ASPECTS OF 7D AND 6D GAUGED SUPERGRAVITIES

A Dissertation

by

DER-CHYN JONG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2007

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ABSTRACT

Aspects of 7D and 6D Gauged Supergravities. (December 2007)

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Chair of Advisory Committee: Dr. Ergin Sezgin

We determine the conditions under which half-maximal matter coupled gauged supergravity in seven dimensions admits a chiral circle reduction to yield a matter coupled gauged supergravity in six dimensions with 8 real supersymmetry. Solving these conditions we find that the $SO(2,2)$ and $SO(3,1)$ gauged $7D$ supergravities give a $U(1)_R$, and the $SO(2,1)$ gauged $7D$ supergravity gives an $Sp(1)_R$ gauged chiral $6D$ supergravity coupled to certain matter multiplets. In the $6D$ models obtained, with or without gauging, we show that the scalar fields of the matter sector parametrize the coset $SO(p+1,4)/SO(p+1)\times SO(4)$, with the $(p+3)$ axions corresponding to its abelian isometries.

We then derive the necessary and sufficient conditions for the existence of a Killing spinor in $N=(1,0)$ gauge $6D$ supergravity coupled to a single tensor multiplet, vector multiplets and hypermultiplets. We show that these conditions imply most of the field equations. We also determine the remaining equations that need to be satisfied by an exact solution. In this framework, we find a novel $1/8$ supersymmetric dyonic string solution with nonvanishing hypermultiplet scalars. The activated scalars parametrize a 4 dimensional submanifold of a quaternionic hyperbolic ball. The key point is that we employ an identity map between this submanifold and the internal space transverse to the string worldsheet, thereby finding a higher dimensional generalization of Gell-Mann-Zweibach tear-drop solution.

To my parents

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TABLE OF CONTENTS

CHAPTER Page

LIST OF TABLES

CHAPTER I

INTRODUCTION*

An impressively large number of string/M theory vacua admit a low energy supergravity description in diverse dimensions. An important class of such theories are gauged supergravities. Study of such theories provides valuable information about string/M theory. In particular, gauge supergravities have played an important role in phenomena such as the anti-de-Sitter Space/conformal field theory correspondence as well as the domain-wall/quantum field theory correspondence, among other phenomena. However, starting directly from a given supergravity theory, it is not always clear what, if any, string/M theory origin it may have. For example, an important class of such theories, where string/M-theory origins are still not known, are the anomaly free gauged minimal supergravities in six dimensions. The requirement of anomaly freedom leads to highly restrictive conditions in $6D$ which single out a small number of consistent quantum models. While the full classification of all possible anomaly-free gauged supergravities in 6D is not available, it is interesting to understand their string/M-theory origin.

Given that the $6D$ models of interest may be related to certain seven dimensional gauged supergravity theories which in turn may be embedded in string/M theory, we are motivated to study in this dissertation various aspects of gauged 6D and 7D supergravity theories.

We reduce the half-maximal $7D$ supergravity with specific noncompact gaugings cou-

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The journal model is *Classical and Quantum Gravity*.

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pled to a suitable number of vector multiplets, on a circle to $6D$ and chirally truncated it to $N = (1,0)$ supergravity such that a R-symmetry gauging survives. These are referred to as the $SO(3,1), SO(2,1)$ and $SO(2,2)$ models, in which these groups refer to isometries of manifolds parametrized by the scalar fields that arise in the $7D$ theory. The $6D$ models we obtain describe coupling of $N = (1,0)$ supergravity to $p+1$ hypermultiplets and n vector multiplets where $p = 1,2$ and $n = 1,2,3$ depending on the model. We then exhibit in the full model, including the fermionic contributions, how the scalar fields can be combined to parametrize an enlarged coset $SO(p+1,4)/SO(p+1) \times SO(4)$.

We next derive the necessary and sufficient conditions for the existence of arbitrary number of Killing spinor in $N = (1,0)$, $6D$ gauge supergravity coupled to vector multiplets and hypermultiplets. We then determine all the integrability conditions and the precise set of field equations that are satisfied automatically as a result. This approach provides a powerful method for finding general supersymmetric solutions, and is known the G-structure method. Here the G-structure refers to a geometric structure, such as Kahler structures, that arise in a submanifold of spacetime. We find that the existence of a null Killing vector suggests a $2+4$ split of spacetime. Thus it is natural to search for a string solution, possibly dyonic one, namely that which carries electric and magnetic charges. Indeed, we have found a new $1/8$ supersymmetric dyonic string solution with novel properties.

Below, we shall summarize in some more detail our results. The full technical details will be presented in chapter II and III.

A. The Noncompact Gaugings and Chiral Reduction of 7D Gauged Supergravities

We start with the $7D$ *ungauged* half-maximal supergravity coupled to n vector multiplets [1], in which the interactions of the scalar fields are governed by the coset $SO(n,3)/SO(n) \times SO(3)$. Due to the conditions that arise from the consistency of gauging with supersymmetry, it turns out that the noncompact gauge groups must have up to 3 compact or up to 3 noncompact generators. Considering only *semi-simple* gauge groups, it turns out that the allowed gauge groups are of the form

$$G_0 \times H \subset SO(n,3) \tag{1.1}$$

with G_0 is one of the six groups listed in (2.7) and H is a semi-simple compact Lie group with $dimH \leq (n + 3 - dimG_0)$. The models of special interest are those in which the chiral truncation of the $7D$ gauged theory gives rise to an R-symmetry gauged theory in $6D$. As we shall see, the gauged chiral $6D$ supergravities arise from half-maximal $7D$ supergravities with *noncompact* gaugings. While noncompact gauging is necessary, it is not sufficient for obtaining R-symmetry gauging in $6D$. For example we find that the $SL(3, R)$ gauged $7D$ model does not allow a consistent chiral reduction to gauged $6D$ supergravity. With mild assumption we determine that the $7D$ models with noncompact gauge groups whose chiral circle reduction do yield gauge $6D$ gauged supergravities with matter multiplets are:

- **The $SO(3,1)$ model:**

  This model is obtained form the $SO(3,1)$ gauged half-maximal $7D$ supergravity coupled to 3 vector multiplets, with $SO(3,3)/SO(3) \times SO(3)$ scalar sector. Its chiral reduction gives a $U(1)_R$ gauged supergravity coupled to a hypermultiplet.

- **The $SO(2,1)$ model:**

This model is obtained form the $SO(2,1)$ gauged half-maximal $7D$ supergravity coupled to a single vector multiplet, with $SO(3,1)/SO(3)$ scalar sector. Its chiral reduction gives rise to an $Sp(1)_R$ gauged supergravity coupled to a hypermultiplet.

- **The $SO(2,2)$ model:** This model is obtained form the $SO(2,2)$ gauged half-maximal $7D$ supergravity coupled to 3 vector multiplets, with $SO(3,3)/SO(3)$ scalar sector. Its chiral reduction gives a $U(1)_R$ gauged theory coupled to an additional Maxwell multiplet, and two hypermultiplets.

The $SO(2,2)$ and $SO(3,1)$ models can be obtained from a reduction of the $N = 1, D = 10$ supergravity on the noncompact hyperboloidal 3-manifolds $H_{2,2}$ and $H_{3,1}$, respectively [2, 3, 4]*. The hyperboloidal manifold, $H_{p,q}$, is the locus of points whose coordinates satisfy

$$x^a x^b \eta_{ab} = 1, \quad \eta_{ab} = (\overbrace{+...+}^{p}, \overbrace{-...-}^{q}) \tag{1.2}$$

with the metric

$$ds^2 = dx^a dx^b \delta_{ab}. \tag{1.3}$$

These models can also be obtained from analytical continuation of an $SO(4)$ gauged $7D$ supergravity [5] which, in turn, can be obtained from an S^3 compactification of Type IIA supergravity [6], or a limit of an S^4 reduction of $D = 11$ supergravity which reduces to a compactification on $S^3 \times R$ [7]. With regard to matter coupled gauged $7D$ supergravities, we note that the heterotic string on T^3 gives rise to half-maximal $7D$ supergravity coupled to 19 Maxwell multiplets, which, in turn, is dual to M-theory on $K3$.

*These reductions can straightforwardly be lifted to $D = 11$. Note also that the spaces $H_{p,q}$ can be constructed from embedding into a (p, q) signature plane.

We shall show that in the full model, including the fermionic contributions, how the scalar fields can be combined to parametrize an enlarged coset $SO(p+1,4)/SO(p+1) \times SO(4)$. This will be studied both in the symmetric gauge and in the Iwasawa gauge. The latter makes use of the Iwasawa decomposition which for a semisimple Lie group G generalizes the way a square real matrix can be written as a product of an orthogonal matrix ,K, a diagonal matrix with positive diagonal entries, A, and a unit upper triangular matrix ,N, as follows:

$$G = KAN. \tag{1.4}$$

Finally, as a remark, we also shown that in a formulation of the $7D$ supergravity that uses a 3-form potential, vector multiplet coupling are possible even in the presence of a topological mass term, contrary to a claim made in the literature [8].

B. Supersymmetric Solutions of Gauged 6D Supergravity

Once gauged $6D$ supergravity is embedded in string/M-theory, it would be useful to know its supersymmetric solutions. Supersymmetric solutions of supergravity theories are of particular importance in string theory because such solutions often have certain stability and non-renormalization properties that are not possessed by non-supersymmetric solutions. For example, it has been possible to give a microscopic description of certain supersymmetric black holes. Several supersymmetric solutions of gauged $6D$ supergravity have already been discovered. However, it is tempting to suspect that these new solutions are just the tip of the iceberg, and that many more surprises will be found in $6D$. Thus, we would like to know the general nature of supersymmetric solutions of supergravity theories.

To do so, we will employ the elegant and powerful method that use G-structures. This involves the derivation of the necessary and sufficient conditions for the existence of a Killing spinor. We will study them for the $N = (1, 0)$, $6D$ gauge supergravity coupled to vector multiplets and hypermultiplets. This generalizes the analysis of [9] and [10] by the inclusion of the hypermultiplets. The existence of the Killing spinor implies that the metric admits a null Killing vector. This is in contrast to some other dimensions such as $D = 4, 5$ where time-like and space-like Killing vectors arise in addition to the null one. The Killing spinor existence conditions and their integrability are shown to imply most of the equations of motion. This simplifies greatly the search for exact solutions. We will show the remaining equations to be solved are (i) the Yang-Mills equation in the null direction, (ii) the field equation for the 2-form potential, (iii) the Bianchi identities for the Yang-Mills curvature and the field strength of the 2-form potential, and (iv) the Einstein equation in the double null direction.

The most symmetric solution in $6D$ supersymmetric Einstein-Maxwell theory with $U(1)$ gauge group, known as the Salam-Sezgin model, is $R^{1,3} \times S^2$ which has been shown [4] to be the *unique* maximally symmetric solution of such model. The model by itself is anomalous but it can be embedded into an anomaly-free model with suitable Yang-Mills and hypermultiplet couplings. To find the string/M-theory origin of the anomaly free models, it is then a natural attempt to a classification of the general form of supersymmetric solutions of $N = (1, 0)$, $U(1)$ and $SU(2)$ gauged $6D$ supergravity [9, 10]. In recent years there also has been a lot of interest in models with branes embedded in higher dimensions. One particular motivation is the hope of finding a solution to the notorious cosmological constant problem. From this point of view, six-dimensional models with codimension-two branes are especially interesting. The authors in [11, 12] have found the general warped solutions with maximally symmetric four-dimensions and conical branes for the $6D$ Salam-Sezgin supergravity.

A solution of the matter coupled $N = (1, 0)$, $6D$ gauged supergravity called the 'superswirl' has been found in [13] where they did not use G-structure method and only activate two hyperscalars. Moreover, conditions for Killing spinors and general form of the $N = 2, D = 5$ supersymmetric solutions in matter coupled gauged supergravities have also been investigated. The authors in [14] recently used the G-structure method to construct supersymmetric solutions of $N = 2, D = 5$ gauged supergravity coupled to two vector multiplets and three hypermultiplets. However, they only consider the first-order equations for supersymmetric solutions that preserve a time-like Killing vector, but not a null Killing vector. Moreover, though various dyonic string solutions of $N = (1, 0), 6D$ supergravities exist in the literature [15, 16, 17, 18], none of them employ the hypermultiplets. In this dissertation, we activate 4 hyperscalar fields which parametrize a 4 dimensional submanifold of a quaternionic hyperbolic ball. We employ an identity map between this submanifold and the internal space transverse to the string worldsheet. By solving the remaining equations, we then find a new 1/8 supersymmetric dyonic string solution with novel properties.

While we will study the general theory, including vector multiplets and hypermultiplets, a particular subset for a certain field content will be free from all anomalies; gravitational, gauged and mixed. The requirement of anomaly freedom puts especially restrictive conditions on the *gauged* supergravities. We conclude our introduction by a summary of what is known so far about the anomaly-free models which satisfy these conditions. At present, the only known "naturally" anomaly-free gauged supergravities in $6D$ are:

- the $E_7 \times E_6 \times U(1)_R$ invariant model in which the hyperfermions are in the $(912, 1, 1)$ representation of the gauge group. This is a well-known model, first found by Randibar-Daemi, Salam, Sezgin and Strathdee [19] in 1985.

- the $E_7 \times G_2 \times U(1)_R$ invariant model with hyperfermions in the $(56, 14, 1)$ representation of the gauge group [20], and
- the $F_4 \times Sp(9) \times U(1)_R$ invariant model with hyperfermions in the $(52, 18, 1)$ representation of the gauge group [21].

These models have the shared features that (i) the hypermultiplets transform in nontrivial representations, (ii) there are no singlet hypermultiplets and (iii) the representations involve half-hypermultiplets. If one considers a large factor of $U(1)$ groups, and tune their $U(1)$ charges in a rather ad-hoc way [21], or considers only products of $SU(2)$ and $U(1)$ factors with a large number of hyperfermions, and tune their $U(1)$ charges again in an ad-hoc way, infinitely many possible anomaly-free combinations arise [22]. These models appear to be "unnatural" at this time.

In the remaining part of the dissertation, we will present the technical details of the ideas and results summarized above.

CHAPTER II

THE NONCOMPACT GAUGINGS AND CHIRAL REDUCTION OF 7D GAUGED SUPERGRAVITIES*

In this chapter we first recall the gauged half-maximal $7D$ supergravity couple to n vector multiplets. In particular, we list the possible non-compact gaugings in the theory. We then determine the conditions that must be satisfied by the requirement of chiral supersymmetry in $6D$, both, for the gauged and ungauged $7D$ theory. After solving these conditions, we obtain the $6D$ supergravity for the fields that survive the chiral circle reduction and their supersymmetry rules. Moreover, we exhibit the hidden quaternionic Kahler coset structure that given the couplings of the matter multiplets in $6D$ by an extensive use of the Iwasawa decomposition. This is not surprising for the bosonic sector of the ungauged supergravity theory; however, here not only we include the fermionic sector but we also exhibit the hidden symmetry in the *gauged* supergravity theory.

A. The Gauged $7D$ Model with Matter Couplings

Half-maximal supergravity in $D = 7$ coupled to n vector multiplets has the field content

$$\begin{aligned} \text{Supergravity multiplet} &: \quad (e_\mu{}^m, B_{\mu\nu}, A_{\mu j}{}^i, \sigma, \psi_\mu, \chi) \\ \text{Vector multiplet} &: \quad (A_\mu^r, \phi^\alpha, \lambda^r) \end{aligned} \tag{2.1}$$

where the fermions $\psi_\mu, \chi, \lambda^r$ are symplectic Majorana and they all carry $Sp(1)$ doublet indices which have been suppressed. Moreover, we will combine a triplet of vector

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fields, $A_{\mu j}{}^{i}$, in the supergravity multiplet to n of vector fields, A^r_μ, in vector multiplet as A^I_μ for later convenience. The $3n$ scalars $\phi^\alpha (\alpha = 1, 2, .., 3n)$ parametrize the coset

$$\frac{SO(n,3)}{SO(n) \times SO(3)} \ . \tag{2.2}$$

The gauge fermions λ^r $(r = 1, ..., n)$ transform in the vector representation of $SO(n)$, while the vector fields A^I_μ $(I = 1, ..., n+3)$ transform in the vector representation of $SO(n,3)$. The 2-form potential $B_{\mu\nu}$ and the dilaton σ are real. It is useful to define a few ingredients associated with the scalar coset manifold as they arise in the Lagrangian. We first introducing the coset representative

$$L = (L_I{}^i, L_I{}^r) \ , \qquad I = 1, ..., n+3, \quad i = 1, 2, 3 \ , \qquad r = 1, ..., n \ , \tag{2.3}$$

which forms an $(n+3) \times (n+3)$ matrix that obeys the relation

$$-L_I{}^i L^i_J + L_I{}^r L_J{}^r = \eta_{IJ} \ , \tag{2.4}$$

where $\eta_{IJ} = \mathrm{diag}(- - - + + ... +)$. The contraction of the $SO(n)$ and $SO(3)$ indices is with the Kronecker deltas δ_{rs} and δ_{ij} while the raising and lowering of the $SO(n,3)$ indices will be with the $SO(n,3)$ invariant metric η_{IJ}. Given that the $SO(3)$ indices are raised and lowered by the Kronecker delta, it follows that, in our conventions,

$$L^i_I = L_{Ii} \ , \qquad L^i_I L^I_j = -\delta^i_j \ , \qquad L^i_I L^{Ij} = -\delta^{ij} \ .$$

Note also that the inverse coset representative L^{-1} is given by $L^{-1} = \left(L^I{}_i, L^I{}_r\right)$ where $L^I{}_i = \eta^{IJ} L_{Ji}$ and $L^I{}_r = \eta^{IJ} L_{Jr}$. In the gauged matter coupled theory of [1], a key building block is the gauged Maurer-Cartan form

$$\begin{aligned} P^{ir}_\mu &= L^{Ir} \left(\partial_\mu \delta^K_I + f_{IJ}{}^K A^J_\mu\right) L^i_K \ , \\ Q^{ij}_\mu &= L^{Ij} \left(\partial_\mu \delta^K_I + f_{IJ}{}^K A^J_\mu\right) L^i_K \ , \end{aligned}$$

$$Q_\mu^{rs} = L^{Ir}\left(\partial_\mu \delta_I^K + f_{IJ}{}^K A_\mu^J\right) L_K^s \ , \tag{2.5}$$

where $f_{IJ}{}^K$ are the structure constants of the not necessarily simple group $K \subset SO(n,3)$ of dimension $n+3$, and the gauge coupling constants are absorbed into their definition of the structure constants. The K-invariance of the theory requires that the adjoint representation of K leaves η_{IJ} invariant:

$$f_{IK}{}^L \eta_{LJ} + f_{JK}{}^L \eta_{LI} = 0 \ . \tag{2.6}$$

It follows that for each simple subgroup of K, the corresponding part of η_{IJ} must be a multiple of its Cartan-Killing metric. Since η_{IJ} contains an arbitrary number of positive entries, K can be an arbitrarily large compact group. On the other hand, as η_{IJ} has only three negative entries, K can have 3 or less compact generators, or 3 or less noncompact generators*. The three real simple noncompact groups satisfying these restrictions are listed in Table I.

Table I. The three real simple noncompact groups with 3 or less compact generators, or 3 or less noncompact generators.

Group	Compact Dimensions	Noncompact Dimensions
SO(3,1)	3	3
SO(2,1)	1	2
SL(3,R)	3	5

*This is similar to the reasoning in [23] where the gauging of $N = 4, D = 4$ supergravity coupled to n vector multiplets is considered. In this case, the relevant η is the $SO(n,6)$ invariant tensor and the resulting noncompact simple gauge groups have been listed in [23].

Thus, the allowed semi-simple gauge groups are of the form $G_0 \times H \subset SO(n,3)$ where G_0 is one of the following

$$
\begin{array}{rl}
(I) & SO(3) \\
(II) & SO(3,1) \\
(III) & SL(3,R) \\
(IV) & SO(2,1) \\
(V) & SO(2,1)\times SO(2,1) \\
(VI) & SO(2,1)\times SO(2,1)\times SO(2,1)
\end{array}
\qquad (2.7)
$$

and H is a semi-simple compact Lie group with $\dim H \le (n+3-\dim G_0)$. Of these cases, only (I) with $H = SO(3)$ corresponding to SO(4) gauged supergravity, (II) and (V) are known to have a ten- or eleven-dimensional origin. Though the cases (III)–(VI) are not mentioned explicitly in [1], the Lagrangian provided there is valid for all the cases listed above. The Lagrangian of [1], up to quartic fermion terms, is given by *.

$$
\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F \qquad (2.8)
$$

$$
\begin{aligned}
e^{-1}\mathcal{L}_B &= \tfrac{1}{2}R - \tfrac{1}{4}e^{\sigma}\left(F^i_{\mu\nu}F^{\mu\nu}_i + F^r_{\mu\nu}F^{\mu\nu}_r\right) - \tfrac{1}{12}e^{2\sigma}G_{\mu\nu\rho}G^{\mu\nu\rho} \\
&\quad -\tfrac{5}{8}\partial_\mu\sigma\partial^\mu\sigma - \tfrac{1}{2}P^{ir}_\mu P^\mu_{ir} - \tfrac{1}{4}e^{-\sigma}\left(C^{ir}C_{ir} - \tfrac{1}{9}C^2\right) \ , \qquad (2.9)
\end{aligned}
$$

$$
e^{-1}\mathcal{L}_F = -\tfrac{i}{2}\bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \tfrac{5i}{2}\bar\chi\gamma^\mu D_\mu\chi - \tfrac{i}{2}\bar\lambda^r\gamma^\mu D_\mu\lambda_r - \tfrac{5i}{4}\bar\chi\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\sigma
$$

---

*We follow the conventions of [1]. In particular, $\eta_{\mu\nu} = \mathrm{diag}(-++\cdots+)$, the spinors are symplectic Majorana, $C^T = C$ and $(\gamma^\mu C)^T = -\gamma^\mu C$. Thus, $\bar\psi\gamma^{\nu_1\cdots\nu_n}\lambda = (-1)^n\bar\psi\gamma^{\nu_n\cdots\nu_1}\lambda$, where the $Sp(1)$ doublet indices are contracted and suppressed. Here we also use $X^A{}_B = \frac{1}{\sqrt{2}}(\sigma^i)^A{}_B X^i$, and further conventions are: $X^A = \epsilon^{AB}X_B,\ X_A = X^B\epsilon_{BA},\ \epsilon^{AB}\epsilon_{BC} = -\delta^A_C,\ \bar\psi\lambda = \psi^A\lambda_A,\ \bar\psi\sigma^i\lambda = \bar\psi^A(\sigma^i)_A{}^B\epsilon_B$.

$$-\tfrac{1}{2}\bar{\lambda}^r\sigma^i\gamma^\mu\gamma^\nu\psi_\mu P_{\nu ri} + \tfrac{i}{24\sqrt{2}}e^\sigma G_{\mu\nu\rho}X^{\mu\nu\rho} + \tfrac{1}{8}e^{\sigma/2}F^i_{\mu\nu}X^{\mu\nu}_i - \tfrac{i}{4}e^{\sigma/2}F^r_{\mu\nu}X^{\mu\nu}_r$$

$$-\tfrac{i\sqrt{2}}{24}e^{-\sigma/2}C\left(\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + 2\bar{\psi}_\mu\gamma^\mu\chi + 3\bar{\chi}\chi - \bar{\lambda}^r\lambda_r\right)$$

$$+\tfrac{1}{2\sqrt{2}}e^{-\sigma/2}C_{ir}\left(\bar{\psi}_\mu\sigma^i\gamma^\mu\lambda^r - 2\bar{\chi}\sigma^i\lambda^r\right) + \tfrac{1}{2}e^{-\sigma/2}C_{rsi}\bar{\lambda}^r\sigma^i\lambda^s \ , \qquad (2.10)$$

where the fermionic bilinears are defined as

$$\begin{aligned} X^{\mu\nu\rho} &= \bar{\psi}^\lambda\gamma_{[\lambda}\gamma^{\mu\nu\rho}\gamma_{\tau]}\psi^\tau + 4\bar{\psi}_\lambda\gamma^{\mu\nu\rho}\gamma^\lambda\chi - 3\bar{\chi}\gamma^{\mu\nu\rho}\chi + \bar{\lambda}^r\gamma^{\mu\nu\rho}\lambda_r \ , \\ X^{\mu\nu}_i &= \bar{\psi}^\lambda\sigma^i\gamma_{[\lambda}\gamma^{\mu\nu}\gamma_{\tau]}\psi^\tau - 2\bar{\psi}_\lambda\sigma^i\gamma^{\mu\nu}\gamma^\lambda\chi + 3\bar{\chi}\sigma^i\gamma^{\mu\nu}\chi - \bar{\lambda}^r\sigma^i\gamma^{\mu\nu}\lambda_r \ , \\ X^{\mu\nu}_r &= \bar{\psi}_\lambda\gamma^{\mu\nu}\gamma^\lambda\lambda^r + 2\bar{\chi}\gamma^{\mu\nu}\lambda^r \ . \end{aligned} \qquad (2.11)$$

The field strengths and the covariant derivatives are defined as

$$G_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} - \tfrac{3}{\sqrt{2}}\omega^0_{\mu\nu\rho} \ , \qquad \omega^0_{\mu\nu\rho} = F^I_{[\mu\nu}A^J_{\rho]}\eta_{IJ} - \tfrac{1}{3}f_{IJ}{}^K A^I_\mu A^J_\nu A_{\rho K} \ ,$$

$$F^I_{\mu\nu} = 2\partial_{[\mu}A^I_{\nu]} + f_{JK}{}^I A^J_\mu A^K_\nu \ , \qquad F^i_{\mu\nu} = F^I_{\mu\nu}L^i_I \ , \qquad F^r_{\mu\nu} = F^I_{\mu\nu}L^r_I \ , \qquad (2.12)$$

$$D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2\sqrt{2}}Q^i_\mu\,\sigma^i \ , \qquad Q^i_\mu = \frac{i}{\sqrt{2}}\ \epsilon^{ijk}Q_{\mu jk} \ , \qquad (2.13)$$

and the C-functions are given by [1]

$$\begin{aligned} C &= -\tfrac{1}{\sqrt{2}}f_{IJ}{}^K L^I_i L^J_j L_{Kk}\,\epsilon^{ijk} \ , \\ C_{ir} &= \tfrac{1}{\sqrt{2}}f_{IJ}{}^K L^I_j L^J_k L_{Kr}\,\epsilon^{ijk} \ , \\ C_{rsi} &= f_{IJ}{}^K L^I_r L^J_s L_{Ki} \ . \end{aligned} \qquad (2.14)$$

The local supersymmetry transformation rules read [1]

$$\begin{aligned} \delta e_\mu{}^m &= i\bar{\epsilon}\gamma^m\psi_\mu \ , \\ \delta\psi_\mu &= 2D_\mu\epsilon - \tfrac{1}{60\sqrt{2}}e^\sigma G_{\rho\sigma\tau}\left(\gamma_\mu\gamma^{\rho\sigma\tau} + 5\gamma^{\rho\sigma\tau}\gamma_\mu\right)\epsilon \end{aligned}$$

$$
\begin{aligned}
&\quad -\tfrac{i}{20}e^{\sigma/2}F^i_{\rho\sigma}\,\sigma^i\left(3\gamma_\mu\gamma^{\rho\sigma}-5\gamma^{\rho\sigma}\gamma_\mu\right)\epsilon-\tfrac{\sqrt{2}}{30}e^{-\sigma/2}C\gamma_\mu\epsilon\ ,\\
\delta\chi &= -\tfrac{1}{2}\gamma^\mu\partial_\mu\sigma\epsilon-\tfrac{i}{10}e^{\sigma/2}F^i_{\mu\nu}\,\sigma^i\gamma^{\mu\nu}\epsilon-\tfrac{1}{15\sqrt{2}}e^\sigma G_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon+\tfrac{\sqrt{2}}{30}e^{-\sigma/2}C\epsilon\ ,\\
\delta B_{\mu\nu} &= i\sqrt{2}e^{-\sigma}\left(\bar\epsilon\gamma_{[\mu}\psi_{\nu]}+\bar\epsilon\gamma_{\mu\nu}\chi\right)-\sqrt{2}A^I_{[\mu}\delta A^J_{\nu]}\eta_{IJ}\ ,\\
\delta\sigma &= -2i\bar\epsilon\chi\ ,\\
\delta A^I_\mu &= -e^{-\sigma/2}\left(\bar\epsilon\sigma^i\psi_\mu+\bar\epsilon\sigma^i\gamma_\mu\chi\right)L^I_i+ie^{-\sigma/2}\bar\epsilon\gamma_\mu\lambda^r L^I_r\ ,\\
\delta L^r_I &= \bar\epsilon\sigma^i\lambda^r L^i_I\ ,\qquad \delta L^i_I=\bar\epsilon\sigma^i\lambda_r L^r_I\ ,\\
\delta\lambda^r &= -\tfrac{1}{2}e^{\sigma/2}F^r_{\mu\nu}\gamma^{\mu\nu}\epsilon+i\gamma^\mu P^{ir}_\mu\sigma^i\epsilon-\tfrac{i}{\sqrt{2}}e^{-\sigma/2}C^{ir}\sigma^i\epsilon\ .
\end{aligned}
\tag{2.15}
$$

For purposes of the next section, we exhibit the gauge field dependent part of the gauged Maurer-Cartan forms:

$$
\begin{aligned}
P^{ir}_\mu &= P^{ir(0)}_\mu-\tfrac{1}{2\sqrt{2}}\,\epsilon^{ijk}\,C^{jr}A^k_\mu-C^{irs}A^s_\mu\ ,\\
Q^{ij}_\mu &= Q^{ij(0)}_\mu+\tfrac{1}{3\sqrt{2}}\,\epsilon^{ijk}\,C\,A^k_\mu-\tfrac{1}{2\sqrt{2}}\,\epsilon^{ijk}\,C^{kr}\,A^r_\mu\ ,
\end{aligned}
\tag{2.16}
$$

where the zero superscript indicates the gauge field independent parts.

B. Chiral Reduction on a Circle

1. Reduction Conditions

Here we shall consider all the $7D$ quantities of the previous section such as fields, world and Lorentz indices to be hatted, and the corresponding $6D$ quantities to be unhatted ones. We parametrize the $7D$ metric as

$$
d\hat{s}^2=e^{2\alpha\phi}\,ds^2+e^{2\beta\phi}(dy-\mathcal{A})^2\ . \tag{2.17}
$$

In order to obtain the canonical Hilbert-Einstein term, $\frac{1}{4}\sqrt{-g}R$, in $D = 6$, we choose

$$\alpha = -\sqrt{\frac{n}{2(D+n-2)(D-2)}} = -\frac{1}{2\sqrt{10}} \ , \qquad \beta = -\frac{D-2}{n}\alpha = -4\alpha \ . \tag{2.18}$$

where $D = 6$ and $n = 1$ in this dimensional reduction. We shall work with the natural vielbein basis

$$\hat{e}^a = e^{\alpha\phi} e^a \ , \qquad \hat{e}^7 = e^{\beta\phi}(dy - \mathcal{A}) \ . \tag{2.19}$$

It is also convenient to work with the Latin connection $\hat{\omega}_{\hat{m}\hat{n}\hat{p}} = \hat{e}^{\hat{\mu}}_{\hat{m}} \hat{\omega}_{\hat{\mu}\hat{n}\hat{p}}$, which is a scalar under general coordinate transformation and is antisymmetric in its last two indices. In the second-order formalism, we have

$$\hat{\omega}_{\hat{m}\hat{n}\hat{p}} = -\hat{\Omega}_{\hat{m}\hat{n},\hat{p}} + \hat{\Omega}_{\hat{n}\hat{p},\hat{m}} - \hat{\Omega}_{\hat{p}\hat{m},\hat{n}} \tag{2.20}$$

where

$$\hat{\Omega}_{\hat{m}\hat{n},\hat{p}} = \frac{1}{2}(\hat{e}^{\hat{\mu}}_{\hat{m}} \hat{e}^{\hat{\nu}}_{\hat{n}} - \hat{e}^{\hat{\nu}}_{\hat{m}} \hat{e}^{\hat{\mu}}_{\hat{n}}) \partial_{\hat{\nu}} \hat{e}_{\hat{\mu}\hat{p}}. \tag{2.21}$$

it turns out that the nonzero components of the spin structure are

$$\hat{\omega}_{cab} = e^{-\alpha\phi} \left(\omega_{cab} + 2\alpha \eta_{c[a} \partial_{b]} \phi \right) \ , \quad \hat{\omega}_{77a} = \beta e^{-\alpha\phi} \partial_a \phi \ . \tag{2.22}$$

Next, we analyze the constraints that come from the requirement of circle reduction followed by chiral truncation retaining $N = (1, 0)$ supersymmetry. Let us first set to zero the $7D$ gauge coupling constant and deduce the consistent chiral truncation conditions. At the end of the section we shall then re-introduce the coupling constant and determine the additional constraints that need to be satisfied. The gravitino field in seven dimensions splits into a left handed and a right handed gravitino in six dimensions upon reduction in a compact direction. Chiral truncation means that we set one of them to zero, say,

$$\widehat{\psi}_{a-} = 0 \ . \tag{2.23}$$

This condition with chirality properties, used in the supersymmetry variation of the 7D vielbein, gives the following supersymmetry variation of the 6D vielbein,

$$\delta e_\mu{}^m = i\bar{\epsilon}_+\gamma^m\psi_{\mu+} + \frac{i}{4}\bar{\epsilon}_+\gamma^m\gamma_\mu\psi_{7-}. \tag{2.24}$$

It turns out that one of the components of the supersymmetric parameter, ϵ_-, vanishes. Note that the second term on the right-hand side can be removed by performing a compensating local Lorentz transformation with parameter, $\Lambda_\mu{}^m = \frac{i}{4}\bar{\epsilon}_+\gamma^m\gamma_\mu\psi_{7-}$. Moreover, the supersymmetry variation of the 7D gravitino and the field $\widehat{\chi}$ gives

$$\begin{aligned}
\delta\psi_{\mu+} &= D_\mu\epsilon_+ - \tfrac{1}{24}e^{\sigma-2\alpha\phi}G_{\rho\sigma\tau}\gamma^{\rho\sigma\tau}\gamma_\mu\epsilon_+ + \tfrac{i}{\sqrt{2}}e^{-\sigma/2-\beta\phi}\widehat{F}^{\hat{I}}_{\mu 7}\,L_{\hat{I}}{}^i\sigma^i\epsilon_+,\\
\delta\psi_{7-} &= -\frac{1}{\sqrt{10}}\gamma^\mu\partial_\mu\phi\epsilon_+ - \tfrac{1}{30}e^{\sigma-2\alpha\phi}G_{\rho\sigma\tau}\gamma^{\rho\sigma\tau}\epsilon_+ - \tfrac{4i}{5\sqrt{2}}e^{\sigma/2-\beta\phi}\gamma^\mu\widehat{F}^{\hat{I}}_{\mu 7}\,L_{\hat{I}}{}^i\sigma^i\epsilon_+,\\
\delta\chi_- &= -\tfrac{1}{4}\gamma^\mu\partial_\mu\sigma\epsilon_+ - \tfrac{1}{30}e^{\sigma-2\alpha\phi}G_{\rho\sigma\tau}\gamma^{\rho\sigma\tau}\epsilon_+ + \tfrac{i}{5\sqrt{2}}e^{\sigma/2-\beta\phi}\gamma^\mu\widehat{F}^{\hat{I}}_{\mu 7}\,L_{\hat{I}}{}^i\sigma^i\epsilon_+,
\end{aligned}$$

which follows further conditions

$$\widehat{F}^{\hat{I}}_{ab}\,L_{\hat{I}}{}^i = 0\ , \tag{2.25}$$

$$\mathcal{A}_a = 0\ , \qquad \widehat{G}_{ab7} = 0\ , \qquad \widehat{\psi}_{7+} = 0\ , \qquad \widehat{\chi}_+ = 0. \tag{2.26}$$

To see how we can satisfy the condition (2.25), it is useful to consider an explicit realization of the $SO(n,3)/SO(n)\times SO(3)$ coset representative. A convenient such parametrization is given by

$$\widehat{L} = \begin{pmatrix} \frac{1+\phi^t\phi}{1-\phi^t\phi} & \frac{2}{1-\phi^t\phi}\phi^t \\ & \\ \phi\frac{2}{1-\phi^t\phi} & 1+\phi\frac{2}{1-\phi^t\phi}\phi^t \end{pmatrix} \tag{2.27}$$

where ϕ is a $n\times 3$ matrix $\phi_{\hat{r}i}$. Note that this is symmetric, and as such, we shall refer to this as the coset representative in the *symmetric gauge*. Now, we observe that to satisfy (2.25), we can split the index

$$\hat{I} = \{I, I'\} \ , \qquad I = 1, ..., p+3 \ , \qquad I' = p+4, ..., n+3 \ , \tag{2.28}$$

and set

$$\widehat{A}_a^I = 0 \ , \qquad L_{I'}{}^i = 0 \ . \tag{2.29}$$

Note that $0 \leq p \leq n$, and in particular, for $p = n$, all vector fields $A_\mu^I, I = 1, ..., n+3$ vanish (i.e. there are no $A_\mu^{r'}$ fields) while all the coset scalars $\phi^{\hat{r}i}$ are nonvanishing *. For $p < n$, however, as we shall see below, $(n-p)$ vector fields survive, and these, in turn, will play a role in obtaining a gauged supergravity in $6D$. The second condition in (2.29) amounts to setting $\phi_{r'i} = 0$ and consequently, introducing the notation

$$\widehat{r} = \{r, r'\} \ , \qquad r = 1, ..., p \ , \qquad r' = p+1, ..., n \ , \tag{2.30}$$

we have

$$L_I{}^{r'} = 0 \ , \qquad L_{I'}{}^r = 0 \ , \qquad L_{I'}{}^{r'} = \delta_{I'}{}^{r'} \ . \tag{2.31}$$

Thus the surviving scalar fields are

$$\left(\widehat{L}_I{}^i, \widehat{L}_I{}^r\right) \equiv \left(L_I{}^i, L_I{}^r\right) \ , \qquad I = 1, ..., p+3 \ , \ \ i = 1,2,3 \ , \ \ r = 1, ..., p \ . \tag{2.32}$$

This is the coset representative of $SO(p,3)/SO(p) \times SO(3)$. From the supersymmetric variations of the vanishing coset representatives $(L_{I'}{}^i, L_I{}^{r'}, L_{I'}{}^r)$, on the other hand, we find that

$$\widehat{\lambda}_+^r = 0 \ , \quad \widehat{\lambda}_-^{r'} = 0 \ . \tag{2.33}$$

---

*Note also that for $p = 0$, all coset scalars $\phi^{\hat{r}i}$ vanish while n vector field $A_\mu^{r'}$ survive.

Using these results in the supersymmetry variation of $\widehat{A}_7^{I'}$, in turn, immediately gives

$$\widehat{A}_7^{I'} = 0 \ . \tag{2.34}$$

Next, defining

$$\widehat{B} = B_{\mu\nu}\ dx^\mu \wedge dx^\nu + B_\mu\ dx^\mu \wedge dy \ , \tag{2.35}$$

the already found conditions $\widehat{G}_{ab7} = \mathcal{A}_a = \widehat{A}_a^I = 0$ gives

$$\begin{aligned} 0 &= \widehat{G}_{ab7} \\ &= 3\partial_{[a}\hat{B}_{b7]} - \tfrac{3}{\sqrt{2}}\hat{A}^I_{[a}\hat{F}^J_{b7]}\eta_{IJ} + \tfrac{1}{\sqrt{2}}f_{IJ}{}^K\hat{A}^I_a\hat{A}^J_b\hat{A}_{7K} \\ &= \partial_a B_b - \partial_b B_a \end{aligned} \tag{2.36}$$

which implies that

$$B_\mu = 0 \ . \tag{2.37}$$

In summary, the surviving bosonic fields are

$$\Big(g_{\mu\nu},\ \phi,\ B_{\mu\nu},\ \widehat{\sigma},\ \phi_{ir},\ \widehat{A}_7^I\ ,\ A_\mu^{I'} \Big)\ , \tag{2.38}$$

and the surviving fermionic fields are

$$\Big(\widehat{\psi}_{\mu+},\ \widehat{\psi}_{7-},\ \widehat{\chi}_-,\ \widehat{\lambda}^r_-,\ \widehat{\lambda}^{r'}_+ \Big)\ . \tag{2.39}$$

We will show in the next section that suitable combinations of these fields (see Eq. (2.62)) form the following supermultiplets:

$$\begin{aligned} &(g_{\mu\nu}, B_{\mu\nu}, \sigma, \psi_\mu, \chi)\ ,\quad (A_\mu^{I'}, \lambda^{r'})\ ,\quad (\phi_{ir}, \Phi^I, \varphi, \lambda^r, \psi)\ , \\ &I = 1, ..., p+3\ ,\quad I' = p+4, ..., n+3\ , \\ &r = 1, ..., p\ ,\quad r' = p+1, ..., n\ ,\quad i = 1, 2, 3\ . \end{aligned} \tag{2.40}$$

The last multiplet represents a fusion of p linear multiplets and one special linear multiplet, as explained in the introduction. In particular, the $(p+3)$ axionic scalars Φ^I can be dualized to 4-form potentials. Further truncations are possible. Setting $\phi_{ir}=0$ gives one special linear multiplet with fields (Φ^i,φ,ψ) while setting $\Phi^I=0$ eliminates all the (special) linear multiplets.

a. Extra Conditions due to Gauging

Extra conditions emerge upon turning on the $7D$ gauge coupling constants. They arise from the requirement that the gauge coupling constant dependent terms in the supersymmetry variations of $(\widehat{\psi}_{a-},\widehat{\psi}_{7+},\widehat{\lambda}^r_+,\widehat{\lambda}^{r'}_-)$ vanish. These conditions are

$$C=0\ ,\qquad C^{ir}=0\ ,$$
$$C^{irs}\Phi^s=0\ ,\qquad C^{ir's'}A^{s'}_\mu=0\ , \tag{2.41}$$

where $\Phi^r=\Phi^I L_I^r$ and $A^{s'}_\mu=A^{I'}_\mu L^{r'}_{I'}$. More explicitly, these conditions take the form

$$\widehat{f}_{IJK}\,L_i^I L_j^J L_k^K=0\ ,\qquad \widehat{f}_{IJK}\,L_i^I L_j^J L_r^K=0\ , \tag{2.42}$$

$$\widehat{f}_{IJK}\,L_i^I L_r^J \Phi^K=0\ ,\qquad \widehat{f}_{Ir's'}\,L_i^I\,A^{s'}_\mu=0\ . \tag{2.43}$$

Solving these conditions, while keeping all $A^{r'}_\mu$ and Φ^I, results in a chiral gauged supergravity theory with the multiplets shown in (2.40) and gauge group $K'\subset SO(n,3)$ with structure constants $\widehat{f}_{r's't'}$. The scalars Φ^I transform in a $(p+3)$ dimensional representation of K', and there are $3p$ scalars which parametrize the coset $SO(p,3)/SO(p)\times SO(3)$. The nature of the R-symmetry gauge group can be read off from

$$D_\mu\epsilon=D^{(0)}_\mu\epsilon+\tfrac{1}{2\sqrt{2}}\sigma^i\,C^{ir'}\,A^{r'}_\mu\epsilon\ . \tag{2.44}$$

Note that the $6D$ model is R-symmetry gauged provided that $C^{ir'}$ does not vanish upon setting all scalars to zero. Moreover, an abelian R-symmetry group can arise when $\widehat{f}_{r's't'}$ vanishes with $C^{kr'} \neq 0$. Next, we show how to solve the conditions (2.42) and (2.43).

2. Solution to the Reduction Conditions

The conditions (2.42) and (2.43) can be solved by setting

$$\widehat{f}_{IJ}{}^{K} = 0 \ , \qquad \widehat{f}_{I'J'}{}^{K} = 0 \ . \tag{2.45}$$

Moreover, the structure constants of the $7D$ gauge group $G_0 \times H \subset SO(n,3)$ must satisfy the condition (2.6):

$$\widehat{f}_{\hat{I}\hat{K}}{}^{\hat{L}}\, \eta_{\hat{L}\hat{J}} + \widehat{f}_{\hat{J}\hat{K}}{}^{\hat{L}}\, \eta_{\hat{L}\hat{I}} = 0 \ . \tag{2.46}$$

Given the $7D$ gauge groups listed in (2.7), we now check case by case when and how these conditions can be satisfied. To begin with, we observe that given the $G_0 \times H \subset SO(n,3)$ gauged supergravity theory, the H sector can always be carried over to $6D$ dimension to give the corresponding Yang-Mills sector whose H-valued gauge fields do not participate in a possible R-symmetry gauging. Therefore, we shall consider the G_0 part of the $7D$ gauge group in what follows.

(I) $SO(3)$

In this model, the $7D$ gauge group is $SO(3)$ with structure constants

$$\widehat{f}_{\hat{I}\hat{J}\hat{K}} = (g\, \epsilon_{IJK}, 0) \ . \tag{2.47}$$

To satisfy (2.45), we must set $g = 0$. Thus, we see that *a chiral truncation to a gauged $6D$ theory is not possible in this case.*

(II) $SO(3,1)$

The smallest $7D$ scalar manifold that can accommodate this gauging is $SO(3,3)/SO(3)\times SO(3)$. In the $7D$ theory, the gauge group $G_0 = SO(3,1)$ can be embedded in $SO(3,3)$ as follows. Denoting the $SO(3,3)$ generators by $T_{AB} = (T_{ij}, T_{rs}, T_{ir})$, we can embed $SO(3,1)$ by choosing the generators (T_{rs}, T_{3r}) which obey the commutation rules of the $SO(3,1)$ algebra. These generators can be relabeled as

$$(T_{34}, T_{35}, T_{36}, T_{45}, T_{56}, T_{64}) = (T_1, T_2, T_3, T_4, T_5, T_6) \equiv (T_I, T_{I'}) \ , \qquad (2.48)$$

with $I = 1,2,3$ and $I' = 4,5,6$. The algebra of these generators is given by

$$[T_I, T_J] = f_{IJ}{}^{K'}\, T_{K'} \ , \qquad [T_{I'}, T_J] = f_{I'J}{}^{K}\, T_K \ , \qquad [T_{I'}, T_{J'}] = f_{I'J'}{}^{K'}\, T_{K'} \ . \qquad (2.49)$$

Thus, the conditions (2.45) are satisfied. Furthermore, the Cartan-Killing metric associated with this algebra is $(+++---)$ and it satisfies the condition (2.46). In this case, all the coset scalars are vanishing and the surviving matter scalar fields are (Φ^i, φ) which are the bosonic fields of a special linear multiplet. This sector will be shown to be described by the quaternionic Kahler coset $SO(4,1)/SO(4)$ in section 5. We thus obtain an $Sp(1,R)$ *gauged supergravity in* $6D$ *coupled to a single hypermultiplet*. In summary, we have the following chain of chiral circle reduction and hidden symmetry in this case:

$$\frac{SO(3,3)}{SO(3)\times SO(3)} \quad \leftrightarrow \quad (\Phi^i, \varphi) \quad \leftrightarrow \quad \frac{SO(4,1)}{SO(4)} \qquad (2.50)$$

Note that the $7D$ theory we start with has $64_B + 64_F$ physical degrees of freedom, while the resulting $6D$ theory has $24_B + 24_F$ physical degrees of freedom. We will show that the $6D$ field contents can be rewritten as

$$\begin{aligned} \text{Supergravity multiplet} &: \quad (g_{\mu\nu}, B_{\mu\nu}, \sigma, \psi_\mu, \chi) \\ \text{Hypermultiplet} &: \quad (\Phi^I, \psi), \quad I = 1, ..., 4. \end{aligned} \tag{2.51}$$

(III) $SL(3,R)$

The minimal $7D$ scalar manifold to accommodate this gauging is $SO(5,3)/SO(5) \times SO(3)$. In the $7D$ theory, the gauge group is $SL(3,R)$, which has 3 compact and 5 noncompact generators. The condition (2.46) can be satisfied with $\eta = \text{diag}(- - - + + + + +)$ by making a particular choice of the generators of $SL(3,R)$ such as

$$(i\lambda_2, i\lambda_5, i\lambda_7, \lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_8) = (T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8) = (T_I, T_{I'}) \ , \tag{2.52}$$

where $\lambda_1, ..., \lambda_8$ are the standard Gell-Mann matrices, and $I = 1, ..., p+3, I' = p+4, ..., 8$ with $0 \leq p \leq 5$. However, the condition (2.45) is clearly not satisfied since $[T_1, T_2] = T_3$ and thus $\widehat{f}_{IJ}{}^K \neq 0$. Therefore, we conclude that the *chiral truncation to a gauged* $6D$ *theory is not possible in this case.*

(IV) $SO(2,1)$

For this gauging, the minimal $7D$ scalar manifold is $SO(3,1)/SO(3)$. Let us denote the generators of $SO(3,1)$ by $T_{AB} = (T_{ij}, T_{4i})$ where $i = 1,2,3$. The $7D$ gauge group $SO(2,1)$ can be embedded into this $SO(3,1)$ by picking out the generators (T_{41}, T_{42}, T_{12}), where the last generator is compact and the other two are noncompact. Thus,

$$\widehat{f}_{\hat{I}\hat{J}\hat{K}} = \Big(g\, \epsilon_{\underline{ijk}}, 0\Big) \ , \quad \underline{i} = 1,2,4 \ , \tag{2.53}$$

where (T_{41}, T_{42}, T_{12}) correspond to (T_1, T_2, T_4), respectively. The $SO(3,1)$ vector index, on the other hand, is labeled as $I = 1,2,3$ and $I' = 4$. Thus, the conditions

(2.45) and (2.46) are satisfied and the resulting $6D$ theory is *a* $U(1)_R$ *gauged supergravity coupled to one special linear multiplet.* The gauge field is A^4_μ, and the special linear multiplet lends itself to a description in terms of the quaternionic Kahler coset $SO(4,1)/SO(4)$. We thus obtain an $U(1)_R$ *gauged supergravity in* $6D$ *coupled to one hypermultiplet.* This model is similar to the $Sp(1)_R$ gauged model obtained from the $SO(3,1)$ gauged $7D$ supergravity described above, the only difference being that the gauge group is now $U(1)_R$. In summary, we have the following chain of chiral circle reduction and hidden symmetry:

$$\frac{SO(3,1)}{SO(3)} \quad \leftrightarrow \quad (\Phi^i,\varphi) \quad \leftrightarrow \quad \frac{SO(4,1)}{SO(4)} \tag{2.54}$$

In this case, the $7D$ theory we start with has $64_B + 64_F$ physical degrees of freedom, while the resulting $6D$ theory has $24_B + 24_F$ physical degrees of freedom. We will show that the $6D$ field contents can be rewritten as

$$\begin{aligned} \text{Supergravity multiplet} \quad &: \quad (g_{\mu\nu}, B_{\mu\nu}, \sigma, \psi_\mu, \chi) \\ \text{Hypermultiplet} \quad &: \quad (\Phi^I, \psi), \;\; I = 1, ..., 4. \end{aligned} \tag{2.55}$$

(V) $SO(2,2)$

This case is of considerable interest as it can be obtained from a reduction of $N = 1$ supergravity in ten dimensions on a certain manifold $H_{2,2}$ as shown in [4], where its chiral circle reduction has been studied. As we shall see below, their result is a special case of a more general such reduction. The minimal model that can accommodate the $SO(2,2)$ gauging is $SO(3,3)/SO(3) \times SO(3) \sim SL(4,R)/SO(4)$. To solve the conditions (2.45), we embed the $SO(2,2)$ in $SO(3,3)$ by setting

$$\begin{aligned} &\widehat{f}_{\hat{I}\hat{J}}{}^{\hat{K}} = (g_1\; \epsilon_{\underline{ij\ell}}\; \eta^{\underline{k\ell}}\;, g_2\; \epsilon_{\underline{rst}}\; \eta^{\underline{tq}}\;)\;, \qquad \underline{i} = 1,2,6\;, \quad \underline{r} = 3,4,5\;, \\ &\eta_{\underline{ij}} = \text{diag}\;(--+)\;, \qquad \eta_{\underline{rs}} = \text{diag}\;(-++)\;, \end{aligned} \tag{2.56}$$

where (g_1, g_2) are the gauge coupling constants for $SO(2,1) \times SO(2,1) \sim SO(2,2)$. These structure constants can be checked to satisfy the condition (2.46). Furthermore, the conditions (2.45) are satisfied since $I = 1,2,3,4$ and $I' = 5,6$. The resulting $6D$ theory is *a* $U(1)_R$ *gauged supergravity coupled to one external Maxwell multiplet (in addition to the Maxwell multiplet that gauges the R-symmetry) and two hypermultiplets.* The two hypermultiplets consist of the fields shown in the last group in (2.40) with $p = 1, n = 3$. The $U(1)_R$ is gauged by the vector field A_μ^6. The vector field A_μ^5, which corresponds to $O(1,1)$ rotations, resides in the Maxwell multiplet. In this model, the surviving $SO(3,1)/SO(3)$ sigma model sector in $6D$, gets enlarged with the help of the axionic fields to become the quaternionic Kahler coset $SO(4,2)/SO(4) \times SO(2)$, as will be shown in section 5. In summary, we have the following chain of chiral circle reduction and hidden symmetry

$$\frac{SO(3,3)}{SO(3) \times SO(3)} \quad \leftrightarrow \quad \frac{SO(3,1)}{SO(3)} \quad \leftrightarrow \quad \frac{SO(4,2)}{SO(4) \times SO(2)} \tag{2.57}$$

It is also worth noting that the Cvetic-Gibbons–Pope reduction [4] that gave rise to the $U(1)_R$ gauged 6D supergravity is a special case of our results that can be obtained by setting to zero all the scalar fields of the $SO(3,1)/SO(3)$ sigma model, the gauge field A_μ^5 and their fermionic partners. This model was studied in the language of the $SL(4,R)/SO(4)$ coset structure. In Appendix B, we give the map between this coset and the $SO(3,3)/SO(3) \times SO(3)$ coset used here. Note that, in this case the $7D$ theory we start with has $64_B + 64_F$ physical degrees of freedom, and the resulting $6D$ theory has half as many, namely, $32_B + 32_F$ physical degrees of freedom. We will show that the $6D$ field contents can be rewritten as

$$\begin{aligned} \text{Supergravity multiplet} &: \ (g_{\mu\nu}, B_{\mu\nu}, \sigma, \psi_\mu, \chi) \\ \text{Vector multiplet} &: \ (A_\mu^{I'}, \lambda^{I'}), \ \ I' = 1,2 \\ \text{Hypermultiplet} &: \ (\Phi^{Ir}, \psi^r), \ \ I = 1,...,4, \ \ r = 1,2. \end{aligned} \tag{2.58}$$

(VI) $SO(2,2) \times SO(2,1)$

In this case, the minimal $7D$ sigma model sector is based on $SO(6,3)/SO(6) \times SO(3)$. To solve the condition (2.45) in such a way to obtain an R-symmetry gauged $6D$ supergravity, we embed the $7D$ gauge group $SO(2,2) \times SO(2,1)$ in $SO(6,3)$ by setting

$$\begin{aligned} &\widehat{f}_{\hat{I}\hat{J}}{}^{\hat{K}} = \Big(g_1 \ \epsilon_{\underline{ij\ell}} \ \eta^{\underline{k\ell}} \ , g_2 \ \epsilon_{\underline{rst}} \ \eta^{\underline{tq}} \ , \ g_3 \ \epsilon_{i'j'\ell'} \eta^{k'\ell'} \Big) \ , \ \underline{i} = 1,4,5, \ \underline{r} = 2,6,7, \ i' = 3,8,9, \\ &\eta_{\underline{ij}} = \text{diag} \ (-++) \ , \ \eta_{\underline{rs}} = \text{diag} \ (-++), \ \eta_{i'j'} = \text{diag} \ \ (-++), \end{aligned} \tag{2.59}$$

where (g_1, g_2, g_3) are the gauge coupling constants for $SO(2,1) \times SO(2,1) \times SO(2,1)$. The conditions (2.45) are satisfied since $I = 1,2,3,5,7,9$ and $I' = 4,6,8$. *The resulting* $6D$ *theory has a local* $O(1,1)^3$ *gauge symmetry, and hence three Maxwell multiplets but no gauged* R *symmetry, and three hypermultiplets.* The gauge fields are $(A_\mu^4, A_\mu^6, A_\mu^8)$, and the hypermultiplets consist of the fields shown in the last group in (2.40) with $p = 3, n = 6$. In this model, the surviving $SO(3,3)/SO(3) \times SO(3)$ sigma model in $6D$ gets enlarged to the quaternionic Kahler $SO(4,4)/SO(4) \times SO(4)$ with the help of the axionic fields, as will be described in section 5. In summary, we have the following chain of chiral circle reduction and hidden symmetry:

$$\frac{SO(6,3)}{SO(6) \times SO(3)} \quad \leftrightarrow \quad \frac{SO(3,3)}{SO(3) \times SO(3)} \quad \leftrightarrow \quad \frac{SO(4,4)}{SO(4) \times SO(4)} \tag{2.60}$$

Moreover, we will show that the resulting $6D$ field contents can be rewritten as

$$\begin{array}{rcl} \text{Supergravity multiplet} & : & (g_{\mu\nu}, B_{\mu\nu}, \sigma, \psi_\mu, \chi) \\ \text{Vector multiplet} & : & (A_\mu^{I'}, \lambda^{I'}),\ \ I' = 1, 2, 3 \\ \text{Hypermultiplet} & : & (\Phi^{Ir}, \psi^r),\ \ I = 1, ..., 4,\ \ r = 1, 2, 3. \end{array} \tag{2.61}$$

To summarize, we have found that the $SO(3,1)$ and $SO(2,2)$ gauged $7D$ models give rise to $U(1)_R$ gauged supergravity, and the $SO(2,1)$ gauged $7D$ model yields an $Sp(1)_R$ gauged chiral supergravity, coupled to specific matter multiplets in six dimensions.

C. The $6D$ Lagrangian and Supersymmetry Transformations

The chiral reduction on a circle along the lines described above requires, as usual, the diagonalization of the kinetic terms for various matter fields. This is achieved by defining:

$$\begin{array}{rcl} \sigma & = & (\hat{\sigma} - 2\alpha\phi)\ , \qquad\qquad \varphi = \frac{1}{2}(\hat{\sigma} + 8\alpha\phi)\ , \\ \chi & = & \sqrt{2}e^{\alpha\phi/2}\left(\hat{\chi} + \frac{1}{4}\hat{\psi}_7\right)\ , \qquad \psi = \frac{1}{\sqrt{2}}\,e^{\alpha\phi/2}\left(\hat{\psi}_7 - \hat{\chi}\right)\ , \\ \psi_a & = & \frac{1}{\sqrt{2}}\,e^{\alpha\phi/2}\left(\hat{\psi}_a - \frac{1}{4}\gamma_a\hat{\psi}_7\right)\ , \qquad \psi^r = \frac{1}{\sqrt{2}}\,e^{\alpha\phi/2}\hat{\lambda}^r\ , \\ \Phi^I & = & \widehat{A}_7^I\ , \qquad \lambda^{r'} = \frac{1}{\sqrt{2}}\,e^{\alpha\phi/2}\hat{\lambda}^{r'}\ , \qquad \widehat{\epsilon} = \frac{1}{\sqrt{2}}\,e^{\alpha\phi/2}\epsilon\ . \end{array} \tag{2.62}$$

The $6D$ supergravity theory obtained by the reduction scheme describe above has the Lagrangian $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$ where*

$$
\begin{aligned}
e^{-1}\mathcal{L}_B &= \tfrac{1}{4}R - \frac{1}{4}(\partial_\mu\sigma)^2 - \tfrac{1}{12}e^{2\sigma}G_{\mu\nu\rho}G^{\mu\nu\rho} - \tfrac{1}{8}e^{\sigma}F^{r'}_{\mu\nu}F^{\mu\nu r'} \\
&\quad -\tfrac{1}{4}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{4}P^{ir}_\mu P^\mu_{ir} - \tfrac{1}{4}\left(P^r_\mu P^\mu_r + \mathcal{P}^i_\mu\mathcal{P}^\mu_i\right) \\
&\quad -\tfrac{1}{8}e^{-\sigma}\left(C^{ir'}C_{ir'} + 2S^{ir'}S_{ir'}\right) \ ,
\end{aligned}
\tag{2.63}
$$

$$
\begin{aligned}
e^{-1}\mathcal{L}_F &= -\tfrac{i}{2}\bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \tfrac{i}{2}\bar\chi\gamma^\mu D_\mu\chi - \tfrac{i}{2}\bar\lambda^{r'}\gamma^\mu D_\mu\lambda_{r'} \\
&\quad -\tfrac{i}{2}\bar\psi\gamma^\mu D_\mu\psi - \tfrac{i}{2}\bar\psi^r\gamma^\mu D_\mu\psi^r - \tfrac{i}{2}\bar\chi\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\sigma \\
&\quad -\tfrac{1}{2}\bar\psi^r\gamma^\mu\gamma^\nu\sigma_i\psi_\mu P^{ir}_\nu + \tfrac{i}{2}\bar\psi\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\varphi \\
&\quad -\tfrac{1}{2}\bar\psi\gamma^\mu\gamma^\nu\sigma_i\psi_\mu\mathcal{P}^i_\nu - \frac{i}{2}\bar\psi^r\gamma^\mu\gamma^\nu\psi_\mu P^r_\nu - \tfrac{1}{4}\mathcal{P}^i_\mu X^\mu_i \\
&\quad -iP^r_\mu X^\mu_r + \tfrac{i}{24}e^\sigma G_{\mu\nu\rho}X^{\mu\nu\rho} - \tfrac{i}{4}e^{\sigma/2}F^{r'}_{\mu\nu}X^{\mu\nu}_{r'} \\
&\quad +e^{-\sigma/2}\left(-C_{irr'}\bar\lambda^{r'}\sigma^i\psi^r + iS_{rr'}\bar\lambda^{r'}\psi^r - S_{ir'}\bar\lambda^{r'}\sigma^i\psi\right) \\
&\quad +\tfrac{1}{2\sqrt{2}}e^{-\sigma/2}\bar\lambda^{r'}\sigma^i\gamma^\mu\psi_\mu\left(C_{ir'} - \sqrt{2}S_{ir'}\right) \\
&\quad +\tfrac{1}{2\sqrt{2}}e^{-\sigma/2}\bar\lambda^{r'}\sigma^i\chi\left(C_{ir'} - \sqrt{2}S_{ir'}\right) \ ,
\end{aligned}
\tag{2.64}
$$

and where

$$
\begin{aligned}
X^{\mu\nu\rho} &= \bar\psi^\lambda\gamma_{[\lambda}\gamma^{\mu\nu\rho}\gamma_{\tau]}\psi^\tau + \bar\psi_\lambda\gamma^{\mu\nu\rho}\gamma^\lambda\chi - \bar\chi\gamma^{\mu\nu\rho}\chi + \bar\lambda^{r'}\gamma^{\mu\nu\rho}\lambda_{r'} + \bar\psi^r\gamma^{\mu\nu\rho}\psi_r + \bar\psi\gamma^{\mu\nu\rho}\psi \ , \\
X^\mu_i &= \bar\psi^\rho\gamma_{[\rho}\gamma^\mu\gamma_{\tau]}\sigma_i\psi^\tau + \bar\chi\gamma^\mu\sigma_i\chi + \bar\lambda^{r'}\gamma^\mu\sigma_i\lambda_{r'} - \bar\psi^r\gamma^\mu\sigma_i\psi_r - \bar\psi\gamma^\mu\sigma_i\psi \ ,
\end{aligned}
$$

---

*In order to make contact with more standard conventions in $6D$, we have redefined $G_{\mu\nu\rho} \to \sqrt{2}G_{\mu\nu\rho}$ and multiplied the Lagrangian by a factor of $1/2$. The spacetime signature is $(-+++++)$, the spinors are symplectic Majorana-Weyl, $C^T = -C$ and $(\gamma^\mu C)^T = -\gamma^\mu C$. Thus, $\bar\psi\gamma^{\nu_1\cdots\nu_n}\lambda = (-1)^n\bar\psi\gamma^{\nu_n\cdots\nu_1}\lambda$, where the $Sp(1)$ doublet indices are contracted and suppressed. We also use the convention: $\gamma_{\mu_1\cdots\mu_6} = e\epsilon_{\mu_1\cdots\mu_6}\gamma_7$.

$$
\begin{aligned}
X^{\mu\nu}_{r'} &= \bar{\psi}_\rho \gamma^{\mu\nu}\gamma^\rho \lambda_{r'} + \bar{\chi}\gamma^{\mu\nu}\lambda_{r'} \ , \\
X^{\mu}_{r} &= \bar{\psi}\gamma^{\mu}\psi_r \ .
\end{aligned} \tag{2.65}
$$

The action is invariant under the following $6D$ supersymmetry transformations

$$
\begin{aligned}
\delta e^m_\mu &= i\bar{\epsilon}\gamma^m \psi_\mu \ , \\
\delta\psi_\mu &= D_\mu \epsilon - \tfrac{1}{24} e^\sigma \gamma^{\rho\sigma\tau}\gamma_\mu G_{\rho\sigma\tau}\,\epsilon - \tfrac{i}{2}\mathcal{P}^i_\mu \sigma^i \epsilon \ , \\
\delta\chi &= -\tfrac{1}{2}\gamma^\mu \partial_\mu \sigma \epsilon - \tfrac{1}{12} e^\sigma \gamma^{\rho\sigma\tau} G_{\rho\sigma\tau}\,\epsilon \ , \\
\delta B_{\mu\nu} &= i e^{-\sigma}\left(\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} + \tfrac{1}{2}\bar{\epsilon}\gamma_{\mu\nu}\chi\right) - A^{r'}_{[\mu}\delta A^{r'}_{\nu]}, \\
\delta\sigma &= -i\bar{\epsilon}\chi \ , \\
\delta A^{r'}_\mu &= i e^{-\sigma/2}\bar{\epsilon}\gamma_\mu \lambda^{r'} \ , \\
\delta\lambda^{r'} &= -\tfrac{1}{4} e^{\sigma/2}\gamma^{\mu\nu}F^{r'}_{\mu\nu}\epsilon - \tfrac{i}{2\sqrt{2}} e^{-\sigma/2}\left(C^{ir'} - \sqrt{2}S^{ir'}\right)\sigma^i \epsilon \ , \\
L^r_I \delta\phi^I &= -i e^{-\varphi}\bar{\epsilon}\psi^r \ , \\
L^i_I \delta\phi^I &= e^{-\varphi}\bar{\epsilon}\sigma^i \psi \ , \\
L^I_i \delta L^r_I &= -\bar{\epsilon}\sigma_i \psi^r \ , \\
\delta\varphi &= i\bar{\epsilon}\psi, \\
\delta\psi &= \tfrac{i}{2}\gamma^\mu\left(\mathcal{P}^i_\mu \sigma_i - i\partial_\mu \varphi\right)\epsilon, \\
\delta\psi^r &= \tfrac{i}{2}\gamma^\mu\left(P^{ir}_\mu \sigma_i + iP^r_\mu\right)\epsilon \ .
\end{aligned} \tag{2.66}
$$

Several definitions are in order. Firstly, the gauged Maurer-Cartan form is associated with the coset $SO(p,3)/SO(p)\times SO(3)$ and it is defined as

$$\begin{aligned}
P_\mu^{ir} &= L^{Ii}\left(\partial_\mu\delta_I^J - f_{r'I}{}^J A_\mu^{r'}\right) L_J^r \ , \\
Q_\mu^{ij} &= L^{Ii}\left(\partial_\mu\delta_I^J - f_{r'I}{}^J A_\mu^{r'}\right) L_J^j \ , \\
Q_\mu^{rs} &= L^{Ir}\left(\partial_\mu\delta_I^K - f_{r'I}{}^J A_\mu^{r'}\right) L_J^s \ .
\end{aligned} \tag{2.67}$$

Various quantities occurring above are defined as follows:

$$\begin{aligned}
&G_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} - \tfrac{3}{2}\left(F_{[\mu\nu}^{r'}A_{\rho]}^{r'} - \tfrac{1}{3}f_{r's'}{}^{t'}A_\mu^{r'}A_\nu^{s'}A_{\rho t'}\right) \ , \\
&F_{\mu\nu}^{r'} = 2\partial_{[\mu}A_{\nu]}^{r'} + f_{s't'}{}^{r'}A_\mu^{s'}A_\nu^{t'} \ , \\
&D_\mu\epsilon = \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \tfrac{1}{2\sqrt{2}}Q_\mu^i\,\sigma^i\right)\epsilon \ , \\
&Q_\mu^i = \tfrac{i}{\sqrt{2}}\,\epsilon^{ijk}Q_{\mu jk} = \epsilon^{ijk}(L^{-1}\partial_\mu L)_{jk} + C^{ir'}A_\mu^{r'} \ .
\end{aligned} \tag{2.68}$$

The axion field strengths are defined as

$$\begin{aligned}
&\mathcal{P}_\mu^i = e^{\varphi}(D_\mu\phi^I)L_I^i \ , \\
&P_\mu^r = e^{\varphi}(D_\mu\phi^I)L_I^r \ , \\
&D_\mu\Phi^I = \partial_\mu\Phi^I + f_{r'J}{}^I\,A_\mu^{r'}\,\Phi^J \ ,
\end{aligned} \tag{2.69}$$

and the gauge functions as

$$\begin{aligned}
&C_{kr'} = \tfrac{1}{\sqrt{2}}\epsilon_{kij}f_{r'I}{}^J\,L^{Ii}L_J^j \ , \qquad C_{irr'} = f_{r'I}{}^J L_i^I L_{Jr} \ , \\
&S_{ir'} = -e^{\varphi}\,f_{r'I}{}^J\;\Phi_J\,L_i^I \ , \qquad S_{rr'} = -e^{\varphi}\,f_{r'I}{}^J\;\Phi_J\,L_r^I \ .
\end{aligned} \tag{2.70}$$

Note that the S^2 term in the potential in (2.63) comes from the $P_7^{ir'}P_7^{ir'}$ term since $P_7^{ir'}\sim S^{ir'}$. The above results cover all the chiral reduction schemes that yield gauged supergravities in $6D$. We simply need to take the appropriate structure constants and the relevant values of p in the $SO(p,3)/SO(p)\times SO(3)$ cosets involved. It is always easiest to evaluate the gauge algebra on bosonic fields since one do not need to Fierz rearrange, while the commutators close on the bosonic fields. These commutators are

$$
\begin{aligned}
[\delta_1,\delta_2]e_\mu^m &= \delta_{G.C.}(\xi^\lambda)+\delta_Q(-\xi^\lambda\psi_\lambda)+\delta_{L.L}(\xi^\lambda\omega_\lambda^{mn}+\tfrac{1}{2}e^\sigma\xi_\lambda G^{\lambda mn}),\\
[\delta_1,\delta_2]B_{\mu\nu} &= \delta_{G.C.}(\xi^\lambda)+\delta_G(-\xi^\lambda B_{\lambda\nu}-\tfrac{1}{2}e^{-\sigma}\xi_\nu)\\
[\delta_1,\delta_2]A_\mu^{r'} &= \delta_{G.C.}(\xi^\lambda)+\delta_G(-\xi^\lambda A_\lambda^{r'}),\\
[\delta_1,\delta_2]\sigma &= \xi^\mu\partial_\mu\sigma,\\
[\delta_1,\delta_2]\varphi &= \xi^\mu\partial_\mu\varphi,\\
[\delta_1,\delta_2]\Phi^I &= \xi^\mu\partial_\mu\Phi^I,
\end{aligned}
\tag{2.71}
$$

where $\xi_\mu = i\bar{\epsilon}_2\gamma^\mu\epsilon_1$. In the case of the models (II) and (IV), with $7D$ gauge groups $SO(3,1)$ and $SO(2,1)$, respectively, we have $p=0$, which means that the coset representative becomes an identity matrix and

$$
\begin{aligned}
f_{rIJ} &\to \epsilon_{r'ij}\ , \qquad C_{ijr'} \to \sqrt{2}\epsilon_{r'ij}\ , \qquad C_{irr'} \to 0\,,\\
S_{ir'} &\to -\epsilon_{r'ij}e^\varphi\Phi^j\ , \quad S_{rr'} \to 0\ .
\end{aligned}
\tag{2.72}
$$

By an untwisting procedure, which will be described in the next section, the scalar fields (Φ^i,φ) can be combined to describe the quaternionic Kahler manifold $SO(4,1)/SO(4)$ that governs the couplings of a single hypermultiplet. In the case of Model (V), we have $p=1$, which means that in the $6D$ model presented above, the relevant sigma model is $SO(3,1)/SO(3)$. This gets enlarged with the help of axionic fields to the quaternionic Kahler coset $SO(4,2)/SO(4)\times SO(2)$ that governs the couplings of one external Maxwell multiplet and two hypermultiplets. In the case of Model (VI), we have $p = 3$, which implies the sigma model is $SO(3,3)/SO(3) \times SO(3)$. This gets enlarged with the help of axionic fields to become the quaternionic Kahler coset $SO(4,4)/SO(4) \times SO(4)$ that govern the couplings of three Maxwell multiplets and three hypermultiplets.

D. The Hidden Quaternionic Kahler Coset Structure

It is well known that the ten dimensional $N = 1$ supergravity theory coupled to N Maxwell multiplets when reduced on a k-dimensional torus down to D dimensions gives rise to half-maximal supergravity coupled to $(N + k)$ vector multiplets with an underlying $SO(N + k, k)/SO(N + k) \times SO(k)$ sigma model sector. This means an $SO(N+3,3)/SO(N+3)\times SO(3)$ sigma model in $7D$. In the notation of the previous sections, we have $N + 3 = n$. A circle reduction of this ungauged theory is then expected to exhibit an $SO(N + 4, 4)/SO(N + 4) \times SO(4)$ coset structure. This is a well known phenomenon which has been described in several papers but primarily in the bosonic sector. In this section, we shall exhibit this phenomenon in the fermionic sector as well, including the supersymmetry transformations. Moreover, we shall describe the hidden symmetry of the gauged $6D$ models obtained from a consistent chiral reduction of the gauged $7D$ models, in which case the $SO(p+3,3)/SO(p+3)\times SO(3)$ coset is enlarged to $SO(p+4,4)/SO(p+4) \times SO(4)$. Here, we have redefined $p \to p+3$ compared to the notation of the previous section, for convenience. The key step in uncovering the hidden symmetry is to first rewrite the Lagrangian in Iwasawa gauge. This gauge is employed by parametrizing the coset $SO(p+3,3)/SO(p+3) \times SO(3) \equiv C(p + 3, 3)$ by means of the $3(p + 3)$ dimensional solvable subalgebra K_s of $SO(p + 3, 3)$. The importance of this gauge lies in the fact that it enables one to absorb the $(p+6)$ axions that come from the $7D$ Maxwell fields, and a single dilaton that comes from the $7D$ metric, into the representative of the coset $C(p+3,3)$ to form the representative of the enlarged coset $C(p+4,4)^*$. To do so, we shall first show, in section 1, how various quantities *formally* combine to give the enlarged coset structure. This will involve identifications such as those in (2.78) below. These identifications by themselves do not furnish a proof of the enlarged coset structure, since one still has to construct explicitly a parametrization of the enlarged coset which produces these identifications. In section 2, we shall provide the proof by exploiting the Iwasawa gauge. In the following we will present the basic idea for the proof of enlarged hidden symmetry. More details will be given later. To begin with, the Maurer-Cartan form $d\mathcal{V}\mathcal{V}^{-1}$ can be decomposed into two parts:

$$P = d\mathcal{V}\mathcal{V}^{-1} + \left(d\mathcal{V}\mathcal{V}^{-1}\right)^T \ , \qquad Q = d\mathcal{V}\mathcal{V}^{-1} - \left(d\mathcal{V}\mathcal{V}^{-1}\right)^T \ . \tag{2.73}$$

where $\mathcal{V}$ is the coset representative given in (2.88). The bosonic sector of the $6D$ Lagrangian that contains the $(4p+4)$ scalar fields, which we call $\mathcal{L}_{G/H}$, will then take the form

$$\begin{aligned} e^{-1}\mathcal{L}_{G/H} &= -\tfrac{1}{4}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{8}\mathrm{tr}\ \left(d\mathcal{V}\mathcal{V}^{-1}\right)\left(d\mathcal{V}\mathcal{V}^{-1} + (d\mathcal{V}\mathcal{V}^{-1})^T\right) \\ &\quad -\tfrac{1}{4}e^{2\varphi}(\mathcal{V}\partial_\mu\Phi)^T(\mathcal{V}\partial^\mu\Phi) \ . \end{aligned} \tag{2.74}$$

where φ is a dilaton and Φ are $p+3$ axions. More explicitly, this can be written as (2.98). The idea is now to combine the dilaton and axionic scalar field strengths with the scalar field strengths for $SO(p+3)/SO(p)\times SO(3)$ to express them all as the scalar field strengths of the enlarged coset $SO(p+4,4)/SO(p+4)\times SO(4)$. The resulting Lagrangian $\mathcal{L}_{G/H}$ then can be rewritten as the $SO(p+4,4)/SO(p+4)\times SO(4)$ sigma

---

*In general, the solvable subalgebra $\widehat{K}_s \subset SO(p+k+1,k+1)$ decomposes into the generators $K_s \subset SO(p+k,k)$, and $(p+2k)$ generators corresponding to axions and a single generator corresponding to a dilaton.

model

$$e^{-1}\mathcal{L}_{G/H} = -\tfrac{1}{8}\text{tr}\ \left(d\widehat{\mathcal{V}}\widehat{\mathcal{V}}^{-1}\right)\left(d\widehat{\mathcal{V}}\widehat{\mathcal{V}}^{-1} + (d\widehat{\mathcal{V}}\widehat{\mathcal{V}}^{-1})^T\right) \tag{2.75}$$

where $\widehat{\mathcal{V}}$ is defined as $\mathcal{V}$ with the 3-valued indices replaced by 4-valued ones, (2.89). This can in turn be written as the second line of (2.102). As to the fermionic sector, for example, when we look at the covariant derivative

$$D_\mu\chi = \left(D_\mu(\omega) + \tfrac{1}{4}Q_\mu^{ij}\ \sigma_{ij} + \tfrac{i}{2}\mathcal{P}_\mu^i\sigma_i\right)\chi \tag{2.76}$$

which follows from the circle reduction of the $7D$ theory, we observe that it contains the composite $SO(3)_R$ connections shifted by the *positive torsion* term $(\frac{i}{2}\mathcal{P}_\mu^i\sigma^i)$ where $\mathcal{P}_\mu^i = \hat{P}_\mu^{i,N+4}$. By introducing the $SO(4)$ Dirac matrices, $\Gamma_{\hat{i}\hat{j}}$, as defined in (2.85) the covariant derivative are then given by

$$D_\mu\chi = \left(\nabla_\nu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \tfrac{1}{4}\widehat{Q}_\mu^{\hat{i}\hat{j}}\ \Gamma_{\hat{i}\hat{j}}\right)\chi, \tag{2.77}$$

which transform under the $SO(4)_R$ symmetry group.

1. Hidden Symmetry in the Symmetric Gauge

The structure of the Lagrangian and transformation rules presented above readily suggest the identifications $\widehat{P}^{ir} = P^{ir}$, $\widehat{Q}^{ij} = Q^{ij}$, $\widehat{Q}^{rs} = Q^{rs}$ and

$$\begin{pmatrix} \widehat{P}^{4r} \\ \widehat{P}^{i,N+4} \\ \widehat{P}^{4,N+4} \end{pmatrix} = \begin{pmatrix} P^r \\ \mathcal{P}^i \\ -\partial\varphi \end{pmatrix}, \qquad \begin{pmatrix} \widehat{Q}^{4i} \\ \widehat{Q}^{N+4,r} \end{pmatrix} = \begin{pmatrix} \mathcal{P}^i \\ P^r \end{pmatrix} \tag{2.78}$$

for the components of the Maurer-Cartan form, $\widehat{C}^{ijr'} = C^{ijr'}$, $\widehat{C}^{irr'} = C^{irr'}$ and the following identifications

$$\begin{pmatrix} \widehat{C}^{4ir'} \\ \widehat{C}^{4rr'} \\ \widehat{C}^{i,N+4,r'} \\ \widehat{C}^{4,N+4,r'} \end{pmatrix} = \begin{pmatrix} S^{ir'} \\ S^{rr'} \\ -S^{ir'} \\ 0 \end{pmatrix} , \tag{2.79}$$

where $\widehat{C}^{ijr'} = \frac{1}{\sqrt{2}} \epsilon^{ijk} \, C^{kr'}$, for the gauge functions. *Note that the hat notation here does not refer to higher dimensions but rather they denote objects which transform under the enlarged symmetry groups.* With these identifications the Lagrangian simplifies dramatically. The bosonic part takes the form

$$\begin{aligned} e^{-1}\mathcal{L}_B &= \tfrac{1}{4}R - \tfrac{1}{4}(\partial_\mu \sigma)^2 - \tfrac{1}{12}e^{2\sigma}G_{\mu\nu\rho}G^{\mu\nu\rho} - \tfrac{1}{8}e^{\sigma}F^{r'}_{\mu\nu}F^{\mu\nu r'} \\ &\quad -\tfrac{1}{4}\widehat{P}^{\hat{i}\hat{r}}_{\mu}\widehat{P}^{\mu}_{\hat{i}\hat{r}} - \tfrac{1}{8}e^{-\sigma}\widehat{C}^{\hat{i}\hat{j}r'}\widehat{C}_{\hat{i}\hat{j}r'} \ , \end{aligned} \tag{2.80}$$

and the fermionic part is given by

$$e^{-1}\mathcal{L}_F = -\tfrac{i}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \tfrac{i}{2}\bar{\chi}\gamma^\mu D_\mu\chi - \tfrac{i}{2}\bar{\lambda}^{r'}\gamma^\mu D_\mu\lambda_{r'} - \tfrac{i}{2}\bar{\psi}^{\hat{r}}\gamma^\mu D_\mu\psi^{\hat{r}} \tag{2.81}$$

$$\begin{aligned} &-\tfrac{i}{2}\bar{\chi}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\sigma - \tfrac{1}{2}\bar{\psi}^{\hat{r}}\gamma^\mu\gamma^\nu\bar{\Gamma}_{\hat{i}}\psi_\mu\widehat{P}^{\hat{i}\hat{r}}_\nu + \tfrac{i}{24}e^{\sigma}G_{\mu\nu\rho}X^{\mu\nu\rho} \\ &-\tfrac{i}{4}e^{\sigma/2}F^{r'}_{\mu\nu}X^{\mu\nu}_{r'} - e^{-\sigma/2}\widehat{C}_{\hat{i}\hat{r}r'}\bar{\lambda}^{r'}\Gamma^{\hat{i}}\psi^{\hat{r}} \\ &-\tfrac{i}{4}e^{-\sigma/2}\widehat{C}^{\hat{i}\hat{j}r'}\left(\bar{\lambda}^{r'}\Gamma_{\hat{i}\hat{j}}\gamma^\mu\psi_\mu + \bar{\lambda}^{r'}\Gamma_{\hat{i}\hat{j}}\chi\right) \ , \end{aligned} \tag{2.82}$$

where $\hat{i} = 1, ..., 4$, $\hat{r} = 1, ..., p+4$, we have defined $\psi^{N+4} = \psi$, and

$$\begin{aligned} X^{\mu\nu\rho} &= \bar{\psi}^\lambda\gamma_{[\lambda}\gamma^{\mu\nu\rho}\gamma_{\tau]}\psi^\tau + \bar{\psi}_\lambda\gamma^{\mu\nu\rho}\gamma^\lambda\chi - \bar{\chi}\gamma^{\mu\nu\rho}\chi + \bar{\lambda}^{r'}\gamma^{\mu\nu\rho}\lambda_{r'} + \bar{\psi}^{\hat{r}}\gamma^{\mu\nu\rho}\psi^{\hat{r}} \ , \\ X^{\mu\nu}_{r'} &= \bar{\psi}_\rho\gamma^{\mu\nu}\gamma^\rho\lambda_{r'} + \bar{\chi}\gamma^{\mu\nu}\lambda_{r'} \ . \end{aligned} \tag{2.83}$$

The covariant derivatives are defined as

$$D_\mu \begin{pmatrix} \psi_\mu \\ \chi \\ \lambda^{r'} \end{pmatrix} = \left(\nabla_\nu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \tfrac{1}{4}\widehat{Q}_\mu^{\hat{i}\hat{j}}\ \Gamma_{\hat{i}\hat{j}}\right) \begin{pmatrix} \psi_\nu \\ \chi \\ \lambda^{r'} \end{pmatrix} \ ,$$

$$D_\mu \psi^{\hat{r}} = \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \tfrac{1}{4}\widehat{Q}_\mu^{\hat{i}\hat{j}}\ \bar{\Gamma}_{\hat{i}\hat{j}}\right)\psi^{\hat{r}} + \widehat{Q}_\mu^{\hat{r}\hat{s}}\ \psi^{\hat{s}} \ . \tag{2.84}$$

The $SO(4)$ Dirac matrices have been introduced in the above formula with the conventions

$$\Gamma_{\hat{i}} = (\sigma_i, -i) \ , \qquad \bar{\Gamma}_{\hat{i}} = (\sigma_i, i) \ , \qquad \Gamma_{\hat{i}\hat{j}} = \Gamma_{[\hat{i}}\bar{\Gamma}_{\hat{j}]} \ , \qquad \bar{\Gamma}_{\hat{i}\hat{j}} = \bar{\Gamma}_{[\hat{i}}\Gamma_{\hat{j}]} \ . \tag{2.85}$$

It is useful to note that $\bar{\psi}^{\hat{r}}\bar{\Gamma}^{\hat{i}}\epsilon = -\bar{\epsilon}\Gamma^{\hat{i}}\psi^{\hat{r}}$. Looking more closely at the covariant derivatives

$$D_\mu \chi = \left(D_\mu(\omega) + \tfrac{1}{4}Q_\mu^{ij}\ \sigma_{ij} + \tfrac{i}{2}\mathcal{P}^i_\mu\sigma_i\right)\chi \ ,$$

$$D_\mu \psi^{\hat{r}} = \left(D_\mu(\omega) + \tfrac{1}{4}Q_\mu^{ij}\ \sigma_{ij} - \tfrac{i}{2}\mathcal{P}^i_\mu\sigma_i\right)\psi^{\hat{r}} + \widehat{Q}_\mu^{\hat{r}\hat{s}}\ \psi^{\hat{s}} \ , \tag{2.86}$$

we observe that they transform covariantly under the composite local $Sp(1)_R$ transformations inherited from $7D$, and that they contain the composite $Sp(1)_R$ connections shifted by the *positive torsion* term $(\frac{i}{2}\mathcal{P}^i\sigma^i)$ in the case of fermions that are doublets under the true $Sp(1)_R$ symmetry group in $6D$, namely $(\psi_\mu, \chi, \lambda^{r'})$, and the *negative torsion* term $(-\frac{i}{2}\mathcal{P}^i\sigma^i)$ in the case of $\psi^{\hat{r}}$, which are singlets under this symmetry. By true $Sp(1)_R$ symmetry group in $6D$ we mean the $SO(3)_R$ symmetry group that emerges upon the recognition of the scalar field couplings as being described by the quaternionic Kahler coset $SO(p+1,4)/SO(p+1)\times SO(4)$ in which $SO(4)\sim SO(3)\times SO(3)_R$. The action of this group is best seen by employing the Iwasawa gauge, as we shall see in the next subsection. The action is invariant under the following $6D$ supersymmetry transformations

$$
\begin{aligned}
\delta e_\mu^m &= i\bar{\epsilon}\gamma^m\psi_\mu, \\
\delta\psi_\mu &= D_\mu\epsilon - \tfrac{1}{24}e^\sigma\gamma^{\rho\sigma\tau}\gamma_\mu G_{\rho\sigma\tau}\,\epsilon\ , \\
\delta\chi &= -\tfrac{1}{2}\gamma^\mu\partial_\mu\sigma\epsilon - \tfrac{1}{12}e^\sigma\gamma^{\rho\sigma\tau}G_{\rho\sigma\tau}\,\epsilon\ , \\
\delta B_{\mu\nu} &= ie^{-\sigma}\left(\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} + \tfrac{1}{2}\bar{\epsilon}\gamma_{\mu\nu}\chi\right) - A^{r'}_{[\mu}\delta A^{r'}_{\nu]}, \\
\delta\sigma &= -i\bar{\epsilon}\chi\ , \\
\delta A_\mu^{r'} &= ie^{-\sigma/2}\bar{\epsilon}\gamma_\mu\lambda^{r'}, \\
\delta\lambda^{r'} &= -\tfrac{1}{4}e^{\sigma/2}\gamma^{\mu\nu}F^{r'}_{\mu\nu}\epsilon + \tfrac{1}{2}e^{-\sigma/2}\widehat{C}^{\hat{i}\hat{j}r'}\Gamma_{\hat{i}\hat{j}}\epsilon\ , \\
\hat{L}_{\hat{i}}{}^{\hat{I}}\delta\hat{L}_{\hat{I}}{}^{\hat{r}} &= -\bar{\epsilon}\Gamma_{\hat{i}}\psi^{\hat{r}}\ , \\
\delta\psi^{\hat{r}} &= \tfrac{i}{2}\gamma^\mu\widehat{P}_\mu^{\hat{i}\hat{r}}\bar{\Gamma}_{\hat{i}}\epsilon\ .
\end{aligned} \tag{2.87}
$$

The relation between the supersymmetric variation of the enlarged coset representative and those involving the $SO(n,3)/SO(n)\times SO(3)$ coset representative, the dilaton and axions is similar to the relations in (3.32) for the corresponding Maurer-Cartan forms, and field strengths, since $L^{-1}dL$ has the same decomposition as $L^{-1}\delta L$. Finally, we note that the above results for the matter coupled gauged $N=(1,0)$ supergravity in $6D$ are in accordance with the results given in [24].

2. Hidden Symmetry in the Iwasawa Gauge

a. The Ungauged Sector

In order to comply with the standard notation for the Iwasawa decomposition of $SO(p,q)$, we switch from our coset representative to its transpose as $L = \mathcal{V}^T$. Following [25, 26], we then parametrize the coset $SO(p+3,3)/SO(p+3)\times SO(3)$ as

$$\mathcal{V} = e^{\frac{1}{2}\vec{\varphi}\cdot\vec{H}} e^{C^i{}_j E_i{}^j}\; e^{\frac{1}{2}A_{ij}V^{ij}}\; e^{B^{i\underline{r}}U_{i\underline{r}}}\;, \qquad i=1,...,3\;, \quad \underline{r}=1,...,p\;, \tag{2.88}$$

where $\left(U_{i\underline{r}}, V^{ij}, E_i{}^j, \vec{H}\right)$ with $i<j$ and $V^{ij}=-V^{ji}$, are the generators of the $3(p+3)$ dimensional solvable subalgebra of $SO(p+3,3)$ multiplying the corresponding scalar fields and $\vec{\varphi}\cdot\vec{H}$ stands for $\varphi^i H_i$. Using the commutation rules of the generators given in Appendix D, one finds [25]

$$\mathcal{V} = \left(\begin{array}{c|c|c} e^{\frac{1}{2}\vec{c}_i\cdot\vec{\varphi}}\,\gamma^j{}_i & e^{\frac{1}{2}\vec{c}_i\cdot\vec{\varphi}}\,\gamma^k{}_i\,B^{\underline{s}}{}_k & e^{\frac{1}{2}\vec{c}_i\cdot\vec{\varphi}}\,\gamma^k{}_i\,\left(A_{kj}+\frac{1}{2}B^{\underline{q}}{}_k\,B^{\underline{q}}{}_j\right) \\ \hline 0 & \delta_{\underline{rs}} & B^{\underline{r}}{}_j \\ \hline 0 & 0 & e^{-\frac{1}{2}\vec{c}_i\cdot\vec{\varphi}}\,\widetilde{\gamma}^i{}_j \end{array}\right)\;, \tag{2.89}$$

where $\vec{c}_i$ is defined in (D.5), and

$$\widetilde{\gamma}^i{}_j = \delta^i_j + C^i{}_j\;, \qquad \gamma^i{}_k\,\widetilde{\gamma}^k{}_j\; = \delta^i_j\;. \tag{2.90}$$

The inverse of $\mathcal{V}$ can be computed from the defining relation $\mathcal{V}^T\Omega\mathcal{V}=\Omega$ and is given by:

$$\mathcal{V}^{-1} = \left(\begin{array}{c|c|c} e^{-\frac{1}{2}\vec{c}_j\cdot\vec{\varphi}}\,\widetilde{\gamma}^j{}_i & -B^{\underline{s}}{}_i & e^{\frac{1}{2}\vec{c}_j\cdot\vec{\varphi}}\,\gamma^k{}_j\,\left(A_{ki}+\frac{1}{2}B^{\underline{q}}{}_k\,B^{\underline{q}}{}_i\right) \\ \hline 0 & \delta_{\underline{rs}} & -e^{\frac{1}{2}\vec{c}_j\cdot\vec{\varphi}}\,\gamma^k{}_j\,B^{\underline{r}}{}_k \\ \hline 0 & 0 & e^{\frac{1}{2}\vec{c}_j\cdot\vec{\varphi}}\,\gamma^i{}_j \end{array}\right)\;. \tag{2.91}$$

In equations (2.89) and (2.91) the indices $(i,\underline{r})$ label the rows and $(j,\underline{s})$ label the column. The Iwasawa gauge means setting the scalars corresponding to the maximal compact subalgebra equal to zero. Under the action of the global G transformations from the right, the coset representative will not remain in the Iwasawa gauge but can be brought back to that form by a compensating h transformation from the left, namely, $\mathcal{V}g = h\mathcal{V}'$. The Maurer-Cartan form $d\mathcal{V}\mathcal{V}^{-1}$ can be decomposed into two parts one of which transforms homogeneously under h and the other one transforms as an h-valued gauge field:

$$\begin{aligned} P &= d\mathcal{V}\mathcal{V}^{-1} + \left(d\mathcal{V}\mathcal{V}^{-1}\right)^T \ : \qquad P \ \rightarrow \ hPh^{-1} \ , \\ Q &= d\mathcal{V}\mathcal{V}^{-1} - \left(d\mathcal{V}\mathcal{V}^{-1}\right)^T \ : \qquad Q \ \rightarrow \ dhh^{-1} + hQh^{-1} \ . \end{aligned} \tag{2.92}$$

Both of these are, of course, manifestly invariant under the global g transformations. The key building block in writing down the action in Iwasawa gauge is the Maurer-Cartan form [25]

$$\partial_\mu \mathcal{V}\mathcal{V}^{-1} = \tfrac{1}{2}\partial_\mu \vec{\varphi} \cdot \vec{H} + \sum_{i<j} \left(e^{\frac{1}{2}\vec{a}_{ij}\cdot\vec{\varphi}} F_{\mu ij} V^{ij} + e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\varphi}} \mathcal{F}^i_{\mu j} E_i{}^j \right) + \sum_{i,\underline{r}} F_{\mu i\underline{r}} U^{i\underline{r}} \ , \tag{2.93}$$

where $\vec{a}_{ij}$ and $\vec{b}_{ij}$ are defined in (D.5) and

$$\begin{aligned} F_{\mu ij} &= \gamma^k{}_i \gamma^\ell{}_j \left(\partial_\mu A_{k\ell} - B^{\underline{q}}{}_k \partial_\mu B^{\underline{q}}{}_\ell \right) \ , \\ F_{\mu i\underline{r}} &= \gamma^j{}_i \partial_\mu B_{j\underline{r}} \ , \\ \mathcal{F}^i_{\mu j} &= \gamma^k{}_j \partial_\mu C^i{}_k \ , \end{aligned} \tag{2.94}$$

and it is understood that $i < j$. Other building blocks for the action are the field strengths for the axions defined as

$$\mathcal{V}\partial_\mu \Phi = \begin{pmatrix} F_{\mu i} \\ F^{\underline{r}}_\mu \\ \mathcal{F}^i_\mu \end{pmatrix} \ , \qquad \Phi = \begin{pmatrix} A_i \\ B^{\underline{r}} \\ C^i \end{pmatrix} \ , \tag{2.95}$$

where A_i and $B^{\underline{r}}$ form a $(p{+}3)$ dimensional representation of $SO(p{+}3) \subset SO(p{+}3, 3)$, and

$$F_{\mu i} = \gamma^j{}_i \left(\partial_\mu A_j + B^{\underline{r}}{}_j \partial_\mu B^{\underline{r}} + A_{jk} \partial_\mu C^k + \tfrac{1}{2} B^{\underline{r}}{}_j B^{\underline{r}}{}_k \partial_\mu C^k \right) \ ,$$

$$F_\mu^{\underline{r}} = \partial_\mu B^{\underline{r}} + B^{\underline{r}}{}_i \partial_\mu C^i \ ,$$

$$\mathcal{F}_\mu^i = \widetilde{\gamma}^i{}_j \ \partial_\mu C^j \ . \qquad (2.96)$$

With these definitions, the bosonic part of our $6D$ Lagrangian that contains the $(4p+4)$ scalar fields, which we shall call $\mathcal{L}_{G/H}$, takes the form

$$e^{-1}\mathcal{L}_{G/H} = -\tfrac{1}{4}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{8}\mathrm{tr}\ \left(d\mathcal{V}\mathcal{V}^{-1}\right)\left(d\mathcal{V}\mathcal{V}^{-1} + (d\mathcal{V}\mathcal{V}^{-1})^T\right)$$
$$-\tfrac{1}{4}e^{2\varphi}(\mathcal{V}\partial_\mu\Phi)^T(\mathcal{V}\partial^\mu\Phi) \ . \qquad (2.97)$$

More explicitly, this can be written as [25]

$$e^{-1}\mathcal{L}_{G/H} = -\tfrac{1}{4}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{8}\sum_{i<j}\left(e^{\vec{a}_{ij}\cdot\vec{\varphi}}F_{\mu ij}F^{\mu ij} + e^{\vec{b}_{ij}\cdot\vec{\varphi}}\mathcal{F}_\mu{}^i{}_j\mathcal{F}^\mu{}_i{}^j\right) - \sum_{i,r}\tfrac{1}{8}e^{\vec{c}_i\cdot\vec{\varphi}}F_{\mu i\underline{r}}F^{\mu i\underline{r}}$$
$$-\tfrac{1}{4}e^{2\varphi}\left(e^{\vec{c}_i\cdot\vec{\varphi}}F_{\mu i}F^{\mu i} + e^{-\vec{c}_i\cdot\vec{\varphi}}\mathcal{F}_\mu^i\mathcal{F}_i^\mu + F_\mu^{\underline{r}}F_{\underline{r}}^\mu\right) \ . \qquad (2.98)$$

The idea is now to combine the dilaton and axionic scalar field strengths (2.96) with the scalar field strengths for $SO(p+3)/SO(p)\times SO(3)$ defined in (2.94) to express them all as the scalar field strengths of the enlarged coset $SO(p+4,4)/SO(p+4)\times SO(4)$. As is well known, this is indeed possible and to this end we need to make the identifications

$$F_{\mu i} = \tfrac{1}{\sqrt{2}}\,F_{\mu i4} \ ,$$

$$F_\mu^{\underline{r}} = \tfrac{1}{\sqrt{2}}\,F_{\mu 4}^{\underline{r}} \ ,$$

$$\mathcal{F}_\mu^i = \tfrac{1}{\sqrt{2}}\,\mathcal{F}_{\mu 4}^i \ . \qquad (2.99)$$

The quantities on the right hand side are restrictions of the Maurer-Cartan form based on the enlarged coset $SO(p+4,4)/SO(p+4)\times SO(3)$ defined as

$$\partial_\mu \widehat{\mathcal{V}}\widehat{\mathcal{V}}^{-1} = \tfrac{1}{2}\partial_\mu\varphi^\alpha H_\alpha + \sum_{\alpha<\beta}\left(e^{\frac{1}{2}\vec{a}_{\alpha\beta}\cdot\vec{\varphi}}F_{\mu\alpha\beta}V^{\alpha\beta} + e^{\frac{1}{2}\vec{b}_{\alpha\beta}\cdot\vec{\varphi}}\mathcal{F}^{\alpha}_{\mu\,\beta}E_\alpha{}^\beta\right) + \sum_{\alpha,r} e^{\frac{1}{2}\vec{c}_\alpha\cdot\vec{\varphi}}F_{\mu\alpha\underline{r}}U^{\alpha\underline{r}} \;, \tag{2.100}$$

where $\widehat{\mathcal{V}}$ is defined as in (2.89) and $(F_{\mu\alpha\beta}, \mathcal{F}^{\alpha}_{\mu\,\beta}, F_{\mu\alpha\underline{r}})$ as in (2.94), and $\vec{a}_{\alpha\beta}, \vec{b}_{\alpha\beta}, \vec{c}_\alpha$ as in (D.5), with the *3-valued indices replaced by the 4-valued indices everywhere.* Equations (3.34) have a solution given by

$$\begin{aligned}
A_i &= \tfrac{1}{\sqrt{2}}\left(A_{i4} - A_{ij}\gamma^j{}_k C^k{}_4 - \tfrac{1}{2}B^{\underline{r}}{}_i B^{\underline{r}}{}_4 + \tfrac{1}{2}B^{\underline{r}}{}_i B^{\underline{r}}{}_j \gamma^j{}_k C^k{}_4\right)\;,\\
B^{\underline{r}} &= \tfrac{1}{\sqrt{2}}\left(B^{\underline{r}}{}_4 - B^{\underline{r}}{}_i\gamma^i{}_j C^j{}_4\right)\;,\\
C^i &= \tfrac{1}{\sqrt{2}}\gamma^i{}_j C^j{}_4\;,\\
\varphi &= \tfrac{1}{\sqrt{2}}\,\varphi_4\;.
\end{aligned} \tag{2.101}$$

The identifications (3.34) (where $r \to \underline{r}, i$), together with (2.69), (2.95), (2.100), (2.93), (2.67) (with $A_\mu = 0$), (2.92) and (D.5) (defined for 3-valued and 4-valued indices similarly), provide the proof of (3.32) used to show the hidden symmetry. Using (3.34), the Lagrangian $\mathcal{L}_{G/H}$ can be written as the $SO(p+4,4)/SO(p+4)\times SO(4)$ sigma model:

$$\begin{aligned}
e^{-1}\mathcal{L}_{G/H} &= -\tfrac{1}{8}\mathrm{tr}\,\left(d\widehat{\mathcal{V}}\widehat{\mathcal{V}}^{-1}\right)\wedge\left(d\widehat{\mathcal{V}}\widehat{\mathcal{V}}^{-1} + (d\widehat{\mathcal{V}}\widehat{\mathcal{V}}^{-1})^T\right)\\
&\leftrightarrow -\tfrac{1}{8}\partial_\mu\varphi^\alpha\partial^\mu\varphi_\alpha - \tfrac{1}{8}\sum_{\alpha<\beta}\left(e^{\vec{a}_{\alpha\beta}\cdot\vec{\varphi}}F^{\alpha\beta}_{\mu}F^{\mu}_{\alpha\beta} + e^{\vec{b}_{\alpha\beta}\cdot\vec{\varphi}}\mathcal{F}_{\mu_\alpha}{}^{\beta}\mathcal{F}^{\mu\alpha}{}_{\beta}\right)\\
&\quad -\tfrac{1}{8}\sum_{\alpha,r} e^{\vec{c}_\alpha\cdot\vec{\varphi}}F_{\mu\alpha\underline{r}}F^{\mu\alpha\underline{r}}\;,
\end{aligned} \tag{2.102}$$

where $a_{\alpha\beta}, b_{\alpha\beta}, c_\alpha$ are defined as in (D.5) with the indices $i, j = 1, 2, 3$ replaced by $\alpha, \beta = 1, ..., 4$.

b. Gauging and the C-functions

In order to justify the identifications (2.79) of the S-functions as certain components of the C-functions associated with the enlarged coset space, we need to study these functions in the Iwasawa gauge. Of the four supergravity models in $6D$ that have nonvanishing gauge functions, two of them, namely models (II) and (IV), see section 3.2, are coupled to one special linear multiplet and as such they deserve separate treatment. In both of these cases, the gauge functions obey the relations (2.72). We begin by showing how these relations follow from the C-functions associated with the $U(1)_R$ or $Sp(1)_R$ gauged $SO(4,1)/SO(4)$ sigma model. The coset representative for $SO(4,1)/SO(4)$ in the Iwasawa gauge takes the form

$$\mathcal{V} = \left(\begin{array}{c|c|c} e^{\varphi} & e^{\varphi}\Phi^i & \frac{1}{2}e^{\varphi}\Phi^2 \\ \hline 0 & \delta_{ij} & \Phi^i \\ \hline 0 & 0 & e^{-\varphi} \end{array}\right) , \tag{2.103}$$

where $\Phi^2 = \Phi^i\Phi_i$. Note that we have made the identification $B_{1r} \rightarrow \Phi_i$ already. Given that the $Sp(1)_R$ or $U(1)_R$ generators $T^{r'}$ are of the form

$$T^{r'} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & T^{r'} & 0 \\ \hline 0 & 0 & 0 \end{array}\right) , \tag{2.104}$$

we find that the C function based on the enlarged coset is given by

$$C^{r'} = \mathcal{V}T^{r'}\mathcal{V}^{-1} = \left(\begin{array}{c|c|c} 0 & -T^{r'}_{ij}\Phi^j & 0 \\ \hline 0 & T^{r'}_{ij} & -T^{r'}_{ij}\Phi^j \\ \hline 0 & 0 & 0 \end{array}\right) . \tag{2.105}$$

Comparing with the relations given in (2.72), and recalling that $T^{r'}_{ij} \sim \epsilon_{r'ij}$, we see that indeed the projection of the C function based on the enlarged coset as defined above does produce the C and S functions obtained from the chiral reduction, as was assumed in the previous section in (2.79). In the notation of Appendix D, the C function is obtained from projection by T_{ij}, and the S-function from projection by U_i. Next, we consider the remaining two supergravities with nontrivial gauge function, namely models (V) and (VI), see section 3.2, with hidden symmetry chains shown in (2.57) and (2.60). Prior to uncovering the hidden symmetry, the C and S functions occurring in the Lagrangian (2.63) and in the supersymmetry transformations (2.66) are $C^{ijr'}, C^{irr'}, S^{ir'}$ and $S^{rr'}$. Using (2.70), and the fact that $f_{r'I}{}^J \sim (T_{r'})_I{}^J$, we deduce the definitions

$$\begin{aligned} \vec{C}^{r'}(H) &= \operatorname{tr}\left(\mathcal{V}T^{r'}\mathcal{V}^{-1}\right)\vec{H} \;, \\ C_i{}^{jr'}(E) &= \operatorname{tr}\left(\mathcal{V}T^{r'}\mathcal{V}^{-1}\right)E_i^{\;j} \;, \\ C_{ij}{}^{r'}(V) &= \operatorname{tr}\left(\mathcal{V}T^{r'}\mathcal{V}^{-1}\right)V^{ij} \;, \\ C_{i\underline{r}}{}^{r'}(U) &= \operatorname{tr}\left(\mathcal{V}T^{r'}\mathcal{V}^{-1}\right)U_{i\underline{r}} \;, \end{aligned} \tag{2.106}$$

for the C functions in the coset direction,

$$C^{ijr'}(X) = \operatorname{tr}\left(\mathcal{V}T^{r'}\mathcal{V}^{-1}\right)X^{ij} \;, \tag{2.107}$$

for the C function in the $SO(3)$ direction, and

$$S^{r'} = \begin{pmatrix} \mathcal{S}_i^{r'} \\ S^{rr'} \\ S^{ir'} \end{pmatrix} = \sqrt{2}\, e^{\varphi}\, \mathcal{V}T^{r'}\Phi \;, \tag{2.108}$$

for the S functions. For $SO(p,q)/SO(p)\times SO(q)$ with $p\geq q$, we have $i=1,...,q$ and $\underline{r}=1,...,p-q$. To show that the C and S functions defined above combine to give the $\hat{C}$ functions for the enlarged coset, we begin with the observation that the gauge group lies in the H^i and $T_{\underline{rs}}$ directions. Thus we can denote the full gauge symmetry generator that acts on the enlarged coset representative $\mathcal{V}$ as

$$\mathcal{T}^{r'} = \left(\begin{array}{c|c|c} H^{r'} & 0 & 0 \\ \hline 0 & T^{r'} & 0 \\ \hline 0 & 0 & H^{r'} \end{array}\right) , \tag{2.109}$$

where $H^{r'}_{\alpha\beta}$ $(\alpha,\beta=1,...,q{+}1)$ is symmetric and $T^{r'}_{rs}(r,s=1,...,p{-}q)$ is antisymmetric. Defining the C functions for the coset $SO(p+1,q+1)/SO(p+1)\times SO(q+1)$ as in (2.106) with the index $i=1,...,q$ replaced by $\alpha=1,...,q+1$ and $\mathcal{T}^{r'}$ defined in (2.109), we find

$$\begin{aligned}
\vec{C}^{r'}(H) &= \gamma^{\gamma}{}_{\alpha}\, H^{r'\delta}_{\gamma}\, \tilde{\gamma}^{\alpha}{}_{\delta}\, \vec{c}_{\alpha} \ , \\
C^{\beta}{}_{\alpha}{}^{r'}(E) &= e^{\frac{1}{2}(\vec{c}_{\alpha}-\vec{c}_{\beta})\cdot\vec{\varphi}}\, \gamma^{\gamma}{}_{\alpha}\, H^{r'\delta}_{\gamma}\, \tilde{\gamma}^{\beta}{}_{\delta} \ , \\
\mathcal{C}_{\alpha\beta r'}(V) &= e^{\frac{1}{2}(\vec{c}_{\alpha}-\vec{c}_{\beta})\cdot\vec{\varphi}}\, \gamma^{\gamma}{}_{[\alpha}\, \gamma^{\delta}{}_{\beta]} \left(H^{r'\eta}_{\gamma}(A_{\delta\eta} + \tfrac{1}{2} B^{\underline{r}}_{\delta} B^{\underline{r}}_{\eta}) - T^{r'}_{\underline{rs}} B^{\underline{r}}_{\gamma} B^{\underline{s}}_{\delta} \right) \ , \\
C_{\alpha\underline{r}}{}^{r'}(U) &= -e^{\frac{1}{2}\vec{c}_{\alpha}\cdot\vec{\varphi}}\, \gamma^{\beta}{}_{\alpha} \left(H^{r'\gamma}_{\beta}\, B_{\gamma\underline{r}} + T^{r'}_{\underline{rs}} B_{\beta}{}^{\underline{s}} \right) \ .
\end{aligned} \tag{2.110}$$

Using (2.101), the above quantities reduce to those for the $SO(p+3,3)/SO(p+3)\times SO(3)$ coset upon restriction of the 4-valued α,β indices to 3-valued (i,j) indices, $C^{4r'}(H)=0$ and

$$\begin{aligned}
C^{r'}_{4i}(E) &= \sqrt{2} e^{\varphi} \mathcal{S}^{r'}{}_{i} \ , \\
C^{r'}_{4i}(V) &= \sqrt{2} e^{\varphi} S^{r'}{}_{i} \ ,
\end{aligned}$$

$$C^{r'}_{4r}(U) \;=\; -\sqrt{2}e^{\varphi}S^{r'}{}_{r}\ . \tag{2.111}$$

These identifications, upon comparing the definitions (2.106), (2.107) and (2.108) with (2.70), provide the proof of the relations (2.79) used in showing the hidden symmetry. In doing so, note that $C^{irr'} \rightarrow (C^{i\underline{r}r'}, C^{i,jr'})$ and $S^{rr'} \rightarrow (S^{\underline{r}r'}, \mathcal{S}^{ir'})$ and that $C^{i,jr'}$ has components in the $(\vec{H}, E, V, U, X)$ directions. Note also that, having proven the relations for the Maurer-Cartan form, (3.32), and the C-functions, (2.79), it follows that not only the *bosonic* part of the $6D$ Lagrangian, (2.80), but also its part that contains the *fermionic* part, namely (2.82), exhibits correctly the enlarged coset structure. Moreover, the relation between the supersymmetric variation of the enlarged coset representative and those involving the $SO(n,3)/SO(n)\times SO(3)$ coset representative, the dilaton and axions is similar to the relations in (3.32) for the corresponding Maurer-Cartan forms, and field strengths, since $L^{-1}dL$ has the same decomposition as $L^{-1}\delta L$. This concludes the demonstration of the enlarged coset structure in the $6D$ models with nontrivial gauge functions. Although the bosonic field contents and field equations of supergravity theory has been formulated by solvable Lie algebra [27, 28, 29], the feature of enlarged hidden symmetry using the Iwasawa gauge has not been shown as far as we know. Here we have shown in the full model, including the fermionic sector, how the scalar fields can be combined to parametrize an enlarged coset $SO(p+1,4)/SO(p+1)SO(4)$.

E. Comments

We have reduced the half-maximal $7D$ supergravity with specific noncompact gaugings coupled to a suitable number of vector multiplets on a circle to $6D$ and chirally truncated it to $N=(1,0)$ supergravity such that a R-symmetry gauging survives. These are referred to as the $SO(3,1), SO(3,2)$ and $SO(2,2)$ models, and their field content and gauge symmetries are summarized in Chapter I. These models, in particular, feature couplings to p hypermultiplets whose scalar fields parametrize the coset $SO(p,3)/SO(p) \times SO(3)$, a dilaton and $(p+3)$ axions, for $p \leq 1$. The value of p is restricted in the case of chiral circle reductions that maintain R-symmetry gauging, but it is arbitrary otherwise. We have exhibited in the full model, including the fermionic contributions, how these fields can be combined to parametrize an enlarged coset $SO(p+1,4)/SO(p+1) \times SO(4)$ whose abelian isometries correspond to the $(p+3)$ axions. Our results for the R-symmetry gauged reduction of certain noncompact gauged $7D$ supergravities are likely to play an important role in finding the string/M-theory origin of the *gauged and anomaly-free* $N = (1,0)$ supergravities in $6D$ which has been a notoriously challenging problem so far. This is due to the fact that at least two of the $7D$ models we have encountered, namely the $SO(3,1)$ and $SO(2,2)$ gauged $7D$ models, are known to have a string/M-theory origin. Therefore, what remains to be understood is the introduction of the matter couplings in $6D$ that are needed for anomaly freedom. A natural approach for achieving this is to associate our chiral reduction with boundary conditions to be imposed on the fields of the $7D$ model formulated on a manifold with boundary [30].

CHAPTER III

SUPERSYMMETRIC SOLUTIONS OF GAUGED 6D SUPERGRAVITY*

The issue to be addressed in this chapter is to find the general supersymmetric solutions in general matter coupled 6D, $N = (1, 0)$ supergravity, including hypermultiplets, which has been studied in [9, 10], in the absence of hypermultiplets. A useful strategy is to use G-structure method to classify supersymmetric solutions in supergravity theories. To do so, we use at least one nonvanishing Killing spinor to begin with. The Killing spinor is defined to be the spinor of the supersymmetry transformations which satisfies the vanishing of the supersymmetric variations of all the spinors. The advantage of seeking such Killing spinors is that they lead to first order equations, which are much easier than the second order field equations, and their integrability conditions imply most of the field equations and Bianchi identities satisfied automatically. In the following, we will present the basic idea of the G-structure method. More details will be given later. We first construct nonvanishing fermionic bilinears from *commuting* symplectic-Majorana Killing spinor ϵ^A

$$\bar{\epsilon}^A \Gamma_\mu \epsilon^B \equiv V_\mu^{AB}, \tag{3.1}$$

$$\bar{\epsilon}^A \Gamma_{\mu\nu\rho} \epsilon^B \equiv X^r_{\mu\nu\rho} T_r^{AB}, \tag{3.2}$$

where T^r are $SU(2)$ generators. These bilinears prove to be convenient in analyzing the necessary and sufficient conditions for the existence of Killing spinors. From the Fierz identity $\Gamma_{\mu(\alpha\beta}\Gamma^\mu_{\ \gamma)\delta} = 0$, it follows that $V^\mu V_\mu = 0$, so V_μ is null. Multiplying $\delta\psi_\mu = 0$, (3.34), which defines Killing spinors, with $\bar{\epsilon}\Gamma_\nu$ and using (3.1), we find

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$\nabla_{(\mu}V_{\nu)} = 0$. Thus, V_μ is a null Killing vector admitted by the metric which is part of a supersymmetric solution. This then helps in parametrizing the metric as

$$ds^2 = 2H^{-1}(du+\beta)\left(dv+\omega+\frac{\mathcal{F}}{2}(du+\beta)\right)+Hds_B^2 \ , \tag{3.3}$$

where ds_B^2 is the metric on the base space $\mathcal{B}$, β and ω are 1-forms on $\mathcal{B}$, and H is a harmonic function. Examining the other vanishing of the supersymmetry variations of the other fermions, as we will show in section B, they lead to the 2-form field strength*

$$F^I = -e^{-\frac{1}{2}\varphi}\, C^{Ir} I^r + \widetilde{F}^I + V\wedge\omega^I \ , \tag{3.4}$$

where $\widetilde{F}^I$ is self-dual, and the anti-selfdual part of three-form field strength

$$e^{\frac{1}{2}\varphi}\, G^- = \tfrac{1}{2}(1-\star)\left[V\wedge e^-\wedge d\varphi + V\wedge K\right] \ , \tag{3.5}$$

where $\star$ is the Hodge-dual, K is self-dual. Moreover, the differential conditions for hyperscalars and dilaton are given by (3.58) and (3.59) respectively. We next show that with the integrability conditions, (3.66)-(3.69), for the existence of killing spinors, most of the equations of motion are satisfied automatically, except the following

$$R_{++} = J_{++} \ , \qquad D_\mu(e^{\frac{1}{2}\varphi}F^{I\mu}{}_+) = J_+^I \ , \qquad D_\mu(e^{\varphi}G^{\mu\nu\rho}) = 0 \ , \tag{3.6}$$

as well as the Bianchi identities

$$DF^I = 0, \qquad dG = \tfrac{1}{2}F^I\wedge F^I. \tag{3.7}$$

We then apply above framework to find new exact solution which is particularly interesting because the scalars of a hypermultiplet are nonvanishing. Indeed, we

*The Hodge-dual of a p-form, $F = \frac{1}{p!}\, dx^{\mu_1}\wedge\cdots dx^{\mu_p}F_{\mu_1\ldots\mu_p}$, is calculated using, $*(dx^{\mu_1}\wedge\cdots dx^{\mu_p}) = \frac{1}{(D-p)!}\,\epsilon_{\nu_1\ldots\nu_{D-p}}{}^{\mu_1\ldots\mu_p}\, dx^{\nu_1}\cdots dx^{\nu_{D-p}}$.

activate only four hyperscalars, and set all the rest equal to zero. The model effectively reduces to one in which the hyperscalars are described by 4-hyperboloid $H_4 = SO(4,1)/SO(4)$. The most important key step in constructing the solution is the identity map between the 4-dimensional scalar coset quaternionic structures and the transverse internal space quaternionic structures,

$$\tilde{J}^r = J^r. \tag{3.8}$$

The idea behind this identity map is the same as the 2-dimensional analog of the Gell-Mann-Zwiebach tear-drop solution [31]. In $N = 2, D = 10$ supergravity theory, the relevant field equations are

$$R_{\hat{\mu}\hat{\nu}}(g) = \frac{1}{2} h_{\alpha\beta}\partial_{\hat{\mu}}\phi^\alpha \partial_{\hat{\nu}}\phi^\beta, \qquad \hat{\mu} = 0, 1, ..., 9 \tag{3.9}$$

$$\nabla_{\hat{\mu}}(g)\partial^{\hat{\mu}}\phi^\alpha = 0, \qquad \alpha = 1, 2 \tag{3.10}$$

where $x^{\hat{\mu}} = (x^\mu, y^i)$, $i = 1, 2$, and $h_{\alpha\beta}(\phi)$ is the scalar manifold metric which can be put into the conformally flat form

$$h_{\alpha\beta}(\phi) = \frac{4\delta_{\alpha\beta}}{[1 - (\phi_1)^2 - (\phi_2)^2]^2}. \tag{3.11}$$

The Gell-Mann-Zwiebach solution uses the identity map,

$$\phi^\alpha = y^i \delta_i^\alpha. \tag{3.12}$$

Since $h_{\alpha\beta}$ is a metric of constant negative curvature and the background metric g_{ij} is positively curved, instead of equating the two metrics h and g, Gell-Mann and Zwiebach proceed to set

$$g_{ij} = \Omega^2 \delta_{ij}, \tag{3.13}$$

where Ω^2 is a conformal factor. They find that (3.9) and (3.10) are satisfied provided that $\Omega^2 = 4a^2(1 - y_1^2 - y_2^2)$ where a is not determined by equations of motion. Thus, the total metric is

$$ds^2 = dx^\mu dx^\nu \eta_{\mu\nu} + 4a^2(1 - y_1^2 - y_2^2) dy^i dy^j \delta_{ij}. \tag{3.14}$$

The internal space which has an axial symmetry is known as the tear-drop space. In our case, dyonic string solution in $6D$, the tear-drop is four-dimensional, and the full solution is

$$\begin{aligned} ds^2 &= e^{-\frac{1}{2}\varphi_+} e^{-\frac{1}{2}\varphi_-}(-dt^2 + dx^2) + e^{\frac{1}{2}\varphi_+} e^{\frac{1}{2}\varphi_-} \left(\frac{b}{y^2}\right)^2 h^{2/3}\, dy^\alpha dy^\beta\, \delta_{\alpha\beta} \;, && (3.15) \\ e^\varphi &= e^{\varphi_+}/e^{\varphi_-} \;, \qquad \phi^\alpha = \frac{a y^\alpha}{y^2} \;, && (3.16) \\ A_\alpha^r &= \frac{4}{3y^2}\, \rho^r_{\alpha\beta} y^\beta \;, && (3.17) \\ G_{\alpha\beta\gamma} &= \frac{8}{27(y^2)^2}\, \epsilon_{\alpha\beta\gamma\delta}\, y^\delta \;, \qquad G_{+-\alpha} = -\partial_\alpha e^{-\varphi_+} \;, && (3.18) \end{aligned}$$

where a and b are integration constants, $\varphi_\pm$ and h are given in (3.117) and (3.118) respectively. Moreover, while in Gell-Mann-Zwiebach solution only the metric and 2-scalars are activated, we emphasize that in our solution we activate dilaton, 4 hyperscalars, two-form and three-form field strengths as well. An interesting property of our dyonic string solution is that while its electric charge is arbitrary, its magnetic charge is fixed in Planckian units, and hence it is necessarily non-vanishing. We also determine the source term needed to balance a delta function type singularity at the origin. Moreover, the solution is shown to have 1/4 supersymmetric $AdS_3 \times S^3$ near horizon limit where the radii are proportional to the electric charge. Now, we turn to a detailed description of our results outlined above.

A. The Model

1. Field Content and the Quaternionic Kahler Scalar Manifold

The six-dimensional gauged supergravity model we shall study involves the combined $N=(1,0)$ supergravity plus anti-selfdual supermultiplet $(g_{\mu\nu}, B_{\mu\nu}, \varphi, \psi^A_{\mu+}, \chi^A_-)$, Yang-Mills multiplet (A_μ, λ^A_+) and hypermultiplet (ϕ^α, ψ^a_-). All the spinors are symplectic Majorana-Weyl, $A=1,2$ label the doublet of the R symmetry group $Sp(1)_R$ and $a=1,...,2n_H$ labels the fundamental representation of $Sp(n_H)$. The chiralities of the fermions are denoted by $\pm$*. The hyperscalars ϕ^α, $\alpha=1,...,4n_H$ parameterize the coset $Sp(n_H,1)/Sp(n_H)\otimes Sp(1)_R$. This choice is due to its notational simplicity. Our formulae can straightforwardly be adapted to more general quaternionic coset spaces G/H whose list can be found, for example in [32]. In this paper, we gauge the group

$$K\times Sp(1)_R\subset Sp(n_H,1)\ ,\qquad K\subseteq Sp(n_H)\ . \tag{3.19}$$

The group K is taken to be semi-simple, and the $Sp(1)_R$ part of the gauge group can easily be replaced by its $U(1)_R$ subgroup. We proceed by defining the basic building blocks of the model constructed in [33] in an alternative notation. The vielbein V_α^{aA}, the $Sp(n_H)$ composite connection Q_α^{ab} and the $Sp(1)_R$ composite connection Q_α^{AB} on the coset are defined via the Maurer-Cartan form as

$$L^{-1}\partial_\alpha L = V_\alpha^{aA}T_{aA}+\tfrac{1}{2}\,Q_\alpha^{ab}T_{ab}+\tfrac{1}{2}\,Q_\alpha^{AB}T_{AB}\ , \tag{3.20}$$

*We use the spacetime signature $(-+++++)$ and set $\epsilon^{+-ijkl}=\epsilon^{ijkl}$. We define $\Gamma_7=\Gamma^{012345}$. The supersymmetry parameter has the positive chirality: $\Gamma_7\,\epsilon=\epsilon$. Thus, $\Gamma_{\mu\nu\rho}=\frac{1}{6}\,\epsilon_{\mu\nu\rho\sigma\lambda\tau}\,\Gamma^{\sigma\lambda\tau}\,\Gamma_7$, and for a self-dual 3-form we have $S_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\epsilon=0$.

where L is the coset representative, $(T_{ab}, T_{AB}, iT_{aA}) \equiv T_{\widehat{A}\widehat{B}}$ obey the $Sp(n_H, 1)$ algebra

$$
\begin{aligned}
&[T_{\widehat{A}\widehat{B}}, T_{\widehat{C}\widehat{D}}] = -\Omega_{\widehat{B}\widehat{C}} T_{\widehat{A}\widehat{D}} - \Omega_{\widehat{A}\widehat{C}} T_{\widehat{B}\widehat{D}} - \Omega_{\widehat{B}\widehat{D}} T_{\widehat{A}\widehat{C}} - \Omega_{\widehat{A}\widehat{D}} T_{\widehat{B}\widehat{C}} \ , \\
&\Omega_{\widehat{A}\widehat{B}} = \begin{pmatrix} \epsilon_{AB} & 0 \\ 0 & \Omega_{ab} \end{pmatrix} \ .
\end{aligned} \tag{3.21}
$$

The generator T_{aA} is hermitian and (T_{AB}, T_{ab}) are anti-hermitian. The vielbeins obey the following relations:

$$
g_{\alpha\beta} V^{\alpha}_{aA} V^{\beta}_{bB} = \Omega_{ab} \epsilon_{AB} \ , \qquad V^{\alpha}_{aA} V^{\beta aB} + \alpha \leftrightarrow \beta = g^{\alpha\beta} \delta^{B}_{A} \ , \tag{3.22}
$$

where $g_{\alpha\beta}$ is the metric on the coset. Another useful definition is that of the three quaternionic Kahler structures given by

$$
V^{A}_{\alpha a} V^{aB}_{\beta} - A \leftrightarrow B = 2 J^{AB}_{\alpha\beta} \ . \tag{3.23}
$$

Next, we define the components of the gauged Maurer-Cartan form as

$$
L^{-1} D_\mu L = P^{aA}_{\mu} T_{aA} + \tfrac{1}{2} Q^{ab}_{\mu} T_{ab} + \tfrac{1}{2} Q^{AB}_{\mu} T_{AB} \ , \tag{3.24}
$$

where

$$
D_\mu L = \left(\partial_\mu - A^I_\mu T^I \right) L \ , \tag{3.25}
$$

A^I_μ are the gauge fields of $K \times Sp(1)_R$. All gauge coupling constants are set equal to unity for simplicity in notation. They can straightforwardly be re-instated. We also use the notation

$$
T^I = (T^{I'}, T^r) \ , \qquad T_r = 2 T^{AB}_r T_{AB} \ , \qquad T^r_{AB} = -\tfrac{i}{2} \sigma^r_{AB} \ , \quad r = 1,2,3 \ . \tag{3.26}
$$

The components of the Maurer-Cartan form can be expressed in terms of the covariant derivative of the scalar fields as follows [34]

$$P_\mu^{aA} = (D_\mu\phi^\alpha)V_\alpha^{aA}\ , \qquad Q_\mu^{ab} = (D_\mu\phi^\alpha)Q_\alpha^{ab} - A_\mu^{ab}\ , \qquad Q_\mu^{AB} = (D_\mu\phi^\alpha)Q_\alpha^{AB} - A_\mu^{AB}\ , \tag{3.27}$$

where

$$D_\mu\phi^\alpha = \partial_\mu\phi^\alpha - A_\mu^I K^{I\alpha}\ , \tag{3.28}$$

and $K^I(\phi)$ are the Killing vectors that generate the $K \times Sp(1)_R$ transformations on G/H. Other building blocks to define the model are certain C-functions on the coset. These were defined in [35], and studied further in [34] where it was shown that they can be expressed as

$$L^{-1}T^I L \equiv C^I = C^{IaA}T_{aA} + \tfrac{1}{2}C^{IAB}T_{AB} + \tfrac{1}{2}C^{Iab}T_{ab}\,. \tag{3.29}$$

Differentiating and using the algebra (3.21) gives the useful relation

$$D_\mu C^I = \left(P_\mu^a{}_B C^{IAB} + P_{\mu b}{}^A C^{Iab}\right) T_{aA} + P_\mu^{aA} C_a^{IB}\, T_{AB} + P_\mu^{aA} C^{Ib}{}_A\, T_{ab}\ . \tag{3.30}$$

Moreover, using (3.24) and (3.27) we learn that

$$K^{I\alpha}V_\alpha^{aA} = C^{IaA}\ , \qquad K^{I\alpha}Q_\alpha^{ab} = C^{Iab} - \delta^{II'}T_{I'}^{ab}\ , \qquad K^{I\alpha}Q_\alpha^{AB} = C^{IAB} - \delta^{Ir}\,T_r^{AB}\ . \tag{3.31}$$

Finally, it is straightforward and useful to derive the identities

$$D_{[\mu}P_{\nu]}^{aA} = -\tfrac{1}{2}\,F_{\mu\nu}^I C^{IaA}\ , \tag{3.32}$$

$$P_{[\mu}^{aA}P_{\nu]A}^b = \tfrac{1}{2}\,Q_{\mu\nu}^{ab} + \tfrac{1}{2}F_{\mu\nu}^I C^{Iab}\ , \tag{3.33}$$

$$P_{[\mu}^{aA}P_{\nu]a}{}^B = \tfrac{1}{2}\,Q_{\mu\nu}^{AB} + \tfrac{1}{2}F_{\mu\nu}^I C^{IAB}\ . \tag{3.34}$$

2. Field Equations and Supersymmetry Transformation Rules

The Lagrangian for the anomaly free model we are studying can be obtained from [33] or [35]. We shall use the latter in the absence of Lorentz Chern-Simons terms and Green-Schwarz anomaly counterterms. Thus, the bosonic sector of the Lagrangian is given by [35]

$$e^{-1}\mathcal{L} = R - \tfrac{1}{4}(\partial\varphi)^2 - \tfrac{1}{12}e^{\varphi}\, G_{\mu\nu\rho}G^{\mu\nu\rho} - \tfrac{1}{4}\, e^{\frac{1}{2}\varphi}\, F^I_{\mu\nu}\, F^{I\mu\nu} - 2P^{aA}_\mu\, P^\mu_{aA} - 4\, e^{-\frac{1}{2}\varphi}\, C^I_{AB}C^{IAB}\ , \tag{3.35}$$

where the Yang-Mills field strength is defined by $F^I = dA^I + \frac{1}{2}f^{IJK}A^J \wedge A^K$ and G obeys the Bianchi identity

$$dG = \tfrac{1}{2}F^I \wedge F^I\ . \tag{3.36}$$

The bosonic field equations following from the above Lagrangian are [35]

$$\begin{aligned}
R_{\mu\nu} &= \tfrac{1}{4}\partial_\mu\varphi\,\partial_\nu\varphi + \tfrac{1}{2}e^{\frac{1}{2}\varphi}\,(F^2_{\mu\nu} - \tfrac{1}{8}F^2\, g_{\mu\nu}) + \tfrac{1}{4}e^{\varphi}\,(G^2_{\mu\nu} - \tfrac{1}{6}G^2\, g_{\mu\nu}) \\
&\quad -2P^{aA}_\mu P_{\nu aA} + e^{-\frac{1}{2}\varphi}(C^I_{AB}C^{IAB})\; g_{\mu\nu}\ , \\
\Box\,\varphi &= \tfrac{1}{4}e^{\frac{1}{2}\varphi}\, F^2 + \tfrac{1}{6}e^{\varphi}\, G^2 - 4\, e^{-\frac{1}{2}\varphi}\, C^I_{AB}C^{IAB} \\
D_\rho(e^{\frac{1}{2}\varphi}\, F^{I\rho}{}_\mu) &= \tfrac{1}{2}e^{\varphi}\, F^{I\rho\sigma}G_{\rho\sigma\mu} + 4P^{aA}_\mu C^I_{aA}\ , \\
\nabla_\rho\,(e^{\varphi}\, G^\rho{}_{\mu\nu}) &= 0\ , \\
D_\mu P^{\mu aA} &= 4e^{-\frac{1}{2}\varphi}C^{IAB}C^{Ia}{}_B\ ,
\end{aligned} \tag{3.37}$$

where we have used a notation $V^2_{\mu\nu} = V_{\mu\lambda_2\ldots\lambda_p}V_\nu{}^{\lambda_2\ldots\lambda_p}$ and $V^2 = g^{\mu\nu}V_{\mu\nu}$ for a p-form V, and $F^2 = F^I_{\mu\nu}F^{\mu\nu I}$. The local supersymmetry transformations of the fermions, up to cubic fermion terms that will not effect our results for the Killing spinors, are given by [35]

$$\delta\psi_\mu = D_\mu\varepsilon + \tfrac{1}{48}e^{\frac{1}{2}\varphi}G^{+}_{\nu\sigma\rho}\,\Gamma^{\nu\sigma\rho}\,\Gamma_\mu\,\varepsilon\ , \tag{3.38}$$

$$\delta\chi = \tfrac{1}{4}\left(\Gamma^\mu\partial_\mu\varphi - \tfrac{1}{6}e^{\frac{1}{2}\varphi}G^{-}_{\mu\nu\rho}\,\Gamma^{\mu\nu\rho}\right)\varepsilon\ , \tag{3.39}$$

$$\delta\lambda^I_A = -\tfrac{1}{8}F^I_{\mu\nu}\Gamma^{\mu\nu}\varepsilon_A - e^{-\frac{1}{2}\varphi}C^I_{AB}\,\varepsilon^B\ , \tag{3.40}$$

$$\delta\psi^a = P^{aA}_\mu\Gamma^\mu\varepsilon_A\ , \tag{3.41}$$

where $D_\mu\varepsilon_A = \nabla_\mu\varepsilon_A + Q_{\mu A}{}^B\varepsilon_B$, with ∇_μ containing the standard torsion-free Lorentz connection only. The transformation rules for the gauge fermions differ from those in [33] by a field redefinition.

B. Killing Spinor Conditions

The Killing spinor in the present context is defined to be the spinor of the supersymmetry transformations which satisfies the vanishing of the supersymmetric variations of all the spinors in the model. The well known advantage of seeking such spinors is that the necessary and sufficient conditions for their existence are first order equations which are much easier than the second order field equations, and moreover, once they are solved, the integrability conditions for their existence can be shown to imply most of the field equations automatically. In deriving the necessary and sufficient conditions for the existence of Killing spinors, it is convenient to begin with the construction of the nonvanishing fermionic bilinears, which provide a convenient tool for analyzing these conditions. In this section, firstly the construction and analysis of the fermionic bilinears are given, and then all the necessary and sufficient conditions for the existence of Killing spinor are derived.

1. Fermionic Bilinears and Their Algebraic Properties

There are only two nonvanishing fermionic bilinears that can be constructed from *commuting* symplectic-Majorana spinor ϵ^A. These are:

$$\begin{aligned} \bar{\epsilon}^A \Gamma_\mu \epsilon^B &\equiv V_\mu^{AB} \ , \\ \bar{\epsilon}^A \Gamma_{\mu\nu\rho} \epsilon^B &\equiv X^r_{\mu\nu\rho} T_r^{AB} \ . \end{aligned} \tag{3.42}$$

Note that X^r is a self-dual three-form due to chirality properties. From the Fierz identity $\Gamma_{\mu(\alpha\beta}\Gamma^\mu{}_{\gamma)\delta} = 0$, it follows that

$$V^\mu V_\mu = 0 \ , \qquad i_V X^r = 0 \ . \tag{3.43}$$

Introducing the orthonormal basis

$$ds^2 = 2e^+ e^- + e^i e^i \ , \tag{3.44}$$

and identifying

$$e^+ = V \ , \tag{3.45}$$

the equation $i_V X^r = 0$ and self-duality of X^r yield

$$X^r = 2V \wedge I^r \ , \tag{3.46}$$

where

$$I^r = \tfrac{1}{2} I^r_{ij}\, e^i \wedge e^j \tag{3.47}$$

is anti-self dual in the 4-dimensional metric $ds_4^2 = e^i e^i$. Straightforward manipulations involving Fierz identities imply that I^r are quaternionic structures obeying the defining relation

$$(I^r)^i{}_k \, (I^s)^k{}_j = \epsilon^{rst} (I^t)^i{}_j - \delta^{rs} \delta^i_j \ . \tag{3.48}$$

Finally, using the Fierz identity $\Gamma_{\mu(\alpha\beta}\Gamma^{\mu}_{\ \gamma)\delta} = 0$ once more, one finds that

$$V_\mu \Gamma^\mu \epsilon = \Gamma^+ \epsilon = 0 \ . \tag{3.49}$$

If there exists more than one linearly independent Killing spinor, one can construct as many linearly independent null vectors. In this case (3.49) is obeyed by each Killing spinor and the corresponding null vector, i.e. $V^1_\mu \Gamma^\mu \epsilon_1 = 0$, $V^2_\mu \Gamma^\mu \epsilon_2 = 0$, but it may be that $V^1_\mu \Gamma^\mu \epsilon_2 \neq 0$ and/or $V^2_\mu \Gamma^\mu \epsilon_1 \neq 0$. In that case, (3.49) should be relaxed since ϵ should be considered as a linear combination of ϵ_1 and ϵ_2.

2. Conditions From $\delta\lambda^I = 0$

Multiplying (3.40) with $\bar\epsilon^B \Gamma^\rho$, we obtain

$$\begin{aligned} i_V F^I &= 0 \ , & (3.50) \\ F^{Iij} I^r_{ij} &= 4 e^{-\frac{1}{2}\varphi}\, C^{Ir} \ . & (3.51) \end{aligned}$$

The second has been simplified by making use of (3.50) and (3.46). Multiplying (3.40) with $\bar\epsilon^B \Gamma_{\lambda\tau\rho}$, on the other hand, gives

$$F^I \wedge V + \star(F^I \wedge V) + 2 e^{\frac{1}{2}\varphi}\, C^{Ir} X^r = 0 \ , \tag{3.52}$$

$$\tfrac{3}{4} F^{I\sigma}{}_{[\mu} X^r_{\nu\rho]\sigma} + \tfrac{1}{2} \epsilon^{rst} e^{-\frac{1}{2}\varphi} C^{Is} X^t_{\mu\nu\rho} = 0 \ . \tag{3.53}$$

One can show that these two equations are identically satisfied upon the use of (3.50) and (3.51), which, in turn imply that F must take the form

$$F^I = -e^{-\frac{1}{2}\varphi}\, C^{Ir} I^r + \widetilde{F}^I + V \wedge \omega^I \ , \tag{3.54}$$

where $\widetilde{F}^I = \frac{1}{2} \widetilde{F}^I_{ij}\, e^i \wedge e^j$ is self-dual, and $\omega^I = \omega^I_i\, e^i$. Reinstating the gauge coupling constants, we note that the C-function dependent term will be absent when the index I points in the direction of a subgroup of $K \subset Sp(2n_H)$ under which all the hyperscalars are neutral. Substituting (3.54) into the supersymmetry transformation rule, and recalling (3.49), one finds that (3.40) gives the additional conditions on the Killing spinor

$$\left(\tfrac{1}{8} I^r_{ij} \Gamma^{ij} \delta^A_B - T^{rA}{}_B\right) \epsilon^B = 0 \ . \tag{3.55}$$

The contribution from $\widetilde{F}$ drops out due to chirality-duality properties involved. Writing this equation as $\mathcal{O}^r \epsilon = 0$, one can check that $[\mathcal{O}^r, \mathcal{O}^s] = \epsilon^{rst} \mathcal{O}^t$. Thus, any two projection imply the third one. In summary, the necessary and sufficient conditions for $\delta\lambda^I = 0$ are (3.54) and (3.55).

3. Conditions From $\delta\psi^a = 0$

This time multiplying (3.40) with $\bar{\epsilon}^B$ and $\bar{\epsilon}^B \Gamma_{\lambda\tau}$ gives rise to four equations which can be shown to imply

$$V^\mu P^{aA}_\mu = 0 \ , \tag{3.56}$$

$$P^{aA}_i = 2(I^r)_i{}^j \, (T^r)^A{}_B \, P^{aB}_j \ . \tag{3.57}$$

Using (3.23) and (3.27), we can equivalently reexpress the second equation above as

$$D_i \phi^\alpha = (I^r)_i{}^j \, (J^r)_\beta{}^\alpha \, D_j \phi^\beta \ . \tag{3.58}$$

Writing (3.57) as $P^a = \mathcal{O} P^a$, we find that $(\mathcal{O} - 1)(\mathcal{O} - 3) = 0$. Thus, (3.57) implies that P^a is an eigenvector of $\mathcal{O}$ with eigenvalue one. Moreover, using (3.57) directly in the supersymmetry transformation rule (3.41), and using the projection condition (3.55), we find that $\delta\psi^a = 3\delta\psi^a$, and hence vanishing. In summary, the necessary and sufficient conditions for $\delta\psi^a = 0$ are (3.56), (3.57) (or equivalently (3.58)), together with the projection condition (3.55).

4. Conditions From $\delta\chi = 0$

The analysis for this case is identical to that given in [10], so we will skip the details, referring to this paper. Multiplying (3.39) with $\bar{\epsilon}^B$ and $\bar{\epsilon}^B\Gamma_{\lambda\tau}$ gives four equations which can be satisfied by

$$V^\mu \partial_\mu \varphi = 0 \ , \tag{3.59}$$

and parametrizing G^- as

$$e^{\frac{1}{2}\varphi}\, G^- = \tfrac{1}{2}(1-\star)\left[V \wedge e^- \wedge d\varphi + V \wedge K\right] \ , \tag{3.60}$$

where $\star$ is the Hodge-dual, $K = \frac{1}{2}K_{ij}\, e^i \wedge e^j$ is self-dual. In fact, these two conditions are the necessary and sufficient conditions for satisfying $\delta\chi = 0$.

5. Conditions From $\delta\psi_\mu = 0$

Multiplying (3.38) with $\bar{\epsilon}\Gamma_\nu$, we find

$$\nabla_\mu V_\nu = -\tfrac{1}{2} e^{\frac{1}{2}\varphi}\, G^+_{\mu\nu\rho} V^\rho \ , \tag{3.61}$$

which implies that V^μ is a Killing vector. Similarly, multiplying (3.38) with $\bar{\epsilon}\Gamma_{\nu\rho\sigma}$ gives an expression for $\nabla_\sigma X^r_{\mu\nu\rho}$. Using (3.61) one finds that this expression is equivalent to

$$D_\mu I^r_{ij} = e^{\frac{1}{2}\varphi} G^{+k}{}_{\mu[i}\, I^r_{j]k} \ , \tag{3.62}$$

where $D_\mu I^r \equiv \nabla_\mu I^r + \epsilon^{rst} Q^s_\mu I^t$. One can use (3.62) to fix the composite $Sp(1)_R$ connection as follows

$$Q^r_\mu = \tfrac{1}{4} e^\varphi G^{(+)}_{\mu ij} I^{rij} - \tfrac{1}{8}\epsilon^{rst} I^{sij} \nabla_\mu I^t_{ij} \ . \tag{3.63}$$

Manipulations similar to those in [10] shows that, using (3.55) and (3.61), the variation $\delta\psi_\mu = 0$ is directly satisfied, with ϵ constant, in a frame where I^r_{ij} are constants. In summary, the necessary and sufficient conditions for $\delta\psi_\mu = 0$ are (3.61), (3.62), together with the projection condition (3.55).

C. Integrability Conditions for the Existence of a Killing Spinor

Assuming the Killing spinor conditions derived in the previous section, the attendant integrability conditions can be used to show that certain field equations are automatically satisfied. Since the field equations are complicated second order equations, it is therefore convenient to determine those which follow from the integrability, and identify the remaining equations that need to be satisfied over and above the Killing spinor conditions. Let us begin by introducing the notation

$$\delta\psi_\mu = \widetilde{D}_\mu\epsilon \ , \qquad \delta\chi = \tfrac{1}{4}\Delta\epsilon \ , \qquad \delta\lambda^I = e^{-\frac{1}{2}\varphi}\Delta^I\epsilon \ , \qquad \delta\psi^a = \Delta^{aA}\epsilon_A \ , \tag{3.64}$$

for the supersymmetry variations and

$$R_{\mu\nu} = J_{\mu\nu} \ , \quad \Box\varphi = J \ , \quad D_\mu(e^{\frac{1}{2}\varphi}F^{I\mu\nu}) = J^{I\nu} \ , \quad D_\mu P^{\mu aA} = J^{aA} \ , \tag{3.65}$$

for bosonic field equations. Then we find that

$$\begin{aligned}
\Gamma^\mu[\widetilde{D}_\mu, \Delta^I]\epsilon^A \;&=\; 2\left[D_\mu(e^{\frac{1}{2}\varphi}F^{I\mu\nu}) - J^{I\nu}\right]\Gamma_\nu\epsilon^A \\
&\quad + e^{\frac{1}{2}\varphi}\left(D_\mu F^I_{\nu\rho}\right)\Gamma^{\mu\nu\rho}\epsilon^A - 8\Gamma^\mu\left(D_\mu C^{IAB} + 2C^{Ia(A}P_{\mu a}{}^{B)}\right)\epsilon_B \\
&\quad -2[\Delta,\Delta^I]\epsilon^A + 2e^{\frac{1}{2}\varphi}F^I_{\mu\nu}\Gamma^{\mu\nu}\left(\delta\chi^A\right) + 16C^{IaA}\left(\delta\psi_a\right) \ , \\
&\quad +8e^{\frac{1}{2}\varphi}f^{IJK}A^J_\mu\Gamma^\mu(\delta\lambda^{KA}) \ ,
\end{aligned} \tag{3.66}$$

$$\begin{aligned}
\Gamma^\mu[\widetilde{D}_\mu, \Delta^{aA}]\epsilon_A \;&=\; \left(D_\mu P^{\mu aA} - J^{aA}\right)\epsilon_A \\
&\quad +\Gamma^{\mu\nu}\left(D_\mu P^{aA}_\nu - \tfrac{1}{2}F^I_{\mu\nu}C^{IaA}\right)\epsilon_A
\end{aligned}$$

$$
\begin{aligned}
& -4C^{IaA}(\delta\lambda^I_A) - \tfrac{1}{24}e^{\frac{1}{2}\varphi}G_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\left(\delta\psi^a\right) \, , \qquad (3.67)\\
\Gamma^\mu[\widetilde{D}_\mu, \Delta]\epsilon_A \; = \; & \left(\Box\varphi - J\right)\epsilon_A - \tfrac{1}{2}e^{-\frac{1}{2}\varphi}D_\mu(e^\varphi G^\mu{}_{\nu\rho})\,\Gamma^{\nu\rho}\epsilon_A \\
& -\tfrac{1}{6}e^{\frac{1}{2}\varphi}\Gamma^{\mu\nu\rho\sigma}\left(\nabla_\mu G_{\nu\rho\sigma} - \tfrac{3}{4}F^I_{\mu\nu}F^I_{\rho\sigma}\right)\epsilon_A \\
& -\left(e^{\frac{1}{2}\varphi}F^I_{\mu\nu}\Gamma^{\mu\nu}\epsilon_{AB} + 8C^I_{AB}\right)\delta\lambda^{IB} + \tfrac{1}{6}e^{\frac{1}{2}\varphi}G_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\left(\delta\chi_A\right) , \qquad (3.68)\\
\Gamma^\nu[\widetilde{D}_\mu, \widetilde{D}_\nu]\epsilon^A \; = \; & \tfrac{1}{2}\left(R_{\mu\nu} - J_{\mu\nu}\right)\Gamma^\nu\epsilon^A + \tfrac{1}{16}e^{-\frac{1}{2}\varphi}\nabla^\nu(e^\varphi G_{\nu\rho\sigma})\,\Gamma^{\rho\sigma}\Gamma_\mu\epsilon^A \\
& +\tfrac{1}{48}e^{\frac{1}{2}\varphi}\Gamma^{\rho\sigma\lambda\tau}\Gamma_\mu\left(\nabla_\rho G_{\sigma\lambda\tau} - \tfrac{3}{4}F^I_{\rho\sigma}F^I_{\lambda\tau}\right)\epsilon^A \\
& +\left(Q^{AB}_{\mu\nu} + F^I_{\mu\nu}C^{IAB} - 2P^{aA}_{[\mu}P_{\nu]a}{}^B\right)\Gamma^\nu\epsilon_B \\
& +\tfrac{1}{2}\left[\partial_\mu\varphi + \tfrac{1}{12}e^{\frac{1}{2}\varphi}G_{\nu\rho\sigma}\Gamma^{\nu\rho\sigma}\Gamma_\mu\right]\delta\chi^A + 2P^{aA}_\mu(\delta\psi_a) \\
& -\tfrac{1}{8}e^{\frac{1}{2}\varphi}\left[(\Gamma^{\nu\rho}\Gamma_\mu - 4\delta^\nu_\mu\Gamma^\rho)F^I_{\nu\rho}\epsilon^{AB} - \Gamma_\mu C^{IAB}\right]\delta\lambda^I_B \, . \qquad (3.69)
\end{aligned}
$$

If one makes the ansatz for the potentials directly , then the Bianchi identities and the relations (3.30) and (3.32)–(3.34) are automatically satisfied. Otherwise, all of these equations must be checked. Assuming that these are satisfied, from (3.66) it follows that the Yang-Mills field equation $K_\mu = 0$, *except for* $K_+ = 0$, is automatically satisfied, as can be seen by multiplying $K_\mu\Gamma^\mu\epsilon^A = 0$ by $\bar\epsilon^B$ and $K_\nu\Gamma^\nu$, recalling $\Gamma^+\epsilon = 0$ and further simple manipulations. Similarly, from (3.67) it follows that the hyperscalar field equation $K^{aA} = 0$ is automatically satisfied as can be seen by multiplying $K^{aA}\epsilon_A = 0$ by $\bar\epsilon_B\Gamma^\mu$. Finally, from (3.68) and (3.69), it follows that the dilaton and Einstein equation $E_{\mu\nu} = 0$, *except* $E_{++} = 0$, are automatically satisfied, provided that we also impose the G-field equation. This can be seen by multiplying $E_{\mu\nu}\Gamma^\nu\epsilon_A = 0$ with $\bar\epsilon_B$ and $E_{\mu\rho}\Gamma^\rho$ and simply manipulations that make use of $\Gamma^+\epsilon = 0$. In summary, once the Killing spinor conditions are obeyed, all the field equations are automatically satisfied as well, except the following,

$$R_{++} = J_{++} \ , \qquad D_\mu(e^{\frac{1}{2}\varphi} F^{I\mu}{}_+) = J^I_+ \ , \qquad D_\mu(e^{\varphi} G^{\mu\nu\rho}) = 0 \ , \tag{3.70}$$

and the Bianchi identities $DF^I = 0$ and $dG = \frac{1}{2} F^I \wedge F^I$. It is useful to note that in the case of gravity coupled to a non-linear sigma model, the scalar field equation follows from the Einstein's equation and the contracted Bianchi identity only when the scalar map is a submersion (i.e. when the rank of the matrix $\partial_\mu \phi^\alpha$ is equal to the dimension of the scalar manifold). In our model, however, the scalar field equation is automatically satisfied as a consequence of the Killing spinor integrability conditions, without having to impose such requirements. This is all the more remarkable given the fact that there are contributions to the energy-momentum tensor from fields other than the scalars. Finally, in analyzing the set of equations summarized above for finding a supersymmetric solution, it is convenient to parametrize the metric, which admits a null Killing vector, in general as [9]

$$ds^2 = 2H^{-1}(du+\beta)\left(dv + \omega + \frac{\mathcal{F}}{2}(du+\beta)\right) + H ds_B^2 \ , \tag{3.71}$$

with

$$\begin{aligned} e^+ &= H^{-1}(du+\beta) \ , \\ e^- &= dv + \omega + \tfrac{1}{2}\mathcal{F} H e^+ \ , \\ e^i &= H^{1/2} \tilde{e}_\alpha{}^i dy^\alpha \ , \end{aligned} \tag{3.72}$$

where $ds_B^2 = h_{\alpha\beta} dy^\alpha dy^\beta$ is the metric on the base space $\mathcal{B}$, and we have $\beta = \beta_\alpha dy^\alpha$ and $\omega = \omega_\alpha dy^\alpha$ as 1-forms on $\mathcal{B}$. These quantities as well as the functions H and $\mathcal{F}$

depend on u and y but not on v. Now, as in [9], defining the 2-forms on $\mathcal{B}$ by

$$\tilde{J}^r = H^{-1} I^r \ , \tag{3.73}$$

these obey

$$(\tilde{J}^r)^\alpha{}_\gamma \, (\tilde{J}^s)^\gamma{}_\beta = \epsilon^{rst} (\tilde{J}^t)^\alpha{}_\beta - \delta^{rs} \delta^\alpha_\beta \ , \tag{3.74}$$

where raising and lowering of the indices is understood to be made with $h_{\alpha\beta}$. Note that the index $\alpha = 1, ..., 4$ labels the coordinates y^α on the base space $\mathcal{B}$. This should not be confused with the index $\alpha = 1, ..., n_H$ that labels the coordinates ϕ^α of the scalar manifold! A geometrically significant equation satisfied by $\tilde{J}^r$ can be obtained from (3.62), and with the help of (3.61) it takes the form [10],

$$\tilde{\nabla}_i \tilde{J}^r_{jk} + \epsilon^{rst} Q^s_i \tilde{J}^t_{jk} - \beta_i \dot{\tilde{J}}{}^r_{jk} - \dot{\beta}_{[j} \tilde{J}^r_{k]i} + \delta_{i[j} \dot{\beta}^m \tilde{J}^r_{k]m} = 0 \ , \tag{3.75}$$

where $\tilde{\nabla}_i$ is the covariant derivative on the base space $\mathcal{B}$ with the metric ds^2_B and $\dot{\beta} \equiv \partial_u \beta$.

D. The Dyonic String Solution

For the string solution we shall activate only four hyperscalars, setting all the rest equal to zero. In the quaternionic notation of Appendix B, this means

$$t = \begin{pmatrix} \phi \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.76}$$

In what follows we shall use the map

$$\phi = \phi^{A'A} = \phi^\alpha (\sigma_\alpha)^{A'A} \ , \tag{3.77}$$

where $\sigma_\alpha = (1, -i\vec{\sigma})$ are the constant van der Wardeen symbols for $SO(4)$. Moreover, we shall chose the gauge group K such that

$$T^{I'}\, t = 0\ . \tag{3.78}$$

This condition can be easily satisfied by taking K to be a subgroup of $Sp(n_H - 1)$ which evidently leaves t given in (3.76) invariant. Finally, we set

$$A_\mu^{I'} = 0\ . \tag{3.79}$$

Then, supersymmetry condition (3.54) in I' direction is satisfied by setting $\widetilde{F}^{I'} = 0 = \omega^{I'}$ and noting that $C^{I'r} = 0$ in view of (3.78) (see (E.10)). The supersymmetry condition (3.57) is also satisfied along the directions in which the hyperscalars are set to zero. Therefore, the model effectively reduces to one in which the hyperscalars are described by $Sp(1,1)/Sp(1) \times Sp(1)$, which is equivalent to a 4-hyperboloid $H_4 = SO(4,1)/SO(4)$. Using (3.77) in the definition of $D_\mu t$ given in (E.8), we obtain

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha - \tfrac{1}{2} A_\mu^r (\rho^r)^\alpha{}_\beta\ \phi^\beta\ , \tag{3.80}$$

where the 't Hooft symbols ρ^r are constant matrices defined as

$$\rho^r_{\alpha\beta} = \mathrm{tr}\,(\sigma_\alpha\, T^r\, \bar{\sigma}_\beta)\ , \qquad \eta^{r'}_{\alpha\beta} = \mathrm{tr}\,(\bar{\sigma}_\alpha\, T^{r'}\, \sigma_\beta)\ , \tag{3.81}$$

where $\sigma_\alpha = (1, -i\vec{\sigma})$ are the constant van der Wardeen symbols for $SO(4)$. These are real and antisymmetric matrices. It is easily verified that $\rho^r_{\alpha\beta}$ is anti-selfdual, while $\eta^{r'}_{\alpha\beta}$ is selfdual. Their further properties are

$$\rho^r_{\alpha\gamma}\, (\rho^s)^\gamma{}_\beta = -\delta^{rs}\delta_{\alpha\beta} + \epsilon^{rst}\, \rho^t_{\alpha\beta}\ , \qquad\qquad \text{idem } \eta^{r'}_{\alpha\beta}\ ,$$

$$\rho^r_{\alpha\beta}\rho^r_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} - \epsilon_{\alpha\beta\gamma\delta}\ ,$$

$$\eta^{r'}_{\alpha\beta}\eta^{r'}_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} + \epsilon_{\alpha\beta\gamma\delta} \ ,$$

$$\epsilon^{trs}(\rho^r)_{\alpha\beta} \ (\rho^s)_{\gamma\delta} = \delta_{\beta\gamma} \ (\rho^t)_{\alpha\delta} + 3 \text{ more} \ , \qquad \text{idem } \eta^{r'}_{\alpha\beta} \ . \tag{3.82}$$

For $SU(2)$ triplets, we use the notation:

$$X^{AB} = X^r \, T^r_{AB} \ , \qquad X^r = \tfrac{1}{2} X^{AB} \, T^r_{AB}. \tag{3.83}$$

For the metric we choose

$$\beta = 0 \ , \qquad \omega = 0 \ , \qquad \mathcal{F} = 0 \ , \qquad h_{\alpha\beta} = \Omega^2 \delta_{\alpha\beta} \ , \tag{3.84}$$

in the general expression (3.71), so that our ansatz takes the form

$$ds^2 = 2H^{-1} \, dudv + H ds^2_B \ , \qquad ds^2_B = \Omega^2 dy^\alpha dy^\beta \delta_{\alpha\beta} \ , \tag{3.85}$$

where Ω is a function of $y^2 \equiv y^\alpha y^\beta \delta_{\alpha\beta}$. We also choose the null basis as

$$e^+ = V = H^{-1} du \ , \qquad e^- = dv \ . \tag{3.86}$$

Thus, $V^\mu \partial_\mu = \partial/\partial v$. Moreover, in the rest of this section, *we shall take all the fields to be independent of* u *and* v. Given that $\beta = 0$, it also follows from (3.75) that

$$\tilde{\nabla}_i \tilde{J}^r_{jk} + \epsilon^{rst} Q^s_i \tilde{J}^t_{jk} = 0 \ . \tag{3.87}$$

Next, in the general form of $G^{(-)}$ given in (3.60), we choose

$$K = 0 \ . \tag{3.88}$$

Then, from (3.60) and (3.61) we can compute all the components of G^+ and G^-, which yield for $G = G^+ + G^-$ the result

$$G = e^{-\varphi/2} \left(e^+ \wedge e^- \wedge d\varphi_+ + \star_4 \, d\varphi_- \right) \ , \tag{3.89}$$

where $\star_4$ refers to Hodge dual on the transverse space with metric

$$ds_4^2 = H ds_B^2 \ , \tag{3.90}$$

and we have defined

$$\varphi_\pm \ := \ \pm\tfrac{1}{2}\varphi + \ln\, H \ . \tag{3.91}$$

Next, we turn to the supersymmetry condition (3.58) in the hyperscalar sector. With our ansatz described so far, it can now be written as

$$D_i\phi^{\underline{\alpha}} = (\tilde{J}^r)_i{}^j\,(J^r)_{\underline{\beta}}{}^{\underline{\alpha}}\,D_j\phi^{\underline{\beta}} \ , \tag{3.92}$$

where

$$D_i\phi^{\underline{\alpha}} \equiv D_i\phi^\alpha\,V_\alpha{}^{\underline{\alpha}} \ , \tag{3.93}$$

and $V_\alpha{}^{\underline{\alpha}}$ is the vielbein on H_4, and the above equations are in the basis

$$\tilde{e}^i = \delta^i_\alpha\,\Omega\,dy^\alpha \ , \tag{3.94}$$

referring to the base space $\mathcal{B}$. We also note that

$$J^r_{\underline{\alpha}\underline{\beta}} = \rho^r_{\alpha\beta}\,\delta^\alpha_{\underline{\alpha}}\,\delta^\beta_{\underline{\beta}} \ , \tag{3.95}$$

which follows from rom (F.2) and (F.3). Recall that the 't Hooft matrices $\rho^r_{\alpha\beta}$ are constants. Next, we choose the components of $\tilde{J}^r_{ij}$ to be constants and make the identification

$$\tilde{J}^r = J^r \ . \tag{3.96}$$

Using the quaternion algebra, we can now rewrite (3.92) as

$$D_i\phi_{\underline{\beta}} = (\delta_{i\underline{\alpha}}\delta_{j\underline{\beta}} - \delta_{j\underline{\alpha}}\delta_{i\underline{\beta}} - \epsilon_{ij\underline{\alpha}\underline{\beta}})\ D_j\phi_{\underline{\alpha}} \ . \tag{3.97}$$

Symmetric and antisymmetric parts in i and $\underline{\beta}$ give

$$D_i\phi^i = 0\ , \qquad \phi^i \equiv \phi^{\underline{\alpha}}\,\delta^i_{\underline{\alpha}}\ , \tag{3.98}$$

$$D_i\phi_j - D_j\phi_i = -\epsilon_{ijk\ell}D_k\phi_\ell\ . \tag{3.99}$$

To solve these equations, we make the ansatz

$$\phi^\alpha = f y^\alpha\ , \qquad A^r_\alpha = g\,\rho^r_{\alpha\beta}\,y^\beta\ , \tag{3.100}$$

where f and g are functions of y^2. This ansatz, in particular, implies that the function ω^r arising in the general form of F^r given in (3.54) vanishes. Assuming that the map ϕ^α is 1-1, one can actually use diffeomorphism invariance to set (at least locally) $f = 1$. However, since we have already fixed the form of the metric as in (3.85), chosen a basis as in (3.94), and identified the components of the quaternionic structures $\tilde{J}^r_{ij}$ referring to this orthonormal basis, the reparametrization invariance has been lost. Therefore it is important to keep the freedom of having an arbitrary function in the map (3.100). Using (3.100) we find that (3.99) is identically satisfied and (3.98) implies

$$g = \frac{4f'y^2 + 8f}{3fy^2}\ , \tag{3.101}$$

where prime denotes derivative with respect to argument, i.e. y^2. Next, the computation of the Yang-Mills field strength from the potential (3.100) gives the result

$$F^r = F^{r(+)} + F^{r(-)}\ , \qquad F^{r\pm} = \pm \star_4 F^{r\pm}\ , \tag{3.102}$$

$$F^{r(-)}_{\alpha\beta} = (-2g - g'y^2 + \tfrac{1}{2}g^2y^2)\,\rho^r_{\alpha\beta}\ ,$$

$$F^{r(+)}_{\alpha\beta} \equiv \widetilde{F}^r_{\alpha\beta} = (2g' + g^2)\,\left(2y_{[\alpha}y^\delta\,\rho^r_{\beta]\delta} + \tfrac{1}{2}y^2\,\rho^r_{\alpha\beta}\right)\ .$$

Comparing these results with the general form of F^I given in (3.54), we obtain

$$e^{\varphi_-} = \frac{\eta}{\Omega^2} \ , \tag{3.103}$$

where

$$\eta \equiv \left(g' y^2 + 2g - \tfrac{1}{2} g^2 y^2\right)\left(1 - f^2 y^2\right) \ . \tag{3.104}$$

Here we have used the fact that $C^{r,s} = \delta^{rs}/(1-\phi^2)$ as it follows from the formula (E.9). Finally using the composite connection (F.4) in (3.87) we obtain

$$\frac{\Omega'}{\Omega} = \frac{(2f^2 - g)}{2(1 - f^2 y^2)} \ . \tag{3.105}$$

This equation can be integrated with the help of (3.101), yielding

$$\Omega = \frac{b}{y^2}\left(\frac{1 - f^2 y^2}{f^2 y^2}\right)^{1/3} \ , \tag{3.106}$$

where b is an integration constant. One can now see that all necessary and sufficient conditions for the existence of a Killing spinor on this background are indeed satisfied. As shown in the previous section, the integrability conditions for the existence of a Killing spinor imply all field equations except (3.70) and the Bianchi identities on F^I and G. It is easy to check that (3.70) is identically satisfied by our ansatz, except for the G-field equation. Furthermore, the Yang-Mills Bianchi identity is trivial since we give the potential. Thus, the only remaining equations to be checked are the G-Bianchi identity and the G-field equation. To this end, it is useful to record the result

$$\frac{\epsilon^{\alpha\beta\gamma\delta}}{\sqrt{g_4}} F^r_{\alpha\beta} F^r_{\gamma\delta} = \frac{16 Q'}{y^2 H^2 \Omega^4} \ , \tag{3.107}$$

where g_4 is the determinant of the metric for the line element ds_4^2, and

$$Q \equiv (g y^2)^2 (g y^2 - 3) + c \ , \tag{3.108}$$

where c is an integration constant. Interestingly, this term is proportional to the sum of F^2 and C^2 terms that arise in the dilaton field equation, up to an overall constant. We now impose the G-field equation $d(e^{\varphi} \star G) = 0$ and the G-Bianchi identity $dG = \frac{1}{2} F^r \wedge F^r$. The G-field equation gives

$$\Box_4 \varphi_+ + \tfrac{1}{2}\partial_\alpha \varphi \partial^\alpha \varphi_+ \;=\; 0 \;, \tag{3.109}$$

and the G-Bianchi identity amounts to

$$\Box_4 \varphi_- - \tfrac{1}{2}\partial_\alpha \varphi \, \partial^\alpha \varphi_- \;=\; \frac{-2Q'}{y^2 H^2 \Omega^4} \;, \tag{3.110}$$

where the Laplancian is defined with respect to the metric (3.90). These equations can be integrated once to give

$$\varphi_+' = \frac{\nu e^{-\varphi}}{(y^2)^2 \eta} \;, \qquad \varphi_-' = \frac{(\lambda - \frac{1}{2}Q)}{(y^2)^2 \eta} \;, \tag{3.111}$$

where ν, λ are the integration constants, c has been absorbed into the definition of λ, and (3.103) has been used in the form $H\Omega^2 = \eta e^{\varphi/2}$. These equation can be rewritten as

$$(e^{\varphi_+})' \;=\; \frac{\nu}{b^2} \left(\frac{f^2 y^2}{1 - f^2 y^2} \right)^{2/3} \;, \tag{3.112}$$

$$(e^{\varphi_-})' \;=\; \frac{\lambda - \frac{1}{2}Q}{b^2} \left(\frac{f^2 y^2}{1 - f^2 y^2} \right)^{2/3} \;, \tag{3.113}$$

by recalling $\varphi = \varphi_+ - \varphi_-$, exploiting (3.103) and using the solution (3.106) for Ω. It is important to observe that the second equation in (3.111), has to be consistent with (3.103). Differentiating the latter and comparing the two expressions, we obtain a third order differential equation for the function f:

$$\eta' - \left(\frac{2f^2 - g}{1 - f^2 y^2} \right) \eta = \frac{\lambda - \frac{1}{2}Q}{(y^2)^2} \;. \tag{3.114}$$

In summary, any solution of this equation for f determines also the functions (φ, H, Ω, g), and therefore fixes the solution completely. This is a highly complicated equation, however, and we do not know its general solution at this time. Nonetheless, it is remarkable that an ansatz of the form

$$f = \frac{a}{y^2} \ , \tag{3.115}$$

with a a constant, which gives $g = 4/(3y^2)$ from (3.101), does solve (3.114), and moreover, it fixes the integration constant

$$\lambda = -\tfrac{4}{3} \ . \tag{3.116}$$

Furthermore, it follows from (3.106), (3.103), (3.104) and (3.112) that

$$\Omega = \frac{b}{y^2}\, h^{1/3} \ , \qquad e^{\varphi_-} = \left(\frac{2a}{3b}\right)^2 h^{1/3} \ , \qquad e^{\varphi_+} = 3\nu \left(\frac{a}{b}\right)^2 h^{1/3} + \nu_0 \ , \tag{3.117}$$

where ν_0 is an integration constant and

$$h \equiv \frac{y^2}{a^2} - 1 \ . \tag{3.118}$$

Thus, the full solution takes the form

$$ds^2 = e^{-\frac{1}{2}\varphi_+} e^{-\frac{1}{2}\varphi_-}(-dt^2 + dx^2) + e^{\frac{1}{2}\varphi_+} e^{\frac{1}{2}\varphi_-} \left(\frac{b}{y^2}\right)^2 h^{2/3}\, dy^\alpha dy^\beta\, \delta_{\alpha\beta} \ , \tag{3.119}$$

$$e^{\varphi} = e^{\varphi_+}/e^{\varphi_-} \ , \qquad \phi^\alpha = \frac{a y^\alpha}{y^2} \ , \tag{3.120}$$

$$A^r_\alpha = \frac{4}{3y^2}\, \rho^r_{\alpha\beta} y^\beta \ , \tag{3.121}$$

$$G_{\alpha\beta\gamma} = \frac{8}{27(y^2)^2}\, \epsilon_{\alpha\beta\gamma\delta}\, y^\delta \ , \qquad G_{+-\alpha} = -\partial_\alpha e^{-\varphi_+} \ , \tag{3.122}$$

with $\varphi_\pm$ given in (3.117). The form of h dictates that $a^2 < y^2 < \infty$, covering outside of a disk of radius a. The hyperscalars map this region into H^4 which can be viewed as the interior of the disk defined by $\phi^2 < 1$. These scalars are gravitating in the sense that their contribution to the energy momentum tensor, which takes the form $(\mathrm{tr}P_iP_j - \frac{1}{2}g_{ij}\mathrm{tr}P^2)$, does not vanish since the solution gives

$$P_i^{A'A} = \frac{a}{3y^2\left(1-\frac{a^2}{y^2}\right)}\left(\delta_i^\alpha - 4\frac{y_i y^\alpha}{y^2}\right)\sigma_\alpha^{A'A} \ . \tag{3.123}$$

It is possible to apply a coordinate transformation and map the base space into the disc by defining

$$z^\alpha \equiv \frac{ay^\alpha}{y^2}. \tag{3.124}$$

In z^α coordinates the solution becomes

$$ds^2 = e^{-\frac{1}{2}\varphi_+}e^{-\frac{1}{2}\varphi_-}(-dt^2+dx^2) + L^2 e^{\frac{1}{2}\varphi_+}e^{\frac{1}{2}\varphi_-}\, h^{2/3}\,(dr^2 + r^2 d\Omega_3^2) \tag{3.125}$$

$$e^\varphi = e^{\varphi_+}/e^{\varphi_-} \ , \tag{3.126}$$

$$G = \tfrac{8}{27}\,\Omega_3 - dt\wedge dx\wedge de^{-\varphi_+} \ , \tag{3.127}$$

$$A^r = \tfrac{2}{3}\,r^2\sigma_R^r \ , \tag{3.128}$$

$$\phi^\alpha = z^\alpha \ , \tag{3.129}$$

where

$$r = \sqrt{z^\alpha z^\beta \delta_{\alpha\beta}}\ , \qquad \Omega_3 = \sigma_R^1\wedge\sigma_R^2\wedge\sigma_R^3\ , \qquad h = \frac{1}{r^2} - 1\ , \tag{3.130}$$

$$e^{\varphi_+} = \frac{3\nu h^{1/3}}{L^2} + \nu_0\ , \qquad e^{\varphi_-} = \frac{4h^{1/3}}{9L^2}\ , \tag{3.131}$$

and $L \equiv b/a$. Here, σ_R^r are the right-invariant one-forms satisfying

$$d\sigma_R^r = \tfrac{1}{2}\epsilon^{rst}\,\sigma_R^s\wedge\sigma_R^t\ , \tag{3.132}$$

and Ω_3 is the volume form on S^3. We have also used the definitions

$$z^\alpha = r\, n^\alpha \ , \qquad n^\alpha n^\beta \delta_{\alpha\beta} = 1 \ , \tag{3.133}$$

where dn^α are orthogonal to the unit vectors n^α on the 3-sphere, and satisfy

$$dn^\alpha = \tfrac{1}{2}\rho^{r\alpha}{}_\beta\, \sigma^r_R\, n^\beta \ , \qquad dn^\alpha dn^\beta \delta_{\alpha\beta} = \tfrac{1}{4} d\Omega_3^2 \ . \tag{3.134}$$

Given the form of A^r, it is easy to see that the Yang-Mills 2-form $F^r = dA^r - \frac{1}{2}\epsilon^{rst} A^s \wedge A^t$ is not (anti)self-dual, as it is given by

$$F^r = \tfrac{4}{3}\, r dr \wedge \sigma^r_R + \tfrac{1}{3} r^2 \left(1 - \tfrac{2}{3} r^2\right) \epsilon^{rst} \sigma^s_R \wedge \sigma^t_R \ . \tag{3.135}$$

The field strength $P_i^{A'A}$ on the other hand, takes the form

$$P_i^{A'A} = \frac{1}{1-r^2} \left[(1 - \tfrac{2}{3} r^2) \delta_i^\alpha + \tfrac{2}{3} r^2 n_i n^\alpha \right] \sigma_\alpha^{A'A} \ . \tag{3.136}$$

We emphasize that, had we started with the identity map $\phi^\alpha = z^\alpha$ from the beginning, the orthonormal basis in which $\tilde{J}^r_{ij}$ are constants would be more complicated than the one given in (3.94). Consequently, (3.105) would change since it uses (3.87) that requires the computation of the spin connection in the new orthonormal basis.

E. Properties of the Solution

1. Dyonic Charges and Limits

To begin with, we observe that the solution we have presented above is a dyonic string with *fixed* magnetic charge given by

$$Q_m = \int_{S^3} G = \frac{8}{27}\, vol_{S^3} \ . \tag{3.137}$$

The electric charge, however, turns out to be proportional to the constant parameter ν as follows:

$$Q_e = \int_{S^3} \star e^{\varphi} G = 2\nu \, vol_{S^3} \ . \tag{3.138}$$

Next, let us compare our solution with that of [17] where a dyonic string solution of the $U(1)_R$ gauged model in the absence of hypermatter has been obtained. We shall refer to this solution as the GLPS dyonic string [17]. To begin with, the GLPS solution has two harmonic functions with two arbitrary integration constants, as opposed to our single harmonic function h with a fixed and negative integration constant. In our solution, this is essentially due to the fact that we have employed an identity map between a hyperbolic negative constant curvature scalar manifold and space transverse to the string worldsheet. Next, the transverse space metric ds_4^2 in the GLPS solution is a warped product of a *squashed* 3-sphere with a real line, while in our solution it is conformal to R^4. In GLPS solution the deviation from the round 3-sphere is proportional to a product of $U(1)_R$ gauge constant and monopole flux due to the $U(1)_R$ gauge field. Thus, assuming that we are dealing with a gauged theory, the round 3-sphere limit would require the vanishing of the monopole flux, which is not an allowed value in GLPS solution. As for the 3-form charges, the electric charge is arbitrary in the GLPS as well as our solution. However, while the magnetic charge in the GLPS solution is proportional to $k\xi/g_R$ where k is the monopole flux, g_R is the $U(1)_R$ coupling constant and ξ is the squashing parameter, and therefore arbitrary, in our solution the magnetic charge is fixed in Planckian units and therefore it is necessarily non-vanishing. This is an interesting property of our solution that results from the interplay between the sigma model manifold whose radius is fixed in units of Plank length, which is typical in supergravities with a sigma model sector, and the four dimensional space transverse to the string worldsheet. Our solution has

$SO(1,1)\times SO(4)$ symmetry corresponding to Poincaré invariance in the string world-sheet and rotational invariance in the transverse space*. The metric components exhibit singularities at $r=0$ and $r=1$. Too see the coordinate invariant significance of these points, we compute the Ricci scalar as

$$R=\frac{48(\Delta+\mu_0)^2+\mu_0^2}{r^6\left(\frac{\Delta}{3\nu}\right)^{\frac{17}{18}}(\Delta+\mu_0)^{\frac{5}{2}}}\ , \tag{3.139}$$

where $\Delta\equiv 3\nu(\frac{1}{r^2}-1)$ and $\mu_0\equiv\nu_0 L^2$. We see that, near the boundary $r\to 1$, the Ricci scalar diverges, and there is a genuine singularity there. Near the origin $r=0$, however, the situation depends on the parameter ν. If $\nu\neq 0$, then as $r\to 0$ the Ricci scalar approaches the constant value $8/\sqrt{3\nu}$. The metric is perfectly regular in this limit, and indeed, we find that it takes the form

$$ds^2\to\frac{L^2}{R_0^2}r^{2/3}(-dt^2+dx^2)+\frac{R_0^2dr^2}{r^2}+R_0^2d\Omega_3^2\ , \tag{3.140}$$

which is $AdS_3\times S_3$ with $R_0=\sqrt{4\nu/3}$. This is to be contrasted with the GLPS solution which approaches the product of AdS_3 with a squashed 3-sphere. The $r=0$ point can be viewed as the horizon, and as is usually the case, our solution also has a factor of two enhancement of supersymmetry near the horizon. This is due to the fact that the condition (3.49), which reads $H^{-1}\Gamma^+\epsilon=0$ has to be relaxed since H^{-1} vanishes in the $r\to 0$ limit. Note, however, that our solution at generic point has 1/8 supersymmetry to begin with, as opposed to 1/4 supersymmetry of the GLPS solution. For $\nu=0$, the $r\to 0$ limit of the metric is

$$ds^2\to\frac{3L}{2\sqrt{\nu_0}}r^{1/3}(-dt^2+dx^2)+\frac{2L\sqrt{\nu_0}}{3}r^{-5/3}(dr^2+r^2d\Omega_3^2)\ , \tag{3.141}$$

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*It is clear that if one makes an $SO(4)$ rotation in z^α coordinates, the same transformation should be applied to hyperscalars and 't Hooft symbols $\rho^r_{\alpha\beta}$ to preserve the structure of the solution.

Defining furthermore $du = dr/r^{5/6}$ the metric becomes

$$ds^2 \sim u^2(-dt^2 + dx^2 + d\Omega_3^2) + du^2. \tag{3.142}$$

Ignoring x and Ω_3 directions, this describes the Rindler wedge which is the near horizon geometry of the Schwarzchild black hole. The "horizon", which has the topology $R \times \Omega_3$, shrinks to the zero size at $u = 0$ and this gives the singularity in the dyonic string. Next, consider the boundary limit in which $r \to 1$. First, assuming that $\nu_0 \neq 0$, we find that in the limit $r \to 1$ the metric takes the form

$$ds^2 \sim \frac{1}{u^{1/3}} \left(-dt^2 + dx^2 + u^4 (\, du^2 + \frac{1}{u^2}\, d\Omega_3^2) \right) \qquad \text{for} \;\; \nu_0 \neq 0 \;, \tag{3.143}$$

where we have defined the coordinate $u = h^{1/2}$ and rescaled the string worldsheet coordinates by a constant. For $\nu_0 = 0$, on the other hand, the $r \to 1$ limit of the metric is given by

$$ds^2 \sim \frac{1}{u^{2/3}} \left(-dt^2 + dx^2 \right) + u^4 \left(du^2 + \frac{1}{u^2}\, d\Omega_3^2 \right) \qquad \text{for} \;\; \nu_0 = 0 \;, \tag{3.144}$$

where, again, we have defined $u = h^{1/2}$ and rescaled coordinates by constants.

2. Coupling of Sources

Since the solution involves the harmonic function h, there is also a possibility of a delta function type singularity at the origin since

$$\partial_\alpha \partial^\alpha \, h \sim -4\pi^2 \delta(\vec{z})\,. \tag{3.145}$$

The presence of such a singularity requires addition of extra sources to supergravity fields to get a proper solution. As it is not known how to write down the coupling of a dyonic string to sources, and as we cannot turn off the magnetic charge, we consider the coupling of the magnetic string to sources. Thus setting $\nu = 0$, from (3.125), (3.126) and (3.129) the dangerous fields that can possibly yield a delta function via (3.145) are the metric, the dilaton ϕ and the three form field G. Indeed from (3.129) we see that

$$dG \sim \delta(\vec{z})\, dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4\,, \tag{3.146}$$

therefore extra (magnetically charged) sources are needed for G at $\vec{z}=0$. For the dilaton we find that the candidate singular term near $\vec{z}=0$ behaves as

$$\Box\varphi \sim\; z^{11/3}\,\delta(\vec{z}) \rightarrow 0\;, \tag{3.147}$$

thus there is no problem at $\vec{z}=0$. Finally for the Ricci tensor expressed in the coordinate basis we find

$$R_{tt} \;=\; -R_{xx} \sim z^4\delta(\vec{z}) \rightarrow 0\;, \tag{3.148}$$

$$R_{\alpha\beta} \;\sim\; z^2\delta(\vec{z})\,\delta_{\alpha\beta} \rightarrow 0\;. \tag{3.149}$$

Contracting with the metric one can see that the possible singular part in the Ricci scalar becomes

$$R \sim\; z^{11/3}\,\delta(\vec{z}) \rightarrow 0\;, \tag{3.150}$$

and thus there appears no extra delta function singularity. The above results can be understood by coupling to supergravity fields a magnetically charged string located at $r=0$ with its action given by

$$S = -\int d^2\sigma e^{\varphi/2}\sqrt{-\gamma} + \int \widetilde{B}\;, \tag{3.151}$$

where γ is the determinant of the induced worldsheet metric and $\widetilde{B}$ is the 2-form potential whose field strength is dual to G. This coupling indeed produces exactly the behavior (3.146) in the Bianchi identity. The source terms in (3.147) and (3.148) are also produced, while the contribution to the right hand side of (3.149) vanishes identically (which does not causes a problem since $z^2\delta(\vec{z})$ vanishes at $z = 0$ as well).

3. Base Space as a Tear Drop

In (3.125) the four dimensional base space for our solution (3.125) is

$$\begin{aligned} ds_B^2 &= L^2\left(\frac{1}{r^2}-1\right)^{2/3}\left(dr^2+r^2d\Omega_3^2\right) \\ &= \frac{(1-r^2)^{8/3}}{2r^{4/3}}\,ds_{H_4}^2\ , \end{aligned} \tag{3.152}$$

where $ds_{H_4}^2 = 2(dr^2 + r^2 d\Omega_3^2)/(1-r^2)^2$ is the metric on H_4. Although the overall conformal factor blows at $r = 0$, the total volume of this space turns out to have a finite value $(4\pi^3 L^4)/(9\sqrt{3})$. To that extent, our solution can be viewed as the analog of the Gell-Mann-Zwiebach teardrop solution, though the latter is regular at $r = 0$ as well. The analogy with Gell-Mann-Zwiebach tear-drop is also evident in the fact that the scalar metric has been conformally rescaled by a factor that vanishes at the boundary. The curvature scalar of the base manifold is also singular at $r = 0$, as it is given by

$$R_{\mathcal{B}} = \frac{16}{3L^2}\,\frac{1}{r^2}\,\frac{r^{4/3}}{(1-r^2)^{8/3}}\ . \tag{3.153}$$

Since the total volume in the base space is finite, one would expect that the singularity at $r = 0$ can be reached by physical particles at a finite proper time. We have checked that this is indeed the case. Another tear-drop like feature here is that the base space metric is conformally related to that of H_4 which has negative constant curvature, and that the curvature scalar of the bases space becomes positive due to the conformal factor. This switching of the sign is crucial for satisfying Einstein equation in the internal direction, just as in the case of 2-dimensional Gell-Mann-Zwiebach teardrop. The base space $\mathcal{B}$ that emerges in the $2 + 4$ split of the $6D$ spacetime is quaternionic manifold, as it admits a quaternionic structure. To decide whether it is Quaternionic

Kahler (QK), however, the standard definition that relies on the holonomy group being contained in $Sp(n) \times Sp(1) \sim SO(4)$ becomes vacuous in $4D$ since all $4D$ Riemann manifolds have holonomy group $Sp(1) \times Sp(1)$. Nonetheless, there exists a generally accepted and natural definition of QK manifolds in four dimensions, which states that an oriented $4D$ Riemann manifold is QK if the metric is self-dual and Einstein (see [36] for a review). According to this definition, our base space $\mathcal{B}$ is not QK since it is neither self-dual nor Einstein.

4. Reduction of Metric to Five Dimensions

Finally, we would like to note the 5-dimensional metric that can be obtained by a Kaluza-Klein reduction along the string direction. The 6-dimensional metric is parametrized in terms of the 5-dimensional metric as

$$ds_6^2 = e^{2\alpha\hat{\phi}} ds_5^2 + e^{2\beta\hat{\phi}} dx^2 \qquad (3.154)$$

where $\beta = -3\alpha$ and $\hat{\phi}$ is the Kaluza-Klein scalar. From (3.125) one finds

$$ds_5^2 = -e^{-\frac{2}{3}\varphi_+} e^{-\frac{2}{3}\varphi_-} \, dt^2 + L^2 e^{\frac{1}{3}\varphi_+} e^{\frac{1}{3}\varphi_-} h^{2/3} (dr^2 + d\Omega_3^2), \qquad (3.155)$$

where the functions are still given in (3.131). The metric (3.155) is singular at $r = 0$. For $\nu = 0$ looking at the metric near the singularity one finds

$$ds_5^2 \sim u^2(-dt^2 + d\Omega_3^2) + du^2, \qquad (3.156)$$

where $du = dr/r^{7/9}$. The geometry is like the Rindler space but the candidate spherical "horizon" shrinks to zero size at $u = 0$ which produces a singularity. When $\nu \neq 0$, one finds near $r = 0$ that

$$ds_5^2 \sim -r^{8/9} dt^2 + r^{-16/9} dr^2 + r^{2/9} d\Omega_3^2 \qquad (3.157)$$

which is again singular at $r = 0$. This singularity is resolved by dimensional *oxidation* which is a well known feature of some black-brane solutions [37].

F. Comments

In this chapter, we have derived the necessary and sufficient conditions for the existence of a Killing spinor in $N = (1, 0)$, $6D$ gauge supergravity coupled to a single tensor multiplet, vector multiplets and hypermultiplets. The Killing spinor existence conditions and their integrability are shown to imply most of the equations of motion. The remaining equations to be solved are (i) the Yang-Mills equation in the null direction, (ii) the field equation for the 2-form potential, (iii) the Bianchi identities for the Yang-Mills curvature and the field strength of the 2-form potential, and (iv) the Einstein equation in the double null direction. We parametrize the most general form of a supersymmetric solution which involves a number of undetermined functions. However, we do not write explicitly the equations that these functions must satisfy. These can be straightforwardly derived from the equations just listed. The existence of a null Killing vector suggests a $2 + 4$ split of spacetime, and search for a string solution, possibly dyonic. As a natural application of the general framework presented here, we have then focused on finding a dyonic string solution in which the hyperscalars have been activated. Indeed, we have found a 1/8 supersymmetric such a dyonic string. The activated scalars parametrize a 4 dimensional submanifold of a quaternionic hyperbolic ball of unit radius, characterized by the coset $Sp(n_H, 4)/Sp(n_H) \times Sp(1)_R$. A key step in the construction of the solution is an identity map between the 4-dimensional scalar submanifold and internal space transverse to the string worldsheet. The spacetime metric turns out to be a warped product of the string worldsheet and a 4-dimensional analog of the

Gell-Mann-Zwiebach tear-drop which is noncompact with finite volume. Unlike the Gell-Mann-Zwiebach tear-drop, ours is singular at the origin. There is also a delta function type singularity that comes from the Laplancian acting on a harmonic function present in the solution. This does not present any problem, however, as we place a suitable source which produces contributions to the field equations that balance the delta function terms. An interesting property of our dyonic string solution is that while its electric charge is arbitrary, its magnetic charge is fixed in Planckian units, and hence it is necessarily non-vanishing. This interesting feature results from the interplay between the sigma model manifold whose radius is fixed in units of Planckian length, as it is the case in almost all supergravities that contain sigma models, and the four dimensional space transverse to the string worldsheet through the identity map. The tear-drop is quaternionic but not quaternionic Kahler, since its metric is neither self-dual nor Einstein. The metric is conformally related to that of H_4 which has negative constant curvature, and its curvature scalar becomes positive due to the conformal factor. This switching of the sign is crucial for satisfying Einstein equation in the internal direction, just as in the case of 2-dimensional Gell-Mann-Zwiebach teardrop. We have also shown to have 1/4 supersymmetric $AdS_3 \times S^3$ near horizon limit where the radii are proportional to the electric charge. This is in contrast with the 1/4 supersymmetric GLPS dyonic string that approaches the product of AdS_3 times a squashed 3-sphere with 1/2 supersymmetry. In GLPS solution the squashing is necessarily non-vanishing for non-vanishing gauge coupling constant, while in our case the round 3-sphere emerges even in presence of nonvanishing gauge coupling.

CHAPTER IV

CONCLUSIONS

In this dissertation, we have presented results on various aspects of supergravity theories in six and seven dimensions. The remarkable properties of anomaly free gauged supergravities in six dimensions, and their possible connection with supergravities in seven dimensions have primarily motivated our work.

In seven dimensional spacetime, we have first determined the possible noncompact gaugings and we have used them to find the most general circle reduction that can yield chiral and *gauged* supergravity in six dimensions. We have obtained the resulting theories in six dimensions and furthermore we have made explicit the realization of the maximal symmetries in these theories. We have also shown that in a formulation of the $7D$ supergravity that uses a 3-form potential, vector multiplet coupling are possible even in the presence of a topological mass term, contrary to a claim made in the literature [8].

Our results are likely to be used in constructing anomaly-free models by means of a Horava-Witten-type formulation [38] of seven-dimensional supergravity on a spacetime with two boundaries. In this case, the symmetries of the theory impose chiral conditions for the fermions on the boundaries yielding a chiral $6D$ theory with an anomalous spectrum. Anomalies are canceled by introducing extra fields living on the boundaries and applying a special version of the Green-Schwarz mechanism [39]. The $6D$ theory of interest could possibly emerge in the limit of coinciding boundaries.

In the second part of the dissertation we have found the general form of the supersymmetric solutions in gauged $6D$, $N = (1,0)$ supergravity coupled to Yang-Mills and hypermultiplets. We do so by using G-structure method which involves a study of the Killing spinors and their integrability conditions. The attendant integrability conditions are then used to show that most of the field equations are satisfied automatically. In this framework, the existence of a null Killing vector suggests a 2+4 split of spacetime. We have determined exactly the remaining equations that need to be satisfied. Next, we have activated 4 hyperscalars parametrizing a 4 dimensional submanifold of a quaternionic hyperbolic ball and then employed an identity map between this submanifold and the internal space transverse to the string worldsheet. Thus, we have found a new 1/8 supersymmetric dyonic string solution with novel properties.

Our results may be used in classifying the supersymmetric solutions of gauged and matter coupled $N = (1, 0)$, $6D$ supergravity theories. Moreover, some of those solutions may play roles in brane-world [11, 40] and cosmology model building [41].

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APPENDIX A

THE DUAL GAUGED 7D MODEL WITH MATTER COUPLINGS AND TOPOLOGICAL MASS TERM*

The $7D$ gauged and matter coupled supergravity we have used in this dissertation is the theory with 2-form potential. For completeness, however, we will show the dual formulation of the gauged and matter coupled supergravity, in which the 2-form potential is dualized to a 3-form potential, even in the presence of couplings to an arbitrary number of vector multiplets. A further motivation for presenting our results here is to show that it is possible to perform this dualization contrary to a claim made in the literature [8]. We begin by adding a total derivative term to obtain the new Lagrangian as follows:

$$\mathcal{L}_3 = \mathcal{L} - \tfrac{1}{144}\epsilon^{\mu_1\cdots\mu_7} H_{\mu_1\cdots\mu_4}\left(G_{\mu_5\cdots\mu_7} + \tfrac{3}{\sqrt{2}}\omega^0_{\mu_5\cdots\mu_7}\right)\ , \tag{A.1}$$

where

$$H_{\mu\nu\rho\sigma} = 4\partial_{[\mu}C_{\nu\rho\sigma]}\ . \tag{A.2}$$

We can treat G as an independent field because the C-field equation will impose the correct Bianchi identity that implies the correct form of G given in the previous section. Thus, treating G as an independent field, its field equation gives

$$G_{\mu\nu\rho} = -\tfrac{1}{24}e^{-2\sigma}e\epsilon_{\mu\nu\rho\sigma_1\cdots\sigma_4}H^{\sigma_1\cdots\sigma_4} + \tfrac{i}{4\sqrt{2}}e^{-\sigma}X_{\mu\nu\rho}\ . \tag{A.3}$$

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Using this result in the Lagrangian given in (2.8), one finds

$$\begin{aligned}\mathcal{L}_3 &= \mathcal{L}' - \tfrac{1}{48} e e^{-2\sigma} H_{\mu\nu\rho\sigma} H^{\mu\nu\rho\sigma} - \tfrac{1}{48\sqrt{2}} \epsilon^{\mu_1\cdots\mu_7} H_{\mu_1\cdots\mu_4} \omega^0_{\mu_5\cdots\mu_7} \\ &\quad - \tfrac{i}{576\sqrt{2}} e^{-\sigma} \epsilon^{\mu_1\cdots\mu_7} H_{\mu_1\cdots\mu_4} X_{\mu_5\cdots\mu_7} \ , \qquad \text{(A.4)}\end{aligned}$$

where $\mathcal{L}'$ is the G-independent part of (2.8). For the readers convenience, we explicitly give the dual Lagrangian $\mathcal{L}_3 = \mathcal{L}_{3B} + \mathcal{L}_{3F}$ where

$$\begin{aligned}e^{-1}\mathcal{L}_{3B} &= \tfrac{1}{2} R - \tfrac{1}{4} e^{\sigma} a_{IJ} F^I_{\mu\nu} F^{\mu\nu J} - \tfrac{1}{48} e e^{-2\sigma} H_{\mu\nu\rho\sigma} H^{\mu\nu\rho\sigma} - \tfrac{1}{48\sqrt{2}} \epsilon^{\mu_1\cdots\mu_7} H_{\mu_1\cdots\mu_4} \omega^0_{\mu_5\cdots\mu_7} \\ &\quad - \tfrac{5}{8} \partial_\mu\sigma \partial^\mu\sigma - \tfrac{1}{2} P^{ir}_\mu P^\mu_{ir} - \tfrac{1}{4} e^{-\sigma} \left(C^{ir} C_{ir} - \tfrac{1}{9} C^2 \right) \ , \qquad \text{(A.5)}\end{aligned}$$

$$\begin{aligned}e^{-1}\mathcal{L}_{3F} &= -\tfrac{i}{2} \bar\psi_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \tfrac{5i}{2} \bar\chi \gamma^\mu D_\mu \chi - \tfrac{5i}{4} \bar\chi \gamma^\mu \gamma^\nu \psi_\mu \partial_\nu \sigma - \tfrac{1}{2} \bar\lambda^r \sigma^i \gamma^\mu \gamma^\nu \psi_\mu P_{\nu ri} \\ &\quad - \tfrac{i}{2} \bar\lambda^r \gamma^\mu D_\mu \lambda_r + \tfrac{i}{96\sqrt{2}} e^{\sigma} H_{\mu\nu\rho\sigma} X^{\mu\nu\rho\sigma} + \tfrac{1}{8} e^{\sigma/2} F^i_{\mu\nu} X^{\mu\nu}_i - \tfrac{i}{4} e^{\sigma/2} F^r_{\mu\nu} X^{\mu\nu}_r \\ &\quad - \tfrac{i\sqrt{2}}{24} e^{-\sigma/2} C \left(\bar\psi_\mu \gamma^{\mu\nu} \psi_\nu + 2\bar\psi_\mu \gamma^\mu \chi + 3\bar\chi\chi - \bar\lambda^r \lambda_r \right) \\ &\quad + \tfrac{1}{2\sqrt{2}} e^{-\sigma/2} C_{ir} \left(\bar\psi_\mu \sigma^i \gamma^\mu \lambda^r - 2\bar\chi \sigma^i \lambda^r \right) + \tfrac{1}{2} e^{-\sigma/2} C_{rsi} \bar\lambda^r \sigma^i \lambda^s \ , \qquad \text{(A.6)}\end{aligned}$$

and where the fermionic bilinears are defined as

$$\begin{aligned}X^{\mu\nu\rho\sigma} &= \bar\psi^\lambda \gamma_{[\lambda} \gamma^{\mu\nu\rho\sigma} \gamma_{\tau]} \psi^\tau + 4\bar\psi_\lambda \gamma^{\mu\nu\rho\sigma} \gamma^\lambda \chi - 3\bar\chi \gamma^{\mu\nu\rho\sigma} \chi + \bar\lambda^a \gamma^{\mu\nu\rho\sigma} \lambda_a \ , \\ X^{i\mu\nu} &= \bar\psi^\lambda \sigma^i \gamma_{[\lambda} \gamma^{\mu\nu} \gamma_{\tau]} \psi^\tau - 2\bar\psi_\lambda \sigma^i \gamma^{\mu\nu} \gamma^\lambda \chi + 3\bar\chi \sigma^i \gamma^{\mu\nu} \chi - \bar\lambda^r \sigma^i \gamma^{\mu\nu} \lambda_r \ , \\ X^{r\mu\nu} &= \bar\psi_\lambda \gamma^{\mu\nu} \gamma^\lambda \lambda^r + 2\bar\chi \gamma^{\mu\nu} \lambda^r \ . \qquad \text{(A.7)}\end{aligned}$$

The supersymmetry transformation rules are

$$\begin{aligned}\delta e_\mu{}^m &= i\bar\epsilon \gamma^m \psi_\mu \ , \\ \delta\psi_\mu &= 2D_\mu \epsilon - \tfrac{\sqrt{2}}{30} e^{-\sigma/2} C \gamma_\mu \epsilon \\ &\quad - \tfrac{1}{240\sqrt{2}} e^{-\sigma} H_{\rho\sigma\lambda\tau} \left(\gamma_\mu \gamma^{\rho\sigma\lambda\tau} + 5\gamma^{\rho\sigma\lambda\tau} \gamma_\mu \right) \epsilon - \tfrac{i}{20} e^{\sigma/2} F^i_{\rho\sigma} \sigma^i \left(3\gamma_\mu \gamma^{\rho\sigma} - 5\gamma^{\rho\sigma} \gamma_\mu \right) \epsilon \ ,\end{aligned}$$

$$\begin{aligned}
\delta\chi &= -\tfrac{1}{2}\gamma^\mu\partial_\mu\sigma\epsilon - \tfrac{i}{10}e^{\sigma/2}F^i_{\mu\nu}\,\sigma^i\gamma^{\mu\nu}\epsilon - \tfrac{1}{60\sqrt{2}}e^{-\sigma}H_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}\epsilon + \tfrac{\sqrt{2}}{30}e^{\sigma/2}C\epsilon\ , \\
\delta C_{\mu\nu\rho} &= e^\sigma\left(\tfrac{3i}{\sqrt{2}}\bar\epsilon\gamma_{[\mu\nu}\psi_{\rho]} - i\sqrt{2}\bar\epsilon\gamma_{\mu\nu\rho}\chi\right)\ , \qquad \text{(A.8)} \\
\delta\sigma &= -2i\bar\epsilon\chi\ ,
\end{aligned}$$

$$\begin{aligned}
\delta A^I_\mu &= -e^{-\sigma/2}\left(\bar\epsilon\sigma^i\psi_\mu + \bar\epsilon\sigma^i\gamma_\mu\chi\right)L^I_i + ie^{-\sigma/2}\bar\epsilon\gamma_\mu\lambda^r L^I_r\ , \\
\delta L^r_I &= \bar\epsilon\sigma^i\lambda^r L^i_I\ , \qquad \delta L^i_I = \bar\epsilon\sigma^i\lambda_r L^r_I\ , \qquad \text{(A.9)} \\
\delta\lambda^r &= -\tfrac{1}{2}e^{\sigma/2}F^r_{\mu\nu}\gamma^{\mu\nu}\epsilon + i\gamma^\mu P^{ir}_\mu\sigma^i\epsilon - \tfrac{i}{\sqrt{2}}e^{-\sigma/2}C^{ir}\sigma^i\epsilon\ .
\end{aligned}$$

The supersymmetry transformation rule for the 3-form potential can be obtained from the supersymmetry of the G-field equation (A.3). Indeed, it is sufficient to check the cancelation of the $\partial_\mu\epsilon$ terms to determine the supersymmetry variation of the 3-form potential. If we set to zero all the vector multiplet fields, the above Lagrangian and transformation rules become those of $SU(2)$ gauged pure half-maximal supergravity [42], which in turn admits a topological mass term for the 3-form potential in a supersymmetric fashion that involves a new constant parameter [42]. In [8], it has been argued that the gauged theory in presence of the coupling to vector multiplets does not admit a topological mass term. However, we have found that this is not the case. Indeed, we have found that one can add the following Lagrangian to $\mathcal{L}_3$ given in (A.4):

$$\begin{aligned}
e^{-1}\mathcal{L}_h &= \frac{he^{-1}}{36}\,\epsilon^{\mu_1\cdots\mu_7}H_{\mu_1\cdots\mu_4}C_{\mu_5\cdots\mu_7} + \tfrac{4\sqrt{2}}{3}he^{3\sigma/2}C - 16ih^2e^{4\sigma} \\
&\quad ihe^{2\sigma}\left(-\bar\psi_\mu\gamma^{\mu\nu}\psi_\nu + 8\bar\psi_\mu\gamma^\mu\chi + 27\bar\chi\chi - \bar\lambda^r\lambda_r\right)\ . \qquad \text{(A.10)}
\end{aligned}$$

Note that the coupling of matter to the model with topological mass term has led to the dressing up of the term $he^{3\sigma/2}$ present in that model by C as shown in the second term on the right hand side of (A.10). The second ingredient to make the supersymmetry work is the term $he^{2\sigma}\bar{\lambda}^r\lambda_r$ in (A.10)*. The action for the total Lagrangian

$$\mathcal{L}_{new} = \mathcal{L}_3 + \mathcal{L}_h \tag{A.11}$$

is invariant under the supersymmetry transformation rules described above with the following new h-dependent terms:

$$\begin{aligned} \delta_h\psi_\mu &= -\tfrac{4}{5}he^{2\sigma}\gamma_\mu\epsilon \ , \\ \delta_h\chi &= -\tfrac{16}{5}he^{2\sigma}\epsilon \ . \end{aligned} \tag{A.12}$$

For comparison with [42], we extract the potential and all the mass terms, and write it as

$$\begin{aligned} \Delta\mathcal{L} &= 60m^2 - 10\left(m + 2he^{2\sigma}\right)^2 + \tfrac{5im}{2}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu - 5i\left(m + 2he^{2\sigma}\right)\bar{\psi}\gamma^\mu\chi \\ &\quad +5i\left(\tfrac{3}{2}m + 6he^{2\sigma}\right)\bar{\chi}\chi - \tfrac{i}{2}\left(5m + 4he^{2\sigma}\right)\bar{\lambda}^r\lambda_r - \tfrac{1}{4}e^{-\sigma}C^{ir}C_{ir} \\ &\quad +\tfrac{1}{2\sqrt{2}}e^{-\sigma/2}C_{ir}\left(\bar{\psi}_\mu\sigma^i\gamma^\mu\lambda^r - 2\bar{\chi}\sigma^i\lambda^r\right) + \tfrac{1}{2}e^{-\sigma/2}C_{rsi}\bar{\lambda}^r\sigma^i\lambda^s \ , \end{aligned} \tag{A.13}$$

where we have defined

$$m = -\tfrac{1}{30\sqrt{2}}Ce^{-\sigma/2} - \tfrac{2}{5}he^{2\sigma} \ , \tag{A.14}$$

so that

$$\begin{aligned} \delta'\psi_\mu &= 2m\gamma_\mu\epsilon \ , \\ \delta'\chi &= -2(m + 2he^{2\sigma})\epsilon \ . \end{aligned} \tag{A.15}$$

In the absence of matter couplings, the above result has exactly the same structure as that of [42] but the coefficients differ, even after taking into account the appropriate

*The obstacle reported in [8] in coupling matter in presence of the topological terms may be due to the fact that these ingredients were not considered.

constant rescalings of fields and parameters due to convention differences.

APPENDIX B

THE MAP BETWEEN $SL(4,R)/SO(4)$ AND $SO(3,3)/SO(3)\times SO(3)^*$

Two of our noncompact gauged $7D$ supergravities, namely the $SO(2,1)$ and $SO(2,2)$ gauged models, have $SO(3,3)/SO(3)\times SO(3)$ σ-model sector. In particular, the $SO(2,2)$ gauged model has been reduced [4] to $6D$ to obtain Salam-Sezgin model. In the work of [4] the $SL(4,R)/SO(4)$ parametrization of the σ-model is used. Given that $SL(3,R)\sim SO(3,3)$ and $SO(4)\sim SO(3)\times SO(3)$, to compare our results, and for other possible future uses, it is useful to exhibit the relation between the two parametrizations. To do so, let us denote the $SL(4,R)/SO(4)$ coset representative by $\mathcal{V}_\alpha^R$ which is a 4×4 uniocular real matrix with inverse $\mathcal{V}_R^\alpha$:

$$\mathcal{V}_R^\alpha \mathcal{V}_\alpha^S = \delta_R^S\ , \qquad \alpha=1,...4, \quad R=1,...,4\ . \tag{B.1}$$

The map between $\mathcal{V}_\alpha^R$ and the $SO(3,3)/SO(3)\times SO(3)$ coset representative L_I^A can be written as

$$L_I^A = \tfrac{1}{4}\,\Gamma_I^{\alpha\beta}\,\eta_{RS}^A\,\mathcal{V}_\alpha^R\,\mathcal{V}_\beta^S \equiv \tfrac{1}{4}\,\mathcal{V}\Gamma_I\eta^A\mathcal{V}\ , \tag{B.2}$$

where Γ^I and η^A are the chirally projected $SO(3,3)$ Dirac matrices which satisfy [43]

$$(\Gamma^I)_{\alpha\beta}(\Gamma^J)^{\alpha\beta} = -4\eta^{IJ}\ , \qquad (\Gamma^I)_{\alpha\beta}(\Gamma_I)_{\gamma\delta} = -2\epsilon_{\alpha\beta\gamma\delta}\ , \tag{B.3}$$

where η_{IJ} as well as η_{AB} have signature $(---+++)$. Similar identities are satisfied by $(\eta^A)_{RS}$. Both Γ^I and η^A are antisymmetric. Pairs of antisymmetric indices are

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raised and lowered by the ϵ tensor:

$$V^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}V_{\gamma\delta} \ , \qquad V_{\alpha\beta} = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}V^{\gamma\delta} \ . \tag{B.4}$$

Since $\mathcal{V}$ is real, the Γ and η-matrices must be real as well. A convenient such representation is given by

$$\Gamma^I \equiv (\Gamma^I)_{\alpha\beta} = \begin{pmatrix} \alpha^i \\ \beta^r \end{pmatrix} \ , \quad (\Gamma^I)^{\alpha\beta} = \begin{pmatrix} \alpha^i \\ -\beta^r \end{pmatrix} \ , \tag{B.5}$$

where α^i and β^r are real antisymmetric 4×4 matrices that satisfy

$$\begin{aligned} \alpha_i\alpha_j &= \epsilon_{ijk}\ \alpha_k - \delta_{ij}\,\mathbb{1} \ , \qquad (\alpha^i)_{\alpha\beta} = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\,(\alpha^i)_{\gamma\delta} \ , \\ \beta_r\beta_s &= \epsilon_{rst}\ \beta_t - \delta_{rs}\,\mathbb{1} \ , \qquad (\beta^r)_{\alpha\beta} = -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\,(\beta^r)_{\gamma\delta} \ . \end{aligned} \tag{B.6}$$

Further useful identities are

$$(\alpha^i)_{\alpha\beta}\ (\alpha^i)_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} + \epsilon_{\alpha\beta\gamma\delta} \ , \tag{B.7}$$

$$\epsilon^{ijk}(\alpha^j)_{\alpha\beta}\ (\alpha^k)_{\gamma\delta} = \delta_{\beta\gamma}\ (\alpha^i)_{\alpha\delta} + 3 \text{ more} \ , \tag{B.8}$$

$$(\beta^r)_{\alpha\beta}\ (\beta^r)_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} - \epsilon_{\alpha\beta\gamma\delta} \ , \tag{B.9}$$

$$\epsilon^{trs}(\beta^r)_{\alpha\beta}\ (\beta^s)_{\gamma\delta} = \delta_{\beta\gamma}\ (\beta^t)_{\alpha\delta} + 3 \text{ more} \ . \tag{B.10}$$

Using the above relations and recalling that $\mathcal{V}$ is uniocular, it simple to verify that

$$L_I^I L_J^B \eta_{AB} = \eta_{IJ} \ , \qquad L_I^A L_J^B \eta^{IJ} = \eta^{AB} \ . \tag{B.11}$$

As a further check, let us compare the potential

$$V = \tfrac{1}{4}e^{-\sigma}\left(C^{ir}C_{ir} - \tfrac{1}{9}C^2\right) \tag{B.12}$$

for the $SO(4)$ gauged theory with that of [5] where it is represented in terms of the $SL(4,R)$ coset representative. To begin with, the function C can be written as

$$\begin{aligned} C &= -\tfrac{1}{\sqrt{2}} f_{IJ}{}^{K} L_i^I L_j^J L_{Kk}\, \epsilon^{ijk} \;, \\ &= -\tfrac{1}{64\sqrt{2}} f_{IJK} \left(\mathcal{V}\Gamma^I \eta_i \mathcal{V}\right) \left(\mathcal{V}\Gamma^J \eta_j \mathcal{V}\right) \left(\mathcal{V}\Gamma^K \eta_k \mathcal{V}\right)\, \epsilon^{ijk} \\ &= \tfrac{1}{8\sqrt{2}} f_{IJK} \left[(\Gamma^{IJK})_{\alpha\beta}\, T^{\alpha\beta} + (\Gamma^{IJK})^{\alpha\beta}\, T_{\alpha\beta} \right] \;, \end{aligned} \tag{B.13}$$

where

$$T_{\alpha\beta} = \mathcal{V}_\alpha^R\, \mathcal{V}_\beta^S\, \delta_{RS} \;, \qquad T^{\alpha\beta} = \mathcal{V}_R^\alpha\, \mathcal{V}_S^\beta\, \delta_{RS} \;. \tag{B.14}$$

In the last step we have used (B.8). In fact, the expression (3.59) is valid for any gauging, not withstanding the fact that the $SO(4)$ invariant tensor δ_{RS} occurs in (B.14). However, only for $SO(4)$ gauging in which the f_{IJK} refers to the $SO(4)$ structure constants, (3.59) simplifies to give a direct relation between C and $T = T^{\alpha\beta}\delta_{\alpha\beta}$ that is manifestly $SO(4)$ invariant, as will be shown below. To obtain a similar relation for gaugings other than $SO(4)$, for example $SO(2,2)$, we would need to construct the Γ and η matrices in a $SO(2,1)\times SO(2,1)$ basis with suitable changes in (C.6). In that case, the $SO(2,2)$ invariant tensor η_{RS} would replace the $SO(4)$ invariant tensor δ_{RS} in (B.14) and we could get a manifestly $SO(2,2)$ invariant direct relation between C and T. In the case of $SO(4)$ gauging we have $f_{IJK} = (\epsilon_{ijk}, -\epsilon_{rst})$. Using this in (3.59) we find that the ϵ_{ijk} term gives a contribution of the form $(\delta_{\alpha\beta}T^{\alpha\beta} + \delta^{\alpha\beta}T_{\alpha\beta})$, while the ϵ_{rst} term gives a contribution of the form $(\delta_{\alpha\beta}T^{\alpha\beta} - \delta^{\alpha\beta}T_{\alpha\beta})$. The $\delta^{\alpha\beta}T_{\alpha\beta}$ contributions cancel and we are left with

$$C = -\tfrac{3}{2\sqrt{2}}\, T \;, \qquad T \equiv T^{\alpha\beta}\, \delta_{\alpha\beta} \;. \tag{B.15}$$

Similarly, it follows from the definition of C^{ir} and the orthogonality relations satisfied by L_I^A that

$$C^{ir}C_{ir} = f_{IJK}f_{MN}{}^K\, L_i^I L_j^J L_i^M L_j^N + \tfrac{1}{3}C^2 \ . \tag{B.16}$$

Thus, it suffices to compute

$$\begin{aligned} f_{IJK}f_{MN}{}^K\, L_i^I L_j^J L_i^M L_j^N &= -\tfrac{1}{4}\left(\mathcal{V}\Gamma^i\eta^j\mathcal{V}\right)\left(\mathcal{V}\Gamma_i\eta_j\mathcal{V}\right) + 6 \\ &= \tfrac{1}{2}T_{RS}T^{RS} - \tfrac{1}{2}T^2 \ . \end{aligned} \tag{B.17}$$

Using the results (B.15), (B.16) and (B.17) in (B.12), we find

$$V = \tfrac{1}{8}e^{-\sigma}\left(T_{RS}T^{RS} - \tfrac{1}{2}T^2\right) \ , \tag{B.18}$$

which agrees with the result of [5]. In the case of $Sp(1)_R$ gauged $6D$ supergravity obtained from the $SO(3,1)$ gauged supergravity in $7D$, i.e. model II in section 1.2.2, we have $f_{IJK} = (-\epsilon_{rst}, -\epsilon_{ijr})$, where ϵ_{ijr} is totally antisymmetric and $\epsilon_{124} = \epsilon_{235} = \epsilon_{316} = 1$. For this case, the C-function has a more complicated form in terms of the $SL(4,R)$ coset representative $\mathcal{V}$. However, setting the scalar fields equal to zero, which is required for model II at hand, $\mathcal{V}$ becomes a unit matrix and the C-function vanishes. This is easily seen in the first line of (3.59), while it can be seen from the last line of (3.59) by noting that the ϵ_{rst} term gives the contribution $(\delta_{\alpha\beta}T^{\alpha\beta} - \delta^{\alpha\beta}T_{\alpha\beta})$, and the ϵ_{rsi} term give the structure $(\vec{\alpha}\cdot\vec{\beta})_{\alpha\beta}T^{\alpha\beta} + (\vec{\alpha}\cdot\vec{\beta})^{\alpha\beta}T_{\alpha\beta}$, where $\vec{\beta}$ refers to β_{r-3}, both of which vanish when $\mathcal{V}$ is taken to be a unit matrix. In the second term this is due to the fact that $\vec{\alpha}\cdot\vec{\beta}$ is traceless.

APPENDIX C

DUALIZATION OF THE AXIONS IN THE UNGAUGED $6D$ MODEL*

Our $SO(3,1), SO(3,2)$ and $SO(2,2)$ models in $7D$, when reduced to $6D$, feature couplings of $6D, N=(1,0)$ supergravity to p hypermultiplets whose scalar fields parametrize the coset $SO(p,3)/SO(p)\times SO(3)$, a dilaton and $(p+3)$ axions, for $p\leq 1$. We have shown how these fields can be combined to parametrize an enlarged coset $SO(p+1,4)/SO(p+1)\times SO(4)$ describing the scalars of $(p+1)$ hypermultiplets. Instead, one can proceed, at least in the ungauged model† by dualizing the $(p+3)$ axionic scalars to 4-form potentials to obtain the coupling of $(p+1)$ linear multiplets‡. In this appendix we will present this dualization. We start by adding the suitable total derivative term to this Lagrangian to define

$$\mathcal{L}_4 = \mathcal{L}_B + \mathcal{L}_F + \tfrac{1}{5!\sqrt{6}}\,\epsilon^{\mu_1\cdots\mu_6}\left(-H^i_{\mu_1\cdots\mu_5}\mathcal{P}^i_{\mu_6} + H^r_{\mu_1\cdots\mu_5}\mathcal{P}^r_{\mu_6}\right)e^{-\varphi}\ , \tag{C.1}$$

where the definitions (2.69) are to be used without the gauge coupling constants. Recalling that (2.4) holds, the Φ^I field equation implies $dH^I_5=0$ with $H^i_5 = H^I_5 L_I^i$ and $H^r_5 = H^I_5 L^r_I$, which means that locally

$$H^I_{\mu_1\cdots\mu_5} = 5\partial_{[\mu_1}C^I_{\mu_2\cdots\mu_5]}\ , \qquad I=1,...,p+3\ . \tag{C.2}$$

---

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†In gauged model, an obstacle occurs to dualization since the axions are charged and therefore minimally couple to gauge fields. See Eq. (2.69).

‡A linear multiplet consists of a 4-form potential, 3 scalars and one symplectic-Majorana-Weyl spinor.

Solving for $(\mathcal{P}^i_\mu, P^r_\mu)$ gives

$$\begin{aligned}
\mathcal{P}^i_\mu &= \tfrac{\sqrt{2}}{5!}\,\epsilon^{\mu\nu_1\cdots\nu_5} H^i_{\nu_1\cdots\nu_5} - \bar{\psi}\gamma^\nu\gamma_\mu\sigma^i\psi_\nu - \tfrac{1}{2}X^i_\mu \ , \\
\mathcal{P}^r_\mu &= -\tfrac{\sqrt{2}}{5!}\,\epsilon^{\mu\nu_1\cdots\nu_5} H^i_{\nu_1\cdots\nu_5} + i\bar{\psi}^r\gamma^\nu\gamma_\mu\sigma^i\psi_\nu - 2iX^r_\mu \ . \qquad \text{(C.3)}
\end{aligned}$$

Substituting these back into the Lagrangian (C.1), we get

$$\begin{aligned}
\mathcal{L}_4 &= \mathcal{L}' - \tfrac{1}{2\times 5!}e^{-2\varphi}H^i_{\mu_1\cdots\mu_5}H^{i\mu_1\cdots\mu_5} - \tfrac{1}{2\times 5!}e^{-2\varphi}H^r_{\mu_1\cdots\mu_5}H^{r\mu_1\cdots\mu_5} \\
&\quad -\tfrac{1}{5!\sqrt{2}}e^{-\varphi}\epsilon^{\mu\nu_1\cdots\nu_5}H^i_{\nu_1\cdots\nu_5}\left(\bar{\psi}\gamma^\nu\gamma_\mu\sigma^i\psi_\nu + \tfrac{1}{2}X^i_\mu\right) \\
&\quad -\tfrac{1}{5!\sqrt{2}}e^{-\varphi}\epsilon^{\mu\nu_1\cdots\nu_5}H^r_{\nu_1\cdots\nu_5}\left(i\bar{\psi}^r\gamma^\nu\gamma_\mu\sigma^i\psi_\nu - 2iX^r_\mu\right) \ , \qquad \text{(C.4)}
\end{aligned}$$

where $\mathcal{L}'$ is the $(\mathcal{P}^i_\mu, P^r_\mu)$ independent part of $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$ with $\mathcal{L}_B$ and $\mathcal{L}_F$ given in (2.63) and (2.64). Thus, we have $\mathcal{L}_4 = \mathcal{L}_{4B} + \mathcal{L}_{4F}$ with

$$\begin{aligned}
e^{-1}\mathcal{L}_{4B} &= \tfrac{1}{4}R - \frac{1}{4}(\partial_\mu\sigma)^2 - \tfrac{1}{12}e^{2\sigma}G_{\mu\nu\rho}G^{\mu\nu\rho} - \tfrac{1}{8}e^\sigma F^{r'}_{\mu\nu}F^{\mu\nu r'} \qquad \text{(C.5)} \\
&\quad -\tfrac{1}{4}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{8}P^{ir}_\mu P^\mu_{ir} - \tfrac{1}{2\times 5!}e^{-2\varphi}a_{IJ}H^I_{\mu_1\cdots\mu_5}H^{J\mu_1\cdots\mu_5} \ , \\
e^{-1}\mathcal{L}_F &= -\tfrac{i}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \tfrac{i}{2}\bar{\chi}\gamma^\mu D_\mu\chi - \tfrac{i}{2}\bar{\lambda}^{r'}\gamma^\mu D_\mu\lambda_{r'} \\
&\quad -\tfrac{i}{2}\bar{\psi}\gamma^\mu D_\mu\psi - \tfrac{i}{2}\bar{\psi}^r\gamma^\mu D_\mu\psi^r - \tfrac{i}{2}\bar{\chi}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\sigma \\
&\quad -\tfrac{1}{2}\bar{\psi}^r\gamma^\mu\gamma^\nu\sigma_i\psi_\mu P^{ir}_\nu + \tfrac{i}{2}\bar{\psi}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\varphi \\
&\quad -\tfrac{1}{5!\sqrt{2}}e^{-\varphi}H^I_{\mu_1\cdots\mu_5}\left(\bar{\psi}\gamma^\nu\gamma_{\mu_1\cdots\mu_5}\sigma^i\psi_\nu\, L^i_I + \bar{\psi}^r\gamma^\nu\gamma_{\mu_1\cdots\mu_5}\sigma^i\psi_\nu\, L^r_I\right) \\
&\quad +\tfrac{i}{24}e^\sigma G_{\mu\nu\rho}X^{\mu\nu\rho} - \tfrac{i}{4}e^{\sigma/2}F^{r'}_{\mu\nu}X^{\mu\nu}_{r'} + \tfrac{1}{2\sqrt{2}\times 5!}e^{-\varphi}H^I_{\mu_1\cdots\mu_5}X^{\mu_1\cdots\mu_5}_I \ , \qquad \text{(C.6)}
\end{aligned}$$

where the structure constants (hence the C-functions as well) are to be set to zero in the definitions (2.67) and (2.68), and

$$
\begin{aligned}
X^{\mu\nu\rho} &= \bar{\psi}^{\lambda}\gamma_{[\lambda}\gamma^{\mu\nu\rho}\gamma_{\tau]}\psi^{\tau} + 2\bar{\psi}_{\lambda}\gamma^{\mu\nu\rho}\gamma^{\lambda}\chi - 2\bar{\chi}\gamma^{\mu\nu\rho}\chi + \bar{\lambda}^{r'}\gamma^{\mu\nu\rho}\lambda_{r'} + \bar{\psi}^{r}\gamma^{\mu\nu\rho}\psi_{r} + \bar{\psi}\gamma^{\mu\nu\rho}\psi \ , \\
X^{\mu\nu}_{r'} &= \bar{\psi}_{\rho}\gamma^{\mu\nu}\gamma^{\rho}\lambda_{r'} + \bar{\chi}\gamma^{\mu\nu}\lambda_{r'} \ , \\
X^{\mu_1\cdots\mu_5}_{I} &= L^{i}_{I}\,(\bar{\psi}^{\rho}\gamma_{[\rho}\gamma^{\mu_1\cdots\mu_5}\gamma_{\tau]}\sigma_i\psi^{\tau} + 2\bar{\chi}\gamma^{\mu_1\cdots\mu_5}\sigma_i\chi + \bar{\lambda}^{r'}\gamma^{\mu_1\cdots\mu_5}\sigma_i\lambda_{r'} \\
&\quad -\bar{\psi}^{r}\gamma^{\mu_1\cdots\mu_5}\sigma_i\psi_r - \bar{\psi}\gamma^{\mu_1\cdots\mu_5}\sigma_i\psi) - 4iL^{r}_{I}\,\bar{\psi}\gamma^{\mu_1\cdots\mu_5}\psi_r \ . \qquad \text{(C.7)}
\end{aligned}
$$

The action is invariant under the following supersymmetry transformations:

$$
\begin{aligned}
\delta e^{m}_{\mu} &= i\bar{\epsilon}\gamma^{m}\psi_{\mu} \ , \\
\delta\psi_{\mu} &= D_{\mu}\epsilon - \tfrac{1}{24}e^{\sigma}\gamma^{\rho\sigma\tau}\gamma_{\mu}G_{\rho\sigma\tau}\,\epsilon - \tfrac{i}{5!\sqrt{6}}e^{-\varphi}\gamma^{\mu\nu_1\cdots\nu_5}H^{i}_{\nu_1\cdots\nu_5}\epsilon \ , \\
\delta\chi &= -\tfrac{1}{2}\gamma^{\mu}\partial_{\mu}\sigma\epsilon - \tfrac{1}{12}e^{\sigma}\gamma^{\rho\sigma\tau}G_{\rho\sigma\tau}\,\epsilon \ , \\
\delta B_{\mu\nu} &= ie^{-\sigma}\left(\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} + \tfrac{1}{2}\bar{\epsilon}\gamma_{\mu\nu}\chi\right) - A^{r'}_{[\mu}\delta A^{r'}_{\nu]}, \\
\delta\sigma &= -i\bar{\epsilon}\chi \ , \\
\delta A^{r'}_{\mu} &= ie^{-\sigma/2}\bar{\epsilon}\gamma_{\mu}\lambda^{r'} \ , \\
\delta\lambda^{r'} &= -\tfrac{1}{4}e^{\sigma/2}\gamma^{\mu\nu}F^{r'}_{\mu\nu}\epsilon \ , \\
\delta C^{I}_{\mu_1\cdots\mu_4} &= -\tfrac{1}{\sqrt{2}}\left(\bar{\epsilon}\gamma_{\mu_1\cdot\mu_4}\sigma^{i}\psi + 4\bar{\epsilon}\gamma_{[\mu_1\cdot\mu_3}\sigma^{i}\psi_{\mu_4]}\right)L^{I}_{i} - \tfrac{i}{\sqrt{2}}\bar{\epsilon}\gamma_{\mu_1\cdot\mu_4}\psi^{r}L^{I}_{r} \ , \\
L^{I}_{i}\delta L^{r}_{I} &= -\bar{\epsilon}\sigma_i\psi^{r} \ , \\
\delta\varphi &= i\bar{\epsilon}\psi, \\
\delta\psi &= \tfrac{1}{2}\gamma^{\mu}\partial_{\mu}\varphi\epsilon + \tfrac{i}{5!\sqrt{2}}e^{-\varphi}\gamma^{\mu_1\cdots\mu_5}H^{i}_{\mu_1\cdots\mu_5}\epsilon \ , \\
\delta\psi^{r} &= \tfrac{i}{2}\gamma^{\mu}P^{ir}_{\mu}\sigma_i - \tfrac{i}{5!\sqrt{2}}e^{-\varphi}\gamma^{\mu_1\cdots\mu_5}H^{r}_{\mu_1\cdots\mu_5}\epsilon \ . \qquad \text{(C.8)}
\end{aligned}
$$

The supersymmetry transformation rule for $C^I_{\mu_1\cdots\mu_4}$ is derived from the requirement of supercovariance of (C.3), which requires the cancelation of the $\partial_\mu\epsilon$ terms.

APPENDIX D

THE IWASAWA DECOMPOSITION OF $SO(P,Q)^*$

This appendix contains some useful formula on the Iwasawa decomposition of $SO(p,q)$ that is used in showing the hidden quaternionic Kahler coset structure in six dimensional model. We begin with the Iwasawa decomposition of the $SO(n+3,3)$ algebra as $g = h \oplus a \oplus n$ where

$$\begin{aligned} h: &\qquad X_{ij}\ , Y_{ij}\ , Z_{ir}\ , T_{rs}\ , \\ a: &\qquad H_i\ , \\ n: &\qquad E_i{}^j\ , V^{ij}\ , U_{ir}\ , \quad i>j\ . \end{aligned} \tag{D.1}$$

Here $X = E - E^T$, $Y = V - V^T$, $Z = U - U^T$, together with the $SO(n)$ generators $T_{rs} = -T_{sr}$ form the maximal compact subalgebra $\{h\}$ of $SO(n+3) \times SO(3)$. Furthermore, $\{a\}$ are the noncompact Cartan generators and $\{n\}$ are the remaining noncompact generators of $SO(n+3,3)$. The generators $a \oplus n$ form the solvable subalgebra of $SO(n+3,3)$, and can be represented as (see, for example, [25])

$$\vec{H} = \left(\begin{array}{c|c|c} \sum_i \vec{c}_i\, e_{ii} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & -\sum_i \vec{c}_i\, e_{ii} \end{array}\right)\ , \qquad E_i{}^j = \left(\begin{array}{c|c|c} -e_{ji} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & e_{ij} \end{array}\right)\ ,$$

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$$V^{ij} = \left(\begin{array}{c|c|c} 0 & 0 & e_{ij} - e_{ji} \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) , \qquad U_r^i = \left(\begin{array}{c|c|c} 0 & e_{ir} & 0 \\ \hline 0 & 0 & e_{ri} \\ \hline 0 & 0 & 0 \end{array}\right) . \tag{D.2}$$

The maximal compact subalgebra generators are then represented as

$$X_{ij} = \left(\begin{array}{c|c|c} e_{ij} - e_{ji} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & e_{ij} - e_{ij} \end{array}\right) , \qquad Y_{ij} = \left(\begin{array}{c|c|c} 0 & 0 & e_{ij} - e_{ji} \\ \hline 0 & 0 & 0 \\ \hline e_{ij} - e_{ji} & 0 & 0 \end{array}\right) ,$$

$$Z_{ir} = \left(\begin{array}{c|c|c} 0 & e_{ir} & 0 \\ \hline -e_{ri} & 0 & e_{ri} \\ \hline 0 & -e_{ir} & 0 \end{array}\right) , \qquad T_{rs} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & e_{rs} - e_{sr} & 0 \\ \hline 0 & 0 & 0 \end{array}\right) . \tag{D.3}$$

Each e_{ab} is defined to be a matrix of the appropriate dimensions that has zeros in all its entries except for a 1 in the entry at row a and column b. These satisfy the matrix product rule $e_{ab}\, e_{cd} = \delta_{bc}\, e_{ad}$. The solvable subalgebra of $SO(n+3,3)$ has the nonvanishing commutators

$$\begin{aligned}
&[\vec{H}, E_i{}^j] = \vec{b}_{ij}\ E_i{}^j\ , \qquad [\vec{H}, V^{ij}] = \vec{a}_{ij}\ V^{ij}\ , \qquad [\vec{H}, U_r^j] = \vec{c}_i\ U_r^i\ , \\
&[E_i{}^j, E_k{}^\ell] = \delta_k^j E_i{}^\ell - \delta_i^\ell E_k{}^j\ , \\
&[E_i{}^j, V^{k\ell}] = -\delta_i^k V^{j\ell} - \delta_i^\ell V^{kj}\ , \qquad [E_i{}^j, U_r^k] = -\delta_i^k U_r^j\ , \\
&[U_r^i, U_s^j] = \delta_{rs} V^{ij}\ ,
\end{aligned} \tag{D.4}$$

where the structure constants are given by

$$\vec{b}_{ij} = \sqrt{2}\ (-\vec{e}_i + \vec{e}_j)\ , \qquad \vec{a}_{ij} = \sqrt{2}\ (\vec{e}_i + \vec{e}_j)\ , \qquad \vec{c}_i = \sqrt{2}\ \vec{e}_i\ . \tag{D.5}$$

The nonvanishing commutation rules of the maximal compact subalgebra $SO(n+3) \oplus SO(3)$ are

$$[X_{ij}, X_{k\ell}] = \delta_{jk} X_{i\ell} + 3\ perms\ , \qquad [T_{pq}, T_{rs}] = \delta_{qr} T_{ps} + 3\ perms\ , \tag{D.6}$$

$$[X_{ij}, Y_{k\ell}] = \delta_{jk} Y_{i\ell} + 3\ perms\ , \qquad [X_{ij}, Z_{kr}] = \delta_{jk} Z_{ir} - \delta_{ik} Z_{jr}\ ,$$

$$[Y_{ij}, Y_{k\ell}] = \delta_{jk} X_{i\ell} + 3\ perms\ , \qquad [T_{pq}, Z_{ir}] = \delta_{qr} Z_{ip} - \delta_{pr} Z_{iq}\ ,$$

$$[Z_{ir}, Z_{js}] = -\delta_{rs} X_{ij} + \delta_{rs} Y_{ij} - 2\delta_{ij} T_{rs}\ , \qquad [Y_{ij}, Z_{kr}] = -\delta_{jk} Z_{ir} + \delta_{ik} Z_{jr}\ .$$

APPENDIX E

THE GAUGED MAURER-CARTAN FORM AND THE C-FUNCTIONS*

For the purposes of chapter III, section D, where our dyonic string solution is described, we describe in this appendix various aspects of the coset $Sp(n_H,1)/Sp(n_H)\times Sp(1)$. A convenient choice for the coset representative L is [44]

$$L = \gamma^{-1}\begin{pmatrix} 1 & t^\dagger \\ & \\ t & \Lambda \end{pmatrix} \tag{E.1}$$

where t is an n_H-component quaternionic vector t^p $(p = 1, ..., n_H)$, and

$$\gamma = (1 - t^\dagger\, t)^{1/2}\ , \qquad \Lambda = \gamma\,(I - t\, t^\dagger)^{-1/2}\ . \tag{E.2}$$

Here, I is an $n_H \times n_H$ unit matrix, and $\dagger$ refers to quaternionic conjugation, and it can be verified that $\Lambda t = t$. The gauged Maurer-Cartan form is defined as

$$L^{-1}D_\mu L = \begin{pmatrix} Q_\mu & P_\mu^\dagger \\ & \\ P_\mu & Q'_\mu \end{pmatrix}\ , \tag{E.3}$$

where $D_\mu L$ is given in (3.25), with T^r representing three anti-hermitian quaternions (in the matrix representation of quaternions $T^r = -i\,\sigma^r/2$) obeying

$$[T^r, T^s] = \epsilon^{rst} T^t \tag{E.4}$$

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and $T^{I'}$ represents a subset of $n_H \times n_H$ quaternion valued anti-hermitian matrices spanning the algebra of the subgroup $K \subset Sp(n_H)$ that is being gauged. A direct computation gives

$$Q_\mu = \frac{1}{2}\gamma^{-2}\left(D_\mu t^\dagger t - t^\dagger D_\mu t\right) - A_\mu^r T^r \tag{E.5}$$

$$Q'_\mu = \gamma^{-2}\left(-tD_\mu t^\dagger + \Lambda D_\mu \Lambda + \tfrac{1}{2}\partial_\mu(t^\dagger t)I\right) - A_\mu^{I'} T^{I'} \ , \tag{E.6}$$

$$P_\mu = \gamma^{-2}\Lambda D_\mu t \ , \tag{E.7}$$

where

$$D_\mu t = \partial_\mu t + t\, T^r A_\mu^r - A_\mu^{I'} T^{I'}\, t \ . \tag{E.8}$$

The C functions are easily computed to yield

$$C^r = L^{-1} T^r L = \gamma^{-2}\begin{pmatrix} T^r & T^r t^\dagger \\ -tT^r & -tT^r t^\dagger \end{pmatrix} \tag{E.9}$$

$$C^{I'} = L^{-1} T^{I'} L = \gamma^{-2}\begin{pmatrix} -t^\dagger T^{I'} t & -t^\dagger T^{I'} \Lambda \\ \Lambda T^{I'} t & \Lambda T^{I'} \Lambda \end{pmatrix} \tag{E.10}$$

APPENDIX F

THE MODEL FOR $SP(1,1)/SP(1)\times SP(1)_R$*

In chapter III, section D, we activated four hyperscalars which parametrize the coset $Sp(1,1)/Sp(1)\times Sp(1)_R$ in finding our dyonic string solution. Here, we summarize the relevant part of the Lagrangian and supersymmetry transfornation rules in which the scalar couplings are governed by this coset. This coset, which is equivalent to $SO(4,1)/SO(4)$, represents a 4-hyperboloid H_4. In this case we have a single quaternion $t=\phi^\alpha\,\sigma_\alpha$, and the vielbein becomes

$$V_\alpha^{A'A}=\gamma^{-2}\,\sigma_\alpha^{A'A}\ . \tag{F.1}$$

It follows from the definitions (3.22) and (3.23) that

$$g_{\alpha\beta}=\frac{2}{(1-\phi^2)^2}\,\delta_{\alpha\beta}\ ,\qquad J^r_{\alpha\beta}=\frac{2\,\rho^r_{\alpha\beta}}{(1-\phi^2)^2}\ . \tag{F.2}$$

We also introduce a basis in the tangent space of H_4

$$V_\alpha{}^{\underline{\alpha}}=\frac{\sqrt{2}}{1-\phi^2}\,\delta_\alpha^{\underline{\alpha}}\ . \tag{F.3}$$

The $Sp(1)_R$ connection Q^r_μ can be found from (E.5) as

$$Q^r_\mu=-2\,\mathrm{tr}\,(Q_\mu T^r)\ =\frac{1}{1-\phi^2}\,\left(2\rho^r_{\alpha\beta}\partial_\mu\phi^\alpha\,\phi^\beta-A^r_\mu\right)\ . \tag{F.4}$$

With the above results at hand, the Lagrangian can be written as

$$e^{-1}\mathcal{L}\ =\ R-\tfrac14(\partial\varphi)^2-\tfrac12 e^{\varphi}\,G_{\mu\nu\rho}G^{\mu\nu\rho}-\tfrac14\,e^{\frac12\varphi}\,F^r_{\mu\nu}\,F^{r\mu\nu}\ -\tfrac14\,e^{\frac12\varphi}\,F^{r'}_{\mu\nu}\,F^{r'\mu\nu}$$

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$$-\frac{4}{(1-\phi^2)^2}\, D_\mu\phi^\alpha D^\mu\phi^\beta\,\delta_{\alpha\beta} - \frac{6e^{-\frac{1}{2}\varphi}}{(1-\phi^2)^2}\,\left[g_R^2 + g'^2(\phi^2)^2\right]\ , \tag{F.5}$$

where the covariant derivatives are defined as

$$D_\mu\phi^\alpha = \partial_\mu\phi^\alpha - \tfrac{1}{2}g_R A_\mu^r(\rho^r)^\alpha{}_\beta\ \phi^\beta - \tfrac{1}{2}g' A_\mu^{r'}(\eta^{r'})^\alpha{}_\beta\ \phi^\beta, \tag{F.6}$$

and we have re-introduced the gauge coupling constants g_R and g'. The supersymmetry transformation rules are

$$\delta\psi_\mu = D_\mu\varepsilon + \tfrac{1}{48}e^{\frac{1}{2}\varphi}G^+_{\nu\sigma\rho}\,\Gamma^{\nu\sigma\rho}\,\Gamma_\mu\,\varepsilon\ , \tag{F.7}$$

$$\delta\chi = \tfrac{1}{4}\left(\Gamma^\mu\partial_\mu\varphi - \tfrac{1}{6}e^{\frac{1}{2}\varphi}G^-_{\mu\nu\rho}\,\Gamma^{\mu\nu\rho}\right)\varepsilon\ , \tag{F.8}$$

$$\delta\lambda^r_A = -\tfrac{1}{8}F^r_{\mu\nu}\Gamma^{\mu\nu}\varepsilon_A - g_R\frac{e^{-\frac{1}{2}\varphi}}{1-\phi^2}\,T^r_{AB}\ \varepsilon^B\ , \tag{F.9}$$

$$\delta\lambda^{r'}_A = -\tfrac{1}{8}F^{r'}_{\mu\nu}\Gamma^{\mu\nu}\varepsilon_A + g'e^{-\frac{1}{2}\varphi}\frac{\phi^\alpha\phi^\beta}{1-\phi^2}\,(\bar\sigma_\alpha T^{r'}\sigma_\beta)_{AB}\ \varepsilon^B\ , \tag{F.10}$$

$$\delta\psi^{A'} = \frac{1}{1-\phi^2}\,D_\mu\phi^\alpha\,\sigma_a^{A'A}\,\varepsilon_A\ , \tag{F.11}$$

where $D_\mu\varepsilon_A = \nabla_\mu\varepsilon_A + Q^r_\mu(T^r)_A{}^B\varepsilon_B$, with ∇_μ containing the standard torsion-free Lorentz connection only, and Q^r is defined in (F.4).

VITA

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