Research Article

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ΛCDM and the principle of equivalence

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Abstract: There is growing evidence that the net acceleration of the Universe over its entire history is essentially zero. This finding is critical in light of a recent examination of the Friedmann–Lemaître–Robertson–Walker (FLRW) metric using the local flatness theorem (LFT) in general relativity, which argues that FLRW is consistent with the principle of equivalence only if the total energy density ρ and pressure p satisfy the zero active mass condition, $\rho + 3p = 0$. This equation-of-state produces zero acceleration, and significantly mitigates the growing tension between lambda cold dark matter (ΛCDM) and the ever-improving observations. This article takes an alternative approach to this critical issue and directly tests the expansion rate predicted by the standard model against the requirements of the LFT. It demonstrates that ΛCDM simply does not satisfy the principle of equivalence. Some of the many important consequences of this outcome are discussed in the conclusions.

Keywords: classical theories of gravity, cosmological theory, early universe, inflation, dark energy

1 Introduction

In spite of its success in accounting for many cosmological observations over the past three decades, the standard model lambda cold dark matter (ΛCDM) [1] is imperfect. Indeed, the tension between its predictions and the data grows as the measurement precision continues to improve. For example, the Hubble constant, H_0, which characterizes the cosmic expansion rate and its absolute distance scale, cannot be determined self-consistently from measurements of the cosmic microwave background (67.4 ± 0.5 km s^{-1} Mpc^{-1}; [2]) and local Type Ia supernovae calibrated with the Cepheid distance ladder (74.03 ± 1.42 km s^{-1} Mpc^{-1}; [3]). Each new generation of instruments seems to compound this 4–5σ disparity rather than eliminate it.

ΛCDM is comprehensively based on the cosmological principle, whose symmetries inform the Friedmann–Lemaître–Robertson–Walker (FLRW) metric (see, *e.g.*, [4,5]), one of the most famous solutions to Einstein's equations. As a special member of the class of spherically symmetric spacetimes often used in problems of gravitational collapse or expansion [6–9], FLRW shares many of its brethren's characteristics, except for one critical difference: the dynamical equations used to describe the Universe's expansion, such as the Raychaudhuri [10] (or "acceleration") Eq. (2), are derived *after* homogeneity and isotropy are introduced to greatly simplify the metric. This procedure thus ignores the possible dependence of at least some of the coefficients on the chosen stress–energy tensor.

FLRW is conventionally written in the form

$$ds^2 = c^2dt^2 - a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2d\Omega^2\right), \tag{1}$$

where $a(t)$ is the expansion factor as a function of cosmic time t, the comoving spatial coordinates (r, θ, ϕ) remain "fixed" for all particles moving with the Hubble flow, and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. The spatial curvature constant k, which can take on the value +1 for a closed universe, 0 for a flat universe, and −1 for an open universe, is now believed to be very close to – if not exactly – zero [2].

The most notable feature of Eq. (1) that we shall address in this article is the so-called "lapse" function, g_{tt} which, quite remarkably, is assumed to be one, reflecting *free-fall* conditions throughout the cosmos, without any confirmation that this choice is consistent with an accelerated expansion ($\ddot{a} \neq 0$). But ΛCDM predicts various phases of acceleration and deceleration, so the Hubble flow is not inertial in this model. One should, therefore, question whether FLRW – with a constant $g_{tt} = 1$ – can adequately handle the dynamics in standard cosmology. We shall demonstrate here that it actually does not.

2 The local flatness theorem (LFT)

The motivation for seriously raising this issue now stems from the combined assessment of observational and theoretical clues pointing to an expansion history of the

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Universe consistent with net zero acceleration. By now, over 27 different tests have been completed, showing that ΛCDM is a better fit to the data with the inclusion of the so-called zero active mass condition in general relativity, *i.e.*, $\rho + 3p = 0$, in terms of the total energy density ρ and pressure p. A summary of this work is provided in Table 2 of the study by Melia [11]. A more complete compilation and discussion are available in the study by Melia [5]. As one can easily see from the Raychaudhuri equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3p), \tag{2}$$

zero active mass produces zero acceleration, *i.e.*, $\ddot{a} = 0$.

From a theoretical standpoint, a careful examination [12] of the LFT in general relativity seems to suggest that the use of FLRW is in fact valid only when $\rho + 3p = 0$ (and the much less relevant case of $\rho = p = 0$, consistent with Minkowski space). The LFT is a mathematical formulation of the Principle of Equivalence (PoE) [4], stating that there exists – at each spacetime point x_0^μ – a local, inertial (*i.e.*, free-falling) frame, $\xi^\mu(x_0)$, against which one may "measure" the spacetime curvature in the observer's frame. The coordinates x^μ and ξ^μ must satisfy the equations [4]

$$\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} = \Gamma^\lambda{}_{\mu\nu}\frac{\partial \xi^\alpha}{\partial x^\lambda}, \tag{3}$$

in terms of the Christoffel symbols,

$$\Gamma^\lambda{}_{\mu\nu} \equiv \frac{1}{2}g^{\alpha\lambda}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\alpha}\right\}, \tag{4}$$

describing the spacetime curvature.

It is not difficult to understand why Eq. (3) must be satisfied in order for a metric (such as FLRW) to be consistent with the chosen stress–energy tensor, $T^{\mu\nu}$. The metric coefficients account for the spacetime curvature expressed in Einstein's equations (the "fine marble" on the left-hand side, as he described it) generated by the source $T^{\mu\nu}$. As is well known, absolute velocity is not measurable in general relativity, but, in contrast, acceleration is known absolutely with respect to a local inertial frame. Unfortunately, the identification of the latter often gives rise to some confusion, because the coordinates ξ^α appearing in Eq. (3) are defined continuously throughout spacetime, not just locally at any given point, x_0^μ.

What is meant here, however, is not that the coordinates ξ^α represent the local free-falling frame everywhere but that, while they are continuously defined everywhere, they nevertheless correspond to the local free-falling frame only at x_0^μ. The only known FLRW solution for which this is not true, *i.e.*, for which a single set of ξ^μ coordinates does in fact constitute a unique free-falling frame validly defined everywhere, is the Milne universe. Only in this case is there a single definition of ξ^μ independent of the chosen point x_0^μ.

But Eq. (3) needs to be satisfied only at the point x_0^μ where the defined coordinates ξ^μ represent the local free-falling frame. This equation simply states that the spacetime curvature at that point must be measurable relative to the free-falling frame. Without this condition, the effects of gravity would not be equivalent to a transformation between accelerated frames, thereby violating the fundamental premise behind the derivation of Einstein's equations. For this reason, there does not appear to be any possibility of the FLRW metric being a viable description of cosmic spacetime while not satisfying Eq. (3).

According to the LFT, g_{tt} in FLRW is not automatically one. It depends on the equation-of-state and must instead satisfy the following constraint [12]:

$$\int^{ct}\sqrt{g_{tt}(t')}\,\mathrm{d}(ct') = cg_{tt}(t)\frac{a}{\dot{a}}. \tag{5}$$

(Note that we have corrected the missing square-root sign in the original expression.) The choice $g_{tt} = 1$ is therefore consistent only with the expansion profile $a(t) \propto t$ (and the much less relevant Minkowski space solution with $a=$ constant).

With the outcome expressed in Eq. (5), it would therefore appear that any expansion profile with $\ddot{a} \neq 0$ would be inconsistent with the LFT and ought to be rejected on the basis that it does not satisfy the PoE. The goal of this article is to demonstrate this result in reverse. We shall focus specifically on ΛCDM, take its predicted form of $a(t)$, and prove that it is inconsistent with Eq. (3).

3 ΛCDM and the LFT

Throughout its evolutionary history, the equation-of-state in ΛCDM has been dominated by one component, either radiation (and possibly an inflaton field) early on, or matter up to the present, and perhaps a cosmological constant into our future. The effect on $a(t)$ due to these changing conditions may easily be seen in Figure 1, which shows $a(t)$ *vs* t for the standard model parameters, including the hypothesized inflationary spurt at ~10^{-37} s. At least for radiation (with $p_\mathrm{r} = \rho_\mathrm{r}/3$) and matter (with $p_\mathrm{m} \approx 0$), the corresponding dynamical expansion predicted by Eq. (2) is well represented by the quantity

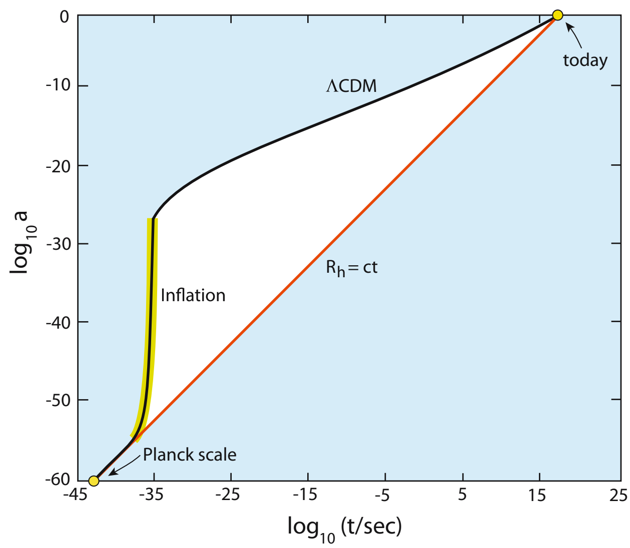

Figure 1: The expansion factor, $a(t)$, as a function of t for the standard model parameters (black), in comparison with a linear expansion (red) corresponding to $\beta = 1$ in Eq. (6). The label R_h is the Hubble radius, defined as c/H, where the Hubble parameter is $H = 1/t$ when $\beta = 1$.

$$a(t) = \left(\frac{t}{t_0}\right)^{\beta}, \tag{6}$$

where $\beta = 1/2$ for radiation and $\beta = 2/3$ for matter. This normalization for a in terms of the present age, t_0, of the Universe is consistent with spatial flatness, *i.e.*, $k = 0$.

Let us first convince ourselves that the case $\beta = 1$, corresponding to $\ddot{a} = 0$ (the straight, red line in Figure 1), does in fact satisfy the LFT. The transformation that takes us into the local free-falling frame at any spacetime point x^μ may be written as follows:

$$\begin{aligned} \xi^0 &= ct\eta \\ \xi^i &= a(t)x^i. \end{aligned} \tag{7}$$

The use of one's chosen ξ^μ needs to satisfy Eq. (3) only in the vicinity[1] of each selected spacetime point. Thus, as is customary when using the LFT, we assume that $\eta(x)$ is approximately constant for coordinates close to x^μ.

Therefore, adopting the Minkowski form of the line element,

$$ds^2 = (d\xi^0)^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2, \tag{8}$$

we recover Eq. (1) if

$$\eta(x)^2 = 1 + \frac{1}{(ct_0)^2}\frac{d}{dt}(tr^2). \tag{9}$$

If we make an additional reasonable assumption that the ξ^μ coordinates correspond to the Hubble flow – which is, after all, what the condition $g_{tt} = 1$ would require – we may simplify this further to

$$\eta(x)^2 = 1 + \frac{r^2}{(ct_0)^2}. \tag{10}$$

It is not difficult to see that the only nonzero Christoffel symbols consistent with Eqs. (1) and (6), written in Cartesian coordinates, are

$$\begin{aligned} \Gamma^0{}_{ii} &= \frac{\beta a^2}{ct} \\ \Gamma^i{}_{i0} = \Gamma^i{}_{0i} &= \frac{\beta}{ct}, \end{aligned} \tag{11}$$

where $i = 1, 2$ or 3. With these, one can thus show that the ξ^μ coordinates in Eq. (7) are fully consistent with Eq. (3). For example, the component $\alpha = \mu = i$ and $\nu = 0$ in this equation gives $1/ct_0$ on both the left- and right-hand sides. The component $\alpha = \mu = \nu = i$ gives 0 on the left-hand side and $ax^i/(ct_0)^2$ on the right-hand side. But $t/t_0 \leq 1$, and $x^i/(ct_0)^2 \ll 1$, so the right-hand side is also ≈ 0.

The situation with $\beta < 1$, however, is completely different. In this case,

$$\eta(x)^2 = 1 + \frac{\beta^2 a^2 r^2}{(ct)^2}. \tag{12}$$

When we now consider the component $\alpha = i$ and $\mu = \nu = 0$ in Eq. (3), we find that the right-hand side is 0, while the left-hand side is

$$\frac{\partial^2 \xi^i}{\partial (ct)^2} = \frac{\beta(\beta - 1)}{(ct)^2}\xi^i, \tag{13}$$

which can become arbitrarily large for small values of t, so the left-hand side of the LFT equation blows up. Similarly, the component $\alpha = \mu = \nu = i$ gives 0 on the left-hand side, while the right-hand side becomes

$$\Gamma^\lambda{}_{ii}\frac{\partial \xi^i}{\partial x^\lambda} = \frac{\beta \xi^i}{ct_0}\left(\frac{t_0}{t}\right)^{1-\beta}, \tag{14}$$

which similarly diverges for small values of t and $\beta < 1$.

Of course, there is nothing mysterious about this dichotomy in the outcome of the LFT for different values of β, because it ultimately originates from the choice of g_{tt}. When $\beta = 1$, there is no acceleration and therefore no implied time dilation in the observer's frame. The lapse function should then correctly be unity, and the LFT equations confirm that the coordinate transformation in Eq. (7) is correct. But when $\beta \neq 1$, the observer's frame is

1 As noted earlier, the Milne universe is unique among FLRW cosmologies, because the coordinates ξ^μ for this model actually refer to a single inertial frame throughout the cosmos; see, *e.g.*, [5]. But this is not the case in general, and one must find the appropriate inertial frame separately at each new spacetime point, as we do here.

decelerating, and they should then see a time dilation relative to the local inertial frame. It is not correct for them to force the lapse function to be unity in that case, which is reflected in the failure of the x^{μ} and ξ^{μ} coordinates to satisfy the LFT equations.

4 Conclusion

In this article, we have affirmed the constraint imposed by the LFT on g_{tt}, as shown in Eq. (5). We have done this specifically for ΛCDM, adopting its predicted expansion factor $a(t)$, and demonstrating that setting $g_{tt} = 1$ in the standard model violates the PoE.

The consequences of this outcome are quite significant, of course. Some of the issues raised by the inconsistency of ΛCDM with the PoE have been highlighted elsewhere (notably [5,12,13]). Specifically, the formal constraint that g_{tt} must satisfy relative to the chosen stress–energy tensor in Einstein's equations has been fully described in the study by Melia [12], leading to the expression in (Eq. 5). When the FLRW metric is written in terms of comoving coordinates (Eq. 1), the LFT does not leave any room for choices of g_{tt} other than those selected by this condition.

There have been some previous attempts at altering the form of FLRW with suitable choices of coordinate transformations, notably with the inclusion of conformal time into the FLRW description. In some cases, these efforts typically begin with the conventional form of the metric given in Eq. (1) and then alter the metric coefficients *via* the transformation to the new time coordinate $d\eta \equiv dt/a(t)$ [14–16]. In the majority of these cases, however, the choice of lapse function $g_{tt} = 1$ at the beginning still imposes a zero time dilation condition on the Hubble flow, so the transformed metric cannot alter the constraint that there should be zero acceleration in this frame.

The work that may be more relevant to the present manuscript is that of Vavryčuk [17], which also begins with $g_{tt} = 1$ and then transforms the FLRW metric into its conformal form, but goes one step further by suggesting that the physical time coordinate should not be the cosmic time t but, rather, the conformal time $d\eta = dt/a(t)$. This approach certainly produces variations from the standard procedure, but one must remember that $a(t)$ itself changes form, now becoming a function of the conformal time η rather than t. Nevertheless, the physics represented by the conformal FLRW metric is identical to that represented by its standard form (based on cosmic time), so this transformation cannot create a time dilation between the accelerated frame and the local inertial frame if zero time dilation, *i.e.*, $g_{tt} = 1$, is assumed from the beginning.

The overall success of ΛCDM in accounting for many kinds of data is slowly being tempered by the growing tension seen between its predictions and the ever-improving measurements. We are seeing growing concerns that the time *vs* redshift relation in the standard model fails to explain the formation of structure [18]. The anisotropies in the cosmic microwave background are not fully consistent with the predictions of inflation, and the persistent disparity between the observed elemental abundances and those predicted in big bang nucleosynthesis all suggest that, at a minimum, refinements are needed in the basic picture [19]. This is quite evident on large scales, but observations on smaller scales ~1 Mpc amplify the concerns because dark energy and dark matter, as assumed in ΛCDM, apparently produce galaxy distributions, halo demographics, and other statistical features at odds with the measurements, calling into question whether the basic idea underlying the nature of dark energy and/or dark matter is even sustainable [20,21].

There is some indication from the observations that the zero active mass condition is a necessary ingredient, and our conclusion that the FLRW spacetime is consistent only with such an equation-of-state certainly supports this view.

A comparison of the two curves in Figure 1 adds to our suspicion that the parametrization in ΛCDM is merely an empirical approximation designed to account for the data as best as possible, without necessarily having a solid theoretical foundation. This plot highlights what is arguably the most glaring coincidence in cosmology, *i.e.*, that in spite of the many phases of acceleration and deceleration (black curve) experienced by the ΛCDM Universe, the expansion factor today is precisely what it would have been anyway (red curve) with a constant expansion. Given the complex formulation of ΛCDM, the crossover at the point labeled "today" could happen only once in the entire history of the Universe, and it is occurring right now, when we happen to be looking, following 60 magnitudes of expansion out of the initial Planck scale. The probability of this happening randomly is, of course, effectively zero. The proof we have provided in this article is fully consistent with this argument.

One of the most anticipated and influential observational campaigns over the next few years will measure the real-time redshift drift of distant sources [22,23]. The redshift drift should be zero everywhere if $a(t) \propto t$, and nonzero otherwise [24]. If all goes well, a confidence level of ~3σ in this measurement should be attainable in only 5 years; ~5σ could be reached over a baseline of about 20 years. Therefore, we may not have to wait very long to see a compelling confirmation of the work reported in this article.

Looking farther afield, an FLRW cosmology with $a(t) \propto t$ completely avoids all horizon problems [25,26], so we may eventually find that inflation is not needed after all. The 40-year struggle to find a consistent model of the inflaton field without much success may therefore be an indication that this hypothesized brief period of acceleration in the early Universe was unnecessary and simply never happened.

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