Scattering Equations, Soft Theorems, and Amplitudes on the Celestial Sphere

by

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Chapter 1

Introduction

Calculation of scattering amplitudes is required to understand the different kinds of scattering processes of elementary particles that can happen according to the rules of an underlying Quantum Field Theory (QFT), as well as to determine the overall probabilities for these processes to occur. It is of interest to calculate these probabilities in order to learn more about fundamental behavior of elementary particles in nature.

Conventionally, scattering amplitudes are constructed by Feynman rules derived from a local Lagrangian of the QFT in question. However, the sum over all contributing Feynman diagrams quickly becomes combinatorially intractable as we consider higher number of participating particles in a scattering process. Amazingly, oftentimes scattering amplitudes exhibit dramatic simplifications once all Feynman diagrams have been taken into account, a prime example being the famous Parke-Taylor formula for maximally helicity violating scattering of any number of gluons [1]. This suggests that computing Feynman diagrams is not the most convenient approach and other techniques are likely to exist that make the hidden simplicity more manifest during the calculation. The following are a few theoretical tools that make some progress in this direction.

In four spacetime dimensions the so called spinor-helicity formalism allows for the simplest formulation of massless scattering, since it removes gauge redundancies and encodes physical degrees of freedom only (see [2] for a review).

Another powerful tool is the so called Britto-Cachazo-Feng-Witten (BCFW) recursion [3, 4], that allows to find higher point amplitudes in terms of a sum over known lower point amplitudes that are completely fixed by scaling and dimensional analysis arguments. In this sense BCFW recursion circumvents Feynman diagram calculation all together, directly providing the final result in relatively simple form.

Recently, amplitudes in Cachazo-He-Yuan (CHY) representation [5] have emerged, which compactly describe scattering amplitudes of an arbitrary number of particles in an arbitrary number of spacetime dimensions and in various theories, at the expense of requiring an integration over a moduli space to be performed localizing so called scattering equation constraints.

In the next sections of this introduction we will introduce relevant notions and describe the tools and techniques mentioned above in more detail.

1.1 Why amplitudes?

From classical physics, we are familiar with the concept of scattering cross-section σ, which in essence is the cross-sectional area of an extended object as perceived by a beam of smaller particles that are scattered off it. Naturally, in a setup where the target is at rest the cross-section is then given by

$$\sigma = \frac{\text{number of scattered particles}}{\text{time}\times\text{velocity of the beam}\times\text{number density of the beam}}, \tag{1.1.1}$$

which in other words is simply the number of scattered particles N per incoming flux Φ and per time T. In quantum mechanics incident particles only have a probability to interact with the target, which leads us to generalize the notion of cross-section to more abstractly refer to the strength of interaction between scattered particles. With this the differential cross-section for a two particle collision becomes

$$d\sigma = \frac{V}{T|\vec{v}_1 - \vec{v}_2|} dP, \tag{1.1.2}$$

where we used that the flux is the difference of velocities of the two particles per volume V, and dP is the differential quantum mechanical probability of scattering. The quantity dP can be formally expressed by introducing the notion of quantum mechanical states describing the initial $|i\rangle$ and the final $|f\rangle$ particle configurations in the system before and after the scattering takes place respectively. As explained in any introductory text on Quantum Field Theory (see for instance [6]), dP is given by

$$dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} d\Pi \quad \text{and} \quad d\Pi = \prod_j \frac{V}{(2\pi)^3} d^3 p_j\,, \tag{1.1.3}$$

with the region of final state momenta $d\Pi$ integrating to one. Here S is the so called scattering matrix which encodes all intermediate processes that can connect the initial and final states:

$$S = \mathbb{1} + i(2\pi)^4\delta^4(\textstyle\sum p)\mathcal{A}. \tag{1.1.4}$$

The non-trivial part of the scattering matrix contains momentum conservation delta functions and the amplitude matrix $\mathcal{A}$. Finally, after a bit of algebra and making use of $\langle s|s\rangle = \prod_j(2E_jV)$ and $\delta^4(0) = \frac{TL^3}{(2\pi)^4}$ in a finite volume $V = L^3$, we can send the scattering volume and time period to infinity and find in the non-trivial case $|f\rangle \neq |i\rangle$:

$$d\sigma = \frac{|\langle f|\mathcal{A}|i\rangle|^2}{4E_1E_2|\vec{v}_1 - \vec{v}_2|} d\Pi_{LIPS}, \tag{1.1.5}$$

where E_i are particle energies and $d\Pi_{LIPS}$ is the Lorentz-invariant phase space

$$d\Pi_{LIPS} = \prod_{\text{particles } j\in|f\rangle} \frac{d^3p_j}{(2\pi)^3} \frac{(2\pi)^4\delta^4(\sum p)}{2E_j}. \tag{1.1.6}$$

This demonstrates that in order to calculate the differential cross-section of two scattering particles, which provides an intuitive measure for the interaction strength between them and can shed light on their properties such as mass, spin and internal quantum numbers, we first have to obtain the so called scattering amplitude

$$A \equiv \langle f|\mathcal{A}|i\rangle. \tag{1.1.7}$$

It will be the subject of this thesis to investigate soft factorization properties of amplitudes in a select group of Quantum Field Theories, and to develop a general evaluation technique for amplitudes based on scattering equations in the so called Cachazo-He-Yuan (CHY) formulation.

1.2 Scattering equations based amplitudes

Conventionally, Feynman rules[1] are used to compute scattering amplitudes from a graph theoretic point of view. In this approach expressions for propagators and interactions are extracted from a local Lagrangian of a QFT, and play the role of edges and vertices respectively. To obtain the scattering amplitude, Feynman rules prescribe to sum all possible diagrams that can be composed out of the

[1] This thesis assumes familiarity of the reader with standard textbook Quantum Field Theory material, such as Feynman rules. Thus technical details on this are omitted.

given edge and vertex expressions leading to appropriate external states. Further complications appear when edges form closed loops, with extra minus signs to keep track of for each fermionic loop, and loop momenta integrations over the entire phase space. But even for tree-level graphs, where no loops appear, the number of Feynman diagrams grows exponentially with the number of particles participating in the scattering process, such that an explicit summation becomes a practically impossible task. For this reason, any combinatorially tractable formulation of amplitudes is a valuable tool. One such formulation is described below.

At the heart of the Cachazo-He-Yuan representation of scattering amplitudes are the so called scattering equations, which map configurations of momenta k_i^μ of scattering particles $i = 1, 2, ..., n$ to an auxiliary moduli σ_i space on a Riemann sphere by demanding the following equality for all a

$$f_a \equiv \sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}} = 0. \tag{1.2.1}$$

The CHY formulation ([7, 5, 8], and later [9, 10]) produces tree level n-point scattering amplitudes for massless particles in arbitrary dimension by means of $(n-3)$ moduli integrations localizing the scattering equations (1.2.1):[2]

$$A = \int d\mu \mathcal{I} \qquad \text{with} \qquad d\mu = \frac{d^n\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \sigma_{ij}\sigma_{jk}\sigma_{ki} \prod_{a \neq i,j,k} \delta\left(f_a\right), \tag{1.2.2}$$

where $\sigma_{ij} = \sigma_i - \sigma_j$, and $\mathcal{I}$ is determined by the particular Quantum Field Theory (we will encounter a few explicit examples of $\mathcal{I}$ in what follows), while $d\mu$ is a universal purely kinematic integration measure. Since all σ_i live on a Riemann sphere, any valid $\mathcal{I}$ has to cancel the $\mathrm{SL}(2,\mathbb{C})$ transformation weight of $d\mu$ such that overall the amplitude is $\mathrm{SL}(2,\mathbb{C})$ invariant on the moduli space.

The scattering equations could also be reformulated in a polynomial form by Dolan and Goddard [11, 12]. This transforms the CHY measure $d\mu$ as

$$d\mu = \frac{d^n\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \left(\prod_{1 \leq i < j \leq n} \sigma_{ij} \right) \left(\prod_{a=2}^{n-2} \delta\left(\tilde{h}_a\right) \right), \tag{1.2.3}$$

where the polynomial scattering equations now read

$$\tilde{h}_i \equiv \sum_{\{q_1,\ldots,q_i\} \subset \{1,2,\ldots,n\}} \mathfrak{s}_{q_1,\ldots,q_i} \prod_{j=1}^{i} \sigma_{q_j} = 0 \qquad \text{with} \qquad \mathfrak{s}_{q_1,\ldots,q_i} = \frac{1}{2} \left(\sum_{j=1}^{i} k_{q_j} \right)^2. \tag{1.2.4}$$

[2]Note that the usual gauge fixing e.g. $\frac{d^n\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} = \left(\prod_{c=4}^{n} d\sigma_c\right) \sigma_{12}\sigma_{23}\sigma_{31}$ reduces the number of integrations to $n-3$.

Here the summation is over all unordered subsets of i elements out of the sequence of numbers from 1 to n. Despite a non-trivial transformation having been applied to achieve the polynomial form, the $\tilde{h}_i = 0$ have exactly the same set of solutions as $f_a = 0$, which makes the particular choice a matter of convenience. We will prefer the polynomial form of scattering equations when we develop our evaluation procedure for CHY amplitudes in a later chapter.

Since each chapter of this thesis aims to be as self-contained as possible, we will recall and repeat the CHY formulation on several occasions throughout this thesis, with emphasis on specific features and theories that are relevant within the respective chapter.

1.3 Spinor-helicity formalism

Physicists are most used to parametrizing physical processes of particles using Minkowski space momenta p^μ and polarization vectors ϵ^μ familiar from special relativity and electro-dynamics. Naturally, scattering amplitudes are often written as functions of these variables $A(p_i^\mu, \epsilon_i^\mu)$. However, these familiar variables are not always the best choice to describe the physics most concisely. For instance, when talking about scattering of massless photons, we know that only two transversal polarizations are possible: plus or minus helicity states. Since a generic polarization vector ϵ^μ has enough components to encode transversal as well as longitudinal polarizations, it has more structure than is required to describe massless photons. Additionally, momenta p^μ appear as variables seemingly independent from ϵ^μ polarizations. We can perform a change of variables that removes the unnecessary degrees of freedom in our description and therefore provides a much more useful description (see e.g. [2]).

Start with a momentum four-vector p^μ and consider multiplying it with a vector composed of the usual Pauli matrices σ^μ:

$$p_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} p_\mu \qquad \text{where} \qquad \sigma^\mu_{\alpha\dot{\alpha}} = (\mathbb{1}, \vec{\sigma})_{\alpha\dot{\alpha}}. \tag{1.3.1}$$

The resulting quantity $p_{\alpha\dot{\alpha}}$ for $\alpha, \dot{\alpha} \in \{1,2\}$ is a 2×2 matrix. The massless condition $p_\mu p^\mu = 0$ translates into $\det p_{\alpha\dot{\alpha}} = 0$, which means that the matrix $p_{\alpha\dot{\alpha}}$ has reduced rank. Any 2×2 matrix of reduced rank can be written as a dyadic product of two vectors

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}. \tag{1.3.2}$$

The vectors λ_α and λ_α are also called Weyl spinors. For real momenta p^μ the two Weyl spinors are complex conjugates of each other. Considering that the Pauli matrices are $SU(2)$ generators, we can contract the α and $\dot{\alpha}$ indices with the completely anti-symmetric tensor $\epsilon^{\alpha\beta}$, for which we can define a bracket notation:

$$\langle ij\rangle = \epsilon^{\alpha\beta}\lambda_{i\alpha}\lambda_{j\beta} \qquad \text{and} \qquad [ij] = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{i\dot{\alpha}}\tilde{\lambda}_{j\dot{\beta}}, \tag{1.3.3}$$

such that the product of two four-momenta is given by $2p_i^\mu p_{j\mu} = \langle ij\rangle[ij]$.

Polarization vectors can similarly be expressed in this $SU(2)$ notation

$$\epsilon_{i\alpha\dot{\alpha}} = \frac{\lambda_{i\alpha}\tilde{r}_{i\dot{\alpha}}}{[i\,\tilde{r}_i]} \qquad , \qquad \tilde{\epsilon}_{i\alpha\dot{\alpha}} = \frac{r_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}}{\langle r_i\, i\rangle}\,, \tag{1.3.4}$$

where r_i and $\tilde{r}_i$ explicitly encode the usual gauge degrees of freedom and are called reference spinors. They can be chosen arbitrarily so long as the resulting $\epsilon_{i\alpha\dot{\alpha}}$ and $\tilde{\epsilon}_{i\alpha\dot{\alpha}}$ are finite.

Thus, with the above choice of parametrization the amplitude can be written as a function of spinors $A(\lambda_i, \tilde{\lambda}_i)$ alone. Since $\lambda_i = (\tilde{\lambda}_i)^*$ has only two independent components, the spinors encode the two helicity states in a minimal fashion and automatically include the kinematic information about the motion of a particle. This compact spinor-helicity language oftentimes yields huge simplifications for expressions that are rather unwieldy in the original momentum and polarization vector language.

1.4 BCFW recursion

Another powerful tool in the amplitudes toolbox is Britto-Cachazo-Feng-Witten (BCFW) recursion [3, 4]. To set up BCFW recursion, we introduce complex deformations of amplitudes $A(\lambda_i, \tilde{\lambda}_i)$. In particular, consider deforming two of the spinor-helicity variables i, j by a complex parameter z as follows

$$\tilde{\lambda}_i^{\dot{\alpha}} \to \tilde{\lambda}_i^{\dot{\alpha}} + z\tilde{\lambda}_j^{\dot{\alpha}} \equiv \hat{\tilde{\lambda}}_i^{\dot{\alpha}} \quad , \quad \tilde{\lambda}_j^{\dot{\beta}} \to \tilde{\lambda}_j^{\dot{\beta}}\,, \tag{1.4.1}$$

$$\lambda_j^{\alpha} \to \lambda_j^{\alpha} - z\lambda_i^{\alpha} \equiv \hat{\lambda}_j^{\alpha} \quad , \quad \lambda_i^{\beta} \to \lambda_i^{\beta}\,. \tag{1.4.2}$$

This implies that the deformed spinors are not complex conjugates of each other for $z \neq 0$. After this deformation, we can trivially recover the original amplitude $A(\lambda_i, \tilde{\lambda}_i)$ from the deformed one

$A(\hat{\lambda}_i, \hat{\tilde{\lambda}}_i)$ by performing the following contour integral

$$A(\lambda_i, \tilde{\lambda}_i) = \oint_{\substack{z=|\varepsilon| \\ \varepsilon\to 0^+}} \frac{dz}{2\pi i} \frac{A(\hat{\lambda}_i, \hat{\tilde{\lambda}}_i)}{z}, \tag{1.4.3}$$

and collecting a residue from a simple pole at $z = 0$. However, we also can consider deforming the integration contour away from the initial locus around zero all the way out to infinity. As the contour is deformed, it encounters and wraps around other poles in the integrand. This is where our general knowledge about constructing amplitudes from Feynman diagrams comes into play. Even though we do not attempt to write down all possible diagrams contributing to a particular amplitude, we still know that at tree level all poles within an amplitude must be due to denominators of propagators that are present in the diagrams. As the contour in z localizes these denominators, the propagators go on-shell producing a divergence and a corresponding residue. Whenever we have propagators that are on-shell, we can think of the propagating intermediate particle as an external particle and the amplitude factorizes into two sub-amplitudes connected by this on-shell bridge. If we denote such factorization locations by z_I, we can therefore write

$$A(\lambda_i, \tilde{\lambda}_i) = -\sum_{z_I} \mathrm{Res}_{z=z_I} \frac{A(\hat{\lambda}_i, \hat{\tilde{\lambda}}_i)}{z} = \sum_{z_I} A_L(z_I) \frac{1}{P_I^2} A_R(z_I), \tag{1.4.4}$$

where A_L and A_R are lower point amplitudes and P_I^2 is the square of the sum of unshifted external momenta entering either the left or the right lower point amplitude.[3]

Since A_L and A_R have a lower number of effectively external legs compared to the original amplitude A, with BCFW recursion we therefore have obtained a way to systematically construct higher point amplitudes from lower point amplitudes iteratively without having to write down all Feynman diagrams from scratch. This is a very powerful tool, which we will employ to study soft factorization properties of amplitudes in the following.

[3]Note that $z_I = \infty$ also could be a valid contributing pole in cases where numerators of Feynman diagrams feature non-trivial momentum dependence. However, in practice such poles at infinity can be circumvented in many cases of interest.

1.5 What are soft theorems?

Soft theorems refer to analytic properties of scattering amplitudes under soft kinematics, meaning the Minkowski momentum of one or more particles involved in the scattering process tends to zero

$$k_i^\mu \to \epsilon k_i^\mu \qquad , \qquad \epsilon \to 0^+ \qquad , \qquad i = 1, 2, ..., m \,, \tag{1.5.1}$$

whereby the scattering amplitude reduces to an amplitude with a lower number of scattering particles times a so called soft factor that is uniquely determined by the nature of the particles with soft momenta

$$A_n \to \left(S_m^{(0)} \epsilon^q + S_m^{(1)} \epsilon^{q+1} + S_m^{(2)} \epsilon^{q+2} + ... \right) A_{n-m} . \tag{1.5.2}$$

where the initial power q is such that the first few terms tend to be divergent. Thanks to this divergence, the soft structure is factorized from the remaining amplitude and becomes universal. Universality in this context means that $S_m^{(i)}$ is independent of the remaining lower point amplitude A_{n-m}, such that $S_m^{(i)}$ is always the same whenever the same types of m external particles are taken soft within any original amplitude A_n. Due to this universality, soft theorems are a powerful tool to verify the validity of different representations of scattering amplitudes, since all of them must reproduce exactly the same soft factors. Soft theorems also can help elucidate the group structure of the moduli space of vacua in some appropriate QFTs [13].

While the leading soft term $S_m^{(0)}$ is just an overall factor depending on polarizations and momenta, the following $S_m^{(1)}, S_m^{(2)}, ...$ are operator valued and feature the angular momentum operator.

In this thesis we will verify the sub-sub-leading soft graviton theorem in arbitrary dimension and derive a general expression for leading m-soft factors in various QFTs from the CHY formulation of scattering amplitudes.

1.6 Example theories

The polynomial reduction procedure for evaluating CHY amplitudes, which we will introduce in this thesis, works in general independently of any particular CHY integrand. In contrast to that, scattering amplitudes in different Quantum Field Theories feature different soft factors. Therefore, we will not be able to keep the discussion completely generic and will have to introduce some example QFTs in order to investigate the different soft theorems which arise in them. The following is our arbitrary but fixed choice of examples.

Bi-adjoint scalar ϕ^3 theory

The bi-adjoint scalar theory is a slightly more involved variant of physicists farourite and simplest looking QFT toy model. Its Lagrangian density is given by [14]

$$\mathcal{L} = \frac{1}{2}\partial^\mu \Phi^{aa'} \partial_\mu \Phi^{aa'} + \frac{g}{3} f^{abc} \tilde{f}^{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'}. \tag{1.6.1}$$

The g is a coupling constant and $\Phi^{aa'}$ are components of a matrix valued field $\mathbf{\Phi} = \Phi^{aa'} \mathbf{T}^a \tilde{\mathbf{T}}^{a'}$, where $\mathbf{T}^a$ and $\tilde{\mathbf{T}}^{a'}$ are generators of possibly distinct Lie algebras

$$[\mathbf{T}^a, \mathbf{T}^b] = i f^{abc} \mathbf{T}^c \quad , \quad [\tilde{\mathbf{T}}^a, \tilde{\mathbf{T}}^b] = i \tilde{f}^{abc} \tilde{\mathbf{T}}^c \tag{1.6.2}$$

with structure constants f^{abc} and $\tilde{f}^{abc}$. Note that if the two Lie algebras are decoupled $\Phi^{aa'} \equiv \phi^a \tilde{\phi}^{a'}$, then the interaction term in the Lagrangian density vanishes identically. Therefore, we will consider the case where the two Lie algebras are the same and fully coupled.

While the presence of the Lie algebra generators introduces some amount of clutter and provides each component field $\Phi^{aa'}$ with a separate "color" which we have to keep track of, this actually works in our favor when we consider scattering amplitudes in this theory. Using relations like $f^{abc} f^{a'bc} = \mathrm{tr}[\mathbf{T}^a \mathbf{T}^{a'}]$ we can rewrite combinations of structure constants in the amplitudes in terms of traces of generator products, and then consider each trace contribution separately. Since the positions of generators in the trace directly correspond to the ordering of fields that enter the remaining amplitude expression, we can therefore concentrate on calculating so called color ordered partial amplitudes and later obtain the full amplitude as their sum multiplied with the respective traces of generator products. It turns out that the color ordered partial amplitudes in bi-adjoint scalar theory can be written in the CHY formulation (1.2.2) by use of the simple integrand factor

$$\mathcal{I} = \frac{1}{(\sigma_{12}\sigma_{23}...\sigma_{n1})^2}, \tag{1.6.3}$$

with moduli differences abbreviated as $\sigma_{ij} = \sigma_i - \sigma_j$. We say the integrand factor consists of a Parke-Taylor like factor squared. The particular sequence of moduli indices that appears in these differences determines the particular color ordering.

Yang-Mills theory

Yang-Mills theory describes gauge bosons like gluons. The Lagrangian density of Yang-Mills is given by [6]

$$\mathcal{L} = -\mathrm{tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right), \tag{1.6.4}$$

where $F^{\mu\nu}$ is the non-abelian field strength, composed of vector fields with matrix-valued components

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu] \quad , \quad A_\mu = A_\mu^A \mathbf{T}^A, \tag{1.6.5}$$

with coupling constant g. As in the previous example, $\mathbf{T}^A$ is a Lie algebra generator. The trace in the Lagrangian makes sure that gauge invariance is satisfied in the non-abelian case. The gauge covariant derivative is defined to be $D_\mu = \partial_\mu - igA_\mu$.

The resulting partial amplitudes are color ordered and are written in the CHY formalism by (1.2.2) with the integrand factor, involving the so called Pfaffian of an anti-symmetric matrix

$$\mathcal{I} = \frac{2\frac{(-1)^{p+q}}{\sigma_{pq}}\mathrm{Pf}(\Psi^{pq}_{pq})}{\sigma_{12}\sigma_{23}...\sigma_{n1}}. \tag{1.6.6}$$

Moduli differences are abbreviated as $\sigma_{ab} \equiv \sigma_a - \sigma_b$ and the matrix Ψ is given by

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad A = \begin{cases} \frac{k_a\cdot k_b}{\sigma_{ab}} & ; a \neq b \\ 0 & ; a = b \end{cases}, \quad B = \begin{cases} \frac{\epsilon_a\cdot \epsilon_b}{\sigma_{ab}} & ; a \neq b \\ 0 & ; a = b \end{cases}, \quad C = \begin{cases} \frac{\epsilon_a\cdot k_b}{\sigma_{ab}} & ; a \neq b \\ -\sum_{\substack{c=1 \\ c\neq a}}^{n} \frac{\epsilon_a\cdot k_c}{\sigma_{ac}} & ; a = b \end{cases}, \tag{1.6.7}$$

with $a, b \in \{1, 2, ..., n\}$. The k^μ are momenta of scattering particles and ϵ^μ contain the corresponding polarization data. The indices $1 \leq i < j < k \leq n$ as well as $1 \leq p < q \leq n$ in (4.1.2) are chosen arbitrarily but fixed. Upper and lower indices on matrix Ψ denote removed columns and rows respectively.

Yang-Mills-Scalar theory

The so called Yang-Mills-Scalar (YMS) theory we will be interested in is a generalization of the usual Yang-Mills theory. The Lagrangian density of YMS is given by [10]

$$\mathcal{L} = -\mathrm{tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^\mu\phi^I D_\mu\phi^I - \frac{g^2}{4}\sum_{I\neq J}[\phi^I, \phi^J]^2\right), \tag{1.6.8}$$

where $F^{\mu\nu}$ is the non-abelian field strength, D^μ is the gauge covariant derivative and the scalar fields ϕ^I carry a flavor $SO(M)$ index I.

The resulting partial amplitudes are again color ordered and can be described in the CHY formalism by (1.2.2) with the following integrand factor

$$\mathcal{I}_n = \frac{2}{\sigma_{12}\sigma_{23}...\sigma_{n1}} \mathrm{Pf}(\chi) \frac{(-1)^{i+j}}{\sigma_{ij}} \mathrm{Pf}(\Psi^{i,j,n+1,n+2,...,n+q}_{i,j,n+1,n+2,...,n+q}). \tag{1.6.9}$$

As before, $1 \leq i < j \leq n$ can be selected arbitrarily. Matrix χ is $q \times q$ dimensional

$$\chi = \begin{cases} \frac{\delta^{I_a, I_b}}{\sigma_{ab}} & ; a \neq b \\ 0 & ; a = b \end{cases}, \tag{1.6.10}$$

and I_a, I_b are flavor indices for scalar fields involved in the scattering process. This corresponds to the first q of the scattering particles being scalars and the remaining $n - q$ being gluons.

Gravity

To consider graviton scattering, one can start with the usual Einstein-Hilbert Lagrangian [6]

$$\mathcal{L} = M_{Pl}^2 \sqrt{-\det(g)} R, \tag{1.6.11}$$

with Ricci scalar R and the determinant of the metric tensor $g_{\mu\nu}$ denoted by $\det(g)$. We then express the metric tensor in terms of a deviation from a flat Minkowski space metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{Pl}} h_{\mu\nu}, \tag{1.6.12}$$

where M_{Pl} is the Planck mass. For plain wave solutions, $h_{\mu\nu}$ can be considered as a dynamical field, called the graviton field, living in the flat Minkowski space background. Expanding the Einstein-Hilbert Lagrangian for small fluctuations yields the following kinetic Lagrangian for the graviton field

$$\mathcal{L}_{kin} = \frac{1}{2} h_{\mu\nu} \partial^2 h_{\mu\nu} - h_{\mu\nu} \partial_\mu \partial_\alpha h_{\nu\alpha} + h \partial_\mu \partial_\nu h_{\mu\nu} - \frac{1}{2} h \partial^2 h, \tag{1.6.13}$$

with $h = \eta^{\mu\nu} h_{\mu\nu}$. The interaction part of the Lagrangian is even more involved. Luckily, we will not be considering Feynman diagrams and therefore will not require an explicit Lagrangian of the theory.

In CHY representation, a scattering amplitude involving n gravitons at tree level is simply described by (1.2.2) with the integrand factor [5]

$$\mathcal{I} = \frac{4}{(\sigma_{mw})^2} \det\left(\Psi^{m,w}_{m,w}\right). \tag{1.6.14}$$

As before, upper indices on the matrix Ψ denote removed columns and lower indices denote removed rows. Values of indices m, w can be chosen arbitrarily without changing the result. The $2(n+1)$ dimensional matrix Ψ is the same as in the Yang-Mills case. There is no gauge group for gravity, and the amplitudes are not color ordered.

Non-Linear Sigma Model

The last example theory in which we will consider soft theorems is the so called Non-Linear Sigma Model (NLSM). While scalar, vector and graviton fields are rather intuitive objects, the fields in NLSM are a bit more involved. Group theoretically, NLSM describes what happens when a chiral Lie group $G_L \times G_R$ with two identical product groups $G_{L,R} = G$ is spontaneously broken to its diagonal subgroup $G_V = G$ [15]. This means that original group elements $(g_L, g_R) \in G_L \times G_R$ are restricted to $g_L = g_R$. We can describe the diagonal subgroup G_V by the symmetric group $G_L \times G_R / G_V$ which is isomorphic to it. This isomorphism can be realised by restricting the following combination of original group elements

$$g_R g_L^{-1} \equiv U \tag{1.6.15}$$

to $G_L \times G_R / G_V$. Considering that the vacuum little group G_V we are interested in describing is invariant under $(g_L, g_R) \leftrightarrow (g_R, g_L)$, we just made a choice to induce the action of the chiral group $G_L \times G_R$ on $G_L \times G_R / G_V$ through left multiplication. The resulting element transforms linearly under the action of elements (V_L, V_R) of the original chiral group

$$U \to V_R U V_L^{-1}. \tag{1.6.16}$$

Promoting U to a field, which then corresponds to the collection of Goldstone bosons [6] resulting from the spontaneous symmetry breaking $G_L \times G_R \to G_V$, we can write down an effective Lagrangian to leading order in the decay constant F of the Goldstone bosons

$$\mathcal{L} = \frac{F^2}{4} \mathrm{tr}\left(\partial_\mu U \partial^\mu U^{-1}\right). \tag{1.6.17}$$

A very convenient parametrization to of the field is the so called Cayley parametrization

$$U = \frac{1 + \frac{i}{\sqrt{2}F}\phi}{1 - \frac{i}{\sqrt{2}F}\phi} \tag{1.6.18}$$

which facilitates the expansion of scattering amplitudes in NLSM by means of a current algebra obtained from usual Feynman rules. While this iterative process is fairly straightforward, it is still rather involved. Luckily, we will not have to consider that, since as it turns out an n-point NLSM amplitude can be written in CHY representation by means of (1.2.2) with the following integrand factor

$$\mathcal{I}_n^{NLSM} = \frac{1}{\sigma_{12}\sigma_{23}...\sigma_{n1}} \frac{4}{(\sigma_{ij})^2} \text{Pf}(A_{i,j}^{i,j})^2, \tag{1.6.19}$$

where $A_{i,j}^{i,j}$ is the same matrix A as in the gluons case, with rows and columns i, j removed, and $1 \leq i < j \leq n$ can be selected arbitrarily [10]. As we expect, the presence of the Parke-Taylor like factor confirms that the amplitudes are color ordered.

Chapter 2

Sub-sub-leading soft-graviton theorem in arbitrary dimension

This chapter is based on the publication [16].

A lot of work has been done on soft theorems in the past, based on local on-shell gauge invariance [17, 18, 19, 20, 21, 22, 23, 24, 25]. The leading soft-graviton behavior was found by Weinberg in 1965 [22], and the sub-leading soft-graviton theorem was first investigated by Gross and Jackiw in 1968 [26]. Recently, active interest in soft theorems has been reawakened in [27, 28, 29, 30], as Strominger and collaborators discovered that soft-graviton behavior can be extracted from extended BMS symmetry [31, 32, 33, 34, 35]. For four dimensions, Cachazo and Strominger provided a proof for the universality of tree level sub-leading and sub-sub-leading corrections [30] to Weinbergs soft-graviton factor [22], making use of spinor helicity formalism and BCFW recursion [3, 4]. The soft-graviton factor refers to the factorization property of an $(n+1)$-point tree level scattering amplitude when the momentum of one external particle, conventionally the $(\mathrm{n}+1)^{\mathrm{th}}$ particle, is going to zero[1]

$$\mathcal{M}_{n+1}(k_1, k_2, ..., \varepsilon k_{n+1}) = \left(\frac{1}{\varepsilon} S^{(0)} + S^{(1)} + \varepsilon S^{(2)} + \mathcal{O}(\varepsilon^2)\right) \mathcal{M}_n(k_1, k_2, ..., k_n). \tag{2.0.1}$$

[1] Substitute $k_{n+1} \to \varepsilon k_{n+1}$ and expand around $\varepsilon = 0$.

In case of gravity, these soft factors read [30]

$$S^{(0)} = \sum_{a=1}^{n} \frac{\epsilon_{n+1\,\mu\nu} k_a^\mu k_a^\nu}{k_{n+1} \cdot k_a} \tag{2.0.2a}$$

$$S^{(1)} = \sum_{a=1}^{n} \frac{\epsilon_{n+1\,\mu\nu} k_a^\mu (k_{n+1\,\lambda} J_a^{\lambda\nu})}{k_{n+1} \cdot k_a} \tag{2.0.2b}$$

$$S^{(2)} = \frac{1}{2} \sum_{a=1}^{n} \frac{\epsilon_{n+1\,\mu\nu} (k_{n+1\,\rho} J_a^{\rho\mu})(k_{n+1\,\lambda} J_a^{\lambda\nu})}{k_{n+1} \cdot k_a}, \tag{2.0.2c}$$

where $\epsilon_{n+1\,\mu\nu}$ is the polarization tensor of the $(n+1)^{\text{th}}$ particle, k_a^μ are momenta and $J_a^{\mu\nu}$ are angular momentum operators. Subsequently, these soft-graviton theorems are being investigated with the restriction to four dimensions lifted. In arbitrary number of dimensions, the leading factor (2.0.2a) was addressed in [5] and the sub-leading factor (2.0.2b) was explicitly confirmed in [36, 37]. Considering Poincaré and gauge invariance in arbitrary number of dimensions, as well as expected formal structure, Broedel, de Leeuw, Plefka and Rosso fixed the orbital part of the sub-leading and sub-sub-leading factors completely and constrained their polarization parts up to one numerical constant for every order of expansion and each hard leg [38], in agreement with (2.0.2). Following Low's example [20], Bern, Davies, Di Vecchia and Nohle used on-shell gauge invariance to fully determine and confirm the first two sub-leading soft-graviton behaviors in D dimensions [39].

Further work on soft factors in general was, for instance, done for Yang-Mills amplitudes in [40],[41]. Several advances in gauge and gravity theories at loop level appeared in [42, 43].[2] Cachazo and Yuan proposed a modification of the usual soft limit procedure to cope with corrections appearing at loop level [44]. For a comment on this procedure, see [39]. Sub-leading soft theorems in gauge and gravity theory were confirmed from a diagrammatic approach in [45, 46]. Soft theorem in QED was revisited in [47, 48]. Stringy soft theorems appeared in [49, 50], and a more general investigation of soft theorems in a broader set of theories was conducted in [51].

In this note we will contribute an additional proof of the validity of the sub-sub-leading tree level soft factor (2.0.2c) by explicit computation in arbitrary dimension, making use of the CHY formula [5]. This note is structured as follows. Section 2.1 recalls the CHY formula. In section 2.2 we outline the computational steps for the higher point expansion in the soft limit. Section 2.3 contains the computation of lower point construction and comparison of the two results. Appendices 2.4, 2.5 and 2.6 contain all terms resulting from higher point expansion, which are compared with and are found to be equal to the result of lower point construction.

[2] Nontrivial corrections are expected at loop level.

2.1 The CHY formula

In order to explicitly prove the sub-sub-leading factor in the soft-graviton expansion, we will make use of the CHY formula for tree level gravity scattering amplitudes with $(n+1)$ external legs, which is valid in any number of dimensions [5]

$$\mathcal{M}_{n+1} = \int \left[\prod_{\substack{c=1 \\ c\neq p,q,r}}^{n+1} d\sigma_c \right] \frac{4(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki})}{(\sigma_{mw})^2} \left[\prod_{\substack{a=1 \\ a\neq i,j,k}}^{n+1} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n+1} \frac{k_a \cdot k_b}{\sigma_{ab}} \right) \right] \det\left(\Psi^{m,w}_{m,w}\right).$$

Here we use the abbreviation $\sigma_{ij} \equiv (\sigma_i - \sigma_j)$. Upper indices on the matrix Ψ denote removed columns and lower indices denote removed rows. Values of indices p, q, r, i, j, k, m and w can be chosen arbitrarily without changing the result. The $2(n+1)$ dimensional matrix Ψ is given by

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix},$$

where the $(n+1)$ dimensional sub-matrices are given by

$$A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}} & , \ a \neq b \\ 0 & , \ a = b \end{cases} \quad , \quad B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}} & , \ a \neq b \\ 0 & , \ a = b \end{cases} \quad , \quad C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{ab}} & , \ a \neq b \\ -\sum_{\substack{c=1 \\ c\neq a}}^{n+1} \frac{\epsilon_a \cdot k_c}{\sigma_{ac}} & , \ a = b \end{cases}.$$

Here k_a^μ is the momentum of the a^{th} particle, and ϵ_a^μ is part of its polarization tensor. The values for all σ_i in the integration are fixed by the product of delta functions which enforce the scattering equations. The momentum of the $(n+1)^{\text{th}}$ leg will be sent to zero in the soft-graviton expansion.

2.2 Higher point expansion

In the higher point expansion we start with the momentum conservation stripped tree level amplitude for $n+1$ external gravitons, substitute $k^\mu_{n+1} \to \varepsilon k^\mu_{n+1}$ and expand around $\varepsilon = 0$. In the sub-sub-leading case we are interested in the order $\mathcal{O}(\varepsilon^1)$ terms of this expansion. Subsequently, we integrate out the σ_{n+1} dependance to obtain the result which we expect to recover from lower point construction by acting with the corresponding soft factor on an amplitude with one fewer external leg in section 2.3. All solutions for σ_{n+1} are fixed by the scattering equations. However, since we are dealing with tree level amplitudes, the functional dependance does not feature any branch cuts such that we will be able to deform the integration contour and pick up a different set of residues in σ_{n+1} as in [36, 37] in order to obtain the same result, effectively avoiding having to solve the scattering equations.

For convenience we set $i = 1$, $j = n$, $m = 2$ and $w = 3$ so that the momentum conservation stripped tree level amplitude for $n+1$ external gravitons is given by

$$\mathcal{M}_{n+1} = \int \left[\prod_{\substack{c=1 \\ c\neq p,q,r}}^{n+1} d\sigma_c \right] \frac{4(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{1n}\sigma_{nk}\sigma_{k1})}{(\sigma_{2,3})^2} \left[\prod_{\substack{a=2 \\ a\neq k,n}}^{n+1} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n+1} \frac{k_a \cdot k_b}{\sigma_{ab}} \right) \right] \det\left(\Psi^{2,3}_{2,3}\right).$$

Since tree level amplitudes do not feature branch cuts, a delta distribution can be mapped to a single pole term

$$\delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n+1} \frac{k_a \cdot k_b}{\sigma_{ab}} \right) \to \frac{1}{\sum_{\substack{b=1 \\ b\neq a}}^{n+1} \frac{k_a \cdot k_b}{\sigma_{ab}}}$$

while the integration contour is deformed to pick up the residue associated with this pole as in [36, 37]. This naturally yields the same result for the amplitude. Therefore, we can substitute one delta function that has index $a = (n+1)$ by a simple pole, take $k_{n+1} \to \varepsilon k_{n+1}$ and expand around $\varepsilon = 0$ as follows:

$$\begin{aligned}
&\prod_{\substack{a=2 \\ a\neq k,n}}^{n+1} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n+1} \frac{k_a \cdot k_b}{\sigma_{ab}} \right) = \qquad (2.2.1)\\
&= \frac{1}{\sum_{c=1}^{n} \frac{k_{n+1}\cdot k_c}{\sigma_{n+1,c}}} \left[\frac{1}{\varepsilon} \prod_{\substack{a=2 \\ a\neq k}}^{n-1} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}} \right) + \sum_{\substack{r=2 \\ r\neq k}}^{n-1} \frac{k_{n+1}\cdot k_r}{\sigma_{n+1,r}} \delta^{(1)}\left(\sum_{\substack{q=1 \\ q\neq r}}^{n} \frac{k_r \cdot k_q}{\sigma_{rq}} \right) \prod_{\substack{a=2 \\ a\neq k,r}}^{n-1} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}} \right) \right.\\
&\quad + \frac{\varepsilon}{2} \sum_{\substack{r=2 \\ r\neq k}}^{n-1} \frac{k_{n+1}\cdot k_r}{\sigma_{n+1,r}} \delta^{(1)}\left(\sum_{\substack{q=1 \\ q\neq r}}^{n} \frac{k_r \cdot k_q}{\sigma_{rq}} \right) \sum_{\substack{s=2 \\ s\neq k,r}}^{n-1} \frac{k_{n+1}\cdot k_s}{\sigma_{n+1,s}} \delta^{(1)}\left(\sum_{\substack{t=1 \\ t\neq s}}^{n} \frac{k_s \cdot k_t}{\sigma_{st}} \right) \prod_{\substack{a=2 \\ a\neq k,r,s}}^{n-1} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}} \right)\\
&\quad \left. + \frac{\varepsilon}{2} \sum_{\substack{r=2 \\ r\neq k}}^{n-1} \left(\frac{k_{n+1}\cdot k_r}{\sigma_{n+1,r}} \right)^2 \delta^{(2)}\left(\sum_{\substack{q=1 \\ q\neq r}}^{n} \frac{k_r \cdot k_q}{\sigma_{rq}} \right) \prod_{\substack{a=2 \\ a\neq k,r}}^{n-1} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}} \right) + \mathcal{O}(\varepsilon^2) \right]\\
&\equiv \frac{1}{\varepsilon}\delta^0 + \delta^1 + \varepsilon\delta^2 + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Here we have introduced abbreviations δ^i to denote the expansion coefficients of order ε^{i-1}. Similarly, we can expand the determinant $\det(\Psi^{2,3}_{2,3})$ to make its ε dependance explicit. For that end we employ the usual recursive formula[3]

$$\det(A) = \sum_{i=1}^{2(n+1)} (-1)^{i+k} a_{ki} \det(A^i_k), \qquad (2.2.2)$$

where a_{ki} are elements of matrix A and the choice of row k is arbitrary.[4] If certain rows and columns

[3] In this case we are dealing with a $2(n+1) \times 2(n+1)$ matrix.
[4] Naturally, an analogous expansion can also be done along a column instead of a row.

are initially missing from the matrix A such that it is less than $2(n+1)\times 2(n+1)$ dimensional before the expansion (2.2.2) is applied, those corresponding values of missing rows and columns have to be skipped in the summation over the expansion index i. Additionally, a jump by ± 1 has to be introduced in the exponent of $(-1)^{i+k}$ whenever such a missing row or column is crossed. This will be accomplished with help of the Heaviside step function $\theta(a,b)\equiv\theta(a-b)$. More explicitly, when an additional row i (or column k) is removed from matrix A, one step function has to be introduced for each of the rows u (or columns v) that were already missing, so that we add[5] $\sum_u \theta(i,u)$ (or $\sum_v \theta(k,v)$) to the exponent of $(-1)^{i+k}$. This ensures that each summand in the expansion (2.2.2) appears with the correct sign.

As in [36], we make use of the gauge condition $(k_{n+1}\cdot\epsilon_i)=0$ for all i to conveniently reduce the number of appearing terms. We realize that with this all the ε dependance is located along the $(n+1)^{\text{th}}$ row and column of $\Psi^{2,3}_{2,3}$. Therefore, we apply the expansion to the $(n+1)^{\text{th}}$ row and column in succession:

$$\begin{aligned}\det\left(\Psi^{2,3}_{2,3}\right) =& \left(\sum_{c=1}^{n}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{n+1,c}}\right)^2 \det\left(\Psi^{2,3,n+1,2(n+1)}_{2,3,n+1,2(n+1)}\right)\\ &+2\varepsilon\sum_{\substack{i=1\\ i\neq 2,3}}^{n}\sum_{c=1}^{n}(-1)^i\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{n+1,c}}\frac{k_{n+1}\cdot k_i}{\sigma_{n+1,i}}\det\left(\Psi^{2,3,n+1,i}_{2,3,n+1,2(n+1)}\right)\\ &+\varepsilon^2\sum_{\substack{i=1\\ i\neq 2,3}}^{n}\sum_{\substack{j=1\\ j\neq 2,3}}^{n}(-1)^{i+j}\frac{k_{n+1}\cdot k_i}{\sigma_{n+1,i}}\frac{k_j\cdot k_{n+1}}{\sigma_{n+1,j}}\det\left(\Psi^{2,3,n+1,i}_{2,3,n+1,j}\right).\end{aligned} \tag{2.2.3}$$

Here and in later equations the Heaviside step functions involving arguments 2, 3, $n+1$ and $2(n+1)$ are suppressed. However, to keep track of the signs we should agree to always order the argument of each step function according to the order in which removed rows or columns appear in the determinant. In particular,

$$\begin{aligned}(-1)^{\ldots+\theta(a,b)+\ldots+\theta(c,d)+\ldots}\det(\Psi^{\ldots,b,a,\ldots}_{\ldots,d,c,\ldots}) &= -(-1)^{\ldots+\theta(b,a)+\ldots+\theta(c,d)+\ldots}\det(\Psi^{\ldots,a,b,\ldots}_{\ldots,d,c,\ldots})\\ &= (-1)^{\ldots+\theta(b,a)+\ldots+\theta(d,c)+\ldots}\det(\Psi^{\ldots,a,b,\ldots}_{\ldots,c,d,\ldots}).\end{aligned} \tag{2.2.4}$$

In cases where more than two rows (columns) are removed from Ψ, there will be one step function for each way an unordered pair of removed rows (columns) can be selected. Therefore, with our agreement (2.2.4) we can think of the step functions as being attached to the determinant, facilitating the property of making the exchange of two neighboring indices of removed rows (or columns)

[5] Note that the newly removed index i (or k) is in the first argument of each respective step function and is attached to the determinant at the far right. This introduces a natural initial index-ordering and will be relevant in the following.

antisymmetric. Furthermore, this ensures that the order of arguments of all step functions is in one to one correspondence to the order of removed row (column) indices in the determinant, allowing us to ignore the step functions and concentrate on comparing determinants. This convenient property yields a slight simplification to the algebraic steps that later will be required in order to show the equality of the higher point expansion and lower point construction results.[6]

Note that the order in which the indices of removed rows and columns appear in the determinants in (2.2.3) is different from the straightforward order which emerges from the expansion. We reordered these indices according to (2.2.4) to ensure proper sign in comparison to the terms of lower point construction computed in the next section. For later convenience we define the abbreviation:

$$\det\left(\Psi'\right) \equiv \det\left(\Psi^{2,3,n+1,2(n+1)}_{2,3,n+1,2(n+1)}\right). \tag{2.2.5}$$

We wish to make the entire σ_{n+1} dependance explicit to be able to integrate it out. Only $(n+1)^{\text{th}}$ and $2(n+1)^{\text{th}}$ rows and columns in the matrix Ψ depend on σ_{n+1}. Therefore, we expand $\det(\Psi^{2,3,n+1,i}_{2,3,n+1,2(n+1)})$ along the $2(n+1)^{\text{th}}$ column, as well as $\det(\Psi^{2,3,n+1,i}_{2,3,n+1,j})$ along the $2(n+1)^{\text{th}}$ row and column in succession. Again, here and in all further steps we make use of the gauge condition $(k_{n+1}\cdot\epsilon_i)=0$ for all i, such that:

$$\begin{aligned}\det\left(\Psi^{2,3}_{2,3}\right) &= \left(\sum_{c=1}^{n}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{n+1,c}}\right)^2\det\left(\Psi'\right) + 2\varepsilon\sum_{\substack{i=1\\i\neq 2,3}}^{n}\sum_{c=1}^{n}(-1)^i\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{n+1,c}}\frac{k_{n+1}\cdot k_i}{\sigma_{n+1,i}}\times\\
&\quad\times\left(\sum_{\substack{j=1\\j\neq 2,3}}^{n}(-1)^j\frac{\epsilon_{n+1}\cdot k_j}{\sigma_{n+1,j}}\det\left(\Psi'^{i}_{j}\right) - \sum_{j=1}^{n}(-1)^{j+n+1}\frac{\epsilon_{n+1}\cdot\epsilon_j}{\sigma_{n+1,j}}\det\left(\Psi'^{i}_{j+n+1}\right)\right)\\
&\quad+\varepsilon^2\sum_{\substack{i=1\\i\neq 2,3}}^{n}\sum_{\substack{j=1\\j\neq 2,3}}^{n}(-1)^{i+j}\frac{k_{n+1}\cdot k_i}{\sigma_{n+1,i}}\frac{k_j\cdot k_{n+1}}{\sigma_{n+1,j}}\times\\
&\quad\times\left(\sum_{\substack{u=1\\u\neq 2,3,i}}^{n}\sum_{\substack{p=1\\p\neq 2,3,j}}^{n}(-1)^{u+p+\theta(u,i)+\theta(p,j)}\frac{\epsilon_{n+1}\cdot k_u}{\sigma_{n+1,u}}\frac{\epsilon_{n+1}\cdot k_p}{\sigma_{n+1,p}}\det\left(\Psi'^{i,u}_{j,p}\right)\right.\\
&\quad\quad-2\sum_{\substack{u=1\\u\neq 2,3,i}}^{n}\sum_{p=1}^{n}(-1)^{u+p+n+1+\theta(u,i)+\theta(p+n+1,j)}\frac{\epsilon_{n+1}\cdot k_u}{\sigma_{n+1,u}}\frac{\epsilon_{n+1}\cdot\epsilon_p}{\sigma_{n+1,p}}\det\left(\Psi'^{i,u}_{j,p+n+1}\right)\\
&\quad\quad\left.+\sum_{u=1}^{n}\sum_{p=1}^{n}(-1)^{u+p+\theta(u+n+1,i)+\theta(p+n+1,j)}\frac{\epsilon_{n+1}\cdot\epsilon_u}{\sigma_{n+1,u}}\frac{\epsilon_{n+1}\cdot\epsilon_p}{\sigma_{n+1,p}}\det\left(\Psi'^{i,u+n+1}_{j,p+n+1}\right)\right)\\
&\equiv \det{}_0+\varepsilon\det{}_1+\varepsilon^2\det{}_2.\end{aligned} \tag{2.2.6}$$

[6] Some of the appearing step functions can never yield a change of sign and it might be tempting to evaluate them right away and get rid of them. However, this would break the agreement (2.2.4) and the convenient general antisymmetry property of the determinant under exchange of two neighboring removed row (column) indices, thus making a more tedious case by case distinction for index-ordering necessary.

Here, again we defined abbreviations $\det_i$ to denote the coefficients of ε^i. The ordering of the indices of removed rows and columns in the determinants was again done in accordance with (2.2.4) to ensure proper signs. In the sub-sub-leading case at hand only terms of overall order $\mathcal{O}(\varepsilon^1)$ are of interest. Therefore, we restrict our attention to:

$$\prod_{\substack{a=2\\a\neq k,n}}^{n+1} \delta\left(\sum_{\substack{b=1\\b\neq a}}^{n+1} \frac{k_a \cdot k_b}{\sigma_{ab}}\right) \det\left(\Psi^{2,3}_{2,3}\right) = \left(\frac{1}{\varepsilon}\delta^0 + \delta^1 + \varepsilon\delta^2 + \mathcal{O}(\varepsilon^2)\right)\left(\det_0 + \varepsilon \det_1 + \varepsilon^2 \det_2\right)$$
$$= \varepsilon\left(\delta^2 \det_0 + \delta^1 \det_1 + \delta^0 \det_2\right) + \ldots\ , \tag{2.2.7}$$

where ... denotes other terms of different order in ε. In fact, the terms given explicitly in (2.2.7) are the only terms of order $\mathcal{O}(\varepsilon^1)$ in the amplitude which depend on σ_{n+1}. Other multiplicative terms and integrals are merely spectators and can be suppressed when we integrate out σ_{n+1} and compare the result to the lower point construction.

As in [36, 37], it is trivial to see that there is no pole and therefore no residue at infinity in σ_{n+1}. Therefore, the integration contour can be reversed to pick up the residues at $\sigma_{n+1} = \sigma_i$ for all $i \neq n+1$ instead. Poles of higher order will occur in the computation, so that we will use Cauchy's integral formula to obtain the respective residues:

$$\text{Res}\left(\frac{f(z)}{(z-z_0)^n}\ ,\ z = z_0\right) = \frac{1}{(n-1)!} f^{(n-1)}(z_0), \tag{2.2.8}$$

where $f^{(n-1)}(z)$ is the $(n-1)^{\text{th}}$ derivative of $f(z)$.

The technical steps necessary to obtain the residues from all the terms of order $\mathcal{O}(\varepsilon^1)$ appearing in (2.2.7) are identical. Let us illustrate the procedure on one expression from $\delta^2 \det_0$:

$$\frac{1}{2}\frac{\left(\sum_{b=1}^{n} \frac{\epsilon_{n+1}\cdot k_b}{\sigma_{n+1,b}}\right)^2}{\sum_{c=1}^{n} \frac{k_{n+1}\cdot k_c}{\sigma_{n+1,c}}} \sum_{\substack{r=2\\r\neq k}}^{n-1} \left(\frac{k_{n+1}\cdot k_r}{\sigma_{n+1,r}}\right)^2 \delta^{(2)}\left(\sum_{\substack{t=1\\t\neq r}}^{n} \frac{k_r \cdot k_t}{\sigma_{r,t}}\right) \prod_{\substack{a=2\\a\neq k,r}}^{n-1} \delta\left(\sum_{\substack{p=1\\p\neq a}}^{n} \frac{k_a \cdot k_p}{\sigma_{a,p}}\right) \det\left(\Psi'\right). \tag{2.2.9}$$

First, we suppress the product of delta functions and the determinant since they are just spectators independent of σ_{n+1}, and we abbreviate $\delta_r^{(2)} \equiv \delta^{(2)}(\sum_{\substack{t=1\\t\neq r}}^{n} \frac{k_r\cdot k_t}{\sigma_{r,t}})$ for convenience. To investigate the residues at $\sigma_{n+1} = \sigma_i$ for all $i \neq n+1$, it is natural to distinguish between two cases of $\sigma_{n+1} = \sigma_q$ where $\sigma_q \in \{\sigma_1, \sigma_k, \sigma_n\}$ and $\sigma_q \notin \{\sigma_1, \sigma_k, \sigma_n\}$. In the first case, where $\sigma_q \in \{\sigma_1, \sigma_k, \sigma_n\}$ we find only first order poles in (2.2.9):

$$\frac{1}{\sigma_{n+1,q}}\left(\frac{1}{2}\frac{\left(\epsilon_{n+1}\cdot k_q+\sigma_{n+1,q}\sum_{\substack{b=1\\b\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_b}{\sigma_{n+1,b}}\right)^2}{k_{n+1}\cdot k_q+\sigma_{n+1,q}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{k_{n+1}\cdot k_c}{\sigma_{n+1,c}}}\sum_{\substack{r=2\\r\neq k}}^{n-1}\left(\frac{k_{n+1}\cdot k_r}{\sigma_{n+1,r}}\right)^2\delta_r^{(2)}\right),$$

so that the sum of corresponding residues is trivially given by using the Cauchy integral formula (2.2.8) with $n=1$ and summing over q:

$$\frac{1}{2}\sum_{q=1,k,n}\frac{(\epsilon_{n+1}\cdot k_q)^2}{k_{n+1}\cdot k_q}\sum_{\substack{r=2\\r\neq k}}^{n-1}\left(\frac{k_{n+1}\cdot k_r}{\sigma_{q,r}}\right)^2\delta_r^{(2)}.$$

In the second case $\sigma_q\notin\{\sigma_1,\sigma_k,\sigma_n\}$ we find first, second and third order poles in (2.2.9):

$$\begin{aligned}&\frac{1}{\sigma_{n+1,q}}\left(\frac{1}{2}\frac{\left(\sum_{\substack{b=1\\b\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{n+1,c}}\right)^2(k_q\cdot k_{n+1})^2\delta_q^{(2)}}{k_{n+1}\cdot k_q+\sigma_{n+1,q}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{k_{n+1}\cdot k_c}{\sigma_{n+1,c}}}+\frac{1}{2}\frac{(\epsilon_{n+1}\cdot k_q)^2\sum_{\substack{r=2\\r\neq k,q}}^{n-1}\left(\frac{k_r\cdot k_{n+1}}{\sigma_{n+1,r}}\right)^2\delta_r^{(2)}}{k_{n+1}\cdot k_q+\sigma_{n+1,q}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{k_{n+1}\cdot k_c}{\sigma_{n+1,c}}}\right)\\&+\frac{1}{(\sigma_{n+1,q})^2}\left(\frac{(\epsilon_{n+1}\cdot k_q)(k_q\cdot k_{n+1})^2\delta_q^{(2)}\sum_{\substack{b=1\\b\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_b}{\sigma_{n+1,b}}}{k_{n+1}\cdot k_q+\sigma_{n+1,q}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{k_{n+1}\cdot k_c}{\sigma_{n+1,c}}}\right)\\&+\frac{1}{(\sigma_{n+1,q})^3}\left(\frac{1}{2}\frac{(\epsilon_{n+1}\cdot k_q)^2(k_q\cdot k_{n+1})^2\delta_q^{(2)}}{k_{n+1}\cdot k_q+\sigma_{n+1,q}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{k_{n+1}\cdot k_c}{\sigma_{n+1,c}}}\right).\end{aligned}$$

The sum of all simple pole residues again is trivially obtained by using the Cauchy integral formula (2.2.8) with $n=1$ and summing over q:

$$\frac{1}{2}\sum_{\substack{q=2\\q\neq k}}^{n-1}\left((k_q\cdot k_{n+1})\left(\sum_{\substack{b=1\\b\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{q,c}}\right)^2\delta_q^{(2)}+\frac{(\epsilon_{n+1}\cdot k_q)^2}{k_{n+1}\cdot k_q}\sum_{\substack{r=2\\r\neq k,q}}^{n-1}\left(\frac{k_r\cdot k_{n+1}}{\sigma_{q,r}}\right)^2\delta_r^{(2)}\right).$$

To obtain the sum over second order pole residues we make use of the Cauchy integral formula (2.2.8) with $n=2$. This yields:

$$-\sum_{\substack{q=2\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)\left((k_q\cdot k_{n+1})\sum_{\substack{b=1\\b\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_b}{(\sigma_{q,b})^2}\delta_q^{(2)}+\sum_{\substack{b=1\\b\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_b}{\sigma_{q,b}}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{k_{n+1}\cdot k_c}{\sigma_{q,c}}\delta_q^{(2)}\right).$$

Finally, to obtain the sum over the third order pole residues we use the Cauchy integral formula

(2.2.8) with $n = 3$:

$$\frac{1}{2}\sum_{\substack{q=2\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)^2\left(\frac{1}{k_{n+1}\cdot k_q}\left(\sum_{\substack{b=1\\b\neq q}}^{n}\frac{k_{n+1}\cdot k_b}{\sigma_{q,b}}\right)^2\delta_q^{(2)}+\sum_{\substack{b=1\\b\neq q}}^{n}\frac{k_{n+1}\cdot k_b}{(\sigma_{q,b})^2}\delta_q^{(2)}\right).$$

The residues of all further terms appearing in (2.2.7) are computed in exactly the same way. Without showing every single step explicitly, we will give a list of pole orders appearing in the respective terms. Additionally, the results for all residues will be gathered in appendices.

Apart from the computation presented above, the term $\delta^2\det_0$ contains one additional expression. It has only first order poles for $\sigma_q \in \{\sigma_1,\sigma_k,\sigma_n\}$, and it has first and second order poles for $\sigma_q \notin \{\sigma_1,\sigma_k,\sigma_n\}$. All residues associated with the term $\delta^2\det_0$ are presented in appendix 2.4.

The residues of the term $\delta^1\det_1$ are obtained from three different cases. In the case of $\sigma_q \notin \{\sigma_1,\sigma_2,\sigma_3,\sigma_k,\sigma_n\}$ there are first, second and third order poles. In the case of $\sigma_q \in \{\sigma_1,\sigma_k,\sigma_n\}$ there are first and second order poles. And in the case of $\sigma_q \in \{\sigma_2,\sigma_3\}$ there are first and second order poles. All residues associated with the term $\delta^1\det_1$ are presented in appendix 2.5.

The residues of the term $\delta^0\det_2$ are obtained from two different cases. In the case of $\sigma_q \notin \{\sigma_2,\sigma_3\}$ there are first, second and third order poles. And in the case of $\sigma_q \in \{\sigma_2,\sigma_3\}$ there are only first order poles. All residues associated with the term $\delta^0\det_2$ are presented in appendix 2.6.

With this, all relevant terms from higher point expansion are obtained and we can proceed with the computation of lower point construction.

2.3 Lower point construction

In the lower point construction we start with the momentum conservation stripped tree level amplitude for n external particles. We set $i = 1$, $j = n$, $m = 2$ and $w = 3$ and invoke the gauge condition $(k_{n+1}\cdot\epsilon_u) = 0$ for all $u \in \{1,2,...,n+1\}$, such that[7]:

$$M_n = \int\left[\prod_{\substack{c=1\\c\neq p,q,r}}^{n} d\sigma_c\right]\frac{4(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{1n}\sigma_{nk}\sigma_{k1})}{(\sigma_{2,3})^2}\left[\prod_{\substack{a=2\\a\neq k}}^{n-1}\delta\left(\sum_{\substack{b=1\\b\neq a}}^{n}\frac{k_a\cdot k_b}{\sigma_{ab}}\right)\right]\det(\Psi'), \qquad (2.3.1)$$

where we used the abbreviation defined in (2.2.5). First we notice that only the product of delta functions and the determinant are relevant for our considerations, and all remaining multiplicative factors and integrals are exactly the same spectators which we suppressed in the higher point expansion case. Therefore, here we again suppress these spectator terms, such that the expression we

[7]The gauge condition ensures that there is no remaining k_{n+1} and σ_{n+1} dependance in $\det(\Psi')$.

should compare to the higher point expansion is given by:

$$S^{(2)}\left[\prod_{\substack{c=2\\c\neq k}}^{n-1}\delta\left(\sum_{\substack{b=1\\b\neq c}}^{n}\frac{k_c\cdot k_b}{\sigma_{cb}}\right)\right]\det\left(\Psi'\right). \tag{2.3.2}$$

As already stated in the introduction, the sub-sub-leading factor $S^{(2)}$ is expected to be given by:

$$S^{(2)}=\frac{1}{2}\sum_{a=1}^{n}\frac{\epsilon_{n+1\mu\nu}(k_{n+1\rho}J_a^{\rho\mu})(k_{n+1\lambda}J_a^{\lambda\nu})}{k_{n+1}\cdot k_a}, \tag{2.3.3}$$

where the action of the angular momentum operators by their orbital or spin part on momenta or polarization vectors is given by [36]

$$J_a^{\mu\nu}k_b^{\beta}=\left(k_a^{\mu}\frac{\partial}{\partial k_{a\nu}}-k_a^{\nu}\frac{\partial}{\partial k_{a\mu}}\right)k_b^{\beta} \tag{2.3.4a}$$

$$J_a^{\mu\nu}\epsilon_b^{\beta}=\left(\eta^{\nu\beta}\delta_\sigma^{\mu}-\eta^{\mu\beta}\delta_\sigma^{\nu}\right)\epsilon_b^{\sigma}. \tag{2.3.4b}$$

Naively, the two angular momentum operators in the sub-sub-leading factor (2.3.3) could act on each other. However, it is trivial to show that the interaction vanishes due to the $(n+1)^{\text{th}}$ particle being massless $k_{n+1}^2=0$, therefore having only transverse polarization modes $k_{n+1}\cdot\epsilon_{n+1}=0$, and the polarization being light-like such that $\epsilon_{n+1}^2=0$. With this we can conclude that we will have to match the resulting terms to the higher point expansion in the following way

$$\frac{1}{2}\sum_{a=1}^{n}\frac{\epsilon_\mu\epsilon_\nu q_\rho q_\lambda}{q\cdot k_a}\left(J_a^{\rho\mu}J_a^{\lambda\nu}\prod_{\substack{c=2\\c\neq k}}^{n-1}\delta\left(\sum_{\substack{b=1\\b\neq c}}^{n}\frac{k_c\cdot k_b}{\sigma_{cb}}\right)\right)\det\left(\Psi'\right)\quad\Leftrightarrow\quad\sum_i\text{Res}_i(\delta^2\det{}_0) \tag{2.3.5a}$$

$$\sum_{a=1}^{n}\frac{\epsilon_\mu\epsilon_\nu q_\rho q_\lambda}{q\cdot k_a}\left(J_a^{\rho\mu}\prod_{\substack{c=2\\c\neq k}}^{n-1}\delta\left(\sum_{\substack{b=1\\b\neq c}}^{n}\frac{k_c\cdot k_b}{\sigma_{cb}}\right)\right)\left(J_a^{\lambda\nu}\det\left(\Psi'\right)\right)\quad\Leftrightarrow\quad\sum_i\text{Res}_i(\delta^1\det{}_1) \tag{2.3.5b}$$

$$\frac{1}{2}\sum_{a=1}^{n}\frac{\epsilon_\mu\epsilon_\nu q_\rho q_\lambda}{q\cdot k_a}\prod_{\substack{c=2\\c\neq k}}^{n-1}\delta\left(\sum_{\substack{b=1\\b\neq c}}^{n}\frac{k_c\cdot k_b}{\sigma_{cb}}\right)\left(J_a^{\rho\mu}J_a^{\lambda\nu}\det\left(\Psi'\right)\right)\quad\Leftrightarrow\quad\sum_i\text{Res}_i(\delta^0\det{}_2), \tag{2.3.5c}$$

where we used the abbreviations

$$\epsilon_\mu\equiv\epsilon_{n+1\mu}\text{ and }q_\mu\equiv k_{n+1\mu}, \tag{2.3.6}$$

and the sum in i is over all residues picked up when integrating out σ_{n+1}.

To compute the lower point construction for (2.3.5a), only the orbital part of the angular momentum operator (2.3.4a) is involved, since the scattering equation delta functions depend on momenta only. Therefore, the object of interest is

$$\frac{1}{2}\sum_{a=1}^{n}\left((q\cdot k_a)\epsilon^{\mu}\epsilon^{\nu}\frac{\partial}{\partial k_a^{\mu}}\frac{\partial}{\partial k_a^{\nu}}-2(\epsilon\cdot k_a)\epsilon^{\mu}q^{\lambda}\frac{\partial}{\partial k_a^{\mu}}\frac{\partial}{\partial k_a^{\lambda}}+\frac{(\epsilon\cdot k_a)^2}{q\cdot k_a}q^{\rho}q^{\lambda}\frac{\partial}{\partial k_a^{\rho}}\frac{\partial}{\partial k_a^{\lambda}}\right)\prod_{\substack{c=2\\c\neq k}}^{n-1}\delta\left(\sum_{\substack{b=1\\b\neq c}}^{n}\frac{k_c\cdot k_b}{\sigma_{cb}}\right).$$

Carrying out the partial derivatives as usual, then suppressing the remaining product of delta functions and abbreviating the derivatives of delta functions in the same way as after (2.2.9), we obtain the same result as from the higher point expansion given in appendix 2.4. The only type of reshaping needed to recover the exact same set of terms (apart from trivial cancellation and (2.2.4)), is to combine expressions which have a similar structure up to σ_{ij}'s appearing in denominators, such that a simplification occurs as in:

$$\frac{1}{\sigma_{jk}\sigma_{ij}}-\frac{1}{\sigma_{jk}\sigma_{ik}}=\frac{1}{\sigma_{ij}\sigma_{ik}}. \tag{2.3.7}$$

These steps eventually demonstrate the equality of both sides in (2.3.5a).

To compute the lower point construction for (2.3.5b), both parts of the angular momentum operator (2.3.4a) and (2.3.4b) are needed. Furthermore, to obtain the derivative of a determinant, we use the chain rule and straightforwardly obtain:

$$\frac{d}{dx}\det(A)=\sum_{q=1}^{2(n+1)}\sum_{i=1}^{2(n+1)}(-1)^{q+i}\left(\frac{da_{qi}}{dx}\right)\det(A_q^i). \tag{2.3.8}$$

First, we compute the action of a single angular momentum operator on the product of scattering equation delta functions:

$$\begin{aligned}\frac{\epsilon_{n+1\mu}k_{n+1\rho}}{k_{n+1}\cdot k_a}J_a^{\rho\mu}\prod_{\substack{c=2\\c\neq k}}^{n-1}\delta\left(\sum_{\substack{b=1\\b\neq c}}^{n}\frac{k_c\cdot k_b}{\sigma_{cb}}\right)=&\sum_{\substack{c=2\\c\neq k}}^{n-1}\sum_{\substack{b=1\\b\neq c}}^{n}\delta_{c,a}\frac{\epsilon_{n+1}\cdot k_b}{\sigma_{cb}}\delta_c^{(1)}+\sum_{\substack{c=2\\c\neq k}}^{n-1}\sum_{\substack{b=1\\b\neq c}}^{n}\delta_{b,a}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{cb}}\delta_c^{(1)}\\&-\frac{(\epsilon_{n+1}\cdot k_a)}{k_{n+1}\cdot k_a}\sum_{\substack{c=2\\c\neq k}}^{n-1}\sum_{\substack{b=1\\b\neq c}}^{n}\delta_{c,a}\frac{k_{n+1}\cdot k_b}{\sigma_{cb}}\delta_c^{(1)}\\&-\frac{(\epsilon_{n+1}\cdot k_a)}{k_{n+1}\cdot k_a}\sum_{\substack{c=2\\c\neq k}}^{n-1}\sum_{\substack{b=1\\b\neq c}}^{n}\delta_{b,a}\frac{k_{n+1}\cdot k_c}{\sigma_{cb}}\delta_c^{(1)},\end{aligned} \tag{2.3.9}$$

where $\delta_{i,j}$ is the Kronecker delta, and where on the right hand side we suppressed the remaining product of delta functions and abbreviated the derivative of the delta function in the same way as after (2.2.9). Next, we compute the action of a single angular momentum operator on the determinant:

$$\begin{aligned}\epsilon_{n+1\nu}k_{n+1\lambda}J_a^{\lambda\nu}\det\left(\Psi'\right) = &\sum_{\substack{j=1\\ j\neq 2,3}}^{n}\sum_{\substack{i=1\\ i\neq 2,3,j}}^{n}(-1)^{i+j}\left(\epsilon_{n+1\nu}k_{n+1\lambda}J_a^{\lambda\nu}\frac{k_j\cdot k_i}{\sigma_{ji}}\right)\det\left(\Psi'^{i}_{j}\right)\\ &+\sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq j}}^{n}(-1)^{i+j}\left(\epsilon_{n+1\nu}k_{n+1\lambda}J_a^{\lambda\nu}\frac{\epsilon_j\cdot\epsilon_i}{\sigma_{ji}}\right)\det\left(\Psi'^{i+n+1}_{j+n+1}\right)\\ &-2\sum_{\substack{j=1\\ j\neq 2,3}}^{n}\sum_{\substack{i=1\\ i\neq j}}^{n}(-1)^{i+j+n+1}\left(\epsilon_{n+1\nu}k_{n+1\lambda}J_a^{\lambda\nu}\frac{k_j\cdot\epsilon_i}{\sigma_{ji}}\right)\det\left(\Psi'^{i+n+1}_{j}\right)\\ &-2\sum_{\substack{j=1\\ j\neq 2,3}}^{n}\sum_{\substack{c=1\\ c\neq j}}^{n}(-1)^{n+1}\left(\epsilon_{n+1\nu}k_{n+1\lambda}J_a^{\lambda\nu}\frac{k_c\cdot\epsilon_j}{\sigma_{jc}}\right)\det\left(\Psi'^{j+n+1}_{j}\right).\end{aligned} \tag{2.3.10}$$

To make the terms more explicit, we invoke the usual gauge from before $k_{n+1}\cdot\epsilon_i = 0$ for all i and obtain:

$$\epsilon_{n+1\nu}k_{n+1\lambda}J_a^{\lambda\nu}\frac{k_j\cdot k_i}{\sigma_{ji}} = (\delta_{a,j}-\delta_{a,i})\frac{(k_{n+1}\cdot k_j)(\epsilon_{n+1}\cdot k_i)-(k_{n+1}\cdot k_i)(\epsilon_{n+1}\cdot k_j)}{\sigma_{ji}} \tag{2.3.11a}$$

$$\epsilon_{n+1\nu}k_{n+1\lambda}J_a^{\lambda\nu}\frac{k_j\cdot \epsilon_i}{\sigma_{ji}} = (\delta_{a,j}-\delta_{a,i})\frac{(k_{n+1}\cdot k_j)(\epsilon_{n+1}\cdot \epsilon_i)}{\sigma_{ji}} \tag{2.3.11b}$$

$$\epsilon_{n+1\nu}k_{n+1\lambda}J_a^{\lambda\nu}\frac{\epsilon_j\cdot \epsilon_i}{\sigma_{ji}} = 0. \tag{2.3.11c}$$

Plugging (2.3.11) into (2.3.10), multiplying with (2.3.9) and summing over $a = 1,\ldots,n$ gives the same result as from higher point expansion in appendix 2.5. To recover the exact same set of terms in order to prove the equality, we use simplifications like (2.2.4) and (2.3.7). Additionally, we realize that for an antisymmetric $2(n+1)\times 2(n+1)$ matrix A we have:

$$\det\left(A^{a_1,a_2,\ldots,a_m}_{b_1,b_2,\ldots,b_m}\right) = (-1)^m\det\left(A^{b_1,b_2,\ldots,b_m}_{a_1,a_2,\ldots,a_m}\right). \tag{2.3.12}$$

Making use of these steps, the demonstration of the equality of both sides in (2.3.5b) becomes straightforward.

Finally, to compute the lower point construction for (2.3.5c), again both parts of the angular momentum operator (2.3.4a) and (2.3.4b) are needed. We start with (2.3.10) and act with the angular momentum operator a second time. The case where both angular momentum operators hit the expansion coefficient in each line vanishes due to the same arguments as the vanishing of the self-interaction of the two angular momentum operators. Therefore, only the case remains where the second angular momentum operator acts on the determinant in each line. Combining (2.3.10) with (2.3.11) and using the abbreviations (2.3.6), this results in:

$$\begin{aligned}
\frac{1}{2}\sum_{a=1}^{n}\frac{\epsilon_\mu\epsilon_\nu q_\rho q_\lambda}{q\cdot k_a}J_a^{\rho\mu}J_a^{\lambda\nu}\det\left(\Psi'\right) = &\sum_{\substack{m=1\\m\neq 2,3}}^{n}\sum_{\substack{i=1\\i\neq 2,3,m}}^{n}(-1)^{i+m}\frac{(\epsilon\cdot k_i)}{\sigma_{mi}}\left(q_\rho\epsilon_\mu J_m^{\rho\mu}\det\left(\Psi'^{i}_{m}\right)\right) \\
&-\sum_{\substack{m=1\\m\neq 2,3}}^{n}\sum_{\substack{i=1\\i\neq 2,3,m}}^{n}(-1)^{i+m}\frac{(\epsilon\cdot k_m)(q\cdot k_i)}{(q\cdot k_m)\sigma_{mi}}\left(q_\rho\epsilon_\mu J_m^{\rho\mu}\det\left(\Psi'^{i}_{m}\right)\right) \\
&+\sum_{\substack{m=1\\m\neq 2,3}}^{n}\sum_{\substack{i=1\\i\neq m}}^{n}(-1)^{i+m+n+1}\frac{(\epsilon\cdot\epsilon_i)(q\cdot k_m)}{(q\cdot k_i)\sigma_{mi}}\left(q_\rho\epsilon_\mu J_i^{\rho\mu}\det\left(\Psi'^{i+n+1}_{m}\right)\right) \\
&-\sum_{\substack{m=1\\m\neq 2,3}}^{n}\sum_{\substack{i=1\\i\neq m}}^{n}(-1)^{i+m+n+1}\frac{(\epsilon\cdot\epsilon_i)}{\sigma_{mi}}\left(q_\rho\epsilon_\mu J_m^{\rho\mu}\det\left(\Psi'^{i+n+1}_{m}\right)\right) \\
&+\sum_{\substack{m=1\\m\neq 2,3}}^{n}\sum_{\substack{i=1\\i\neq m}}^{n}(-1)^{n+1}\frac{(\epsilon\cdot\epsilon_m)(q\cdot k_i)}{(q\cdot k_m)\sigma_{mi}}\left(q_\rho\epsilon_\mu J_m^{\rho\mu}\det\left(\Psi'^{m+n+1}_{m}\right)\right) \\
&-\sum_{\substack{m=1\\m\neq 2,3}}^{n}\sum_{\substack{i=1\\i\neq m}}^{n}(-1)^{n+1}\frac{(\epsilon\cdot\epsilon_m)}{\sigma_{mi}}\left(q_\rho\epsilon_\mu J_i^{\rho\mu}\det\left(\Psi'^{m+n+1}_{m}\right)\right).
\end{aligned} \tag{2.3.13}$$

The action of the angular momentum operator on the determinants in each of these six lines is then expanded further analogously to (2.3.10). The only difference is, that now the expansion summations have to omit one removed row and column more in each case, and we have to explicitly display the corresponding step functions in the exponent of (-1). Since the product of scattering equation delta functions is untouched by the operators in this case, it can be suppressed as a spectator completely, so that the terms resulting from a further expansion of (2.3.13) correspond to the higher point expansion result given in appendix 2.6. Again, making use of simplifications (2.2.4), (2.3.7) and (2.3.12), it is then straightforward to reshape the finding to obtain the exact same set of terms listed in appendix 2.6, which proves the equality of both sides in (2.3.5c).

This concludes the computation of the lower point construction and its comparison with the higher point expansion. Both yield the same result, which confirms that the sub-sub-leading factor (2.0.2c) in the soft-graviton expansion of tree level scattering amplitudes is indeed valid in arbitrary dimension.

2.4 Residues of $\delta^2\text{det}_0$

The following are all residues obtained from $\delta^2\text{det}_0$ in (2.2.7) by integrating out the σ_{n+1} dependance. Multiplicative spectator terms and integrals which are trivially the same in the lower point construction are suppressed.[8] Additionally, the product of scattering equation delta functions is suppressed and the derivative of delta function is abbreviated as

$$\delta_j^{(i)} \equiv \delta^{(i)}\left(\sum_{\substack{a=1\\a\neq j}}^{n} \frac{k_a \cdot k_j}{\sigma_{aj}}\right). \tag{2.4.1}$$

With this the residues are:

$$\begin{aligned}
&2\sum_{\substack{q=2\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)\delta_q^{(1)}\sum_{\substack{r=2\\r\neq k,q}}^{n-1}\frac{k_r\cdot k_{n+1}}{\sigma_{qr}}\delta_r^{(1)}\sum_{\substack{b=1\\b\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_b}{\sigma_{qb}}\\
&+\frac{1}{2}\sum_{\substack{r=2\\r\neq k}}^{n-1}(k_r\cdot k_{n+1})\delta_r^{(1)}\sum_{\substack{q=1\\q\neq r}}^{n}\frac{1}{k_{n+1}\cdot k_q}\frac{(\epsilon_{n+1}\cdot k_q)^2}{\sigma_{qr}}\sum_{\substack{t=2\\t\neq k,r,q}}^{n-1}\frac{k_t\cdot k_{n+1}}{\sigma_{qt}}\delta_t^{(1)}\\
&-\sum_{\substack{q=2\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)^2\delta_q^{(1)}\left(\sum_{\substack{r=2\\r\neq k,q}}^{n-1}\frac{k_r\cdot k_{n+1}}{(\sigma_{qr})^2}\delta_r^{(1)}+\frac{1}{k_{n+1}\cdot k_q}\sum_{\substack{r=2\\r\neq k,q}}^{n-1}\frac{k_r\cdot k_{n+1}}{\sigma_{qr}}\delta_r^{(1)}\sum_{\substack{b=1\\b\neq q}}^{n}\frac{k_{n+1}\cdot k_b}{\sigma_{qb}}\right)\\
&+\frac{1}{2}\sum_{\substack{q=2\\q\neq k}}^{n-1}(k_q\cdot k_{n+1})\left(\sum_{\substack{b=1\\b\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{q,c}}\right)^2\delta_q^{(2)}+\frac{1}{2}\sum_{q=1}^{n}\frac{(\epsilon_{n+1}\cdot k_q)^2}{k_{n+1}\cdot k_q}\sum_{\substack{r=2\\r\neq k,q}}^{n-1}\left(\frac{k_r\cdot k_{n+1}}{\sigma_{q,r}}\right)^2\delta_r^{(2)}\\
&-\sum_{\substack{q=2\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)\left((k_q\cdot k_{n+1})\sum_{\substack{b=1\\b\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_b}{(\sigma_{q,b})^2}\delta_q^{(2)}+\sum_{\substack{b=1\\b\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_b}{\sigma_{q,b}}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{k_{n+1}\cdot k_c}{\sigma_{q,c}}\delta_q^{(2)}\right)\\
&+\frac{1}{2}\sum_{\substack{q=2\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)^2\left(\frac{1}{k_{n+1}\cdot k_q}\left(\sum_{\substack{b=1\\b\neq q}}^{n}\frac{k_{n+1}\cdot k_b}{\sigma_{q,b}}\right)^2\delta_q^{(2)}+\sum_{\substack{b=1\\b\neq q}}^{n}\frac{k_{n+1}\cdot k_b}{(\sigma_{q,b})^2}\delta_q^{(2)}\right).
\end{aligned}$$

2.5 Residues of $\delta^1\text{det}_1$

The following are all residues obtained from $\delta^1\text{det}_1$ in (2.2.7) by integrating out the σ_{n+1} dependance. Multiplicative spectator terms and integrals which are trivially the same in the lower point construction are suppressed. Additionally, the product of scattering equation delta functions is suppressed

[8] In this particular case the determinant $\det(\Psi')$ is also suppressed, since it is also a multiplicative spectator term in $\delta^2\text{det}_0$.

and the derivative of delta function is abbreviated as (2.4.1). With this the residues are:

$$
\begin{aligned}
&-2\sum_{\substack{q=2\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)\delta_q^{(1)}\sum_{\substack{i=1\\i\neq 2,3,q}}^{n}\frac{k_{n+1}\cdot k_i}{\sigma_{qi}}\sum_{\substack{j=1\\j\neq 2,3,q}}^{n}(-1)^{i+j}\frac{\epsilon_{n+1}\cdot k_j}{\sigma_{qj}}\det\left(\Psi_j'^i\right)\\
&-2\sum_{\substack{q=4\\q\neq k}}^{n-1}(k_{n+1}\cdot k_q)\delta_q^{(1)}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{qc}}\sum_{\substack{j=1\\j\neq 2,3,q}}^{n}(-1)^{q+j}\frac{\epsilon_{n+1}\cdot k_j}{\sigma_{qj}}\det\left(\Psi_j'^q\right)\\
&-2\sum_{\substack{q=4\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)\delta_q^{(1)}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{qc}}\sum_{\substack{i=1\\i\neq 2,3,q}}^{n}(-1)^{i+q}\frac{k_{n+1}\cdot k_i}{\sigma_{qi}}\det\left(\Psi_q'^i\right)\\
&+2\sum_{\substack{q=4\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)\delta_q^{(1)}\sum_{\substack{b=1\\b\neq q}}^{n}\frac{k_{n+1}\cdot k_b}{\sigma_{qb}}\sum_{\substack{j=1\\j\neq 2,3,q}}^{n}(-1)^{q+j}\frac{\epsilon_{n+1}\cdot k_j}{\sigma_{qj}}\det\left(\Psi_j'^q\right)\\
&+2\sum_{\substack{q=4\\q\neq k}}^{n-1}\frac{(\epsilon_{n+1}\cdot k_q)^2}{k_{n+1}\cdot k_q}\delta_q^{(1)}\sum_{\substack{b=1\\b\neq q}}^{n}\frac{k_{n+1}\cdot k_b}{\sigma_{qb}}\sum_{\substack{i=1\\i\neq 2,3,q}}^{n}(-1)^{i+q}\frac{k_{n+1}\cdot k_i}{\sigma_{qi}}\det\left(\Psi_q'^i\right)\\
&-2\sum_{\substack{q=1\\q\neq 2,3}}^{n}(\epsilon_{n+1}\cdot k_q)\sum_{\substack{r=2\\r\neq k,q}}^{n-1}\frac{k_{n+1}\cdot k_r}{\sigma_{qr}}\delta_r^{(1)}\sum_{\substack{j=1\\j\neq 2,3,q}}^{n}(-1)^{q+j}\frac{\epsilon_{n+1}\cdot k_j}{\sigma_{qj}}\det\left(\Psi_j'^q\right)\\
&-2\sum_{\substack{q=1\\q\neq 2,3}}^{n}\frac{(\epsilon_{n+1}\cdot k_q)^2}{(k_{n+1}\cdot k_q)}\sum_{\substack{r=2\\r\neq k,q}}^{n-1}\frac{k_{n+1}\cdot k_r}{\sigma_{qr}}\delta_r^{(1)}\sum_{\substack{i=1\\i\neq 2,3,q}}^{n}(-1)^{i+q}\frac{k_{n+1}\cdot k_i}{\sigma_{qi}}\det\left(\Psi_q'^i\right)\\
&+2\sum_{\substack{q=4\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)(k_{n+1}\cdot k_q)\delta_q^{(1)}\sum_{\substack{j=1\\j\neq 2,3,q}}^{n}(-1)^{q+j}\frac{\epsilon_{n+1}\cdot k_j}{(\sigma_{qj})^2}\det\left(\Psi_j'^q\right)\\
&+2\sum_{\substack{q=4\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k)^2\delta_q^{(1)}\sum_{\substack{i=1\\i\neq 2,3,q}}^{n}(-1)^{i+q}\frac{k_{n+1}\cdot k_i}{(\sigma_{qi})^2}\det\left(\Psi_q'^i\right)\\
&+2\sum_{\substack{q=2\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)\delta_q^{(1)}\sum_{\substack{i=1\\i\neq 2,3,q}}^{n}\frac{k_{n+1}\cdot k_i}{\sigma_{qi}}\sum_{\substack{j=1\\j\neq q}}^{n}(-1)^{i+j+n+1}\frac{\epsilon_{n+1}\cdot \epsilon_j}{\sigma_{qj}}\det\left(\Psi_{j+n+1}'^i\right)\\
&+2\sum_{\substack{q=4\\q\neq k}}^{n-1}k_{n+1}\cdot k_q\delta_q^{(1)}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{qc}}\sum_{\substack{j=1\\j\neq q}}^{n}(-1)^{q+j+n+1}\frac{\epsilon_{n+1}\cdot \epsilon_j}{\sigma_{qj}}\det\left(\Psi_{j+n+1}'^q\right)\\
&+2\sum_{\substack{q=2\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot \epsilon_q)\delta_q^{(1)}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{qc}}\sum_{\substack{i=1\\i\neq 2,3,q}}^{n}(-1)^{i+q+n+1}\frac{k_{n+1}\cdot k_i}{\sigma_{qi}}\det\left(\Psi_{q+n+1}'^i\right)\\
&-2\sum_{\substack{q=4\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)\delta_q^{(1)}\sum_{\substack{b=1\\b\neq q}}^{n}\frac{k_{n+1}\cdot k_b}{\sigma_{qb}}\sum_{\substack{j=1\\j\neq q}}^{n}(-1)^{q+j+n+1}\frac{\epsilon_{n+1}\cdot \epsilon_j}{\sigma_{qj}}\det\left(\Psi_{j+n+1}'^q\right)\\
&-2\sum_{\substack{q=2\\q\neq k}}^{n-1}\frac{(\epsilon_{n+1}\cdot k_q)(\epsilon_{n+1}\cdot \epsilon_q)}{(k_{n+1}\cdot k_q)}\delta_q^{(1)}\sum_{\substack{b=1\\b\neq q}}^{n}\frac{k_{n+1}\cdot k_b}{\sigma_{qb}}\sum_{\substack{i=1\\i\neq 2,3,q}}^{n}(-1)^{i+q+n+1}\frac{k_{n+1}\cdot k_i}{\sigma_{qi}}\det\left(\Psi_{q+n+1}'^i\right)
\end{aligned}
$$

$$+\,2\sum_{\substack{q=1\\q\neq 2,3}}^{n}(\epsilon_{n+1}\cdot k_q)\sum_{\substack{r=2\\r\neq k,q}}^{n-1}\frac{k_{n+1}\cdot k_r}{\sigma_{qr}}\delta_r^{(1)}\sum_{\substack{j=1\\j\neq q}}^{n}(-1)^{q+j+n+1}\frac{\epsilon_{n+1}\cdot\epsilon_j}{\sigma_{qj}}\det\left(\Psi'^{q}_{j+n+1}\right)$$

$$+\,2\sum_{q=1}^{n}\frac{(\epsilon_{n+1}\cdot k_q)(\epsilon_{n+1}\cdot\epsilon_q)}{(k_{n+1}\cdot k_q)}\sum_{\substack{r=2\\r\neq k,q}}^{n-1}\frac{k_{n+1}\cdot k_r}{\sigma_{qr}}\delta_r^{(1)}\sum_{\substack{i=1\\i\neq 2,3,q}}^{n}(-1)^{i+q+n+1}\frac{k_{n+1}\cdot k_i}{\sigma_{qi}}\det\left(\Psi'^{i}_{q+n+1}\right)$$

$$-\,2\sum_{\substack{q=4\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)(k_{n+1}\cdot k_q)\delta_q^{(1)}\sum_{\substack{j=1\\j\neq q}}^{n}(-1)^{q+j+n+1}\frac{\epsilon_{n+1}\cdot\epsilon_j}{(\sigma_{qj})^2}\det\left(\Psi'^{q}_{j+n+1}\right)$$

$$-\,2\sum_{\substack{q=2\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)(\epsilon_{n+1}\cdot\epsilon_q)\delta_q^{(1)}\sum_{\substack{i=1\\i\neq 2,3,q}}^{n}(-1)^{i+q+n+1}\frac{k_{n+1}\cdot k_i}{(\sigma_{qi})^2}\det\left(\Psi'^{i}_{q+n+1}\right)$$

$$-\,2\sum_{\substack{q=4\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot\epsilon_q)(k_{n+1}\cdot k_q)\delta_q^{(1)}\sum_{\substack{c=1\\c\neq q}}^{n}(-1)^{n+1}\frac{\epsilon_{n+1}\cdot k_c}{(\sigma_{qc})^2}\det\left(\Psi'^{q}_{q+n+1}\right)$$

$$-\,2\sum_{\substack{q=4\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot\epsilon_q)\delta_q^{(1)}\sum_{\substack{c=1\\c\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{qc}}\sum_{\substack{b=1\\b\neq q}}^{n}(-1)^{n+1}\frac{k_{n+1}\cdot k_b}{\sigma_{qb}}\det\left(\Psi'^{q}_{q+n+1}\right)$$

$$+\,2\sum_{\substack{q=1\\q\neq 2,3}}^{n}(\epsilon_{n+1}\cdot\epsilon_q)\sum_{\substack{c=1\\c\neq q}}^{n}\frac{\epsilon_{n+1}\cdot k_c}{\sigma_{qc}}\sum_{\substack{r=2\\r\neq k,q}}^{n-1}(-1)^{n+1}\frac{k_{n+1}\cdot k_r}{\sigma_{qr}}\delta_r^{(1)}\det\left(\Psi'^{q}_{q+n+1}\right)$$

$$+\,2\sum_{\substack{q=4\\q\neq k}}^{n-1}\frac{(\epsilon_{n+1}\cdot k_q)(\epsilon_{n+1}\cdot\epsilon_q)}{(k_{n+1}\cdot k_q)}\delta_q^{(1)}\left(\sum_{\substack{b=1\\b\neq q}}^{n}\frac{k_{n+1}\cdot k_b}{\sigma_{qb}}\right)^2(-1)^{n+1}\det\left(\Psi'^{q}_{q+n+1}\right)$$

$$+\,2\sum_{\substack{q=4\\q\neq k}}^{n-1}(\epsilon_{n+1}\cdot k_q)(\epsilon_{n+1}\cdot\epsilon_q)\delta_q^{(1)}\sum_{\substack{b=1\\b\neq q}}^{n}(-1)^{n+1}\frac{k_{n+1}\cdot k_b}{(\sigma_{qb})^2}\det\left(\Psi'^{q}_{q+n+1}\right)$$

$$-\,2\sum_{\substack{q=1\\q\neq 2,3}}^{n}\frac{(\epsilon_{n+1}\cdot k_q)(\epsilon_{n+1}\cdot\epsilon_q)}{(k_{n+1}\cdot k_q)}\sum_{\substack{b=1\\b\neq q}}^{n}\frac{k_{n+1}\cdot k_b}{\sigma_{qb}}\sum_{\substack{r=2\\r\neq k,q}}^{n-1}(-1)^{n+1}\frac{k_{n+1}\cdot k_r}{\sigma_{qr}}\delta_r^{(1)}\det\left(\Psi'^{q}_{q+n+1}\right)$$

$$-\,2\sum_{\substack{q=1\\q\neq 2,3}}^{n}(\epsilon_{n+1}\cdot k_q)(\epsilon_{n+1}\cdot\epsilon_q)\sum_{\substack{r=2\\r\neq k,q}}^{n-1}(-1)^{n+1}\frac{k_{n+1}\cdot k_r}{(\sigma_{qr})^2}\delta_r^{(1)}\det\left(\Psi'^{q}_{q+n+1}\right)$$

2.6 Residues of $\delta^0\det_2$

The following are all residues obtained from $\delta^0\det_2$ in (2.2.7) by integrating out the σ_{n+1} dependance. Multiplicative spectator terms and integrals which are trivially the same in the lower point construction are suppressed. Additionally, the product of scattering equation delta functions is suppressed.

With this the residues are:

$$\sum_{\substack{q=1\\q\neq 2,3}}^{n} (k_{n+1}\cdot k_q) \sum_{\substack{u=1\\u\neq 2,3,q}}^{n} \sum_{\substack{j=1\\j\neq 2,3,q}}^{n} (-1)^{u+j+\theta(u,q)+\theta(j,q)} \frac{\epsilon_{n+1}\cdot k_u}{\sigma_{qu}} \frac{\epsilon_{n+1}\cdot k_j}{\sigma_{qj}} \det\left(\Psi'^{q,u}_{q,j}\right)$$

$$+2\sum_{\substack{q=1\\q\neq 2,3}}^{n} (\epsilon_{n+1}\cdot k_q) \sum_{\substack{j=1\\j\neq 2,3,q}}^{n} \sum_{\substack{u=1\\u\neq 2,3,q}}^{n} (-1)^{j+u+\theta(u,q)+\theta(q,j)} \frac{k_{n+1}\cdot k_j}{\sigma_{qj}} \frac{\epsilon_{n+1}\cdot k_u}{\sigma_{qu}} \det\left(\Psi'^{q,u}_{j,q}\right)$$

$$+\sum_{\substack{q=1\\q\neq 2,3}}^{n} \frac{(\epsilon_{n+1}\cdot\epsilon_q)^2}{(k_{n+1}\cdot k_q)} \sum_{\substack{i=1\\i\neq 2,3,q}}^{n} \sum_{\substack{j=1\\j\neq 2,3,q}}^{n} (-1)^{i+j+\theta(q,i)+\theta(q,j)} \frac{k_{n+1}\cdot k_i}{\sigma_{qi}} \frac{k_{n+1}\cdot k_j}{\sigma_{qj}} \det\left(\Psi'^{i,q}_{j,q}\right)$$

$$-2\sum_{\substack{q=1\\q\neq 2,3}}^{n} (\epsilon_{n+1}\cdot\epsilon_q) \sum_{\substack{j=1\\j\neq 2,3,q}}^{n} \sum_{\substack{u=1\\u\neq 2,3,q}}^{n} (-1)^{j+u+n+1+\theta(u,q)+\theta(q+n+1,j)} \frac{k_{n+1}\cdot k_j}{\sigma_{qj}} \frac{\epsilon_{n+1}\cdot k_u}{\sigma_{qu}} \det\left(\Psi'^{q,u}_{j,q+n+1}\right)$$

$$-2\sum_{\substack{q=1\\q\neq 2,3}}^{n} (\epsilon_{n+1}\cdot\epsilon_q) \sum_{\substack{i=1\\i\neq 2,3,q}}^{n} \sum_{\substack{u=1\\u\neq 2,3,q,i}}^{n} (-1)^{i+u+n+1+\theta(u,i)+\theta(q+n+1,q)} \frac{k_{n+1}\cdot k_i}{\sigma_{qi}} \frac{\epsilon_{n+1}\cdot k_u}{\sigma_{qu}} \det\left(\Psi'^{i,u}_{q,q+n+1}\right)$$

$$+2\sum_{\substack{q=1\\q\neq 2,3}}^{n} (\epsilon_{n+1}\cdot\epsilon_q) \sum_{\substack{b=1\\b\neq q}}^{n} \frac{k_{n+1}\cdot k_b}{\sigma_{qb}} \sum_{\substack{u=1\\u\neq 2,3,q}}^{n} (-1)^{q+u+n+1+\theta(u,q)+\theta(q+n+1,q)} \frac{\epsilon_{n+1}\cdot k_u}{\sigma_{qu}} \det\left(\Psi'^{q,u}_{q,q+n+1}\right)$$

$$-2\sum_{\substack{q=1\\q\neq 2,3}}^{n} \frac{(\epsilon_{n+1}\cdot k_q)(\epsilon_{n+1}\cdot\epsilon_q)}{(k_{n+1}\cdot k_q)} \sum_{\substack{i=1\\i\neq 2,3,q}}^{n} \sum_{\substack{j=1\\j\neq 2,3,q}}^{n} (-1)^{i+j+n+1+\theta(q,i)+\theta(q+n+1,j)} \frac{k_{n+1}\cdot k_i}{\sigma_{qi}} \frac{k_{n+1}\cdot k_j}{\sigma_{qj}} \det\left(\Psi'^{i,q}_{j,q+n+1}\right)$$

$$-2\sum_{\substack{q=1\\q\neq 2,3}}^{n} (k_{n+1}\cdot k_q) \sum_{\substack{u=1\\u\neq 2,3,q}}^{n} \sum_{\substack{j=1\\j\neq q}}^{n} (-1)^{u+j+n+1+\theta(u,q)+\theta(j+n+1,q)} \frac{\epsilon_{n+1}\cdot k_u}{\sigma_{qu}} \frac{\epsilon_{n+1}\cdot \epsilon_j}{\sigma_{qj}} \det\left(\Psi'^{q,u}_{q,j+n+1}\right)$$

$$-2\sum_{\substack{q=1\\q\neq 2,3}}^{n} (\epsilon_{n+1}\cdot k_q) \sum_{\substack{i=1\\i\neq 2,3,q}}^{n} \sum_{\substack{j=1\\j\neq q}}^{n} (-1)^{i+j+n+1+\theta(q,i)+\theta(j+n+1,q)} \frac{k_{n+1}\cdot k_i}{\sigma_{qi}} \frac{\epsilon_{n+1}\cdot \epsilon_j}{\sigma_{qj}} \det\left(\Psi'^{i,q}_{q,j+n+1}\right)$$

$$+2\sum_{\substack{q=1\\q\neq 2,3}}^{n} \frac{(\epsilon_{n+1}\cdot k_q)(\epsilon_{n+1}\cdot\epsilon_q)}{(k_{n+1}\cdot k_q)} \sum_{\substack{b=1\\b\neq q}}^{n} \frac{k_{n+1}\cdot k_b}{\sigma_{qb}} \sum_{\substack{i=1\\i\neq 2,3,q}}^{n} (-1)^{i+q+n+1+\theta(q,i)+\theta(q+n+1,q)} \frac{k_{n+1}\cdot k_i}{\sigma_{qi}} \det\left(\Psi'^{i,q}_{q,q+n+1}\right)$$

$$+2\sum_{\substack{q=1\\q\neq 2,3}}^{n} (k_{n+1}\cdot k_q)(\epsilon_{n+1}\cdot\epsilon_q) \sum_{\substack{u=1\\u\neq 2,3,q}}^{n} (-1)^{q+u+n+1+\theta(u,q)+\theta(q+n+1,q)} \frac{\epsilon_{n+1}\cdot k_u}{(\sigma_{qu})^2} \det\left(\Psi'^{q,u}_{q,q+n+1}\right)$$

$$+2\sum_{\substack{q=1\\q\neq 2,3}}^{n} (\epsilon_{n+1}\cdot\epsilon_q)(\epsilon_{n+1}\cdot k_q) \sum_{\substack{i=1\\i\neq 2,3,q}}^{n} (-1)^{i+q+n+1+\theta(q,i)+\theta(q+n+1,q)} \frac{k_{n+1}\cdot k_i}{\sigma_{qi}} \det\left(\Psi'^{i,q}_{q,q+n+1}\right)$$

$$+\sum_{\substack{q=1\\q\neq 2,3}}^{n} (\epsilon_{n+1}\cdot\epsilon_q)^2 \sum_{\substack{b=1\\b\neq q}}^{n} \frac{k_{n+1}\cdot k_b}{(\sigma_{qb})^2} (-1)^{\theta(q+n+1,q)+\theta(q+n+1,q)} \det\left(\Psi'^{q,q+n+1}_{q,q+n+1}\right)$$

$$+\sum_{\substack{q=1\\q\neq 2,3}}^{n} \frac{(\epsilon_{n+1}\cdot\epsilon_q)^2}{(k_{n+1}\cdot k_q)} \left(\sum_{\substack{b=1\\b\neq q}}^{n} \frac{k_{n+1}\cdot k_b}{\sigma_{qb}}\right)^2 (-1)^{\theta(q+n+1,q)+\theta(q+n+1,q)} \det\left(\Psi'^{q,q+n+1}_{q,q+n+1}\right)$$

$$
\begin{aligned}
&- 2 \sum_{\substack{q=1\\q\neq 2,3}}^{n} \frac{(\epsilon_{n+1}\cdot\epsilon_q)^2}{(k_{n+1}\cdot k_q)} \sum_{\substack{b=1\\b\neq q}}^{n} \sum_{\substack{i=1\\i\neq 2,3,q}}^{n} (-1)^{i+q+\theta(q+n+1,i)+\theta(q+n+1,q)} \frac{k_{n+1}\cdot k_i}{\sigma_{qi}} \det\left(\Psi'^{i,q+n+1}_{q,q+n+1}\right)\\
&- 2 \sum_{\substack{q=1\\q\neq 2,3}}^{n} (\epsilon_{n+1}\cdot\epsilon_q) \sum_{\substack{b=1\\b\neq q}}^{n} \frac{k_{n+1}\cdot k_b}{\sigma_{qb}} \sum_{\substack{u=1\\u\neq q}}^{n} (-1)^{u+q+\theta(u+n+1,q)+\theta(q+n+1,q)} \frac{\epsilon_{n+1}\cdot\epsilon_u}{\sigma_{qu}} \det\left(\Psi'^{q,u+n+1}_{q,q+n+1}\right)\\
&- 2 \sum_{\substack{q=1\\q\neq 2,3}}^{n} (\epsilon_{n+1}\cdot\epsilon_q)^2 \sum_{\substack{i=1\\i\neq 2,3,q}}^{n} (-1)^{i+q+\theta(q+n+1,i)+\theta(q+n+1,q)} \frac{k_{n+1}\cdot k_i}{(\sigma_{qi})^2} \det\left(\Psi'^{i,q+n+1}_{q,q+n+1}\right)\\
&- 2 \sum_{\substack{q=1\\q\neq 2,3}}^{n} (k_{n+1}\cdot k_q)(\epsilon_{n+1}\cdot\epsilon_q) \sum_{\substack{u=1\\u\neq q}}^{n} (-1)^{u+q+\theta(u+n+1,q)+\theta(q+n+1,q)} \frac{\epsilon_{n+1}\cdot\epsilon_u}{(\sigma_{qu})^2} \det\left(\Psi'^{q,u+n+1}_{q,q+n+1}\right)\\
&+ 2 \sum_{\substack{q=1\\q\neq 2,3}}^{n} (\epsilon_{n+1}\cdot\epsilon_q) \sum_{\substack{i=1\\i\neq 2,3,q}}^{n} \sum_{\substack{u=1\\u\neq q}}^{n} (-1)^{i+u+\theta(u+n+1,i)+\theta(q+n+1,q)} \frac{k_{n+1}\cdot k_i}{\sigma_{qi}} \frac{\epsilon_{n+1}\cdot\epsilon_u}{\sigma_{qu}} \det\left(\Psi'^{q,q+n+1}_{i,u+n+1}\right)\\
&+ 2 \sum_{\substack{q=1\\q\neq 2,3}}^{n} (\epsilon_{n+1}\cdot\epsilon_q) \sum_{\substack{i=1\\i\neq 2,3,q}}^{n} \sum_{\substack{u=1\\u\neq q}}^{n} (-1)^{i+u+\theta(u+n+1,i)+\theta(q+n+1,q)} \frac{k_{n+1}\cdot k_i}{\sigma_{qi}} \frac{\epsilon_{n+1}\cdot\epsilon_u}{\sigma_{qu}} \det\left(\Psi'^{i,q+n+1}_{q,u+n+1}\right)\\
&+ \sum_{\substack{q=1\\q\neq 2,3}}^{n} (k_{n+1}\cdot k_q) \sum_{\substack{u=1\\u\neq q}}^{n} \sum_{\substack{j=1\\j\neq q}}^{n} (-1)^{u+j+\theta(u+n+1,q)+\theta(j+n+1,q)} \frac{\epsilon_{n+1}\cdot\epsilon_u}{\sigma_{qu}} \frac{\epsilon_{n+1}\cdot\epsilon_j}{\sigma_{qj}} \det\left(\Psi'^{q,u+n+1}_{q,j+n+1}\right)\\
&+ \sum_{q=1}^{n} \frac{(\epsilon_{n+1}\cdot\epsilon_q)^2}{(k_{n+1}\cdot k_q)} \sum_{\substack{i=1\\i\neq 2,3,q}}^{n} \sum_{\substack{j=1\\j\neq 2,3,q}}^{n} (-1)^{i+j+\theta(q+n+1,i)+\theta(q+n+1,j)} \frac{k_{n+1}\cdot k_i}{\sigma_{qi}} \frac{k_{n+1}\cdot k_j}{\sigma_{qj}} \det\left(\Psi'^{i,q+n+1}_{j,q+n+1}\right)
\end{aligned}
$$

Note

C. Kalousios and F. Rojas published similar results in [52].

Chapter 3

Double Soft Theorems in Gauge and String Theories

This chapter is based on the publication [53].

Recently there has been a resurrection of interest in studying various low energy limits of scattering amplitudes. Of particular interest are situations which exhibit universal behavior; that is, when the limiting behavior of an amplitude factors into a product of a universal "soft factor" times a lower-point amplitude independent of the soft particles. Such cases are called "soft theorems", the most famous of which may be Weinberg's classic soft (photon, gluon, or graviton) theorems [22]. Other theorems include [20, 23, 26, 54, 24, 25] as well as, much more recently, the subleading and sub-subleading graviton theorems of Cachazo and Strominger [30] (see [40, 41, 38, 44, 50, 39, 46, 55, 56, 57, 58, 59, 57, 60, 61]for further developments and applications).

Strominger and collaborators [62, 63, 64, 65, 48, 47, 29, 28, 27] have argued that all of the known soft and subleading soft theorems may be understood as consequences of large gauge transformations. That is, transformations which fall off sufficiently rapidly at infinity such that they must be considered consistent with the asymptotic boundary conditions defining the theory, while sufficiently slowly that they act nontrivially on asymptotic scattering states. In the case of gravity, the relevant "gauge transformations" are of course diffeomorphisms, and the relevant asymptotic symmetry group (in four-dimensional Minkoswki space) is the Bondi, van der Burg, Metzner, Sachs group [31, 32, 33, 66, 34]. It has been shown using the CHY scattering equations [5] that the subleading and sub-subleading graviton soft theorems hold for tree-level graviton amplitudes in any number of space-time dimensions, suggesting that an analog of the BMS symmetry should be relevant more generally [37, 36, 16, 52, 62, 63, 64, 65, 48, 47, 29, 28, 27]. Perturbative theories at null infinity realizing these symmetries have been proposed in [67, 49, 68, 69]. The issues regarding possible loop corrections to the subleading soft theorems were studied in [42, 43, 44, 51].

Double-soft limits (where two particles are taken to have very low energy) have also received a lot of attention in the literature, both in the earlier works [70, 71] and more recently. For example, Arkani-Hamed et. al. [36] have shown that the double soft limit of scalars in $\mathcal{N} = 8$ supergravity exhibit the expected $E_{7(7)}$ symmetry of the scalar moduli space, in a manner analogous to the classic soft-pion theorem of [72, 71]. This result was recently extended to the four-dimensional supergravity theories with $\mathcal{N} < 8$ supersymmetry, and the $\mathcal{N} = 16$ supergravity in three dimensions in [73]. Furthermore, supergravity amplitudes in both four and three dimensions with two soft fermions were studied in [74], and new soft theorems were proposed. New double-soft leading and subleading theorems for scalars (and leading for photons) were also studied in various theories such as DBI, Einstein-Maxwell-scalar, NLSM, and Yang-Mills-scalar in [75]. Kac-Moody structure has been found for the four dimensional Yang-Mills at null infinity [62], where double soft limits play another important role.

In this chapter we derive several new soft theorems for tree-level scattering amplitudes in gauge and string theories with more than one soft particle. We derive the universal behavior of amplitudes with two or three soft gluons. It is known that when the soft gluons have identical helicities, the result can be obtained simply by setting the gluons to be soft one by one, thus we focus on the non-trivial cases when the soft gluons have different helicities. Indeed we find that for these cases the soft factors are a product of the individual soft-gluon factors with certain non-trivial corrections. We first derive theorems from the BCFW formula in four dimensions [4, 3], and further extend our results with double-soft gluons for gauge theories in any number of dimensions by using CHY formula [5]. We check that our results are consistent with the fact that if the soft limit is taken in order then the soft factors reduce to a product of the single-soft factors given by Weinberg. We also note that, in contrast to the gluon case, amplitudes with multiple soft gravitons can always be obtained by simply taking the gravitons to be soft one by one.

We then proceed to study amplitudes in $\mathcal{N} = 4$ and pure $\mathcal{N} = 2$ Super Yang-Mills theory (SYM) with two soft scalars or two soft fermions. We find that the double soft behavior is governed by R-symmetry generators acting on a lower-point amplitude, resembling the results of supergravity theories found in [36, 73, 74], although the vacuum structure of SYM is quite different from that of supergravity theories. Finally, we consider double-soft scalars in the open superstring theory. Unlike the double-soft-scalar theorem in $\mathcal{N} = 8$ supergravity, which would receive α' corrections if one tried to extend it to closed superstring theory, we find that open superstring amplitudes satisfy exactly the same double-soft-scalar theorem of SYM at $\alpha' = 0$. Given the similarity of the double-soft theorems of SYM and those of supergravity theories, it would be very interesting to understand if any of these theorems could have an interpretation as hidden symmetries.

This chapter is organized as follows: In section 3.1 we derive the double-soft-gluon theorem for tree-level amplitudes in gauge theories using the BCFW recursion relations formula (3.1.10), which may be recast into a different form (3.1.12), and we further extend the results to arbitrary dimensions resulting in formula (3.1.28). Then, in section 3.2 amplitudes with three soft gluons are considered. In the following section 3.3 we comment on multi-soft gravitons. Subsequently, in section 3.4, we explore the universal behavior of amplitudes with two soft scalars or two soft fermions in supersymmetric gauge theories, including $\mathcal{N} = 4$ SYM as well as pure $\mathcal{N} = 2$ SYM, with main results given by (3.4.13) and (3.4.28). Finally, in section 3.5, we prove that the newly discovered double-soft-scalar theorem in SYM can be extended to the open superstring theory without any α' corrections.

Note added: After finishing this work, we became aware of a related work by Klose, McLoughlin, Nandan, Plefka and Travaglini, which has some overlap with this chapter [76].

3.1 Double-soft gluons

3.1.1 Double-soft gluons from BCFW recursions

We start by considering color-stripped amplitudes in gauge theories with two adjacent gluons taken to be soft. It is straightforward to see that if the two gluons have the same helicity, then the two gluons may be taken soft one at a time. Moreover it is evident from

$$\begin{aligned}
\lim_{p_2\to 0}\lim_{p_1\to 0} A(1^+,2^+,3,\ldots,n) &\to \frac{\langle n2\rangle}{\langle n1\rangle\langle 12\rangle}\frac{\langle n3\rangle}{\langle n2\rangle\langle 23\rangle}A(3,\ldots,n) \\
\lim_{p_1\to 0}\lim_{p_2\to 0} A(1^+,2^+,3,\ldots,n) &\to \frac{\langle 13\rangle}{\langle 12\rangle\langle 23\rangle}\frac{\langle n3\rangle}{\langle n1\rangle\langle 13\rangle}A(3,\ldots,n)
\end{aligned} \tag{3.1.1}$$

that the result is independent of the order in which the two gluons are taken soft.

A similar simple calculation shows that if the two gluons have different helicities, then the result cannot be given by a product of two single soft factors obtained by taking the gluons to be soft one by one. Therefore this is the non-trivial case we are interested in, namely we would like to study the amplitude $A(1^+, 2^-, 3, \ldots, n)$ in the double-soft limit

$$p_{1,2} \to \tau p_{1,2} \qquad \text{with} \qquad \tau \to 0. \tag{3.1.2}$$

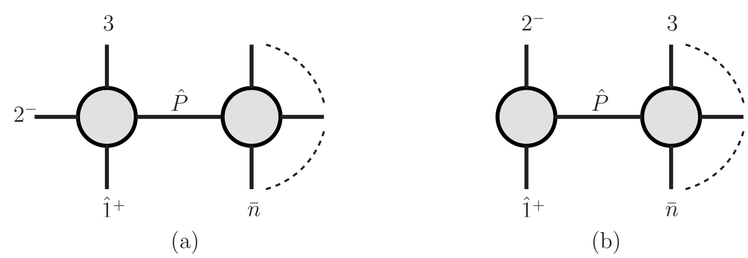
We will use the standard spinor-helicity formalism for the four-dimensional massless particles throughout this chapter:

$$p_{\alpha,\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}, \quad \langle ij\rangle = \epsilon_{\alpha\beta}\lambda_i^\alpha \lambda_j^\beta, \quad [ij] = \epsilon_{\dot\alpha\dot\beta}\tilde\lambda_i^{\dot\alpha}\tilde\lambda_j^{\dot\beta} \tag{3.1.3}$$

and realize the soft limit by taking

$$\lambda_{1,2} \to \sqrt{\tau}\lambda_{1,2} \quad \text{and} \quad \tilde\lambda_{1,2} \to \sqrt{\tau}\tilde\lambda_{1,2}. \tag{3.1.4}$$

Using the BCFW recursion relations [4, 3], it is straightforward to see that the two dominant diagrams that contribute in the limit at hand are



(3.1.5)

with the following BCFW shifts

$$\lambda_{\hat 1} = \lambda_1 + z\lambda_n\,, \quad \tilde\lambda_{\bar n} = \tilde\lambda_n - z\tilde\lambda_1\,. \tag{3.1.6}$$

Let us now analyse the two contributions separately. First for diagram (a) we have

$$A_{(\mathrm{a})} = \frac{[13]^4}{[12][23][3\hat P][\hat P 1]s_{123}} A_{n-2} \to \frac{[13]^3\langle n3\rangle}{[12][23]\langle n|1+2|3]s_{123}} A_{n-2}\,, \tag{3.1.7}$$

where we have used the fact that $\hat P \to p_3$ in the limit; hence this result is independent of whether particle 3 has positive or negative helicity (in the above calculation we have chosen it to be positive). Now, the second diagram (b) gives

$$A_{(\mathrm{b})} = \frac{[\hat P 1]^3}{[12][2\hat P]s_{12}} A_{n-1}(\hat P, 3, \ldots, \bar n) \to \frac{[\hat P 1]^3}{[12][2\hat P]s_{12}} \frac{[n3]}{[n\hat P][\hat P 3]} A_{n-2}(3, \ldots, n)\,, \tag{3.1.8}$$

where in the second expression we have used the fact that $p_{\bar n} = p_n$ in the limit, and we also applied the single-soft theorem for the soft leg $\hat{P}$. After some simplification, we find

$$A_{(\mathrm{b})} \to \frac{\langle n2\rangle^3[n3]}{\langle n1\rangle\langle 12\rangle\langle n|1+2|3]s_{n12}}A_{n-2}\,. \tag{3.1.9}$$

Adding the contributions from the two diagrams together, we obtain the final result for two soft gluons having different helicities,

$$\lim_{p_1\sim p_2\to 0} A(1^+,2^-,3,\ldots,n) \to \frac{1}{\langle n|1+2|3]}\left(\frac{[13]^3\langle n3\rangle}{[12][23]s_{123}} + \frac{\langle n2\rangle^3[n3]}{\langle n1\rangle\langle 12\rangle s_{n12}}\right)A_{n-2}\,. \tag{3.1.10}$$

As typical for amplitudes computed from the BCFW recursion relations, the result contains a spurious pole $\frac{1}{\langle n|1+2|3]}$. We will show that it indeed cancels out between the two terms at leading order of the soft limit, as should be the case. Now, if on the other hand we take the soft limit in succession, namely say take p_1 to be soft first, then the first term in the soft factor is subleading, and the second term simplifies to

$$\frac{1}{\langle n|1+2|3]}\left(\frac{[13]^3\langle n3\rangle}{[12][23]s_{123}} + \frac{\langle n2\rangle^3[n3]}{\langle n1\rangle\langle 12\rangle s_{n12}}\right) \to 0 + \frac{\langle n2\rangle}{\langle n1\rangle\langle 12\rangle}\frac{[n3]}{[32][2n]}\,, \tag{3.1.11}$$

which is precisely the product of two soft factors of a positive gluon and a negative gluon, with the positive gluon p_1 being taken soft first.

Although the above result (3.1.10) is very compact and nicely reduces to a product of two soft factors, if we take the soft limits in succession, it is specific to four dimensions and as a natural property of using the BCFW recursion it contains a spurious pole. In the next section we will use the CHY formula for pure Yang-Mills tree level scattering amplitudes [5] to derive a further formula for the universal double-soft-gluon factor. This result will be valid in any dimension, for any helicity combination of the soft gluons, and it will be manifestly free of unphysical poles. When the two soft gluons have opposite helicity, the comparison of the result obtained from the CHY formula and (3.1.10) will yield agreement and provide us with the intuition to recast the above into the following equivalent form

$$\lim_{p_1\sim p_2\to 0} A(1^+,2^-,3,\ldots,n) \to \frac{\langle n2\rangle}{\langle n1\rangle\langle 12\rangle}\frac{[13]}{[12][23]}\left(1 + \frac{\langle n1\rangle[13]\langle 32\rangle}{s_{123}\langle n2\rangle} + \frac{[1n]\langle n2\rangle[23]}{s_{n12}[13]}\right)A_{n-2}\,. \tag{3.1.12}$$

Therefore, we see that the alternating helicity double soft gluon factor is composed of the product of two single soft gluon factors plus a non-trivial correction.

3.1.2 Double-soft gluons from CHY

As we mentioned earlier, in this section we will reconsider the double-soft-gluon limit making use of the CHY formula for tree-level scattering amplitudes in pure Yang-Mills, valid in arbitrary dimensions [5]. The CHY formula for an n-point gluon scattering amplitude is given by

$$\mathcal{A}_n = \int \left(\prod_{\substack{c=1 \\ c\neq p,q,r}}^{n} d\sigma_c \right) \frac{(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki})}{\sigma_{12}\sigma_{23}...\sigma_{n,1}} \left(\prod_{\substack{a=1 \\ a\neq i,j,k}}^{n} \delta\left(f_a\right) \right) \frac{2(-1)^{m+w}}{\sigma_{mw}} \text{Pf}\left(\Psi^{m,w}_{m,w}\right),$$

where $\sigma_{ij} \equiv (\sigma_i - \sigma_j)$ and $f_a = \sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{ab}}$. Upper and lower indices on the matrix Ψ denote removed columns and rows respectively. The indices p, q, r, i, j, k, m and w can be fixed arbitrarily without changing the result. The $2n \times 2n$ dimensional matrix Ψ is given by

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix},$$

where the $n \times n$ dimensional sub-matrices are

$$A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}} \,, & a \neq b \\ 0 \,, & a = b \end{cases} \quad , \quad B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}} \,, & a \neq b \\ 0 \,, & a = b \end{cases} \quad , \quad C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{ab}} \,, & a \neq b \\ -\sum_{\substack{c=1 \\ c\neq a}}^{n} \frac{\epsilon_a \cdot k_c}{\sigma_{ac}} \,, & a = b \end{cases} .$$

Here k_a^μ are external leg momenta, and the ϵ_a^μ are corresponding polarization vectors. The product of delta functions enforces the scattering equations and saturates all integrals. With this the integration reduces to a sum over all solutions to the scattering equations.

We want to make the external gluon momenta k_1^μ and k_2^ν soft by substituting $k_1^\mu \to \tau k_1^\mu$ and $k_2^\nu \to \tau k_2^\nu$ and considering $\tau \to 0$. It is essential to send both momenta to zero simultaneously in order to capture the double soft factor structure. We choose not to erase indices (1) and (2). With this we have to isolate the extra terms in $\mathcal{A}_n$ as compared to $\mathcal{A}_{n-2}$ and integrate out σ_1 and σ_2. While doing so we will only keep the leading contribution in the $\tau \to 0$ limit, to obtain the leading double soft gluon factor.

First we notice that at leading order in τ the entire σ_1 and σ_2 dependence in $\mathcal{A}_n$ apart from the pfaffian $\text{Pf}\left(\Psi^{m,w}_{m,w}\right)$ is contained in

$$\int d\sigma_1 d\sigma_2 \frac{\sigma_{n,3}}{\sigma_{n,1}\sigma_{1,2}\sigma_{2,3}} \delta(f_1)\delta(f_2). \tag{3.1.13}$$

Another $\sigma_{n,3}$ term in the denominator is suppressed, which will help restore the proper Parke-Taylor factor for the $(n-2)$-point amplitude case. As in [75], we can make the convenient variable transformation

$$\sigma_1 = \rho - \xi/2, \quad \sigma_2 = \rho + \xi/2,$$
$$d\sigma_1 d\sigma_2 \delta(f_1)\delta(f_2) = -2 d\rho d\xi \delta(f_1 + f_2)\delta(f_1 - f_2), \tag{3.1.14}$$

and immediately integrate out $\delta(f_1 - f_2)$ using the variable ξ. This will introduce a summation over all solutions ξ for the equation $f_1 - f_2 = 0$, and an overall factor of $1/F(\xi)$, where

$$F(\xi) = \frac{d}{d\xi}(f_1 - f_2) \tag{3.1.15}$$
$$= \frac{1}{2}\frac{1}{k_1 \cdot k_2}\left(\sum_{b=3}^{n}\left(\frac{k_1 \cdot k_b}{\rho - \frac{\xi}{2} - \sigma_b} - \frac{k_2 \cdot k_b}{\rho + \frac{\xi}{2} - \sigma_b}\right)\right)^2 + \frac{1}{2}\sum_{c=3}^{n}\left(\frac{\tau k_1 \cdot k_c}{\left(\rho - \frac{\xi}{2} - \sigma_c\right)^2} + \frac{\tau k_2 \cdot k_c}{\left(\rho + \frac{\xi}{2} - \sigma_c\right)^2}\right).$$

Here we used that on the support of $f_1 - f_2 = 0$ we can always substitute

$$\xi = \tau \frac{2k_1 \cdot k_2}{\sum_{b=3}^{n}\left(\frac{k_1 \cdot k_b}{\rho - \frac{\xi}{2} - \sigma_b} - \frac{k_2 \cdot k_b}{\rho + \frac{\xi}{2} - \sigma_b}\right)}. \tag{3.1.16}$$

Making use of this, (3.1.13) becomes

$$\sum_{\text{solutions}\,\xi} \int d\rho \frac{\delta(f_1 + f_2)}{\tau(k_1 \cdot k_2)F(\xi)} \sum_{b=3}^{n}\left(\frac{k_1 \cdot k_b}{\rho - \frac{\xi}{2} - \sigma_b} - \frac{k_2 \cdot k_b}{\rho + \frac{\xi}{2} - \sigma_b}\right)\frac{\sigma_{n,3}}{\sigma_{n,1}\sigma_{2,3}}. \tag{3.1.17}$$

Before we rewrite $\int d\rho\delta(f_1 + f_2)$ as a contour integral over poles and deform the contour as usual, we should also extract the extra terms depending on ρ and ξ from the pfaffian factor $\mathrm{Pf}\left(\Psi^{m,w}_{m,w}\right)$ in order to reduce it to the $(n-2)$-point amplitude case. To do that, we will use the recursive definition of a pfaffian for an anti-symmetric $2n \times 2n$ matrix A:

$$\mathrm{Pf}\left(A\right) = \sum_{\substack{j=1 \\ j \neq i}}^{2n} (-1)^{i+j+1+\theta(i-j)} a_{ij} \mathrm{Pf}\left(A^{ij}_{ij}\right), \tag{3.1.18}$$

where a_{ij} is an element of matrix A, $\theta(x)$ is the Heaviside step function, and index i can be chosen arbitrarily. If rows and/or columns are missing from matrix A before the expansion is applied, the respective indices have to be skipped in the summation. Since we are ultimately interested in the leading double soft gluon factor, for convenience we will only keep the leading in τ terms in the expansion of $\mathrm{Pf}\left(\Psi^{m,w}_{m,w}\right)$. In order to isolate the leading terms, we recall that the summation over the solutions ξ in (3.1.17) features two types of solutions: non-degenerate solutions for which $\xi = O(1)$, and a unique degenerate solution for which $\xi = O(\tau)$ [75].

Let us first consider the non-degenerate (nd) solutions. The (nd) case is non-trivial, since equation (3.1.16) seemingly has to be solved in ξ for the full non-linear constraint imposed by the scattering equations involved, yet polynomial roots can be obtained in closed form for low degree polynomials only. It is possible to derive non-degenerate solution contributions in this case employing a somewhat cumbersome procedure. However, this will not be required in the following and will be addressed in more generality in a future work. Instead, investigation of the soft factor integrand reveals that the necessity of non-degenerate solutions computation can be avoided here at the expense of fixing a particular polarization gauge for the two gluons going soft.[1] This argument works as follows.
In the (nd) case it is straightforward to see that the only leading term in the pfaffian expansion is given by

$$\begin{aligned}\mathrm{Pf}\left(\Psi^{m,w}_{m,w}\right)_{(\mathrm{nd})} &= -C_{1,1}C_{2,2}\mathrm{Pf}\left(\Psi^{m,w,1,2,n+1,n+2}_{m,w,1,2,n+1,n+2}\right) + O(\tau)\\ &= -\sum_{b=3}^{n}\frac{\epsilon_1\cdot k_b}{\rho-\frac{\xi}{2}-\sigma_b}\sum_{c=3}^{n}\frac{\epsilon_2\cdot k_c}{\rho+\frac{\xi}{2}-\sigma_c}\mathrm{Pf}\left(\Psi'^{m,w}_{m,w}\right)+O(\tau),\end{aligned} \tag{3.1.19}$$

where for convenience we define the abbreviation

$$\mathrm{Pf}\left(\Psi'^{m,w}_{m,w}\right) \equiv \mathrm{Pf}\left(\Psi^{m,w,1,2,n+1,n+2}_{m,w,1,2,n+1,n+2}\right). \tag{3.1.20}$$

Combining (3.1.17) with (3.1.19), writing $\int d\rho\delta(f_1+f_2)$ as a contour integral

$$\int d\rho\delta(f_1+f_2) \to \oint \frac{d\rho}{2\pi i}\frac{1}{f_1+f_2}, \tag{3.1.21}$$

and deforming the contour to wrap around all other poles in ρ instead, immediately reveals that there is no pole at infinity and the only residues come from poles at $(\rho+\xi/2-\sigma_3)\to 0$ and/or $(\rho-\xi/2-\sigma_n)\to 0$ due to the term $\sigma_{n,3}/(\sigma_{n,1}\sigma_{2,3})$ remaining from the Parke-Taylor factor. Keeping (3.1.19) in mind, this tells us that for any of the non-degenerate solutions $\xi_{(nd)}$, at leading order in τ these residues will always be proportional to $\epsilon_2\cdot k_3$ and/or $\epsilon_1\cdot k_n$. Therefore, we select the following polarization gauge for the external legs going soft

$$\epsilon_2\cdot k_3 = 0 \qquad , \qquad \epsilon_1\cdot k_n = 0. \tag{3.1.22}$$

In this gauge all the non-degenerate solution contributions to the leading double soft gluon factor

[1]The lost gauge invariance in the final result is recovered once we convert it to spinor helicity formalism.

vanish, such that we can concentrate on the degenerate solution only.

Now we compute the degenerate (d) solution contribution. Using (3.1.16) we can straightforwardly expand the degenerate solution $\xi_{(d)}$ to leading order

$$\xi_{(d)} = \tau \frac{2k_1 \cdot k_2}{\sum_{b=3}^{n} \frac{(k_1-k_2)\cdot k_b}{\rho-\sigma_b}} + O(\tau^2). \tag{3.1.23}$$

All the terms appearing in (3.1.17) are expanded to leading order in τ analogously. The expansion of the pfaffian features three leading terms in this case:

$$\begin{aligned} \mathrm{Pf}\left(\Psi^{m,w}_{m,w}\right)_{(\mathrm{d})} &= \left(B_{1,2}A_{1,2} + C_{1,2}C_{2,1} - C_{1,1}C_{2,2}\right)\mathrm{Pf}\left(\Psi'^{m,w}_{m,w}\right) + O(\tau) \\ &= \frac{1}{4}\left[\left(\frac{\epsilon_1\cdot\epsilon_2}{k_1\cdot k_2} - \frac{(\epsilon_2\cdot k_1)(\epsilon_1\cdot k_2)}{(k_1\cdot k_2)^2}\right)S^2 + \right. \\ &\quad \left. + \left(\frac{\epsilon_1\cdot k_2}{k_1\cdot k_2}S - 2\sum_{i=3}^{n}\frac{\epsilon_1\cdot k_i}{\rho-\sigma_i}\right)\left(\frac{\epsilon_2\cdot k_1}{k_1\cdot k_2}S + 2\sum_{j=3}^{n}\frac{\epsilon_2\cdot k_j}{\rho-\sigma_j}\right)\right]\mathrm{Pf}\left(\Psi'^{m,w}_{m,w}\right) + \\ &\quad + O(\tau), \end{aligned} \tag{3.1.24}$$

where $S = \sum_{b=3}^{n} \frac{(k_1-k_2)\cdot k_b}{\rho-\sigma_b}$, and we used the abbreviation (3.1.20). Again, we combine (3.1.17) with (3.1.24), write $\int d\rho\delta(f_1+f_2)$ as a contour integral (3.1.21) and deform the contour to wrap around all other poles in ρ instead. Analogously to the non-degenerate case we see that there is no pole at infinity, and the only two contributing residues come from poles at $\rho-\sigma_3 \to 0$ and $\rho-\sigma_n \to 0$. Dropping $\mathrm{Pf}\left(\Psi'^{m,w}_{m,w}\right)$, which is part of the $(n-2)$-point amplitude and not the double soft gluon factor, both these residues are of the following type at leading order in τ:

$$\begin{aligned} R^q_{i,i+1} =& \frac{1}{2}\frac{(k_i-k_{i+1})\cdot k_q}{(k_i+k_{i+1})\cdot k_q}\left[\frac{\epsilon_i\cdot\epsilon_{i+1}}{k_i\cdot k_{i+1}} - \frac{(\epsilon_{i+1}\cdot k_i)(\epsilon_i\cdot k_{i+1})}{(k_i\cdot k_{i+1})^2} + \right. \\ & \left. + \left(\frac{\epsilon_i\cdot k_{i+1}}{k_i\cdot k_{i+1}} - \frac{2\epsilon_i\cdot k_q}{(k_i-k_{i+1})\cdot k_q}\right)\left(\frac{\epsilon_{i+1}\cdot k_i}{k_i\cdot k_{i+1}} + \frac{2\epsilon_{i+1}\cdot k_q}{(k_i-k_{i+1})\cdot k_q}\right)\right]. \end{aligned} \tag{3.1.25}$$

With this we conclude that the leading double soft gluon factor for legs i and $i+1$ going soft is given by

$$\begin{aligned} S^{(0)}_{i,i+1} =& (R^{i+2}_{i,i+1} - R^{i-1}_{i,i+1}) \\ =& \frac{1}{(k_i+k_{i+1})\cdot k_{i+2}}\left(\frac{1}{2}\frac{\epsilon_i\cdot\epsilon_{i+1}}{k_i\cdot k_{i+1}}(k_i-k_{i+1})\cdot k_{i+2} - \frac{\epsilon_{i+1}\cdot k_i}{k_i\cdot k_{i+1}}\epsilon_i\cdot k_{i+2}\right) \\ & - \frac{1}{(k_i+k_{i+1})\cdot k_{i-1}}\left(\frac{1}{2}\frac{\epsilon_i\cdot\epsilon_{i+1}}{k_i\cdot k_{i+1}}(k_i-k_{i+1})\cdot k_{i-1} + \frac{\epsilon_i\cdot k_{i+1}}{k_i\cdot k_{i+1}}\epsilon_{i+1}\cdot k_{i-1}\right), \end{aligned} \tag{3.1.26}$$

valid in the polarization gauge

$$\epsilon_i \cdot k_{i-1} = 0 \quad , \quad \epsilon_{i+1} \cdot k_{i+2} = 0. \tag{3.1.27}$$

Despite its first glance appearance, the double-soft factor (3.1.26) is not manifestly anti-symmetric under $i+2 \leftrightarrow i-1$, since this symmetry is broken by the gauge choice (3.1.27). This is consistent with the results of the previous section.

Therefore, the particular computation above for legs 1 and 2 going soft in an n-point amplitude gives the following factorization in the double soft gluon limit

$$\begin{aligned}\lim_{p_1 \sim p_2 \to 0} \mathcal{A}_n \to & S^{(0)}_{1,2} \mathcal{A}_{n-2} \\ = & \left[\frac{1}{(k_1+k_2)\cdot k_3} \left(\frac{1}{2} \frac{\epsilon_1 \cdot \epsilon_2}{k_1 \cdot k_2} (k_1 - k_2) \cdot k_3 - \frac{\epsilon_2 \cdot k_1}{k_1 \cdot k_2} \epsilon_1 \cdot k_3 \right) \right. \\ & \left. - \frac{1}{(k_1+k_2)\cdot k_n} \left(\frac{1}{2} \frac{\epsilon_1 \cdot \epsilon_2}{k_1 \cdot k_2} (k_1 - k_2) \cdot k_n + \frac{\epsilon_1 \cdot k_2}{k_1 \cdot k_2} \epsilon_2 \cdot k_n \right) \right] \mathcal{A}_{n-2},\end{aligned} \tag{3.1.28}$$

valid in the gauge (3.1.22). Here we emphasize again that since the above result is obtained from the CHY formula, it features only physical poles, it holds in arbitrary dimension and for all helicity combinations of the two soft gluons.

Let us now compare (3.1.28) to the result (3.1.10) obtained from BCFW. Specifying to four dimensions and selecting $(1^+, 2^-)$ helicities for the soft gluons, we use the following standard dictionary to translate $R^3_{1,2}$ and $R^n_{1,2}$ into spinor helicity formalism:

$$k_i \cdot k_j = \frac{1}{2} \langle ij \rangle [ji] \quad , \quad \epsilon_1^+ \cdot k_i = \frac{[1i]\langle in \rangle}{\sqrt{2}\langle n1 \rangle} \quad , \quad \epsilon_2^- \cdot k_i = \frac{\langle 2i \rangle [i3]}{\sqrt{2}[23]} \quad , \quad \epsilon_1^+ \cdot \epsilon_2^- = \frac{\langle 2n \rangle [13]}{[23]\langle n1 \rangle}. \tag{3.1.29}$$

Here we have selected proper reference spinors to account for the gauge (3.1.22).[2] Anticipating that $R^3_{1,2}$ and $R^n_{1,2}$ roughly correspond to the two terms that are summed in (3.1.10), we notice that $R^3_{1,2}$ already features an $s_{123} \approx 2k_3 \cdot (k_1 + k_2)$ and $R^n_{1,2}$ an $s_{n12} \approx 2k_n \cdot (k_1 + k_2)$ in the denominator. So in both cases we introduce an extra factor of $\langle n|1+2|3]$ in numerator and denominator, and expand the numerators. The Schouten identity then yields a slight simplification such that the terms in the numerators separate into an expected part and a part proportional to s_{123} or s_{n12} in the two cases respectively. Finally, subtracting the resulting $R^n_{1,2}$ from $R^3_{1,2}$ displays some cancellation and we are

[2] Note that the specific choice of reference spinors merely facilitates the proper conversion of the result (3.1.28) to spinor helicity formalism. Once the conversion is done, full gauge invariance is recovered for the final result in spinor helicity language i.e. (3.1.12).

left with exactly the terms appearing in (3.1.10).[3]

Similarly, we can show that selecting the soft gluons to be of the same helicity, i.e. $(1^+, 2^+)$, the double soft gluon factor (3.1.28) reduces to the product of two single soft factors. Here we also use:

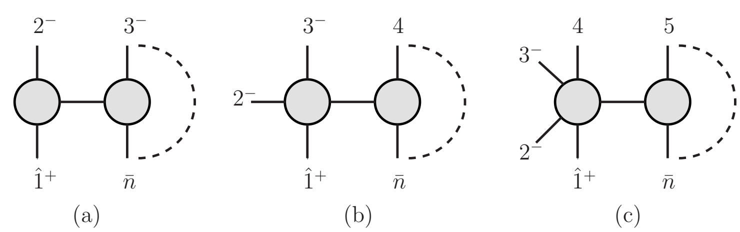
$$\epsilon_2^+ \cdot k_i = \frac{[2i]\langle i3\rangle}{\sqrt{2}\langle 32\rangle} \quad , \quad \epsilon_1^+ \cdot \epsilon_2^+ = \frac{\langle n3\rangle[21]}{\langle n1\rangle\langle 32\rangle}. \tag{3.1.30}$$

In this case no strategic term manipulations are needed. $R^3_{1,2}$ directly reduces to half of the expected result and $R^n_{1,2}$ to minus half of it, so that $(R^3_{1,2} - R^n_{1,2})$ properly gives what we expect.

3.2 Triple-soft gluons

With results of the double-soft limit at hand, we can go on to study the universal behavior of scattering amplitudes with multiple gluons being soft. Here we will take a look at the triple-soft limit, which is a natural next step beyond the double-soft limit. Again the non-trivial cases occur when all soft gluons are adjacent. Beside the straightforward case of all soft gluons having the same helicity, there are two helicity configurations of interest: $A(1^+, 2^-, 3^-, \ldots)$ and $A(1^+, 2^-, 3^+, \ldots)$, where $1, 2$ and 3 are the soft legs.

Let us begin with the first case, $A(1^+, 2^-, 3^-, \ldots)$. It is easy to see that the following BCFW diagrams are dominant in the soft limit



(3.2.1)

Since the calculation is similar to that of the double-soft limit, we will be brief here. The contribution from diagram (a) to the soft factor gives

$$\mathcal{S}^{+--}_{(a)} = \frac{[\hat{P}1]^3}{[12][2\hat{P}]s_{12}} \frac{[\bar{n}3]}{[\bar{n}\hat{P}][\hat{P}3]} \frac{[\bar{n}4]}{[\bar{n}3][34]}, \tag{3.2.2}$$

where we have used the fact that $\hat{P}$ is soft, as well as the result of the double-soft limit with two

[3]One should keep in mind that in spinor helicity formalism factors of $\sqrt{2}$ from the amplitude are absorbed into the coupling constant in front. In case of the double soft gluon factor this amounts to an overall extra factor of 2 which is suppressed in (3.1.10).

negative-helicity gluons. Specifying $\hat{P}$ in terms of external momenta, the soft factor simplifies to

$$\mathcal{S}^{+--}_{(\mathrm{a})} = \frac{\langle n2\rangle^3[n4]}{\langle n1\rangle\langle 12\rangle[34]\langle n|K_{12}|3]s_{n12}}\,, \tag{3.2.3}$$

where $K_{i\ldots j} = k_i + \ldots + k_j$. Similarly, diagram (b) gives

$$\mathcal{S}^{+--}_{(\mathrm{b})} = \frac{[\hat{P}1]^3}{[12][23][3\hat{P}]s_{123}}\frac{[\bar{n}4]}{[\bar{n}\hat{P}][\hat{P}4]} = \frac{[4n]\langle n|K_{23}|1]^3}{[12][23]\langle n|K_{12}|3]\langle n|K_{123}|4]s_{123}s_{n123}}\,. \tag{3.2.4}$$

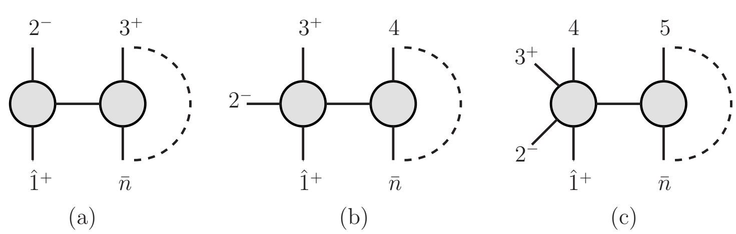
Finally, for diagram (c) a couple of remarks are in order. First we note that if gluon 4 has positive helicity, there are two allowed cases for the helicity of the internal line in the BCFW diagram. However, it is clear that the one diagram with an NMHV five-point amplitude on the left-hand side is dominant in the soft limit. Second, just as in the case of the double-soft limit, the result is independent of the helicity of gluon 4. Therefore, we can choose it to be negative and conclude

$$\mathcal{S}^{+--}_{(\mathrm{c})} = \frac{[\hat{P}1]^3}{[12][23][34][4\hat{P}]s_{1234}} = \frac{[14]^3\langle n4\rangle}{[12][23][34]\langle n|K_{123}|4]s_{1234}}\,. \tag{3.2.5}$$

Summing over the three contributions, we obtain the universal behavior of amplitudes with three adjacent soft gluons

$$A(1^+,2^-,3^-,4,\ldots,n)\big|_{p_1\sim p_2\sim p_3\to 0} \to \left(\mathcal{S}^{+--}_{(\mathrm{a})} + \mathcal{S}^{+--}_{(\mathrm{b})} + \mathcal{S}^{+--}_{(\mathrm{c})}\right)A_{n-3}\,. \tag{3.2.6}$$

Now we go on to consider the second case of interest, $A(1^+,2^-,3^+,\ldots)$. The result is given by the same set of BCFW diagrams, but now with the helicity of gluon 3 changed



(3.2.7)

From diagram (a) we have

$$\mathcal{S}^{+-+}_{(\mathrm{a})} = \frac{[\hat{P}1]^3}{[12][2\hat{P}]s_{12}}\frac{1}{\langle 4|\hat{P}+3|\bar{n}]}\left(\frac{\langle\hat{P}4\rangle^3[\bar{n}4]}{\langle\hat{P}3\rangle\langle 34\rangle s_{\hat{P}34}} + \frac{[\bar{n}3]^3\langle n4\rangle}{[\bar{n}\hat{P}][\hat{P}3]s_{\bar{n}\hat{P}3}}\right)\,, \tag{3.2.8}$$

where we have applied the alternating helicity double-soft gluon theorem (3.1.10) to the right sub-amplitude in the BCFW diagram. After some further simplifications taking the soft limit into account, we obtain

$$
\begin{aligned}
\mathcal{S}_{(\mathrm{a})}^{+-+} &= \frac{\langle 2n\rangle^3}{\langle n1\rangle\langle 12\rangle\langle 2|K_{n1}K_{n123}|4\rangle} \\
&\times \left(\frac{\langle 24\rangle^3[n4]}{\langle 23\rangle\langle 34\rangle\langle 2|K_{34}K_{1234}|n\rangle} + \frac{\langle n2\rangle\langle n4\rangle[n3]^3}{\langle n|K_{12}|3]s_{n12}s_{n123}}\right).
\end{aligned}
\tag{3.2.9}
$$

Note that it is not allowed to discard the soft momenta k_1, k_2 and k_3 in $\langle 2|K_{n1}K_{n123}|4\rangle$ and $\langle 2|K_{34}K_{1234}|4\rangle$ to further simplify the above expressions in the soft limit. For the diagram (b) we find

$$
\mathcal{S}_{(\mathrm{b})}^{+-+} = \frac{[13]^4}{[12][23][3\hat{P}][\hat{P}1]s_{123}}\frac{\langle n4\rangle}{\langle n\hat{P}\rangle\langle\hat{P}4\rangle} = \frac{[13]^4\langle n4\rangle}{[12][23]\langle n|K_{12}|3]\langle 4|K_{23}|1]s_{123}}. \tag{3.2.10}
$$

Finally, diagram (c) gives

$$
\mathcal{S}_{(\mathrm{c})}^{+-+} = \frac{\langle 24\rangle^4}{\langle\hat{1}2\rangle\langle 23\rangle\langle 34\rangle\langle 4\hat{P}\rangle\langle\hat{P}\hat{1}\rangle s_{1234}} = \frac{\langle 24\rangle^4[41]^3\langle n4\rangle}{\langle 23\rangle\langle 34\rangle\langle 4|K_{23}|1]\langle n|K_{1234}K_{34}|2\rangle s_{234}s_{1234}}. \tag{3.2.11}
$$

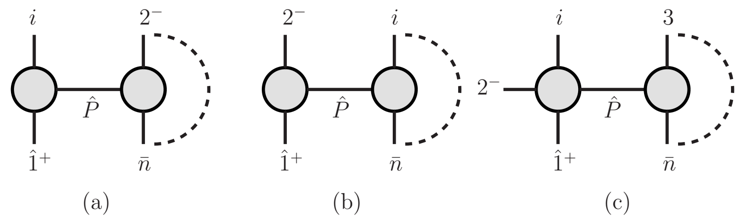
In conclusion, we obtain the following soft theorem for three adjacent soft gluons with alternating helicities

$$
A(1^+,2^-,3^+,4,\ldots,n)\big|_{p_1\sim p_2\sim p_3\to 0} \to \left(\mathcal{S}_{(\mathrm{a})}^{+-+} + \mathcal{S}_{(\mathrm{b})}^{+-+} + \mathcal{S}_{(\mathrm{c})}^{+-+}\right)A_{n-3}. \tag{3.2.12}
$$

Before we close this section, we would like to remark that both soft factors $\sum_{i=(\mathrm{a}),(\mathrm{b}),(\mathrm{c})}\mathcal{S}_i^{+--}$ and $\sum_{i=(\mathrm{a}),(\mathrm{b}),(\mathrm{c})}\mathcal{S}_i^{+-+}$ nicely reduce to a product of a single-soft factor and a double-soft factor if we take any one of the three soft gluons to be soft first. Finally, we note that all the unphysical poles appear in pairs, and we have checked numerically that they all precisely cancel at leading order in the soft limit.

3.3 Multi-soft gravitons

In this section we comment that, unlike in the case of two soft gluons, the double-soft-graviton limit is simply given by the product of two single-soft gravitons, independent of their helicity configuration. For instance, let us consider soft gravitons of opposite helicity g_1^{++} and g_2^{--}. Similar to the case of double-soft gluons from BCFW recursion, one needs to consider the following three diagrams:



(3.3.1)

In fact, a simple analysis of three- and four-point amplitudes reveals that only the diagram (a) will contribute at leading order in the double-soft limit. A simple way to obtain the result for diagram (a) is to view it as an "inverse-soft" diagram [77, 78], where leg 1^+ is considered as being added to an $(n{-}1)$-point amplitude making use of

$$M_n^{(\mathrm{a})} \quad = \quad \sum_{i\neq 2} \mathcal{S}_{1^+}(i) M_{n-1}(i', \ldots, 2^-, \ldots, n'). \tag{3.3.2}$$

Here the soft factor $\mathcal{S}_{1^+}$ is defined as

$$\mathcal{S}_{1^+}(i) = \frac{\langle ni\rangle^2 [i1]}{\langle n1\rangle^2 \langle i1\rangle}\,. \tag{3.3.3}$$

In this diagram the shifted legs are $p_{i'}$ and $p_{n'}$, which are given by

$$\tilde\lambda_{i'} = \tilde\lambda_i + \frac{\langle 1n\rangle}{\langle in\rangle}\tilde\lambda_1\,, \quad \tilde\lambda_{n'} = \tilde\lambda_n + \frac{\langle 1i\rangle}{\langle ni\rangle}\tilde\lambda_1\,. \tag{3.3.4}$$

In the soft limit we simply have $p_{i'} \to p_i$ and $p_{n'} \to p_n$. Since p_2 is soft as well, it follows from the single-soft graviton theorem that the above expression reduces to

$$M_n \quad \to \quad \sum_{i\neq 2} \mathcal{S}_{1^+}(i) \sum_{j\neq 1} \mathcal{S}_{2^-}(j) M(3, \ldots, n) \tag{3.3.5}$$

with

$$\mathcal{S}_{2^-}(j) = \frac{[xj][yj]\langle j2\rangle}{[x2][y2][j2]} \tag{3.3.6}$$

for any choices of x and y. Considering that $M_n^{(\mathrm{a})}$ is the dominant diagram at leading order, we have replaced it by the full tree-level amplitude M_n. Finally, we note that the terms $\mathcal{S}_{1^+}(2)$ and $\mathcal{S}_{2^-}(1)$, which are missing in the summation in (3.3.5), are subleading in the limit. Thus the result can be alternatively written as

$$\begin{aligned} M_n &\to \sum_i \mathcal{S}_{1^+}(i) \sum_{j\neq 1} \mathcal{S}_{2^-}(j) M(3,\ldots,n) \\ &\sim \sum_j \mathcal{S}_{2^-}(j) \sum_{i\neq 2} \mathcal{S}_{1^+}(i) M(3,\ldots,n) \\ &\sim \sum_{i,j} \mathcal{S}_{1^+}(i) \mathcal{S}_{2^-}(j) M(3,\ldots,n)\,, \end{aligned} \tag{3.3.7}$$

being simply the product of two single-soft factors. As mentioned earlier, this confirms that the leading double-soft-graviton limit can be obtained by taking the gravitons to be soft in succession, in either order, unlike the case of double-soft gluons. Given the result of double-soft gravitons, it is straightforward to see that it can be extended to the case of multiple soft gravitons, such that the soft factor of multiple-soft gravitons should be given by the product of multiple single-soft-graviton factors for any number of soft gravitons.

3.4 Double-soft limits in supersymmetric gauge theories

In this section, we move on to study the universal behavior of scattering amplitudes in supersymmetric gauge theories (in particular $\mathcal{N}=4$ SYM and pure $\mathcal{N}=2$ SYM) in the limit with the momenta of two scalars or two fermions being soft. The double-soft-scalar limit was first studied in $\mathcal{N}=8$ supergravity in [36], where the 70 scalar fields in the theory parametrize the coset space $E_{7(7)}/\mathrm{SU}(8)$. Thus these scalar fields behave as “pions". As pointed out in [36], amplitudes in this theory vanish in the single-soft-scalar limit consistent with the famous “Adler's zero" [72], and behave universally in the double-soft-scalar limit in a manner analogous to the soft-pion theorem

$$\lim_{\tau\to 0} M_n\left(\phi^{II_1I_2I_3}(\tau p_1),\ \phi_{JI_1I_2I_3}(\tau p_2), 3, \cdots, n\right) \to \frac{1}{2}\sum_{i=3}^{n} \frac{p_i\cdot(p_1-p_2)}{p_i\cdot(p_1+p_2)} (R_i)^I{}_J M_{n-2}\,, \tag{3.4.1}$$

where $(R_i)^I{}_J$ is the generator for SU(8) rotations on particle i

$$(R_i)^I{}_J = \eta_i^I \partial_{\eta_i^J}\,. \tag{3.4.2}$$

Recently, this result was extended to more general supersymmetric gravity theories [73], including $4 \leq \mathcal{N} < 8$ supergravity theories in four dimensions as well as $\mathcal{N}=16$ supergravity in three dimensions. Soft-scalar theorems have been very useful in determining the UV counter terms in supergravity theories [79, 80, 73]. It is known that for supersymmetric gauge theories (in particular $\mathcal{N}=4$ SYM), a generic vacuum has mostly massive particles, and the massless S-matrix only exists at the origin of moduli space. Thus one should not expect that the scalars would behave as “pions". Indeed it is easy to see that the amplitudes in $\mathcal{N} = 4$ SYM do not vanish in the single-soft-scalar limit, in contrast to supergravity theories. However, as in [74], one can still ask whether the amplitudes in SYM exhibit some universal behavior in certain soft limits. This is what we will explore in this section.

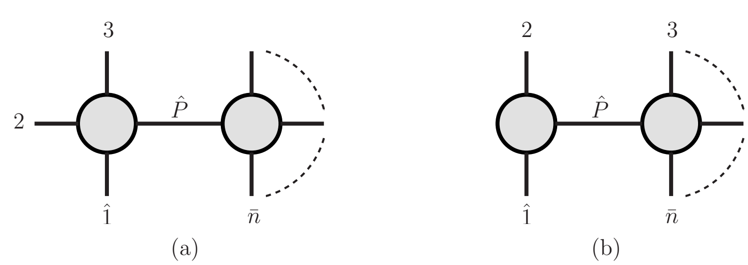
3.4.1 Double-soft scalars in $\mathcal{N} = 4$ SYM

The on-shell fields in $\mathcal{N} = 4$ SYM can be nicely packaged into a superfield [81],

$$\mathcal{A}(\eta) = g^+ + \eta^A \psi_A + \frac{1}{2!}\eta^A\eta^B\phi_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C\psi_{ABC} + (\eta^1\eta^2\eta^3\eta^4)g^- , \tag{3.4.3}$$

where g^+ is the positive-helicity gluon, ψ_A is the spin $+1/2$ gluino, and so on. In this section, we will consider the limit with two scalars ϕ_{AB} becoming soft. First of all, as we mentioned previously, it is easy to see that amplitudes in $\mathcal{N} = 4$ SYM behave as $\mathcal{O}(\tau^0)$ in the single-soft-scalar limit.

Let us now consider the double-soft-scalar limit. First we note that when the two soft scalars are not adjacent, the amplitude is not singular, and thus it cannot behave universally under the soft limit. So we will only consider the case where the two soft scalars are adjacent, which is singular and therefore universal. To be precise, we take p_1 and p_2 to be soft. Furthermore, if two scalars have no common SU(4) index, they form a singlet and the leading singular result should simply be given by the single-soft gluon limit. However, as pointed out in [73] for supergravity theories, one can extract interesting information about this case by introducing suitably anti-symmetrised amplitudes. This is particularly relevant to pure $\mathcal{N} = 2$ SYM where two scalars can only form a singlet, which will be discussed in the next section. Here we will focus on the case where two scalars do not form a singlet, as was considered in [36] for $\mathcal{N} = 8$ supergravity. For this configuration it is easy to see that the leading contribution arises when two soft scalars have one and only one common SU(4) index. In terms of the BCFW representation of the amplitude, there are two leading contributions in the double-soft limit:



(3.4.4)

After integrating out $\eta_{\hat{P}}$, the contribution (a) is given by

$$\begin{aligned} A_{(\mathrm{a})} &= \int d^4\eta_1 d^4\eta_2 \eta_1^A \eta_1^B \eta_2^B \eta_2^C \frac{\langle \hat{1}\hat{P}\rangle^4 \delta^{(4)}(\eta_1 + \frac{\langle \hat{P}2\rangle}{\langle \hat{P}\hat{1}\rangle}\eta_2 + \frac{\langle \hat{P}3\rangle}{\langle \hat{P}\hat{1}\rangle}\eta_3)}{\langle \hat{1}2\rangle\langle 23\rangle\langle 3\hat{P}\rangle\langle \hat{P}\hat{1}\rangle s_{123}} \exp\left(-\frac{\langle \hat{1}2\rangle}{\langle \hat{1}\hat{P}\rangle}\eta_2 \frac{\partial}{\partial \eta_3}\right) \\ &\times \ \exp\left(z_P \eta_1 \frac{\partial}{\partial \eta_n}\right) A_{n-2} \end{aligned} \tag{3.4.5}$$

where the integration over η's selects the soft legs 1 and 2 to be scalars. Note that, as mentioned above, we are interested in the case where two scalars have one common SU(4) index. We have applied the super BCFW recursion relations [36, 82], with shifts chosen as

$$\lambda_{\hat{1}} = \lambda_1 - z\lambda_n \,, \quad \tilde{\lambda}_{\bar{n}} = \tilde{\lambda}_n + z\tilde{\lambda}_1 \,, \quad \eta_{\bar{n}} = \eta_n + z\eta_1 \,. \tag{3.4.6}$$

Finally, we have written the shifts in A_{n-2} in an exponentiated form and only kept the leading terms. There are two possible ways to get a leading contribution above, one is by expanding η_2 from $\exp\left(-\frac{\langle \hat{1}2\rangle}{\langle \hat{1}\hat{P}\rangle}\eta_2 \frac{\partial}{\partial\eta_3}\right)$, and another one is by expanding η_1 from $\exp\left(z_P\eta_1 \frac{\partial}{\partial \eta_n}\right)$. In the first case we get one η_2 from the exponent, thus from the fermionic delta-function $\delta^{(4)}$ we have one η_2, two η_1's and one η_3. Thus we obtain,

$$A_{(\mathrm{a}),1} = \frac{\langle \hat{1}\hat{P}\rangle^4}{\langle \hat{1}2\rangle\langle 23\rangle\langle 3\hat{P}\rangle\langle \hat{P}\hat{1}\rangle s_{123}} \frac{\langle \hat{1}2\rangle}{\langle \hat{1}\hat{P}\rangle} \frac{\langle \hat{P}2\rangle}{\langle \hat{P}\hat{1}\rangle} \frac{\langle \hat{P}3\rangle}{\langle \hat{P}\hat{1}\rangle} \eta_3^B \frac{\partial}{\partial \eta_3^D} A_{n-2} \,, \tag{3.4.7}$$

where an extra minus due to the fermionic integral has been included. In the soft limit the above expression simplifies to

$$A_{(\mathrm{a}),1} \to \frac{1}{2p_3 \cdot (p_1 + p_2)} \eta_3^B \frac{\partial}{\partial \eta_3^D} A_{n-2} \,. \tag{3.4.8}$$

Analogously, we obtain the second contribution, which is given by

$$\begin{aligned} A_{(\mathrm{a}),2} &= \frac{\langle \hat{1}\hat{P}\rangle^4}{\langle \hat{1}2\rangle\langle 23\rangle\langle 3\hat{P}\rangle\langle \hat{P}\hat{1}\rangle s_{123}} z_P \left(\frac{\langle \hat{P}2\rangle}{\langle \hat{P}\hat{1}\rangle}\right)^2 \frac{\langle \hat{P}3\rangle}{\langle \hat{P}\hat{1}\rangle} \eta_3^B \frac{\partial}{\partial \eta_n^D} A_{n-2} \\ &\to \frac{1}{\langle n|1+2|3]} \eta_3^B \frac{\partial}{\partial \eta_n^D} A_{n-2} \,, \end{aligned} \tag{3.4.9}$$

where we used the on-shell solution $z_P = -\frac{s_{123}}{\langle n|3+2|1]} \sim -\frac{s_{123}}{\langle n3\rangle[31]}$.

Let us now consider the diagram (b), for which a similar consideration leads to

$$A_{(\mathrm{b})} = \frac{\delta^{(4)}\left([12]\eta_P + [2\hat{P}]\eta_1 + [\hat{P}1]\eta_2\right)}{[12][2\hat{P}][\hat{P}1]s_{12}} \exp\left(z_P \eta_1 \frac{\partial}{\partial \eta_n}\right) A_{n-1}(\hat{P}, \ldots, n) \,. \tag{3.4.10}$$

Now, using the fact that $\hat{P}$ is also soft, one can apply the supersymmetric single-soft theorem to $A_{n-1}(\hat{P},\ldots,n)$. Thus we have

$$A_{n-1}(\hat{P},3,\ldots,n)\big|_{\hat{P}\to 0} \to \frac{[n3]}{[\hat{P}3][n\hat{P}]}\delta^{(4)}\left(\eta_P + \frac{[n\hat{P}]}{[3n]}\eta_3 + \frac{[\hat{P}3]}{[3n]}\eta_n\right)A_{n-2}\,. \tag{3.4.11}$$

Substituting this result into eq. (3.4.10), integrating out η_P, and selecting the scalar components we find

$$\begin{aligned} A_{(\mathrm{b})} &= -\frac{[2\hat{P}][\hat{P}1]^2 z_P}{[12][2\hat{P}][\hat{P}1]s_{12}}\frac{[n3]}{[\hat{P}3][n\hat{P}]}\left(\frac{[12][n\hat{P}]}{[3n]}\eta_3^B + \frac{[12][\hat{P}3]}{[3n]}\eta_n^B\right)\frac{\partial}{\partial\eta_n^D}A_{n-2} \\ &\to -\left(\frac{1}{\langle n|1+2|3]}\eta_3^B + \frac{1}{2p_n\cdot(p_1+p_2)}\eta_n^B\right)\frac{\partial}{\partial\eta_n^D}A_{n-2}\,. \end{aligned} \tag{3.4.12}$$

We observe that the unphysical pole cancels out. In particular, the first term in $A_{(\mathrm{b})}$ cancels $A_{(\mathrm{a}),2}$, and we obtain the double soft-scalar theorem in $\mathcal{N}=4$ SYM

$$A_n((\phi_1)_{CD},(\phi_2)^{BC},\ldots) \to \left(\frac{1}{2p_3\cdot(p_1+p_2)}\eta_3^B\partial_{\eta_3^D} - \frac{1}{2p_n\cdot(p_1+p_2)}\eta_n^B\partial_{\eta_n^D}\right)A_{n-2}\,, \tag{3.4.13}$$

where $\phi^{BC} = \epsilon^{ABCD}\phi_{DA}$. Note the appearance of the R-symmetry generators $\eta^B\partial_{\eta^D}$. As mentioned earlier, although scalars in SYM are not Goldstone bosons, we find that our result very much resembles what has been found in $\mathcal{N}=8$ supergravity. Furthermore, as we will see, the double-soft-scalar theorem is exact even when we consider amplitudes in open superstring theory, meaning that it does not receive any α' corrections from string theory. Finally, we remark that the subleading order of this limit will be finite and thus not universal, since general BCFW diagrams start to contribute. This is the same for the double-soft limit of scalars in $\mathcal{N}=8$ supergravity.

3.4.2 Double-soft scalars in pure $\mathcal{N}=2$ SYM

In this section we consider the double-soft-scalar limit for pure $\mathcal{N}=2$ SYM. Due to the fact that it is not a maximally supersymmetric theory, the on-shell fields in $\mathcal{N}=2$ SYM are separated into two distinct mulitplets. These multiplets can be nicely obtained from $\mathcal{N}=4$ SYM by SUSY truncation [83],

$$\mathcal{A}^{\mathcal{N}=2}(\eta) = \mathcal{A}^{\mathcal{N}=4}(\eta)\big|_{\eta^3,\eta^4\to 0}\,,\quad \bar{\mathcal{A}}^{\mathcal{N}=2}(\eta) = \int d\eta^3 d\eta^4 \mathcal{A}^{\mathcal{N}=4}(\eta)\,, \tag{3.4.14}$$

where $\mathcal{A}^{\mathcal{N}=4}(\eta)$ is the superfield in $\mathcal{N}=4$ SYM that we defined in the previous section. Therefore, we see that the scalar in $\mathcal{A}^{\mathcal{N}=2}(\eta)$ corresponds to ϕ_{12} in $\mathcal{N}=4$ SYM, while the scalar in $\bar{\mathcal{A}}^{\mathcal{N}=2}(\eta)$ corresponds to ϕ_{34} in $\mathcal{N}=4$ SYM. Thus they form a singlet.

Since the scattering amplitudes in pure $\mathcal{N}=2$ SYM can be obtained from amplitudes in $\mathcal{N}=4$ SYM via SUSY reduction, we will use the same strategy as in [73]: instead of studying the amplitudes in $\mathcal{N}=2$ SYM directly we will study the relevant amplitude in $\mathcal{N}=4$ SYM first, and then reduce it to $\mathcal{N}=2$ SYM via the SUSY reduction. Now, in contrast with the case we studied in the previous section, here we are interested in precisely the amplitudes with the two soft scalars forming a singlet $A((\phi_1)_{12},(\phi_2)_{34},\ldots)$, and with the following anti-symmetrization as introduced in [73]:

$$A((\phi_1)_{12},(\phi_2)_{34},\ldots)-A((\phi_1)_{34},(\phi_2)_{12},\ldots)\,. \tag{3.4.15}$$

Let us focus on $A((\phi_1)_{12},(\phi_2)_{34},\ldots)$. As before, in the soft limit the dominant contributions are given by the diagrams shown in Fig.(3.4.4). The diagram (a) is given by a similar expression to that used above, but now we select different species of scalars

$$\begin{aligned} A_{(\mathrm{a})} &= \int d^4\eta_1 d^4\eta_2 \eta_1^1\eta_1^2\eta_2^3\eta_2^4 \frac{\langle\hat{1}\hat{P}\rangle^4\delta^{(4)}(\eta_1+\frac{\langle\hat{P}2\rangle}{\langle\hat{P}\hat{1}\rangle}\eta_2+\frac{\langle\hat{P}3\rangle}{\langle\hat{P}\hat{1}\rangle}\eta_3)}{\langle\hat{1}2\rangle\langle23\rangle\langle3\hat{P}\rangle\langle\hat{P}\hat{1}\rangle s_{123}}\exp\left(-\frac{\langle\hat{1}2\rangle}{\langle\hat{1}\hat{P}\rangle}\eta_2\frac{\partial}{\partial\eta_3}\right) \\ &\times\ \exp\left(z_P\eta_1\frac{\partial}{\partial\eta_n}\right)A_{n-2}(\hat{3},\ldots,\hat{n})\,, \end{aligned} \tag{3.4.16}$$

here we keep BCFW shifted legs (shifting the momenta p_3 and p_n) in A_{n-2}, since we select different scalars, the leading term now comes from taking two η_1's as well as two η_2's from the fermionic delta-function, these shifted legs contribute in the subleading orders. However, all these contributions vanish after the anti-symmetrization (3.4.15). In fact, all the terms with all η_1's and η_2's from the fermionic delta-function vanish after the anti-symmetrization. Thus we will focus on terms with one η_1 or η_2 from the exponent, and the calculation proceeds as outlined in the previous section. Therefore, we only quote the results

$$A_{(a)}=\sum_{A=1}^{2}\left(\frac{1}{2p_3\cdot(p_1+p_2)}\eta_3^A\frac{\partial}{\partial_{\eta_3^A}}+\frac{1}{\langle n|1+2|3]}\eta_3^A\frac{\partial}{\partial_{\eta_n^A}}\right)A_{n-2}\,. \tag{3.4.17}$$

Similarly from diagram (b), we find

$$A_{(\mathrm{b})}=-\sum_{A=1}^{2}\left(\frac{1}{\langle n|1+2|3]}\eta_3^A+\frac{1}{2p_n\cdot(p_1+p_2)}\eta_n^A\right)\frac{\partial}{\partial\eta_n^A}A_{n-2}\,. \tag{3.4.18}$$

Summing over all contributions, we find that after the anti-symmetrization we end up with

$$A_n^{\mathcal{N}=4}((\phi_1)_{12},(\phi_2)_{34},\ldots)-A_n^{\mathcal{N}=4}((\phi_1)_{34},(\phi_2)_{12},\ldots)\big|_{p_1\sim p_2\to 0}$$

$$= \left(\mathcal{S}^{\mathcal{N}=4}_{12;3} - \mathcal{S}^{\mathcal{N}=4}_{12;n} - \mathcal{S}^{\mathcal{N}=4}_{34;3} + \mathcal{S}^{\mathcal{N}=4}_{34;n}\right) A_{n-2}\,, \tag{3.4.19}$$

where the double-soft factor $\mathcal{S}^{\mathcal{N}=4}_{ij;k}$ is defined as

$$\mathcal{S}^{\mathcal{N}=4}_{ij;k} = \sum_{A=i,j} \frac{1}{2p_k \cdot (p_1 + p_2)} \eta^A_k \partial_{\eta^A_k}\,. \tag{3.4.20}$$

Now we have to project this to $\mathcal{N} = 2$ SUSY. The soft factor $\mathcal{S}^{\mathcal{N}=4}_{12;k}$ is unchanged, while $\mathcal{S}^{\mathcal{N}=4}_{34;k}$ depends on whether particles 3 and n are in the $\mathcal{A}^{\mathcal{N}=2}(\eta)$ or the $\bar{\mathcal{A}}^{\mathcal{N}=2}(\eta)$ multiplet. If they are in $\mathcal{A}^{\mathcal{N}=2}(\eta)$, then the contribution from $\mathcal{S}^{\mathcal{N}=4}_{34;k}$ should be discarded, since we set η^3 and η^4 to 0. If they are in $\mathcal{A}^{\mathcal{N}=2}(\eta)$, then integrating out η^3 and η^4 the contribution from $\mathcal{S}^{\mathcal{N}=4}_{34;k}$ simply reduces to 2. The result can be summarized as

$$A^{\mathcal{N}=2}_n(\phi_1, \bar{\phi}_2, \ldots) - A^{\mathcal{N}=2}_n(\bar{\phi}_1, \phi_2, \ldots)\big|_{p_1 \sim p_2 \to 0} = \left(R^{\mathcal{N}=2}_3 - R^{\mathcal{N}=2}_n\right) A_{n-2}\,, \tag{3.4.21}$$

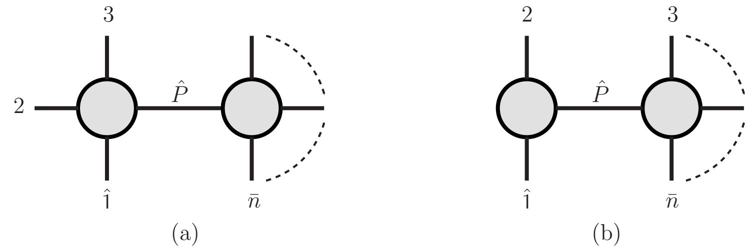
where the U(1) generator $R^{\mathcal{N}=2}_i$ is defined as

$$R^{\mathcal{N}=2}_i = \sum_{I=1}^{2} \eta^I_i \frac{\partial}{\partial_{\eta^I_i}} - 2\,, \quad (\text{for } i \in \mathcal{A}^{\mathcal{N}=2})\,, \quad R^{\mathcal{N}=2}_i = \sum_{I=1}^{2} \eta^I_i \frac{\partial}{\partial_{\eta^I_i}}\,, \quad (\text{for } i \in \bar{\mathcal{A}}^{\mathcal{N}=2})\,, \tag{3.4.22}$$

which precisely correspond to the U(1) part of the R-symmetry generators in pure $\mathcal{N} = 2$ SYM.

3.4.3 Double-soft fermions in $\mathcal{N} = 4$ and pure $\mathcal{N} = 2$ SYM

In a similar fashion one can study the limit with two soft fermions in $\mathcal{N} = 4$ SYM as well as pure $\mathcal{N} = 2$ SYM. As before, the interesting case occurs when the two fermions are adjacent. Because the (anti)-symmetrization procedure does not work for the double-soft fermions since they have different helicities [74], we will only consider the case when two fermions do not form a singlet. Thus the leading singular terms arise from adjacent fermions having one and only one common SU(4) index. To be precise we take soft particles as $(\psi_1)_D$ and $(\psi_2)_{BCD}$. The calculation in terms of BCFW recursion relations is very similar to the case of double-soft scalars, and again the relevant BCFW diagrams are shown in Fig.(3.4.4). Let us quote them here for convenience



(3.4.23)

As before, any other generic BCFW diagrams are subleading, since they are diagrams with a single-soft fermion and behave as $1/\sqrt{\tau}$ in our soft limit. In contrast, the dominant diagrams above behave as $1/\tau$. The contribution from diagram (a) is given by

$$\begin{aligned} A_{(a)} &= \int d^4\eta_1 d^4\eta_2 \eta_1^A \eta_1^B \eta_1^C \eta_2^A \frac{\langle \hat{1}\hat{P}\rangle^4 \delta^{(4)}(\eta_1 + \frac{\langle \hat{P}2\rangle}{\langle \hat{P}\hat{1}\rangle}\eta_2 + \frac{\langle \hat{P}3\rangle}{\langle \hat{P}\hat{1}\rangle}\eta_3)}{\langle \hat{1}2\rangle\langle 23\rangle\langle 3\hat{P}\rangle\langle \hat{P}\hat{1}\rangle s_{123}} \exp\left(-\frac{\langle \hat{1}2\rangle}{\langle \hat{1}\hat{P}\rangle}\eta_2 \frac{\partial}{\partial \eta_3}\right) \\ &\times \exp\left(z_P \eta_1 \frac{\partial}{\partial \eta_n}\right) A_{n-2} \,. \end{aligned} \tag{3.4.24}$$

Now the integration on η's is such that the soft legs 1 and 2 are the soft fermions of interest. Following the analysis of double-soft scalars, we find two kinds of contributions from diagram (a). One of them is given by

$$\begin{aligned} A_{(a),1} &= \frac{\langle \hat{1}\hat{P}\rangle^4}{\langle \hat{1}2\rangle\langle 23\rangle\langle 3\hat{P}\rangle\langle \hat{P}\hat{1}\rangle s_{123}} \frac{\langle \hat{1}2\rangle}{\langle \hat{1}\hat{P}\rangle} \frac{\langle \hat{P}3\rangle}{\langle \hat{P}\hat{1}\rangle} \left(\frac{\langle \hat{P}2\rangle}{\langle \hat{P}\hat{1}\rangle}\right)^2 \eta_3^A \frac{\partial}{\partial \eta_3^D} A_{n-2} \\ &= -\frac{1}{2p_3 \cdot (p_1 + p_2)} \frac{[31]}{[32]} \eta_3^A \frac{\partial}{\partial \eta_3^D} A_{n-2} \,, \end{aligned} \tag{3.4.25}$$

and the other contribution is

$$\begin{aligned} A_{(a),2} &= \frac{\langle \hat{1}\hat{P}\rangle^4}{\langle \hat{1}2\rangle\langle 23\rangle\langle 3\hat{P}\rangle\langle \hat{P}\hat{1}\rangle s_{123}} z_P \left(\frac{\langle \hat{P}2\rangle}{\langle \hat{P}\hat{1}\rangle}\right)^3 \frac{\langle \hat{P}3\rangle}{\langle \hat{P}\hat{1}\rangle} \eta_3^A \frac{\partial}{\partial \eta_n^D} A_{n-2} \\ &= -\frac{1}{\langle n|1+2|3]} \frac{[31]}{[32]} \eta_3^A \frac{\partial}{\partial \eta_n^D} A_{n-2} \,. \end{aligned} \tag{3.4.26}$$

Similarly, from diagram (b) we find

$$\begin{aligned} A_{(b)} &= -\frac{[\hat{P}1]^3 z_P}{[12][2\hat{P}][\hat{P}1] s_{12}} \frac{[n3]}{[\hat{P}3][n\hat{P}]} \left(\frac{[12][n\hat{P}]}{[3n]} \eta_3^A + \frac{[12][\hat{P}3]}{[3n]} \eta_n^A\right) \frac{\partial}{\partial \eta_n^D} A_{n-2} \\ &\to -\frac{\langle n2\rangle}{\langle n1\rangle} \left(\frac{1}{\langle n|1+2|3]} \eta_3^A + \frac{1}{2p_n \cdot (p_1 + p_2)} \eta_n^A\right) \frac{\partial}{\partial \eta_n^D} A_{n-2} \,. \end{aligned} \tag{3.4.27}$$

Adding the results of the two diagrams together, we finally obtain the double soft-fermion theorem in $\mathcal{N}=4$ SYM,

$$A_n((\psi_1)_D,(\psi_2)^A,\ldots)\to-\frac{1}{[23]\langle n1\rangle}\left(\frac{\langle n2\rangle[23]}{2p_n\cdot(p_1+p_2)}\eta_n^A\partial_{\eta_n^D}+\frac{\langle n1\rangle[13]}{2p_3\cdot(p_1+p_2)}\eta_3^A\partial_{\eta_3^D}+\eta_3^A\frac{\partial}{\partial\eta_n^D}\right)A_{n-2}\,, \tag{3.4.28}$$

Unlike the case of double-soft scalars, the cross term $\eta_3^A\frac{\partial}{\partial\eta_n^D}$ does not cancel anymore. However, all the unphysical poles cancel out manifestly. Note that the fermions in $\mathcal{N}=2$ SYM are not required to form a singlet like the scalars. The extension to the fermions in $\mathcal{N}=2$ SYM is straightforward via SUSY truncation as we discussed in the previous section.

3.5 Double-soft limit in open superstring theory

It is known that the soft-scalar theorems in $\mathcal{N}=8$ supergravity are violated in the closed superstring theory if α' corrections are included [79, 80]. It is then natural to ask whether the newly established double-soft-scalar theorems in SYM would receive any α' corrections for scattering amplitudes in open superstring theory. We find remarkably that amplitudes in open superstring theory satisfy exactly the same double-soft-scalar theorems as in SYM theory.

A general n-point color-ordered open string superamplitude of SYM vector multiplet at tree level can be very nicely expressed in terms of a basis of $(n-3)!$ SYM amplitudes [84, 85],

$$\mathcal{A}(1,2,\ldots,n)=\sum_{\sigma\in S_{n-3}}F^{(2_\sigma,\ldots,(n-2)_\sigma)}A_{\rm SYM}(1,2_\sigma,\ldots,(n-2)_\sigma,n-1,n) \tag{3.5.1}$$

where $A_{\rm SYM}(1,2_\sigma,\ldots,(n-2)_\sigma,n-1,n)$ is the color-ordered tree-level amplitude of SYM, and the multiple hypergeometric functions are given as

$$F^{(2,\ldots,n-2)}=(-1)^{n-3}\int_{0<z_i<z_{i+1}}^1\prod_{j=2}^{n-2}dz_j\left(\prod_{i<l}|z_{il}|^{s_{il}}\right)\left(\prod_{k=2}^{[n/2]}\sum_{m=1}^{k-1}\frac{s_{mk}}{z_{mk}}\right)\left(\prod_{k=[n/2]+1}^{n-2}\sum_{m=k+1}^{n-1}\frac{s_{km}}{z_{km}}\right). \tag{3.5.2}$$

The Mandelstam variables are defined as $s_{ij}\equiv\alpha'(k_i+k_j)^2$. Here we have fixed the $\mathrm{SL}(2,\mathbb{C})$ symmetry by choosing $z_1=0, z_{n-1}=1$ and $z_n=\infty$. Explicit expressions for the multiple hypergeometric functions in terms of α' expansion may be found in [85, 84]. For instance, at four points we have,

$$\begin{aligned} F^{(2)} &= -\int_0^1 dz_2\, z_2^{s_{12}}(1-z_2)^{s_{23}}\,\frac{s_{12}}{z_{12}} = \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} \\ &= 1-\zeta_2 s_{12}s_{23}+\zeta_3 s_{12}s_{13}s_{23}+\cdots . \end{aligned} \tag{3.5.3}$$

Let us start with the six-point amplitude as a simple example, the string amplitude is given as

$$\mathcal{A}(1,2,\ldots,6) = \sum_{\sigma\in S_3} F^{(2_\sigma,3_\sigma,4_\sigma)} A_{\text{SYM}}(1,2_\sigma,3_\sigma,4_\sigma,5,6)\,. \tag{3.5.4}$$

It turns out to be convenient to take the soft limit on legs 3 and 4, more generally for a n-point amplitude, we take p_{n-3} and p_{n-2} to be soft. From the definition of $F^{(2,\ldots,n-2)}$ (for $n=6$, the explicit expressions for $F^{(2_\sigma,3_\sigma,4_\sigma)}$ in α' expansion can be found in eq.(2.29) in [84]), it is easy to see that for six points only $F^{(2,3,4)}$ contributes in the limit, and it simply becomes $F^{(2)}$. Thus we have

$$\mathcal{A}(1,2,\ldots,6) \to F^{(2)}\mathcal{S}_{ij}A_{\text{SYM}}(1,2,5,6) = \mathcal{S}_{ij}\mathcal{A}(1,2,5,6)\,, \tag{3.5.5}$$

where for the convenience of following discussion we defined the soft factor $\mathcal{S}_{ij}$

$$\mathcal{S}_{ij} = \frac{1}{2p_i\cdot(p+q)}\eta_i^I\partial_{\eta_i^J} - \frac{1}{2p_j\cdot(p+q)}\eta_j^I\partial_{\eta_j^J}\,, \tag{3.5.6}$$

with p and q being the soft legs. In the above case these are p_3 and p_4. The amplitude with a general multiplicity can be considered similarly by following a proof of single-soft-gluon theorem in [84]. First of all, we note that only those permutations $\sigma\in S_{n-3}$ where indices $(n-3)$ and $(n-2)$ are adjacent may contribute, since otherwise the amplitudes in SYM would be finite and therefore subleading. By the property of hypergeometric functions F, the position $(n-4)$ should always be on the left of $(n-3)$ and $(n-2)$. Furthermore, $(n-3)$ and $(n-2)$ should be in the canonical order, meaning that $(n-3)$ should be on the left of $(n-2)$. Otherwise, for all the above cases the multiple hypergeometric function F is vanishing. For such σ's we find the following configurations:

- $\sigma\in S_{n-5}$ with $(n-4)_\sigma = n-4$, we have

$$\begin{aligned} &F_n^{(\sigma)}A_{\text{SYM}}(1,2_\sigma,\ldots,(n-4),(n-3),(n-2),(n-1),n) \\ &\to \mathcal{S}_{n-4,n-1}F_{n-2}^{(\sigma)}A_{\text{SYM}}(1,2_\sigma,\ldots,(n-4),(n-1),n) \end{aligned} \tag{3.5.7}$$

In the following, we then consider the cases with $(n-4)_\sigma \neq n-4$.

- $\sigma \in S_{n-5}$ with $(n-4)_\sigma \neq n-4$, we have

$$\begin{aligned}&F_n^{(\sigma)} A_{\mathrm{SYM}}(1, 2_\sigma, \ldots, (n-4)_\sigma, (n-3), (n-2), (n-1), n)\\&\to \mathcal{S}_{(n-4)_\sigma, n-1} F_{n-2}^{(\sigma)} A_{\mathrm{SYM}}(1, 2_\sigma, \ldots, (n-4)_\sigma, (n-1), n)\end{aligned} \tag{3.5.8}$$

- Finally, we have the non-vanishing contribution with $\sigma \in S_{n-5}$ with $(n-4)_\sigma \in \{2_\sigma, \ldots, i_\sigma\}$,

$$\begin{aligned}&F_n^{(\sigma)} A_{\mathrm{SYM}}(1, 2_\sigma, \ldots, i_\sigma, (n-3), (n-2), (i+1)_\sigma, \ldots, (n-4)_\sigma, (n-1), n)\\&\to \mathcal{S}_{i_\sigma, (i+1)_\sigma} F_{n-2}^{(\sigma)} A_{\mathrm{SYM}}(1, 2_\sigma, \ldots, (n-4)_\sigma, (n-1), n)\end{aligned} \tag{3.5.9}$$

Using the definition of the soft factor $\mathcal{S}_{ij}$ (in particular its antisymmetric property), we find that the results of the second the third cases combine nicely,

$$\text{eq.}(3.5.8) + \text{eq.}(3.5.9) = \mathcal{S}_{n-4,n-1} F_{n-2}^{\sigma} A_{\mathrm{SYM}}(1, 2_\sigma, \ldots, (n-4)_\sigma, (n-1), n)\,. \tag{3.5.10}$$

Combining with the result of (3.5.7), this concludes the proof that the amplitudes in open superstring theory satisfy the same double-soft-scalar theorem as in SYM theory.

Chapter 4

Leading multi-soft limits from scattering equations

This chapter is based on the publication [86].

Investigation of soft factors has a rich history, reaching back to the contributions of Low, Weinberg and others [17, 18, 19, 20, 21, 22, 23, 26, 24, 25]. Soft factorization is a universal property of scattering amplitudes. An n-point scattering amplitude A_n depends on external momenta k_i^μ of the $i = 1, 2, ..., n$ ingoing and outgoing scattering particles. If a subset of adjacent external momenta k_j^μ for $\forall j = 1, 2, ..., m$ with $m < n - 3$ is taken to zero, for example parametrized as $k_j^\mu \to \tau k_j^\mu$ and $\tau \to 0$, the amplitude is expected to factorize at leading order in τ into a soft factor S_m times a lower point amplitude A_{n-m}:

$$A_n \to S_m A_{n-m} + \text{sub-leading in } \tau. \tag{4.0.1}$$

Universality in this context means that S_m is independent of the remaining lower point amplitude A_{n-m}, such that S_m is always the same whenever the same types of m external particles are taken soft within any original amplitude A_n.

More recently, interest in investigation of soft theorems was refueled [27, 28, 29, 30] as Strominger et al. showed that soft-graviton theorems can be understood from the point of view of BMS symmetry [31, 32, 33, 34, 35]. Further study of leading and sub-leading soft theorems in Yang-Mills, gravity, string and supersymmetric theories ensued [5, 37, 36, 42, 43, 38, 39, 40, 41, 46, 16, 52, 56, 68, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103], partly based on the amplitude formulation due to Cachazo, He and Yuan (CHY) [5]. Double soft theorems have been considered in [71, 70, 72], and more recently [13, 73, 74, 75, 53, 76, 104, 105, 106, 107, 61, 50, 51, 58, 108, 109, 110, 111, 112, 110, 113]. Construction rules for soft factors with multiple soft particles in $\mathcal{N}=4$ SYM theory appeared in [114]. Work on related topics was also done, like sub-leading collinear limits [115] and investigation of the current algebra at null infinity induced by soft gluon limits [116].

In this note we use the CHY formulation of scattering amplitudes [5, 10] to derive the leading m-soft factor S_m for gluons, bi-adjoint scalar ϕ^3, Yang-Mills-scalar and non-linear sigma model.

We find the m-soft gluon factor in the case when external legs $1,2,...,m$ are soft to be given by the CHY type formula (4.2.19, 4.2.20, 4.2.21, 4.2.22). We then consider explicit examples, obtain analytic results in cases $m=1,2,3$, and check the cases $m=2,3,4$ numerically via amplitude ratios in four dimensions obtained from the GGT package [117]. Based on these explicit examples, we infer and conjecture a general pattern for the m-soft gluon factor:

$$S_m^{gluon}=\sum_{r=1}^{m+1}(-1)^{r+1}P^{(m+1-r)}_{r,r+1,\dots,m,m+1}P^{(r-1)}_{r-1,r-2,\dots,1,n}\,, \tag{4.0.2}$$

where $P^{(0)}_{m+1}=P^{(0)}_n\equiv 1$, and $P^{(i)}_{1,2,\dots,i,i+1}$, with $d\nu_1$ and $\psi^{(i+1)}_{[1,i]}$ defined in (4.2.20) and (4.2.22), is[1]

$$P^{(i)}_{1,2,\dots,i,i+1}=\int d\nu_1\frac{1}{\prod_{c=2}^{i+1}\bar{\sigma}_{c-1,c}}\mathrm{Pf}\left(\psi^{(i+1)}_{[1,i]}\right). \tag{4.0.3}$$

If all $P^{(i)}_{1,2,\dots,i,i+1}$ with $i<m$ are known from calculations of lower soft factors, then $P^{(m)}_{1,2,\dots,m,m+1}$ is the only new contribution that has to be computed to construct S_m at a given m.

The leading m-soft factor in bi-adjoint scalar ϕ^3, Yang-Mills-scalar and non-linear sigma model theories involves the same integration measure $d\nu_r$ as in (4.2.19), while the integrands are different: (4.4.4), (4.4.9) and (4.4.12).

As an alternative in four dimensions, we also develop a CSW type [118] automated recursive procedure that gives the leading m-soft gluon factor (compare with construction rules in [114]). Finally, we use BCFW recursion [3] to obtain all leading four-soft gluon factors with analytically distinct helicity combinations in four dimensions.

This work is organized as follows. In section 4.1 we recall the CHY formalism and introduce the soft limit. In section 4.2 we demonstrate the soft factorization of gluons at any m and obtain our general result. Explicit examples are worked out in section 4.3 and a simpler evaluation formula is conjectured. Multi-soft factors in scalar ϕ^3, Yang-Mills-scalar and non-linear sigma model are

[1] The cases $P^{(m+1-r)}_{r,r+1,\dots,m,m+1}$ and $P^{(i)}_{i,i-1,\dots,2,1,n}$ are obtained by simple index exchange after integration.

discussed in section 4.4. Appendix 4.5 contains a CSW type recursive procedure for m-soft factors in four dimensions. Appendix 4.6 contains BCFW results for four-soft gluon factors in four dimensions.

4.1 The CHY formulation of Yang-Mills and the soft limit

We start with the usual n-point formula for the tree-level gluon amplitude [5]:

$$A_n = \int d\mu_n \, \mathcal{I}_n^{YM}, \tag{4.1.1}$$

where the CHY integration measure $d\mu_n$ and the Yang-Mills CHY integrand $\mathcal{I}_n^{YM}$ are

$$d\mu_n = \int d^n\sigma \frac{\sigma_{ij}\sigma_{jk}\sigma_{ki}}{\text{vol}\,(SL(2,C))} \prod_{\substack{a=1 \\ a\neq i,j,k}}^{n} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a\cdot k_b}{\sigma_{ab}}\right) \quad , \quad \mathcal{I}_n^{YM} = \frac{2\frac{(-1)^{p+q}}{\sigma_{pq}}\text{Pf}(\Psi_{pq}^{pq})}{\sigma_{12}\sigma_{23}\ldots\sigma_{n1}}. \tag{4.1.2}$$

Moduli differences are abbreviated as $\sigma_{ab} \equiv \sigma_a - \sigma_b$ and the matrix Ψ is given by

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad A = \begin{cases} \frac{k_a\cdot k_b}{\sigma_{ab}} & ; a\neq b \\ 0 & ; a=b \end{cases}, \quad B = \begin{cases} \frac{\epsilon_a\cdot\epsilon_b}{\sigma_{ab}} & ; a\neq b \\ 0 & ; a=b \end{cases}, \quad C = \begin{cases} \frac{\epsilon_a\cdot k_b}{\sigma_{ab}} & ; a\neq b \\ -\sum_{\substack{c=1 \\ c\neq a}}^{n} \frac{\epsilon_a\cdot k_c}{\sigma_{ac}} & ; a=b \end{cases}, \tag{4.1.3}$$

with $a,b \in \{1,2,\ldots,n\}$. The k^μ are momenta of scattering particles and ϵ^μ contain the corresponding polarization data. The indices $1\leq i<j<k\leq n$ as well as $1\leq p<q\leq n$ in (4.1.2) are chosen arbitrarily but fixed. Upper and lower indices on matrix Ψ denote removed columns and rows respectively. We would like to consider the case where m external legs with $m<n-3$ are going soft simultaneously:

$$k_q^\mu \to \tau k_q^\mu, \quad \tau\to 0, \quad \text{for} \quad q\in\{1,2,\ldots,m\}. \tag{4.1.4}$$

As we take $\tau\to 0$, it is clear from the structure of matrix Ψ that at leading order in τ the Pfaffian factorizes as:[2]

$$\text{Pf}(\Psi_{pq}^{pq}) \to \text{Pf}(\psi)\text{Pf}(\Psi_{p,q,1,2,\ldots,m,n+1,n+2,\ldots,n+m}^{p,q,1,2,\ldots,m,n+1,n+2,\ldots,n+m}|_{\tau=0}) + \text{subleading in } \tau, \tag{4.1.5}$$

possibly up to an overall sign. The $2m\times 2m$ matrix ψ in the first Pfaffian on the right hand side of (4.1.5) is defined the same way as Ψ, except the indices a,b in the sub-matrices A,B,C are

[2]To see this, make the substitution (4.1.4) and expand the Pfaffian along rows and/or columns $1,2,\ldots,m,n+1,n+2,\ldots,n+m$. Retain only leading summands under $\tau\to 0$, keeping in mind that solutions with $\sigma_{ab}=O(\tau)$ or $\sigma_{ab}=O(1)$ for $a,b\in\{1,2,\ldots,m\}$ are possible. Finally, reassemble the remaining coefficients into $\text{Pf}(\psi)$.

restricted to the subset $a,b \in \{1,2,...,m\}$. Here, to do the expansion along rows we employed the usual recursive formula for the Pfaffian of an anti-symmetric $2n \times 2n$ matrix M:

$$\text{Pf}\,(M) = \sum_{\substack{j=1\\ j\neq i}}^{2n} (-1)^{i+j+1+\theta_{i-j}} m_{ij} \text{Pf}\left(M_{ij}^{ij}\right), \tag{4.1.6}$$

where m_{ij} are elements of matrix M, $\theta_x \equiv \theta(x)$ is the Heaviside step function, and index i can be freely chosen.

Alternatively, we could have noticed that $\tau \to 0$ reduces matrix Ψ_{pq}^{pq} at leading order to a block matrix structure, with several blocks equal to zero. Factorization (4.1.5) then directly follows from trivial Pfaffian factorization identities for block matrices.

Note that $\text{Pf}(\psi)$ contains terms leading and/or sub-leading in τ, depending on whether it is evaluated on degenerate ($\sigma_{ab} = O(\tau)$ for some a,b) or non-degenerate ($\sigma_{ab} = O(1)$ for all a,b) solutions to the scattering equations. However, for our purposes it is only important that for all types of solutions $\text{Pf}(\psi)$ contains all leading contributions.

The second Pfaffian on the right hand side of (4.1.5) is the one we expect in an $(n-m)$-point amplitude as we take $\tau \to 0$. Furthermore, we can trivially rewrite

$$\frac{1}{\sigma_{12}\sigma_{23}...\sigma_{n1}} = \frac{\sigma_{n,m+1}}{\sigma_{n1}\sigma_{12}...\sigma_{m,m+1}} \cdot \frac{1}{\sigma_{n,m+1}\sigma_{m+1,m+2}...\sigma_{n-1,n}}, \tag{4.1.7}$$

and observe the following behavior in scattering equation delta functions

$$\prod_{a=1}^{n} \delta\left(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}}\right) = \prod_{a=1}^{m} \delta\left(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}}\right) \prod_{c=m+1}^{n} \delta\left(\sum_{\substack{b=m+1\\ b\neq c}}^{n} \frac{k_c \cdot k_b}{\sigma_{cb}} + O(\tau)\right). \tag{4.1.8}$$

The last equation holds since we necessarily have $\sigma_{cb} = O(1)$ for $m+1 \le c \le n$ due to the kinematics in all k_c^μ being generic and therefore producing non-degenerate configurations of σ_c, while all $k_b = O(\tau)$ for the soft particles $1 \le b \le m$ tend to zero. The behavior of the first $1 \le a \le m$ delta functions in (4.1.8) is more subtle, since we can have $\sigma_{ab} = O(1)$ or $\sigma_{ab} = O(\tau)$ in this case. It will be investigated in detail in the next section.

Considering the above, we can structurally rewrite (4.1.1) at leading order in $\tau \to 0$ as

$$A_n \to \int d\mu_{n-m}\; S_m\; \mathcal{I}_{n-m}^{YM} + \text{sub-leading in } \tau, \tag{4.1.9}$$

$$S_m = \int d^m\sigma \prod_{a=1}^{m} \delta\left(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}}\right) \frac{\sigma_{n,m+1}}{\sigma_{n1}\sigma_{12}...\sigma_{m,m+1}} \text{Pf}(\psi), \tag{4.1.10}$$

where $d\mu_{n-m}$ and $\mathcal{I}^{YM}_{n-m}$ are based on objects with indices in the range $\{m+1, m+2, ..., n\}$.

Of course this alone does not provide a factorization yet, since S_m still depends on σ_n and σ_{m+1}, and the delta functions within still depend on all n momenta and σ-moduli. In the following we show that for any m the $\sigma_{m+1}, ..., \sigma_n$ dependence in S_m drops out at leading order in τ and the amplitude indeed factorizes as $A_n \to S_m A_{n-m}+$ sub-leading in τ. Furthermore, we find that S_m only depends on polarizations $\epsilon^\mu_1, \epsilon^\mu_2, ..., \epsilon^\mu_m$ as well as momenta $k^\mu_n, k^\mu_1, k^\mu_2, ..., k^\mu_{m+1}$, and establish a CHY type formula for evaluating S_m independently of the remaining factored amplitude A_{n-m}.

4.2 Factorization of S_m for Yang-Mills and the general result

Starting with S_m in (4.1.10) we apply several transformations in order to more conveniently work with this expression. First we rewrite the delta functions making use of the general identity

$$\int d^m x \prod_{i=1}^{m} \delta\left(f_i(\vec{x})\right) \bullet \ = \int d^m x \det(M) \prod_{i=1}^{m} \delta\left(\sum_{j=1}^{m} M_{ij} f_j(\vec{x})\right) \bullet \ , \tag{4.2.1}$$

where $\bullet$ is a placeholder for some test function and we employ the specific $m \times m$ matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & ... & 1 & 1 & 1 \\ 1 & -1 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & -1 & ... & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & ... & 1 & -1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 & -1 \end{pmatrix}, \qquad \det(M) = (-1)^{m+1} m, \tag{4.2.2}$$

which for our particular variables and functions of interest yields the effective relation

$$\prod_{a=1}^{m} \delta\left(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}}\right) = (-1)^{m+1} m\, \delta\left(\sum_{a=1}^{m} \sum_{b=m+1}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}}\right) \prod_{q=1}^{m-1} \delta\left(h_q\right), \tag{4.2.3}$$

$$h_q = \sum_{\substack{a=1\\ a\neq q}}^{n} \frac{k_q \cdot k_a}{\sigma_{qa}} - \sum_{\substack{b=1\\ b\neq q+1}}^{n} \frac{k_{q+1} \cdot k_b}{\sigma_{q+1,b}}. \tag{4.2.4}$$

Furthermore, we transform the moduli σ_a into a new set of variables ρ and ξ_i:

$$\sigma_q = \rho - \sum_{a=1}^{q-1} \frac{\xi_a}{2} + \sum_{b=q}^{m-1} \frac{\xi_b}{2}, \tag{4.2.5}$$

which leads to a change of the integration measure as

$$d\sigma_1 \wedge d\sigma_2 \wedge ... \wedge d\sigma_m = (-1)^{m+1} d\rho \wedge d\xi_1 \wedge d\xi_2 \wedge ... \wedge d\xi_{m-1}. \tag{4.2.6}$$

The transformation (4.2.5) is convenient, since $\sigma_{a,a+1} = \xi_a$ allows for more direct access to degenerate solutions $\sigma_{a,a+1} = O(\tau)$ in the new ξ_a variables. To keep expressions short, we will maintain the σ_a notation while implying the substitution (4.2.5). With the above changes, S_m becomes

$$S_m = \int d\rho\, d^{m-1}\xi\ m\, \delta\left(\sum_{a=1}^{m}\sum_{b=m+1}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}}\right) \prod_{q=1}^{m-1} \delta\left(h_q\right) \frac{\sigma_{n,m+1}}{\sigma_{n1}\sigma_{12}...\sigma_{m,m+1}} \mathrm{Pf}(\psi). \tag{4.2.7}$$

Now consider keeping ρ fixed and integrating out the $q = 1, 2, ..., m-1$ constraints $h_q = 0$ (which we will denote as $\{h\} = 0$) in the ξ variables. This introduces a Jacobian $\det(H)^{-1}$ with derivative matrix $H_{ij} = \partial_{\xi_i} h_j$ and a summation over all solutions to the set of $m-1$ equations $\{h\} = 0$ in the ξ variables:

$$S_m = \sum_{\substack{\{h\}=0\\ \text{solutions}}} \int d\rho \frac{m}{\det(H)}\, \delta\left(\sum_{a=1}^{m}\sum_{b=m+1}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}}\right) \frac{\sigma_{n,m+1}}{\sigma_{n1}\sigma_{12}...\sigma_{m,m+1}} \mathrm{Pf}(\psi). \tag{4.2.8}$$

Clearly, here all expressions in the integrand can be effectively thought of as functions of the single variable ρ, since $\sigma_a = \sigma_a\left(\rho, \{\xi(\rho)\}\right)$ for $a \in \{1, 2, ..., m\}$ for each solution of $\{h\} = 0$ in ξ variables. Therefore, we can now map the single remaining delta function to a simple pole

$$S_m = \sum_{\substack{\{h\}=0\\ \text{solutions}}} \oint \frac{d\rho}{2\pi i} \frac{m}{\det(H)} \frac{1}{\sum_{a=1}^{m}\sum_{b=m+1}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}}} \frac{\sigma_{n,m+1}}{\sigma_{n1}\sigma_{12}...\sigma_{m,m+1}} \mathrm{Pf}(\psi), \tag{4.2.9}$$

and consider contour deformations away from the initial locus $\sum_{a=1}^{m}\sum_{b=m+1}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}} = 0$ in ρ.

By simple power counting of poles we see that there is no pole and therefore no residue at infinity in ρ. As we deform the contour in ρ, the expressions $\{h\}$ change dynamically since they depend on ρ directly and through $\xi(\rho)$ variables. When we localize ρ at a pole contained in the integrand, the $\{h\} = 0$ constraints can get rescaled and simplified. However, since we are summing over the solutions, the set of constraints $\{h\} = 0$ has to stay analytic to leading order at the poles in ρ at all times. This implies i.e. that the Jacobian $\det(H)^{-1}$ can get rescaled and simplified due to the contour deformation, but may never diverge. This is a powerful constraint that allows us to find all integrand poles in ρ as follows.

Structurally, the only type of poles that exists in the integrand is of the shape $1/\sigma_{ab}$. As one such pole becomes localized, corresponding terms in the set of expressions $\{h\}$ start to diverge. Maintaining analyticity at leading order of the divergence in one of the $\{h\} = 0$ constraints then demands that at least one different independent $1/\sigma_{cd}$ pole must become localized as well simultaneously and at the same rate.[3] This second pole then threatens the analyticity in another $\{h\} = 0$ constraint which is affected only by this new divergence, etc. In this fashion a chain of relations occurs demanding that more and more poles must be localized at the same rate simultaneously until it is ensured that analyticity in all $\{h\} = 0$ constraints at leading order in the poles is preserved. Overall we realize that whenever a $1/\sigma_{ab}$ pole is localized due to the d.o.f. in ρ, the ρ dependence in contributing $\{\xi(\rho)\}$ solutions must be such that other $(m-1)$ independent poles become localized as well simultaneously to maintain analyticity in all the $\{h\} = 0$ constraints at leading order of divergence.

Equipped with the above observations, we must consider simultaneously localizing subsets of m independent $1/\sigma_{ab}$ poles in the integrand, with $a \neq b$ pairs $a, b \in \{n, 1, 2, ..., m, m+1\}$. Regardless of which m independend poles we choose to localize, the scattering equation expressions $\{h\}$ as well as the pfaffian $\mathrm{pf}(\psi)$ develop poles in $\rho \to 0$ of a certain power completely homogeneously (independent of indices chosen). The only part of the integrand which can diverge more or less dependent on the choice of localized poles is the Parke-Talor like factor. In the following we consider the case of highest divergence, where combinations of m poles in the Parke-Taylor like factor are localized.[4] There are $\binom{m+1}{m} = m+1$ such pole combinations.

In the ρ and ξ_i variables the Parke-Taylor-like factor reads:

$$\frac{\sigma_{n,m+1}}{\sigma_{n1}\sigma_{12}...\sigma_{m,m+1}} = \frac{\sigma_{n,m+1}}{\left(\sigma_n - \rho - \sum_{i=1}^{m-1}\frac{\xi_i}{2}\right)\xi_1\xi_2...\xi_{m-1}\left(\rho - \sum_{i=1}^{m-1}\frac{\xi_i}{2} - \sigma_{m+1}\right)}. \tag{4.2.10}$$

Structurally, there are three different classes of m-poles combinations that can occur, namely where all appearing poles are localized except for:

$$\begin{aligned} &1.)\text{ the pole } \frac{1}{\sigma_n - \rho - \sum_{i=1}^{m-1}\frac{\xi_i}{2}},\\ &2.)\text{ xor a single pole } \frac{1}{\xi_i} \text{ out of } i \in \{1, 2, ..., m-1\},\\ &3.)\text{ xor the pole } \frac{1}{\rho - \sum_{i=1}^{m-1}\frac{\xi_i}{2} - \sigma_{m+1}}. \end{aligned} \tag{4.2.11}$$

[3]This is the case since $\frac{k_a \cdot k_b}{\bar{\sigma}_{ab}} = 0$ for generic momenta only has the solution $|\bar{\sigma}_{ab}| = \infty$, which is non-analytic, while in the case of at least two summands $\frac{k_a \cdot k_b}{\bar{\sigma}_{ab}} + \frac{k_c \cdot k_d}{\bar{\sigma}_{cd}} = 0$ finite solutions for the $\bar{\sigma}_i$ exist such that analyticity is preserved.

[4]We will see that this leads to a simple pole overall, such that any lower polynomial degree would not develop a divergence or residue and thus does not contribute. Therefore, localizing m pole combinations in the Parke-Taylor like factor gives the only non-vanishing contributions.

We choose to parametrize the m localized poles in the above three cases by a parameter $\bar{\rho} \to 0$ as follows:

$$1.)\ \rho = \bar{\rho} + \sigma_{m+1}, \quad \xi_j = \bar{\rho}\,\bar{\xi}_j \text{ for all } j \in \{1,2,...,m-1\},$$
$$2.)\ \rho = \bar{\rho} + \frac{1}{2}(\sigma_{m+1} + \sigma_n), \quad \xi_i = \sigma_n - \sigma_{m+1} + \bar{\rho}\,\bar{\xi}_i, \quad \text{and} \quad \xi_j = \bar{\rho}\,\bar{\xi}_j \text{ for all } j \neq i, \tag{4.2.12}$$
$$3.)\ \rho = \bar{\rho} + \sigma_n, \quad \xi_j = \bar{\rho}\,\bar{\xi}_j \text{ for all } j \in \{1,2,...,m-1\}.$$

The new variables $\bar{\xi}_i$ account for the original degrees of freedom of ξ_i variables at leading order after localizing $\bar{\rho} \to 0$. Note that in all three cases we have $d\rho = d\bar{\rho}$, and the one pole that is not localized always directly reduces to $1/\sigma_{n,m+1}$ under $\bar{\rho} \to 0$, which cancels the numerator in (4.2.10). In general, if we define [5]

$$\bar{\sigma}_n = \bar{\sigma}_0 = \bar{\sigma}_{m+1} \equiv 0 \quad \text{and} \quad \bar{\sigma}_q = 1 - \sum_{a=1}^{q-1} \frac{\bar{\xi}_a}{2} + \sum_{b=q}^{m-1} \frac{\bar{\xi}_b}{2} \quad \text{for} \quad q \in \{1,2,...,m\}, \tag{4.2.13}$$

$$d\bar{\xi}_1 \wedge d\bar{\xi}_2 \wedge ... \wedge d\bar{\xi}_{m-1} = 2\, d\bar{\sigma}_1 \wedge d\bar{\sigma}_2 \wedge ... \wedge d\bar{\sigma}_{m-1} \tag{4.2.14}$$

then, for all possible pole combinations, the behavior of (4.2.10) for $\bar{\rho} \to 0$ is parametrized as

$$\frac{\sigma_{n,m+1}}{\sigma_{n1}\sigma_{12}...\sigma_{m,m+1}} = \frac{1}{\bar{\rho}^m \prod_{\substack{a=1 \\ a\neq r}}^{m+1} \bar{\sigma}_{a-1,a}} + O\left(\frac{1}{\bar{\rho}^{m-1}}\right), \tag{4.2.15}$$

where index $r \in \{1,2,...,m+1\}$ labels which one of the $m+1$ poles in the denominator of (4.2.10) is not being localized. Similarly, for all $m+1$ possible pole combinations we obtain

$$\frac{1}{\sum_{a=1}^{m}\sum_{b=m+1}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}}} = \frac{1}{\bar{\rho}\left(\sum_{a=1}^{r-1} \frac{k_a \cdot k_n}{\bar{\sigma}_a} + \sum_{b=r}^{m} \frac{k_b \cdot k_{m+1}}{\bar{\sigma}_b}\right)} + O(\bar{\rho}^0), \tag{4.2.16}$$

with the same index r. Depending on the particular value of r we also get[6]

$$\frac{1}{\det(H)} = \frac{\bar{\rho}^{2m-2}}{\det(H_r)} + O\left(\bar{\rho}^{2m-1}\right) \quad \text{and} \quad \text{Pf}(\psi) = \frac{1}{\bar{\rho}^m}\text{Pf}(\psi_r) + O\left(\frac{1}{\bar{\rho}^{m-1}}\right), \tag{4.2.17}$$

where now H_r and ψ_r only contain terms supported on the localized poles appearing in the Parke-Taylor-like factor (4.2.10) for each r. It is only at this point that the scattering equations $\{h_r\} = 0$,

[5] Note that only $m-1$ of the $\bar{\sigma}_q$ are now linearly independent since we have $\bar{\sigma}_m = 2 - \bar{\sigma}_1$.

[6] Recall that H is the derivative matrix of scattering equations. This means it is composed of elements $k_i \cdot k_j/\sigma_{ij}^2$ and their sums. While $i,j \in \{1,2,...,n\}$ initially, localizing the poles from the Parke-Taylor-like factor (4.2.10) as described above removes all dependence on $\sigma_{m+1},...,\sigma_n$. This factorizes the scattering equations and their Jacobian from the remaining $(n-m)$-point amplitude.

their Jacobian $1/\det(H_r)$ and all other terms become completely factorized from the remaining $(n-m)$-point amplitude A_{n-m}. This means H_r and ψ_r only depend on momenta $k_n^\mu, k_1^\mu, k_2^\mu, ..., k_{m+1}^\mu$ and polarizations $\epsilon_1^\mu, \epsilon_2^\mu, ..., \epsilon_m^\mu$, as expected.

Plugging the above findings into (4.2.9) and collecting the overall power of $\bar{\rho}$ we observe

$$S_m = \sum_{r=1}^{m+1} \sum_{\substack{\{h_r\}=0 \\ \text{solutions}}} \oint \frac{d\bar{\rho}}{2\pi i} \left(\frac{1}{\bar{\rho}} \frac{m}{\det(H_r)} \frac{1}{\sum_{a=1}^{r-1} \frac{k_a \cdot k_n}{\bar{\sigma}_a} + \sum_{b=r}^{m} \frac{k_b \cdot k_{m+1}}{\bar{\sigma}_b}} \frac{1}{\prod_{\substack{c=1 \\ c \neq r}}^{m+1} \bar{\sigma}_{c-1,c}} \mathrm{Pf}(\psi_r) + O(\bar{\rho}^0) \right),$$

so that it is now trivial to compute the residues in $\bar{\rho}$, since for all r we just have a single simple pole at $\bar{\rho} = 0$. The result is

$$S_m = \sum_{r=1}^{m+1} \sum_{\substack{\{h_r\}=0 \\ \text{solutions}}} \frac{m}{\det(H_r)} \frac{1}{\sum_{a=1}^{r-1} \frac{k_a \cdot k_n}{\bar{\sigma}_a} + \sum_{b=r}^{m} \frac{k_b \cdot k_{m+1}}{\bar{\sigma}_b}} \frac{1}{\prod_{\substack{c=1 \\ c \neq r}}^{m+1} \bar{\sigma}_{c-1,c}} \mathrm{Pf}(\psi_r). \tag{4.2.18}$$

Under closer inspection we note that the Pfaffian factorizes as $\mathrm{Pf}(\psi_r) = \mathrm{Pf}\left(\psi_{[1,r-1]}^{(n)}\right) \mathrm{Pf}\left(\psi_{[r,m]}^{(m+1)}\right)$ with definitions (4.2.22), again due to trivial factorization properties of Pfaffians of block matrices with some zero blocks.

In principle, (4.2.18) is already the final completely factorized result. For convenience, we can rewrite it by reassembling the Jacobian and the sum over solutions back into a shape of delta function integrations. This leads to our final general formula:[7]

$$S_m = \sum_{r=1}^{m+1} \int d\nu_r \frac{1}{\prod_{\substack{c=1 \\ c \neq r}}^{m+1} \bar{\sigma}_{c-1,c}} \mathrm{Pf}\left(\psi_{[1,r-1]}^{(n)}\right) \mathrm{Pf}\left(\psi_{[r,m]}^{(m+1)}\right), \tag{4.2.19}$$

$$d\nu_r \equiv \prod_{i=1}^{m-1} d\bar{\sigma}_i \prod_{q=1}^{m-1} \delta(h_{q,r}) \frac{2m}{\sum_{a=1}^{r-1} \frac{k_a \cdot k_n}{\bar{\sigma}_a} + \sum_{b=r}^{m} \frac{k_b \cdot k_{m+1}}{\bar{\sigma}_b}}, \tag{4.2.20}$$

where, identifying $k_0^\mu \equiv k_n^\mu$ and keeping $\bar{\sigma}_0 \equiv \bar{\sigma}_n = \bar{\sigma}_{m+1} = 0$ and $\bar{\sigma}_m = 2 - \bar{\sigma}_1$ in mind, we have

$$h_{q,r} = \sum_{a=q}^{q+1} \sum_{\substack{b=0 \\ b \neq a}}^{m+1} (-1)^{a-q} \frac{k_a \cdot k_b}{\bar{\sigma}_{ab}} \theta_{(r-a-\frac{1}{2})(r-b-\frac{1}{2})}, \tag{4.2.21}$$

with $\theta_x \equiv \theta(x)$ being the Heaviside step function. We call the constraints $h_{q,r} = 0$ the soft scattering

[7]Note the convention $\mathrm{Pf}\left(\psi_{[i,j]}^{(w)}\right) \equiv 1$ when $i > j$.

equations. The $2(j-i+1)\times 2(j-i+1)$ matrix $\psi^{(w)}_{[i,j]}$ can be written explicitly as

$$\psi^{(w)}_{[i,j]} = \begin{pmatrix} A_{[i,j]} & -(C^{(w)}_{[i,j]})^T \\ C^{(w)}_{[i,j]} & B_{[i,j]} \end{pmatrix}, \quad \text{with } (j-i+1)\times(j-i+1) \text{ sub-matrices} \tag{4.2.22}$$

$$A_{[i,j]} = \begin{cases} \frac{k_a\cdot k_b}{\bar\sigma_{ab}} & ; a\neq b \\ 0 & ; a=b \end{cases}, \quad B_{[i,j]} = \begin{cases} \frac{\epsilon_a\cdot \epsilon_b}{\bar\sigma_{ab}} & ; a\neq b \\ 0 & ; a=b \end{cases}, \quad C^{(w)}_{[i,j]} = \begin{cases} \frac{\epsilon_a\cdot k_b}{\bar\sigma_{ab}} & ; a\neq b \\ -\frac{\epsilon_a\cdot k_w}{\bar\sigma_a} - \sum_{\substack{q=1\\ q\neq a}}^{n} \frac{\epsilon_a\cdot k_q}{\bar\sigma_{aq}} & ; a=b \end{cases},$$

and with indices in the range $a,b\in\{i,i+1,\ldots,j\}$. This is the final result for the m-soft gluon theorem in CHY formulation. We emphasize that the result is correct to leading order in $\tau\to 0$. However, since (4.2.21) admits different solutions of types $\bar\sigma_{a,b}=O(1)$ and $\bar\sigma_{a,b}=O(\tau)$, the integrations in (4.2.19) have to be evaluated before the result can be systematically expanded to leading order in τ.

4.3 Explicit examples and general pattern

In this section we work out examples for the first few soft factors S_m. The factors S_1, S_2 and S_3 are obtained analytically. The factor S_4 (and higher) involves solutions to soft scattering equations that cannot be solved in terms of radicals, therefore we verify the validity of S_4 numerically. Based on the considered examples, we infer a non-trivial structural pattern for the m-soft factors which we conjecture to hold for any m.

4.3.1 One-soft gluon factor S_1

For $m=1$ there are no soft scattering equations (4.2.21) and no delta functions to integrate. The result is just directly given by the sum over r in (4.2.19):[8]

$$S_1 = 2\frac{\epsilon_1\cdot k_2}{s_{12}} - 2\frac{\epsilon_1\cdot k_n}{s_{1n}}, \tag{4.3.1}$$

which clearly is the correct Weinberg soft factor.[9] We see that the soft factor is composed out of two pieces such as:

$$P^{(1)}_{1,2} \equiv 2\frac{\epsilon_1\cdot k_2}{s_{12}}. \tag{4.3.2}$$

[8]Recall that we imply $\bar\sigma_m = 2-\bar\sigma_1$, which for $m=1$ reduces to $\bar\sigma_1=1$.
[9]The $s_{ij}=(k_i+k_j)^2$ is the usual Mandelstam variable.

Anticipating the structure of higher m-soft factors, we also define

$$P_{m+1}^{(0)} = P_n^{(0)} \equiv 1. \tag{4.3.3}$$

Using (4.3.3) and (4.3.2) we can structurally write the Weinberg soft factor (4.3.1) as

$$S_1 = P_{1,2}^{(1)} P_n^{(0)} - P_2^{(0)} P_{1,n}^{(1)}. \tag{4.3.4}$$

Based on this and further explicit results of this section, we propose in (4.3.21) that this structure generalizes and persists for all higher m-soft factors.

Restricting to four dimensions, we can convert the soft factor S_1 to spinor helicity formalism. We use the following standard dictionary to convert expressions of given helicity:

$$k_i \cdot k_j = \frac{1}{2}\langle ij\rangle[ji], \qquad \epsilon_i^+ \cdot k_j = \frac{[ij]\langle jr_i\rangle}{\sqrt{2}\langle r_i i\rangle}, \qquad \epsilon_i^- \cdot k_j = \frac{\langle ij\rangle[jr_i]}{\sqrt{2}[ir_i]}, \tag{4.3.5}$$

$$\epsilon_i^+ \cdot \epsilon_j^- = \frac{\langle jr_i\rangle[ir_j]}{[jr_j]\langle r_i i\rangle}, \qquad \epsilon_i^+ \cdot \epsilon_j^+ = \frac{\langle r_i r_j\rangle[ji]}{\langle r_i i\rangle\langle r_j j\rangle}, \qquad \epsilon_i^- \cdot \epsilon_j^- = \frac{\langle ij\rangle[r_j r_i]}{[ir_i][jr_j]}, \tag{4.3.6}$$

where r_i and r_j label reference spinors assigned to spinor i and j respectively. With an appropriate choice of reference spinor, we see in four dimensions:

$$S_1^+ = \frac{\langle n2\rangle}{\langle n1\rangle\langle 12\rangle}, \tag{4.3.7}$$

which is the expected familiar single soft factor in spinor helicity formalism. For real momenta, S_1^- is given by complex conjugation of S_1^+. Here we have suppressed an overall factor of $\sqrt{2}$ in S_1^+ per usual spinor helicity convention.

4.3.2 Two-soft gluons factor S_2

For $m = 2$, there is one soft scattering equation (4.2.21) for each r, and the number of solutions organizes as follows for the different solution types and different values of r:

solution type	$r = 1$	$r = 2$	$r = 3$
$\bar{\xi}_1 \sim O(1)$	1	1	1
$\bar{\xi}_1 \sim O(\tau)$	1	0	1

. (4.3.8)

Adding up the contributions of all 5 solutions and expanding to leading order in τ, we obtain the following expression for S_2:

$$S_2 = P^{(2)}_{1,2,3} P^{(0)}_n - P^{(1)}_{2,3} P^{(1)}_{1,n} + P^{(0)}_3 P^{(2)}_{2,1,n}. \tag{4.3.9}$$

This agrees with the generalization (4.3.21) for $m = 2$. The quantities $P^{(0)}_i$ and $P^{(1)}_{i,j}$ are given by (4.3.3), (4.3.2), and the new contribution of type $P^{(2)}_{i,j,l}$ reads:[10]

$$P^{(2)}_{1,2,3} = \frac{s_{13}\epsilon_1 \cdot \epsilon_2}{s_{123}s_{12}} - \frac{s_{23}\epsilon_1 \cdot \epsilon_2}{s_{123}s_{12}} - \frac{4\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_1}{s_{123}s_{12}} + \frac{4\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3}{s_{123}s_{12}} + \frac{4\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3}{s_{123}s_{23}}. \tag{4.3.10}$$

Counting the powers of k_1 and k_2 we see that this expression diverges as τ^{-2}, as we expect for the two-soft gluon factor. The result (4.3.9) is gauge independent and reduces to the gauge fixed result found in [53] when we select the gauge $\epsilon_2 \cdot k_3 = 0,\ \epsilon_1 \cdot k_n = 0$.

Restricting to four dimensions, converting to spinor helicity formalism by use of (4.3.5) and (4.3.6), and choosing appropriate reference spinors we get the following expression for the non-trivial helicity combination $(+-)$ after some simplification via Schouten identities:

$$S_2^{+-} = \frac{\langle n2\rangle}{\langle n1\rangle\langle 12\rangle} \frac{[13]}{[12][23]} \left(1 + \frac{\langle n1\rangle[13]\langle 32\rangle}{s_{123}\langle n2\rangle} + \frac{[1n]\langle n2\rangle[23]}{s_{n12}[13]}\right), \tag{4.3.11}$$

which naturally agrees with the result found in [53]. The trivial helicity combination $(++)$ reduces to the product of single soft factors $S_2^{++} = \frac{\langle n3\rangle}{\langle n1\rangle\langle 12\rangle\langle 23\rangle}$ as expected. Again, an overall factor of $(\sqrt{2})^2$ is suppressed in the above expressions per spinor helicity convention and the other helicity combinations can be obtained by complex conjugation.

We can additionally numerically test the above result in four dimensions. Making use of the GGT package provided in [117] to generate explicit lower point amplitudes, we can form amplitude ratios that correspond to the soft factor in appropriate soft kinematics.[11] Keeping in mind the overall powers of $\sqrt{2}$ that are suppressed in spinor helicity, we expect to find the following relation at leading order in τ:

$$|S_m| = |\frac{(\sqrt{2})^m A_n(1, 2, ..., n)}{A_{n-m}(m + 1, m + 2, ..., n)}|. \tag{4.3.12}$$

Indeed, if we generate a numeric kinematic point where k_1^μ, k_2^μ have soft entries of order 10^{-10} while

[10] Here, for brevity we use that $2(k_1 + k_2) \cdot k_3 \approx s_{123}$ at leading order in τ.

[11] Note that there is a `Chop` command in one of the routines of the GGT package, which does not work well with soft limit numerics and therefore needs to be removed.

the rest of the momenta have hard entries of order 10^0, we can check that i.e.

$$|S_2^{++}| = |\frac{2A_6(1^+,2^+,3^+,4^+,5^-,6^-)}{A_4(3^+,4^+,5^-,6^-)}|, \quad \text{or} \quad |S_2^{+-}| = |\frac{2A_6(1^+,2^-,3^+,4^+,5^-,6^-)}{A_4(3^+,4^+,5^-,6^-)}|, \tag{4.3.13}$$

hold at least to first 10 digits, reflecting that the leading soft factor receives a first correction at the next polynomially sub-leading power in τ.[12] Naturally, ratios of more complicated amplitudes yield the same agreement.

4.3.3 Three-soft gluons factor S_3

For $m = 3$, there are two soft scattering equations (4.2.21) for each r, and the number of solutions organizes as follows for the different solution types and different values of r:

solution type	$r = 1$	$r = 2$	$r = 3$	$r = 4$
$\bar{\xi}_1 \sim \bar{\xi}_2 \sim O(1)$	2	1	1	2
$\bar{\xi}_i \sim O(1),\ \bar{\xi}_j \sim O(\tau)$	2	1	1	2
$\bar{\xi}_1 \sim \bar{\xi}_2 \sim O(\tau)$	2	0	0	2

, (4.3.14)

where we imply $i \neq j$ and $i,j \in \{1,2\}$. Adding up the contributions of all 16 solutions and expanding to leading order in τ, we obtain the following expression for S_3:

$$S_3 = P^{(3)}_{1,2,3,4}P^{(0)}_n - P^{(2)}_{2,3,4}P^{(1)}_{1,n} + P^{(1)}_{3,4}P^{(2)}_{2,1,n} - P^{(0)}_4 P^{(3)}_{3,2,1,n}. \tag{4.3.15}$$

This agrees with the generalization (4.3.21) for $m = 3$. As before, expressions of type $P^{(0)}_i$, $P^{(1)}_{i,j}$ and $P^{(2)}_{i,j,l}$ are given by (4.3.3), (4.3.2) and (4.3.10), while the new contribution of type $P^{(3)}_{i,j,l,t}$ can still be analytically computed to be:[13]

$$\begin{aligned} P^{(3)}_{1,2,3,4} = &\frac{1}{s_{12}}(w_{312} - u_{312} - u_{213} - v_{312} - v_{213}) + \frac{1}{s_{23}}(w_{231} - u_{231} - u_{132} - v_{231} - v_{132}) + \\ &+ \left(\frac{1}{s_{12}} + \frac{1}{s_{23}}\right)(u_{123} + u_{321} + v_{123} + v_{321} - w_{123}) + \frac{8\epsilon_1 \cdot k_4 \epsilon_2 \cdot k_4 \epsilon_3 \cdot k_4}{s_{34}s_{234}s_{1234}} + \\ &+ \frac{8\epsilon_1 \cdot k_4 \left(\epsilon_2 \cdot k_3 \epsilon_3 \cdot k_4 - \epsilon_3 \cdot k_2 \epsilon_2 \cdot k_4\right)}{s_{23}s_{234}s_{1234}} + \frac{8\left(\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_4 - \epsilon_2 \cdot k_1 \epsilon_1 \cdot k_4\right)\epsilon_3 \cdot k_4}{s_{12}s_{34}s_{1234}} + \\ &+ \frac{2\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot k_4}{s_{12}s_{1234}}\left(\frac{2s_{13}}{s_{123}} + \frac{2s_{14}}{s_{34}} - \frac{s_{1234}}{s_{34}}\right) + \frac{4\epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot k_4}{s_{23}s_{1234}}\left(\frac{s_{13}}{s_{123}} - \frac{s_{34}}{s_{234}}\right) + \frac{4\epsilon_3 \cdot \epsilon_1 \epsilon_2 \cdot k_4}{s_{123}s_{1234}}, \end{aligned} \tag{4.3.16}$$

[12] To make sure that the comparison works properly, we use the same spinor conventions as the GGT package: $\lambda^1_i = \sqrt{k^0_i + k^3_i}$, $\lambda^2_i = (k^1_i + ik^2_i)/\sqrt{k^0_i + k^3_i}$ and $\tilde{\lambda}_i = (\lambda_i)^*$.

[13] Again, we use that $2(k_1 + k_2 + k_3) \cdot k_4 \approx s_{1234}$ and similar at leading order in τ to keep notation short.

where we used the abbreviations

$$u_{ijl} \equiv \frac{4\epsilon_i \cdot k_j \epsilon_j \cdot \epsilon_l}{s_{ijl}} \left(\frac{1}{3} - \frac{s_{l4}}{s_{ijl4}} \right), \quad v_{ijl} \equiv \frac{8\epsilon_i \cdot k_j \epsilon_j \cdot k_l \epsilon_l \cdot k_4}{s_{ijl} s_{ijl4}}, \quad w_{ijl} \equiv \frac{8\epsilon_i \cdot k_j \epsilon_j \cdot k_4 \epsilon_l \cdot k_j}{s_{ijl} s_{ijl4}}.$$

Counting the powers of k_1, k_2 and k_3 we see that this expression diverges as τ^{-3}, as we expect for the three-soft gluon factor.

Again, we can use (4.3.5) and (4.3.6) to pass to spinor helicity formalism if we restrict to four dimensions. In particular, the two non-trivial independent polarization combinations are $(-+-)$ and $(+--)$. For the case $(-+-)$ we obtain, with appropriate choice of reference spinors and after some simplification via Schouten identities:

$$\begin{aligned} S_3^{-+-} =& \frac{[n2]}{[n1][12]} \frac{\langle 13\rangle}{\langle 12\rangle\langle 23\rangle} \frac{[24]}{[23][34]} \left(1 - \left[\frac{\langle 1n\rangle[n2]\langle 23\rangle}{s_{n123}\langle 13\rangle} + \frac{[2n]\langle n|k_1+k_3|2]\langle 23\rangle[34]}{s_{123}s_{n123}[24]} \right.\right. \\ & \left.\left. + \frac{[n1]\langle 13\rangle[32]}{s_{123}[n2]} + \frac{\langle 1n\rangle[n2]\langle 23\rangle\ [23]\langle 3n\rangle[n4]}{\langle 13\rangle s_{n12}s_{n123}[24]} + \left\{ \begin{smallmatrix} n \leftrightarrow 4 \\ 1 \leftrightarrow 3 \end{smallmatrix} \right\} \right] \right). \end{aligned} \tag{4.3.17}$$

Similarly, the case $(+--)$ with an appropriate choice of reference spinors and after some simplification via Schouten identities yields

$$\begin{aligned} S_3^{+--} =& \frac{\langle n2\rangle}{\langle n1\rangle\langle 12\rangle} \frac{[14]}{[12][23][34]} \left(1 - \frac{\langle n1\rangle[14]\langle 42\rangle}{s_{1234}\langle n2\rangle} - \frac{[1n]\langle n|k_2+k_3|4]}{s_{n123}[14]} - \frac{[1n]\langle n2\rangle[23]\langle 3n\rangle[n4]}{s_{n12}s_{n123}[14]} \right. \\ & \left. - \frac{\langle n1\rangle[1|k_2+k_3|4]\langle 43]\langle 32\rangle}{s_{123}s_{1234}\langle n2\rangle} - \frac{s_{n1}[12]\langle 23\rangle[34]}{s_{123}s_{n123}[14]} + \frac{\langle n1\rangle[13]\langle 32\rangle\ [1n]\langle n3\rangle[34]}{\langle n2\rangle s_{123}s_{n123}[14]} \right). \end{aligned} \tag{4.3.18}$$

The trivial helicity configuration $(+++)$ as expected reduces to $S_3^{+++} = \frac{\langle n4\rangle}{\langle n1\rangle\langle 12\rangle\langle 23\rangle\langle 34\rangle}$, and all other helicity configurations are obtained from the above by symmetry and complex conjugation. An overall factor of $2^{3/2}$ is suppressed in the above expressions per spinor helicity convention.

As before, (4.3.12) is expected to hold. Making use of the GGT package [117] to generate explicit lower point amplitudes we can form ratios that correspond to the soft factor in appropriate soft kinematics. Generating a numeric kinematic point such that k_1^μ, k_2^μ and k_3^μ have soft entries of order 10^{-10} while the rest of the momenta have hard entries of order 10^0, we observe that i.e.

$$|S_3^{-++}| = \left|\frac{2^{3/2}A_7(1^-,2^+,3^+,4^+,5^+,6^-,7^-)}{A_4(4^+,5^+,6^-,7^-)}\right|, \quad |S_3^{+-+}| = \left|\frac{2^{3/2}A_7(1^+,2^-,3^+,4^+,5^+,6^-,7^-)}{A_4(4^+,5^+,6^-,7^-)}\right|, \quad \text{etc.}$$

hold to at least the first 10 digits, after which the first sub-leading correction in τ becomes important. Again, ratios of more complicated amplitudes yield the same agreement.

4.3.4 Four-soft gluons factor S_4 and beyond

For $m = 4$, there are three soft scattering equations (4.2.21) for each r, and the number of solutions organizes as follows for the different solution types and different values of r:

solution type	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
$\bar{\xi}_1 \sim \bar{\xi}_2 \sim \bar{\xi}_3 \sim O(1)$	5	2	1	2	5
$\bar{\xi}_i \sim \bar{\xi}_j \sim O(1),\ \bar{\xi}_l \sim O(\tau)$	8	2	2	2	8
$\bar{\xi}_i \sim O(1),\ \bar{\xi}_j \sim \bar{\xi}_l \sim O(\tau)$	5	2	1	2	5
$\bar{\xi}_1 \sim \bar{\xi}_2 \sim \bar{\xi}_3 \sim O(\tau)$	6	0	0	0	6

, (4.3.19)

where we imply $i \neq j,\ i \neq l,\ j \neq l$ and $i, j, l \in \{1, 2, 3\}$. With the generalization (4.3.21) in mind, we expect that the contributions for cases $r = 2, 3, 4$ can be constructed from previously determined quantities (4.3.2), (4.3.10) and (4.3.16). That is easily verified numerically by obtaining and summing over explicit approximate solutions to the soft scattering equations (4.2.21) in some example kinematics. This confirms that the structure

$$S_4 = P^{(4)}_{1,2,3,4,5} P^{(0)}_{n} - P^{(3)}_{2,3,4,5} P^{(1)}_{1,n} + P^{(2)}_{3,4,5} P^{(2)}_{2,1,n} - P^{(1)}_{4,5} P^{(3)}_{3,2,1,n} + P^{(0)}_{5} P^{(4)}_{4,3,2,1,n} \tag{4.3.20}$$

continues to hold. Trying to obtain $P^{(4)}_{1,2,3,4,5}$ for $r = 1$ (and $r = 5$) we discover that finding the 12 solutions of the type $\bar{\xi}_1 \sim \bar{\xi}_2 \sim \bar{\xi}_3 \sim O(\tau)$ is equivalent to solving for the roots of two 6th degree polynomials. Therefore, an analytic solution cannot be obtained in this direct fashion.

Based on the knowledge of previous analytic results found so far, we could try to infer the pole structure of all the different terms appearing in $P^{(4)}_{1,2,3,4,5}$, effectively constructing the result without solving the soft scattering equations. This works reasonably well for some of the appearing terms such as $\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_4 \epsilon_4 \cdot k_5$, for which the correct contribution can be guessed (and numerically checked) to be given by:

$$16 \frac{\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_4 \epsilon_4 \cdot k_5}{s_{1234} s_{12345}} \left(\left(\frac{1}{s_{12}} + \frac{1}{s_{23}} \right) \frac{1}{s_{123}} + \frac{1}{s_{12} s_{34}} + \left(\frac{1}{s_{23}} + \frac{1}{s_{34}} \right) \frac{1}{s_{234}} \right),$$

or terms like $\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot \epsilon_4$ with the correct guess for the contribution being:

$$8 \frac{\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot \epsilon_4}{s_{1234}} \left(\frac{1}{4} - \frac{s_{45}}{s_{12345}} \right) \left(\left(\frac{1}{s_{12}} + \frac{1}{s_{23}} \right) \frac{1}{s_{123}} + \frac{1}{s_{12} s_{34}} + \left(\frac{1}{s_{23}} + \frac{1}{s_{34}} \right) \frac{1}{s_{234}} \right).$$

However, $P^{(4)}_{1,2,3,4,5}$ also contains terms such as $\epsilon_3 \cdot \epsilon_4 \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_5$ or $\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4$ for which the pole structure is unclear since these patterns did not appear before. Even though an analytic solution is thus not available, we can still check numerically that (4.2.19) is correct.

Using (4.3.5) and (4.3.6) to pass to spinor helicity formalism in four dimensions, (4.3.12) is again expected to hold. Therefore, we generate a numeric kinematic point such that $k_1^\mu, k_2^\mu, k_3^\mu$ and k_4^μ have soft entries of order 10^{-10} while the rest of the momenta have hard entries of order 10^0. Now we can solve (4.2.21) numerically and obtain the numeric soft factor S_4 as a sum over all 64 solutions. Subsequently, making use of the GGT package [117], we can generate explicit amplitude ratios and observe that e.g.

$$|S_4^{-+++}| = |\frac{4A_8(1^-,2^+,3^+,4^+,5^+,6^+,7^-,8^-)}{A_4(5^+,6^+,7^-,8^-)}|, \ |S_4^{-+-+}| = |\frac{4A_8(1^-,2^+,3^-,4^+,5^+,6^+,7^-,8^-)}{A_4(5^+,6^+,7^-,8^-)}|, \text{ etc.}$$

hold to at least the first 10 digits, after which the first sub-leading correction in τ becomes important. As before, ratios of more complicated amplitudes yield the same agreement.

For even higher m, the soft scattering equations (4.2.21) become more and more complicated, so that even numeric evaluation becomes increasingly harder to do. However, in principle the m-soft gluon factor is always given by the CHY type expression summarized by (4.2.19), (4.2.21) and (4.2.22), valid to leading order in τ.

4.3.5 Conclusion and general structural pattern

The above findings are of interest since they prove the existence of a universal soft factor for any number of soft adjacent gluons and in principle provide a way to calculate these soft factors in arbitrary dimension. As a byproduct we obtained an explicit analytic result for the three-soft gluon factor for arbitrary polarizations and in arbitrary dimension, which to our knowledge is a new result.

Considering the particular results for $m = 1, 2, 3, 4$ discussed above, we can infer a generalization for the structural pattern at arbitrary m to be given by:

$$S_m = \sum_{r=1}^{m+1} (-1)^{r+1} P^{(m+1-r)}_{r,r+1,\dots,m,m+1} P^{(r-1)}_{r-1,r-2,\dots,1,n}. \tag{4.3.21}$$

In essence, if all soft factors S_a with $a < m$ for a fixed m are known, then all contributions to S_m with $1 < r < m+1$ are constructed from the lower point results, while the summand[14] $r = 1$ equals

[14] Or alternatively the summand $r = m + 1$, which is related by simple index exchange.

the only previously unknown contribution $P^{(m)}_{1,2,\ldots,m,m+1}$. In general we define $P^{(0)}_{m+1} = P^{(0)}_{n} \equiv 1$ and

$$P^{(i)}_{1,2,\ldots,i,i+1} = \int d\nu_1 \frac{1}{\prod_{c=2}^{i+1} \bar{\sigma}_{c-1,c}} \mathrm{Pf}\left(\psi^{(i+1)}_{[1,i]}\right). \tag{4.3.22}$$

In this sense, it suffices to evaluate only the $r = 1$ summand of (4.2.19) to obtain all new information at a given m.[15]

The above conjecture is inferred empirically, and it seems to be highly non-trivial to demonstrate the factorization of each summand of (4.2.19) into (4.3.21) analytically. While the structure of the Pfaffian admits such a factorization, the Parke-Taylor like factor as well as the multiplicative term remaining from the contour deformation in ρ are not convenient. This implies the necessity of a transformation along the lines of (4.2.1) with a non-trivial Jacobian, which is not easily guessed. We leave a general proof of the conjecture (4.3.21), (4.3.22) to future work.

4.4 Multi-soft factors in other theories

It is possible to directly apply the procedure described above to several other theories in CHY formulation. An important feature that largely governs the computations is the presence of at least one Parke-Taylor factor

$$\mathcal{C} \equiv \frac{1}{\sigma_{12}\sigma_{23}\ldots\sigma_{n1}} \tag{4.4.1}$$

in the CHY integrand of the amplitude, such that the amplitude in question is color ordered. The theories considered in this section have this same feature. As further building blocks we will require the sub-matrix A defined in (4.1.3), the matrix $\Psi^{n+1,n+2,\ldots,n+q}_{n+1,n+2,\ldots,n+q}$ which is (4.1.3) with rows and columns $n+1, n+2, \ldots, n+q$ dropped, and the matrix

$$\chi = \begin{cases} \frac{\delta^{I_a,I_b}}{\sigma_{ab}} & ; a \neq b \\ 0 & ; a = b \end{cases}, \tag{4.4.2}$$

where I_a, I_b are some internal space indices for scalar fields involved in the scattering process [10]. Since these indices have no non-trivial effect on the momentum dependence of soft factors, we will consider the simplest case where $I_a = I_b$ for all particle labels a, b , such that $\delta^{I_a,I_b} = 1$.

[15]There seems to be no obstruction to assuming that a similar pattern should appear for soft theorems e.g. in the other theories discussed below as well, where appropriate.

4.4.1 Multi-soft factors in bi-adjoint scalar ϕ^3 theory

The CHY formula for tree level scattering in bi-adjoint scalar ϕ^3 theory can be written as (4.1.1) [5] with $\mathcal{I}_n^{YM}$ replaced by

$$\mathcal{I}_n^{\phi^3} = \mathcal{C}^2. \tag{4.4.3}$$

Starting with this integrand, the considerations in sections 4.1 and 4.2 go through in the same manner, such that we are left with the following general expression for the m-soft scalar factor with particles $1, 2, ..., m$ going soft:

$$S_m^{\phi^3} = \sum_{r=1}^{m+1} \int d\nu_r \frac{1}{\prod_{\substack{c=1 \\ c \neq r}}^{m+1} (\bar{\sigma}_{c-1,c})^2}, \tag{4.4.4}$$

with $d\nu_r$ given in (4.2.20), and the identification $\sigma_0 \equiv \sigma_n$. As in the gluon case, the soft scattering equations contained in $d\nu_r$ can be explicitly solved for the cases $m = 1, 2, 3$, with exactly the same solutions. At leading order in the soft limit this leads to

$$S_1^{\phi^3} = \frac{1}{k_n \cdot k_1} + \frac{1}{k_1 \cdot k_2}, \tag{4.4.5}$$

$$S_2^{\phi^3} = \frac{1}{k_1 \cdot k_2} \left(\frac{1}{k_n \cdot (k_1 + k_2)} + \frac{1}{(k_1 + k_2) \cdot k_3} \right), \tag{4.4.6}$$

$$S_3^{\phi^3} = \frac{2}{s_{123}} \left(\frac{1}{k_1 \cdot k_2} + \frac{1}{k_2 \cdot k_3} \right) \left(\frac{1}{k_n \cdot (k_1 + k_2 + k_3)} + \frac{1}{(k_1 + k_2 + k_3) \cdot k_4} \right). \tag{4.4.7}$$

It is worth noticing that all contributions to the soft factors at leading order in the soft limit are due to the two summands $r = 1$ and $r = m + 1$ only, while the intermediate summands are subleading. As before, the general expression $S_m^{\phi^3}$ can be used to evaluate $S_4^{\phi^3}$ and higher soft factors numerically. We tested the results numerically against amplitude ratios in CHY formulation and found agreement.

4.4.2 Multi-soft factors in Yang-Mills-scalar theory

The CHY formula for tree level scattering in Yang-Mills-scalar theory is (4.1.1) with $\mathcal{I}_n^{YM}$ replaced by

$$\mathcal{I}_n^{YMS} = 2\mathcal{C}\,\mathrm{Pf}(\chi) \frac{(-1)^{i+j}}{\sigma_{ij}} \mathrm{Pf}(\Psi^{i,j,n+1,n+2,\ldots,n+q}_{i,j,n+1,n+2,\ldots,n+q}), \tag{4.4.8}$$

where matrix χ is $q \times q$ dimensional (4.4.2), and $1 \le i < j \le n$ can be selected arbitrarily [10]. This corresponds to the first q of the scattering particles being scalars and the remaining $n-q$ being gluons.

Starting with this integrand, the considerations in sections 4.1 and 4.2 go through in the same manner. Soft gluon factors in this theory are exactly the same as in pure Yang-Mills. The general expression for the m-soft scalar factor with particles $1, 2, ..., m$ going soft amounts to:[16]

$$S_m^{YMS} = \sum_{r=1}^{m+1} \int d\nu_r \frac{1}{\prod_{\substack{c=1 \\ c \neq r}}^{m+1} \bar{\sigma}_{c-1,c}} \text{Pf}(\chi_{[1,r-1]})\text{Pf}(\chi_{[r,m]})\text{Pf}(A_{[1,r-1]})\text{Pf}(A_{[r,m]}), \tag{4.4.9}$$

with $d\nu_r$ given in (4.2.20), and the identification $\sigma_0 \equiv \sigma_n$. The matrix $A_{[i,j]}$ was defined in (4.2.22), and the matrix $\chi_{[i,j]}$ relates to χ in (4.4.2) the same as $A_{[i,j]}$ relates to A in (4.1.3). As in the gluon case, the soft scattering equations contained in $d\nu_r$ can be explicitly solved for the cases $m = 1, 2, 3$, with exactly the same solutions. However, since $\text{Pf}(\chi_{[i,j]})$ vanishes when $\chi_{[i,j]}$ is of odd dimension, only soft factors with an even number m of soft scalars are non-zero and only summands of odd r contribute. At leading order in the soft limit this leads to

$$S_2^{YMS} = \frac{1}{2k_1 \cdot k_2} \left(\frac{k_n \cdot (k_2 - k_1)}{k_n \cdot (k_1 + k_2)} + \frac{(k_1 - k_2) \cdot k_3}{(k_1 + k_2) \cdot k_3} \right). \tag{4.4.10}$$

This agrees with the result in [75]. As before, the general expression S_m^{YMS} can be used to evaluate S_4^{YMS} and higher soft factors numerically. We tested the results numerically against amplitude ratios in CHY formulation and found agreement.

4.4.3 Multi-soft factors in non-linear sigma model

The CHY formula for tree level scattering in non-linear sigma model is (4.1.1) with $\mathcal{I}_n^{YM}$ replaced by

$$\mathcal{I}_n^{NLSM} = \mathcal{C} \frac{4}{(\sigma_{ij})^2} \text{Pf}(A_{i,j}^{i,j})^2, \tag{4.4.11}$$

where $A_{i,j}^{i,j}$ is the matrix A defined in (4.1.3) with rows and columns i, j removed, and $1 \le i < j \le n$ can be selected arbitrarily [10].

Starting with this integrand, the considerations in sections 4.1 and 4.2 go through in the same

[16] Again, we introduce the convention $\text{Pf}(\chi_{[i,j]}) = \text{Pf}(A_{[i,j]}) \equiv 1$ when $i > j$.

manner. The general expression for the m-soft factor with particles $1, 2, ..., m$ going soft amounts to:

$$S_m^{NLSM} = \sum_{r=1}^{m+1} \int d\nu_r \frac{1}{\prod_{\substack{c=1 \\ c \neq r}}^{m+1} \bar{\sigma}_{c-1,c}} \text{Pf}(A_{[1,r-1]})^2 \text{Pf}(A_{[r,m]})^2, \tag{4.4.12}$$

with $d\nu_r$ given in (4.2.20), and the identification $\sigma_0 \equiv \sigma_n$. The matrix $A_{[i,j]}$ was defined in (4.2.22). As in the gluon case, the soft scattering equations contained in $d\nu_r$ can be explicitly solved for the cases $m = 1, 2, 3$, with exactly the same solutions. However, since $\text{Pf}(A_{[i,j]})$ vanishes when $A_{[i,j]}$ is of odd dimension, only soft factors with an even number m of soft particles are non-zero and only summands of odd r contribute. At leading order in the soft limit this leads to

$$S_2^{NLSM} = \frac{1}{2} \left(\frac{k_n \cdot (k_2 - k_1)}{k_n \cdot (k_1 + k_2)} + \frac{(k_1 - k_2) \cdot k_3}{(k_1 + k_2) \cdot k_3} \right). \tag{4.4.13}$$

This agrees with the result in [75]. As before, the general expression S_m^{NLSM} can be used to evaluate S_4^{NLSM} and higher soft factors numerically. We tested the results numerically against amplitude ratios in CHY formulation and found agreement. Additionally, our S_4^{NLSM} numerically agrees with the result found in [119].[17]

4.5 CSW recursion for multi-gluon soft-factors in four dimensions

As an alternative to the construction rules presented in [114], we can set up a CSW type recursion [118] for the m-soft factors in four dimensions as follows. We start with the amplitude $A^{(m)}(k_n^{+1}, k_1^{h_1}, ..., k_m^{h_m}, k_{m+1}^{+1})$, where $k_i^{h_i}$ denotes the external momentum of the i-th particle with helicity $h_i \in \{+1, -1\}$. Here we have cyclically rotated the n-th position to be the first, and suppressed all entries $k_j^{h_j}$ with $m + 1 < j < n$ since they do not enter the soft factor that we want to extract from this amplitude. Since the helicities of particle n and $m+1$ do not enter the soft factor, we can choose these helicities to be + without loss of generality. The superscript (m) keeps track of the number of adjacent external momenta that are taken soft.

In order to obtain the soft factor from CSW recursion, we have to generate all possible diagrams

[17] Note a typo in eq. (4.10) of [119]: The numerator of last expression on the first line should involve $q_5 \cdot k_1$ instead of $q_4 \cdot k_1$.

in MHV expansion. To do this recursively, we introduce the following two functions:

$$\mathbf{S}\left(A^{(m)}(k_{q_1}^{h_{q_1}},k_{q_2}^{h_{q_2}},...,k_{q_l}^{h_{q_l}})\right)= \tag{4.5.1}$$
$$=\begin{cases} \displaystyle\sum_{\nu=\pm1}\sum_{i=1}^{l-1}\sum_{\substack{j=i+1\\ j-i<l-1}}^{l}\mathbf{H}\left(A_{j-i+2}(k_{q_i}^{h_{q_i}},...,k_{q_j}^{h_{q_j}},k_{p(q_i,...,q_j)}^{-\nu})\right)\frac{1}{P^2_{q_i,...,q_j}}\times \\ \qquad\times\mathbf{S}\left(A^{(m)}(k_{q_1}^{h_{q_1}},...,k_{q_{i-1}}^{h_{q_{i-1}}},k_{p(q_i,...,q_j)}^{+\nu},k_{q_{j+1}}^{h_{q_{j+1}}},...,k_{q_l}^{h_{q_l}})\right) & ;\quad \text{if } \displaystyle\sum_{a=1}^{l}h_{q_a}<l, \\ A^{(m)}(k_{q_1}^{h_{q_1}},k_{q_2}^{h_{q_2}},...,k_{q_l}^{h_{q_l}}) & ;\quad \text{otherwise,}\end{cases}$$

as well as, making use of $\mu(x)\equiv \mathrm{mod}(x-1,l)+1$, the function:

$$\mathbf{H}\left(A_l(k_{q_1}^{h_{q_1}},k_{q_2}^{h_{q_2}},...,k_{q_l}^{h_{q_l}})\right)= \tag{4.5.2}$$
$$=\begin{cases} \displaystyle\sum_{i=1}^{l}\sum_{j=i+1}^{i+l-3}\mathbf{H}\left(A_{j-i+2}(k_{q_{\mu(i)}}^{h_{q_{\mu(i)}}},...,k_{q_{\mu(j)}}^{h_{q_{\mu(j)}}},k_{p(q_{\mu(i)},...,q_{\mu(j)})}^{-1})\right)\frac{1}{P^2_{q_{\mu(i)},...,q_{\mu(j)}}}\times \\ \qquad\times\mathbf{H}\left(A_{l+i-j}(k_{q_{\mu(j+1)}}^{h_{q_{\mu(j+1)}}},...,k_{q_{\mu(l+i-1)}}^{h_{q_{\mu(l+i-1)}}},k_{p(q_{\mu(j+1)},...,q_{\mu(l+i-1)})}^{+1})\right) & ;\quad \text{if } \displaystyle\sum_{a=1}^{l}h_{q_a}<l-4, \\ A_l(k_{q_1}^{h_{q_1}},k_{q_2}^{h_{q_2}},...,k_{q_l}^{h_{q_l}}) & ;\quad \text{otherwise.}\end{cases}$$

We supplement the above functions with the following resolution properties:

$$p(i,...,j,p(a,...,b),u,...,v) = p(i,...,j,a,...,b,u,...,v), \tag{4.5.3}$$
$$p(i,...,j,r,a,...,b,r,u,...,v) = p(i,...,j,a,...,b,u,...,v), \tag{4.5.4}$$
$$P^2_{i,...,j,p(a,...,b),u,...,v} = P^2_{i,...,j,a,...,b,u,...,v}, \tag{4.5.5}$$
$$P^2_{i,...,j,r,a,...,b,r,u,...,v} = P^2_{i,...,j,a,...,b,u,...,v}, \tag{4.5.6}$$

which ensure that the explicit propagator momenta always are properly resolved in terms of external momenta. Naturally, the order of indices $i,...,j$ appearing in $p(i,...,j)$ and $P^2_{i,...,j}$ is irrelevant and can be assumed to be sorted to make it easier to identify and group together identical expressions.

It is important to note that the sums in the functions (4.5.1) and (4.5.2) may contain summands that immediately vanish due to trivial helicity configurations of sub-amplitudes involved that enter the **H** function.[18] Setting such summands to zero directly without allowing for any recursion depth in such terms greatly speeds up the calculation.

[18]By trivial helicity configuration we mean amplitudes with none, or only one negative helicity gluon, as well as amplitudes with none, or only one positive helicity gluon (special care is required for 3-point amplitudes due to special kinematics).

Recursion by means of (4.5.1) and (4.5.2) with the above supplements will generate all possible diagrams in MHV expansion that contribute to leading order in the soft limit. However, the simple summation employed here comes at the expense of multiple counting for some of the resulting diagrams. The easiest way to remove the over-counting is to simply set the integer coefficient in front of each overall summand to 1 after the recursion has been completed and all terms have been properly grouped together:

$$\mathbf{S}' \equiv \mathbf{S} \text{ with multiplicity of each overall summand set to 1,} \tag{4.5.7}$$

which implies that invariance of amplitudes under cyclic permutation of external legs is used to identify and group together equivalent terms in the expansion. This, as well as the entire recursive procedure, can be easily automated i.e. in *Mathematica*, such that the m-soft factor S_m for any helicity configuration is automatically generated by the input:

$$S_m = \mathbf{S}'\left(A^{(m)}(k_n^{+1}, k_1^{h_1}, ..., k_m^{h_m}, k_{q_{m+1}}^{+1})\right). \tag{4.5.8}$$

Finally, to evaluate the soft factor we use the substitutions

$$A^{(m)}(k_{q_1}^{+1}, k_{q_2}^{+1}, ..., k_{q_l}^{+1}) \to \frac{\langle n-1, n\rangle\langle n, m+1\rangle\langle m+1, m+2\rangle}{\langle n-1, q_1\rangle\left(\prod_{i=1}^{l-1}\langle q_i, q_{i+1}\rangle\right)\langle q_l, m+2\rangle}, \tag{4.5.9}$$

$$A_l(k_{q_1}^{+1}, ..., k_{q_{i-1}}^{+1}, k_{q_i}^{-1}, k_{q_{i+1}}^{+1}, ..., k_{q_{j-1}}^{+1}, k_{q_j}^{-1}, k_{q_{j+1}}^{+1}, ..., k_{q_l}^{+1}) \to \frac{\langle q_i, q_j\rangle^4}{\langle q_l, q_1\rangle \prod_{i=1}^{l-1}\langle q_i, q_{i+1}\rangle}, \tag{4.5.10}$$

where entries like $|p(i, ..., j)\rangle$ are evaluated by the usual CSW replacement $P_{i,...,j}|X]$ with reference spinor $|X]$. Superficially, due to (4.5.9) it might seem that the soft factor depends on $(n-1)$-st and $(m+2)$-nd external momentum as well. However, just as in [114], this dependence always cancels out upon the CSW replacement of the shifted spinors at leading order in τ.

We have tested the above recursive procedure for soft factors $S_1,\ S_2, ..., S_7$ with various helicity configurations against appropriate amplitude ratios obtained from the GGT package [117], and found numerical agreement at leading order in τ. For example, our recursion takes about two minutes to generate the 2277 different analytic terms in the $S_7^{------+}$ soft factor. If required, a trivial further expansion in τ can be used to isolate leading terms only.

4.6 Four-soft gluons from BCFW

Naturally, it is also possible to apply BCFW recursion relations [3] to compute higher soft factors. Here we demonstrate the four-soft gluon calculation. We pick gluons $1, 2, 3, 4$ to be soft and perform a $[23\rangle$ BCFW shift, so that $2 \to \hat{2}$ and $3 \to \hat{3}$ with

$$|\hat{2}\rangle = |2\rangle \quad , \quad |\hat{2}] = |2] + z|3] \quad , \quad |\hat{3}\rangle = |3\rangle - z|2\rangle \quad , \quad |\hat{3}] = |3]. \tag{4.6.1}$$

It is trivial to see that under this shift only the following four diagrams could possibly contribute to the leading soft factor with any helicity configuration:

$$S_{4,A} = A_4(n, 1, \hat{2}, -\hat{P}_{n12}) \frac{1}{s_{n12}} S_2(\hat{P}_{n12}, \hat{3}, 4, 5), \tag{4.6.2}$$

$$S_{4,B} = A_3(1, \hat{2}, -\hat{P}_{12}) \frac{1}{s_{12}} S_3(n, \hat{P}_{12}, \hat{3}, 4, 5), \tag{4.6.3}$$

$$S_{4,C} = A_4(-\hat{P}_{345}, \hat{3}, 4, 5) \frac{1}{s_{345}} S_2(n, 1, \hat{2}, \hat{P}_{345}), \tag{4.6.4}$$

$$S_{4,D} = A_3(-\hat{P}_{34}, \hat{3}, 4) \frac{1}{s_{34}} S_3(n, 1, \hat{2}, \hat{P}_{34}, 5), \tag{4.6.5}$$

while the complete four-soft gluon factor is given by

$$S_4 = S_{4,A} + S_{4,B} + S_{4,C} + S_{4,D} \tag{4.6.6}$$

in each case. Here, A_3, A_4 are mostly-soft-leg sub-amplitudes factored by BCFW, and S_2, S_3 are two- and three-soft gluon factors that are extracted from the mostly-hard-leg sub-amplitudes factored by BCFW. The usual on-shell constraints $\hat{P}^2_{...} = 0$ provide the following z values to leading order in the soft limit:[19]

$$z_A = \frac{-s_{n12}}{\langle 2n\rangle [n3]} \quad , \quad z_B = -\frac{[12]}{[13]} \quad , \quad z_C = \frac{s_{345}}{\langle 25\rangle [53]} \quad , \quad z_D = \frac{\langle 34\rangle}{\langle 24\rangle}. \tag{4.6.7}$$

In case when all four soft gluons have the same helicity, the four-soft factor trivially reduces to a product of consecutive soft factors. In the following, we specify explicit helicity configurations and obtain the results for all analytically distinct non-trivial helicity configurations.

[19] We use the convention $s_{ij} = \langle ij\rangle [ji]$, which with our spinor contraction conventions ($\langle ij\rangle = \lambda_i^1 \lambda_j^2 - \lambda_i^2 \lambda_j^1$ and $[ij] = \tilde{\lambda}_i^2 \tilde{\lambda}_j^1 - \tilde{\lambda}_i^1 \tilde{\lambda}_j^2$) corresponds to $(+,-,-,-)$ Minkowski metric signature.

Helicity configuration $(-+++)$**:**

For the helicity configuration of soft gluons $(1^-, 2^+, 3^+, 4^+)$ we find:

$$S_{4,A}^{-+++} = \frac{[3\,n]^3\langle 1\,n\rangle^3\langle 5\,n\rangle}{s_{n12}s_{n123}\langle 1\,2\rangle\langle 4\,5\rangle\langle n|k_{12}|3]\langle 4|k_{n123}k_{n1}|2\rangle}, \tag{4.6.8}$$

$$S_{4,B}^{-+++} = \frac{[2\,3]^3\langle n\,5\rangle}{s_{123}[1\,2]\langle 4\,5\rangle\langle 4|k_{23}|1]\langle n|k_{12}|3]}, \tag{4.6.9}$$

$$S_{4,C}^{-+++} = 0, \tag{4.6.10}$$

$$\begin{aligned} S_{4,D}^{-+++} = & \frac{\langle n\,5\rangle\langle 4|k_{23}|n]^3}{\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle[n\,1]\langle 4|k_{23}|1]\langle 4|k_{123}|n]\langle 2|k_{n1}k_{n123}|4\rangle} + \\ & + \frac{\langle 1\,5\rangle^3[n\,5]}{s_{12345}\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5|k_{1234}|n]} + \frac{\langle n\,5\rangle\langle 1|k_{234}|n]^3}{s_{1234}s_{n1234}\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4|k_{123}|n]\langle 5|k_{1234}|n]}. \end{aligned} \tag{4.6.11}$$

To see that the diagram C is zero, we use the fact that the soft factor is independent of the helicity of particle 5, thus we can choose it to be 5^+ which leads to no non-vanishing helicity configurations for A_4. In all other diagrams only one helicity configuration is non-vanishing. We tested the above result numerically against amplitude ratios and found agreement.

Helicity configuration $(+-++)$**:**

For the helicity configuration of soft gluons $(1^+, 2^-, 3^+, 4^+)$ we find:

$$S_{4,A}^{+-++} = \frac{[3\,n]^3\langle 2\,n\rangle^4\langle 5\,n\rangle}{s_{n12}s_{n123}\langle 1\,2\rangle\langle 4\,5\rangle\langle 1\,n\rangle\langle n|k_{12}|3]\langle 4|k_{n123}k_{n1}|2\rangle}, \tag{4.6.12}$$

$$S_{4,B}^{+-++} = \frac{[1\,3]^4\langle n\,5\rangle}{s_{123}[1\,2][2\,3]\langle 4\,5\rangle\langle 4|k_{23}|1]\langle n|k_{12}|3]}, \tag{4.6.13}$$

$$S_{4,C}^{+-++} = 0, \tag{4.6.14}$$

$$\begin{aligned} S_{4,D}^{+-++} = & \frac{\langle 5\,n\rangle}{\langle 2\,3\rangle\langle 3\,4\rangle\langle 5|k_{234}|1]}\left(\frac{[1\,5]^3\langle 2\,5\rangle^4}{s_{12345}s_{2345}\langle 4\,5\rangle\langle 2|k_{345}k_{12345}|n\rangle} + \frac{\langle 2|k_{34}|1]^4}{s_{1234}s_{234}\langle 4|k_{23}|1]\langle 2|k_{34}k_{1234}|n\rangle}\right) \\ & + \frac{\langle 2\,n\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 1\,n\rangle\langle 2|k_{n1}k_{n1234}|5\rangle}\left(\frac{\langle 2\,5\rangle^3[n\,5]}{\langle 4\,5\rangle\langle 2|k_{345}k_{12345}|n\rangle} + \frac{\langle 2\,n\rangle\langle 5\,n\rangle\langle 2|k_{34}|n]^3}{s_{n1234}\langle 2|k_{n1}k_{n123}|4\rangle\langle n|k_{1234}k_{34}|2\rangle}\right). \end{aligned} \tag{4.6.15}$$

Diagram C vanishes the same way as described above. In all other diagrams again only one helicity configuration is non-vanishing. We tested the above result numerically against amplitude ratios and found agreement.

Helicity configuration $(+--+)$**:**

For the helicity configuration of soft gluons $(1^+, 2^-, 3^-, 4^+)$ we find:

$$S_{4,A}^{+--+} = \frac{\langle 2\,n\rangle^3}{s_{\rm n12}\langle 1\,2\rangle\langle 1\,n\rangle\langle n|k_{12}|3]\langle 2|k_{\rm n1}k_{\rm n1234}|5\rangle}\left(\frac{[4\,n]^3\langle 2\,n\rangle\langle 5\,n\rangle}{s_{\rm n1234}[3\,4]}+\right. \\ \left. + \frac{[5\,n]\langle 2|k_{\rm n1}k_{\rm n123}|5\rangle^3}{\langle 4\,5\rangle\langle 2|k_{\rm n1}k_{\rm n123}|4\rangle\,(s_{345}[3\,n]\langle 2\,n\rangle + s_{\rm n12}[3\,5]\langle 2\,5\rangle)}\right) \tag{4.6.16}$$

$$S_{4,B}^{+--+} = \frac{\langle n|k_{23}|1]^3}{s_{123}[1\,2][2\,3]\langle n|k_{12}|3]\,(\langle 5|k_{1234}|n]\langle n|k_{23}|1] - \langle 5\,n\rangle[1|k_{23}k_{123}|n])}\times \\ \times\left(\frac{[4\,n]^3\langle 5\,n\rangle\langle n|k_{23}|1]}{s_{\rm n123}s_{\rm n1234}\langle n|k_{123}|4]} + \frac{[5\,n]\langle 5|k_{23}|1]^3}{\langle 4\,5\rangle\langle 4|k_{23}|1]\,([4\,5]\langle 5\,n\rangle\langle 4|k_{23}|1] + \langle 5|k_{23}|1]\langle n|k_{1234}|5])}\right) \\ + \frac{1}{[1\,2][2\,3]\langle 5|k_{234}|1]}\left(\frac{[1\,4]^4\langle 5\,n\rangle}{s_{1234}[3\,4]\langle n|k_{123}|4]}+\right. \\ \left. + \frac{[1\,5]^3\langle n\,5\rangle\langle 5|k_{23}|1]^4}{s_{12345}\langle 4\,5\rangle\langle 4|k_{23}|1][1|k_{2345}k_{45}|3]\,([4\,5]\langle 5\,n\rangle\langle 4|k_{23}|1] + \langle 5|k_{23}|1]\langle n|k_{1234}|5])}\right), \tag{4.6.17}$$

$$S_{4,C}^{+--+} = \frac{[4\,5]^3\langle 2\,5\rangle^3}{s_{345}[3\,4]\langle 2|k_{34}|5]\langle 2|k_{345}k_{12345}|n\rangle}\left(\frac{[1\,5]^3\langle 2\,5\rangle\langle n\,5\rangle}{s_{12345}s_{2345}[1|k_{2345}k_{45}|3]}+\right. \\ \left. + \frac{[n\,5]\langle 2\,n\rangle^3}{\langle 1\,2\rangle\langle 1\,n\rangle\,(s_{345}[3\,n]\langle 2\,n\rangle + s_{\rm n12}[3\,5]\langle 2\,5\rangle)}\right), \tag{4.6.18}$$

$$S_{4,D}^{+--+} = \frac{\langle 2\,3\rangle^3}{s_{234}\langle 3\,4\rangle\langle 4|k_{23}|1]\langle n|k_{1234}|5]}\left(\frac{[n\,5]\langle n|k_{234}|1]^3}{s_{1234}s_{\rm n1234}\langle 2|k_{34}k_{1234}|n\rangle} + \frac{[1\,5]^3\langle n\,5\rangle}{s_{12345}\langle 2|k_{34}|5]}\right) \\ + \frac{\langle 2\,3\rangle^3[n\,5]\langle 2\,n\rangle^3}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 1\,n\rangle\langle 2|k_{34}|5]\langle 4|k_{\rm n123}k_{\rm n1}|2\rangle\langle 2|k_{34}k_{1234}|n\rangle}. \tag{4.6.19}$$

In all diagrams again only one helicity configuration is non-vanishing. We tested the above result numerically against amplitude ratios and found agreement.

Helicity configuration $(--++)$**:**

For the helicity configuration of soft gluons $(1^-, 2^-, 3^+, 4^+)$ we find:

$$S_{4,A}^{--++} = \frac{\langle 1\,2\rangle^3[3\,n]^3\langle 5\,n\rangle}{s_{\rm n12}s_{\rm n123}\langle 4\,5\rangle\langle 1\,n\rangle\langle n|k_{12}|3]\langle 4|k_{\rm n123}k_{\rm n1}|2\rangle}, \tag{4.6.20}$$

$$S_{4,B}^{--++} = \frac{1}{s_{123}[1\,2][2\,3]\langle 4|k_{23}|1]\langle 5|k_{1234}|n]}\left(\frac{\langle n\,5\rangle[n|k_{1234}k_{12}|3]^3}{s_{1234}s_{\rm n1234}\langle 4|k_{123}|n]} + \frac{[5\,n]\langle 5|k_{12}|3]^3}{s_{12345}\langle 4\,5\rangle}\right) \\ + \frac{[3\,n]^3\langle 5\,n\rangle}{s_{\rm n123}[1\,2][2\,3]\langle 4\,5\rangle[1\,n]\langle 4|k_{123}|n]}, \tag{4.6.21}$$

$$S_{4,C}^{--++} = 0, \tag{4.6.22}$$

$$S_{4,D}^{--++} = \frac{[n\,5]}{\langle 2\,3\rangle\langle 3\,4\rangle\langle 5|k_{234}|1]}\left(\frac{\langle 2\,5\rangle^3}{s_{2345}\langle 4\,5\rangle[1\,n]} + \frac{\langle 2|k_{34}k_{1234}|5\rangle^3}{s_{1234}s_{12345}s_{234}\langle 4|k_{23}|1]\langle 5|k_{1234}|n]}\right) \\ + \frac{\langle n\,5\rangle\langle 2|k_{34}|n]^3}{s_{234}s_{\rm n1234}\langle 2\,3\rangle\langle 3\,4\rangle[1\,n]\langle 4|k_{23}|1]\langle 5|k_{1234}|n]}. \tag{4.6.23}$$

Diagram C vanishes the same way as described above. In all other diagrams again only one helicity configuration is non-vanishing. We tested the above result numerically against amplitude ratios and found agreement.

Helicity configuration $(+-+-)$**:**

For the helicity configuration of soft gluons $(1^+, 2^-, 3^+, 4^-)$ we find:

$$S_{4,A}^{+-+-} = \frac{[3\,n]^3\langle 2\,n\rangle^4}{s_{\mathrm{n}12}\langle 1\,2\rangle\langle 1\,n\rangle[5|k_{\mathrm{n}1234}k_{\mathrm{n}12}|3]\langle n|k_{12}|3]}\times \tag{4.6.24}$$
$$\times\left(\frac{[3\,n][5\,n]\langle 4\,n\rangle^3}{s_{\mathrm{n}123}s_{\mathrm{n}1234}\langle 4|k_{\mathrm{n}123}k_{\mathrm{n}1}|2\rangle}+\frac{[3\,5]^3\langle n\,5\rangle}{[3\,4][4\,5]\,(s_{345}[3\,n]\langle 2\,n\rangle+s_{\mathrm{n}12}[3\,5]\langle 2\,5\rangle)}\right),$$

$$S_{4,B}^{+-+-} = \frac{[1\,3]^4[3\,5]^3\langle n\,5\rangle}{[1\,2][2\,3][3\,4][4\,5][3|k_{12}k_{1234}|5][1|k_{2345}k_{45}|3]\langle n|k_{12}|3]}+ \tag{4.6.25}$$
$$+\frac{[1\,3]^4}{s_{123}[1\,2][2\,3]\langle 4|k_{23}|1]\langle n|k_{1234}|5]}\left(\frac{\langle 5\,n\rangle\langle 4|k_{123}|5]^3}{s_{1234}s_{12345}[3|k_{12}k_{1234}|5]}+\frac{[5\,n]\langle 4\,n\rangle^3}{s_{\mathrm{n}1234}\langle n|k_{12}|3]}\right),$$

$$S_{4,C}^{+-+-} = \frac{[3\,5]^4\langle 2\,5\rangle^3}{s_{345}[3\,4][4\,5]\langle 2|k_{34}|5]\langle n|k_{12345}k_{345}|2\rangle}\times \tag{4.6.26}$$
$$\times\left(\frac{[1\,5]^3\langle 2\,5\rangle\langle 5\,n\rangle}{s_{12345}s_{2345}[1|k_{2345}k_{45}|3]}+\frac{[5\,n]\langle 2\,n\rangle^3}{\langle 1\,2\rangle\langle 1\,n\rangle\,(s_{345}[3\,n]\langle 2\,n\rangle+s_{\mathrm{n}12}[3\,5]\langle 2\,5\rangle)}\right),$$

$$S_{4,D}^{+-+-} = \frac{\langle 2\,4\rangle^4[n\,5]\langle 2\,n\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 1\,n\rangle\langle 2|k_{34}|5]\langle 2|k_{\mathrm{n}1}k_{\mathrm{n}123}|4\rangle\langle n|k_{1234}k_{34}|2\rangle}+ \tag{4.6.27}$$
$$+\frac{\langle 2\,4\rangle^4}{s_{234}\langle 2\,3\rangle\langle 3\,4\rangle\langle 4|k_{23}|1]\langle n|k_{1234}|5]}\left(\frac{[n\,5]\langle n|k_{234}|1]^3}{s_{1234}s_{\mathrm{n}1234}\langle 2|k_{34}k_{1234}|n\rangle}+\frac{[1\,5]^3\langle n\,5\rangle}{s_{12345}\langle 2|k_{34}|5]}\right).$$

In all diagrams again only one helicity configuration is non-vanishing. We tested the above result numerically against amplitude ratios and found agreement.

Chapter 5

Polynomial Reduction and Evaluation of Tree- and Loop-Level CHY Amplitudes

This chapter is based on the publication [120].

After the amazing discovery of the relation between perturbative gauge theory and twistor string theory by Witten [121], there have been several developments on computing scattering matrices in various theories from a moduli space on a punctured sphere [122, 123, 124, 125, 126]. Cachazo, He and Yuan (CHY) proposed the equations governing the map from the space of kinematic invariants to the moduli space to be the same in each case and independent of the particular spacetime dimension. This led them to search for a more general formulation of scattering matrices in arbitrary dimension. Deriving some inspiration from a formula for MHV gravity amplitudes due to Hodges [127, 128, 129], CHY went on to discover their new formulation for amplitudes in a range of theories in [7, 5, 8], and later [9, 10]. This so called CHY formulation produces tree level n-point scattering amplitudes for massless particles in arbitrary dimension by means of $(n-3)$ moduli integrations localizing so called scattering equations. The scattering equations first appeared in the work of Fairlie and Roberts [130, 131], and later Gross and Mende [132], as well as more recently Witten [133], and from the string theory classical worldsheet perspective in[1] [134, 135]. Soon after the CHY equations made their appearance, the scalar and gluon cases were proven directly [136] by means of BCFW recursion relations [3, 4]. Subsequently generalizations appeared, extending the formulation

[1]The author thanks P. Caputa for pointing out this last point.

in terms of scattering equations to involve i.e. massive particles [136, 137, 138, 139], fermions [140], supersymmetric theory [141, 142], one-loop amplitudes [143, 144, 145], QCD related amplitudes [146], off-shell amplitudes [147], or comparison to a string theory setting [148, 149, 150].

The most direct approach to evaluate amplitudes in CHY formulation was to try and find solutions to the scattering equations in general [151, 152], or solve at special kinematics [153, 154]. The scattering equations could also be reformulated in a polynomial form [11, 12]. However, it became clear that solving scattering equations is very non-trivial and is not the most convenient way of evaluating amplitudes. Subsequently, techniques that avoid explicit solving of scattering equations started to emerge [155]. Contour deformations in the moduli integrals led to diagrammatic prescriptions that can be used to evaluate separate amplitude building blocks [156, 157, 158, 159, 160]. An algebraic approach to evaluating scattering amplitudes in CHY formulation involving so-called companion matrices was suggested in [161]. For a comparison of this method with an elimination theory based technique see [162]. One further algebraic technique involving polynomial inversion of moduli differences on the support of the ideal spanned by scattering equations, as well as the Bezoutian matrix to evaluate amplitudes was presented in [163]. Elimination theory was applied to scattering equations in polynomial form to obtain single variable polynomials [164, 165]. Loop level integrands have been shown to follow from higher dimensional massless tree-level amplitudes [166, 167]. Some further progress on evaluating CHY amplitudes was made in [168], diagrammatic techniques were generalized to compute higher order poles [169], and a double cover deformation of the moduli space led to evaluation of more general amplitude types as well [170]. Finally, monodromy relations were applied to Yang-Mills amplitudes in CHY representation to facilitate evaluation [171].

In this work we start by developing a polynomial degree reduction procedure for multivariate polynomials in σ-moduli on the support of gauge fixed scattering equations for any n. As a consequence we realize that the most general multivariate polynomial in σ-moduli can be reduced to contain what we call *ladder type* monomials only, with multivariate degree of at most $\frac{(n-3)(n-4)}{2}$ and coefficients rational in kinematic data. We say such a fully reduced polynomial is of *standard form*. Application of Hilbert's strong Nullstellensatz as well as our degree reduction procedure conceptually allows us to find a standard form polynomial expression for rational functions in the σ-moduli. Making use of the above findings, a CHY amplitude integrand of any theory at any n can be converted to a corresponding standard form polynomial. This general structural constraint is one of the main findings of the current work. After the polynomial reduction is carried out, we use the global residue theorem to derive a prescription to evaluate CHY amplitudes by collecting simple residues at infinity only. We note that only highest degree ladder type monomials contribute to any such amplitude integral, and since we find only simple poles the evaluation step is trivial. The difficulty is shifted towards finding standard form polynomial integrands for CHY amplitudes. We demonstrate the prescription on explicit examples of amplitude integrands at tree and one-loop level.

This chapter is organized as follows. In section 5.1 we review the CHY formulation of tree-level scattering amplitudes for massless ϕ^3 scalar theory as an example. As a warm up, section 5.2 shows a five point amplitude calculation to motivate our further investigation in section 5.3. Section 5.3.1 describes the degree reduction of multivariate polynomials to the standard form, and section 5.3.2 extends the reduction procedure to rational functions, on the support of gauge fixed scattering equations. Subsequently, section 5.3.3 describes the global residue theorem based proof for our amplitude evaluation prescription after polynomial reduction is applied to the integrands. In section 5.3.4 we give explicit examples on how amplitudes are evaluated making use of our new method. We go on to consider 1-loop amplitudes in section 5.4, where we determine gauge fixed polynomial scattering equations that are free of singular solutions in the forward limit. Section 5.4.1 contains a few amplitude evaluation examples at 1-loop. We conclude in section 5.5. Appendix 5.6 suggests a simple method to generate real rational on-shell momenta based on Euclid's Pythagorean triple parametrization.

Note

J. Bosma, M. Søgaard and Y. Zhang published a paper with similar results in [172].

5.1 CHY formulation of tree level scattering amplitudes

The Cachazo-He-Yuan (CHY) formulation of tree-level scattering amplitudes for massless particles in arbitrary dimension was introduced in [7, 5]. In CHY representation, the map of kinematic data to the moduli space is governed by the rational scattering equations

$$f_a = \sum_{b=1,b\neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} \qquad \forall a \in \{1, 2, ..., n\}. \tag{5.1.1}$$

Dolan and Goddard transformed the original amplitude expression to involve polynomial scattering equations [11]. In what follows, it will be more convenient for us to work with polynomial scattering equations, therefore we will use the latter form for i.e. an n-point scalar ϕ^3 amplitude in the examples to follow:

$$A_n = \int \left(\prod_{\substack{c=1 \\ c \neq q,p,w}}^{n} d\sigma_c \right) (\sigma_{qp}\sigma_{pw}\sigma_{wq}) \left(\prod_{1 \le i < j \le n} \sigma_{ij} \right) \left(\prod_{a=2}^{n-2} \delta\left(\tilde{h}_a\right) \right) \frac{1}{(\sigma_{12}\sigma_{23}...\sigma_{n1})^2}. \tag{5.1.2}$$

Here the indices $1 \le q < p < w \le n$ are fixed and can be chosen arbitrarily without changing the result. Minkowski momenta of scattering external particles are denoted k_i, and the difference of moduli is abbreviated as $\sigma_{ij} = \sigma_i - \sigma_j$. There are $n - 3$ moduli integrations and the same amount of delta functions, such that the integral reduces to a sum over the solutions to the system of the scattering equations in the delta function arguments

$$\tilde{h}_i \equiv \sum_{\{q_1,...,q_i\} \subset \{1,2,...,n\}} \mathfrak{s}_{q_1,...,q_i} \prod_{j=1}^{i} \sigma_{q_j} = 0. \tag{5.1.3}$$

In this formula the summation is over all possible unordered subsets of i different numbers $\{q_1, ..., q_i\}$ out of the integer sequence from 1 to n. Due to momentum conservation and massless on-shell conditions, the kinematic variables

$$\mathfrak{s}_{q_1,...,q_i} = \frac{1}{2}\left(\sum_{j=1}^{i} k_{q_j} \right)^2 \tag{5.1.4}$$

are only non-zero when at least 2 or at most $n - 2$ indices are provided. Therefore, exactly $n - 3$ scattering equations (5.1.3) from $\tilde{h}_2$ through $\tilde{h}_{n-2}$ are nontrivial.

In the following we will be working with the particular gauge choice $\sigma_1 = \infty$, $\sigma_2 = 0$ and $\sigma_3 = 1$ for convenience. For this purpose we define the gauge fixed polynomial scattering equations:

$$h_i \equiv \left(\lim_{\sigma_1 \to \infty} \frac{1}{\sigma_1} \tilde{h}_{i+1} \right) \Big|_{\substack{\sigma_2 = 0 \\ \sigma_3 = 1}} = 0 \quad , \quad \forall i \in \{1, 2, ..., n - 3\}. \tag{5.1.5}$$

Correspondingly, we will fix the free indices in (5.1.2) as $q = 1, p = 2, w = 3$.

5.2 Warm up: five point tree level scalar amplitude

At five points we have two scattering equations:

$$h_1 = \sigma_4 \mathfrak{s}_{1,4} + \sigma_5 \mathfrak{s}_{1,5} + \mathfrak{s}_{1,3} = 0,$$

$$h_2 = \sigma_4 \sigma_5 \mathfrak{s}_{2,3} + \sigma_5 \mathfrak{s}_{2,4} + \sigma_4 \mathfrak{s}_{2,5} = 0.$$

The gauge fixed scattering amplitude for scalars becomes

$$A_5^{\phi^3} = \oint \frac{d\sigma_4 d\sigma_5}{h_1 h_2} \frac{\sigma_4\sigma_5 (1-\sigma_5)}{(1-\sigma_4)(\sigma_4-\sigma_5)}, \tag{5.2.1}$$

where the delta functions have been mapped to simple poles as usual, and the integration contour is such that both poles are localized. We would like to transform the integrand such that an evaluation via contour deformation becomes simpler. For that end, consider the following equality

$$\sigma_4\sigma_5 (1-\sigma_5) \triangleq (1-\sigma_4)(\sigma_4-\sigma_5) N_5^{\phi^3} \tag{5.2.2}$$

where $\triangleq$ shall denote equivalence on the support of scattering equations. Here $N_5^{\phi^3}$ clearly corresponds to the explicit integrand part of (5.2.1). We claim that (5.2.2) can be realized i.e. by the following Ansatz

$$N_5^{\phi^3} = c_1\sigma_4 + c_2\sigma_5. \tag{5.2.3}$$

To show that this is indeed the case, we can first solve $h_1 = 0$ for either σ_4 or σ_5, and solve $h_2 = 0$ for $\sigma_4\sigma_5$:

$$\sigma_4 = -\frac{\mathfrak{s}_{1,5}}{\mathfrak{s}_{1,4}}\sigma_5 - \frac{\mathfrak{s}_{1,3}}{\mathfrak{s}_{1,4}}, \qquad \sigma_5 = -\frac{\mathfrak{s}_{1,4}}{\mathfrak{s}_{1,5}}\sigma_4 - \frac{\mathfrak{s}_{1,3}}{\mathfrak{s}_{1,5}}, \tag{5.2.4}$$

$$\sigma_4\sigma_5 = -\frac{\mathfrak{s}_{2,4}}{\mathfrak{s}_{2,3}}\sigma_5 - \frac{\mathfrak{s}_{2,5}}{\mathfrak{s}_{2,3}}\sigma_4. \tag{5.2.5}$$

Then we start with (5.2.2) making use of (5.2.3), expand both sides of the equation, and iterate the following substitution rules:

1. Whenever we encounter a monomial featuring both σ_4 and σ_5, we isolate the highest power of $\sigma_4\sigma_5$, substitute in the right hand side of (5.2.5) and expand - this leads to an overall multivariate degree reduction in monomials.

2. Whenever we encounter a monomial featuring σ_4 xor σ_5 to a power higher than one, we isolate a single power of σ_4 xor σ_5 respectively, substitute it by the right hand side of the respective equation in (5.2.4) and expand - this leads either to an overall degree reduction in monomials, or to creation of new $\sigma_4\sigma_5$ terms.

Iterating the above two steps a few times reduces both sides of (5.2.2) to only the two monomials σ_4 and σ_5 with some constant coefficients.[2] Collecting all terms on one side of the equation and demanding that the overall coefficients of monomials σ_4 and σ_5 vanish identically, we obtain a set of two linear equations in two unknowns c_1 and c_2. Solving these equations yields one possible solution for the Ansatz $N_5^{\phi^3}$, i.e.

$$c_1 = \frac{\mathfrak{s}_{1,4}\mathfrak{s}_{2,5}\left(\left(\mathfrak{s}_{1,3}+\mathfrak{s}_{1,5}\right)\mathfrak{s}_{2,4}+\mathfrak{s}_{1,4}\left(\mathfrak{s}_{2,3}+2\mathfrak{s}_{2,4}+\mathfrak{s}_{2,5}\right)\right)}{\left(\left(\mathfrak{s}_{1,3}+\mathfrak{s}_{1,4}\right)\left(\mathfrak{s}_{2,3}+\mathfrak{s}_{2,4}\right)-\mathfrak{s}_{1,5}\mathfrak{s}_{2,5}\right)\left(\mathfrak{s}_{1,3}\mathfrak{s}_{2,3}-\left(\mathfrak{s}_{1,4}+\mathfrak{s}_{1,5}\right)\left(\mathfrak{s}_{2,4}+\mathfrak{s}_{2,5}\right)\right)},$$
$$c_2 = \frac{\mathfrak{s}_{2,4}\left(\mathfrak{s}_{1,3}+\mathfrak{s}_{1,4}\right)}{\mathfrak{s}_{1,5}\mathfrak{s}_{2,5}-\left(\mathfrak{s}_{1,3}+\mathfrak{s}_{1,4}\right)\left(\mathfrak{s}_{2,3}+\mathfrak{s}_{2,4}\right)}+\frac{\mathfrak{s}_{1,5}\mathfrak{s}_{2,4}}{\left(\mathfrak{s}_{1,4}+\mathfrak{s}_{1,5}\right)\left(\mathfrak{s}_{2,4}+\mathfrak{s}_{2,5}\right)-\mathfrak{s}_{1,3}\mathfrak{s}_{2,3}}.$$

Deforming the integration contours to infinity consecutively, we find only simple poles and get [3]

$$A_5^{\phi^3} = \oint d\sigma_4 d\sigma_5 \frac{N_5^{\phi^3}}{h_1 h_2} = \frac{c_1}{\mathfrak{s}_{1,4}\mathfrak{s}_{2,3}} - \frac{c_2}{\mathfrak{s}_{1,5}\mathfrak{s}_{2,3}}. \tag{5.2.6}$$

Using momentum conservation and the fact that all external particles are massless, we can re-express the above in the following familiar form

$$A_5^{\phi^3} = \frac{1}{\mathfrak{s}_{1,2}\mathfrak{s}_{3,4}} + \frac{1}{\mathfrak{s}_{5,1}\mathfrak{s}_{2,3}} + \frac{1}{\mathfrak{s}_{4,5}\mathfrak{s}_{1,2}} + \frac{1}{\mathfrak{s}_{3,4}\mathfrak{s}_{5,1}} + \frac{1}{\mathfrak{s}_{2,3}\mathfrak{s}_{4,5}}, \tag{5.2.7}$$

confirming that the result we found is indeed the correct five point massless scalar amplitude in ϕ^3 theory. In the following section we will generalize the above technique to all n.

5.3 Amplitude structure and evaluation prescription

Our plan is to show that any multivariate polynomial on the support of scattering equations can be written in a specific monomial structure we call the *standard form*. Subsequently, we show that any rational function that is finite and non-vanishing on the support of scattering equations can be written as a standard form polynomial. Lastly, we apply these findings to amplitude integrands, convert them to standard form polynomials and evaluate the amplitude by means of the global residue theorem while collecting simple pole residues at infinity only.

[2] The exact coefficients are not necessarily unique and might depend on the order of substitutions during the reduction.

[3] In what follows, we give more details on this, from the point of view of global residue theorem.

5.3.1 Degree reduction of polynomials to a standard form

In this section we start with an arbitrary multivariate polynomial N in the $n-3$ different σ-moduli that are not gauge fixed (substitute $n \to n+2$ everywhere for 1-loop), and show that any such polynomial can be degree reduced to a very specific form.

Conventions: Consider a generic monomial M within polynomial N separately:

$$M = C\sigma_{q_1}^{p_1}\sigma_{q_2}^{p_2}...\sigma_{q_{m_{max}}}^{p_{m_{max}}}. \tag{5.3.1}$$

C is an overall constant, $q_1 \neq q_2 \neq ... \neq q_{m_{max}}$ label the different σ-moduli appearing in the monomial M, while $p_1, p_2, ..., p_{m_{max}}$ are the corresponding powers of each σ-modulus. We choose to always order all σ-moduli within each monomial such that $p_1 \leq p_2 \leq ... \leq p_{m_{max}}$. For convenience we define $p_0 \equiv 0$ for all M. Since there are at most $n-3$ different non-gauge fixed σ-moduli, we have $0 \leq m_{max} \leq n-3$ in general.[4]

Definition 1: We define a monomial M as introduced in the conventions above to be of *ladder type* if its moduli powers satisfy $0 \leq p_j - p_{j-1} \leq 1$ for all $j \in \{1, 2, ..., m_{max}\}$ when $m_{max} > 0$, and iff additionally the property $0 \leq m_{max} \leq n-4$ is satisfied.

Definition 2: We define a multivariate polynomial in the non-gauge fixed σ-moduli to be of *standard form* if it consists of *ladder type* monomials only, with coefficients rational in kinematic data. See Table 5.1 for some examples of ladder type monomials.

$n=4$	$n=5$	$n=6$
1	1, σ_4, σ_5	1, σ_4, σ_5, σ_6, $\sigma_4\sigma_5$, $\sigma_4\sigma_6$, $\sigma_5\sigma_6$, $\sigma_5\sigma_4^2$, $\sigma_6\sigma_4^2$, $\sigma_6\sigma_5^2$, $\sigma_4\sigma_5^2$, $\sigma_4\sigma_6^2$, $\sigma_5\sigma_6^2$

Table 5.1: Examples of all ladder type monomials for the first few n. ($\sigma_1, \sigma_2, \sigma_3$ gauge fixed.)

Theorem 1: *On the support of the ideal spanned by scattering equations, an arbitrary regular multivariate polynomial N in the $n-3$ non-gauge fixed moduli, with coefficients rational in kinematic data, is equivalent to at least one* standard form *polynomial N' that consists of ladder type monomials only, with coefficients rational in kinematic data.*

Proof: To prove this we use flow arguments induced by scattering equation based transformations in the space of moduli powers within monomials. The arguments consist of the following two steps.

[4] The case $m_{max} = 0$ corresponds to only C being present in (5.3.1).

Step 1: Reduction of monomials to $0 \le p_j - p_{j-1} \le 1$ for all $j \in \{1, 2, ..., m_{max}\}$

Consider a generic monomial of an arbitrary polynomial

$$C \underbrace{\sigma_{q_1}^{p_1} \sigma_{q_2}^{p_2} ... \sigma_{q_{j-1}}^{p_{j-1}}}_{w_1 \text{ terms}} \underbrace{\sigma_{q_j}^{p_j} ... \sigma_{q_{m_{max}}}^{p_{m_{max}}}}_{w_2 \text{ terms}}, \tag{5.3.2}$$

for some fixed $1 \le j \le m_{max}$, ordered as $p_1 \le p_2 \le ... \le p_{m_{max}}$ and such that $p_j - p_{j-1} > 1$, so that the monomial is non-ladder type. Also note that $0 \le (w_1 + w_2 = m_{max}) \le n - 3$. If we want to transform this monomial into a sum over ladder type monomials, we first have to reduce the discrepancy $p_j - p_{j-1} > 1$ to $0 \le p_j - p_{j-1} \le 1$. We employ the scattering equations to do that as follows.

The general structure of gauge fixed polynomial scattering equations $h_a = 0$ for $a = 1, ..., n-3$ is such that h_a features all possible multilinear monomials of degree a and $a - 1$ respectively. Therefore, we can solve the scattering equation $h_{w_2} = 0$ for the monomial $\underbrace{\sigma_{q_j} ... \sigma_{q_{m_{max}}}}_{w_2 \text{ terms}}$:

$$\sigma_{q_j} ... \sigma_{q_{m_{max}}} = \sigma_{q_j} ... \sigma_{q_{m_{max}}} - \frac{h_{w_2}}{(\partial_{\sigma_{q_j}} ... \partial_{\sigma_{q_{m_{max}}}} h_{w_2})}. \tag{5.3.3}$$

The derivatives in the denominator isolate the coefficient of monomial $\sigma_{q_j} ... \sigma_{q_{m_{max}}}$ within h_{w_2}. This coefficient is canceled for the corresponding summand in the numerator and the pure monomial is subtracted. Therefore, the right hand side of (5.3.3) features all possible multilinear monomials of degree $w_2 - 1$ and all multilinear monomials of degree w_2 except for $\sigma_{q_j} ... \sigma_{q_{m_{max}}}$.

We can now isolate $(\sigma_{q_j} ... \sigma_{q_{m_{max}}})^{\lfloor \frac{p_j - p_{j-1}}{2} \rfloor}$ moduli from the w_2 terms in (5.3.2), substitute them by the right hand side of (5.3.3) to the power $\lfloor \frac{p_j - p_{j-1}}{2} \rfloor$ and expand.[5] Since each multilinear monomial of a certain degree is unique up to a constant factor, this has the effect that in each of the resulting terms

- the power of at least one modulus in the w_2 terms is reduced by at least one,
- the power of at least one modulus in the w_1 terms is increased by at least one[6], or the overall degree is reduced.

Since the above guarantees a non-zero flow in the distribution of σ-moduli powers away from w_2 terms either into the w_1 terms or into overall degree reduction, iteration of the substitution rule for all j and each monomial in the resulting terms is bound to reach a fixed point. This fixed point is straightforwardly given by the state where all monomials obey $0 \le p_j - p_{j-1} \le 1$ for all $j \in \{1, 2, ...\}$

[5]The notation $\lfloor x \rfloor$ means the floor function, returning the biggest integer $\le x$.

[6]Note that the power of this modulus could have been zero initially.

within each respective monomial, since then $\lfloor \frac{p_j - p_{j-1}}{2} \rfloor = 0$ for all j and no substitutions can be carried out any more.

Step 2: Reduction of monomials to $m_{max} \leq n-4$

After step 1 is applied to all monomials in a polynomial N, it can still contain monomials with the maximal number of different moduli $m_{max} = n-3$:

$$C\sigma_{q_1}^{p_1}\sigma_{q_2}^{p_2}...\sigma_{q_{n-3}}^{p_{n-3}}, \tag{5.3.4}$$

with $p_1 = 1$ and $0 \leq p_j - p_{j-1} \leq 1$ for all $j \in \{2, 3, ..., n-3\}$. Similar to (5.3.3), we can solve the gauge fixed polynomial scattering equation $h_{n-3} = 0$ for the single highest degree multilinear monomial $\sigma_{q_1}\sigma_{q_2}...\sigma_{q_{n-3}}$. Since that yields only multilinear terms of degree $n-4$, this necessarily leads to a degree reduction. We isolate the highest power of $\sigma_{q_1}\sigma_{q_2}...\sigma_{q_{n-3}}$ from monomials such as (5.3.4), make the substitution obtained from $h_{n-3} = 0$ and expand. Due to the guaranteed degree reduction in this step, we are again bound to iteratively reach a fixed point. This fixed point is trivially given by the condition $m_{max} \leq n-4$ for all resulting monomials M, since then no highest degree multilinear monomial can be isolated within the monomials, and therefore no substitutions can be carried out any more.

Conclusion

Step 1 and 2 above can be applied consecutively and iteratively to an arbitrary multivariate polynomial N. Due to the guaranteed degree reduction in step 2, both fixed points are bound to be reached simultaneously eventually. Therefore, we have shown that any polynomial N on the support of scattering equations can be cast into a *standard form* N' containing only ladder type monomials.[7] Note that the degrees of the ladder type monomials M_{lt} are $0 \leq \deg(M_{lt}) \leq \frac{(n-3)(n-4)}{2}$ at n points. The full set of pure ladder type monomials at any n is symmetric in all moduli. This homogeneity follows from the homogeneity of scattering equations that are used to achieve this form.

5.3.2 Polynomial reduction of rational expressions

Theorem 2: *On the support of the ideal spanned by the scattering equations, any regular*[8] *multivariate rational function $\frac{P}{Q}$ in the $n-3$ non-gauge fixed moduli, where P and Q are polynomials*

[7] The coefficients stay rational in kinematic data since we only used a finite number of additions and multiplications, and the coefficients in the scattering equations are rational as well.

[8] By regular we mean non-infinite and non-zero on all solutions to the scattering equations.

with rational coefficients in kinematic data, is equivalent to at least one standard form *polynomial* N' *that consists of ladder type monomials only, with rational coefficients in kinematic data.*

Proof: Similar to some ideas of [163], we will make use of Hilbert's Nullstellensatz. Consider the following equation involving the set of gauge fixed polynomial scattering equations h_m and multivariate polynomials in the σ-moduli P, Q, a and a_m for $m \in \{1, 2, ..., n-3\}$

$$a\,Q + \sum_{m=1}^{n-3} a_m h_m = P. \tag{5.3.5}$$

The strong version of the Nullstellensatz guarantees that we can always find polynomials a and a_m for given polynomials P and Q such that (5.3.5) is satisfied, as long as the a, a_m, P and Q do not share common roots among themselves and with the set of scattering equation polynomials h_m. Considering the situation at the locus of solutions to the scattering equations, this simplifies to

$$a\,Q \triangleq P, \tag{5.3.6}$$

where we use the symbol $\triangleq$ to denote equivalence modulo the ideal spanned by the scattering equations. Thus, $a \triangleq \frac{P}{Q}$ is a polynomial expression for a rational function.[9] Due to Theorem 1, a standard form polynomial $N' \triangleq a$ must exist, which concludes the proof.

Construction: In the above proof we used the fact that a standard form polynomial $N' \triangleq a$ must exist, however the proof was not constructive. To construct an explicit N' corresponding to a given rational function $\frac{P}{Q}$ we have to work harder. In principle, this step could be realized by various techniques. Here we will make use of an ad hoc procedure as follows.

Since the ladder type monomials span a complete polynomial basis with rational coefficients on the support of scattering equations, we can make an ansatz $\tilde{N}'$ containing all ladder type monomials with unfixed coefficients to parametrize our ignorance of what N' actually is: $\tilde{N}'Q - P \triangleq 0$. Making use of an implementation of the degree reduction procedure for Theorem 1, we can find a standard form polynomial H' such that $H' \triangleq \tilde{N}'Q - P \triangleq 0$. Demanding that the overall coefficient of each monomial in H' vanishes separately, sets up a number of linear equations in (at least) the same number of unknown coefficients of $\tilde{N}'$.[10] Solving this set of equations fixes the coefficients and yields an N'.

[9] Dividing by Q is allowed since it is per assumption non-zero at the locus of solutions to the scattering equations.

[10] Since there is a finite number of ladder type monomials at any n, the number of unfixed coefficients in $\tilde{N}'$ is at least equal to the number of monomials in a most general resulting standard form H'. If H' has fewer than the maximum number of monomials, then the amount of unfixed coefficients is greater than the number of linear equations.

In practice, in many cases of interest the ansatz for $\tilde{N}'$ does not require all ladder type monomials to be present to find a valid standard form polynomial N'. This reduces the dimension and complexity of the linear set of equations one has to solve. Additionally, we will see in the next section that only the coefficients of the highest degree ladder type monomials have a non-vanishing contribution to an amplitude integral.

5.3.3 Collecting residues

In this section we concentrate on tree level amplitudes for concreteness. However, at every step in the following it should be clear that essentially the same logic applies to the loop level integrands. Therefore, the result we find is valid in general.

Theorem 3: *Any amplitude integral of the general shape*[11]

$$A_n = \oint \frac{N(\sigma_4, \sigma_5, ..., \sigma_n)}{\prod_{j=1}^{n-3} h_j} \prod_{i=4}^{n} d\sigma_i, \tag{5.3.7}$$

where h_j for $j = 1, ..., n-3$ are gauge fixed scattering equation polynomials, $N(\sigma_4, \sigma_5, ..., \sigma_n)$ is a standard form *polynomial in the $n-3$ non-gauge fixed moduli and the integration contour is initially localized at the locus of scattering equation solutions, can be evaluated by the following anti-symmetrized sum over the $(n-3)!$ different orders of consecutive infinity residues*[12]

$$A_n = (-1)^{n-3}(n-3)! \operatorname{Res}_{\sigma_{[n}=\infty,\ ...,\ \sigma_5=\infty,\ \sigma_{4]}=\infty} \left[\frac{N}{\prod_{j=1}^{n-3} h_j} \right]. \tag{5.3.8}$$

Note: Instead of calculating the $(n-3)!$ residues to evaluate the amplitude integral, it is possible to employ an integrand deformation in which the h_i's are replaced by their leading homogeneous parts $\mathrm{lt}(h_i)$. With this, the sum over residues equals one single residue at the origin by the transformation law of multivariate residues. This is an efficient alternative approach [173].[13]

Proof of Theorem 3: Starting with (5.3.7), it is straightforward to realize that any contour deformation away from the locus defined by the solutions to the scattering equations can possibly yield other residues only at infinity.

[11] Here, again, we consider the formulation where the delta functions have been mapped to simple poles with appropriate integration contours. Factors of $2\pi i$ are suppressed.

[12] The square brackets in $\sigma_{[n} = \infty,\ ...,\ \sigma_{4]} = \infty$ denote anti-symmetrization with respect to the moduli indices, so that i.e. $\operatorname{Res}_{\sigma_{[5}=\infty,\ \sigma_{4]}=\infty} = \frac{1}{2!}(\operatorname{Res}_{\sigma_5=\infty}\operatorname{Res}_{\sigma_4=\infty} - \operatorname{Res}_{\sigma_4=\infty}\operatorname{Res}_{\sigma_5=\infty})$. The right most residue operation always acts first.

[13] The author thanks the JHEP referee of the original paper for pointing this out.

Decompose the numerator polynomial of the integrand into monomials $N = \sum_i N_i$. By additivity of integrals, consider the contour integral in pieces involving just one monomial $N_q = M \propto \prod_{r=4}^n \sigma_r^{a_r}$ at a time, where the integer powers $a_r \geq 0$ are such that M is a ladder type monomial. Planning to investigate residues at infinity, we perform the substitution $\sigma_i \to 1/\sigma_i$ and $d\sigma_i \to -d\sigma_i/\sigma_i^2$ for $i \in \{4, 5, ..., n\}$, to focus on residues at zero instead, so that

$$\oint \frac{\prod_{r=4}^n \sigma_r^{a_r}}{\prod_{j=1}^{n-3} h_j} \prod_{i=4}^n d\sigma_i \to \oint \frac{(-1)^{n-3}}{\left(\prod_{j=1}^{n-3} \hat{h}_j\right)\left(\prod_{r=4}^n \sigma_r^{a'_r}\right)} \prod_{i=4}^n d\sigma_i, \tag{5.3.9}$$

where $a'_r = (a_r - n + 5)$ is an abbreviation for the new integer exponents, and $\hat{h}_j$ can be conveniently obtained from the gauge fixed scattering equations in the slightly different gauge $\sigma_1 = 0$, $\sigma_2 = \infty$, $\sigma_3 = 1$.[14]

Next we apply the global residue theorem (GRT), as for instance described in detail in [174]. Consider a contour integral in $n-3$ variables over an integrand $1/f_1 f_2 ... f_{n-3}$, such that the contours localize all possible poles in the integrand $f_i = 0, \forall i$. Since all possible residues are collected in this way, it follows from the GRT that the result must be zero:

$$\mathrm{Res}_{\{f_1, f_2, ..., f_{n-4}, f_{n-3}\}} = 0. \tag{5.3.10}$$

Using the above in our integrand of interest in (5.3.9), assign $f_i = \hat{h}_i$ for $i = \{1, 2, ..., n-4\}$ and $f_{n-3} = \hat{h}_{n-3} \prod_{r=4}^n {\sigma_r}^{a'_r}$. This clearly takes all possible poles into consideration, so that eq. (5.3.10) is satisfied. Expand the global residue as a sum over the poles in f_{n-3}:

$$\mathrm{Res}_{\{f_1, f_2, ..., f_{n-4}, f_{n-3}\}} = \mathrm{Res}_{\{f_1, f_2, ..., f_{n-4}, \hat{h}_{n-3}\}} + \sum_{t=4}^n \mathrm{Res}_{\{f_1, f_2, ..., f_{n-4}, {\sigma_t}^{a'_t}\}} = 0. \tag{5.3.11}$$

The first summand corresponds to (5.3.9), so that we can re-express it in terms of the other $n-3$ residues $\mathrm{Res}_{\{f_1, f_2, ..., f_{n-4}, \hat{h}_{n-3}\}} = -\sum_{t=4}^n \mathrm{Res}_{\{f_1, f_2, ..., f_{n-4}, {\sigma_t}^{a'_t}\}}$. Whenever partial poles in a multivariate residue calculation depend on one variable only, single variable complex analysis can be used to integrate out the corresponding residue separately. In our case each $\mathrm{Res}_{\{f_1, f_2, ..., f_{n-4}, {\sigma_t}^{a'_t}\}}$, among other poles, involves a pole $1/{\sigma_t}^{a'_t}$ dependent on a single variable σ_t, which we will now integrate out separately.

Considering that $a'_t = (a_t - n + 5)$ for each t, only highest degree ladder type monomials have a

[14] The set of scattering equations is invariant under simultaneous inversion $\sigma \to 1/\sigma$ of all σ-moduli (up to overall σ-moduli factors that here are accounted for by the powers a'_r), as long as we also invert the values of the gauge fixed moduli.

non-vanishing contribution to the integral, since exactly one of their a_t satisfies $a_t = n-4$ which produces a simple pole as $1/\sigma_t{}^{a'_t}$. For all other ladder type monomials we have $0 \leq a_t < n-4$ such that $a'_t \leq 0$ and $1/\sigma_t{}^{a'_t}$ ceases to be a pole and thus no residue is present.

To keep track of the correct contour orientation in the remaining variables, we anti-commute $d\sigma_t$ to one side in the integration measure $d\sigma_4 \wedge d\sigma_5 \wedge \ldots \wedge d\sigma_n = (\pm)_t d\sigma_t \prod_{i=4, i\neq t}^{n} (\wedge d\sigma_i)$. This produces an overall plus or minus $(\pm)_t$ dependent on the initial position t. Thus, we have

$$\oint \frac{\prod_{i=4}^{n} d\sigma_i}{\left(\prod_{j=1}^{n-3} \hat{h}_j\right)\left(\prod_{r=4}^{n} \sigma_r^{a'_r}\right)} = \begin{cases} -\oint \sum_{t=4}^{n} (\pm)_t \left(\dfrac{\prod_{\substack{i=4\\ i\neq t}}^{n} d\sigma_i}{\left(\prod_{j=1}^{n-3} \hat{h}_j\right)\left(\prod_{\substack{r=4\\ r\neq t}}^{n} \sigma_r^{a'_r}\right)} \right)_{\sigma_t=0} & \text{for } a'_t = 1 \\ 0 & \text{for } a'_t < 1 \end{cases}, \tag{5.3.12}$$

with the saturation $a'_t = 1$ occurring for exactly one of the moduli in each highest degree ladder type monomial (nevertheless, we sum over all $\sum_{t=4}^{n}$ since it is not known a priori which label t is going to yield the contribution).[15]

As $\sigma_t = 0$ in (5.3.12) is set, we find that $\hat{h}_{n-3}$ reduces to a single monomial $\hat{h}_{n-3}|_{\sigma_t=0} \propto \prod_{\substack{j=4\\ j\neq t}}^{n} \sigma_j$ by general structure of scattering equations. Therefore, the non-vanishing contribution schematically becomes

$$\oint \sum_{t=4}^{n} (\pm)_t \left(\frac{\prod_{\substack{i=4\\ i\neq t}}^{n} d\sigma_i}{\left(\prod_{j=1}^{n-3} \hat{h}_j\right)\left(\prod_{\substack{r=4\\ r\neq t}}^{n} \sigma_r^{a'_r}\right)} \right)_{\sigma_t=0} = \oint \sum_{t=4}^{n} (\pm)_t \frac{C_t \prod_{\substack{i=4\\ i\neq t}}^{n} d\sigma_i}{\left(\prod_{j=1}^{n-4} (\hat{h}_j|_{\sigma_t=0})\right)\left(\prod_{\substack{r=4\\ r\neq t}}^{n} \sigma_r^{a''_{r,t}}\right)}. \tag{5.3.14}$$

Here C_t is one over the constant coefficient of the single monomial that survives as we take $\hat{h}_{n-3}|_{\sigma_t=0} \propto \prod_{j=4, j\neq t}^{n} \sigma_j$, while the moduli of this monomial are accounted for by the new powers $a''{}_{r,t}$. The remaining $n-4$ scattering equation denominators $\hat{h}_j|_{\sigma_t=0}$ now have the same monomial structure as scattering equation polynomials at $n-1$ points. Therefore, we can treat each summand in the sum over t in (5.3.14) the same way as the initial expression (5.3.9), except now there is one fewer contour to integrate in each case. Thus, we can iterate. Noticing that by general structure of polynomial

[15] In terms of the expression in original variables on the left hand side of (5.3.9), this structurally means

$$\oint \frac{\prod_{r=4}^{n} \sigma_r^{a_r}}{\prod_{j=1}^{n-3} h_j} \prod_{i=4}^{n} d\sigma_i = -\oint \sum_{u=4}^{n} (\pm)_u \mathrm{Res}_{\sigma_u=\infty} \left[\frac{\prod_{r=4}^{n} \sigma_r^{a_r}}{\prod_{j=1}^{n-3} h_j} \right] \prod_{\substack{i=4\\ i\neq u}}^{n} d\sigma_i, \tag{5.3.13}$$

where we imply that there are at most first order poles at infinity.

scattering equations we always get single monomials as more and more σ_i are set to zero:

$$\hat{h}_{n-3}|_{\sigma_t=0} \propto \prod_{\substack{j=4\\ j\neq t}}^{n} \sigma_j, \qquad \hat{h}_{n-4}|_{\sigma_t=0,\ \sigma_l=0} \propto \prod_{\substack{j=4\\ j\neq t,l}}^{n} \sigma_j, \qquad \hat{h}_{n-5}|_{\sigma_t=0,\ \sigma_l=0,\ \sigma_c=0} \propto \prod_{\substack{j=4\\ j\neq t,l,c}}^{n} \sigma_j, \qquad \text{etc.}$$

ensures that each time a residue in a σ-modulus is collected, the remaining set of non-trivial scattering equation polynomials in the denominators is effectively reduced by one, as one of the scattering equation polynomials reduces to a single monomial and produces simple poles for the next iteration. With this, the above steps may be iterated from (5.3.9) to (5.3.14) $n-3$ times, while always expanding the resulting terms and summing over the process applied to one term at a time. Formally, each iteration adds one more level of signed infinity residue operations to (5.3.13). At the end of the day, when all contours have been treated, we are left with an anti-symmetrized sum over consecutive residue operations

$$\oint \frac{\prod_{r=4}^{n} \sigma_r^{a_r}}{\prod_{j=1}^{n-3} h_j} \prod_{i=4}^{n} d\sigma_i = (-1)^{n-3}(n-3)!\ \mathrm{Res}_{\sigma_{[n}=\infty,\ \ldots,\ \sigma_5=\infty,\ \sigma_{4]}=\infty} \left[\frac{\prod_{r=4}^{n} \sigma_r^{a_r}}{\prod_{j=1}^{n-3} h_j} \right]. \qquad (5.3.15)$$

This straightforwardly yields the full amplitude as we sum over all numerator monomials in the integrand, so that our final result for the amplitude is (5.3.8). This concludes the proof.

Due to the structure of standard form polynomials on the support of scattering equations we could rely on the fact that all residues we collect come from simple poles only. However, a straightforward generalization of the above steps yields the same result (5.3.8) even for cases where N is not a standard form polynomial and higher order residues are present.

It is interesting to note that the above procedure replaces a summation over $(n-3)!$ scattering equation solutions by a summation over the $(n-3)!$ different $(n-3)$-fold consecutive infinity residues in the σ-moduli. When N is a standard form polynomial, all residues come from simple poles, such that the map from the integrand to the final result is trivial. With this the difficulty of the problem is shifted towards finding a standard form polynomial numerator N. Applying the degree reduction procedure described in the previous section this corresponds to solving a linear set on the order of $(n-3)!$ equations.

5.3.4 Tree level amplitude examples

In the following we demonstrate the evaluation prescription (5.3.8) on ϕ^3 scalar amplitudes at tree level. We also consider specific examples that otherwise require the more advanced evaluation techniques in order to be solved.

5.3.4.1 Six point tree level scalar example

At six points the three scattering equations are given by:

$$h_1 = \sigma_4\mathfrak{s}_{1,4} + \sigma_5\mathfrak{s}_{1,5} + \sigma_6\mathfrak{s}_{1,6} + \mathfrak{s}_{1,3} = 0,$$

$$h_2 = \sigma_4\mathfrak{s}_{1,3,4} + \sigma_5\mathfrak{s}_{1,3,5} + \sigma_6\mathfrak{s}_{1,3,6} + \sigma_4\sigma_5\mathfrak{s}_{1,4,5} + \sigma_4\sigma_6\mathfrak{s}_{1,4,6} + \sigma_5\sigma_6\mathfrak{s}_{1,5,6} = 0,$$

$$h_3 = \sigma_4\sigma_5\sigma_6\mathfrak{s}_{2,3} + \sigma_5\sigma_6\mathfrak{s}_{2,4} + \sigma_4\sigma_6\mathfrak{s}_{2,5} + \sigma_4\sigma_5\mathfrak{s}_{2,6} = 0.$$

The gauge fixed scattering amplitude for scalars is given by

$$A_6^{\phi^3} = \oint \frac{d\sigma_4 d\sigma_5 d\sigma_6}{h_1 h_2 h_3} \frac{\sigma_4 (1-\sigma_5)\sigma_5 (1-\sigma_6)(\sigma_4-\sigma_6)\sigma_6}{(1-\sigma_4)(\sigma_4-\sigma_5)(\sigma_5-\sigma_6)}. \tag{5.3.16}$$

Applying partial fraction decomposition as well as transformations by rational scattering equations (5.1.1), we can rewrite the integrand of (5.3.16) as

$$\frac{\sigma_4 (1-\sigma_5)\sigma_5 (1-\sigma_6)(\sigma_4-\sigma_6)\sigma_6}{(1-\sigma_4)(\sigma_4-\sigma_5)(\sigma_5-\sigma_6)} \triangleq \frac{P_1}{\sigma_4-\sigma_5} + P_2 \tag{5.3.17}$$

where P_1 and P_2 are polynomials. To reduce the rational part to a polynomial, we take

$$P_1 \triangleq (\sigma_4-\sigma_5) P_3 \tag{5.3.18}$$

with the following standard form Ansatz:[16]

$$P_3 = c_1\sigma_5\sigma_4^2 + c_2\sigma_4\sigma_5^2 + c_3\sigma_6\sigma_4^2 + c_4\sigma_4\sigma_6^2 + c_5\sigma_6\sigma_5^2 + c_6\sigma_5\sigma_6^2 + c_7\sigma_5\sigma_4 + c_8\sigma_6\sigma_4 + c_9\sigma_5\sigma_6.$$

There are nine constants c_i with $i = 1, 2, ..., 9$ we have to fix. We apply the reduction procedure of section 5.3.1 to both sides of (5.3.18), collect all terms on one side of the equation and demand that the overall coefficient in front of each monomial vanishes. This produces a set of nine linear equations in nine unknowns. Solving the set of linear equations fixes the nine unknown coefficients and thus yields a polynomial P_3. With this, also reducing P_2 to contain ladder type monomials only, a standard form numerator polynomial $N_6^{\phi^3} \triangleq P_2 + P_3$ is obtained. It takes a direct implementation of the polynomial reduction algorithm in *Mathematica* and a linear solver just a few seconds to find a valid analytic $N_6^{\phi^3}$ result, without much effort spent on optimization.[17] We can evaluate the

[16] Ladder type monomials with base length $m_{max} = n-4$ appear to be a sufficient monomial basis.

[17] If we start with the left hand side of eq. (5.3.17) instead, as in $\sigma_4(1-\sigma_5)\sigma_5(1-\sigma_6)(\sigma_4-\sigma_6)\sigma_6 \triangleq (1-\sigma_4)(\sigma_4-\sigma_5)(\sigma_5-\sigma_6) N_6^{\phi^3}$, it takes the polynomial reduction algorithm and linear solver, with a few tweaks,

amplitude making use of prescription (5.3.8):

$$A_6^{\phi^3} = (-1)^3 3!\ \mathrm{Res}_{\sigma_{[6}=\infty,\ \sigma_5=\infty,\ \sigma_{4]}=\infty}\left[\frac{N_6^{\phi^3}}{\prod_{j=1}^{n-3} h_j}\right]. \tag{5.3.19}$$

The result is completely analytic and about one page long. It can be simplified making use of momentum conservation and on-shell conditions by hand, which is somewhat tedious. Instead we set up a basis of physical poles and fix the coefficients by multiple evaluation on different kinematic points as follows.

As in [161], the physical poles are given by $\mathfrak{s}_{1,2}, \mathfrak{s}_{2,3}, \mathfrak{s}_{3,4}, \mathfrak{s}_{4,5}, \mathfrak{s}_{5,6}, \mathfrak{s}_{6,1}, \mathfrak{s}_{1,2,3}, \mathfrak{s}_{2,3,4}$ and $\mathfrak{s}_{3,4,5}$. By dimensional analysis we see that each term in the amplitude should have three different poles. This means the complete basis is given by $\binom{9}{3} = 84$ different triple pole combinations with unknown coefficients. Making use of the procedure described in appendix 5.6, we can generate 84 different rational kinematic points and evaluate the amplitude and the basis 84 times. This sets up a linear set of 84 equations in the same number of unknowns. Solving this set of equations fixes the coefficients (which turn out to be exactly 1 or 0) and yields the simplified 6-point scalar tree level amplitude in terms of physical poles

$$\begin{aligned} A_6^{\phi^3} = -\Bigg(& \frac{1}{\mathfrak{s}_{1,2}\mathfrak{s}_{3,4}\mathfrak{s}_{5,6}} + \frac{1}{\mathfrak{s}_{1,2}\mathfrak{s}_{5,6}\mathfrak{s}_{1,2,3}} + \frac{1}{\mathfrak{s}_{2,3}\mathfrak{s}_{5,6}\mathfrak{s}_{1,2,3}} + \frac{1}{\mathfrak{s}_{1,6}\mathfrak{s}_{2,3}\mathfrak{s}_{2,3,4}} + \frac{1}{\mathfrak{s}_{1,6}\mathfrak{s}_{3,4}\mathfrak{s}_{2,3,4}} \\ & + \frac{1}{\mathfrak{s}_{2,3}\mathfrak{s}_{5,6}\mathfrak{s}_{2,3,4}} + \frac{1}{\mathfrak{s}_{3,4}\mathfrak{s}_{5,6}\mathfrak{s}_{2,3,4}} + \frac{1}{\mathfrak{s}_{1,2}\mathfrak{s}_{3,4}\mathfrak{s}_{3,4,5}} + \frac{1}{\mathfrak{s}_{1,6}\mathfrak{s}_{3,4}\mathfrak{s}_{3,4,5}} + \frac{1}{\mathfrak{s}_{1,6}\mathfrak{s}_{2,3}\mathfrak{s}_{4,5}} \\ & + \frac{1}{\mathfrak{s}_{1,2}\mathfrak{s}_{1,2,3}\mathfrak{s}_{4,5}} + \frac{1}{\mathfrak{s}_{2,3}\mathfrak{s}_{1,2,3}\mathfrak{s}_{4,5}} + \frac{1}{\mathfrak{s}_{1,2}\mathfrak{s}_{3,4,5}\mathfrak{s}_{4,5}} + \frac{1}{\mathfrak{s}_{1,6}\mathfrak{s}_{3,4,5}\mathfrak{s}_{4,5}}\Bigg), \end{aligned} \tag{5.3.20}$$

which is equivalent to summing Feynman diagrams in ϕ^3 theory and agrees with the result found in [161].

5.3.4.2 Six point tree level - first special example

Here we will give an example that is very hard to do with less advanced versions of diagrammatic integration rule techniques.[18] It involves integrating the following terms over the CHY measure

$$\frac{1}{\sigma_{2,3}^4\sigma_{4,5}^4\sigma_{6,1}^4}. \tag{5.3.21}$$

about a minute to obtain a different more complicated analytic version of $N_6^{\phi^3}$.

[18]The author thanks J. Bourjaily for pointing this out and suggesting this test integrand.

Multiplying with the CHY measure and applying our gauge we get

$$U_1 = \oint \frac{d\sigma_4 d\sigma_5 d\sigma_6}{h_1 h_2 h_3} \frac{(1-\sigma_4)\,\sigma_4\,(1-\sigma_5)\,\sigma_5\,(1-\sigma_6)\,(\sigma_4-\sigma_6)\,(\sigma_5-\sigma_6)\,\sigma_6}{(\sigma_4-\sigma_5)^3}. \tag{5.3.22}$$

In order to polynomially reduce the effective rational integrand, we write

$$(1-\sigma_4)\,\sigma_4\,(1-\sigma_5)\,\sigma_5\,(1-\sigma_6)\,(\sigma_4-\sigma_6)\,(\sigma_5-\sigma_6)\,\sigma_6 \triangleq (\sigma_4-\sigma_5)^3\,N \tag{5.3.23}$$

where we use the following standard form polynomial Ansatz

$$N = c_1\sigma_5\sigma_4^2 + c_2\sigma_4\sigma_5^2 + c_3\sigma_6\sigma_4^2 + c_4\sigma_4\sigma_6^2 + c_5\sigma_6\sigma_5^2 + c_6\sigma_5\sigma_6^2 + c_7\sigma_5\sigma_4 + c_8\sigma_6\sigma_4 + c_9\sigma_5\sigma_6.$$

We have to find nine constants $c_1, c_2, ..., c_9$. A completely analytic result is directly accessible applying our procedure, yet not very readable.[19] We do not expect the result to be given by pure physical poles either. Therefore, we will instead demonstrate an explicit exact evaluation of the integral on the following kinematic point, which was generated making use of the procedure described in appendix 5.6:

$$\begin{aligned} k_1^\mu &= (20,\ 20,\ 0,\ 0), & k_4^\mu &= (\ 60, -48,\ 0, -36), \\ k_2^\mu &= (25, -20,\ 15,\ 0), & k_5^\mu &= (-80,\ 48,\ 64,\ 0), \\ k_3^\mu &= (39,\ 0, -15,\ 36), & k_6^\mu &= (-64,\ 0, -64,\ 0). \end{aligned} \tag{5.3.24}$$

First we apply the degree reduction procedure of section 5.3.1 to both sides of equation (5.3.23) and collect all monomials on one side. The vanishing of the overall coefficient of each monomial separately produces a set of linear equations. Solving this set of equations yields

$$c_5 = \frac{7059649218217401322274}{3974168469797996315755}, \quad c_6 = -\frac{5529649875686983344959}{15896673879191985263020}, \quad c_2 = \frac{12838684423}{1662217245}, \quad c_4 = \frac{354818034905}{57180273228}$$

$$c_7 = -\frac{5774994253402805042003591}{2146050973690918010507700}, \quad c_8 = -\frac{466431129022169341083793}{343368155790546881681232}, \quad c_9 = -\frac{70384223902707859416469}{158966738791919852630200}, \quad c_1 = c_3 = 0.$$

[19]Here an analytic N can be obtained from the polynomial reduction algorithm and a linear solver within 1 to 2 minutes. This timing probably could be substantially improved by optimization.

Using this in the Ansatz for N above, we obtain a standard form numerator polynomial and can apply (5.3.8) to evaluate the integral:

$$\begin{aligned} U_1 &= (-1)^3 3!\, \mathrm{Res}_{\sigma_{[6}=\infty,\ \sigma_5=\infty,\ \sigma_{4]}=\infty}\left[\frac{N}{\prod_{j=1}^{n-3} h_j}\right] \\ &= -\frac{c_2}{\mathfrak{s}_{1,5}\mathfrak{s}_{2,3}\mathfrak{s}_{1,4,5}} + \frac{c_4}{\mathfrak{s}_{1,6}\mathfrak{s}_{2,3}\mathfrak{s}_{1,4,6}} + \frac{c_5}{\mathfrak{s}_{1,5}\mathfrak{s}_{2,3}\mathfrak{s}_{1,5,6}} - \frac{c_6}{\mathfrak{s}_{1,6}\mathfrak{s}_{2,3}\mathfrak{s}_{1,5,6}} \\ &= \frac{14174374134763}{408541369353397862400000}. \end{aligned} \tag{5.3.25}$$

Note that indeed properly only the coefficients of highest degree ladder type monomials appear in the final result.

Alternatively, we can solve the scattering equations numerically and obtain a numerical approximation for U_1, which agrees with (5.3.25).

5.3.4.3 Six point tree level - second special example

Another example that is impossible to do with less advanced diagrammatic integration rule techniques involves integrating the following terms over the CHY measure[20]

$$\frac{1}{\sigma_{2,3}^2\sigma_{3,4}^2\sigma_{4,2}^2\sigma_{1,5}^2\sigma_{5,6}^2\sigma_{6,1}^2}. \tag{5.3.26}$$

Combining this with the CHY measure and applying our usual gauge we have

$$U_2 = \oint \frac{d\sigma_4 d\sigma_5 d\sigma_6}{h_1 h_2 h_3} \frac{(1-\sigma_5)(\sigma_4-\sigma_5)\sigma_5(1-\sigma_6)(\sigma_4-\sigma_6)\sigma_6}{(1-\sigma_4)\sigma_4(\sigma_5-\sigma_6)}. \tag{5.3.27}$$

In order to polynomially reduce the effective rational integrand, we write the equation

$$(1-\sigma_5)(\sigma_4-\sigma_5)\sigma_5(1-\sigma_6)(\sigma_4-\sigma_6)\sigma_6 \triangleq (1-\sigma_4)\sigma_4(\sigma_5-\sigma_6)N \tag{5.3.28}$$

where we use the following standard form polynomial Ansatz

$$N = c_1\sigma_5\sigma_4^2 + c_2\sigma_4\sigma_5^2 + c_3\sigma_6\sigma_4^2 + c_4\sigma_4\sigma_6^2 + c_5\sigma_6\sigma_5^2 + c_6\sigma_5\sigma_6^2 + c_7\sigma_5\sigma_4 + c_8\sigma_6\sigma_4 + c_9\sigma_5\sigma_6.$$

So that again there are nine constants $c_1, c_2, ..., c_9$ to be fixed. Just as before, we can proceed completely analytically, yet the result would be too large to report.[21] Therefore, we will illustrate

[20] Again, the author thanks J. Bourjaily for pointing this out and suggesting this test integrand.

[21] Here, again, an analytic N can be obtained from the polynomial reduction algorithm and a linear solver within 1 to 2 minutes. This timing probably could be substantially improved by optimization.

the procedure by evaluating the integral on the kinematic point (5.3.24) instead.

First we apply the degree reduction procedure of section 5.3.1 to both sides of equation (5.3.28) and collect all monomials on one side of the equation. Demanding that the overall coefficient of each monomial vanishes separately provides us with a set of linear equations. Solving the set of equations we obtain

$$c_5 = \frac{162215379551}{1221259549104}, \quad c_6 = \frac{5662761717335}{17097633687456}, \quad c_4 = -\frac{92500}{133623}, \quad c_2 = \frac{39458}{133623},$$
$$c_7 = -\frac{23433636506339}{34195267374912}, \quad c_8 = \frac{329688097714075}{273562138999296}, \quad c_9 = -\frac{3664568494697}{3256692130944}, \quad c_1 = c_3 = 0.$$

Plugging this into the Ansatz for N above, we therefore have obtained a standard form numerator polynomial and can use (5.3.8) to evaluate the integral:

$$\begin{aligned} U_2 &= (-1)^3 3!\ \mathrm{Res}_{\sigma_{[6}=\infty,\ \sigma_5=\infty,\ \sigma_{4]}=\infty} \left[\frac{N}{\prod_{j=1}^{n-3} h_j} \right], \\ &= -\frac{c_2}{\mathfrak{s}_{1,5}\mathfrak{s}_{2,3}\mathfrak{s}_{1,4,5}} + \frac{c_4}{\mathfrak{s}_{1,6}\mathfrak{s}_{2,3}\mathfrak{s}_{1,4,6}} + \frac{c_5}{\mathfrak{s}_{1,5}\mathfrak{s}_{2,3}\mathfrak{s}_{1,5,6}} - \frac{c_6}{\mathfrak{s}_{1,6}\mathfrak{s}_{2,3}\mathfrak{s}_{1,5,6}} \\ &= -\frac{2407}{15692753534976}. \end{aligned} \tag{5.3.29}$$

Note that again properly only the coefficients of the highest degree ladder type monomials enter the final result. Additionally, it is clear that the calculation for this example structurally follows exactly the same steps and has the same level of complexity as the previous two examples, which would have been different from the point of view of applying diagrammatic integration rules to evaluate the integral.

Alternatively, we can solve the scattering equations numerically and obtain a numerical approximation for U_2, which agrees with (5.3.29).

5.3.4.4 Eight point tree level scalar amplitude

At eight points there are five scattering equations. The gauge fixed scattering amplitude for scalars reads[22]

$$A_8^{\phi^3} = \oint \frac{\prod_{i=4}^{8} d\sigma_i}{\prod_{j=1}^{5} h_j} \frac{\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_{3,5}\sigma_{3,6}\sigma_{3,7}\sigma_{3,8}\sigma_{4,6}\sigma_{4,7}\sigma_{4,8}\sigma_{5,7}\sigma_{5,8}\sigma_{6,8}}{\sigma_{3,4}\sigma_{4,5}\sigma_{5,6}\sigma_{6,7}\sigma_{7,8}}. \tag{5.3.30}$$

[22]Where $\sigma_3 = 1$ is implied.

We will demonstrate an explicit evaluation of the amplitude. Making use of the procedure described in appendix 5.6, we generate some on-shell kinematic data

$$\begin{aligned} k_1^\mu &= (\ -54,\ -54,\ \ 0,\ \ 0), & k_5^\mu &= (-85,\ 0,\ 75,\ 40),\\ k_2^\mu &= (-246,\ \ 54,-240,\ \ 0), & k_6^\mu &= (\ 50,\ 0,-30,-40),\\ k_3^\mu &= (\ 260,\ 100,\ 240,\ \ 0), & k_7^\mu &= (-34,\ 0,\ 30,-16),\\ k_4^\mu &= (\ 125,-100,\ -75,\ \ 0), & k_8^\mu &= (-16,\ 0,\ \ 0,\ 16). \end{aligned} \tag{5.3.31}$$

We want to find an effective integral expression

$$A_8^{\phi^3} = \oint \frac{\prod_{i=4}^8 d\sigma_i}{\prod_{j=1}^5 h_j} N_8^{\phi^3}, \tag{5.3.32}$$

where $N_8^{\phi^3}$ is a standard form polynomial satisfying

$$\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_{3,5}\sigma_{3,6}\sigma_{3,7}\sigma_{3,8}\sigma_{4,6}\sigma_{4,7}\sigma_{4,8}\sigma_{5,7}\sigma_{5,8}\sigma_{6,8} \triangleq \sigma_{3,4}\sigma_{4,5}\sigma_{5,6}\sigma_{6,7}\sigma_{7,8}N_8^{\phi^3} \tag{5.3.33}$$

on the support of the ideal spanned by the scattering equations. As an Ansatz for $N_8^{\phi^3}$ we take the 375 different ladder type monomials with $m_{max} = n-4 = 4$. At eight points, polynomially reducing the complete right hand side of (5.3.33) proves to be time consuming. Therefore, we instead perform a much simpler polynomial reduction of the expression $\sigma_i N_{\sigma_i} \to N'_{\sigma_i}$ for $i = 4,\dots,8$ with the same Ansatz for N_{σ_i}.[23] These results can now be straightforwardly linearly combined as in $(\sigma_i - \sigma_j)N \to N'_{\sigma_i} - N'_{\sigma_j} \equiv N'_{\sigma_{ij}}$. Additionally, we can nest them by computing the reduction in steps of one degree at a time $(\sigma_i-\sigma_j)(\sigma_a-\sigma_b)N \to (\sigma_i-\sigma_j)N'_{\sigma_{ab}} \to N''_{\sigma_{ij}\sigma_{ab}}$, where in the second step we treat the complete monomial coefficients of $N'_{\sigma_{ab}}$ as simple unknowns and substitute their structure back in once the reduction has been performed. Clearly, we can apply the nesting as many times as required. Therefore, the polynomial reduction of $\sigma_i N_{\sigma_i}$ is the only building block we need to construct the complete effective numerator polynomial $N_8^{\phi^3}$.

Furthermore, it is more convenient to fractionally decompose the integrand in (5.3.30). The numerators and denominators of each of the resulting fractions have smaller polynomial degree, so that the complexity of finding a polynomial reduction for each of these fractions separately is reduced compared to the original expression.

Once the polynomial reduction is complete, we collect all terms in (5.3.33) on one side of the equation

[23] The resulting polynomial N'_{σ_i} features the same monomials as N_{σ_i}, but with the coefficients mixed by the reduction procedure.

and demand the vanishing of all overall monomial coefficients separately. This gives us 375 linear equations in the same number of unknowns. Solving these equations, we fix the unknown coefficients and obtain the effective standard form numerator polynomial $N_8^{\phi^3}$. With this, prescription (5.3.8) is easily evaluated:

$$A_8^{\phi^3} = (-1)^5 5! \, \mathrm{Res}_{\sigma_{[8}=\infty,\ \sigma_7=\infty,\ \sigma_6=\infty,\ \sigma_5=\infty,\ \sigma_{4]}=\infty} \left[\frac{N_8^{\phi^3}}{\prod_{j=1}^{n-3} h_j} \right] = \frac{1360947997721}{22293435818142720000000000000}.$$

This is an exact result since we did not invoke any floating point calculations at any step. Alternatively, we can approximately solve the scattering equations numerically and evaluate $A_8^{\phi^3}$ on the solutions, which yields agreement.

5.4 CHY formulation of 1-loop level scattering amplitudes

At one loop, n-point scattering equations have been shown to follow from $(n+2)$-point tree level scattering equations with two massive particles by taking the forward limit of the two massive momenta [175]. The tree level scattering equations with two massive particles are given by [137, 138]:

$$E_a = \sum_{\substack{b=1 \\ b\neq a}}^{n+2} \frac{\mathfrak{p}_{a,b}}{\sigma_{ab}} \quad \text{for} \quad a \in \{1,2,...,n\}, \tag{5.4.1}$$
$$E_{n+1} = \sum_{b=1}^{n} \frac{\mathfrak{p}_{n+1,b}}{\sigma_{n+1,b}} + \frac{\mathfrak{p}_{n+1,n+2} + m^2}{\sigma_{n+1,n+2}} \quad , \quad E_{n+2} = \sum_{b=1}^{n} \frac{\mathfrak{p}_{n+2,b}}{\sigma_{n+2,b}} - \frac{\mathfrak{p}_{n+1,n+2} + m^2}{\sigma_{n+1,n+2}},$$

where two particles are massive with the same mass $k_{n+1}^2 = k_{n+2}^2 = m^2$. Here we have introduced a shorthand notation[24]

$$\mathfrak{p}_{\alpha(1),\alpha(2),...,\alpha(q)} \equiv \sum_{\{\beta(1),\beta(2)\}\subset\{\alpha(1),\alpha(2),...,\alpha(q)\}} k_{\beta(1)} \cdot k_{\beta(2)} \qquad \text{for integer } q > 1. \tag{5.4.2}$$

The sum is over all unordered subsets of two numbers out of a set of q numbers. In the context of 1-loop CHY amplitudes, equations (5.4.1) and (5.4.2) also naturally arise from the formalism described in [176], without the need to impose them.[25]

In the following we will require the scattering equations in polynomial form. To obtain them, we can for instance apply an appropriate transformation to (5.4.1). However, we should proceed carefully,

[24] When all momenta are massless and on-shell, we have $\mathfrak{p}_{\alpha(1),\alpha(2),...,\alpha(q)} = \mathfrak{s}_{\alpha(1),\alpha(2),...,\alpha(q)}$ from (5.1.4).

[25] The author thanks C. Cardona and H. Gomez for pointing this out.

since in the forward limit

$$k^{\mu}_{n+1} \to -l^{\mu} \quad , \quad k^{\mu}_{n+2} \to l^{\mu} \tag{5.4.3}$$

the set of equations (5.4.1) admits singular solutions with $\sigma_{ij} \to 0$ for some $i \neq j$, if E_{n+1} and E_{n+2} are taken into consideration. Such singular solutions have no physical contribution to the amplitudes of relevant theories [175, 166]. Therefore, we will use $(n-1)$ independent equations E_a with $a \leq n$ in order to exclude the singular solutions. It is straightforward to check that the transformation we are looking for is given by

$$\tilde{h}^{p,q,v}_a = \sum_{\substack{i=1 \\ i\neq p,q,v}}^{n+2} \sigma_{ip}\sigma_{iq}\sigma_{iv} Y^{a-2}_{p,q,v,i} E_i \quad \text{for} \quad a \in \{2,3,...,n\}, \tag{5.4.4}$$

where

$$Y^{x}_{p,q,v,i} = \begin{cases} \sum_{\{\alpha(1),...,\alpha(x)\}\subset\{1,...,n+2\}/\{p,q,v,i\}} \prod_{j=1}^{x} \sigma_{\alpha(j)} & \text{for } 0 < x \leq n-2, \\ 1 & \text{for } x = 0, \\ 0 & \text{for } x < 0 \text{ and } x > n-2. \end{cases} \tag{5.4.5}$$

The range in the index a is set to correspond to (5.1.3). Indices p, q, v label the three different massive scattering equations (5.4.1) that are dropped. As we expect, $\tilde{h}^{p,n+1,n+2}_a$ yields the same results regardless of the choice of p, so in the following we can consider $\tilde{h}^{1,n+1,n+2}_a$ for convenience. We can compactly write this result as

$$\tilde{h}^{1,n+1,n+2}_a = \sum_{\{\alpha(1),...,\alpha(a)\}\subset\{1,2,...,n+2\}} (\mathfrak{p}_{\alpha(1),...,\alpha(a)} + m^2\delta_{\alpha,\{n+1,n+2\}}) \prod_{j=1}^{a} \sigma_{\alpha(j)} = 0, \tag{5.4.6}$$

for integer $2 \leq a \leq n$. Here we used a generalized Kronecker delta

$$\delta_{\alpha,\{n+1,n+2\}} = \begin{cases} 1 & \text{if} \quad \{n+1, n+2\} \subset \{\alpha(1), ..., \alpha(a)\} \\ 0 & \text{if} \quad \{n+1, n+2\} \not\subset \{\alpha(1), ..., \alpha(a)\} \end{cases}. \tag{5.4.7}$$

As long as we consider $\tilde{h}^{1,n+1,n+2}_a$ in the massive case before taking the forward limit, the scattering equations have the full set of $(n-1)!$ solutions. Knowing that the forward limit is singular in nature, we should check whether any singular solutions resurge in (5.4.6) due to the transformation (5.4.4) having been applied. Indeed, if we choose to gauge fix σ_1, σ_{n+1} and σ_{n+2}, it is straightforward to see that the trivial solution $\sigma_i = \sigma_1$ for $i = 2, 3, ..., n$ is now present in the forward limit,[26] additionally to the $(n-1)! - 2(n-2)!$ expected regular solutions. Luckily, we can remove this trivial solution by fixing the gauge $\sigma_1 = \infty$.[27] For convenience we will also fix $\sigma_{n+1} = 0$, $\sigma_{n+2} = 1$. Thus, we will work with the following representation of gauge fixed polynomial scattering equations with two massive particles

$$h_i \equiv \left(\lim_{\sigma_1 \to \infty} \frac{1}{\sigma_1} \tilde{h}_{i+1}^{1,n+1,n+2} \right) \Big|_{\substack{\sigma_{n+1}=0 \\ \sigma_{n+2}=1}} = 0 \quad , \quad \forall i \in \{1, 2, ..., n-1\}, \tag{5.4.8}$$

which has a smooth forward limit containing only regular solutions of interest.[28] It will be convenient to treat the forward limit as a regulator whenever the kinematics in the limit becomes singular. For $\tilde{h}_a^{1,n+1,n+2}$ the transformation Jacobian is $(-1)^{n+1}[\prod_{i=2}^{n} \sigma_{1i}][\prod_{1<j<q\leq n+2}^{j\leq n} \sigma_{jq}]$. Therefore, possibly up to a minus sign we have the usual CHY measure for polynomial scattering equations

$$d\mu = \left(\prod_{\substack{c=1 \\ c\neq q,p,w}}^{n+2} d\sigma_c \right) (\sigma_{qp}\sigma_{pw}\sigma_{wq}) \left(\prod_{1\leq i<j\leq n+2} \sigma_{ij} \right) \left(\prod_{a=2}^{n} \delta\left(\tilde{h}_a^{1,n+1,n+2}\right) \right). \tag{5.4.9}$$

Recall that we gauge fixed the moduli $q = 1$, $p = n+1$, $w = n+2$. To test our evaluation procedure at one-loop level, we will consider the bi-adjoint scalar ϕ^3 theory as proposed in [175], which can be written as

$$A_n^{1-loop,\phi^3} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2} \lim_{\substack{k_{n+1}\to -l \\ k_{n+2}\to \ l}} \int d\mu \left(\sum_{\gamma \in \text{cyclic}\{1,2,\ldots,n\}} PT(n+2, \gamma, n+1) \right)^2, \tag{5.4.10}$$

where

$$PT(n+2, \gamma, n+1) = \frac{1}{\sigma_{n+2,\gamma(1)}\sigma_{\gamma(1),\gamma(2)} \cdots \sigma_{\gamma(n),n+1}\sigma_{n+1,n+2}}. \tag{5.4.11}$$

However, our evaluation method applies more generally to any integrand that is rational in σ-moduli and is being integrated over the measure $d\mu$.

[26] Setting $\sigma_i = \sigma_1$ for $i = 2, 3, ..., n$ causes all scattering equations to be proportional to $\mathfrak{p}_{1,2,...,n}$, which vanishes in the forward limit.

[27] The fact that the trivial solution can be projected out by a gauge choice indicates that its contribution is not physical.

[28] We use the same symbol h as for tree level scattering equations here, since it is always clear from context which scattering equations are in use.

5.4.1 One-loop amplitude examples

5.4.1.1 Two point 1-loop scalar amplitude

At two points and 1-loop there is one scattering equation, given by[29]

$$h_1 = \sigma_2 \mathfrak{p}_{1,2} + \mathfrak{p}_{2,3} = 0. \tag{5.4.12}$$

The gauge fixed amplitude amounts to

$$A_2^{1-loop,\phi^3} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2} \lim_{\substack{k_3 \to -l \\ k_4 \to \ l}} \oint \frac{d\sigma_2}{h_1} \frac{1}{(1-\sigma_2)\,\sigma_2}. \tag{5.4.13}$$

We require a standard form numerator polynomial $N^{2,\phi^3}_{1\text{-loop}}$ such that $1 \triangleq (1-\sigma_2)\,\sigma_2 N^{2,\phi^3}_{1\text{-loop}}$ with the standard form Ansatz $N^{2,\phi^3}_{1\text{-loop}} = c_1$. Making use of the scattering equation, we polynomially reduce the right hand side, collect all terms on one side of the equation and in doing so obtain one linear equation in one unknown. Solving this equation and applying momentum conservation yields:

$$N^{2,\phi^3}_{1\text{-loop}} = \frac{\mathfrak{p}_{1,2}^2}{\mathfrak{p}_{2,3}\mathfrak{p}_{2,4}}. \tag{5.4.14}$$

Prescription (5.3.8) suggests the calculation

$$(-1)^1(1!)\text{Res}_{\sigma_2=\infty}\left[\frac{1}{h_1}\frac{\mathfrak{p}_{1,2}^2}{\mathfrak{p}_{2,3}\mathfrak{p}_{2,4}}\right] = \frac{\mathfrak{p}_{1,2}}{\mathfrak{p}_{2,3}\mathfrak{p}_{2,4}}. \tag{5.4.15}$$

If we solve the scattering equation instead $\sigma_2 = -\frac{\mathfrak{p}_{2,3}}{\mathfrak{p}_{1,2}}$, we get exactly the same result

$$\sum_{\substack{h=0 \\ \text{solutions}}} \frac{1}{\det\left([\partial_i h_j]\right)} \frac{1}{(1-\sigma_2)\,\sigma_2} = \frac{\mathfrak{p}_{1,2}}{\mathfrak{p}_{2,3}\mathfrak{p}_{2,4}}. \tag{5.4.16}$$

In the forward limit we have $\mathfrak{p}_{1,2} \to 0$ while $\mathfrak{p}_{2,3}$ and $\mathfrak{p}_{2,4}$ stay finite. Therefore, the 1-loop integrand vanishes.

[29]Since the forward limit makes the kinematics singular, we use it as a parametrization.

5.4.1.2 Three point 1-loop scalar amplitude

At three points and 1-loop there are two scattering equations, given by

$$h_1 = \sigma_2 \mathfrak{p}_{1,2} + \sigma_3 \mathfrak{p}_{1,3} + \mathfrak{p}_{1,5} = 0,$$
$$h_2 = \sigma_3 \mathfrak{p}_{2,4} + \sigma_2 \mathfrak{p}_{3,4} + \sigma_2 \sigma_3 \mathfrak{p}_{4,5} = 0.$$

The gauge fixed amplitude can be written as

$$A_3^{1-loop,\phi^3} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2} \lim_{\substack{k_4 \to -l \\ k_5 \to \ l}} \oint \frac{d\sigma_2 d\sigma_3}{h_1 h_2} \frac{-\left(\sigma_2^2 + \sigma_3^2 - (\sigma_2 + 1)\,\sigma_3\right)^2}{(1-\sigma_2)\,\sigma_2\,(1-\sigma_3)\,(\sigma_2 - \sigma_3)\,\sigma_3}.$$

Therefore, we consider the following equality in order to find a standard form effective numerator polynomial $N_{1-\text{loop}}^{3,\phi^3}$

$$-\left(\sigma_2^2 + \sigma_3^2 - (\sigma_2 + 1)\,\sigma_3\right)^2 \mathrel{\hat{=}} (1-\sigma_2)\,\sigma_2\,(1-\sigma_3)\,(\sigma_2 - \sigma_3)\,\sigma_3 N_{1-\text{loop}}^{3,\phi^3},$$

with the standard form Ansatz $N_{1-\text{loop}}^{3,\phi^3} = c_1 \sigma_2 + c_2 \sigma_3$. We apply the reduction procedure of section 5.3.1 to both sides of this equation, collect all terms on one side and demand that the overall coefficient in front of each monomial vanishes separately. This sets up two linear equations in two unknowns c_1, c_2. Solving for the unknowns yields a numerator polynomial $N_{1-\text{loop}}^{3,\phi^3}$. Using prescription (5.3.8) and simplifying via five-point momentum conservation and on-shell conditions with two massive particles we get the result

$$\begin{aligned} &(-1)^2 2!\,\text{Res}_{\sigma_{[3}=\infty,\ \sigma_{2]}=\infty} \left[\frac{N_{1-\text{loop}}^{3,\phi^3}}{h_1 h_2}\right] = \\ &= -\frac{1}{\mathfrak{p}_{1,2}}\left(\frac{1}{\mathfrak{p}_{3,5}} + \frac{1}{\mathfrak{p}_{3,4}}\right) - \frac{1}{\mathfrak{p}_{2,3}}\left(\frac{1}{\mathfrak{p}_{1,5}} + \frac{1}{\mathfrak{p}_{1,4}}\right) - \frac{1}{\mathfrak{p}_{1,3}}\left(\frac{1}{\mathfrak{p}_{2,5}} + \frac{1}{\mathfrak{p}_{2,4}}\right) - \frac{1}{\mathfrak{p}_{1,5}\mathfrak{p}_{2,4}} - \frac{1}{\mathfrak{p}_{2,5}\mathfrak{p}_{3,4}} - \frac{1}{\mathfrak{p}_{1,4}\mathfrak{p}_{3,5}}. \end{aligned} \tag{5.4.17}$$

Alternatively, we can solve the scattering equations and obtain the two solutions $(\sigma_{2,+}, \sigma_{3,+})$ and $(\sigma_{2,-}, \sigma_{3,-})$ with

$$\sigma_{2,\pm} = \frac{\mathfrak{p}_{1,3}\mathfrak{p}_{3,4} - \mathfrak{p}_{1,5}\mathfrak{p}_{4,5}}{2\mathfrak{p}_{1,2}\mathfrak{p}_{4,5}} - \frac{\mathfrak{p}_{2,4}}{2\mathfrak{p}_{4,5}} \pm \frac{\sqrt{(\mathfrak{p}_{1,2}\mathfrak{p}_{2,4} - \mathfrak{p}_{1,3}\mathfrak{p}_{3,4} + \mathfrak{p}_{1,5}\mathfrak{p}_{4,5})^2 - 4\mathfrak{p}_{1,2}\mathfrak{p}_{1,5}\mathfrak{p}_{2,4}\mathfrak{p}_{4,5}}}{2\mathfrak{p}_{1,2}\mathfrak{p}_{4,5}},$$
$$\sigma_{3,\pm} = \frac{\mathfrak{p}_{1,2}\mathfrak{p}_{2,4} - \mathfrak{p}_{1,5}\mathfrak{p}_{4,5}}{2\mathfrak{p}_{1,3}\mathfrak{p}_{4,5}} - \frac{\mathfrak{p}_{3,4}}{2\mathfrak{p}_{4,5}} \mp \frac{\sqrt{(\mathfrak{p}_{1,2}\mathfrak{p}_{2,4} - \mathfrak{p}_{1,3}\mathfrak{p}_{3,4} + \mathfrak{p}_{1,5}\mathfrak{p}_{4,5})^2 - 4\mathfrak{p}_{1,2}\mathfrak{p}_{1,5}\mathfrak{p}_{2,4}\mathfrak{p}_{4,5}}}{2\mathfrak{p}_{1,3}\mathfrak{p}_{4,5}}.$$

Evaluating the integral on these solutions, summing the contributions and simplifying by means of momentum conservation and on-shell conditions directly leads to exactly the same result (5.4.17).
In the forward limit, terms $\mathfrak{p}_{i,4}, \mathfrak{p}_{i,5}$ with $i \in 1,2,3$ stay finite while $\mathfrak{p}_{i,j}$ with $i,j \in \{1,2,3\}$ tend to zero. Therefore, we first rewrite each of the three different terms in parenthesis in (5.4.17) analogously to the following

$$-\frac{1}{\mathfrak{p}_{1,2}}\left(\frac{1}{\mathfrak{p}_{3,5}}+\frac{1}{\mathfrak{p}_{3,4}}\right)=-\frac{1}{\mathfrak{p}_{3,4}\mathfrak{p}_{3,5}}\left(\frac{\mathfrak{p}_{3,4}+\mathfrak{p}_{3,5}}{\mathfrak{p}_{3,4}+\mathfrak{p}_{3,5}+\frac{1}{2}(k_4+k_5)^2}\right). \tag{5.4.18}$$

We may parametrize the forward limit as $k_4^\mu = -(l^\mu + \tau q_4^\mu)$ and $k_5^\mu = (l^\mu + \tau q_5^\mu)$ with $\tau \to 0$ and finite $q_4^\mu \neq q_5^\mu$. With this, at leading order we find

$$\frac{1}{\mathfrak{p}_{3,l+\tau q_4}\mathfrak{p}_{3,l+\tau q_5}}\left(\frac{\tau\mathfrak{p}_{3,q_5}-\tau\mathfrak{p}_{3,q_4}}{\tau\mathfrak{p}_{3,q_5}-\tau\mathfrak{p}_{3,q_4}+\tau^2\frac{1}{2}(q_5-q_4)^2}\right)=\frac{1}{(\mathfrak{p}_{3,l})^2}+O(\tau). \tag{5.4.19}$$

Therefore, the one-loop integrand at three points in bi-adjoint scalar ϕ^3 theory is given by

$$A_3^{1-loop,\phi^3} = \int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2}\left(\frac{1}{\mathfrak{p}_{1,l}\mathfrak{p}_{2,l}}+\frac{1}{\mathfrak{p}_{1,l}\mathfrak{p}_{3,l}}+\frac{1}{\mathfrak{p}_{2,l}\mathfrak{p}_{3,l}}+\frac{1}{\mathfrak{p}_{1,l}^2}+\frac{1}{\mathfrak{p}_{2,l}^2}+\frac{1}{\mathfrak{p}_{3,l}^2}\right) \tag{5.4.20}$$

$$= \int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2}\left(\frac{1}{\mathfrak{p}_{1,l}^2}+\frac{1}{\mathfrak{p}_{2,l}^2}+\frac{1}{\mathfrak{p}_{3,l}^2}\right), \tag{5.4.21}$$

since the first three terms vanish by three-point momentum conservation. Since we might be interested in the 1-loop 3-point amplitude as a vertex correction, it would make sense to consider the momenta k_1, k_2, k_3 to be off-shell – then the above result is non-trivial. In case when k_1, k_2, k_3 are on-shell, all appearing integrals are scaleless.

5.5 Conclusion and outlook

In this work we started with the CHY formulation of scattering amplitudes in arbitrary dimension. We then developed the degree reduction procedure of section 5.3.1 and applied it alongside the strong Nullstellensatz to show that any rational function can be written as a standard form polynomial on the support of scattering equations. Making use of this conversion for CHY amplitude integrands, we derived an evaluation prescription that allows to find an amplitude purely from collecting consecutive simple residues at infinity only.
Summing over all possible ladder type shapes and taking into account the multiplicity due to available subsets of non-gauge fixed moduli that are used to compose the shapes, we realize that the total

number of different ladder type monomials at any n is given by $N_n^{\text{ladd}} = s(n-3)$, where the function $s(x)$ is

$$s(0) = 1, \qquad s(x) = \sum_{i=0}^{x-1} \binom{x}{i} s(i).$$

Upon inspection, the $s(x)$ turn out to be equivalent to so called ordered Bell numbers, or Fubini numbers. For large x these numbers asymptote to $xs(x-1) \approx \ln(2)s(x)$, so that the number of ladder type monomials grows quicker than factorially with n.

In all explicit amplitude examples we studied above, it was sufficient to consider the subset of ladder type monomials with highest base length $m_{max} = n-4$ to find standard form polynomials corresponding to relevant rational functions. By the counting above, at any n there are $N_{m_{max}=n-4}^{\text{ladd}} = (n-3)s(n-4)$ such ladder type monomials.

It is well known that gauge fixed scattering equations have $(n-3)!$ different solutions at tree level [7, 5]. In [164, 165] it was shown that gauge fixed polynomial scattering equations can be transformed to a different form such that $\sigma_i - P_i(\sigma_n) = 0$ for $i \in \{4, 5, ..., n-1\}$ and $P_n(\sigma_n) = 0$, where the $P_i(\sigma_n)$ are univariate polynomials in σ_n. The polynomial $P_n(\sigma_n)$ is of highest degree $(n-3)!$ and accomodates the $(n-3)!$ different solutions. Reducing multivariate polynomials over this transformed system of equations trivially leaves $(n-3)!$ univariate monomials (i.e. $1, \sigma_n, \sigma_n^2, ..., \sigma_n^{(n-3)!-1}$) as a minimal basis for the quotient ring of multivariate polynomials over the ideal spanned by scattering equations $Q = R/\langle h_1, h_2, ..., h_{n-3}\rangle$. Therefore, the dimension of the quotient ring is $\dim_R(Q) = (n-3)!$ and thus, in the present case, we can similarly expect only $(n-3)!$ of ladder type monomials to be linearly independent on the support of the ideal spanned by scattering equations $\langle h_1, h_2, ..., h_{n-3}\rangle$. Here, a natural candidate for such a minimal basis would be the $(n-3)!$ highest degree ladder type monomials. At first glance it might seem that restricting to this minimal basis could increase computational efficiency, since this sets up a minimal linear system of equations in the polynomial construction of rational terms and makes the resulting coefficients unique. However, on a second thought it becomes apparent that modifying the polynomial reduction algorithm such as to eliminate the tail of lower degree ladder type monomials is highly non-trivial and would introduce a large computational overhead before the linear system of equations is set up. Therefore, employing more than the minimal amount of ladder type monomials to keep polynomial reduction simple appears to be more convenient.

One nice feature of the above procedure is that it works in exactly the same fashion at any n and for amplitudes of any theory in CHY formulation due to the inherent structure of CHY integrands:

While the complexity of the kinematic part of a CHY amplitude integrand in a theory like i.e. pure Yang-Mills or gravity is greater compared to massless scalars, the integrand still always is a rational function in the σ-moduli, such that the conceptual steps towards finding the amplitude described in previous sections still remain exactly the same, making the procedure universal. Furthermore, since all relevant residues for any amplitude or partial term in consideration are always collected from simple poles at infinity only, each generic evaluation step is of low complexity and the difficulty is shifted towards finding standard form polynomial expressions for the originally rational amplitude integrands. The polynomial reduction procedure that addresses this problem can be implemented algorithmically in general, so that the amplitude evaluation becomes automated for general input, which is one further strength of the current approach.

One problem that is bound to appear as we choose higher values for n, is the question of efficiency. The number of linear equations and corresponding number of unknowns increases as $(n-3)s(n-4)$ if we apply the construction step of section 5.3.2. Even though other techniques to find the reduced form might exist, this kind of limitation is bound to appear whenever a solution is formulated algorithmically involving a sequence of structural steps leading from a certain input to an output of a different structure. Therefore, as a possible direction for further investigation it might be interesting to search for general n-point integrands of standard polynomial form in various theories of interest directly, eliminating the necessity for the polynomial reduction procedure. Additionally, knowing that only the highest degree ladder type monomials contribute to any integral, finding just the coefficients for the minimal basis of highest degree ladder type monomials based on some general physical arguments would be equivalent to obtaining a direct closed form expression for the amplitude, since the remaining contour integration is trivial.

5.6 Generating real rational on-shell momenta

Pythagorean triples are integers a, b, c such that the relation $c^2 = a^2 + b^2$ is satisfied. The following well known parametrization of all such triples due to Euclid is convenient

$$a = h(u^2 - v^2) \qquad , \qquad b = 2huv \qquad , \qquad c = h(u^2 + v^2), \tag{5.6.1}$$

where h, u, v are arbitrary integers. Thinking of an n-point amplitude, we can consider $n-2$ separate copies of these integers $a_i, b_i, c_i, h_i, u_i, v_i$ with $i \in \{1, 2, ..., n-2\}$. We would like to use the above to parametrize n massless external momenta obeying momentum conservation. For that end, we distribute the integers a_i, b_i, c_i into Minkowski momenta components in a fashion similar to the following.

1) Fill a_1 into k_1^0 (with a random overall sign $\pm$ in front) and k_1^1 components, such that:

$$k_1^\mu = (\pm a_1, a_1, 0, ..., 0)$$

2) Fill a_i, b_i into spatial components and $\pm c_i$ (random sign) into the zero component of vectors k_j^μ for $j \in \{2, 3, ..., n-1\}$ so that a_q and b_{q+1} always appear in consecutive vectors and in the same spatial component but with opposite sign, such that i.e.:

$$\begin{aligned}
k_2^\mu &= (\pm c_1, -a_1, \; b_1, \; 0, \; 0, ..., 0) \\
k_3^\mu &= (\pm c_2, \; 0, -a_2, \; b_2, \; 0, ..., 0) \\
k_4^\mu &= (\pm c_3, \; 0, \; 0, -a_3, b_3, ..., 0) \\
&\vdots \\
k_{n-2}^\mu &= (\pm c_{n-3}, 0, ..., -a_{n-3}, \; b_{n-3}, \; 0) \\
k_{n-1}^\mu &= (\pm c_{n-2}, 0, ..., \; 0, -a_{n-2}, b_{n-2})
\end{aligned}$$

3) Fill b_{n-2} into k_n^0 and k_n^i components, pairing the spatial component of k_{n-1}^i, i.e.:

$$k_n^\mu = (\pm b_{n-2}, 0, ..., 0, 0, -b_{n-2})$$

Since each set of a_i, b_i, c_i integers is internally parametrized by (5.6.1), all momenta defined above are automatically light-like $k_i \cdot k_i = 0$ for $i \in \{1, 2, ..., n\}$. Furthermore, if we ensure that $b_q = a_{q+1}$ for all $q \in \{1, 2, ..., n-3\}$, then all spatial components will sum up to zero, providing spatial momentum conservation. The set of constraints $b_q = a_{q+1}$ can be solved using $n-3$ of the h_i of (5.6.1) and promoting them to variables. Finally, to ensure momentum conservation in the zero-th component, we can solve the equation $\sum_{i=1}^n k_i^0 = 0$ in u_1 of (5.6.1) while promoting it to a variable. The solutions to the constraints above are rational in the unfixed parameters, so that we are guaranteed to obtain rational momenta if we seed integers to the unfixed h_{n-2} and u_i, v_i. However, we should seed the integers carefully since singular configurations exist. In order to avoid most singular results we could for instance fix $u_i = 1$ for all remaining i, while randomly selecting $h_{n-2}, v_i > 1$. Finally, it is clear that the position of the spatial components within a vector can be assigned flexibly as long as the canceling entries, such as b_q and $-a_{q+1}$, always are properly paired. Therefore, we can randomly create real rational on-shell momenta in any spacetime dimension $D > 2$ using the above. Even though this only provides access to a very specific subset of all possible real and rational on-shell momenta, they are nevertheless sufficiently generic for testing purposes. Straightforward slight modifications can also be made to obtain sufficiently generic results even for the four point configuration, or cases involving massive particles.

Chapter 6

Tree-level gluon amplitudes on the celestial sphere

This chapter is based on the publication [177].

The holographic description of bulk physics in terms of a theory living on the boundary has been concretely realised by the AdS/CFT correspondence for spacetimes with global negative curvature. It remains an important outstanding problem to understand suitable formulations of holography for flat spacetime, a goal that has elicited a considerable amount of work from several complementary approaches [178, 179, 180, 181, 182, 29, 30, 183].

Recently, Pasterski, Shao and Strominger [184] studied the scattering of particles in four-dimensional Minkowski space and formulated a prescription that maps these amplitudes to the celestial sphere at infinity. The Lorentz symmetry of four-dimensional Minkowski space acts as the conformal group $SL(2,\mathbb{C})$ on the celestial sphere. It has been shown explicitly that the near-extremal three-point amplitude in massive cubic scalar field theory has the correct structure to be identified as a three-point correlation function of a conformal field theory living on the celestial sphere [184]. The factorization singularities of more general scattering amplitudes in this CFT perspective have been further studied in [185]. The map uses conformal primary wave functions which have been constructed for various fields in arbitrary dimensions in [186]. In [187] it was shown that the change of basis from plane waves to the conformal primary wave functions is implemented by a Mellin transform, which was computed explicitly for three and four-point tree-level gluon amplitudes. The optical theorem in the conformal basis and scattering in three dimensions were studied in [188]. One-loop and two-loop four-point amplitudes have also been considered in [189].

In this note we use the prescription [187] to investigate the structure of CFT correlators corresponding to arbitrary n-point gluon tree-level scattering amplitudes, thus generalizing their three- and four-point MHV results. Gluon amplitudes can be represented in many different ways that exhibit different, complementary aspects of their rich mathematical structure. It is natural to suspect that they may also take a particularly interesting form when written as correlators on the celestial sphere. We find that Mellin transforms of n-point MHV gluon amplitudes are given by Aomoto-Gelfand generalized hypergeometric functions on the Grassmannian $Gr(4,n)$ (6.2.19). For non-MHV amplitudes the analytic structure of the resulting functions is more complicated, and they are given by Gelfand A-hypergeometric functions (6.3.6) and its generalizations. It will be very interesting to explore further the structure of these functions, and possibly make connections to other representations of tree-level amplitudes [5, 174, 122, 190, 191] which we leave for future work.

6.1 Gluon amplitudes on the celestial sphere

We work with tree-level n-point scattering amplitudes of massless particles $\mathcal{A}_{\ell_1\cdots\ell_n}(k_j^\mu)$ which are functions of external momenta k_j^μ and helicities $\ell_j = \pm 1$, where $j = 1, \dots, n$. We want to map these scattering amplitudes to the celestial sphere. To that end we can parametrize the massless external momenta k_j^μ as

$$k_j^\mu = \epsilon_j \omega_j q_j^\mu \equiv \epsilon_j \omega_j (1 + |z_j|^2, z_j + \bar{z}_j, -i(z_j - \bar{z}_j), 1 - |z_j|^2), \tag{6.1.1}$$

where $z_j, \bar{z}_j$ are the usual complex cordinates on the celestial sphere, ϵ_j encodes a particle as incoming ($\epsilon_j = -1$) or outgoing ($\epsilon_j = +1$), and ω_j is the angular frequency associated with the energy of the particle [187]. Therefore, the amplitude $\mathcal{A}_{\ell_1\cdots\ell_n}(\omega_j, z_j, \bar{z}_j)$ is a function of ω_j, z_j and $\bar{z}_j$ under the parametrization (6.1.1).

Usually, we write any massless scattering amplitude in terms of spinor-helicity angle- and square-brackets representing Weyl-spinors (see [2] for a review). The spinor-helicity variables are related to external momenta k_j^μ, so that in turn we can express them in terms of variables on the celestial sphere via [187]:

$$[ij] = 2\sqrt{\omega_i \omega_j}\bar{z}_{ij}\,, \qquad \langle ij \rangle = -2\epsilon_i \epsilon_j \sqrt{\omega_i \omega_j} z_{ij}\,, \tag{6.1.2}$$

where $z_{ij} = z_i - z_j$ and $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$.

In [186, 187] it was proposed that any massless scattering amplitude is mapped to the celestial sphere via a Mellin transform:

$$\tilde{\mathcal{A}}_{J_1\cdots J_n}(\lambda_j, z_j, \bar{z}_j) = \prod_{j=1}^{n} \int_0^\infty d\omega_j\, \omega_j^{i\lambda_j}\, \mathcal{A}_{\ell_1\cdots\ell_n}(\omega_j, z_j, \bar{z}_j)\,. \tag{6.1.3}$$

The Mellin transform maps a plane wave solution for a helicity ℓ_j field in momentum space to a corresponding conformal primary wave function on the boundary with spin J_j, where helicity ℓ_j and spin J_j are mapped onto each other, and the operator dimension takes values in the principal continuous series representation $\Delta_j = 1 + i\lambda_j$ [186]. Therefore, $\tilde{\mathcal{A}}_{J_1\cdots J_n}(\lambda_j, z_j, \bar{z}_j)$ has the structure of a conformal correlator on the celestial sphere, where the symmetry group of diffeomorphisms is the conformal group $SL(2,\mathbb{C})$.

Explicitly, under conformal transformations, we have the following behavior:

$$\omega_j \to \omega_j' = |cz_j + d|^2\omega_j \quad , \quad z_j \to z_j' = \frac{az_j + b}{cz_j + d} \quad , \quad \bar{z}_j \to \bar{z}_j' = \frac{\bar{a}\bar{z}_j + \bar{b}}{\bar{c}\bar{z}_j + \bar{d}}, \tag{6.1.4}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. The transformation for $z_j, \bar{z}_j$ is familiar from the usual action of $SL(2,\mathbb{C})$ on the complex coordinates on a sphere. Concerning ω_j, recall that q_j^μ transforms as $q_j^\mu \to |cz_j + d|^{-2}\Lambda^\mu{}_\nu q_j^\nu$ [186], where $\Lambda^\mu{}_\nu$ is a Lorentz transformation in Minkowski space corresponding to the celestial sphere conformal transformation. Thus, ω_j must transform as in (6.1.4) to ensure that k_j^μ transforms as a Lorentz vector: $k_j^\mu \to \Lambda^\mu{}_\nu k_j^\nu$.

The conformal covariance of $\tilde{\mathcal{A}}_{J_1\cdots J_n}(\lambda_j, z_j, \bar{z}_j)$ on the celestial sphere demands:

$$\tilde{\mathcal{A}}_{J_1\cdots J_n}\left(\lambda_j, \frac{az_j + b}{cz_j + d}, \frac{\bar{a}\bar{z}_j + \bar{b}}{\bar{c}\bar{z}_j + \bar{d}}\right) = \prod_{j=1}^{n}\left[(cz_j + d)^{\Delta_j + J_j}(\bar{c}\bar{z}_j + \bar{d})^{\Delta_j - J_j}\right]\tilde{\mathcal{A}}_{J_1\cdots J_n}(\lambda_j, z_j, \bar{z}_j)\,, \tag{6.1.5}$$

as expected for a correlator of operators with weights Δ_j and spins J_j.

6.2 n-point MHV

The cases of 3- and 4-point gluon amplitudes have been considered in [187]. Here we will map $n \geq 5$-point MHV gluon amplitudes to the celestial sphere.

6.2.1 Integrating out one ω_i

Starting from (6.1.3), we can anchor the integration to one of our variables ω_i by making a change of variables for all $l \neq i$

$$\omega_l \to \frac{\omega_i}{s_i}\omega_l, \tag{6.2.1}$$

where s_i is a constant factor that cancels the conformal scaling of ω_i in (6.1.4), so that the ratio $\frac{\omega_i}{s_i}$ is conformally invariant. One choice which is always possible in Minkowski signature is

$$s_i = \frac{|z_{i-1\ i+1}|}{|z_{i-1\ i}|\,|z_{i\ i+1}|}. \tag{6.2.2}$$

Since gluon scattering amplitudes scale homogeneously under uniform rescalings, collecting all the factors in front, we have

$$\tilde{\mathcal{A}}_{J_1\cdots J_n}(\lambda_j, z_j, \bar{z}_j) = \int_0^\infty \frac{d\omega_i}{\omega_i}\left(\frac{\omega_i}{s_i}\right)^{\sum_{j=1}^n i\lambda_j} s_i^{1+i\lambda_i}\left(\prod_{\substack{a=1\\a\neq i}}^n \int_0^\infty d\omega_a\,\omega_a^{i\lambda_a}\right)\mathcal{A}_{\ell_1\cdots\ell_n}(s_i,\omega_l,z_j,\bar{z}_j)\,, \tag{6.2.3}$$

where we used that the scaling power of dressed gluon amplitudes is $\mathcal{A}_n(\Lambda\omega_i) \to \Lambda^{-n}\mathcal{A}_n(\omega_i)$. We recognize that the integral over ω_i is the Mellin transform of 1, which is given by

$$\int_0^\infty \frac{d\omega_i}{\omega_i}\left(\frac{\omega_i}{s_i}\right)^{iz} = 2\pi\delta(z). \tag{6.2.4}$$

With this we simplify the transformation prescription (6.1.3) to

$$\tilde{\mathcal{A}}_{J_1\cdots J_n}(\lambda_j, z_j, \bar{z}_j) = 2\pi\delta\left(\sum_{j=1}^n \lambda_j\right)s_i^{1+i\lambda_i}\left(\prod_{\substack{a=1\\a\neq i}}^n \int_0^\infty d\omega_a\,\omega_a^{i\lambda_a}\right)\mathcal{A}_{\ell_1\cdots\ell_n}(s_i,\omega_l,z_j\bar{z}_j)\,. \tag{6.2.5}$$

6.2.2 Integrating out momentum conservation δ-functions

For simplicity, we choose the anchor variable above to be ω_1 and use $\omega_{n-3},\ldots,\omega_n$ to localize the momentum conservation δ-functions in the amplitude. These δ-functions can then be equivalently rewritten as follows, compensating the transformation by a Jacobian:

$$\delta^4(\epsilon_1 s_1 q_1 + \sum_{i=2}^n \epsilon_i\omega_i q_i) = \frac{4}{U}\prod_{j=n-3}^n s_j\delta\left(\omega_j - \omega_j^*\right)\mathbf{1}_{>0}(\omega_j^*), \tag{6.2.6}$$

where ω_j^* are solutions to the initial set of linear equations:

$$\omega_j^\star = -s_j\left(\frac{U_{1,j}}{U} + \sum_{i=2}^{n-4}\frac{\omega_i}{s_i}\frac{U_{i,j}}{U}\right). \tag{6.2.7}$$

The U_{ij} and U are minor determinants by Cramer's rule:

$$U_{i,j} = \det(M^{\{n-3,\dots,j\to i,\dots,n\}}), \quad U = \det(M^{\{n-3,\dots,n\}}), \tag{6.2.8}$$

where $j \to i$ means that index j is replaced by index i. $M^{\{a,b,c,d\}}$ denotes the 4×4 matrix

$$M^{\{a,b,c,d\}} = (p_a\, p_b\, p_c\, p_d). \tag{6.2.9}$$

For the purpose of determinant calculation, the column vectors $p_i^\mu = \epsilon_i s_i q_i^\mu$ can be written in a manifestly conformally invariant form:

$$\begin{aligned} &p_1^\mu(z,\bar z) = \epsilon_1(1,0,0,-1) \;\;,\;\; p_2^\mu(z,\bar z) = \epsilon_2(1,0,0,1) \;\;,\;\; p_3^\mu(z,\bar z) = \epsilon_3(2,2,0,0)\,, \\ &p_i^\mu(z,\bar z) = \epsilon_i\frac{1}{|u_i|}(1+|u_i|^2, u_i+\bar u_i, -i(u_i-\bar u_i), 1-|u_i|^2) \quad \text{for} \quad i = 4,5,\dots,n\,, \end{aligned} \tag{6.2.10}$$

in terms of conformal invariant cross-ratios

$$u_i = \frac{z_{31}z_{i2}}{z_{32}z_{i1}} \quad \text{and} \quad \bar u_i = \frac{\bar z_{31}\bar z_{i2}}{\bar z_{32}\bar z_{i1}} \quad \text{for} \quad i = 4,5,\dots,n\,, \tag{6.2.11}$$

but if, and only if, we also specify the explicit choice

$$s_1 = \frac{|z_{3,2}|}{|z_{3,1}|\,|z_{1,2}|}\,, \quad s_2 = \frac{|z_{3,1}|}{|z_{3,2}|\,|z_{2,1}|}\,, \quad \text{and} \quad s_i = \frac{|z_{1,2}|}{|z_{1,i}|\,|z_{i,2}|} \quad \text{for} \quad i = 3,\dots,n. \tag{6.2.12}$$

The indicator functions $\prod_{i=n-3}^n \mathbf{1}_{>0}(\omega_i^*)$ appear due to the integration range in all ω being along the positive real line, such that the δ-functions can only be localized in this region.

Furthermore, in order for all the remaining integration variables ω_j with $j = 2,\dots,n-4$ to be defined on the whole integration range, the indicator functions $\prod_{i=n-3}^n \mathbf{1}_{>0}(\omega_i^*)$ have to demand $\frac{U_{i,j}}{U} < 0$ for all $i = 1,\dots,n-4$ and $j = n-3,\dots,n$, so that we can write them as $\prod_{i,j}\mathbf{1}_{<0}(\frac{U_{i,j}}{U})$.

6.2.3 Integrating the remaining ω_i

In this section we apply (6.2.5) to the usual n-point MHV Parke-Taylor amplitude [1] in spinor-helicity formalism for $n \geq 5$ rewritten via (6.1.2):

$$\mathcal{A}_{--+\ldots+}(s_1, \omega_j, z_j, \bar{z}_j) = \frac{z_{12}^3 s_1 \omega_2 \delta^4(\epsilon_1 s_1 q_1 + \sum_{i=2}^n \epsilon_i \omega_i q_i)}{(-2)^{n-4} z_{23} z_{34} \ldots z_{n1} \omega_3 \omega_4 \ldots \omega_n}. \tag{6.2.13}$$

Making use of the solutions (6.2.6) and performing four of the integrations in (6.2.5), we have:

$$\tilde{\mathcal{A}}_{--+\ldots+}(\lambda_i, z_i, \bar{z}_i) = 2\pi \frac{\delta(\sum_{j=1}^n \lambda_j) z_{12}^3 \, s_1^{i\lambda_1+2}}{(-2)^{n-4} U z_{23} z_{34} \ldots z_{n1}} \prod_{a=2}^{n-4} \int_0^\infty d\omega_a \, \omega_a^{i\lambda_a} \frac{\omega_2 \prod_{b=n-3}^n s_b \omega_b^{*\, i\lambda_b}}{\omega_3 \omega_4 \ldots \omega_n^*} \prod_{i,j} \mathbf{1}_{<0}(\frac{U_{i,j}}{U}). \tag{6.2.14}$$

For convenience, we transform the remaining integration variables as:

$$\omega_i = s_i \frac{U_{1,n}}{U_{i,n}} \frac{u_{i-1}}{1 - \sum_{j=1}^{n-5} u_j} \quad , \quad i = 2, 3, \ldots, n-4 \,, \tag{6.2.15}$$

which leads to

$$\tilde{\mathcal{A}}_{--+\ldots+}(\lambda_i, z_i, \bar{z}_i) \sim \frac{z_{12}^3 s_1^{i\lambda_1+2} s_2^{i\lambda_2+2} s_3^{i\lambda_3} \ldots s_n^{i\lambda_n}}{z_{23} z_{34} \ldots z_{n1} U_{1,n}} \delta(\sum_{j=1}^n \lambda_j) \, \hat{\varphi}(\{\alpha\}, x) \prod_{i,j} \mathbf{1}_{<0}(\frac{U_{i,j}}{U}). \tag{6.2.16}$$

Note that the overall factor in (6.2.16) accounts for proper transformation weight of the resulting correlator under conformal transformations (6.1.5).

Here we recognize a hypergeometric function $\hat{\varphi}(\{\alpha\}, x)$ of type $(n-4, n)$, as defined in section 3.8.1 of [192] and described in appendix 6.5. In particular, here we have:

$$\begin{aligned} \hat{\varphi}(\{\alpha\}, x) &\equiv \int_{\substack{u_1 \geq 0, \ldots, u_{n-5} \geq 0 \\ 1 - \sum_a u_a \geq 0}} \prod_{j=1}^n P_j(u)^{\alpha_j} d\varphi \quad , \quad d\varphi = \frac{dP_2}{P_2} \wedge \ldots \wedge \frac{dP_{n-4}}{P_{n-4}} \,, \\ P_j(u) &= x_{0j} + x_{1j} u_1 + \ldots + x_{n-5\, j} u_{n-5} \quad , \quad 1 \leq j \leq n \,. \end{aligned} \tag{6.2.17}$$

The parameters in (6.2.17) corresponding to (6.2.16) read:[1]

$$\begin{aligned} &\alpha_1 = 1 \,, \ \alpha_2 = 2 + i\lambda_2 \,, \ \alpha_3 = i\lambda_3 \,, \ \ldots \,, \ \alpha_{n-4} = i\lambda_{n-4} \,, \ \alpha_{n-3} = i\lambda_{n-3} - 1 \,, \ \ldots \,, \ \alpha_{n-1} = i\lambda_{n-1} - 1, \\ &\alpha_n = 1 + i\lambda_1 \,, \ x_{0\,i} = \frac{U_{1,i}}{U_{1,n}} \,, \ x_{j-1\,i} = \frac{U_{j,i}}{U_{j,n}} - \frac{U_{1,i}}{U_{1,n}} \,, \ x_{0n} = -\frac{U}{U_{1,n}} \,, \ x_{j-1\,n} = \frac{U}{U_{1,n}} \,, \ x_{01} = 1 \,, \ x_{j-1\,j} = -\frac{U}{U_{j,n}} \,, \end{aligned} \tag{6.2.18}$$

[1] For $n = 5$, the normally different cases $\alpha_2 = 2 + i\lambda_2$ and $\alpha_{n-3} = i\lambda_{n-3} - 1$ are reduced to a single $\alpha_2 = 1 + i\lambda_2$. In this case there also are no integrations so that the result becomes a simple product of factors.

for $i = n-3, n-2, n-1$ and $j = 2,3,...,n-4$, and all other $x_{ab} = 0$.

These kinds of functions are also known as Aomoto-Gelfand hypergeometric functions on the Grassmannian $Gr(n-4,n)$.

Making use of eq. (3.24) and (3.25) from [192], we can write down a dual representation of the same function, which yields a hypergeometric function of type $(4,n)$:

$$\begin{aligned}\hat{\varphi}(\{\alpha\},x) \equiv& \frac{c_2}{c_1}\int_{\substack{u_1\geq 0,\dots,u_3\geq 0\\ 1-\sum_a u_a \geq 0}} \prod_{j=1}^{n} P_j(u)^{\alpha_j} d\varphi \quad , \quad d\varphi = \frac{dP_{n-3}}{P_{n-3}} \wedge \dots \wedge \frac{dP_{n-1}}{P_{n-1}} ,\\ P_j(u) =& x_{0j} + x_{1j}u_1 + x_{2j}u_2 + x_{3j}u_3 \quad , \quad 1 \leq j \leq n .\end{aligned} \tag{6.2.19}$$

In this case, the parameters of (6.2.19) corresponding to (6.2.16) read:

$$\begin{aligned}\alpha_1 =& 1\,,\ \alpha_2 = -2-i\lambda_2\,,\ \alpha_3 = -i\lambda_3\,,\dots,\ \alpha_{n-4} = -i\lambda_{n-4}\,,\ \alpha_{n-3} = 1-i\lambda_{n-3}\,,\dots,\ \alpha_{n-1} = 1-i\lambda_{n-1},\\ \alpha_n =& -i\lambda_n\,,\ x_{0j} = \frac{U_{j,n}}{U_{1,n}}\,,\ x_{ij} = \frac{U_{j,n-4+i}}{U_{1,n-4+i}} - \frac{U_{j,n}}{U_{1,n}}\,,\ x_{0n} = -\frac{U}{U_{1,n}}\,,\ x_{in} = \frac{U}{U_{1,n}}\,,\ x_{01} = 1\,,\\ x_{1\,n-3} =& \frac{-U}{U_{1,n-3}}\,,\ x_{2\,n-2} = \frac{-U}{U_{1,n-2}}\,,\ x_{3\,n-1} = \frac{-U}{U_{1,n-1}}\,,\ \frac{c_2}{c_1} = \frac{\Gamma(2+i\lambda_1)\Gamma(2+i\lambda_2)\prod_{j=3}^{n-4}\Gamma(i\lambda_j)}{\Gamma(1-i\lambda_1)\prod_{i=1}^{3}\Gamma(1-i\lambda_{n-i})}.\end{aligned} \tag{6.2.20}$$

for $i = 1,2,3$ and $j = 2,3,...,n-4$, and all other $x_{ab} = 0$.

The hypergeometric functions $\hat{\varphi}(\{\alpha\},x)$ form a basis of solutions to a Pfaffian form equation which defines a Gauss-Manin connection as described in section 3.8 of [192]. This Pfaffian form equation can be interpreted as a generalized Knizhnik-Zamolodchikov equation satisfied by our correlators [193, 194]. Similar generalized hypergeometric functions appeared in [195] in the context of $\mathcal{N} = 4$ Yang-Mills scattering amplitudes and the deformed Grassmannian.

6.2.4 6-point MHV

In the special case of six gluons there is only one integral in (6.2.17), such that the function reduces to the simpler case of Lauricella function $\hat{\varphi}_D$:

$$\begin{aligned}\hat{\varphi}_D(\{\alpha\},x) =& \left(\frac{-U}{U_{2,6}}\right)^{i\lambda_1+1}\left(\frac{-U}{U_{1,6}}\right)^{i\lambda_2+2}\left(\frac{U_{2,3}}{U_{2,6}}\right)^{i\lambda_3-1}\left(\frac{U_{2,4}}{U_{2,6}}\right)^{i\lambda_4-1}\left(\frac{U_{2,5}}{U_{2,6}}\right)^{i\lambda_5-1}\times\\ &\times\int_0^1 dt\, t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\prod_{i=1}^{3}(1-x_i t)^{-\beta_i},\end{aligned} \tag{6.2.21}$$

with parameters and arguments given by

$$\alpha = 2+i\lambda_2,\ \gamma = 4+i\lambda_1+i\lambda_2,\ \beta_i = 1-i\lambda_{i+2},\ x_i = 1-\frac{U_{1,i+2}U_{2,6}}{U_{1,6}U_{2,i+2}} \quad \text{for} \quad i = 1,2,3. \tag{6.2.22}$$

Note that x_{0j} arguments have been factored out of the integrand to achieve this form.

6.3 n-point NMHV

In this section we will map the n-point NMHV split helicity amplitude $\mathcal{A}_{---++\cdots+}$ to the celestial sphere via (6.2.5). The spinor-helicity expression for $\mathcal{A}_{---++\cdots+}$ can be found e.g. in [196]

$$\mathcal{A}_{---++\cdots+} = \frac{1}{F_{3,1}} \sum_{j=4}^{n-1} \frac{\langle 1|P_{2,j}P_{j+1,2}|3\rangle^3}{P_{2,j}^2 P_{j+1,2}^2} \frac{\langle j+1\ j\rangle}{[2|P_{2,j}|j+1\rangle\langle j|P_{j+1,2}|2]} \equiv \sum_{j=4}^{n-1} \{M_j\} \tag{6.3.1}$$

where $F_{i,j} \equiv \langle i\ i+1\rangle\langle i+1\ i+2\rangle\cdots\langle j-1\ j\rangle$ and $P_{x,y} \equiv \sum_{k=x}^{y} |k\rangle[k|$ where $x < y$ cyclically.

We will work with $\{M_4\}$ for the purpose of our calculations. Using momentum conservation and writing $\{M_4\}$ in terms of spinor-helicity variables, we find

$$\begin{aligned}\{M_4\} =& \frac{1}{\langle 34\rangle\langle 45\rangle\cdots\langle n-1\ n\rangle\langle n1\rangle} \frac{(\langle 12\rangle[24]\langle 43\rangle + \langle 13\rangle[34]\langle 43\rangle)^3}{(\langle 23\rangle[23] + \langle 24\rangle[24] + \langle 34\rangle[34])\langle 34\rangle[34]} \times \\ &\times \frac{\langle 54\rangle}{([23]\langle 35\rangle + [24]\langle 45\rangle)(\langle 43\rangle[32])}.\end{aligned} \tag{6.3.2}$$

Writing this in terms of celestial sphere variables via (6.1.2), we find

$$\{M_4\} = \frac{\frac{\omega_1\omega_4(\epsilon_2 z_{12}\bar{z}_{24}\omega_2 + \epsilon_3 z_{13}\bar{z}_{34}\omega_3)^3}{2^{n-4} z_{56} z_{67}\cdots z_{n-1,n} z_{n1}\bar{z}_{23}\bar{z}_{34}\prod_{j=2,j\neq 4}^{n}\omega_j}}{(\epsilon_3 z_{35}\bar{z}_{23}\omega_3 + \epsilon_4 z_{45}\bar{z}_{24}\omega_4)\left(\epsilon_2\omega_2\left(\epsilon_3|z_{23}|^2\omega_3 + \epsilon_4|z_{24}|^2\omega_4\right) + \epsilon_3\epsilon_4|z_{34}|^2\omega_3\omega_4\right)}. \tag{6.3.3}$$

The following map of the above formula to the celestial sphere will only be strictly valid for $n \geq 8$. We will comment on changes at 6- and 7-points in the next section. We use the map (6.2.5), anchor the calculation about ω_1, make use of solutions (6.2.6) and perform a change of variables

$$\omega_i = s_i \frac{u_{i-1}}{1 - \sum_{j=1}^{n-5} u_j}, \quad i = 2, \ldots, n-4, \tag{6.3.4}$$

to find the resulting term in the n-point NMHV correlator

$$\{\tilde{M}_4\} \sim \delta\left(\sum_{j=1}^{n}\lambda_j\right) \frac{\prod_{i=1}^{n} s_i^{i\lambda_i}}{\bar{z}_{12}\bar{z}_{23}\bar{z}_{13} z_{45} z_{56}\cdots z_{n-1,n} z_{4,n}} \frac{\bar{z}_{12}\bar{z}_{13} z_{45} z_{4,n} s_1^2 s_4^2}{\bar{z}_{34} z_{n1} U} \hat{\mathcal{F}}(\alpha, x) \prod_{i,j} \mathbf{1}_{<0}\left(\frac{U_{i,j}}{U}\right), \tag{6.3.5}$$

with the function $\hat{\mathcal{F}}(\alpha, x)$ being a Gelfand A-hypergeometric function as defined in Appendix 6.5.

In this case it explicitly reads:

$$\hat{\mathcal{F}}(\{\alpha\},x) = \int_{\substack{u_1\geq 0,\dots u_{n-5}\geq 0\\ 1-u_1-\dots-u_{n-5}\geq 0}} \prod_{a=1}^{n-5}\frac{du_a}{u_a}\prod_{j=1}^{n-5}u_j^{i\lambda_{j+1}}u_3^2(u_1u_2x_{10}+u_1u_3x_{20}+u_2u_3x_{30})^{-1} \\ \times\prod_{i=1}^{7}(x_{0i}+u_1x_{1i}+\dots+u_{n-5}x_{n-5,i})^{\alpha_i}, \tag{6.3.6}$$

where parameters are given by

$$\alpha_1 = 3,\ \alpha_2 = -1,\ \alpha_3 = i\lambda_1+1,\ \alpha_4 = i\lambda_{n-3}-1,\ \alpha_5 = i\lambda_{n-2}-1,\ \alpha_6 = i\lambda_{n-1}-1,\ \alpha_7 = i\lambda_n-1, \tag{6.3.7}$$

and function arguments are given by

$$\begin{aligned}
&x_{10} = \epsilon_2\epsilon_3|z_{23}|^2 s_2s_3,\ x_{20} = \epsilon_2\epsilon_4|z_{24}|^2 s_2s_4,\ x_{30} = \epsilon_3\epsilon_4|z_{34}|^2 s_3s_4,\\
&x_{11} = \epsilon_2 z_{12}\bar{z}_{24}s_2,\ x_{21} = \epsilon_3 z_{13}\bar{z}_{34}s_3,\ x_{22} = \epsilon_3 z_{35}\bar{z}_{23}s_3,\ x_{32} = \epsilon_4 z_{45}\bar{z}_{24}s_4,\\
&x_{03} = 1,\ x_{j3} = -1,\ j = 1,\dots,n-5,\ x_{04} = \frac{U_{1,n-3}}{U},\quad x_{j4} = \frac{U_{j,n-3}-U_{1,n-3}}{U},\ j = 1,\dots,n-5,\\
&x_{05} = \frac{U_{1,n-2}}{U},\quad x_{j5} = \frac{U_{j,n-2}-U_{1,n-2}}{U},\ j = 1,\dots,n-5,\\
&x_{06} = \frac{U_{1,n-1}}{U},\quad x_{j6} = \frac{U_{j,n-1}-U_{1,n-1}}{U},\ j = 1,\dots,n-5,\\
&x_{07} = \frac{U_{1,n}}{U},\quad x_{j7} = \frac{U_{j,n}-U_{1,n}}{U},\ j = 1,\dots,n-5.
\end{aligned} \tag{6.3.8}$$

Note that the first fraction in (6.3.5) accounts for the correct transformaton weight of the correlator under conformal tranformation (6.1.5).

6- and 7-point NMHV

In the cases of 6- and 7-point the results in the previous section change somewhat, due to the presence of ω_3 and ω_4 in the denominator of (6.3.3). These variables are fixed by momentum conservation δ-functions in the lower point cases, such that the parameters and function arguments of the resulting Gelfand A-hypergeometric functions change.

For the 6-point case, we find that the resulting correlator part $\{\tilde{M}_4\}$ is proportional to a Gelfand A-hypergeometric function as defined in Appendix 6.5:

$$\hat{\mathcal{F}}(\{\alpha\},x) = \int_{\substack{u_1\geq 0\\ 1-u_1\geq 0}}\frac{du_1}{u_1}u_1^{i\lambda_2}(x_{00}+u_1x_{10}+u_1^2x_{20})^{-1}(1-u_1)^{i\lambda_1+1}\prod_{i=2}^{7}(x_{0i}+u_1x_{1i})^{\alpha_i} \tag{6.3.9}$$

where parameters are given by

$$\alpha_2 = i\lambda_3 - 1,\ \alpha_3 = i\lambda_4 + 1,\ \alpha_4 = i\lambda_5 - 1,\ \alpha_5 = i\lambda_6 - 1,\ \alpha_6 = 3,\ \alpha_7 = -1, \tag{6.3.10}$$

and function arguments x_{ij} depend on $\epsilon_i, z_i, \bar{z}_i$ and U_{ij}. Performing a partial fraction decomposition on the quadratic denominator in (6.3.9), we can reduce the result to a sum of two Lauricella functions.

In the 7-point case, we find that the resulting correlator part $\{\tilde{M}_4\}$ is proportional to a Gelfand A-hypergeometric function as defined in Appendix 6.5:

$$\begin{aligned}\hat{\mathcal{F}}(\{\alpha\}, x) = &\int_{\substack{u_1\geq 0, u_2\geq 0\\ 1-u_1-u_2\geq 0}} \frac{du_1}{u_1}\frac{du_2}{u_2}\, u_1^{i\lambda_2} u_2^{i\lambda_3} (u_1 x_{10} + u_2 x_{20} + u_1 u_2 x_{30} + u_1^2 x_{40} + u_2^2 x_{50})^{-1} \\ &\times \prod_{i=1}^{7} (x_{0i} + u_1 x_{1i} + u_2 x_{2i})^{\alpha_i},\end{aligned} \tag{6.3.11}$$

where parameters are given by

$$\alpha_1 = i\lambda_1 + 1,\ \alpha_2 = i\lambda_4 + 1,\ \alpha_3 = i\lambda_5 - 1,\ \alpha_4 = i\lambda_6 - 1,\ \alpha_5 = i\lambda_7 - 1,\ \alpha_6 = 3,\ \alpha_7 = -1, \tag{6.3.12}$$

and function arguments x_{ij} again depend on $\epsilon_i, z_i, \bar{z}_i$ and U_{ij}.

6.4 n-point N^{k}MHV

In this section we discuss the schematic structure of N^{k}MHV amplitudes with higher k under the Mellin transform (6.2.5).

N^{2}MHV amplitude

In the 8-point N^{2}MHV split helicity case, $\mathcal{A}_{----++++}$, we consider one of the six terms of the amplitude found in e.g. [196] on page 6 as an example:

$$\frac{1}{F_{4,1}\bar{F}_{2,3}} \frac{\langle 1|P_{2,6}P_{7,2}P_{3,5}P_{6,3}|4\rangle^3}{P_{2,6}^2 P_{7,2}^2 P_{3,5}^2 P_{6,3}^2} \frac{\langle 76\rangle[23]\langle 65\rangle}{[2|P_{2,6}|7\rangle\langle 6|P_{7,2}|2][3|P_{3,5}|6\rangle\langle 5|P_{6,3}|3]}, \tag{6.4.1}$$

where $\bar{F}_{i,j}$ is the complex conjugate of $F_{i,j}$. Performing the same sequence of steps as in the previous sections, we find a resulting Gelfand A-hypergeometric function of the form

$$\hat{\mathcal{F}}(\{\alpha\},x) = \int_{\substack{u_1\geq 0, u_2\geq 0, u_3\geq 0\\ 1-u_1-u_2-u_3\geq 0}} \frac{du_1}{u_1}\frac{du_2}{u_2}\frac{du_3}{u_3} u_1^{\alpha_1}u_2^{\alpha_2}u_3^{\alpha_3}\mathcal{P}^3_{\{4\}}\prod_{i=4}^{13}(x_{0i}+u_1x_{1i}+u_2x_{2i}+u_3x_{3i})^{\alpha_i} \tag{6.4.2}$$
$$\times\prod_{j=14}^{17}(x_{0j}+u_1x_{1j}+u_2x_{2j}+u_3x_{3j}+u_1u_2x_{4j}+u_1u_3x_{5j}+u_2u_3x_{6j}+u_1^2x_{7j}+u_2^2x_{8j}+u_3^2x_{9j})^{\alpha_j},$$

for some parameters α_i, where $\mathcal{P}_{\{4\}}$ is a degree four polynomial in u_i, and function arguments x_{ij} again depend on $\epsilon_i, z_i, \bar{z}_i$ and U_{ij}.

N^{k}MHV amplitude

More generally a split helicity N^{k}MHV amplitude $\mathcal{A}_{-\ldots-+\ldots+}$ involves a sum over the terms described in eq. (3.1), (3.2) of [196]. Terms corresponding in complexity to $\{\tilde{M}_4\}$ discussed in the previous section are always present, with constant Laurent polynomial powers at any k. However, for higher k, the most complicated contributing summands result in hypergeometric integrals schematically given by

$$\hat{\mathcal{F}}(\{\alpha\},x) = \int_{\substack{u_1,\ldots,u_{n-4}\geq 0\\ 1-u_2-\cdots-u_{n-4}\geq 0}} \prod_{l=2}^{n-4}\frac{du_l}{u_l}u_l^{\alpha_l}\left(1-\sum_{j=2}^{n-4}u_j\right)^{\alpha_1}\mathcal{P}^3_{\{2k\}}\left(\prod_i(\mathcal{P}^i_{\{1\}})^{\alpha_i}\right)\left(\prod_j(\mathcal{P}^j_{\{2\}})^{\alpha_j}\right) \tag{6.4.3}$$

where α_i are parameters and $\mathcal{P}_{\{d\}}$ is a degree d polynomial in u_a. Here we explicitly see an increase in power of the Laurent polynomials with increasing k in N^{k}MHV. The examples above feature the Gelfand A-hypergeometric function $\hat{\mathcal{F}}$. The increase in Laurent polynomial degree is traced back to the presence of Mandelstam invariants $P^2_{i,j}$ for degree two polynomials, as well as the factors $\langle a|P_{i,j}P_{k,l}...P_{r,t}|b\rangle$ for higher degree polynomials. The length of chains of the $P_{i,j}$ depends on n and k, such that multivariate Laurent polynomials of any positive degree are present at sufficiently high n,k.

Similar generalized hypergeometric functions, or, equivalently, generalized Euler integrals are found in the case of string scattering amplitudes [197, 84]. It will be interesting to explore this connection further.

6.5 Generalized hypergeometric functions

The Aomoto-Gelfand hypergeometric functions of type $(n+1, m+1)$ relevant in this work can be defined as in section 3.5.1 of [192]:

$$\hat{\varphi}(\{\alpha\}, x) \equiv \int_{\substack{u_1 \geq 0, \ldots, u_n \geq 0 \\ 1-\sum_a u_a \geq 0}} \prod_{j=0}^{m} P_j(u)^{\alpha_j} d\varphi \,, \tag{6.5.1}$$

$$d\varphi = \frac{dP_{j_1}}{P_{j_1}} \wedge \ldots \wedge \frac{dP_{j_n}}{P_{j_n}} \quad , \quad 0 \leq j_1 < \ldots < j_n \leq m \,, \tag{6.5.2}$$

$$P_j(u) = x_{0j} + x_{1j} u_1 + \ldots + x_{nj} u_n \quad , \quad 1 \leq j \leq m \,, \tag{6.5.3}$$

where here the parameters α_i collectively describe all the powers for the factors in the integrand. When all α_i are zero, the function reduces to the Aomoto polylogarithm.

The arguments x_{ij} of the hypergeometric function of type $(m+1, n+1)$ in (6.5.3) can be arranged in a matrix:

$$\bar{X} = \begin{pmatrix} x_{00} & \cdots & x_{0m} \\ x_{10} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{n0} & \cdots & x_{nm} \end{pmatrix} . \tag{6.5.4}$$

Each column in this matrix defines a hyperplane in $\mathbb{C}^n$ that appears in the hypergeometric integral as $(x_{0j} + \sum_{i=1}^{n} x_{ij} u_i)^{\alpha_i}$. Furthermore, $(n+1) \times (n+1)$ minor determinants of the matrix can be regarded as Plücker coordinates on the Grassmannian $Gr(n+1, m+1)$ over the space of arguments x_{ij}.

Sometimes it is convenient to transform the argument arrangement (6.5.4) to the following gauge fixed form

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & 0 & -1 & -x_{11} & \ldots & -x_{1\, m-n-1} \\ \vdots & & \ddots & & -1 & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 & -x_{n1} & \ldots & -x_{n\, m-n-1} \end{pmatrix} . \tag{6.5.5}$$

In this case the hypergeometric function can then be written in the following two equivalent ways, eq. (3.24) of [192]:

$$
\begin{aligned}
F((\alpha_i),(\beta_j),\gamma;x) =& c_1 \int_{\substack{u_1\geq 0,\ldots,u_n\geq 0\\ 1-\sum_a u_a\geq 0}} d^n u \prod_{i=1}^{n} u_i^{\alpha_i-1}\cdot(1-\sum_{l=1}^{n} u_l)^{\gamma-\sum_i \alpha_i -1} \prod_{j=1}^{m-n-1}(1-\sum_{i=1}^{n} x_{ij}u_i)^{-\beta_j},\\
c_1 =& \Gamma(\gamma)/\Gamma(\gamma-\sum_{i=1}^{n}\alpha_i)\cdot\prod_{i=1}^{n}\Gamma(\alpha_i),
\end{aligned}
\tag{6.5.6}
$$

and the dual representation in eq. (3.25) of [192]:

$$
\begin{aligned}
F((\alpha_i),(\beta_j),\gamma;x) =& c_2 \int_{\substack{u_1\geq 0,\ldots,u_{m-n-1}\geq 0\\ 1-\sum_a u_a\geq 0}} d^{m-n-1} u \prod_{i=1}^{m-n-1} u_i^{\beta_i-1}\cdot(1-\sum_{l=1}^{m-n-1} u_l)^{\gamma-\sum_i \beta_i -1} \prod_{j=1}^{n}(1-\sum_{i=1}^{m-n-1} x_{ji}u_i)^{-\alpha_j},\\
c_2 =& \Gamma(\gamma)/\Gamma(\gamma-\sum_{i=1}^{m-n-1}\beta_i)\cdot\prod_{i=1}^{m-n-1}\Gamma(\beta_i),
\end{aligned}
\tag{6.5.7}
$$

where the parameters are assumed to satisfy the conditions

$$
\begin{aligned}
&\alpha_i \notin \mathbb{Z},\ 1\leq i\leq n,\ \beta_j\notin\mathbb{Z},\ 1\leq j\leq m-n-1,\\
&\gamma-\sum_{i=1}^{n}\alpha_i\notin\mathbb{Z},\ \gamma-\sum_{j=1}^{m-n-1}\beta_j\notin\mathbb{Z}.
\end{aligned}
\tag{6.5.8}
$$

The hypergeometric functions (6.5.1) comprise a basis of solutions to the defining set of differential equations

$$
\begin{aligned}
&(1)\quad \sum_{i=0}^{n} x_{ij}\frac{\partial\hat{\varphi}}{\partial x_{ij}} = \alpha_j\hat{\varphi}, && 0\leq j\leq m,\\
&(2)\quad \sum_{j=0}^{m} x_{ij}\frac{\partial\hat{\varphi}}{\partial x_{ij}} = -(1+\alpha_i)\hat{\varphi}, && 0\leq i\leq n,\\
&(3)\quad \frac{\partial^2\hat{\varphi}}{\partial x_{ij}\partial x_{pq}} = \frac{\partial^2\hat{\varphi}}{\partial x_{iq}\partial x_{pj}}, && 0\leq i,p\leq n,\quad 0\leq j,q\leq m.
\end{aligned}
\tag{6.5.9}
$$

In cases where factors of the integrand are non-linear in the integration variables, the functions can be generalized further to Gelfand A-hypergeometric functions [198, 199] defined as:

$$
\hat{\mathcal{F}}(\{\alpha\},x) = \int_{\substack{u_1\geq 0,\ldots,u_k\geq 0\\ 1-\sum_a u_a\geq 0}} \prod_i \mathcal{P}_i(u_1,\ldots,u_k)^{\alpha_i} u_1^{\alpha_1}\ldots u_k^{\alpha_k} du_1\ldots du_k,
\tag{6.5.10}
$$

where α_i are complex parameters and $\mathcal{P}_i$ now are Laurent polynomials in $u_1,\ldots,u_k$.

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