the sense that the reduction of the constraints leaves no local degrees of freedom. Therefore, the $U(1)^6$ theory is not a Lagrangian origin of the Hamiltonian $U(1)^3$ theory, which has 4 propagating degrees of freedom.

9.2.2. The Twisted Self-Dual Model

Surprisingly, the following analysis works for both $U(1)^6$ and $SO(4)$ simultaneously. We scrutinise the action

$$S=\frac{1}{2}\int dt\, d^3x\; F^{IJ}_{AB}\,\hat{\Sigma}^{AB}_{IJ}, \qquad \hat{\Sigma}^{AB}_{IJ}=\hat{e}^A_{[I}\,\hat{e}^B_{J]} \tag{9.25}$$

but F^{IJ}_{AB} is not a $U(1)^6$ or $SO(4)$ curvature but instead a *twisted self-dual* $U(1)^6$ or $SO(4)$ curvature, that is,

$$F^{0j}=F^j=\frac{1}{2\gamma}\epsilon_{jkl}\,F^{kl} \tag{9.26}$$

with $F^j=2dA^j$ and $F^j=2dA^j+\epsilon_{jkl}A^k\wedge A^l$ respectively a $U(1)^3$ and $SU(2)$ curvature. Here $\gamma\neq 0$ is similar to, but different from, the Immirzi-Barbero parameter: Note that the condition (9.26) defines the (anti-)self-dual model only for $\gamma=\pm 1$. Assuming the Holst action (9.1) with topological term and $\gamma=\pm 1$, one reaches (9.25), since then the standard curvature of $SO(4)$ or

[2] Note that a safe way to count the degrees of freedom is to use Dirac's algorithm, which in particular ensures that the encountered constraints are algebraically independent. Counting at the Lagrangian level can be tricky: for example, the action [73] reveals 24 configuration degrees of freedom, 4 diffeomorphism gauges, and 3 Yang-Mills type gauges. From the Lagrangian, one cannot readily deduce that this theory has 3 propagating canonical pairs.

$U(1)^6$ is projected into the curvature of the (anti-)self-dual (i.e. $SU(2)$ or $U(1)^3$ respectively) connection. However, if $\gamma \neq \pm 1$, in the usual Holst action the connection stays a genuine $SO(4)$ or $U(1)^6$ connection and one does not reaches (9.25). So for $\gamma \neq \pm 1$ the action is new. In this action the connection is only $SU(2)$ or $U(1)^3$ connection, by definition, which can anyway act on tetrads by imposing that the curvature be constrained by self-duality. Still, as we will show below, it turns out to be equivalent to the Euclidean GR or its $U(1)^3$ shortening. The fact that the action includes a complete co-tetrad e^I_A rather than merely three frame one-forms e^j_A distinguishes it from the $SU(2)$ model introduced by Husain and Kuchar [73], which has no Hamiltonian constraint, although we also have only an $SU(2)$ (or $U(1)^3$) connection. It is perhaps puzzling how an $SU(2)$ or $U(1)^3$ connection acts on $\mathbb{R}^4$, but this happens by the same mechanism as a self-dual connection would act. Notice it is crucial that $\gamma \neq 0, \infty$, otherwise we would arrive at the action of [2] which have too many degrees of freedom (recall section 5.3).

The subsequent analysis turns out to be much simpler than in section 9.2.1. To apprehend the main result for readers who are not interested in the details, we sketch the outcome of the analysis: this time we get among the primary constraints 9 tetrad constraints $T^a_j = \pi^a_j - \sigma^{ta}_j$, 3 momentum constraints π^t_j and the 16 momentum constraints $\hat{P}^I_A$ as before. The stability of π^t_j enforces the Gauß constraint G_j for the corresponding gauge group as a secondary constraint. The stability of $\hat{P}^I_A$ results in 7 secondary constraints C_a, C, D_j where C_a is the spatial diffeomorphism constraint, C is the Hamiltonian constraint, and $D_j = \hat{e}^t_j$ is the *time gauge constraint*. This means that the time gauge in this model is not just a convenient gauge choice but is "dynamically enforced". Moreover, all 9 Lagrange multipliers v^j_a of T^a_j get specified in this process. Finally, the stabilisation of T^a_j fixes 9 out of 16 Lagrange multipliers $\hat{v}^A_I$ of $\hat{P}^I_A$. Then the constraints G_j, C, C_a are already stable, while the stabilisation of D_j specifies another 3 out of 16 of $\hat{v}^A_I$. This completes the stabilisation process.

All Lagrange multipliers except $4 = 16 - 9 - 3$ of the $\hat{v}^A_I$ and all 9 of the v^j_a have been fixed. This means that 4 of the $\hat{P}^I_A$, let us call them $\hat{P}_A$ are first-lass (they are linear combinations of the momenta conjugate to the lapse and shift functions). Moreover, π^t_j, G_j, C, C_a are first-class, while 3 of the $\hat{P}^I_A$, let us call them $\hat{P}^j$, form second-class pairs with D_j, while 9 of the $\hat{P}^I_A$, let us call them $\hat{P}^j_a$, form second class pairs with T^a_j. Accordingly, we have $2 \cdot (3+9) = 24$ second-class constraints $D_j, \hat{P}^j, T^a_j, \hat{P}^j_a$ and $3+3+4+4 = 14$ first-class constraints $\pi^t_j, G_j, C_a, C, \hat{P}_A$. These subtract $24 + 2 \cdot 14 = 52$ of the $2 \cdot (12+16) = 56$ degrees of freedom $A^j_B, \pi^B_j, \hat{e}^A_I, \hat{P}^I_A$ leaving the 4 propagating degrees of freedom of the Hamiltonian $U(1)^3$ or $SU(2)$ theory respectively.

We now sketch the details:
Plugging (9.26) into (9.25) leads to the 3+1 decomposition

$$S = \int dt\, d^3x\, [F^j_{ta}\, \hat{\sigma}^{ta}_j + \frac{1}{2}\, F^j_{ab}\, \hat{\sigma}^{ab}_j], \qquad \hat{\sigma}^{AB}_j := 2\hat{\Sigma}^{AB}_{0j} + \gamma \epsilon^{jkl} \hat{\Sigma}^{AB}_{kl} \tag{9.27}$$

The computation of the conjugate momenta results in the primary constraints

$$\pi^t_j = 0, \qquad T^a_j = \pi^a_j - \hat{\sigma}^{ta}_j = 0, \qquad \hat{P}^I_A = 0 \tag{9.28}$$

and thus the primary Hamiltonian reads with the velocities $v^j_A, \hat{v}^A_I$

$$\begin{aligned} H &= \int d^3x\, [v^j_A \pi^A_j + \hat{v}^A_I\, \hat{P}^I_A - L] \\ &= \int d^3x\, [v^j_t \pi^t_j + v^j_a\, T^a_j + \hat{v}^A_I\, \hat{P}^I_A - A^j_t\, (\nabla_a\, \sigma^{ta}_j) - \frac{1}{2}\, F^j_{ab}\, \sigma^{ab}_j] \end{aligned} \tag{9.29}$$

where ∇ denotes the $U(1)^3$ or $SU(2)$ covariant derivative acting on Lie algebra indices only, i.e. $\nabla_a T_j = \partial_a T_j$ and $\nabla_a T_j = \partial_a T_j + \epsilon_{jkl} A_a^k T_l$ respectively.

Stabilisation of π_j^t gives the Gauß constraint

$$G_j = \nabla_a \sigma_j^{ta} \tag{9.30}$$

Stabilisation of $\hat{P}_A^I$ yields the condition

$$\{H, \hat{P}_A^I(f_I^A)\} = \int d^3x \, \{\hat{P}_A^I(f_I^A), v_a^j \sigma_j^{ta} + \frac{1}{2} F_{ab}^j \hat{\sigma}_j^{ab}\} = 0 \tag{9.31}$$

for all f_I^A. Isolating i. $A = t$, $I = 0$, ii. $A = t$, $I = i$, iii. $A = c$, $I = 0$, iv. $A = c$, $I = i$ coefficients gives the following set of $1 + 3 + 3 + 9 = 16$ conditions

$$\begin{aligned} 0 &= v_a^j \hat{e}_j^a \\ 0 &= v_a^i \, \hat{e}_0^a + \gamma \, \epsilon^{ikl} \, v_a^k \, \hat{e}_l^a \\ 0 &= -\hat{e}_j^t \, v_c^j + F_{cb}^j \, \hat{e}_j^b \\ 0 &= v_c^i \, \hat{e}_0^t + \gamma \, \epsilon^{ikl} \, v_c^k \, \hat{e}_l^t - [F_{cb}^i \, \hat{e}_0^b + \gamma \, \epsilon^{ikl} \, F_{cb}^k \, \hat{e}_l^b] \end{aligned} \tag{9.32}$$

The general solution of the system (9.32) requires a detailed case by case analysis which is provided in the appendix B.

However, a physically motivated solution consists of 7 secondary constraints

$$D_j := \hat{e}_j^t, \qquad C_a := F_{ab}^j \, \hat{e}_j^b, \qquad C := \epsilon_{jkl} F_{ab}^j \, \hat{e}_k^a \, \hat{e}_l^b \tag{9.33}$$

and a restriction on v_a^j

$$v_a^j = \frac{1}{\hat{e}_0^t} \, [F_{ab}^j \, \hat{e}_0^b + \gamma \, \epsilon^{jkl} \, F_{ab}^k \, \hat{e}_l^b] \tag{9.34}$$

where $\hat{e}_0^t \neq 0$ has been assumed. The use of $D_j = 0$ in the fourth equation of (9.32) gives (9.34). Using $D_j = 0$ in the third equation of (9.32) leads to $C_a = 0$. Substituting (9.34) into $\hat{e}_0^t$ times the first equation of (9.32) results in $\hat{e}_0^a C_a + \gamma \, C = 0$, i.e. $C = 0$. Finally, substituting (9.34) into $\hat{e}_0^t$ times the second equation in (9.32), we obtain

$$0 = \gamma^2 \, \epsilon^{ikl} \, \epsilon^{kmn} \, F_{ab}^m \hat{e}_n^b \, \hat{e}_l^a = \gamma^2 (-C_a \, \hat{e}_i^a - F_{ab}^i \, \hat{q}^{ab}) = -\gamma^2 \, C_i \tag{9.35}$$

and is hence already satisfied. Here we have exploited the symmetry of $\hat{q}^{ab} = \delta^{jk} \hat{e}_j^a \hat{e}_k^b$.

Stabilisation of T_j^a gives rise to

$$(\nabla_b \hat{\sigma}_j^{ab}) + \hat{v}_0^t \hat{e}_j^a - \hat{v}_j^t \hat{e}_0^a + \hat{e}_0^t \, \hat{v}_j^a + \epsilon^{jkl} \, \hat{v}_k^t \, \hat{e}_l^a = 0 \tag{9.36}$$

which can be solved for $\hat{v}_j^a$.

Before we do this, let us consider the stabilisation of the secondary constraints. Clearly, the stabilisation of D_j enforces

$$\hat{v}_j^t = 0 \tag{9.37}$$

so that (9.36) simplifies to

$$(\nabla_b \hat{\sigma}_j^{ab}) + \hat{v}_0^t \hat{e}_j^a + \hat{e}_0^t \, \hat{v}_j^a = 0 \tag{9.38}$$

The Gauß constraint can be replaced by $\hat{G}_j = \nabla_a \pi_j^a$ modulo T_j^a, which is already stabilised. In this form it generates Gauß gauge transformations on the sector (A_a^j, π_j^a). Since

$$F_{ab}^j \sigma_j^{ab} = 2 \hat{e}_0^a C_a + C \tag{9.39}$$

depends only on gauge invariants, it has a vanishing Poisson bracket with this part of H. In the Abelian case, it has also vanishing Poisson brackets with all other parts of H but in the non-Abelian case, it does not Poisson commute with T^a_j. In the non-Abelian case, we attach terms to $\hat{G}_j$ linear in $\hat{P}^I_A$, which is already stabilised

$$\hat{G}_j = \nabla_a \pi^a_j - \epsilon_{jkl}\ \hat{P}^A_k\ \hat{e}^A_l \tag{9.40}$$

and now $\hat{G}_j$ also generates $SU(2)$ rotations in the sector $(\hat{e}^A_j, \hat{P}^j_A)$ while leaving the sector $(\hat{e}^A_0, \hat{P}^0_A)$ invariant. In particular, under its action, T^a_j is rotated into itself. As a result, $\hat{G}_j$ is stabilised and hence G_j is also stabilised in the non-Abelian theory.

Next, modulo T^a_j, G_j, which are already stabilised, we can replace C_a by $\hat{C}_a = F^j_{ab}\ \pi^b_j - A^j_a \nabla_b \pi^b_j$, in whose form it generates spatial diffeomorphisms on the variables A^j_a, π^a_j. We add terms linear in the already stabilised $\hat{P}^I_A$ such that it generates spatial diffeomorphisms on all variables (taking into account the half-density weight such that $\hat{e}^t_I$ and $\hat{e}^a_I$ are scalar and vector half-densities, respectively)

$$\hat{C}_a = F^j_{ab}\ \pi^b_j - A^j_a \nabla_b \pi^b_j + \frac{1}{2}(\hat{P}^I_t \hat{e}^t_{I,a} - \hat{P}^I_{t,a}\hat{e}^t_I + 2(\hat{P}^I_a \hat{e}^b_I)_{,b} - \hat{P}^I_{b,a}\hat{e}^b_I + \hat{P}^I_b \hat{e}^b_{I,a}) \tag{9.41}$$

It reflects that $\hat{C}_a$ and hence C_a is already stabilised because all constraints are tensor densities and H is a linear combination of constraints. As for C, we instead consider $\hat{C}$, which is the integrand of H with the fixed v^j_a, $\hat{v}^A_j$. Then, we calculate

$$\{\hat{C}(f), \hat{C}(g)\} = \frac{1}{2}\int d^3x\ d^3y\ [f(x)\ g(y) - f(y)\ g(x)]\ \{\hat{C}(x), \hat{C}(y)\} \tag{9.42}$$

Modulo constraints, only the contributions to the Poisson bracket which give rise to derivatives of the δ-distribution do not vanish in (9.42). The only derivatives within constraints in $\hat{C}$ come from the term $-\frac{1}{2}F^j_{ab}\ \hat{\sigma}^{ab}_j$ which has non-vanishing Poisson brackets, resulting in derivatives of δ-distributions only with the term $v^j_a T^a_j$. It follows

$$\begin{aligned}
\{\hat{C}(f), \hat{C}(g)\} &= -\frac{1}{2}\int d^3x\ d^3y\ [f(x)\ g(y) - f(y)\ g(x)]\ v^j_a(x)\{\pi^a_j(x), F^k_{bc}(y)\}\ \sigma^{bc}_k(y) \\
&= -\int d^3x\ d^3y\ [f(x)\ g(y) - f(y)\ g(x)]\ v^j_a(x)\{\pi^a_j(x), \partial_{[b}\ A^k_{c]}(y)\}\ \sigma^{bc}_k(y) \\
&= \int d^3x\ [f\ g_{,b} - f_{,b}\ g]\ v^j_c\ \sigma^{bc}_j k(y) \\
&= \int d^3x\ \omega_b\ (F^j_{cd} + \gamma\epsilon^{jkl}F^k_{cd}\ \hat{e}^d_l)\ (2\hat{e}^b_{[0}\ \hat{e}^c_{j]} + \epsilon^{jmn}\ \hat{e}^b_m\ \hat{e}^c_n) \\
&= \int d^3x\ \omega_b\ \{-C_d\ \hat{e}^d_0\hat{e}^b_0 - (F^j_{cd}\hat{e}^c_0\hat{e}^d_0)\hat{e}^b_j - \gamma\ \hat{e}^b_0\ C \\
&\quad +\gamma\ [\hat{e}^d_0\ F^j_{cd}\epsilon^{jmn}\hat{e}^b_m\hat{e}^c_n - \hat{e}^c_0\ F^k_{cd}\epsilon^{jkl}\hat{e}^d_l\hat{e}^b_j] + \gamma^2\ F^k_{cd}\ \epsilon^{jmn}\epsilon^{jkl}\hat{e}^d_l\hat{e}^b_m\hat{e}^c_n\} \\
&= \gamma\int d^3x\ \omega_b\ \{F^k_{cd}\ \hat{e}^b_m\ [\hat{e}^d_0\ \epsilon^{kmn}\ \hat{e}^c_n - \hat{e}^c_0\ \epsilon^{mkn}\hat{e}^d_n] + \gamma\ F^k_{cd}\ (\hat{e}^b_k\ \hat{q}^{cd} - \hat{e}^c_k\ \hat{q}^{bd}) \\
&= \gamma\int d^3x\ \omega_b\ \{F^k_{cd}\ \hat{e}^b_m\ [\hat{e}^d_0\ \epsilon^{kmn}\ \hat{e}^c_n - \hat{e}^c_0\ \epsilon^{mkn}\hat{e}^d_n] + \gamma\ F^k_{cd}\ (\hat{e}^b_k\ \hat{q}^{cd} - \hat{e}^c_k\ \hat{q}^{bd}) \\
&= \gamma\int d^3x\ \omega_b\ \{F^k_{cd}\ \hat{e}^b_m\ \epsilon^{kmn}\ [\hat{e}^d_0\ \ \hat{e}^c_n + \hat{e}^c_0\ \hat{e}^d_n] + \gamma\ \hat{q}^{bd}\ C_d\} \\
&= \gamma^2\int d^3x\ \omega_b\ \hat{q}^{bd}\ C_d
\end{aligned} \tag{9.43}$$

where, in the first step, we isolated the relevant terms, in the second step, we omitted the term in F^j_{ab} which is quadratic in the connection since its Poisson bracket is ultralocal (this step is avoided in the Abelian case), in the third step, the Poisson bracket is carried out and integration by parts is used, in the fourth step, $\omega_b := \frac{1}{\hat{e}^t_0}(f\ g_{,b} - f_{,b}\ g)$ is introduced to simplify the notation and the definition of $\hat{\sigma}^{ab}_j$ and (9.34) are utilised, in the fifth step, we have explicitly written out the six terms resulting from the multiplication of the round brackets in the previous step, in the sixth step, we have put away the terms proportional to C, C_d and the term in round brackets which is eliminated because of antisymmetry, and in addition we have relabelled indices in the square bracket term and used the summation identities, in the seventh step we wrote out the square bracket term such that it is manifestly symmetric in c, d and applied the definition of C_d and $F^k_{cd}\hat{q}^{cd} = 0$ and in the last step, we used the antisymmetry of F^k_{cd}. Thus, the identity (9.44) is a manifestation of the hypersurface deformation algebra. Selecting $f = 1$ reveals that $\{H, \hat{C}(g)\}$ is weakly vanishing for all g and thus $\hat{C}$ is stabilised. Since $\hat{C} = C$ plus already stabilised constraints, it results that C itself is also stabilised.

Consequently, no tertiary constraints arise and all secondary constraints are stabilised. All v^j_a and all $\hat{v}^A_j$ have been fixed, while $\hat{v}^A_t$ remain free. We arrive at the classification of the primary constraints π^t_j, T^a_j, $\hat{P}^I_A$ and the secondary constraints G_j, C, C_a, D_j.

i. π^t_j

Since all constraints are independent of A^j_t, obviously π^t_j is first-class.

ii. $\hat{G}_j$

This constraint generates Gauß gauge transformations on $\pi^a_j, A^j_a, \hat{e}^A_j, \hat{P}^j_A$. Since all constraints are either invariant or covariant under Gauß transformations, $\hat{G}_j$ is first-class.

iii. $\hat{C}_a$

This constraint generates spatial diffeomorphisms on all variables. Since all constraints are tensor densities, $\hat{C}_a$ is first-class.

iv. $\hat{C}$

Since $\hat{C}$ is the integrand of H which stabilises all constraints, it follows that $\hat{C}$ has weakly vanishing Poisson brackets with all constraints except possibly those whose Poisson brackets involve derivatives of δ-distributions, since $H = \hat{C}(f = 1)$ and not the general $\hat{C}(f)$. These are the brackets with $\hat{C}(g)$ and with $T^a_j(g^a_j)$. The first bracket was verified in (9.42) and the second gives the same result as with H, except that the integral of the resulting Poisson bracket also involves f as an underived factor. Thus, $\hat{C}$ is first-class.

v. $\hat{P}^0_A$:

We only need to check its Poisson brackets with $\hat{P}^I_B, T^a_j, D_j$. It is clear that the brackets with $\hat{P}^I_B$, D_j vanish exactly, while

$$\{\hat{P}^0_A(f^A), T^j_a(g^a_j)\} = -\int d^3x\ f_A\ g^a_j\ \frac{\partial \hat{\sigma}^{ta}_j}{\partial \hat{e}^A_0} = -\int d^3x\ g^a_j\ (f^0 \hat{e}^a_j - f^a D_j) \tag{9.44}$$

It results that $\hat{P}^0_a$ is first-class. We replace $\hat{P}^0_t$ by $\hat{P}'^0_t = \hat{P}^0_t - \frac{\hat{e}^a_j}{\hat{e}^t_0}\hat{P}^j_a$. Note that as this quantity is Gauß invariant and a tensor density we only need to check its Poisson brackets with $\hat{P}^B_I, D_j, T^a_I$. With D_j they vanish exactly and with $\hat{P}^B_I$ weakly, while

$$\{\hat{P}'^0_t(f), T^j_a(g^a_j)\} = \gamma \int d^3x\ f\ \frac{\hat{e}^a_j}{\hat{e}^t_0}\ g^a_k\ \epsilon_{jkl} D_l \tag{9.45}$$

Thus $\hat{P}'^0_t$ is also first-class.

vi. $\hat{P}^j_t$, D_k:

These clearly form a second-class pair

$$\{\hat{P}_t^j(x), D_k(y)\} = \delta_{jk}\delta(x,y) \tag{9.46}$$

vii. $T_j^a, \hat{P}_b^k$:
These also form a second class-pair

$$\{\hat{T}_a^j(x), \hat{P}_b^k(y)\} = \{\hat{P}_b^k(y), \sigma_j^{ta}(x)\} = \hat{e}_0^t\delta(x,y)\delta_b^a\delta_j^k \tag{9.47}$$

modulo a term $\propto D_l$.

Accordingly, we get exactly the constraint structure that we expected. It remains to solve the second-class constraints and compute the Dirac bracket. Solving $D_j = 0, P_A^j = 0$ is trivial. Note that for $D_j = 0$ we have $\sigma_j^{ta} = \hat{e}_0^t\hat{e}_j^a$ thus solving $T_j^a = 0$ is equivalent to $\pi_j^a = \hat{e}_0^t\ \hat{e}_j^a$. Consequently, we turn our attention to the remaining variables $A_a^j, \pi_j^a, \hat{P}_A^0, \hat{e}_0^A$ and the spatial diffeomorphism and the Hamiltonian constraint can be equivalently formulated as

$$C_a = F_{ab}^j\ \pi_j^b, \qquad C = F_{ab}^j\ \epsilon_{jkl}\pi_k^a\ \pi_l^b \tag{9.48}$$

Because of (9.46) and (9.47) the Dirac bracket between phase space functions U, V is

$$\begin{aligned}\{U,V\}^* = \quad & \{U,V\} \pm (\int\ d^3x\ \{U,\pi_t^j(x)\}\ \{D_j(x),V\} - \ U\ \leftrightarrow\ V) \\ & \pm(\int\ d^3x\ \tfrac{1}{\hat{e}_0^t(x)}\ \{U,T_j^a(x)\}\ \{\hat{P}_a^j(x),V\} - \ U\ \leftrightarrow\ V)\end{aligned} \tag{9.49}$$

Restricting U, V to be functions of $A_a^j, \pi_j^a, \hat{P}_A^0, \hat{e}_0^A$, we get certainly $\{U, P_A^j\} = \{V, P_A^j\} = 0$. Hence on those functions, the Dirac bracket coincides with the Poisson bracket. Finally, since we put the second-class constraints strongly to zero we obtain

$$\hat{G}_j = G_j = \nabla_a\pi_j^a, \qquad \hat{P}_t^{0\prime} = \hat{P}_t^0 \tag{9.50}$$

and H is a linear combination of $\hat{P}_A^0, C_a, C, G_j$. Note that when $D_j = 0$, in terms of lapse and shift functions, one gets

$$e_0^t = \frac{1}{N}, \qquad e_0^a = -\frac{N^a}{N} \tag{9.51}$$

and e_j^a is invertible. Then $\det(e_A^I) = N\det(e_a^i)$ and $\hat{P}_A^0$ are actually the momenta conjugate to lapse and shift (modulo a canonical transformation). In this way, we obtain exactly the Hamiltonian formulation of the $U(1)^3$ or $SU(2)$ model (Euclidean GR) respectively, regardless of the value of $\gamma \neq 0$.

9.3. Pure Connection Formulation

As is known in tandem with the Ashtekar-Barbero variables, there exist (almost) pure connection formulations for Lorentzian GR in vacuum. Without the cosmological constant, it is feasible to construct a polynomial action in terms of a self-dual $SL(2,\mathbb{C})$ connection and a density-weighted volume form [269, 270]. With the cosmological constant, it is possible to eliminate the volume form and obtain a non-polynomial pure connection formulation [175, 296]. In this section, we revisit these considerations for the Euclidean signature and arbitrary γ parameter (twisted self-duality) simultaneously for both $U(1)^3$ and $SU(2)$. We closely follow [175, 296], but can go one

step further in the following sense: In [175, 296] it was explained that imposition of the Hamiltonian constraint as a primary constraint, rather than a secondary constraint, into the resulting action, can be employed "in principle" to get a pure connection formulation. However, not only was the equation to be solved not written out in detail, but it was not explicitly shown that one can solve it algebraically (it could be a polynomial equation in the Lagrange multiplier of higher than fourth-order). In this section, we demonstrate that, luckily, the equation to be solved is only a quartic equation. We write it out explicitly in the form that can be solved using the Cardano – Ferrari set of formulae [297, 298]. The consequential Lagrangian is then a "spacetime diffeomorphism covariant and pure connection Lagrangian" for an $SU(2)$ (or the $U(1)^3$) gauge theory that is equivalent to Euclidean GR (or its $U(1)^3$ truncation) with a cosmological constant. This is quite remarkable, as GR becomes linguistically much closer to Yang-Mills theory and opens up new possibilities for path integral formulations. In contrast to Yang-Mills theory, the Lagrangian of GR is not polynomial in the connection. Note that all the observations in this section work also for the Lorentzian signature, with the difference that the curvatures appearing, are genuinely complex-valued. It should be emphasised that some recent and in-depth analysis of pure connection formulations after Peldan was developed in [299, 300, 301, 302, 303]. Nevertheless, our intention here is not to derive the most general action for the $U(1)^3$ model, but to provide a concrete example in which a non-polynomial pure connection Lagrangian can be obtained explicitly.

To avoid confusion about the involved calculations that follow, it is useful to state at the outset that the main result of this section is the attainment of the Lagrangian (9.90). As we will see in detail, the function $\hat{D}$ in the expression of (9.90) depends only on the connection A and $\hat{\omega}$ is the real solution of the equation (9.89) whose coefficients all depend only on A, so does $\hat{\omega}$ itself. Accordingly, as a final result, the Lagrangian (9.90) renders the pure connection formulation of the theory.

We start with the Hamiltonian of the previous section, which contains a cosmological constant

$$H = \int d^3x\, h, \quad h := -A^j_t\, \nabla_a E^a_j + N^a C_a - \frac{\gamma}{2}\, N\, \bar{C} - N\, \Lambda\, [\det(E)]^{1/2},$$
$$C_a = F^j_{ab}\, E^b_j, \qquad \bar{C} = \epsilon_{jkl} F^j_{ab}\, E^a_k\, E^b_l\, [\det(E)]^{-1/2} \tag{9.52}$$

where we expressed $\hat{e}^A_0$ in terms of lapse and shift functions. To obtain an action purely in terms of A^j_A, we first perform the Legendre transform of (9.52) with respect to the momentum E^a_j conjugate to A^j_a. This still leaves us with an expression that depends on N, N^a. We remove these by extremising the action with respect to N, N^a and substituting the corresponding solution back into the action. As explained in [175, 296], this leads to an action from which $C_a = 0$ and $C := \frac{\gamma}{2}\, \bar{C} + \Lambda[\det(E)]^{1/2} = 0$ follow as primary constraints when we go back to the Hamiltonian formulation, rather than as secondary constraints as we concluded in the previous section.

The Legendre transform determines the velocity

$$\partial_t A^j_a = \frac{\delta H}{\delta E^a_j} = \nabla_a A^j_t + N^b\, F^j_{ba} - \tilde{N}\, \gamma \epsilon_{jkl} F^k_{ba}\, E^b_l - \tilde{N}\, \frac{\Lambda}{2} \epsilon_{jkl}\, \epsilon_{abc}\, E^b_k\, E^c_l \tag{9.53}$$

where we have defined the lapse $\tilde{N} = N\, [\det(E)]^{-1/2}$ with density weight -1 as an independent variable. By a misuse of notation, we denote $C[\det(E)]^{1/2}$ as C again. Equation (9.53) can be re-expressed as

$$F^j_{ta} - N^b F^j_{ba} = N\, F^j_{na} = -\tilde{N}\, \epsilon_{jkl} [\gamma F^k_{ba}\, E^b_l + \frac{\Lambda}{2} \epsilon_{abc}\, E^b_k\, E^c_l] \tag{9.54}$$

with the spacetime curvature F^j_{AB} and the normal $n^t = \frac{1}{N}$, $n^a = -\frac{N^a}{N}$. Assuming the magnetic field $B^a_j := \epsilon^{abc}F^j_{bc}/2$ is nondegenerate, one can decompose $E^a_j = B^a_k\ \psi^k_j$ for a matrix ψ and express (9.54) in the equivalent form (assuming the spatial metric is nondegenerate, one also has $\det(\psi) \neq 0$)

$$-\frac{N\ F^j_{na}\ B^a_k}{\tilde{N}\det(B)} = \gamma[\mathrm{Tr}(\psi)\ \delta^j_k - \psi^j_k] + \Lambda\ [[\psi^{-1}]^T]^j_k \det(\psi) \tag{9.55}$$

To recognise which restrictions are imposed on ψ when $C_a = C = 0$ hold, one computes

$$C_a = \epsilon_{abc}B^c_l B^b_k \psi^k_j \delta^{lj} = 0, \qquad C = \det(B)\ \{\frac{\gamma}{2}\Big([\mathrm{Tr}(\psi)]^2 - \mathrm{Tr}(\psi^2)\Big) + \Lambda\det(\psi)\} = 0 \tag{9.56}$$

Thus, moving the internal indices with the Kronecker δ one infers that $\psi = \psi^T$ and that $\mathrm{Tr}(\psi^{-1}) = -\Lambda\gamma^{-1}$. We can utilise the antisymmetric part of ψ_{jk} to remove the antisymmetric piece of the r.h.s. of (9.55). Therefore, in what follows, we take ψ to be symmetric and hence (9.55) becomes

$$-\frac{N\ F^{(j}_{na}\ B^a_l\ \delta^{k)l}}{\tilde{N}\det(B)} = \gamma[\mathrm{Tr}(\psi)\ \delta^j_k - \psi^j_k] + \Lambda\ [\psi^{-1}]^j_k \det(\psi) \tag{9.57}$$

The l.h.s. is associated a symmetric, covariant spacetime scalar density one weighted matrix

$$\kappa^{jk} := \frac{1}{4}\epsilon^{ABCD}\ F^j_{AB}\ F^k_{CD} = F^{(j}_{ta}B^a_l\ \delta^{k)l} = N\ F^{(j}_{na}B^a_l\ \delta^{k)l} \tag{9.58}$$

and the scalar density of weight -1

$$w := -\frac{1}{\tilde{N}\ \det(B)} \tag{9.59}$$

so that we gain the density weight zero matrix identity

$$\Omega := w\kappa = \gamma[\mathrm{Tr}(\psi)\ 1_3 - \psi] + \Lambda\ \det(\psi)\psi^{-1} \tag{9.60}$$

and the Lagrangian becomes

$$\begin{aligned} L &= E^a_j\partial_t A^j_a - h \\ &= \tilde{N}\det(B)\{-\mathrm{Tr}(\Omega\psi) + \frac{\gamma}{2}[[\mathrm{Tr}(\psi)]^2 - \mathrm{Tr}(\psi^2)] + \Lambda\det(\psi)\} \\ &= w^{-1}\{\frac{\gamma}{2}[[\mathrm{Tr}(\psi)]^2 - \mathrm{Tr}(\psi^2)] + 2\Lambda\det(\psi)]\} \end{aligned} \tag{9.61}$$

where in the second step we computed $\mathrm{Tr}(\Omega\psi)$ from (9.60). We have not yet used $C = 0$, since we want to compare with the method in [175, 296], so we defer this to a later stage.

To organise the following straightforward but tedious calculations, we introduce the scalars

$$\begin{aligned} T &:= \mathrm{Tr}(\Omega), \qquad S := T^2 - \mathrm{Tr}(\Omega^2), \qquad D := \det(\Omega), \\ \tau &:= \mathrm{Tr}(\psi), \qquad \sigma := \tau^2 - \mathrm{Tr}(\psi^2), \qquad \delta := \det(\psi) \end{aligned} \tag{9.62}$$

We will also require the Caley-Hamilton identity in three-dimensions

$$\psi^3 = \delta\ 1_3 - \frac{\sigma}{2}\ \psi + \tau\ \psi^2 \tag{9.63}$$

which, in the present symmetric case, can also be checked by elementary means, by passage to the diagonal form.

The strategy of [175, 296] is to

(i) derive three relations between $T, S, D, \tau, \sigma, \delta$ from the master equation (9.60) by taking traces of its powers,

(ii) solve σ, δ in terms of T, D, S,

(iii) insert the solution into the Lagrangian (9.61),

(iv) ask that the Lagrangian is stationary under variation of w, which determines w in terms of κ,

(v) insert this solution into (9.61).

It is quite amazing that one can get even to stage (iii), as in stage (ii) we obtain coupled algebraic equations of order three which, when decoupled, can easily lead to polynomial equations of degree five or higher for which no algebraic solution can be attained. Nevertheless, it is possible to provide an expression for L in terms of S, T, D in closed form. However, in stage (iv) we encounter a "quartic" equation. While we still can find a closed (Ferrari) formula for its solution (which, however, involves solving a cubic equation), its introduction in L, which itself does not depend polynomially or even rationally on w, would consume several pages.

Thus, having done all the steps up to (iv), to illustrate the algebraic complexity that arises, and because a derivation was not given in [175, 296], we return to stage (i) and solve $C = 0$ already at this level. Now one finds three relations between T, S, D and the two parameters τ, δ, since according to (9.56) the constraint $C = 0$ is equivalent to $\sigma = \frac{2}{\gamma}\Lambda\delta$. It follows that there is a constraint among S, T, D that gives rise to a polynomial in w with coefficients depending on κ, that is a depressed quartic equation. In this case, the Lagrangian depends rationally on w, so the ultimate solution is of lower complexity.

Stage (i)
The subsequent computations are extremely simplified by re-expressing (9.60) in terms of the eigenvalues λ_j, μ_j of Ω, ψ respectively

$$\lambda_j = \gamma\,[\tau - \mu_j] + \Lambda\,\frac{\delta}{\mu_j} \tag{9.64}$$

where

$$T = \lambda_1 + \lambda_2 + \lambda_3, \qquad S = 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1), \qquad D = \lambda_1\lambda_2\lambda_3 \tag{9.65}$$

and similar for τ, σ, δ.

Taking the trace of (9.60) results in

$$T = 2\gamma\tau + \frac{\Lambda}{2}\sigma \tag{9.66}$$

Next, multiplying out the r.h.s. of S given in (9.65) after regrouping terms, we get

$$\frac{S}{2} = \gamma^2[\tau^2 + \frac{\sigma}{2}] + \Lambda\gamma[6\delta - \sigma\tau + \tau^3 - \ \mathrm{Tr}(\psi^3)] + \Lambda^2\delta\tau \tag{9.67}$$

Taking the trace of (9.63), one writes the r.h.s. just in terms of τ, σ, δ

$$\frac{S}{2} = \gamma^2[\tau^2 + \frac{\sigma}{2}] + \Lambda\gamma[3\delta + \frac{1}{2}\sigma\tau] + \Lambda^2\delta\tau \tag{9.68}$$

Note that the r.h.s. is only quadratic. If this were not the case, we could not complete even step (ii).

Next, again multiplying out the r.h.s. of D given in (9.65) after regrouping terms, we get

$$D = \gamma^3[-\delta + \frac{1}{2}\tau\sigma] + \gamma\Lambda^2\delta\sigma + \Lambda^3\delta^2 + \gamma^2\Lambda[3\tau\delta + \frac{1}{2}\{[\mathrm{Tr}(\psi^2)]^2 - \mathrm{Tr}(\psi^4)\}] \tag{9.69}$$

Multiplying (9.63) with ψ and taking the trace yields

$$[\mathrm{Tr}(\psi^2)]^2 - \mathrm{Tr}(\psi^4) = \frac{1}{2}\sigma^2 - 4\delta\tau \tag{9.70}$$

so that (9.69) can again be written just in terms of τ, σ, δ

$$D = \gamma^3[-\delta + \frac{1}{2}\tau\sigma] + \gamma\Lambda^2\delta\sigma + \Lambda^3\delta^2 + \gamma^2\Lambda[\delta\tau + \frac{1}{4}\sigma^2] \tag{9.71}$$

Again, it is noteworthy that the r.h.s. of (9.72) is only a quadratic.

The equations (9.66), (9.68) and (9.71) allow in principle to express τ, σ, δ in terms of T, S, D. Still, the fact that these form a coupled system of one linear and two quadratic polynomials might prevent a simple algebraic solution. To see this, assume writing τ in terms of T, σ, using (9.66) in (9.68) and (9.71). Then we get a coupled system of two quadratic equations in terms of σ, δ. One can solve (9.68) (which contains δ linearly) for δ in terms of S, σ, which is a fraction with a quadratic and linear polynomial in σ in the numerator and denominator, respectively. Plugging this solution into (9.71), which contains the square of δ, and multiplying it by the square of the denominator, we arrive at a quartic polynomial in σ, which is in general very intricate to solve.

Fortunately, these complications can be avoided, since we are only interested in the combination of τ, σ that occurs in L. First, let $y := \Lambda\delta + \frac{1}{2}\gamma\sigma$, then

$$\frac{S}{2} = \gamma^2(\tau^2 - \sigma) + 3\gamma y + \Lambda\tau y, \; D = -\gamma^3\delta + \Lambda y^2 + \gamma^2\tau y \tag{9.72}$$

Next, with $z = \Lambda y + \gamma^2\tau$

$$\frac{S}{2} = \tau\, z + 3\gamma\, y - \gamma^2\sigma, \qquad D = -\gamma^3\delta + y\, z \tag{9.73}$$

whence

$$\Lambda D + \frac{\gamma^2}{2}S = \gamma^3\, l + z^2, \qquad l = \frac{1}{2}\gamma\sigma + 2\Lambda\delta \tag{9.74}$$

Recalling (9.61) and (9.66), we can now observe that

$$L = w^{-1}\, l, \qquad z = \frac{1}{2}[\Lambda\, l + \gamma T] \tag{9.75}$$

Consequently, we reach the quadratic equation

$$\Lambda D + \frac{\gamma^2}{4}[2S - T^2] = [\gamma^3 + \frac{1}{2}\gamma\Lambda T]\, l + \frac{\Lambda^2}{4}l^2 \tag{9.76}$$

with the two solutions

$$l = -a \pm \sqrt{b + a^2}, \qquad a = \frac{2\gamma^3 + \gamma\Lambda T}{\Lambda^2}, \qquad b = \frac{4}{\Lambda^2}\{\Lambda D + \frac{\gamma^2}{4}[2S - T^2]\} \tag{9.77}$$

If we insist on a well-defined $\Lambda \to 0$ limit, only the positive sign before the square root is admissible.

Equation (9.77) is as far as [175, 296] went (modulo the fact that there $\gamma = \pm i$ for Lorentzian (anti-)self-dual GR). The possibility of removing the implicit occurrence of w in $L = w^{-1}\, l(w, \kappa)$ was mentioned in [175, 296], but not carried out because of the great complexity of the resulting equations. To observe what would be required, we proceed a little further and introduce the dimensionless quantities (F/Λ is dimensionless like N, E)

$$\hat{\kappa} = \frac{\kappa}{\Lambda^2}, \qquad \tilde{w} = \Lambda^3 w, \qquad \hat{\Omega} = \Lambda\Omega = \tilde{w}\, \hat{\kappa} \tag{9.78}$$

and the corresponding dimension-free and w independent quantities

$$\Lambda\, T = \hat{T}\, \tilde{w}, \qquad \Lambda^2\, S = \hat{S}\, [\tilde{w}]^2, \qquad \Lambda^3\, D = \hat{D}\, [\tilde{w}]^3 \tag{9.79}$$

Then, we can rewrite the Lagrangian as

$$\begin{aligned} L = w^{-1}\, l = [\tilde{w}]^{-1}\, \Lambda^3\, l = \Lambda\, \gamma \hat{l}, \qquad \hat{l} = -\hat{a} \pm \sqrt{\hat{b} + \hat{a}^2}, \\ \hat{a} = 2\, \hat{w}^{-1} + \hat{T}\hat{b} = 4\hat{D}\hat{w} + 2\hat{S} - \hat{T}^2, \qquad \hat{w} = \tfrac{\tilde{w}}{\gamma^2} \end{aligned} \tag{9.80}$$

The action (9.80) is stationary with respect to $\hat{w}$ when

$$L' = \gamma\Lambda[-\hat{a}' \pm \frac{\hat{b}' + 2\hat{a}\hat{a}'}{2\hat{W}}] = 0, \qquad \hat{W} = \sqrt{\hat{b} + \hat{a}^2} \tag{9.81}$$

where the prime denotes a derivative with respect to $\hat{w}$. One can write this as

$$2\hat{a}'[\hat{W} \mp \hat{a}] = \pm\hat{b}' = 2\hat{a}' \frac{\hat{b}}{\hat{W} \pm \hat{a}} \tag{9.82}$$

Solving for $\hat{W}$, we find

$$\pm\, \hat{W} = \hat{a} + \frac{\hat{b}'}{2\hat{a}'} = 2\frac{\hat{a}'\hat{b}}{\hat{b}'} - \hat{a} \tag{9.83}$$

i.e.

$$\hat{a}'(\hat{a}'\hat{b} - \hat{a}\hat{b}') = [\frac{\hat{b}'}{2}]^2 \tag{9.84}$$

Denoting $\alpha = 4\hat{D},\ \beta = 2\hat{S} - \hat{T}^2$, we finally obtain

$$\frac{2}{\hat{w}^2}[\frac{2}{\hat{w}^2}[\alpha\hat{w} + \beta] + \alpha[2\hat{w} - 1 + \hat{T}]] = [\frac{\alpha}{2}]^2 \tag{9.85}$$

After multiplication by $\hat{w}^4$, this turns to a quartic equation in depressed form (with no cubic term), which can be solved using the Ferrari formula. Nonetheless, since this necessitates implicitly solving a non-depressed (the quadratic term is non-vanishing) cubic equation, the explicit formula for the four roots is not only very tedious but also its insertion into (9.80) would be extremely complicated due to the occurrence of the square root term.

The above procedure imposes the Hamiltonian constraint as a secondary constraint by extremising with respect to the Lagrange multiplier w. We naturally obtain an equivalent result by

imposing $C = 0$, i.e. $\frac{\gamma\sigma}{2} + \Lambda\delta = 0$, already in (9.66), (9.68) and (9.71), making the equation to be solved more transparent. First, we find the simplified system

$$\begin{aligned} D &= -\gamma^3\,\delta \\ \frac{S}{2} &= \gamma^2\tau^2 + 2\gamma\Lambda\delta \\ T &= 2\gamma\tau - \frac{\Lambda^2}{2\gamma}\delta \end{aligned} \tag{9.86}$$

Eliminating τ, δ imposes the constraint on T, S, D given by

$$\frac{S}{2} + 2\frac{\Lambda}{\gamma^2}\,D = \frac{1}{4}\,[\frac{\Lambda^2}{\gamma^4}D - T]^2 \tag{9.87}$$

which is equivalent to imposing $C = 0$ i.e. choosing σ appropriately. In terms of the dimension-free quantities

$$\hat{w} = \Lambda^3\,w\,\gamma^{-2}, \qquad \hat{T} = T\,\Lambda^{-2}\,w^{-1}, \qquad \hat{S} = S\,\Lambda^{-4}\,w^{-2}, \qquad \hat{D} = D\,\Lambda^{-6}\,w^{-3}, \tag{9.88}$$

we find

$$\frac{\hat{S}}{2} + 2\hat{w}\,\hat{D} = \frac{1}{4}\,[\hat{w}^2\,\hat{D} - \hat{T}]^2 \tag{9.89}$$

which is again a quartic equation in depressed form for $\hat{w}$. Note that the Lagrangian for $C = 0$ is simply

$$L = -w^{-1}\,\Lambda\,\gamma^{-3}\,D\; = -w^2\,\Lambda^7\,\gamma^{-3}\,\hat{D}\; = -\hat{w}^2\,\Lambda\gamma\,\hat{D} \tag{9.90}$$

which looks deceptively simple, but of course the challenge is to solve the quartic equation (9.89) for $\hat{w}$ in terms of $\hat{T}, \hat{S}, \hat{D}$, which are traces of polynomials in the matrix $\hat{\kappa} = \kappa\Lambda^{-2}$. We can acquire the general solution of (9.89) using the Cardano-Ferrari theory. We do not display it explicitly here since the formulas are quite lengthy. We want to know, however, whether there are real roots among the four, and in the case of an affirmative answer, under what conditions for κ the real solution is guaranteed. Examining its discriminants, we conclude that the equation (9.89) always has a real solution **unless** $\hat{S} < 0$ and

$$\begin{aligned} &\hat{T} < 0, \;\; 0 < \hat{D} < \frac{1}{54}\hat{T}(9\hat{S} - 4\hat{T}^2) + \frac{\sqrt{(2\hat{T}^2 - 3\hat{S})^3}}{27\sqrt{2}} \\ &\text{OR} \\ &\hat{T} > 0, \;\; \frac{1}{54}\hat{T}(9\hat{S} - 4\hat{T}^2) - \frac{\sqrt{(2\hat{T}^2 - 3\hat{S})^3}}{27\sqrt{2}} < \hat{D} < 0. \end{aligned} \tag{9.91}$$

9.3.1. Consistency Check: Hamiltonian Analysis of the Pure Connection Action

The next task is to verify that the Legendre transform of (9.90) renders $C_a = C = 0$ as primary constraints. To make arrangements for the following computations that are rather tedious, we

define

$$\begin{aligned}
\mathfrak{A} &:= -\frac{\gamma\hat{\omega}}{\Lambda(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))},\\
\mathfrak{B} &:= \frac{\gamma}{6(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))},\\
\mathfrak{M}^{ij} &:= \kappa^{ij} - \hat{\omega}\hat{D}(\kappa^{-1})^{ij},\\
\mathfrak{N} &:= \hat{T}+\hat{D}\hat{\omega}^2,\\
\tilde{\mathfrak{N}} &:= 6+4\hat{T}\hat{\omega}-2\hat{D}\hat{\omega}^3
\end{aligned} \tag{9.92}$$

Some of the quantities associated with the matrix $\mathfrak{M}^{ij}$ needed below can be expressed in terms of $\hat{S},\hat{T},\hat{D},\hat{\omega}$

$$\begin{aligned}
\mathrm{Tr}(\mathfrak{M}) &= \hat{T}-\frac{\hat{S}\hat{\omega}}{2}\\
\mathrm{Tr}(\mathfrak{M}^2) &= \hat{T}^2-\hat{S}-6\hat{D}\hat{\omega}+\frac{\hat{S}^2\hat{\omega}^2}{4}-2\hat{T}\hat{D}\hat{\omega}^2\\
[\mathrm{Tr}(\mathfrak{M})]^2-\mathrm{Tr}(\mathfrak{M}^2) &= \hat{S}+(6\hat{D}-\hat{S}\hat{T})\hat{\omega}+2\hat{T}\hat{D}\hat{\omega}^2\\
\det(\mathfrak{M}) &= \hat{D}-\frac{1}{2}(\frac{\hat{S}^2}{2}-4\hat{T}\hat{D})\hat{\omega}+\hat{D}(\hat{T}^2-\hat{S})\hat{\omega}^2-\hat{D}^2\hat{\omega}^3
\end{aligned} \tag{9.93}$$

where in order to derive the second equation, we used $\mathrm{Tr}(\hat{\kappa}^{-2})=\frac{1}{2\hat{D}^2}\left(\frac{\hat{S}^2}{2}-4\hat{T}\hat{D}\right)$ and for the last equation, we took advantage of the relation

$$\det(A+B)=\det(A)+\det(B)+\det(B)\mathrm{Tr}(AB^{-1})+\det(A)\mathrm{Tr}(BA^{-1}) \tag{9.94}$$

which is valid for all 3×3 invertible matrices A,B.

Starting from the Lagrangian $L=-\gamma\Lambda\hat{D}\hat{\omega}^2$, we must first compute the momentum conjugate to A^j_a. Since the Lagrangian depends on $\hat{\omega}$, the conjugate momentum depends on the variation of $\hat{\omega}$ with respect to $\dot{A}^j_a$. Taking the variation of both sides of (9.89) and isolating $\delta\hat{\omega}$, one obtains

$$\begin{aligned}
\delta\hat{\omega} =& \frac{(\hat{T}-\hat{D}\hat{\omega}^2)(\delta\hat{T}-\hat{\omega}^2\delta\hat{D})-4\hat{\omega}\delta\hat{D}-\delta\hat{S}}{2\hat{D}(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))}\\
=& \frac{\hat{T}-\hat{D}\hat{\omega}^2}{2\hat{D}(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))}\delta\hat{T}+\frac{-1}{2\hat{D}(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))}\delta\hat{S}+\frac{-\hat{\omega}^2(\hat{T}-\hat{D}\hat{\omega}^2)-4\hat{\omega}}{2\hat{D}(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))}\delta\hat{D}
\end{aligned} \tag{9.95}$$

Thus, the momentum is obtained as

$$\begin{aligned}
\Pi^a_j :=& \frac{\delta L}{\delta \dot{A}^j_a} = -\gamma\Lambda \frac{\delta(\hat{\omega}^2\hat{D})}{\delta \dot{A}^j_a} = -\gamma\Lambda\left(2\hat{D}\hat{\omega}\frac{\delta\hat{\omega}}{\delta\dot{A}^j_a} + \hat{\omega}^2\frac{\delta\hat{D}}{\delta\dot{A}^j_a}\right)\\
=& -\frac{\gamma\Lambda}{2\hat{D}(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))}\left[2\hat{D}\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2)\frac{\delta\hat{T}}{\delta\dot{A}^j_a} - 2\hat{D}\hat{\omega}\frac{\delta\hat{S}}{\delta\dot{A}^j_a} - 4\hat{D}\hat{\omega}^2\frac{\delta\hat{D}}{\delta\dot{A}^j_a}\right]\\
=& -\frac{\gamma\Lambda\hat{\omega}}{(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))}\left[(\hat{T}-\hat{D}\hat{\omega}^2)\frac{\delta\hat{T}}{\delta\dot{A}^j_a} - \frac{\delta\hat{S}}{\delta\dot{A}^j_a} - 2\hat{\omega}\frac{\delta\hat{D}}{\delta\dot{A}^j_a}\right]\\
=& -\frac{\gamma\hat{\omega}}{\Lambda(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))}\left[(\hat{T}-\hat{D}\hat{\omega}^2)B^a_j - (2\hat{T}B^a_j - 2\kappa^{ij}B^a_i) - 2\hat{\omega}\hat{D}(\kappa^{-1})_{ji}B^a_i\right]\\
=& -\frac{\gamma\hat{\omega}}{\Lambda(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))}\left[2\left(\kappa^{ij}-\hat{\omega}\hat{D}(\kappa^{-1})_{ji}\right)B^a_i - (\hat{T}+\hat{D}\hat{\omega}^2)B^a_j\right]\\
=& \mathfrak{A}\left[2\mathfrak{M}^{ij}B^a_i - \mathfrak{N}B^a_j\right]
\end{aligned} \tag{9.96}$$

where from the third to the fourth line, we have used the following variations

$$\begin{aligned}
\left(\frac{\delta\hat{\kappa}}{\delta\dot{A}^l_c}\right)^{ij} &= \Lambda^{-2}\delta^{(i}_l B^{j)c},\\
\frac{\delta\hat{T}}{\delta\dot{A}^l_c} &= \Lambda^{-2}B^c_l,\\
\frac{\delta\hat{S}}{\delta\dot{A}^l_c} &= 2\hat{T}\frac{\delta\hat{T}}{\delta\dot{A}^l_c} - 2\hat{\kappa}^{ij}\left(\frac{\delta\hat{\kappa}}{\delta\dot{A}^l_c}\right)_{ij} = \Lambda^{-2}(2\hat{T}B^c_l - 2\hat{\kappa}^{il}B^c_i)\\
\frac{\delta\hat{D}}{\delta\dot{A}^l_c} &= \hat{D}(\hat{\kappa}^{-1})_{ij}\left(\frac{\delta\hat{\kappa}}{\delta\dot{A}^l_c}\right)^{ij} = \Lambda^{-2}\hat{D}(\hat{\kappa}^{-1})_{ij}\delta^{(i}_l B^{j)c} = \Lambda^{-2}\hat{D}(\hat{\kappa}^{-1})_{lj}B^c_j
\end{aligned} \tag{9.97}$$

In the last line of (9.97), the Jacobi's formula $d(\det(M)) = \det(M)\mathrm{Tr}(M^{-1}dM)$ has been utilised, that is valid for every invertible matrix M.

As $F^i_{ab} = \epsilon_{cab}B^c_i$, from (9.96), one immediately infer that

$$C_a = F^i_{ab}\Pi^b_i = \mathfrak{A}\epsilon_{cab}B^c_i\left[2\mathfrak{M}^{ji}B^b_j - \mathfrak{N}B^b_i\right] = 0 \tag{9.98}$$

because the matrix $\mathfrak{M}$ is symmetric. This simply unveils that the vector constraint is a primary constraint.

To obtain a similar result for the Hamiltonian constraint more computation is required.

$$\begin{aligned}
C =& \frac{\gamma}{2}\epsilon_{ijk}F^i_{ab}\Pi^a_j\Pi^b_k + \Lambda\det(\Pi)\\
=& \frac{\gamma}{2}\epsilon_{ijk}\epsilon_{cab}B^c_i\Pi^a_j\Pi^b_k + \frac{\Lambda}{6}\epsilon_{ijk}\epsilon_{cab}\Pi^c_i\Pi^a_j\Pi^b_k\\
=& \left(\frac{\gamma}{2}B^c_i + \frac{\Lambda}{6}\Pi^c_i\right)\epsilon_{ijk}\epsilon_{cab}\Pi^a_j\Pi^b_k\\
=& \frac{\gamma}{2}\left(B^c_i - \frac{\hat{\omega}}{3(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))}\left[2\left(\kappa^{li}-\hat{\omega}\hat{D}(\kappa^{-1})_{il}\right)B^c_l - (\hat{T}+\hat{D}\hat{\omega}^2)B^c_i\right]\right)\epsilon_{ijk}\epsilon_{cab}\Pi^a_j\Pi^b_k\\
=& \frac{\gamma}{6(2+\hat{\omega}(\hat{T}-\hat{D}\hat{\omega}^2))}\left((6+4\hat{T}\hat{\omega}-2\hat{D}\hat{\omega}^3)B^c_i - 2\hat{\omega}\left(\kappa^{li}-\hat{\omega}\hat{D}(\kappa^{-1})_{il}\right)B^c_l\right)\epsilon_{ijk}\epsilon_{cab}\Pi^a_j\Pi^b_k\\
=& \mathfrak{B}\left(\tilde{\mathfrak{N}}B^c_i - 2\hat{\omega}\mathfrak{M}^{il}B^c_l\right)\epsilon_{ijk}\epsilon_{cab}\Pi^a_j\Pi^b_k
\end{aligned} \tag{9.99}$$

Using (9.96), we calculate $\epsilon_{ijk}\epsilon_{cab}\Pi^a_j\Pi^b_k$ part of C as

$$\begin{aligned}
\epsilon_{ijk}\epsilon_{cab}\Pi^a_j\Pi^b_k &= \epsilon_{ijk}\epsilon_{cab}\mathfrak{A}^2\left[2\mathfrak{M}^{mj}B^a_m - \mathfrak{N}B^a_j\right]\left[2\mathfrak{M}^{nk}B^b_n - \mathfrak{N}B^b_k\right]\\
&= \epsilon_{ijk}\epsilon_{cab}\mathfrak{A}^2\left[4\mathfrak{M}^{mj}\mathfrak{M}^{nk}B^b_nB^a_m - 4\mathfrak{M}^{mj}\mathfrak{N}B^b_kB^a_m + \mathfrak{N}^2B^a_jB^b_k\right]\\
&= \epsilon_{ijk}\mathfrak{A}^2\det(B)(B^{-1})^l_c\left[4\mathfrak{M}^{mj}\mathfrak{M}^{nk}\epsilon_{lmn} - 4\mathfrak{M}^{mj}\mathfrak{N}\epsilon_{lmk} + \mathfrak{N}^2\epsilon_{ljk}\right]\\
&= \mathfrak{A}^2\det(B)(B^{-1})^l_c\left[4\mathfrak{M}^{mj}\mathfrak{M}^{nk}\epsilon_{ijk}\epsilon_{lmn} - 4\mathfrak{M}^{mj}\mathfrak{N}(\delta^i_l\delta^j_m - \delta^i_m\delta^j_l) + 2\mathfrak{N}^2\delta^i_l\right]\\
&= \mathfrak{A}^2\det(B)(B^{-1})^l_c\left[4\mathfrak{M}^{mj}\mathfrak{M}^{nk}\epsilon_{ijk}\epsilon_{lmn} + 4\mathfrak{M}^{il}\mathfrak{N} + (2\mathfrak{N}^2 - 4\mathfrak{M}^j_j\mathfrak{N})\delta^i_l\right] \qquad (9.100)
\end{aligned}$$

Finally, inserting (9.100) into (9.99) and simplifying, we get

$$\begin{aligned}
C =&\mathfrak{B}\mathfrak{A}^2\det(B)(B^{-1})^l_c\left(\tilde{\mathfrak{N}}B^c_i - 2\hat{\omega}\mathfrak{M}^{ir}B^c_r\right)\left[4\mathfrak{M}^{mj}\mathfrak{M}^{nk}\epsilon_{ijk}\epsilon_{lmn} + 4\mathfrak{M}^{il}\mathfrak{N} + (2\mathfrak{N}^2 - 4\mathfrak{M}^j_j\mathfrak{N})\delta^i_l\right]\\
=&\mathfrak{B}\mathfrak{A}^2\det(B)\left(\tilde{\mathfrak{N}}\delta^l_i - 2\hat{\omega}\mathfrak{M}^{il}\right)\left[4\mathfrak{M}^{mj}\mathfrak{M}^{nk}\epsilon_{ijk}\epsilon_{lmn} + 4\mathfrak{M}^{il}\mathfrak{N} + (2\mathfrak{N}^2 - 4\mathfrak{M}^j_j\mathfrak{N})\delta^i_l\right]\\
=&\mathfrak{B}\mathfrak{A}^2\det(B)\\
&\left(\tilde{\mathfrak{N}}\left[4\mathfrak{M}^{mj}\mathfrak{M}^{nk}\epsilon_{ijk}\epsilon_{imn} + 4\mathfrak{M}^i_i\mathfrak{N} + 3(2\mathfrak{N}^2 - 4\mathfrak{M}^j_j\mathfrak{N})\right]\right.\\
&\left.-2\hat{\omega}\left[4\mathfrak{M}^{il}\mathfrak{M}^{mj}\mathfrak{M}^{nk}\epsilon_{ijk}\epsilon_{lmn} + 4\mathfrak{M}^{il}\mathfrak{M}^{il}\mathfrak{N} + (2\mathfrak{N}^2 - 4\mathfrak{M}^j_j\mathfrak{N})\mathfrak{M}^i_i\right]\right)\\
=&\mathfrak{B}\mathfrak{A}^2\det(B)\\
&\left(\tilde{\mathfrak{N}}\left[4([\mathrm{Tr}(\mathfrak{M})]^2 - \mathrm{Tr}(\mathfrak{M}^2)) - 8\mathfrak{N}\mathrm{Tr}(\mathfrak{M}) + 6\mathfrak{N}^2\right]\right.\\
&\left.-2\hat{\omega}\left[24\det(\mathfrak{M}) + 4(\mathrm{Tr}(\mathfrak{M}^2) - [\mathrm{Tr}(\mathfrak{M})]^2)\mathfrak{N} + 2\mathfrak{N}^2\mathrm{Tr}(\mathfrak{M})\right]\right)\\
=&\mathfrak{B}\mathfrak{A}^2\det(B)\left(4([\mathrm{Tr}(\mathfrak{M})]^2 - \mathrm{Tr}(\mathfrak{M}^2))(\tilde{\mathfrak{N}} + 2\hat{\omega}\mathfrak{N}) - (8\mathfrak{N}\tilde{\mathfrak{N}} + 4\hat{\omega}\mathfrak{N}^2)\mathrm{Tr}(\mathfrak{M}) + 6\tilde{\mathfrak{N}}\mathfrak{N}^2 - 48\hat{\omega}\det(\mathfrak{M})\right)
\end{aligned}$$

Now we can use the relations (9.93) and express the Hamiltonian constraint as a polynomial of $\hat{\omega}$

$$\begin{aligned}
C = \mathfrak{B}\mathfrak{A}^2\det(B)\left[-12\hat{D}^3\hat{\omega}^7 - 6\hat{D}^2\hat{S}\hat{\omega}^6 + 12\hat{D}^2\hat{T}\hat{\omega}^5 + 6(14\hat{D}^2 + 2\hat{D}\hat{S}\hat{T})\hat{\omega}^4 + 6(12\hat{D}\hat{S} + 2\hat{D}\hat{T}^2)\hat{\omega}^3\right.\\
\left.+6(2\hat{S}^2 + 20\hat{D}\hat{T} - \hat{S}\hat{T}^2)\hat{\omega}^2 + 6(16\hat{D} + 4\hat{S}\hat{T} - 2\hat{T}^3)\hat{\omega} + 24\hat{S} - 12\hat{T}^2\right] \qquad (9.101)
\end{aligned}$$

One can deduce that C arises as a primary constraint only when equations (9.101) and (9.89) have a common real solution. Although (9.101) seems too complicated to solve, it is not surprising that one can factorise it into two polynomials

$$C = -6\mathfrak{B}\mathfrak{A}^2\det(B)\left(2\hat{D}\hat{\omega}^3 + \hat{S}\hat{\omega}^2 + 2\hat{T}\hat{\omega} + 2\right)\left(\hat{D}^2\hat{\omega}^4 - 2\hat{D}\hat{T}\hat{\omega}^2 - 8\hat{D}\hat{\omega} + \hat{T}^2 - 2\hat{S}\right) = 0 \qquad (9.102)$$

which vanishes since $\hat{D}^2\hat{\omega}^4 - 2\hat{D}\hat{T}\hat{\omega}^2 - 8\hat{D}\hat{\omega} + \hat{T}^2 - 2\hat{S} = 0$ due to the equation (9.89). This ends the verification that $C = C_a = 0$ emerge as primary constraints.

Summary, Conclusion and Outlook

The main subject of this thesis is to analyse the $U(1)^3$ model of Euclidean quantum gravity as a toy model in order to gain a deeper understanding of the dynamics of loop quantum gravity. It aims at comparing different quantisation approaches of the theory with the hope that an inkling, applicable to the full GR, can be found to shed new light on the path toward an impeccable theory for quantum gravity. It should be emphasised that, in our work, the reduced phase space formulation and correct underlying actions of the $U(1)^3$ model were arrived at for the first time.

10.1. Summary of the Main Results and Conclusions

We have taken *the first steps* toward the reduced phase space [89] and path integral [91] quantisations of the $U(1)^3$ model whose Dirac quantisation has already been derived to some extent in [87, 88]. In fact, at the outset, one must ensure that an "appropriate" classical infrastructure is employed as a jumping platform to dive into the quantisation stage! In the context of these two types of quantisation, the word "appropriate" entails:

In the reduced phase space quantisation:

Finding gauge conditions whose associated reduced phase space admits a fairly simple physical Hamiltonian manageable to be quantised.

In view of this, in [89] (chapter 8) we provided several lines of evidence for the hope that the $U(1)^3$ truncation model for Euclidean vacuum quantum gravity might have a reduced phase space quantisation with an unexpected level of analytic control. We worked with two different sets of canonical variables, namely (A, E) and (B, f). The key feature of the model, making the reduced quantisation possible, is that the constraints are at most linear in momentum. Indeed, in the (A, E) description the spacetime constraints are linear in the Abelian connection and the Gauß constraint in E, while in the (B, f) description all constraints are linear in B. This subtle discrepancy between the two descriptions makes working with the latter even more convenient. We studied various gauges in which the reduced Hamiltonian takes a manageable algebraic form for the respective physical degrees of freedom. In the (A, E) description, we only outlined the quantisation of the resulting physical degrees of freedom and the physical Hamiltonian, but we could already argue that the key feature of the model extremely simplifies the spectral problem.
In this work, a complete analysis of the asymptotic properties of the theory was required, since we had to use the decay behaviour of the phase space variables to select appropriate gauges and Green functions. We performed this task in a separate work [90] (chapter 6), whose main outcome is summarised as

Asymptotically flat boundary conditions:

We studied the asymptotically flat boundary conditions yielding well-defined symplectic structure and finite and integrable charges associated with the asymptotic symmetries. We showed that in the $U(1)^3$ model, the boundary terms that spoil the functional differentiability of the constraints are exact one-forms and therefore all constraints can be improved to differentiable functionals. However, these functionals are not finite for boosts and rotations and we showed that the reason for this is the absence of the non-Abelian term, i.e. $\epsilon_{ijk}A_a^j E_k^a$, in the Gauß constraint of this model compared to that of GR. Therefore, in the $U(1)^3$ model, only asymptotic spacetime translations have well-defined generators. Fortunately, this result turned out to be sufficient for our purposes in [89] (chapter 8), where we were concerned only with spacetime translations.

It is worth noting that concerning the relation between gauge-fixing and gauge-invariant formalisms, there is a one-to-one correspondence between a choice of gauge fixing and a preferred set of gauge-invariant functions that generate the full algebra of gauge-invariant functions [146]. The two formalisms are therefore equivalent at generic points of the reduced phase space where the Dirac matrix (which in any interacting theory is a nontrivial phase-space function) is nonsingular. In the same sense, different gauge fixing conditions are, in general, (locally in phase space) equivalent. As usual, global differences may affect the quantisation in different gauge choices. However, our attitude is that, in quantum gravity, the global non-equivalence of gauge-fixed theories is a second-order concern, one would be pleased to have at least a working quantisation at one's proposal to start with, which can then be further improved. In this sense, the paper [89] (chapter 8) provides the ground for the reduced phase space quantisation of the $U(1)^3$ toy model of LQG.

In the path integral quantisation:

Finding a covariant origin for the theory under consideration.

Note first that the simplification of $SU(2)$ to $U(1)^3$ within the Hamiltonian formulation can be easily done by hand by deleting all non-Abelian terms from the Gauss, spatial diffeomorphism, and Hamiltonian constraints respectively. However, the question arises as to which Lagrangian formulation this theory derives from. For the $SU(2)$ theory it is known that one can choose the Palatini action, the Holst action, or the (anti-)self-dual action (Euclidean signature) as a starting point, all of which lead to equivalent Hamiltonian formulations. Although a Lagrangian for the $U(1)^3$ theory was proposed in [2], we showed in [91] (section 5.2) that it has too many degrees of freedom and cannot be a suitable Lagrangian. We, therefore, had to look for another covariant action.
In [91] (chapter 9) we systematically analysed this question directly for the $U(1)^3$ theory. Surprisingly, it turned out that the Abelian analogue of the Palatini or Holst formulation is a consistent but topological theory without propagating degrees of freedom. On the other hand, a twisted Abelian analogue of the (anti-)self-dual formulation does lead to the desired Hamiltonian formulation.
A new aspect of our derivation is that we worked 1. with semi-density-valued tetrads, which simplifies the analysis, 2. without the simplicity constraint (which admits an undesirable solution that is usually neglected by hand), and 3. without imposing the time gauge from the beginning. As a by-product, we showed that the non-Abelian theory also admits a twisted (anti-)self-dual formulation. Finally, we also derived a pure connection formulation of the Euclidean GR with a cosmological constant by extending earlier work by Capovilla, Dell,

Jacobson, and Peldan. This could prepare an interesting foundation for path integral studies and display (Euclidean) GR as a Yang-Mills theory with non-polynomial Lagrangian. Both Lagrangians can serve as a possible starting point for a path integral quantisation and can be translated into the language of spin foams (see chapter 3).

10.2. Outlook

In the continuation of [89]:

Note that, in contrast to GR, because of the simplicity of the constraints in the $U(1)^3$ model, one does not have to add matter (e.g. dust) to the theory in order to perform the reduced phase space quantisation. It means that if one can derive the quantum theory of the reduced phase space, it will be directly comparable to that obtained by operator quantisation. In future work, we intend to complete the reduced phase space quantisation of the $U(1)^3$ model. Afterwards, we compare the outcome with the results obtained from the Dirac approach [87, 88] (chapter 7). This may extend our technical arsenal to examine the dynamics of the theory. Especially we want to find some clues reducing the ambiguities represented in chapter 4. In the end, by getting some lessons from this toy model, we hope that applying them to the $SU(2)$ case, i.e. full GR, leads to less ambiguous dynamics.

In the continuation of [90]:

We are interested in performing asymptotic quantisation, especially after having access to the BMS group. We wish to find the BMS group of the Euclidean GR written in terms of Ashtekar-Barbero variables, in analogy to what is done in [286] using ADM variables. This is challenging because of the presence of internal gauge freedoms whose asymptotic behaviours have to be determined. This may lead to some interesting results in connection with other approaches to quantum gravity in which asymptotic behaviours are of great importance.

In the continuation of [91]:

As the starting point of covariant quantisation is a Lagrangian, we are now able to apply the techniques of spin foam models (see chapter 3) to the $U(1)^3$ theory. We would call the resulting models *charge foams* in analogy with the charge networks we introduced in chapter 7. The procedure behind this is twofold:

1. Since we have a satisfactory control on the reduced phase space of the $U(1)^3$ model (from chapter 8), one can proceed using the method of section 3.2.1.
2. One can still follow the strategy of the spin foam models sketched in section 3.2.2. This would be an interesting laboratory to test the method of imposing second-class constraints that we explained in section 3.3.4.2.

As the gauge group is Abelian, we expect that the resulting charge foam model from our current work can be much better controlled than their non-Abelian versions, which thus could serve as an interesting test laboratory for the spin foam approach to LQG. Interestingly, our results immediately generalise from $U(1)^3$ to $SU(2)$ so that one can also write a spin foam model for Euclidean General Relativity but with gauge group $SU(2)$ rather than $SO(4)$. This may also lead to major simplifications even in the non-Abelian context.

Solutions of the Constraints in the (B,f) Description

Due to the aforementioned reason in the theorem 8.2.2, the solutions of the constraints are not required to obtain the physical Hamiltonian. However, to ensure that the model is consistent, we need to answer the question of whether the system of equations under consideration admits solutions that satisfy the specified asymptotic behaviour. To this end, in this appendix, we show that by considering the boundary conditions (8.28), (8.29), and (8.30), there exist solutions for the systems of equations (8.37) and (8.41).

A.1. Solutions for the System of Equations (8.37)

We can solve the first and second equations of the system of equations (8.37) for B^x_α as $B^x_\alpha = -\partial_x^{-1}(\partial_y B^y_\alpha + \partial_z B^z_\alpha) + g_\alpha(y,z)$, where g_α are arbitrary functions depending only on y, z. Because B^x_α falls off at infinity due to (8.28), so g_α must vanish. Hence,

$$B^x_\alpha = -\partial_x^{-1}(\partial_y B^y_\alpha + \partial_z B^z_\alpha). \tag{A.1}$$

The third equation of (8.37) is solved for B^z_3 as $B^z_3 = -\partial_z^{-1}\partial_I B^I_3 + g(x,y)$ where g is an arbitrary function in the kernel of ∂_z. Again, exploiting the asymptotic behaviour of B^z_3, one deduces $g = 0$. Hence,

$$B^z_3 = -\partial_z^{-1}\partial_I B^I_3 \tag{A.2}$$

As $H^3_z = \det(H^\alpha_I) \neq 0$, we can simply solve the fourth and fifth equations of (8.37) forming an algebraic system of two equations with two unknowns B^I_3

$$B^I_3 = \epsilon^{IJ}\epsilon^{\alpha\beta} B^z_\alpha H^\beta_J \tag{A.3}$$

Plugging (A.1)-(A.3) into the sixth and seventh equations of (8.37), one gets

$$-Y_2 B^y_1 + Y_1 B^y_2 = \epsilon^{\alpha\beta} H^\beta_x \partial_x^{-1}\partial_z B^z_\alpha \tag{A.4}$$

$$-Y_1 B^y_1 - Y_2 B^y_2 = H^\alpha_x \partial_x^{-1}\partial_z B^z_\alpha + \epsilon^{IJ}\epsilon^{\alpha\beta} H^3_z \partial_z^{-1}\partial_I (H^\beta_J B^z_\alpha) \tag{A.5}$$

where $Y_\alpha := H^\alpha_x \partial_x^{-1}\partial_y - H^\alpha_y$. The inverse of the operator Y_2 can be used to solve (A.4) for B^y_1 as $B^y_1 = Y_2^{-1} Y_1 B^y_2 - Y_2^{-1}\left(\epsilon^{\alpha\beta} H^\beta_x \partial_x^{-1}\partial_z B^z_\alpha\right) + \kappa$ where κ is in the kernel of Y_2 and of the form $\kappa = \sum_{n=3}^{\infty} \kappa_n r^{-n}$, because $B^y_1 = O(r^{-3})$. Since $H^\alpha_I = \delta^\alpha_I + O(r^{-1})$, the highest order term of the defining equation for κ, that is $0 = Y_2\kappa = H^2_x \partial_x^{-1}\partial_y \kappa - H^2_y \kappa$, is κ_3/r^3 which has to vanish

individually. Therefore, $\kappa_3 = 0$ and $\kappa = O(r^{-4})$. Repeating the same reasoning, one infers that $\kappa = 0$. Thus,

$$B_1^y = Y_2^{-1} Y_1 B_2^y - Y_2^{-1} \left(\epsilon^{\alpha\beta} H_x^\beta \partial_x^{-1} \partial_z B_\alpha^z \right) \tag{A.6}$$

Inserting (A.6) in (A.5) results in

$$X B_2^y = Y_1 Y_2^{-1} \left(\epsilon^{\alpha\beta} H_x^\beta \partial_x^{-1} \partial_z B_\alpha^z \right) - H_x^\alpha \partial_x^{-1} \partial_z B_\alpha^z - \epsilon^{IJ} \epsilon^{\alpha\beta} H_z^3 \partial_z^{-1} \partial_I (H_J^\beta B_\alpha^z) \tag{A.7}$$

where $X := Y_1 Y_2^{-1} Y_1 + Y_2$. The inverse of X is used to solve (A.7) for B_2^y as

$$B_2^y = X^{-1} Y_1 Y_2^{-1} \left(\epsilon^{\alpha\beta} H_x^\beta \partial_x^{-1} \partial_z B_\alpha^z \right) - X^{-1} \left(H_x^\alpha \partial_x^{-1} \partial_z B_\alpha^z \right) - \epsilon^{IJ} \epsilon^{\alpha\beta} X^{-1} \left(H_z^3 \partial_z^{-1} \partial_I (H_J^\beta B_\alpha^z) \right) + \bar{\kappa}$$

, where $\bar{\kappa}$ is a member of the kernel of X and decays at infinity as $O(r^{-3})$, since $B_2^y = O(r^{-3})$. In the following, we employ the same method as described in full detail in section 8.3.2.3 in order to specify $\bar{\kappa}$. We note that the fall-off behaviour of H_I^α implies $Y_1 = \partial_x^{-1} \partial_y + O(r^{-1})$. Moreover, if one uses (8.114), it is readily deduced that $Y_2^{-1} = -1 + O(r^{-1})$. Hence, the highest order term of the r.h.s. of the defining equation for $\bar{\kappa}$, i.e. $0 = X\bar{\kappa} = (Y_1 Y_2^{-1} Y_1 + Y_2)\bar{\kappa}$, is $-(\partial_x^{-2} \partial_y^2 + 1) \frac{\bar{\kappa}_3}{r^3}$ that must vanish separately. Applying ∂_x^2, one observes that the highest order term of $\bar{\kappa}$ has to satisfy the 2-dimensional Laplace equation $\Delta(\bar{\kappa}_3 / r^3) = 0$, meaning that $\bar{\kappa}_3 = 0$, as in $\mathbb{R}^3$ the only harmonic function vanishing at infinity is the trivial function. Thus, $\bar{\kappa} = O(r^{-4})$. Iterating the reasoning leads to $\bar{\kappa} = 0$. Accordingly,

$$B_2^y = X^{-1} Y_1 Y_2^{-1} \left(\epsilon^{\alpha\beta} H_x^\beta \partial_x^{-1} \partial_z B_\alpha^z \right) - X^{-1} \left(H_x^\alpha \partial_x^{-1} \partial_z B_\alpha^z \right) - \epsilon^{IJ} \epsilon^{\alpha\beta} X^{-1} \left(H_z^3 \partial_z^{-1} \partial_I (H_J^\beta B_\alpha^z) \right) \tag{A.8}$$

This shows that by solving the constraints, B_i^I and B_3^z can be written in terms of our degrees of freedom, i.e, B_α^z.

A.2. Solutions for the System of Equations (8.41)

We can solve the first and second equations of the system of equations (8.41) for B_α^x as $B_\alpha^x = -\partial_x^{-1} (\partial_y B_\alpha^y + \partial_z B_\alpha^z) + g_\alpha(y, z)$ in which g_α are arbitrary functions in the kernel of ∂_x. Due to the fall-off behaviour (8.28), g_α must vanish. Hence,

$$B_\alpha^x = -\partial_x^{-1} (\partial_y B_\alpha^y + \partial_z B_\alpha^z). \tag{A.9}$$

The third equation of (8.41) can be solved for B_3^z as $B_3^z = -\partial_z^{-1} \partial_I B_3^I + g(x, y)$ where g is an arbitrary function that depends only on x, y. Again, $g = 0$ is concluded from the asymptotic behaviour of B_3^z. Therefore,

$$B_3^z = -\partial_z^{-1} \partial_I B_3^I \tag{A.10}$$

Solving the fourth and fifth equations of (8.41) for B_α^z simply results in

$$B_\alpha^z = H_x^1 \delta_I^\alpha B_3^I - H_z^\alpha \partial_z^{-1} \partial_I B_3^I \tag{A.11}$$

Because $H_x^1 \neq 0$, the last two equations of (8.41) can be re-expressed as

$$B_1^y = B_2^x - \frac{1}{H_x^1} \epsilon^{\alpha\beta} B_\alpha^z H_z^\beta \tag{A.12}$$

$$B_1^x + B_2^y = -\frac{1}{H_x^1} (H_z^\alpha B_\alpha^z + B_3^z) \tag{A.13}$$

If we exploit (A.9)-(A.12) in (A.13) and simplify, the following integro-differential equation emerges

$$\partial_x^{-2}\Delta B_2^y = -\frac{1}{H_x^1}\left(H_z^\alpha B_\alpha^z + B_3^z\right) - \partial_x^{-1}\partial_y\left(\frac{1}{H_x^1}\epsilon^{\alpha\beta}B_\alpha^z H_z^\beta\right) + \partial_x^{-1}\partial_z B_1^z - \partial_x^{-2}\partial_y\partial_z B_2^z \tag{A.14}$$

that can be solved for B_2^y employing the inverse of $\partial_x^{-2}\Delta$. We denote the r.h.s. of (A.14) depending only on B_3^I by $R(B_3^I)$ and observe that $B_2^y = \Delta^{-1}\partial_x^2 R + \kappa$ in which $\partial_x^{-2}\Delta\kappa = 0$ and $\kappa = O(r^{-3})$ due to (8.28). The asymptotic behaviour of κ indicates that $\Delta\kappa$ must vanish and since in $\mathbb{R}^3$ the only harmonic function decaying at infinity is the trivial function, $\kappa = 0$. Thus,

$$B_2^y = \Delta^{-1}\left[-\partial_x^2\left(\frac{1}{H_x^1}\left(H_z^\alpha B_\alpha^z + B_3^z\right)\right) - \partial_x\partial_y\left(\frac{1}{H_x^1}\epsilon^{\alpha\beta}B_\alpha^z H_z^\beta\right) + \partial_x\partial_z B_1^z - \partial_y\partial_z B_2^z\right] \tag{A.15}$$

Accordingly, it has been exhibited that by solving the constraints, B_α^I and B_i^z can be written in terms of our degrees of freedom, i.e, B_3^I.

The General Solution of the System (9.32)

This appendix is devoted to find general solution of the system (9.32). It needs a detailed case-by-case analysis as described below.

Case 1: $\hat{e}_0^t \neq 0$

First, we extract v_a^i from the last equation of (9.32). As $\hat{e}_0^t \neq 0$, we can use its inverse to get

$$v_a^i = \frac{1}{\hat{e}_0^t}(F_{ab}^i \hat{e}_0^b + \gamma \epsilon^{ikl} F_{ab}^k \hat{e}_l^b - \gamma \epsilon^{ikl} v_a^k \hat{e}_l^t) \tag{B.1}$$

Note that the r.h.s. still depends on v_a^k and hence this is not the desired solution. To get rid of it, we simply put (B.1) into itself and simplify the result

$$\begin{aligned}
v_a^i =& \frac{1}{\hat{e}_0^t}\left(F_{ab}^i \hat{e}_0^b + \gamma \epsilon^{ikl} F_{ab}^k \hat{e}_l^b - \frac{\gamma \epsilon^{ikl}}{\hat{e}_0^t}(F_{ab}^k \hat{e}_0^b + \gamma \epsilon^{kmn} F_{ab}^m \hat{e}_n^b - \gamma \epsilon^{kmn} v_a^m \hat{e}_n^t)\hat{e}_l^t\right) \\
=& \frac{1}{\hat{e}_0^t}\left(F_{ab}^i \hat{e}_0^b + \gamma \epsilon^{ikl} F_{ab}^k \hat{e}_l^b - \frac{\gamma \epsilon^{ikl}}{\hat{e}_0^t}(F_{ab}^k \hat{e}_0^b + \gamma \epsilon^{kmn} F_{ab}^m \hat{e}_n^b)\hat{e}_l^t + \frac{\gamma^2}{\hat{e}_0^t}(\delta_n^i \delta_m^l - \delta_m^i \delta_n^l) v_a^m \hat{e}_n^t \hat{e}_l^t\right) \\
=& \frac{1}{\hat{e}_0^t}\left(F_{ab}^i \hat{e}_0^b + \gamma \epsilon^{ikl} F_{ab}^k \hat{e}_l^b - \frac{\gamma \epsilon^{ikl}}{\hat{e}_0^t}(F_{ab}^k \hat{e}_0^b + \gamma \epsilon^{kmn} F_{ab}^m \hat{e}_n^b)\hat{e}_l^t + \frac{\gamma^2}{\hat{e}_0^t}([v_a^l \hat{e}_l^t]\hat{e}_i^t - v_a^i \hat{e}_l^t \hat{e}_l^t)\right) \\
=& \frac{1}{\hat{e}_0^t}\left(F_{ab}^i \hat{e}_0^b + \gamma \epsilon^{ikl} F_{ab}^k \hat{e}_l^b - \frac{\gamma \epsilon^{ikl}}{\hat{e}_0^t}(F_{ab}^k \hat{e}_0^b + \gamma \epsilon^{kmn} F_{ab}^m \hat{e}_n^b)\hat{e}_l^t + \frac{\gamma^2}{\hat{e}_0^t}(F_{ab}^l \hat{e}_l^b \hat{e}_i^t - v_a^i \hat{e}_l^t \hat{e}_l^t)\right)
\end{aligned}$$

where in the last step, the third equation of (9.32) has been used. Now one can move the last term of the r.h.s. to the l.h.s. and isolate v_a^i to obtain

$$\begin{aligned}
v_a^i &= \frac{\hat{e}_0^t}{(\hat{e}_0^t \hat{e}_0^t + \gamma^2 \hat{e}_l^t \hat{e}_l^t)}\left(F_{ab}^i \hat{e}_0^b + \gamma \epsilon^{ikl} F_{ab}^k \hat{e}_l^b - \frac{\gamma \epsilon^{ikl}}{\hat{e}_0^t}(F_{ab}^k \hat{e}_0^b + \gamma \epsilon^{kmn} F_{ab}^m \hat{e}_n^b)\hat{e}_l^t + \frac{\gamma^2}{\hat{e}_0^t} F_{ab}^l \hat{e}_l^b \hat{e}_i^t\right) \\
&= \frac{\hat{e}_0^t}{(\hat{e}_0^t \hat{e}_0^t + \gamma^2 \hat{e}_l^t \hat{e}_l^t)}\left(F_{ab}^i \hat{e}_0^b + \gamma \epsilon^{ikl} F_{ab}^k \hat{e}_l^b - \frac{\gamma \epsilon^{ikl}}{\hat{e}_0^t} F_{ab}^k \hat{e}_0^b \hat{e}_l^t + \frac{\gamma^2}{\hat{e}_0^t}(F_{ab}^i \hat{e}_j^b \hat{e}_j^t - 2F_{ab}^j \hat{e}_{[i}^b \hat{e}_{j]}^t)\right)
\end{aligned} \tag{B.2}$$

Assuming $e^t_j = 0$, (B.2) reduces to (9.34). Now, substituting (B.2) into the second, third and first equations of (9.32), respectively, yields the following constraints

$$\begin{aligned}
\tilde{C}_a &:= \left(1 - \frac{\gamma^2 \hat{e}^t_i \hat{e}^t_i}{\hat{e}^t_0 \hat{e}^t_0 + \gamma^2 \hat{e}^t_k \hat{e}^t_k}\right) F^l_{ab} \hat{e}^b_l - \frac{\hat{e}^t_i \hat{e}^t_0}{\hat{e}^t_0 \hat{e}^t_0 + \gamma^2 \hat{e}^t_k \hat{e}^t_k} (F^i_{ab} \hat{e}^b_0 + \gamma \epsilon^{ikl} F^k_{ab} \hat{e}^b_l) \\
\tilde{C}_i &:= \frac{\gamma^2 \hat{e}^t_0}{(\hat{e}^t_0 \hat{e}^t_0 + \gamma^2 \hat{e}^t_l \hat{e}^t_l)} \left(\frac{\hat{e}^a_0}{\hat{e}^t_0} F^l_{ab} (2 \hat{e}^b_l \hat{e}^t_i - \hat{e}^b_i \hat{e}^t_l) + F^l_{ab} \hat{e}^a_l \hat{e}^b_i + \frac{\gamma}{\hat{e}^t_0} \epsilon^{ikl} \hat{e}^a_l (F^k_{ab} \hat{e}^b_j \hat{e}^t_j - 2 F^j_{ab} \hat{e}^b_{[k} \hat{e}^t_{j]}) \right) \\
\tilde{C} &:= \frac{\hat{e}^t_0}{(\hat{e}^t_0 \hat{e}^t_0 + \gamma^2 \hat{e}^t_l \hat{e}^t_l)} \left(F^i_{ab} \hat{e}^b_0 \hat{e}^a_i + \gamma \epsilon^{ikl} F^k_{ab} \hat{e}^b_l \hat{e}^a_i - \frac{\gamma \epsilon^{ikl}}{\hat{e}^t_0} F^k_{ab} \hat{e}^b_0 \hat{e}^t_l \hat{e}^a_i \right)
\end{aligned} \tag{B.3}$$

We want to verify that, assuming $\hat{e}^t_j = 0$, the above constraints (B.3) reduce to (9.33). We start with $\tilde{C}_a = 0$. It is obvious that substituting $\hat{e}^t_j = 0$ into this equation leads to $F^l_{ab} \hat{e}^b_l = 0$ which is the second constraint of (9.33), i.e. $C_a = 0$.
Since most terms in $\tilde{C}_i$ are proportional to $\hat{e}^t_j$, when the latter is zero the constraint $\tilde{C}_i = 0$ reduces to the simple equation $0 = F^l_{ab} \hat{e}^a_l \hat{e}^b_i = -C_b \hat{e}^b_i$ that is not an independent constraint since we already have $C_a = 0$. Note that if we assume $\hat{e}^t_j = 0$, we actually substitute three constraints $D_j = 0$ for the above constraint $\tilde{C}_j = 0$ and therefore in this case $\tilde{C}_j = 0$ does not provide us with any new independent constraints.
Finally, If we substitute $\hat{e}^t_j = 0$ into the equation $\tilde{C} = 0$, we conclude that $0 = F^i_{ab} \hat{e}^b_0 \hat{e}^a_i + \gamma \epsilon^{ikl} F^k_{ab} \hat{e}^b_l \hat{e}^a_i = -C_b \ \hat{e}^b_0 + \gamma \epsilon^{ikl} F^k_{ab} \hat{e}^b_l \hat{e}^a_i$, whose first term already vanishes. Thus, $\epsilon^{ikl} F^k_{ab} \hat{e}^b_l \hat{e}^a_i = 0$ serves as an independent constraint which is the third constraint in (9.33), i.e. $C = 0$.

Case 2: $\hat{e}^t_0 = 0$

In this case, to prevent degeneracy, at least one of the variables $\hat{e}^t_i$ has to be non-zero. Without loss of generality, one can assume that $\hat{e}^t_1 \neq 0$. A consequence of this choice is the separation of quantities with index $i = 1$ and those with $i = \alpha \in \{2, 3\}$. The Greek alphabet is employed to denote the indices $i \neq 1$.
The last equation of (9.32) is used to get v^α_a as

$$v^\alpha_a = \frac{1}{\gamma \hat{e}^t_1} (\gamma v^1_a \hat{e}^t_\alpha - \epsilon_{\alpha\beta} [F^\beta_{ab} \hat{e}^b_0 + \gamma \epsilon^{\beta kl} F^k_{ab} \hat{e}^b_l]) \tag{B.4}$$

and the third equation is solved for v^1_a

$$v^1_a = \frac{1}{\hat{e}^t_1} (F^j_{ab} \hat{e}^b_j - \hat{e}^t_\alpha v^\alpha_a) \tag{B.5}$$

Now plugging (B.4) into (B.5) renders v^1_a completely in terms of the canonical variables

$$v^1_a = \frac{\hat{e}^t_1}{\hat{e}^t_i \hat{e}^t_i} \left(F^j_{ab} \hat{e}^b_j + \hat{e}^t_\alpha \frac{1}{\gamma \hat{e}^t_1} \epsilon_{\alpha\beta} [F^\beta_{ab} \hat{e}^b_0 + \gamma \epsilon^{\beta kl} F^k_{ab} \hat{e}^b_l] \right) \tag{B.6}$$

and inserting (B.6) in (B.4) terminates the restrictions on Lagrange multipliers v^i_a, as

$$v^\alpha_a = \frac{\hat{e}^t_\alpha}{\hat{e}^t_i \hat{e}^t_i} \left(F^j_{ab} \hat{e}^b_j + \hat{e}^t_\delta \frac{\epsilon_{\delta\beta}}{\gamma \hat{e}^t_1} [F^\beta_{ab} \hat{e}^b_0 + \gamma \epsilon^{\beta kl} F^k_{ab} \hat{e}^b_l] \right) - \frac{\epsilon_{\alpha\beta}}{\gamma \hat{e}^t_1} [F^\beta_{ab} \hat{e}^b_0 + \gamma \epsilon^{\beta kl} F^k_{ab} \hat{e}^b_l]) \tag{B.7}$$

The remaining task is just substitution of v_a^i into the rest of the equation of (9.32) to obtain the constraints. These equations are of the form

$$\begin{aligned} 0 =& v_a^1 \hat{e}_0^a + \gamma \epsilon^{\alpha\beta} v_a^\alpha \hat{e}_\beta^a \\ 0 =& v_a^\alpha \hat{e}_0^a - \gamma \epsilon^{\alpha\beta} v_a^1 \hat{e}_\beta^a + \gamma \epsilon^{\alpha\beta} v_a^\beta \hat{e}_1^a \\ 0 =& \hat{e}_1^a v_a^1 + \hat{e}_\alpha^a v_a^\alpha \end{aligned} \tag{B.8}$$

where the first one is just the component $i = 1$ of the last equation of (9.32). After simplification, the constraints take the following forms

$$\begin{aligned} \tilde{C}_a :=& -\frac{\hat{e}_\alpha^t}{\hat{e}_1^t}[F_{ab}^\alpha \hat{e}_0^b + \gamma \epsilon^{\alpha kl} F_{ab}^k \hat{e}_l^b] \\ \tilde{C}_1 :=& \frac{\hat{e}_1^t}{\hat{e}_i^t \hat{e}_i^t}\left(F_{ab}^j \hat{e}_j^b + \hat{e}_\alpha^t \frac{1}{\gamma \hat{e}_1^t} \epsilon_{\alpha\beta}[F_{ab}^\beta \hat{e}_0^b + \gamma \epsilon^{\beta kl} F_{ab}^k \hat{e}_l^b]\right) \hat{e}_0^a + \gamma \epsilon^{\alpha\beta} v_a^\alpha \hat{e}_\beta^a \\ \tilde{C}_\alpha :=& \frac{\hat{e}_0^a \hat{e}_\alpha^t}{\hat{e}_i^t \hat{e}_i^t}\left(F_{ab}^j \hat{e}_j^b + \hat{e}_\delta^t \frac{\epsilon_{\delta\beta}}{\gamma \hat{e}_1^t}[F_{ab}^\beta \hat{e}_0^b + \gamma \epsilon^{\beta kl} F_{ab}^k \hat{e}_l^b]\right) - \frac{\epsilon_{\alpha\beta}}{\gamma \hat{e}_1^t}[F_{ab}^\beta \hat{e}_0^b + \gamma \epsilon^{\beta kl} F_{ab}^k \hat{e}_l^b] \hat{e}_0^a \\ & - \gamma \epsilon^{\alpha\beta} \hat{e}_\beta^a \frac{\hat{e}_1^t}{\hat{e}_i^t \hat{e}_i^t}\left(F_{ab}^j \hat{e}_j^b + \hat{e}_\delta^t \frac{1}{\gamma \hat{e}_1^t} \epsilon_{\delta\sigma}[F_{ab}^\sigma \hat{e}_0^b + \gamma \epsilon^{\sigma kl} F_{ab}^k \hat{e}_l^b]\right) \\ & + \gamma \epsilon^{\alpha\beta} \hat{e}_1^a \frac{\hat{e}_\beta^t}{\hat{e}_i^t \hat{e}_i^t}\left(F_{ab}^j \hat{e}_j^b + \hat{e}_\delta^t \frac{\epsilon_{\delta\sigma}}{\gamma \hat{e}_1^t}[F_{ab}^\sigma \hat{e}_0^b + \gamma \epsilon^{\sigma kl} F_{ab}^k \hat{e}_l^b]\right) - \gamma \epsilon^{\alpha\beta} \hat{e}_1^a \frac{\epsilon_{\beta\sigma}}{\gamma \hat{e}_1^t}[F_{ab}^\sigma \hat{e}_0^b + \gamma \epsilon^{\sigma kl} F_{ab}^k \hat{e}_l^b])) \\ \tilde{C} :=& \hat{e}_1^a \frac{\hat{e}_1^t}{\hat{e}_i^t \hat{e}_i^t}\left(F_{ab}^j \hat{e}_j^b + \hat{e}_\alpha^t \frac{1}{\gamma \hat{e}_1^t} \epsilon_{\alpha\beta}[F_{ab}^\beta \hat{e}_0^b + \gamma \epsilon^{\beta kl} F_{ab}^k \hat{e}_l^b]\right) \\ & + \hat{e}_\alpha^a \frac{\hat{e}_\alpha^t}{\hat{e}_i^t \hat{e}_i^t}\left(F_{ab}^j \hat{e}_j^b + \hat{e}_\delta^t \frac{\epsilon_{\delta\beta}}{\gamma \hat{e}_1^t}[F_{ab}^\beta \hat{e}_0^b + \gamma \epsilon^{\beta kl} F_{ab}^k \hat{e}_l^b]\right) - \hat{e}_\alpha^a \frac{\epsilon_{\alpha\beta}}{\gamma \hat{e}_1^t}[F_{ab}^\beta \hat{e}_0^b + \gamma \epsilon^{\beta kl} F_{ab}^k \hat{e}_l^b] \end{aligned} \tag{B.9}$$

To summarise, solving 16 equations (9.32), ensuring the stabilisation of $\hat{P}_A^I$, yields fixing 9 Lagrange multipliers v_a^1 and v_a^α via (B.6) and (B.7), respectively, in addition to 7 secondary constraints $\tilde{C}_a, \tilde{C}_1, \tilde{C}_\alpha, \tilde{C}$. This ends the case-by-case analysis.

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