Spectral networks, abelianization, and opers

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To my family

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Abstract

In this thesis we study a number of geometric structures arising in the study of four-dimensional supersymmetric quantum field theories.

We study properties and applications of so-called “spectral networks” on Riemann surfaces C, focusing in particular on the “abelianization map” which we use to produce special coordinate systems on moduli spaces of local systems on C. We generalize the classical Fenchel-Nielsen coordinates and utilize these coordinates to compute superpotentials, following and generalizing a conjecture of Nekrasov-Rosly-Shatashvili.

Our first result is a computation of the higher rank spectral coordinates associated to certain “generalized Fenchel-Nielsen” networks, yielding explicit formulas for the trace functions on the moduli space with two “minimal” and two “maximal” punctures. We use this result to verify the NRS conjecture at the lowest order asymptotics for a prototypical $SU(3)$ theory, and furthermore compute the 1-instanton correction in the $SU(2)$ case, extending previous results. In the final chapter we include some partial results the author has obtained on the existence and uniqueness of abelianizations for certain classes of networks related to Grassmannians.

"All problems in mathematics are psychological."

— P. Deligne

Contents

Chapter 1

Introduction

In this thesis we study a number of aspects of the geometry of meromorphic connections on Riemann surfaces in relation to four-dimensional $\mathcal{N} = 2$ supersymmetric quantum field theories (QFTs), focused around geometric objects known as "spectral networks". To help the reader orient themselves as to what this means, let us give some context.

1.1 Context

Connections on vector bundles are among the most fundamental mathematical objects used in the basic description of reality. Indeed, connections are everywhere — they govern the curvature of spacetime in general relativity, and describe the three fundamental forces (weak, strong, and electromagnetic) of the Standard Model; not to mention the shape of the Earth and other everyday geometries. Our connections will be slightly more abstract than these — meromorphic, and on holomorphic vector bundles over Riemann surfaces — but as we will see, still very much intertwined with physics.

One of the most important special cases is that of *flat* connections — these are constrained enough to be tractable, but retain enough information to tell us something interesting about the underlying space. Given some space X, we can then assemble *all* the possible flat connections on vector bundles over X into a single, reasonably well-behaved, *de Rham moduli space* $\mathcal{M}_{\mathrm{dR}}$ — a space parameterizing all of the flat connections, up to equivalence, on (say, a fixed rank K bundle over) X.

On the other hand, the moduli space of flat connections, somewhat amazingly, appears in the study of four-dimensional $\mathcal{N} = 2$ supersymmetric QFTs from a totally different perspective as well. Here it arises as part of a hyperkähler family (owing to the supersymmetry) describing the low energy dynamics of the theory and the vacua of its reduction to three dimensions. From this fact, numerous constructions from physics can be reinterpreted as geometric structures on $\mathcal{M}_{\mathrm{dR}}$. One such structure is the "abelianization map" arising from a "spectral network" [1, 2, 3, 4]. The abelianization map is a machine for producing very special coordinate systems on $\mathcal{M}_{\mathrm{dR}}$, by relating connections on rank K bundles over C to connections on *line* bundles over a related "spectral" curve Σ.

We are going to study some of the properties, structure, and physical applications of various classes of these coordinates.

1.2 Motivation

Spectral networks were introduced by Gaiotto, Moore, and Neitzke [1, 2, 3] for the purpose of counting "BPS states" in four-dimensional $\mathcal{N} = 2$ QFTs. These are certain collections of paths drawn on a surface with some additional data; they have been shown to have a number of interesting applications, both physical and mathematical, especially to moduli spaces arising in geometry. We will use them to obtain a generalization of "complexified Fenchel-Nielsen coordinates" on $\mathcal{M}_{\mathrm{dR}}$, and then use these coordinates as a tool for obtaining "effective twisted superpotentials" for certain kinds of supersymmetric QFTs. Let us elaborate.

The main physical problem in this thesis consists of studying and generalizing a conjecture of Nekrasov, Rosly, and Shatashvili regarding a geometric approach to computing "effective twisted superpotentials". In the course of studying this conjecture, we are interested in generalizing the classical Fenchel-Nielsen coordinates on Teichmüller space (though of as embedded in the moduli space of flat connections) to higher rank bundles. In [4] it was shown that (complexified) Fenchel-Nielsen coordinates arise as spectral coordinates at a special real locus of the Coulomb branch. By similarly studying networks arising from a special locus in higher rank, we find these coordinates.

Let us describe the NRS conjecture in slightly more detail. In recent years, Nekrasov and collaborators have studied the relationship between quantum integrability and four-dimensional gauge theories. In the process, it was found that one could obtain crucial physical information by studying a certain geometric object, the "brane of opers" sitting inside a moduli space of flat connections. The brane of opers is conjectured to contain essentially all the information of the low-energy physics of a certain reduced theory, provided we study it in the "right" coordinates. This is interesting in its own right, but also leads to the prospect of allowing access to the superpotential for theories not possessing a Lagrangian description, which are not amenable to traditional methods of analysis, and whose superpotential is not known. We should mention that there are numerous examples known of such QFTs, so the problem is quite important from the physical perspective. By mapping the physical problem to a purely geometric one, we can generalize to cases even when we have no Lagrangian from which to extract physics.

Let us describe this "effective twisted superpotential" in further detail. In general, one of the main problems in quantum field theory is to evaluate the partition function Z of a given theory T. A major advance in the case of four-dimensional $\mathcal{N} = 2$ supersymmetric theories came with the advent of the *Nekrasov partition function* [5, 6], which is the partition function of a two-parameter (traditionally denoted ϵ_1 and ϵ_2) deformation of the theory which regularizes the IR divergences arising from instantons running off to infinity. When defined, this can be understood as a perfectly well-defined formal series in the *instanton parameter q* with rational coefficients in the remaining parameters, and can be written explicitly for theories with a Lagrangian description. However, in general, it is not known, particularly for the so-called *non-Lagrangian* theories that have attracted much attention from physicists in recent years. Here, one might seek alternative approaches to the definition that generalize more easily. This motivates understanding the following proposed equivalence.

Let us first simplify the situation slightly — take the *Nekrasov-Shatashvili limit*, resulting in

$$\widetilde{W}^{\text{eff}}(a, m, \epsilon, q) = \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z(a, m, \epsilon, \epsilon_2, q) \tag{1.2.1}$$

called the *effective twisted superpotential* associated to T; here a, m denote some additional parameters. $\widetilde{W}^{\text{eff}}$ can be thought of as a two-dimensional shadow of the original Z. Furthermore, consider only those theories of "class S", which are a large class of theories arising from a choice of simple Lie algebra $\mathfrak{g}$, and a punctured Riemann surface equipped with some extra data at the punctures. It will be important that this is just the data needed to define $\mathcal{M}_{\text{dR}}$.

Nekrasov, Rosly, and Shatashvili [7] proposed a geometric description of this superpotential $\widetilde{W}^{\text{eff}}$. Specifically, they propose to study the *opers* sitting inside an

$\mathcal{M}_{\mathrm{dR}}$ associated to the data we started with. These form a complex Lagrangian subvariety, and so given a Darboux coordinate chart $\{\alpha_i, \beta_i\}$, possess a local *generating function* W^{oper}, defined by the property that $\partial_{\alpha_i} W^{\mathrm{oper}} = \beta_i$. The main claim is that, if the Darboux chart is chosen judiciously,

$$\boxed{W(a, m, q, \epsilon) = W^{\mathrm{oper}}(\alpha, \tilde{m}, q, \epsilon)} \tag{1.2.2}$$

where one identifies the appropriate parameters on the left with the α coordinates on the right, and m, $\tilde{m}$ are parameters related in a simple way. NRS verified this when $K = 2$ using "complexified Fenchel-Nielsen" coordinates, but the right coordinate choice was not at all apparent for $K > 2$. Our goal is to study the higher rank case, find these coordinates, and use them to compute the superpotential.

1.3 Outline

In this thesis we present a number of results related to spectral networks and the NRS conjecture. After some preliminary material, the first step is to define and compute the higher length-twist coordinates that we will need. Chapter 5 describes this part of the joint work [8] with L. Hollands. We first strengthen the $K = 2$ result of [4] by observing that a certain "averaged" spectral coordinate prescription fixes ambiguities otherwise present in the definition of the twist coordinate, whose utility carries over to $K > 2$ as well. The main result of this chapter is a computation of these spectral coordinates arising from $K = 3$ "generalized Fenchel-Nielsen networks of length-twist type", yielding a formula for the trace functions on $\mathcal{M}_{\mathrm{dR}}$ in terms of these coordinates.

In Chapter 6, we use our coordinates to generalize the NRS calculation and give a proposal for the coordinates in which equality holds. More precisely, we find that our higher length-twist coordinates do indeed reproduce, at the lowest asymptotic order in q, the superpotential in the case of the simplest nontrivial $K = 3$ example ("SU(3), $N_f = 6$ theory"). We also improve on previously known results for SU(2), computing the first "instanton correction" which provides further evidence for the conjecture. Furthermore, we were able to put some of the heuristic approximations employed by physicists on firm footing using a slight generalization of known perturbation theory results, summarized in Appendix A. From a mathematical perspective, we have given a description of the monodromy representation of opers on a punctured curve C in a series expansion in its complex structure parameters and verified a prediction for the generating function of a particular interesting Lagrangian subspace inside the moduli space of flat connections.

In the last chapter of this thesis, we move on to study some purely mathematical properties of spectral coordinates in some specific cases – in particular, we study some special classes of spectral networks whose associated coordinates are essentially the cluster coordinates for Grassmannians introduced previously in the literature. We study a question of the uniqueness of the abelianization construction, posed to the author by A. Neitzke. We prove uniqueness in several nontrivial examples, and outline an approach to deal with the problem more generally (without looking at the specific network, of which there are many).

The thesis is organized as follows. Chapters 2 and 3 contain review of background material from both physics and mathematics related to 4d $\mathcal{N} = 2$ theories and moduli spaces of flat connections. Chapter 4 contains an introduction to the basics of spectral networks and abelianization and sets the stage for the main content of the thesis. Chapters 5 and 6 cover the results outlined above of the joint work with L. Hollands "Higher length-twist coordinates, generalized Heun's opers, and twisted superpotentials" [8] — they are essentially a modified version of this preprint, made to fit better the structure of the thesis. In Chapter 7 we move on to study the uniqness problem for "Grassmannian networks", and record some partial results on our work-in-progress.

This work lies at the boundary between pure mathematics and theoretical physics. As such, I have tried to make it understandable to both physicists and mathematicians. For this reason, we include two separate chapters of prerequisite material, trying to assume as little as possible from the reader (which is still probably a lot!). For the preliminary material, we have organized the physics to come before the mathematics for motivational and narrative purposes, though we will make free use of definitions and facts mentioned in the latter. The reader may wish to read the mathematics preliminaries separately.

1.4 Notational conventions

Throughout, by punctured curve we will mean either the compact curve C equipped with the corresponding divisor of poles (perhaps even enhanced to the data of *defects*), or the noncompact $C^{\times}$ (i.e. C with points removed) – it should be clear from the context which is meant. D will always denote the divisor of poles/punctures on C. The number n always refers to the number of punctures.

So-called "mass parameters" m will usually be omitted from notation, but are present in many places and assumed fixed as part of the initial data, satisfying necessary genericity assumptions.

We will sometimes shorten the surface equipped with defects $(C, \mathcal{D})$ to $C_{z_1,\ldots,z_n}$, leaving masses implicit, and underlining so-called "minimal" punctures.

K_C will always denote the canonical bundle of the compact curve, and whenever necessary we assume from that we have fixed a choice of $K_C^{1/2}$.

Chapter 2

Preliminaries: Physics

In this chapter we give a basic introduction to four-dimensional $\mathcal{N} = 2$ supersymmetric QFTs, particularly the theories of "class $\mathcal{S}$" that we will be primarily concerned with. These are the fundamental physical theories which we are interested in studying, from which geometric structures will unravel. We keep prerequisites to a minimum, and try to explain things in a way that a mathematician might understand them. On the other hand, we do not attempt to give a comprehensive treatment, so one should think of this as a "lightning review" for orientation.

2.1 4d $\mathcal{N} = 2$ supersymmetric field theories

2.1.1 $\mathcal{N} = 2$ SUSY Lagrangians

Consider everything on $\mathbb{R}^4$, with either the Minkowski or Euclidean ("Wick-rotated") metric. Fix some compact simple Lie group G, the gauge group.

The $d = 4$, $\mathcal{N} = 2$ ***super Poincaré algebra*** is the super Lie algebra

$$\mathfrak{spoin} := \mathfrak{s}_0 \oplus \mathfrak{s}_1 := [\mathfrak{poin} \oplus \mathbb{C}] \oplus [(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})]$$

where $\mathfrak{s}_0$ is the even-graded part and $\mathfrak{s}_1$ is the odd-graded part. Here $\mathfrak{poin}$ denotes the 4d Poincaré algebra, and $(\mathbf{i}, \mathbf{j})$ label its spinor representations by dimension in the usual way. The extra $\mathbb{C}$ summmand is the "central charge". Our theories will be invariant under an infinitesimal action of this super Lie algebra on the space of fields. With this purpose in mind, the field content is chosen so that the fields take values in various representations of this algebra, as in what follows.

$\mathcal{N} = 2$ supersymmetry is a special case of $\mathcal{N} = 1$ supersymmetry, so it's useful to describe the field content in terms of $\mathcal{N} = 1$ superfields. The basic fields in

$\mathcal{N} = 1$ SUSY field theory, written in superfield notation (which can be found in any textbook on the subject, with conventions usually following [9]) are:

- $V = -\theta\sigma^{\mu}\bar{\theta}A_{\mu} + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D$ "$\mathcal{N} = 1$ vector multiplet"
- $\Phi = \phi + \sqrt{2}\theta\psi + \theta\theta F + \ldots$ "chiral multiplet"

where the "…" denotes terms involving derivatives of ϕ, ψ, F. Here $A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$ is a connection over the trivial G-bundle over spacetime,

- $\lambda \in \Omega^0(\mathbb{R}^4, \mathfrak{g})$ an "adjoint-valued Weyl-spinor"
- $\phi \in \Omega^0(\mathbb{R}^4, R)$ an "R-valued scalar"
- $\psi \in \Omega^0(\mathbb{R}^4, \Pi S_+ \otimes W \otimes R)$ an "R-valued Weyl spinor"

where R is some chosen "matter" representation of G (this can be any complex representation equipped with a hermitian form), S_+ is the usual left-handed spinor module $(\mathbf{2}, \mathbf{1})$, W is a two-dimensional complex vector space, and Π denotes parity reversal of super vector spaces. The D and the F are just ($\mathfrak{g}$ or R-valued) functions called "auxiliary fields" — in a sense, they only exist to make the supersymmetry formulas work out; their equations of motion are algebraic so they can be solved for in terms of the other fields. One often simply writes and thinks of these superfields as tuples written (A, λ, D) and (ϕ, ψ, F).

In $\mathcal{N} = 2$ SUSY we will not bother introducing superfield notation (though we could), and simply list the field content as a tuple:

- $(V, \Phi) = ((A, \lambda, D), (\phi, \psi, F))$ "$\mathcal{N} = 2$ vector multiplet"
- $(Q, \tilde{Q}) = ((q, \psi_q, F_q), (\tilde{q}, \tilde{\psi}_{\tilde{q}}, \tilde{F}_{\tilde{q}}))$ "hypermultiplet"

where now, denoting the complex conjugate representation by $\bar{R}$,

- $(q, \tilde{q}) \in \Omega^0(\mathbb{R}^4, R \oplus \bar{R})$ two scalars, valued in R and $\bar{R}$
- $(\psi_q, \tilde{\psi}_{\tilde{q}}) \in \Omega^0(\mathbb{R}^4, \Pi S_+ \otimes W \otimes R \oplus \bar{R})$ one left and one right-handed Weyl-spinor in R and $\bar{R}$

and F_q and $\tilde{F}_{\tilde{q}}$ are R-valued functions.

Assuming familiarity with $\mathcal{N}=1$ superfield notation and Berezin integration, we assemble these into a Lagrangian as follows:

$$\mathcal{L}_{\text{vmult}} = \frac{\text{Im}\tau}{4\pi}\int \mathrm{d}^4\theta\ \Phi^\dagger e^{[V,\cdot]}\Phi - \frac{i}{8\pi}\tau\int \mathrm{d}^2\theta\ \mathrm{Tr}\, W_\alpha W^\alpha\ +\ \text{c.c.} \tag{2.1.1}$$

where $W_\alpha := \lambda_\alpha + F_{(\alpha\beta)}\theta^\beta + D\theta_\alpha \ldots$, with $F_{\alpha\beta} := \frac{i}{2}\sigma^{\mu\beta}_{\dot\gamma}\bar\sigma^{\nu\dot\gamma}_\alpha F_{\mu\nu}$ the SUSY analogue of the Yang-Mills field strength. Here we have written the complexified coupling constant

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2} \tag{2.1.2}$$

where g is the usual Yang-Mills coupling constant and ϑ is another (real) parameter, the coefficient of the "topological term" $\int \mathrm{Tr}\, F\wedge F$. Classically this is just a fancy way of writing a constant, so it doesn't affect the classical equations of motion, but it plays an important role in the quantum theory, since it affects the weighting of different topological sectors of the field space in the path integral.

The Lagrangian $\mathcal{L}_{\text{vmult}}$ by itself is the Lagrangian for the $\mathcal{N}=2$ analogue of pure gauge theory. We observe that by taking fields valued in representations of $\mathfrak{spin}$, merely having a gauge field in our theory in a natural way necessitates certain other terms and fields, even without putting fermions in by hand.

Writing the Lagrangian out in components, we can get an idea of what's actually going on under the notation:

$$\begin{aligned}\mathcal{L}_{\text{vmult}} = \frac{1}{g^2}\mathrm{Tr}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g^2\frac{\vartheta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - i\lambda\sigma^\mu D_\mu\bar\lambda \\ -i\bar\psi\bar\sigma^\mu D_\mu\psi - i\sqrt{2}[\lambda,\psi]\phi^\dagger - i\sqrt{2}[\bar\lambda,\bar\psi]\phi - \frac{1}{2}[\phi^\dagger,\phi]^2\Big]\end{aligned} \tag{2.1.3}$$

Similarly, we can put hypermultiplets, which represent matter, in the theory by adding terms of the form:

$$\mathcal{L}_{\text{matter}} = \sum_{i=1}^{N_f}\int \mathrm{d}^4\tilde\theta\left(Q_i^\dagger e^{-2V}Q_i + \widetilde{Q}_i e^{2V}\widetilde{Q}_i^\dagger\right) + \int \mathrm{d}^2\tilde\theta\left(\sqrt{2}\widetilde{Q}_i\Phi Q_i + \mu_i\widetilde{Q}_i Q_i\right)\ +\ \text{h.c} \tag{2.1.4}$$

where we have N_f copies (or "flavours") of the representation R (so really, R is their direct sum), and μ_i are some complex ("bare mass") parameters. More generally, we can add terms for different choices of R to introduce various species of particle in the theory.

We can see that there are only a few building blocks for our Lagrangian SUSY theories, namely:

(i) A choice of the gauge group G

(ii) A choice of matter representation(s) R

(iii) The values of the parameters: the coupling constant τ_i associated to each simple factor of G, and the masses μ_i of the hypermultiplets.

Terminology: the representation R is often given as a direct sum of irreducible representations, some of which may appear multiple times. Physicists might thus refer to a theory containing "N_f fundamental hyper(multiplet)s", which then means that R includes as a summand $\mathbf{fund}^{N_f}$.

Example 1. Pure $N = 2$ super-Yang-Mills theory. Here we have

- $G = \mathrm{SU}(2)$
- $R = \mathbf{triv}$ (no matter multiplets)

This is the theory that Seiberg and Witten famously solved exactly in the low energy limit (described below).

Example 2. $\mathrm{SU}(2)$, "$N_f = 4$" theory. Here we have

- $G = \mathrm{SU}(2)$
- $R = (\mathbf{fund})^4$

This is the theory that Gaiotto generalizes to produce his network of S-dual theories in [10]. It is a fundamental building block of the theories we study.

2.1.2 Vacuum moduli space

Quantum field theory is usually "defined" via the path integral. That is, the expectation value of an observable $\mathcal{O}$ is (schematically) defined to be

$$\langle \mathcal{O} \rangle = \int_{\text{fields}} \mathcal{D}\Phi \, \mathcal{O} e^{-S} \tag{2.1.5}$$

One must specify what exactly one means by integrating over the "space of fields"—specifically, on a noncompact spacetime, boundary conditions on the fields to single out a class to integrate over. This can be viewed as a special case, relevant to our theories, of the general principle that we must compute observable quantities relative to a particular choice of quantum vacuum. We will see in a moment that the supersymmetric Lagrangians above possess a continuum of vacua, so that a $\mathcal{N} = 2$ supersymmetric theory automatically produces, by passing to the IR limit, a family of theories determined by the choice of vacuum. The parameter space of this family is called the ***moduli space of vacua*** or ***vacuum moduli space***.

How can we characterize these vacua? We can evaluate the expectation values of various operators in the theory on a given vacuum state to get some numbers, which will serve as coordinates. For example: $u = \langle \operatorname{tr} \Phi^2 \rangle$, where we are abusing notation in a standard way[1]. On fairly general grounds, and certainly for the theories we will be interested in — those of "class $\mathcal{S}$" — it can be argued that the vacuum expectation values (vevs) of non-scalar fields all vanish, so only the scalars[2] are relevant, and thus the equations describing vacua are written purely in terms of ϕ, q_i, and $\tilde{q}_i$

We begin by studying the classical moduli space of vacua. We take the Lagrangian $\mathcal{L} = \mathcal{L}_{\text{vmult}} + \mathcal{L}_{\text{matter}}$ and seek to minimize the "potential", i.e. the part of the Lagrangian polynomial in the scalar fields. A simple calculation yields the following equations:

$$\frac{1}{g^2}[\phi^\dagger, \phi] + (q_i q^{i\dagger} - \tilde{q}_i^\dagger \tilde{q}^i)|_{\text{traceless}} = 0 \tag{2.1.6}$$

$$\phi q_i + \mu_i q_i = \tilde{q}^i \phi + \mu_i \tilde{q}_i = 0 \tag{2.1.7}$$

$$q_i \tilde{q}^i|_{\text{traceless}} = 0 \tag{2.1.8}$$

where $X|_{\text{traceless}} := X - \frac{1}{m}\operatorname{tr} X$ for an $m \times m$ matrix X. This locus, modulo conjugation by elements of G and the action of the Weyl group W, is known as the ***classical moduli space of vacua***.

The moduli space can be divided into three "branches" with different behaviour:

- $\phi = 0$ is known as the "Higgs branch". By general considerations, it is always hyperkähler.
- $q_i = \tilde{q}_i = 0$ is known as the "Coulomb branch" or the "vector multiplet moduli space". It is always Kähler.

[1] Rather than thinking of the classical superfield Φ we mean the corresponding operator that ought to exist in the quantum version of the theory.

[2] That is, the lowest components ϕ or $q, \tilde{q}$ in the multiplets of §2.1.1

- If neither ϕ nor all $q_i, \tilde{q}_i$ vanish, it is called the "mixed branch".

We will be interested in the Coulomb branch, so let us set the hypermultiplets to zero. This means minimizing the potential amounts to the single equation:

$$\mathrm{tr}\,[\phi^\dagger, \phi]^2 = 0 \tag{2.1.9}$$

The left hand side is clearly hermitian, so this equation is equivalent to ϕ commuting with its adjoint. Hence ϕ is diagonalizable and can be taken to lie in the Cartan subalgebra, $\phi \in \mathfrak{t}$. Thus the Coulomb branch $\mathbf{B}$ is

$$\mathbf{B} = \mathfrak{t}/W$$

Let us focus on the case of $\mathrm{SU}(2)$ now for simplicity, though everything generalizes to higher rank and other gauge groups.

In this case any $\phi \in \mathfrak{t}$ has some eigenvalue $\frac{1}{2}a$, and is of the form

$$\phi = \frac{1}{2}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

So that we can parametrize inequivalent vacua using

$$u = \mathrm{tr}\,\phi^2 = \frac{1}{2}a^2. \tag{2.1.10}$$

In the quantum theory, ϕ is promoted to an operator in some appropriate sense, and u is now defined as the expectation value $\langle \mathrm{tr}\,\phi^2 \rangle$. The relationship (2.1.10) is thus modified, but can be thought of as a first approximation that holds asymptotically at weak coupling.

Now, a general fact: generically (away from a lower-dimensional "singular locus" $\mathbf{B}_{\mathrm{sing}}$), the low energy behaviour of the theory corresponding to the vacuum $u \in \mathbf{B} \setminus \mathbf{B}_{\mathrm{sing}}$ is described by a *pure* and *abelian* $\mathcal{N} = 2$ gauge theory; that is, with $G = \mathrm{U}(1)^r$, where r is the rank of the original gauge group, and with no matter ($\mathcal{R} = \mathbf{triv}$). This follows from integrating out massive modes: the matter multiplets go away, and only the components of the vector multiplet scalars contained in the Cartan, which are massless, remain as they cannot be integrated out. This description breaks down at $\mathbf{B}_{\mathrm{sing}}$ because at this locus some of the particles integrated out actually become massless, which invalidates the argument.

Returning to the SU(2) case, the most general physically reasonable[3] low-energy effective action for a $\mathrm{U}(1)^r$, $\mathcal{N}=2$ gauge theory with at most two derivatives is:

$$S_{\text{eff}} = \frac{1}{4\pi}\mathrm{Im}\int \mathrm{d}^4x \left(\int \mathrm{d}^4\theta\, \Phi^{\dagger}\frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} + \int \mathrm{d}^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi^2} W_{\alpha}W^{\alpha} \right) \tag{2.1.11}$$

where $\mathcal{F}$ is holomorphic. Thus all the data about the low-energy theory is contained in a single holomorphic function $\mathcal{F}$ (for example, the metric on the moduli space is computed as $\mathrm{Im}(\frac{\partial^2\mathcal{F}}{\partial a^2}da d\bar{a})$). Thus to "solve" the theory in the low-energy limit is to determine this function, i.e. which pure U(1) theory, of all the possible choices of $\mathcal{F}$ in (2.1.11) describes the low energy behaviour of our theory?

Now, for physical reasons (positive-definiteness of the kinetic energy), it turns out $a(u)$ cannot be globally defined on the whole quantum moduli space – it is multivalued. Thus we need to introduce a new coordinate,

$$a_D := \frac{\partial \mathcal{F}}{\partial a}. \tag{2.1.12}$$

Furthermore, (a, a_D) must satisfy certain monodromy constraints (they form a section of a local system over $\mathbf{B} \setminus \mathbf{B}_{\text{sing}}$). We are looking for locally defined functions (a, a_D) with such properties. Seiberg and Witten produced an answer to this as follows.

Introduce an auxiliary curve Σ_u, and a one-form λ on that curve , both of which must be derived somehow via physical insight. In our case of pure SU(2) theory, Seiberg and Witten proposed the curve Σ_u

$$y^2 = (x+1)(x-1)(x-u)$$

and the one-form

$$\lambda = \frac{y}{x^2-1}\,dx$$

defined for $u \in \mathbf{B} = \mathbb{C}$, with singular locus $\mathbf{B}_{\text{sing}} = \{\pm 1\}$. Σ_u is called the *Seiberg-Witten curve* of the theory and λ is called the *Seiberg-Witten differential*.

Now, choose a symplectic basis $\{A, B\}$ of $H^1(\Sigma_u, \mathbb{Z})$. We claim that

$$a(u) = \int_A \lambda \qquad\qquad a_D(u) = \int_B \lambda$$

[3]There is more freedom here in the Lagrangian than suggested earlier because we are now only talking about an effective theory rather than a fundamental theory — renormalizability doesn't concern us.

satisfies all the necessary properties (for higher rank there will be g A_i's and B_i's and corresponding $(a_i, a_{D,i})$). This has the desired monodromy and a correct $\mathrm{SL}(2,\mathbb{Z})$ ambiguity corresponding to a change in the symplectic basis, and turns out to pass many "tests" of its physical correctness.

Given a symplectic basis, a general cycle can be decomposed as $\gamma = n_e A + n_m B$, where n_e, n_m are thought of as the "electric" and "magnetic" charge of the particle corresponding to γ. Thus in general, the central charge can be written:

$$Z_\gamma(u) = \frac{1}{\pi}\int_\gamma \lambda = \frac{1}{\pi}\int_{\gamma_e(u)} \lambda + \frac{1}{\pi}\int_{\gamma_m(u)} \lambda = n_e a(u) + n_m a_D(u) \tag{2.1.13}$$

Let us give an indication of what happens in general. Given a four-dimensional $\mathcal{N} = 2$ theory, there is an algebraically integrable system over a base $\mathbf{B}$ which describes its low-energy behaviour away from a singular locus $\mathbf{B}_{\text{sing}}$. In particular

1. We have a holomorphic symplectic manifold $\mathcal{M}$, and a holomorphic map $\pi : \mathcal{M} \to \mathbf{B}$ such that if $u \in \mathbf{B} \setminus \mathbf{B}_{\text{sing}}$, then $\pi^{-1}(u)$ is a compact complex Lagrangian torus of dimension r. Integration along the cycles of these tori gives the coordinates $a_i, a_{D,i}$.

2. The IR physics at $u \in \mathbf{B} \setminus \mathbf{B}_{\text{sing}}$ is given by pure abelian $\mathrm{U}(1)^r$ gauge theory in accordance with the above. The complex dimension of $\mathbf{B}$ is r, the rank of the gauge group.

At the singular locus, the tori degenerate and the physics is in general more complicated.

2.2 Theories of class $\mathcal{S}$

2.2.1 Class $\mathcal{S}$

One fairly general method of constructing four-dimensional $\mathcal{N} = 2$ theories is by compactifying from higher dimensions — six, in our case. More precisely, one considers the conjectured "6d $(2,0)$ theory" associated to a Lie algebra $\mathfrak{g}$ and a punctured Riemann surface C with certain "defects" $\mathcal{D}$ associated to the punctures, by taking the spacetime to be

$$M = X \times C, \tag{2.2.1}$$

where X is a (pseudo-)Riemannian 4-manifold and assuming C to be small. Combining this with a so-called "partial topological twist", we produce a 4d $\mathcal{N} = 2$ theory. When $X = \mathbb{R}^4$, we denote the resulting theory by $S[\mathfrak{g}, C, \mathcal{D}]$, or when $\mathfrak{g} = A_{K-1}$

as will often be the case, $T_K[C, \mathcal{D}]$. Theories of this form are called ***theories of class*** $\mathcal{S}$.

As we mentioned in the previous section, every 4d $\mathcal{N} = 2$ theory is expected to have an integrable system associated to it governing its low-energy behaviour. In the case of theories of class $\mathcal{S}$, it turns out that this is the Hitchin system $\mathcal{M} = \mathcal{M}_H$ (described in detail in §3.3). The Coulomb branch is identified with the Hitchin base

$$\mathbf{B} = \bigoplus_i H^0(C, K_C^{\otimes d_i})$$

and the projection $\pi : \mathcal{M}_H \to \mathbf{B}$ is the Hitchin map. The Seiberg-Witten curve is then the spectral curve $\Sigma \subset T^*C$

$$\det(x - \varphi) = 0 \tag{2.2.2}$$

where φ is the Higgs field. The Seiberg-Witten differential λ is then the restriction of the tautological one-form on T^*C.

Remark. Further compactification on S^1 leads to a three-dimensional theory whose low-energy effective Lagrangian is that of a supersymmetric sigma model into $\mathcal{M}_H$. The vacua of a sigma model are simply the constant maps, so we identify the moduli space of vacua as $\mathcal{M}_H$, the *total space* of the Hitchin system. So we can see that the Hitchin system not only arises in field theory as an associated integrable system, but also as a vacuum moduli space.

Without going into the precise details of the construction, let us summarize the data and some properties of a class S theory. Fix a positive integer K, which we call the "rank"[4] and a punctured Riemann surface C. We equip the Riemann surface with a collection

$$\mathcal{D} = \{\mathcal{D}_l\}_{l=1,\ldots,n} \tag{2.2.3}$$

of "regular defects" associated to each puncture z_l. Physically, these are objects filling up the entire $\mathbb{R}^4$ factor of spacetime, thus appearing as a point in the remaining two dimensions, which modify the path integral along their extent. From a physics perspective, defects in QFT are a subject of intense study, and we will content ourselves below with a somewhat restricted definition which will suffice for our

[4] This is the rank of the bundles we will later be studying, rather than the rank of the Lie algebra which is $K - 1$ in our usual case of A_{K-1}

purposes. To each choice of such data $(K, C, \mathcal{D})$ corresponds a four-dimensional $\mathcal{N} = 2$ superconformal field theory $S[A_{K-1}, C, \mathcal{D}]$ of type A_{K-1} with defects $\mathcal{D}$ of "regular" type.

A slightly more precise though less flexible way of saying these things, sufficient for our purposes is as follows. Let C be a Riemann surface equipped with some marked points, or *punctures*.

Definition. A ***regular defect*** at a puncture z_l is the data $(Y_l, \mathfrak{m}_l)$ consisting of a Young diagram Y_l with K boxes and a collection of compatible "mass" parameters $\mathfrak{m}_l = (m_{l,i})_{i=1,\ldots,K}$ satisfying $\operatorname{Tr}\mathfrak{m}_l = \sum_{i=1}^{K} m_{l,i} = 0$. The height of each column in the Young diagram encodes the multiplicities of coincident mass parameters in the obvious way.

We will use the notation that the divisor $D = \sum_{k=1}^{n} 1 \cdot z_k$ keeps track of the locations of the defects. We will only have occasion to deal with regular singularities, so this is all we need, but we could encode "irregular" punctures as well in an analogous fashion. We will call any compact Riemann surface equipped with defects at a finite number of points a ***punctured Riemann surface*** $(C, \mathcal{D})$.

Definition. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, C a compact Riemann surface, and $\mathcal{D} = \{(Y_i, \mathfrak{m}_i)\}$ a collection of (regular) defects at some choice of punctures. We will call a tuple $T = (\mathfrak{g}, C, \mathcal{D})$ a tuple of ***class $\mathcal{S}$ data***.

Given class $\mathcal{S}$ data, we can associate a number of objects that are part of the corresponding physical theory. The ***Seiberg-Witten curve*** (or ***spectral curve***) corresponding to the point $\varphi = (\varphi_2, \ldots \varphi_K) \in \mathbf{B}$ is the curve $\Sigma \subset T^*C$

$$\lambda^K + \lambda^{K-2}\varphi_2 + \ldots \varphi_K = 0 \tag{2.2.4}$$

and the ***Seiberg-Witten differential*** is the pullback of the tautological one-form to Σ.

Definition. The ***mass parameters*** of the theory are the residues of the Seiberg-Witten differential λ,

$$m_{l,j} = \operatorname{res}_{z_l}\lambda_j \tag{2.2.5}$$

Thus whenever we specify class S data, we are implicitly fixing the residues of the Seiberg-Witten curve in accordance with our choice of mass parameters.

The data of the defects determines the Coulomb branch. To state this, we define the space of ***allowed translations*** of the Hitchin base given by restricting the allowed poles of the subleading terms of φ_k at each puncture z_i, as follows:

1. Starting from the topmost row of the Young diagram, write integers $p_i^{(1)} = 0, p_i^{(2)} = 1, \ldots p_i^{(r_1)} = r_1$ from left to right in each box until reaching the last (r_1th) box in the row.

2. In the next row, write integers $p_i^{(r_1+1)} = r_1, p_i^{(r_1+2)} = r_1 + 1, \ldots p_i^{(r_2)} = r_1 + r_2$ until reaching the last (r_2th) box in the row, and repeat this process until all K boxes are filled

3. Ignoring the irrelevant case of $k = 1$, the order of the pole of φ_k at the puncture z_i is $p_i^{(k)}$.

Thus, the allowed translations are $H^0(C, K_C^{\otimes k}(\sum p_k^{(i)} \cdot z_i)$. In particular, for the case of a maximal puncture, we can just write $D = \sum 1 \cdot z_i$, and $H^0(C, K_C(D)^{\otimes K}$. For example, when $K = 2$ there is only one choice, leading φ_2 having a pole of order 1. When $K = 3$ we have two choices, giving either φ_2, φ_3 both with a pole of order 1 (minimal) or φ_2 of order 1, φ_3 of order 2 (maximal).

Definition. The ***Coulomb branch*** of a class $\mathcal{S}$ theory T with regular defects is the affine subspace of the full Hitchin base

$$\mathbf{B} = \mathbf{B}(T) \subset \bigoplus_{i=2}^{K} H^0(C, K_C(D)^{\otimes i}) \tag{2.2.6}$$

where the masses are fixed to have residues $m_{i,j}$ and the subleading terms are allowed translations.

"Definition". The ***theory of class*** $\mathcal{S}$ associated to the class $\mathcal{S}$ data $(\mathfrak{g}, C, \mathcal{D})$ is the partially twisted compactification $T = S[\mathfrak{g}, C, \mathcal{D}]$ described above.

This is a four-dimensional $\mathcal{N} = 2$ supersymmetric QFT which, like almost any other QFT, has no precise mathematical definition at present. But unlike many other QFTs, it does not even have a satisfactory *physical* definition, owing to the mysterious and conjectural nature of the 6d $(2,0)$ theory it came from. Nonetheless, many of its expected properties can be determined and studied.

2.2.2 Generalized quiver gauge theories

In [10] Gaiotto studies *generalized quiver gauge theories*. These are theories $T_{g,n}$ associated to trivalent graphs with g loops and n external legs. Such graphs (once equipped with a cyclic ordering of edges at each vertex) are equivalent to pants decompositions of a Riemann surface $C_{g,n}$ of genus g with n punctures. Pick an integer $K \geq 2$ specifying that our theories will have gauge groups which are products of $\mathrm{SU}(K)$. Each weakly coupled Lagrangian description of $T_{g,n}$ corresponds to a pants decomposition of $C_{g,n}$ (though not conversely!), with the "plumbing parameter" $q = e^{2\pi i \tau}$ (see §3.4.1) identified with the exponentiated coupling of the gauge group at the ith pants curve. For $K = 2$, the construction is:

- Assign to each external leg a hypermultiplet in the representation $R = \mathbf{fund}$, so that the flavor group is $\mathrm{SU}(2)^n$.
- Assign to each internal edge a gauge group $\mathrm{SU}(2)$ so that the total gauge group is $G = \mathrm{SU}(2)^{3g-3+n}$.
- Assign to each vertex with two internal edges hypermultiplet in the "bifundamental" representation $\mathbf{2}_1 \otimes \mathbf{2}_2$, where the subscripts denote the two adjacent gauge groups.
- Assign to each vertex with three internal edges the "trifundamental representation" $\mathbf{2}_1 \otimes \mathbf{2}_2 \otimes \mathbf{2}_3$ where the subscripts denote the three adjacent gauge groups.

Speaking in terms of pants decompositions, what this means is that (in the case of weak coupling) we have a collection of three-punctured spheres connected by long thin tubes. Each tube adds a factor of $\mathrm{SU}(2)$ to the gauge group, each puncture gives a hypermultiplet, and bi- or tri-fundamentals are put between adjacent gauge groups. Physically one should think of these theories as obtained from pairs of pants (which correspond to the theory of four free hypermultiplets) by "weakly gauging a flavour symmetry".

The quiver gauge theories are exactly the weakly coupled Lagrangian descriptions of theories of class $\mathcal{S}$. The theory came from $C = C_\tau$ which has a complex structure τ, and the moduli space of complex structures $\mathcal{M}_{g,n}(C)$ is identified with the parameter space of gauge couplings. If τ is not near a degeneration there is no weakly coupled description of the theory.

Duality. Recall that S-duality in pure $\mathcal{N} = 4$ super Yang-Mills is a "quantum symmetry" of the theory which conjectures the equivalence of theories (that is, that various properties like correlation functions, Coulomb branch, etc. are the same or related in some prescribed way) under the usual $\mathrm{SL}(2, \mathbb{Z})$ action on the coupling constant $\tau \in \mathbb{H}$ by fractional linear transformations. A similar symmetry, still called S-duality, exists for the $N_f = 4$ $\mathrm{SU}(2)$ theory mentioned earlier, but the group is actually $\mathrm{SL}(2, \mathbb{Z}) \ltimes S_3$ which permutes the flavour symmetries associated to the hypermultiplets (via triality i.e. the outer automorphisms of $\mathrm{Spin}(8)$, i.e. permuting its three irreducible representations). Gaiotto extends this to an S-duality on any $\mathrm{SU}(2)$ quiver gauge theory.

Claim: (generalized) S-duality acts transitively on the set of pants decompositions. Here by "generalized" we mean the group is actually $\mathrm{SL}(2, \mathbb{Z}) \ltimes S_3$ which permutes the flavour symmetries associated to the hypermultiplets (via triality i.e. the outer automorphisms of $\mathrm{Spin}(8)$, i.e. permuting its three irreducible representations). This depends on the fact that all $\mathrm{SU}(2)$ theories in class S have a Lagrangian description.

In the $\mathrm{SU}(3)$, $N_f = 6$ theory, there is another duality, called Argyres-Seiberg duality [11], replacing S-duality, but which no longer necessarily takes Lagrangian theories to Lagrangian theories. The constructions above generalize to this case, but now we can have two types of punctures ("maximal" and "minimal"), corresponding to two different possibilities for the defects in the 6d theory. These place constraints on the Coulomb branch (the allowed orders of poles in the differentials). As in the $\mathrm{SU}(2)$ case, Gaiotto extends the known duality to a duality on any $\mathrm{SU}(3)$ quiver gauge theory. However, the claim about the transitive action no longer holds, since we often get so-called "non-Lagrangian" theories. The basic example of this phenomenon is Argyres-Seiberg duality itself, which sends the weakly coupled $\mathrm{SU}(3), N_f = 6$ theory to an $\mathrm{SU}(2)$ theory coupled to a non-Lagrangian E_6 theory, the so-called "Minahan-Nemeschansky E_6 theory".

Similar constructions with a few more elaborations can also be generalized to $\mathrm{SU}(N)$ for arbitrary N. One refers to this as "Gaiotto duality" in general.

We can summarize the identifications described above in the following table, a "dictionary" between the geometry and the physics:

Physics	Geometry
UV curve	Punctured Riemann surface C
Seiberg-Witten curve	Branched covering $\pi : \Sigma \to C$
Coulomb branch	Base of the integrable system $\mathbf{B}$
Duality frame	Symplectic basis of $H_1(\Sigma)$
S-duality group	Mapping class group $MCG(C)$
Gauge coupling τ	Complex structure τ
Space of gauge couplings	$\mathcal{M}_{g,n}(C)$
Weakly-coupled Lagrangian description	Pants decomposition of C

2.2.3 Examples and zoology

In the same way that any punctured Riemann surface C can be glued out of three-punctured spheres, the basic building blocks of theories of class $\mathcal{S}$ are those corresponding to three-punctured spheres. The possible building blocks are specified by the integer K and the choice of defects.

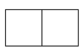
Some building blocks have a standard field theory description in terms of the usual matter multiplets of the $\mathcal{N} = 2$ SUSY algebra in §2.1.1, whereas others are much more mysterious, described as intrinsically strongly coupled (non-Lagrangian) SCFTs.

As per the construction via quiver gauge theories above, none of these building blocks involve vector multiplets. These are only introduced when gluing the three-punctured spheres. On the level of the $\mathcal{N} = 2$ theory this corresponds to "gauging" the corresponding flavour symmetry groups associated to punctures.

As a result, one can in some sense build up an understanding of complicated theories by studying theories on four-punctured spheres, thought of as being glued together from two such building blocks. The generalization to more gluings should be relatively straightforward, at least conceptually, once this basic case is understood. For this reason, our main examples in Chapters 5 and 6 will be the theories $T_K[C, \mathcal{D}]$ where $(C, \mathcal{D})$ is the four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$, with $q \in \mathbb{C} \setminus \{0, 1\}$, with the rank either $K = 2$ or $K = 3$. In the following we briefly review their geometry. For a complete classification of the class $\mathcal{S}$ theories of type A_{K-1} for small K, see [12].

Examples, $K = 2$

When $K = 2$ there is only one possible regular defect, labeled by the Young diagram



(2.2.7)

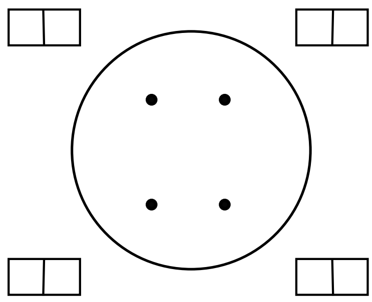


Figure 2.1: The UV curve for the theory $T_2[\mathbb{P}^1_{0,q,1,\infty}]$.

consisting of one row with two boxes. The mass parameters corresponding to this defect are taken to be generic, with $m_{l,2} = -m_{l,1}$. In the corresponding four-dimensional quantum field theory this defect corresponds to an $\mathrm{SU}(2)$ flavour symmetry group. In particular, there is only a single building block $T_2[\mathbb{P}^1_{0,1,\infty}]$.

Example. The theory $T_2[\mathbb{P}^1_{0,1,\infty}]$ describes a half-hypermultiplet in the trifundamental representation of $\mathrm{SU}(2)_0 \times \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_\infty$. Its Coulomb branch $\mathbf{B}$ is a single point corresponding to the quadratic differential

$$\varphi_2(z) = -\frac{m_\infty^2 z^2 - (m_\infty^2 + m_0^2 - m_1^2)z + m_0^2}{4z^2(z-1)^2}(dz)^2, \tag{2.2.8}$$

for fixed values of the parameters m_0, m_1 and m_∞. The combinations $\frac{\pm m_0 \pm m_1 \pm m_\infty}{2}$ correspond to the (bare) masses.

Gauge fields are introduced by gluing three-punctured spheres. The corresponding complex structure parameters q are identified with the gauge couplings $e^{2\pi i \tau}$. The limit $q \to 0$ corresponds to the weakly coupled description of the gauge theory at a cusp of the moduli space. For every pants cycle α there is a Coulomb parameter a_0, which is defined as the period integral $a_0 = \oint_A \lambda$ along a lift A of the pants cycle.

Example. The theory $T_2[\mathbb{P}^1_{0,q,1,\infty}]$ corresponds to the superconformal $\mathrm{SU}(2)$ gauge theory coupled to four hypermultiplets, see Figure 2.1. Its Coulomb branch $\mathbf{B}$ is 1-dimensional and parametrized by the family of quadratic differentials

$$\begin{aligned}\varphi_2(z) = -\Bigg(&\frac{m_0^2}{4z^2} + \frac{m^2}{4(z-q)^2} + \frac{m_1^2}{4(z-1)^2} \\ &+ \frac{m_0^2 + m^2 + m_1^2 - m_\infty^2}{4z(z-1)} - \frac{u}{z(z-q)(z-1)} \Bigg)(dz)^2,\end{aligned} \tag{2.2.9}$$

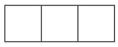
where the parameter u is free and the mass parameters m_0, m, m_1 and m_∞ are fixed. The combinations $\frac{m_0 \pm m}{2}$ and $\frac{m_1 \pm m_\infty}{2}$ correspond to the bare masses of the four hypermultiplets.

The corresponding Seiberg-Witten curve Σ is a genus one (after compactifying) covering of $\mathbb{P}^1_{0,q,1,\infty}$ with four simple branch points. Let A be the lift of the 1-cycle α going counterclockwise around the punctures at $z = 0$ and $z = q$; then the Coulomb parameter $a_0 = a_0(u)$ is defined as the period integral $a_0 = \oint_A \lambda$.

Examples, $K = 3$

The case $K = 3$ is the first in which we encounter different types of punctures. The two types will be referred to as "maximal" and "minimal" punctures[5]. For a maximal puncture z_l the mass parameters $m_{l,i}$ are generic with $m_{l,1} \neq m_{l,2}$, whereas for a minimal puncture $m_{l,1} = m_{l,2}$. In particular:

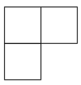
A ***maximal puncture*** is labeled by the Young diagram



(2.2.10)

consisting of one row with three boxes. In the corresponding quantum field theory this defect corresponds to an SU(3) flavour symmetry group.

A ***minimal puncture*** is labeled by the Young diagram



(2.2.11)

consisting of one row with two boxes and one row with a single box. In the corresponding quantum field theory this defect corresponds to a U(1) flavour symmetry group.

In terms of the Seiberg-Witten differential, a maximal puncture at $z = z_l$ turns into a minimal puncture if it satisfies two requirements:

(i) Two of the masses at the puncture coincide:

$$m_{l,1} = m_{l,2} = m_l. \tag{2.2.12}$$

(ii) The discriminant of

$$\lambda^3 + (z - z_l)^2 \varphi_2 \, \lambda + (z - z_l)^3 \varphi_3 \tag{2.2.13}$$

should vanish up to order $(z - z_l)^2$. This enforces two simple branch points of type (ij) of the covering to collide with the puncture at $z = z_l$.

[5] Some use the terminology "full" and "simple" instead, respectively.

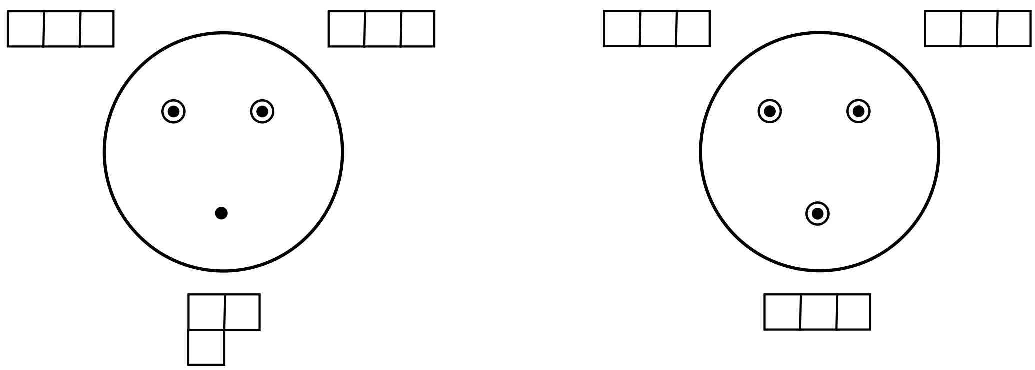


Figure 2.2: Left: the UV curve for the free bifundamental hypermultiplet $T_3[\mathbb{P}^1_{0,1,\infty}]$. Right: the UV curve for the non-Lagrangian E_6 theory $T_3[\mathbb{P}^1_{0,1,\infty}]$.

Example. The theory $T_3[\mathbb{P}^1_{0,1,\infty}]$ with three maximal punctures (Figure 2.2, right), is the so-called E_6 Minahan-Nemeschansky theory [13]. Microscopically (via the class S construction) its flavour symmetry group is $\mathrm{SU}(3)_1 \times \mathrm{SU}(3)_2 \times \mathrm{SU}(3)_3$, enhanced in the low energy limit to E_6. It is a non-Lagrangian theory.

The Coulomb branch $\mathbf{B}$ of the theory $T_3[\mathbb{P}^1_{0,1,\infty}]$ is described by the 1-dimensional family of differentials

$$\varphi_2 = \frac{c_\infty z^2 - (c_0 - c_1 + c_\infty)z + c_0}{z^2(z-1)^2}(dz)^2 \tag{2.2.14}$$

$$\varphi_3 = \frac{d_\infty z^3 + uz^2 + (d_0 + d_1 - d_\infty - u)z - d_0}{z^3(z-1)^3}(dz)^3, \tag{2.2.15}$$

where u is a free parameter, whereas the parameters c_l and d_l are fixed and can be written as combinations of $\mathrm{SU}(3)_1 \times \mathrm{SU}(3)_2 \times \mathrm{SU}(3)_3$ mass parameters. If we choose

$$c_l = \frac{1}{4}(-m_{l,1}^2 - m_{l,1}m_{l,2} - m_{l,2}^2) \tag{2.2.16}$$

$$d_l = \frac{1}{8}\left(m_{l,1}m_{l,2}(m_{l,1} + m_{l,2})\right) \tag{2.2.17}$$

then the residues at the punctures $z = l$ are $\{\frac{m_{l,1}}{2}, \frac{m_{l,2}}{2}, \frac{-m_{l,1}-m_{l,2}}{2}\}$, respectively.

The Seiberg-Witten curve Σ defines a 3-fold ramified covering over the UV curve $\mathbb{P}^1_{0,1,\infty}$, with generically six simple branch points. This implies that Σ is a punctured genus one Riemann surface. In contrast to weakly coupled gauge theories, the Seiberg-Witten curve has no distinguished A-cycle.

Recall that we sometimes write $(C, \mathcal{D})$ as $C_{z_1,\ldots z_n}$, and denote a minimal puncture by underlining the position of the puncture. Mass parameters are left implicit.

Example. The theory $T_3[\mathbb{P}^1_{0,\underline{1},\infty}]$ with two maximal and one minimal puncture (Figure 2.2, left), corresponds to a free hypermultiplet in the bifundamental representation of $\mathrm{SU}(3)_0 \times \mathrm{SU}(3)_\infty$. We find its Coulomb branch by applying the constraints (2.2.12) and (2.2.13) to the family of $T_3[\mathbb{P}^1_{0,1,\infty}]-$differentials described in equation (2.2.14) and (2.2.15) at $z = 1$. The latter constraint cuts down the only parameter we had:

$$u = \left(\frac{m_1}{2}\right)^3 - d_0 - 2d_\infty + \frac{m_1}{2}(c_0 - c_\infty). \tag{2.2.18}$$

so that the Coulomb branch is just a single point $\varphi^{\mathrm{bif}} = (\varphi_2^{\mathrm{bif}}, \varphi_3^{\mathrm{bif}})$.

These two examples provide the possible building blocks for $K = 3$ theories [10, 12]. Vector multiplets are introduced by gluing three-punctured spheres at maximal punctures (gluing elsewhere is not allowed). The (exponentiated) gauge coupling corresponds to the complex structure parameter q, where the gluing is performed in a standard way according to the transition $z_1 z_2 = q$ (see §3.4.1).

Example. The theory $T_3[\mathbb{P}^1_{0,\underline{q},\underline{1},\infty}]$ is the superconformal SU(3) gauge theory coupled to $N_f = 6$ hypermultiplets. It may be obtained by gluing two three-punctured spheres with two maximal and one minimal puncture. Its Coulomb branch $\mathbf{B}$ is parametrized by two parameters u_1 and u_2.

The explicit form of the differentials φ_2 and φ_3 can be obtained as before. First we write down the most general quadratic and cubic differential with regular poles at the punctures. Eight of the twelve parameters are fixed by writing the residues at each punctures in terms of the mass parameters. Two more parameters are fixed by additional requirements at both minimal punctures, analogous to equation (2.2.12) and (2.2.18). The resulting differentials are written down explicitly in Chapter 6.

2.3 Instanton counting

2.3.1 Nekrasov partition function

Formally, the partition function of a quantum field theory with action S is given by the path integral

$$Z = \int_{\mathrm{fields}} \mathcal{D}\Phi \, e^{-S} \tag{2.3.1}$$

where Φ is shorthand for all of the fields involved, and $\mathcal{D}\Phi$ is the heuristic (non-existent) measure on the space of fields.

For theories whose classical solutions are instantons, this expression can be factored as follows:

$$Z = Z_{\text{pert}} Z_{\text{inst}} \tag{2.3.2}$$

Here Z_{pert} is the perturbative (classical/tree and loop contributions coming from Feynman calculus), and Z_{inst} represents the nonperturbative contributions coming from the existence of instantons. In the 4d, $\mathcal{N} = 2$ theories we are interested in, it turns out that only 1-loop corrections occur, so in fact we can write:

$$Z = Z_{\text{class}} Z_{1-\text{loop}} Z_{\text{inst}}$$

The method for making sense of and evaluating these expressions in our setting was pioneered by Nekrasov [5]. The classical and 1-loop contributions had been known for some time, but it was not known was how to make sense of the seemingly infinite contributions from instantons. Nekrasov's answer to this was to introduce the so-called Ω-*deformation*, a two-parameter deformation of the theory. Without going into precise details, these are theories on spacetimes possessing a $\mathbb{T} = U(1) \times U(1)$ isometry group that allow us to render finite the integrals over instanton moduli spaces. The Nekrasov partition function (or instanton partition function) is then defined by

$$Z_{\text{inst}}(a, m, q, \epsilon_1, \epsilon_2) = \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_k} 1 \tag{2.3.3}$$

where 1 is interpreted as the equivariant cohomology class $1 \in H_T^*(\mathcal{M}_k)$, and $\mathcal{M}_k$ is a certain compactification of the moduli space of instantons on $\mathbb{R}^4$ with instanton number k. Here a collectively denotes the Coulomb branch parameters, m the mass parameters of the matter representations, and τ the coupling constants. One often writes $q = e^{2\pi i \tau}$ due to its occurrence in formulas, but really q should be interpreted formally here since we do not know if Z_{inst} converges. The integrals can be interpreted as certain explicit rational functions in $a, m, \epsilon_1, \epsilon_2$ (see [14]), so that Z_{inst} is a well-defined formal series with coefficients in $\mathbb{C}(a, m, \epsilon_1, \epsilon_2)$.

The fundamental claim (proved in [14, 6]) is that the Nekrasov partition function of a 4d, $\mathcal{N} = 2$ theory contains the information of the prepotential of the low-energy effective theory. In particular, it is obtained as the following limit:

$$\mathcal{F}(a, m, q) = \lim_{\epsilon_1, \epsilon_2 \to 0} \epsilon_1 \epsilon_2 \log Z(a, m, q, \epsilon_1, \epsilon_2) \tag{2.3.4}$$

Thus the Ω-deformation and Nekrasov partition function give a two-parameter deformation of the physical setup we started with. In the limit as $\epsilon_1, \epsilon_2 \to 0$, we amazingly recover the information of Seiberg and Witten's low-energy theory despite Z ostensibly having no knowledge at all about periods, differentials, etc., which convinces us of the correctness of their solution.

The Nekrasov partition function can also be understood as a field theory limit of topological strings, where the ϵ_1, ϵ_2 are interpreted as certain deformation parameters in the *refined topological string* (see e.g. [15]).

2.3.2 Localization

In the presence of supersymmetry, it is often possible to argue that the "infinite dimensional" path integral actually reduces to a finite dimensional integral over some "supersymmetric locus". Let $V[\Phi]$ denote a fermionic functional, and Q an odd element of the supersymmetry algebra such that QV has positive-semidefinite bosonic part, and $Q^2V = 0$. Suppose, furthermore, that T is a *topological* quantum field theory[6], which means that the observables have no dependence on the metric of spacetime. Then one can argue (e.g. [16]) that

$$Z_t = \int \mathcal{D}[\Phi] e^{-S-tQV} \tag{2.3.5}$$

is independent of t and therefore can be evaluated in the limit as $t \to \infty$ to compute $Z = Z_0$. In particular, the exponential is suppressed unless $QV = 0$, i.e. unless we are are on the locus where these equations are satisfied. So we expect that in fact the partition function should become an integral over the moduli space of solutions, which in our case are instantons:

$$Z = \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_k} 1 \tag{2.3.6}$$

This is how we can justify the indirect definition given above as correctly computing the path integral.

Terminology. The "semiclassical approximation" refers to summing over perturbations around all critical points of the action i.e. classical extrema. The point is that the semiclassical approximation is exact when we have supersymmetry in a topological theory as above.

[6]This phrase means different things to different people. A quantum field theory that is topological ("topological quantum field theory" to physicists) is as above. A topological QFT (or TQFT) to topologists is a certain kind of functor from a "bordism category" to the category of vector spaces.

The only obstruction to the above procedure for us is that the theory as formulated is not actually topological. Nekrasov circumvented this and applied localization by first using Witten's classic trick of "topological twisting" — roughly speaking, this is a method of modifying the theory by exploiting the R-symmetry group H_R (outer automorphisms fixing the even part) of $\mathfrak{spoin}$. The possible twists are classified by homomorphisms $\rho : \mathrm{Spin}(4) \to H_R$, and in particular Nekrasov chose a particular one known as the "Donaldson twist". The localization applied to the Donaldson twist of our original theory then gives the Nekrasov partition function as above.

2.3.3 Explicit formulas

It turns out that for $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ gauge theories (or more generally, any theories with an explicit Lagrangian description) the partition function can be explicitly evaluated. We will need these expressions to derive the superpotential for the theory we will eventually study. The conventions for the expressions given here are among the most standard in the literature, and can be found in [17].

The classical part is just

$$Z_{\text{clas}} = \exp\left[-\frac{1}{\epsilon_1\epsilon_2}\sum_i (2\pi i)\, a_i^2 \log q_i\right] \tag{2.3.7}$$

The only further perturbative corrections arise at one loop, so we only need:

$$\begin{aligned}
z^{1-\text{loop}}_{\text{vector}}(\vec{a}) &= \prod_{i<j} \exp[-\gamma_{\epsilon_1,\epsilon_2}(a_i - a_j - \epsilon_1) - \gamma_{\epsilon_1,\epsilon_2}(a_i - a_j - \epsilon_2)] \\
z^{1-\text{loop}}_{\text{fund}}(\vec{a},\mu) &= \prod_{i} \exp[\gamma_{\epsilon_1,\epsilon_2}(a_i - \mu)] \\
z^{1-\text{loop}}_{\text{antifund}}(\vec{a},\mu) &= \prod_{i} \exp[\gamma_{\epsilon_1,\epsilon_2}(-a_i + \mu + \epsilon_+)] \\
z^{1-\text{loop}}_{\text{bifund}}(\vec{a},\vec{b},m) &= \prod_{i,j} \exp[\gamma_{\epsilon_1,\epsilon_2}(a_i - b_j - m)]
\end{aligned}$$

Here $\epsilon_+ = \epsilon_1 + \epsilon_2$, and we have used $\gamma_{\epsilon_1,\epsilon_2}(x) = \log\Gamma_2(x+\epsilon_+,\epsilon_1,\epsilon_2)$ where

$$\Gamma_2(x,\epsilon_1,\epsilon_2) = \exp \left.\frac{d}{ds}\right|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t^2} \frac{t^s e^{-tx}}{(1-e^{-\epsilon_1 t})(1-e^{-\epsilon_2 t})} \tag{2.3.8}$$

Finally, the instanton pieces look as follows. It is known that $\mathrm{U}(N)$ instanton configurations centred at the origin of $\mathbb{R}^4$ can be labelled by an N-tuple of Young diagrams whose total number of boxes is the instanton number. The instanton partition function is expressed as a sum over instantons, and thus as a sum over N-tuples of Young diagrams. For linear quivers, they look like (recall that fund and antifund are different for U(2), so both must be included):

$$\begin{aligned} Z_{\text{inst}} = \sum_{\vec{Y}_1 \ldots \vec{Y}_N} & \left(\prod_{i=1}^{N} q_i^{|\vec{Y}_i|} z_{\text{vector}}(\vec{a}_i, \vec{Y}_i) \right) z_{\text{antifund}}(\vec{a}_1, \vec{Y}_1, \mu_1) z_{\text{antifund}}(\vec{a}_1, \vec{Y}_1, \mu_2) \\ \times & \left(\prod_{i=1}^{N-1} z_{\text{bifund}}(\vec{a}_i, \vec{Y}_i; \vec{a}_{i+1}, \vec{Y}_{i+1}) \right) z_{\text{fund}}(\vec{a}_N, \vec{Y}_N, \mu_3) z_{\text{fund}}(\vec{a}_N, \vec{Y}_N, \mu_4) \end{aligned}$$

So, for example, in the U(2) $N_f = 4$ theory we have:

$$\begin{aligned} Z_{\text{inst}}^{U(2), N_f=4} = & \sum_{\vec{Y}} q^{|\vec{Y}|} z_{\text{vector}}(\vec{a}, \vec{Y}) \\ & z_{\text{antifund}}(\vec{a}, \vec{Y}, \mu_1) z_{\text{antifund}}(\vec{a}, \vec{Y}, \mu_2) z_{\text{fund}}(\vec{a}, \vec{Y}, \mu_3) z_{\text{fund}}(\vec{a}, \vec{Y}, \mu_4). \end{aligned}$$

The individual pieces are given by certain simple rational expressions involving the "arm length" and "leg length" of the Young diagram. We refer the reader to [17] for the expressions.

To obtain formulas for SU(N) gauge groups, we take the U(N) results and simply impose the condition of tracelessness: $a_N = -a_1 - \ldots - a_{N-1}$.

2.4 Nekrasov-Shatashvili limit and the twisted superpotential

2.4.1 The cigar theory

So far, all of the constructions above were done on $\mathbb{R}^4$ equipped with the Minkowski or Euclidean metric; in particular, the theory $T_K[C, \mathcal{D}]$ was defined on this spacetime. Let us for a moment consider a slightly different setup, in which instead of $\mathbb{R}^4$, we take as our spacetime

$$X = D^2 \times \mathbb{R}^2, \tag{2.4.1}$$

where D^2 is topologically a disk with a cigar metric $ds^2 = dr^2 + f(r)d\phi^2$ — this means that $f(r) \sim r^2$ for $r \to 0$ and $f(r) \sim R^2$ for $r \to \infty$, for some constant $R > 0$. We should think of D^2 as a "cigar", a degenerate S^1 fibration over the nonnegative real axis $\mathbb{R}_{\geq 0}$, parametrized by $r \geq 0$.

The Ω-deformation still makes sense in this setup, with one of the planar rotations (say, corresponding to ϵ_1) replaced by the rotation of the cigar generated by $\partial/\partial\phi$. Furthermore, we consider the limit in which we turn off the remaining parameter ϵ_2, so the resulting theory T_ϵ is deformed by only one parameter (which we drop the subscript for and call ϵ). It can be shown that T_ϵ is invariant under a $\mathcal{N} = (2,2)$ super-Poincare algebra, and its low-energy behaviour turns out to be characterized by a 2d supersymmetric sigma model. Analogous to the prepotential $\mathcal{F}$ in four dimensions, this low energy theory is then characterized by a single holomorphic function, the "(effective) twisted superpotential" $\widetilde{W}^{\text{eff}}(a, m, q, \epsilon)$. This is the quantity we will be interested in computing — we can think of it as a kind of two-dimensional shadow of the four-dimensional partition function.

2.4.2 Effective twisted superpotential

It is argued in [18] that the effective twisted superpotential governing the low-energy limit of the cigar theory should be given by the expression ("the Nekrasov-Shatashvili limit"):

$$\widetilde{W}^{\text{eff}}(a, m, q, \epsilon) = \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z(a, m, q, \epsilon_1, \epsilon_2) \tag{2.4.2}$$

Since Z is a formal series in q and the first term of Z_{inst} is 1, we can define the log of this formally without trouble, and we see immediately that we have:

$$\widetilde{W}^{\text{eff}}(a, m, \epsilon) = \widetilde{W}^{\text{eff}}_{\text{clas}}(a, m, \epsilon) \log q + \widetilde{W}^{\text{eff}}_{1-\text{loop}}(a, m, \epsilon) + \sum_{k=1}^{\infty} \widetilde{W}^{\text{eff}}_{k}(a, m, \epsilon) q^k \tag{2.4.3}$$

as a formal series in q. Hence, in order to keep things mathematically well-defined, we will *define* $\widetilde{W}^{\text{eff}}$ to be this limit.

Definition. Let $(\mathfrak{g}, C, \mathcal{D})$ be a tuple of class S data. The ***effective twisted superpotential*** associated to $(\mathfrak{g}, C, \mathcal{D})$ is the Nekrasov-Shatashvili limit

$$\widetilde{W}^{\text{eff}}(a, m, q, \epsilon) = \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z(a, m, \epsilon, \epsilon_2, q) \tag{2.4.4}$$

Since it will be important for us later, we record the explicit expressions for the $SU(2)$, $N_f = 4$ theory up to one instanton. While the classical and instanton parts we give next are simply the NS limit of the previous section up to a normalization of a_i, the 1-loop part is more subtle due to the possible differences in regularization scheme one can use to define it. It turns out that only the magnitude of $Z_{1-\text{loop}}$ is canonically defined, leaving an ambiguity in the phase. For agreement with the generating function, we choose to use the 1-loop factor coming from the so-called "Liouville" scheme. Then the expressions are:

$$\widetilde{W}_{\text{clas}}(a,m,\epsilon) = \frac{1}{4\epsilon}a^2 \tag{2.4.5}$$
$$\widetilde{W}_{1-\text{loop}}(a,m,\epsilon) = \widetilde{W}_{\text{vect}}(a,\epsilon) + \widetilde{W}_{\text{hyp}}(a,m,\epsilon) \tag{2.4.6}$$
$$\widetilde{W}_1(a,m,\epsilon) = \prod_{l=1}^{4} \frac{(a+m_l+\epsilon)}{16a(a+\epsilon)} + \prod_{l=1}^{4} \frac{(a-m_l-\epsilon)}{16a(a-\epsilon)}. \tag{2.4.7}$$
$$\tag{2.4.8}$$

with

$$\widetilde{W}_{\text{vect}}(a,\epsilon) = -\frac{1}{2}\Upsilon(-a) - \frac{1}{2}\Upsilon(a) \tag{2.4.9}$$
$$\widetilde{W}_{\text{hyp}}(a,m,\epsilon) = \frac{1}{2}\sum_{l=1}^{4}\Upsilon\left(\frac{\epsilon+a+m_l}{2}\right) + \frac{1}{2}\sum_{l=1}^{4}\Upsilon\left(\frac{\epsilon-a+m_l}{2}\right), \tag{2.4.10}$$

where the special function Υ is given by

$$\Upsilon(x) = \int_{\frac{1}{2}}^{x} du \log\frac{\Gamma(u)}{\Gamma(1-u)}. \tag{2.4.11}$$

2.4.3 The NRS conjecture

If compactified to three dimensions along the S^1-fibre of the cigar D^2, the resulting theory may be studied in the low-energy limit as a three-dimensional $\mathcal{N}=4$ sigma model [7] with worldsheet $\mathbb{R}_+ \times \mathbb{R}^2$ into the Hitchin moduli space $\mathcal{M}$. The boundary condition at $r=0$ is known to be specified by the "brane of opers" **L** [19]. In §3.5 we describe in detail what these are; for now we simply mention they form a certain distinguished complex Lagrangian subvariety of the moduli space of flat connections. As a result, they possess a *generating function* in any Darboux coordinate chart, defined by $y_i = \partial_{x_i} W^{\text{oper}}$. Nekrasov-Rosly-Shatashvili proposed that, as a consequence of the physical picture above,

$$\boxed{\widetilde{W}^{\text{eff}}(a,m,q,\epsilon) = W^{\text{oper}}(a,\tilde{m},q,\epsilon)} \tag{2.4.12}$$

when we identify the Darboux coordinates x_i with the two-dimensional scalars a_i, and $\tilde{m}, m$ are related in a simple prescribed way [7].

[7] That is, with bosonic part a 3d sigma model whose target space metric is hyperkähler

More precisely, they studied this conjecture for $K = 2$, where they introduced a particular Darboux coordinate system on $\mathcal{M}_{\mathrm{dR}}^{\mathcal{C}}(C, \mathrm{SL}_2)$, which we will refer to as the NRS Darboux coordinates and coincide with the "complexified Fenchel-Nielsen coordinates" of Tan [20] and Kourouniotis [21]. They found that the correspondence (2.4.12) holds provided the generating function of the Lagrangian of SL_2-opers is expressed in the NRS Darboux coordinates. Chapters 5 and 6 are devoted to the study and generalization of these coordinates and the NRS conjecture.

Chapter 3

Preliminaries: Mathematics

In this chapter we will review some basic facts about the mathematics of connections and moduli spaces, which will play a fundamental role in the rest of the thesis. We will necessarily be cursory in our treatment — the reader is encouraged to consult the original papers in the bibliography for a complete understanding.

Notation. We will adhere to the notation that sheaves $\mathcal{A}_X^k$ refer to sheaves of smooth k-form valued sections of various objects, and Ω_X to sheaves of holomorphic k-form valued holomorphic sections. If $\mathcal{F}$ is a sheaf, then $\mathcal{F}(U)$ where U is an open set always denotes sections over U. For smooth (resp. holomorphic) sections of a smooth (resp. holomorphic) vector bundle E, we define $\mathcal{A}_X^k(E) := \mathcal{A}_X^k \otimes E$ (resp. $\Omega_X^k(E) := \Omega_X^k \otimes E$). Throughout, all our Lie groups G will be linear algebraic groups $G \subset \mathrm{GL}_K(\mathbb{C})$ or PSL.

3.1 Flat connections

Our main objects of study will be moduli spaces of flat connections on vector bundles over Riemann surfaces. Let us give some of the basic setup before proceeding, to fix notation and orient the reader.

Definition. Let E be a smooth complex vector bundle over a smooth manifold X. A ***connection*** on E is a $\mathbb{C}$-linear map

$$D : \mathcal{A}_X(E) \to \mathcal{A}_X^1(E)$$

satisfying the Leibniz rule:

$$D(fs) = df \otimes s + fDs \tag{3.1.1}$$

for $s \in \mathcal{A}_X(U)(E)$, $f \in \mathcal{A}_X(U)$. If X is a complex manifold, then a ***holomorphic connection*** over a holomorphic vector bundle $\mathcal{E}$ is one in which $\mathcal{A}_X$ is replaced with Ω_X, the holomorphic sections.

Definition. The ***curvature*** of a connection D is the map

$$D \circ D : \mathcal{A}_X(E) \to \mathcal{A}^2_X(E)$$

which can be thought of as a two-form taking values in $\mathrm{End}(E)$. In the holomorphic case, again replace $\mathcal{A}$ with Ω.

It follows immediately that a holomorphic connection on a holomorphic vector bundle over a Riemann surface is automatically flat, since the curvature is necessarily a $(2,0)$-form, of which there are none.

Monodromy Given a flat connection ∇, and a basepoint, x_0, we can define the ***parallel transport map*** around any loop γ based at x_0:

$$T_\gamma : E_{x_0} \to E_{x_0}$$

The fact that this map always exists and is well-defined amounts to the existence and uniqueness theorem for ODEs. Since ∇ is flat, it can be shown that the parallel transport map is independent of the precise curve, and depends only on its homotopy class. Thus, a flat connection yields representation

$$\rho : \pi_1(X, x_0) \to \mathrm{GL}(E_{x_0})$$

called the ***monodromy representation*** of ∇.

We are interested in studying the space of all connections, up to an appropriate notion of equivalence (that is, up to the action of bundle automorphisms or "gauge transformations"). We define the ***de Rham moduli space*** or ***moduli space of flat connections*** on E to be

$$\mathcal{M}_{\mathrm{dR}} = \{(\mathcal{E}, \nabla) \mid \nabla \text{ a connection on } \mathcal{E} \text{ and } F_\nabla = 0\} / \mathcal{G}$$

where $\mathcal{G}$ is the group of gauge transformations i.e. holomorphic sections of $\mathrm{End}(\mathcal{E})$. As written, this space is a quotient of an infinite-dimensional affine space by an infinite-dimensional group. While it is possible to treat this analytically, it turns out that it has a more algebraic avatar as well.

Definition. Let G be a reductive algebraic group. The ***character variety*** or ***Betti moduli space*** is the GIT quotient

$$\mathcal{M}_{\mathrm{B}}(C,G) = \mathrm{Hom}(\pi_1(C),G)/\!/G \tag{3.1.2}$$

where the action of G is by conjugation:

$$(g\cdot\rho)(x) = g\rho(x)g^{-1}$$

This space is an algebraic variety. To see this when $G \subset \mathrm{GL}_K(\mathbb{C})$, pick a set of generators $\{\gamma_1,\dots\gamma_k\}$, allowing us to identify $\mathrm{Hom}(\pi_1(C),G)$ with a subvariety of G^k. If we restrict to certain nice representations, then the categorical quotient is the same as the quotient by conjugation, so that $\mathcal{M}_{\mathrm{B}}$ is an algebraic variety. This algebraic structure can be shown to be independent of the choice of generators and embedding.

There is a correspondence between the character variety for the structure group of a bundle G and the moduli space of flat connections. One direction simply takes a connection to its monodromy: $\nabla \mapsto \mathrm{Hol}_p(\nabla)$ where p is a "basepoint". This map is invertible as follows. Given a representation $\rho : \pi_1(\widetilde{C}) \to G$, consider the principal G-bundle $P = \widetilde{C} \times_{\pi_1(C)} V$ where $\pi_1(C)$ acts on $\widetilde{C}$ by deck transformations, and on V via ρ. There is a canonical flat connection which one obtains from this. One can then check that the map descends to the quotient. The corresponding holomorphic vector bundle equipped with a flat connection is denoted V_ρ. We will blur the distinction between $\mathcal{M}_{\mathrm{dR}}$ and $\mathcal{M}_{\mathrm{B}}$ as a result of this correspondence, and almost always simply stick to the notation $\mathcal{M}_{\mathrm{dR}}$.

Definition. Let X be a complex manifold. A ***(complex, rank K) local system*** on X is a sheaf that is locally isomorphic to the constant sheaf $\underline{\mathbb{C}}^K$. Local systems on X, with morphisms the usual morphisms of sheaves, form a category, which we will denote $\mathrm{Loc}(X)$.

There is a categorical equivalence between local systems and holomorphic bundles equipped with holomorphic connections over a Riemann surface. Given a local section of $\mathbf{V}$, write $s = \sum f_i \mathbf{e}_i$. Define a connection on the bundle defined by $\mathcal{V} = \mathcal{O} \otimes_{\mathbb{C}} \mathbf{V}$ by $\nabla s = \sum df_i \otimes \mathbf{e}_i$. Conversely, a holomorphic connection is flat on a Riemann surface and so we can define the sections over U to be the parallel sections $\nabla s = 0$.

It is not hard to see that the category of rank K local systems and the category of representations of π_1 into GL_K are equivalent. Thus we can see the moduli space of local systems is just the same thing as $\mathcal{M}_{\mathrm{B}}$.

3.1.1 Singularities

We will mostly be concerned with the more interesting case in which connections can have singularities. The object we will be interested in is:

Definition. Let D be an effective divisor. A ***meromorphic connection with poles bounded by*** D is a $\mathbb{C}$-linear morphism of sheaves

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X(D) \tag{3.1.3}$$

The points in D will be called ***singularities*** of the connection. As before, a meromorphic connection on a curve is automatically flat since $\dim C = 1$. Restricting the connection to the complement $C \setminus |D|$, we still get a monodromy representation, but this time it is not enough data to classify them up to isomorphism. The Riemann-Hilbert correspondence in the next section tells us how to restrict our attention so as to keep the classification by monodromy data even when there are singularities[1].

If ∇ has simple poles at p (this is well-defined) we say it is ***logarithmic at*** p. If all the poles of ∇ are simple, then we call the connection itself ***logarithmic*** or ***Fuchsian***. We can define a notion of residue at such points:

Definition. Let ∇ be a connection on a holomorphic vector bundle $\mathcal{E}$, logarithmic at a singularity p. The **residue at** p, denoted $\mathrm{Res}_p\nabla$, is the linear map

$$\mathrm{res}_p\nabla : E_p \to E_p \tag{3.1.4}$$

given by writing in ∇ a local chart

$$\nabla = d - A = d - \left(\frac{A_{-1}}{z} + A_0 + A_1 z + \dots\right) dz \tag{3.1.5}$$

and setting $\mathrm{Res}_p(\nabla) = A_{-1}$. It is easy to check that this is well-defined, independent of the choice of chart.

The following is not hard to verify:

[1] More generally, one should consider *Stokes data* which gives us additional information about asymptotics near the singularities to distinguish between distinct meromorphic connections with the same monodromy.

Proposition 3.1.1. *Let ∇ be a meromorphic connection on $\mathcal{E}$, logarithmic at p. Then the monodromy of ∇ around a small counterclockwise loop γ around p is conjugate to* $\exp(2\pi i \operatorname{Res}_p(\nabla))$. *In particular, the eigenvalues of* $\operatorname{Hol}_\gamma \nabla$ *are* $\exp(2\pi i \cdot)$ *of the eigenvalues of* $\operatorname{Res}_p(\nabla)$.

3.1.2 The Riemann-Hilbert correspondence

The Riemann-Hilbert correspondence relates flat connections to their monodromy data, and also deals with the case of singularities if they are sufficiently nice. DDenote by Conn ($\mathrm{Conn}^{\mathrm{reg}}$) the category whose objects are meromorphic connections (with regular singularities along D) and whose morphisms are $\mathcal{O}_X(D)$-linear maps satisfying $\nabla \circ \varphi = (\mathbf{1} \otimes \varphi) \circ \nabla$. We have:

Theorem 3.1.2. *[22] Suppose X is a complex manifold, D is a divisor, and $Y := X \setminus D$. Then there is an equivalence of categories*

$$\mathrm{Conn}^{\mathrm{reg}}(X, D) \xrightarrow{\sim} \mathrm{Conn}(Y) \tag{3.1.6}$$

induced by the restriction functor $(\mathcal{E}, \nabla) \mapsto (\mathcal{E}, \nabla)|_Y$

Then, from the earlier correspondence, we conclude the following equivalence between regular (flat) meromorphic connections and monodromy data:

Corollary 3.1.3. *There is an equivalence of categories*

$$\mathrm{Conn}^{\mathrm{reg}}(X, D) \xrightarrow{\sim} \mathrm{Loc}(X \setminus D) \tag{3.1.7}$$

In particular, this yields the analytic isomorphism between $\mathcal{M}_{\mathrm{B}} \simeq \mathcal{M}_{\mathrm{dR}}$ that we will need.

3.2 The relative moduli space

Let C be a compact Riemann surface, and let $\{z_1, \ldots, z_n\}$ be n marked points, or ***punctures***. Furthermore, assume for each z_i we are given a Young diagram Y_i.

Let $\pi_1 = \pi_1(C \setminus \{z_1, \ldots, z_n\})$ denote the fundamental group of the punctured surface. Recall that the SL_K***-character variety*** is the GIT quotient

$$\mathcal{M}_{\mathrm{B}}(C, \mathrm{SL}_K) = \mathrm{Hom}\,(\pi_1, \mathrm{SL}_K) // \mathrm{SL}_K. \tag{3.2.1}$$

It is an affine variety with smooth locus given by the irreducible representations. We may take a standard presentation of the fundamental group by generators so that we can identify

$$\operatorname{Hom}(\pi_1, \mathrm{SL}_K) = \left\{(A_1, \ldots, A_g, B_1, \ldots, B_g, G_1, \ldots, G_n) \in \mathrm{SL}_K^{2g+n} \mid [A_1, B_1] \ldots [A_g, B_g]\, G_1 \ldots G_n = \mathbf{1}\right\} \tag{3.2.2}$$

and if we restrict to reductive or irreducible representations, $\mathcal{M}_{\mathrm{B}}^{\mathcal{C}}(C, \mathrm{SL}_K)$ is the usual geometric quotient by simultaneous conjugation.

The character variety is a holomorphic Poisson manifold on its smooth locus, but the Poisson structure is in general degenerate [23, 24, 25, 26]. If we fix the conjugacy classes of the monodromy around punctures, which we will collectively denote by $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)$, we end up on a symplectic leaf. This leaf, the ***relative character variety*** (with the boundary conjugacy classes $\mathcal{C}$) is denoted

$$\mathcal{M}_{\mathrm{dR}}^{\mathcal{C}}(C, \mathrm{SL}_K) \tag{3.2.3}$$

We will study relative character varieties where all $\mathcal{C}_i$ are semisimple and with multiplicities of the eigenvalues encoded by the Young diagram in the obvious way. We can write these as a tuple $\mu = (\mu_1 \ldots, \mu_n)$ of partitions of K, $\mu_i = (\mu_i^{(1)}, \ldots \mu_i^{(r_i)})$ with $\mu_i^{(1)} \geq \ldots \geq \mu_i^{(r_i)}$ and $\sum_{j=1}^{r_i} \mu_i^{(j)} = K$.

Condition 3.2.1. *[27] A tuple of semisimple conjugacy classes $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ in SL_K* ***satisfy the genericity condition*** *if there are no nonzero proper vector subspaces $V \subset \mathbb{C}^K$ stable under some $X_i \in \mathcal{C}_i$ for all i with the property that*

$$\prod_{i=1}^{n} \det\left(X_i|_V\right) = 1. \tag{3.2.4}$$

A moment of thought should convince the reader that this is a genuinely "generic" condition, and it is true in any case that generic conjugacy classes of any type always exist [27]. We can guarantee that we have no singularities to worry about in this case:

Theorem 3.2.2. *[27] If $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)$ is of type μ and satisfies the genericity condition, then $\mathcal{M}_{\mathrm{dR}}^{\mathcal{C}}$ is a smooth affine variety of pure dimension (whenever nonempty)*

$$\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{dR}}^{\mathcal{C}}(C, \mathrm{SL}_K) = (2g - 2 + n)K^2 - (2g - 2) - \sum_{i=1}^{n} |\mu_i|^2 \tag{3.2.5}$$

where for a partition $\mu_i = (\mu_i^{(1)}, \ldots, \mu_i^{(r_i)})$ of K, we set $|\mu_i|^2 := \sum_{j=1}^{r_i} (\mu_i^{(j)})^2$.

Finally, let us define the notion of maximal and minimal punctures for the relative character variety:

Definition. A puncture/conjugacy class is called ***maximal*** if all eigenvalues are distinct, and ***minimal*** if all but one eigenvalue coincide. In other words, they correspond to partitions with Young diagram consisting of either a single row of K boxes, or one column of $K-1$ boxes and one column with one box.

Perhaps the simplest example we can give is the case where $K = 2$ and there are three punctures, all (necessarily) maximal. Then $\mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K)$ is a single point, the unique local system arising from the unique hypergeometric equation (up to meromorphic equivalence) with specified monodromy conjugacies at the punctures. This generalizes when $K > 2$ to the well-known fact that the generalized hypergeometric equation is *rigid*:

Example. Let $K > 2$, and $C = \mathbb{P}^1$, with punctures at $0, 1, \infty$, and let $\mathcal{C}_0, \mathcal{C}_\infty$ be of maximal type, $\mathcal{C}_1$ minimal (no eigenvalues of which are Kth roots of unity). Then $\mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K)$ contains a single point, and this point corresponds to the Kth order generalized hypergeometric equation with appropriate exponents.

Example. Let $K = 2$, and $C = \mathbb{P}^1$, and $q \in \mathbb{C} \setminus \{0, 1\}$. Take punctures at $p = 0, q, 1, \infty$, with $\mathcal{C}_p$ all of one (the only) type $\mu = (1, 1)_{i=1,\ldots,4}, \mathcal{C}_1$. Then the dimension formula gives $\dim_{\mathbb{C}} \mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K) = 2$, which is the number of NRS coordinates as expected.

Example. Let $K = 3$, and $C = \mathbb{P}^1$, with punctures at $0, q, 1, \infty$, and let $\mathcal{C}_0, \mathcal{C}_\infty$ be of maximal type $(1, 1, 1)$, $\mathcal{C}_q, \mathcal{C}_1$ of minimal type $(2, 1)$. Then the dimension formula gives $\dim_{\mathbb{C}} \mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K) = 4$, which is the number of coordinates we will need to construct later on in the thesis.

The following proposition will be useful to us:

Proposition 3.2.3. *Suppose $M \in \mathrm{SL}_K(\mathbb{C})$, has eigenvalues $\lambda_1 \ldots \lambda_K$, and assume none of the λ_i are Kth roots of unity. Then M has $K-1$ coincident eigenvalues if and only if M is a a scalar multiple of a complex reflection matrix, that is,* $\mathrm{rk}(\lambda^{-1}M - \mathbb{I}) = 1$ *for some* $\lambda \in \mathbb{C}^\times$.

Proof. If M has $K-1$ coincident eigenvalues, say $\lambda_1 = \ldots = \lambda_{K-1} = \lambda$, $\lambda_K = \lambda^{-(K-1)}$, then $\mathrm{rk}(\lambda^{-1}M - I) = \mathrm{rk}\left(\mathrm{diag}\left\{1,\ldots,1,\lambda^{-K}\right\} - I\right)$ which is 0 if $\lambda^K = 1$, and 1 under the assumption.

Conversely, if there is $\lambda \in \mathbb{C}^*$ with $\mathrm{rk}\left(\lambda^{-1}M - I\right) = 1$ then $\mathrm{diag}\left\{\frac{\lambda_1}{\lambda} - 1, \ldots \frac{\lambda_{K-1}}{\lambda} - 1, \frac{\lambda_K}{\lambda}\right\}$ has at most one nonzero column, so all but one eigenvalue must coincide and equal λ. □

Thus with our semisimplicity assuption, minimal punctures are characterized exactly by having a multiple of a complex reflection for their local monodromy. In §3.6.4 we will see that a single minimal puncture is essentially the condition characterizing the generalized hypergeometric equation.

3.2.1 Trace functions

The definition of the GIT quotient defining the character variety is such that the coordinate ring

$$\mathbb{C}[\mathcal{M}_{\mathrm{B}}(C, \mathrm{SL}_K)] = \mathbb{C}[x_1, \ldots, x_r]^{\mathrm{SL}_K} \tag{3.2.6}$$

is precisely the set of invariants of the monodromy matrices associated to the curves γ. It turns out that by a theorem of Procesi [28], this ring is generated by the ***trace functions***. That is, the monodromy traces

$$\mathrm{tr}_\gamma : \mathcal{M}_{\mathrm{dR}}(C, \mathrm{SL}_K) \longrightarrow \mathbb{C} \tag{3.2.7}$$

$$[\rho] \longrightarrow \mathrm{tr}\,\rho(\gamma) \tag{3.2.8}$$

form a generating set for the algebraic (holomorphic) functions on $\mathcal{M}_{\mathrm{B}}(C, \mathrm{SL}_K)$, with or without conjugacy classes fixed. Thus, algebraically speaking anything we can study about our character variety can be understood by relating it to the trace functions. Later on, we will do precisely that and find formulas expressing the trace functions in terms of our generalized Fenchel-Nielsen coordinates.

3.3 Hitchin systems

3.3.1 Moduli of Higgs bundles

Higgs bundles were introduced by Hitchin [29] in the study of the dimensionally reduced Yang-Mills equations over a Riemann surface C. Since then, the study of their moduli spaces and related geometry has been a rich source of interest to both physicists and mathematicians. While we will not use Higgs bundles explicitly, they are deeply linked to our story both mathematically and physically. In particular, the ***Hitchin base*** of tuples of k-differentials will play an important role in the rest of the thesis. Thus we will give a brief summary to the reader here. For a fuller understanding, we direct the reader to original references and recommended reviews such as [29, 30, 31, 32, 33].

The following objects will be important throughout the rest of the thesis:

Definition. Let D be an effective divisor on C. A ***meromorphic k-differential with poles bounded by*** D is a section $\varphi \in H^0(C, K_C(D)^{\otimes k})$, where K_C is the canonical bundle of C. If $D = 0$, we say φ is a ***holomorphic k-differential***.

Locally, a meromorphic k-differential can be written as $\varphi_k = u(z)dz^k$ where u is a meromomorphic function. If D is reduced, so that φ_k has a pole of at most order k, we will say φ_k has ***regular singularities***.

Definition. A ***hyperkähler manifold*** is a $4n$ dimensional Riemannian manifold possessing 3 covariantly constant orthogonal endomorphisms I, J, K of the tangent bundle, satisfying the quaternionic relations:

$$I^2 = J^2 = K^2 = IJK = -\mathbf{1}$$

Note: all hyperkähler manifolds are Ricci-flat and therefore Calabi-Yau.

Definition. A ***Higgs bundle*** over a Riemann surface C is a pair $(\mathcal{E}, \Phi)$ where $\pi : \mathcal{E} \to C$ is a holomorphic vector bundle and $\Phi \in H^0(C, \mathrm{End}(\mathcal{E}) \otimes K_C)$.

These arise from the ***Hitchin equations***:

$$F_A + [\Phi, \Phi^*] = 0 \tag{3.3.1}$$

$$\bar{\partial}_A \Phi = 0 \tag{3.3.2}$$

where A is a unitary connection, $F_A \in \Omega^2_C(\mathrm{End}(\mathcal{E}))$ is its curvature, and Φ an adjoint-valued $(1, 0)$-form, and the adjoint is taken with respect to the hermitian metric h making A into its Chern connection. A ***Higgs pair*** is a solution (A, Φ) of the Hitchin equations. There is a one-to-one correspondence [29, 34] between irreducible Higgs pairs and "stable" Higgs bundles – that is, between irreducible solutions of Hitchin's equations and what turn out to be the smooth points in the moduli space. If we drop the condition of irreducibility, we arrive at "semistable"

Higgs bundles, which are still points in the moduli space but correspond to singularities.

The defintion of stability is as follows. Define the ***slope*** of a holomorphic vector bundle $\mathcal{E}$ to be $\mu(\mathcal{E}) := \deg\mathcal{E}/\mathrm{rk}\mathcal{E}$. Then $\mathcal{E}$ is ***stable*** if for every proper holomorphic subbundle $\mathcal{F}$,

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \tag{3.3.3}$$

For Higgs bundles $(\mathcal{E}, \Phi)$, we have the same definition verbatim, except that we demand only that (3.3.3) hold for Φ-invariant holomorphic subbundles. So a Higgs bundle might have an unstable underlying holomorphic bundle $\mathcal{E}$, but still be stable as a Higgs bundle by excluding the subbundles not preserved by Φ.

Then, fixing the determinant of the bundle, the ***moduli space of stable Higgs bundles (with fixed determinant)*** is the space

$$\mathcal{M}_H = \mathcal{A}/\mathcal{G}$$

where $\mathcal{A}$ is the space of all irreducible solutions (A, Φ) and $\mathcal{G}$ is the group of gauge transformations. This space is also called the ***Hitchin moduli space*** after its discoverer, or the ***Dolbeault moduli space*** to emphasize its close relation with the complex structure of C[2].

Some of the most important facts about $\mathcal{M}_H$ are:

(i) It is noncompact.

(ii) It is hyperkähler.

(iii) It is a quasiprojective algebraic variety.

(iv) It is smooth away from reducible solutions.

(v) There is a fibration $p : \mathcal{M}_H \to \mathbf{B}$ with p proper, making it into an integrable system.

(vi) When $G = \mathrm{SU}(K)$, it contains $T^*\mathcal{N}$ as a natural open dense submanifold, where $\mathcal{N}$ is the moduli space of (stable) holomorphic bundles of rank K and trivial determinant.

[2]In contrast, the Betti moduli space is actually independent of the complex structure of C, and encodes topological information in that sense.

If there are no punctures, then $\dim_{\mathbb{C}} \mathcal{M}_H = (2g-2)\dim G$.

The Hitchin moduli space is a noncompact hyperkähler manifold. To see this, one considers the description as a quotient of the infinite-dimensional affine space

$$\mathcal{W} = \left\{ (A, \phi) \mid A \text{ a connection, } \phi \in \Omega^{1,0}_C(\mathrm{End}(\mathcal{E})) \right\}$$

This is a symplectic affine space when equipped with any of the three symplectic structures:

$$\begin{aligned} \omega_I &= -\frac{1}{4\pi} \mathrm{Tr} \int_C \delta A \wedge \delta A - \delta\phi \wedge \delta\phi \\ \omega_J &= \frac{1}{2\pi} \mathrm{Tr} \int_C |dz|^2 \delta\phi_{\bar{z}} \wedge \delta A_z + \delta\phi_z \wedge \delta A_{\bar{z}} \\ \omega_K &= \frac{1}{2\pi} \int_C \mathrm{Tr}\, \delta\phi \wedge \delta A \end{aligned}$$

where $\delta A \in \Omega^1(\Sigma, \mathfrak{g})$ denotes a tangent vector to the infinite-dimensional affine space of connections. One also defines three complex structures I, J, K on this space in a similar way (we omit the formulas). Now, define a flat metric on $\mathcal{W}$ by:

$$ds^2 = -\frac{1}{4\pi} \int_C |dz|^2 \,\mathrm{Tr}\,(\delta A_z \otimes \delta A_{\bar{z}} + \delta A_{\bar{z}} \otimes \delta A_z + \delta\phi_z \otimes \delta\phi_{\bar{z}} + \delta\phi_{\bar{z}} \otimes \delta\phi_z) \quad (3.3.4)$$

The Hitchin equations can be written as the the vanishing of three moment maps (with respect to each of the symplectic structures).

$$\begin{aligned} \mu_I &= -\frac{1}{2\pi} \int_C \mathrm{Tr}\, \epsilon(F_A - \phi \wedge \phi) \\ \mu_J &= -\frac{1}{2\pi} \int_C |dz|^2 \,\mathrm{Tr}\, \epsilon(D_z \phi_{\bar{z}} + D_{\bar{z}} \phi_z) \\ \mu_K &= -\frac{1}{2\pi} \int_C |dz|^2 \,\mathrm{Tr}\, \epsilon(D_z \phi_{\bar{z}} - D_{\bar{z}} \phi_z) \end{aligned}$$

collectively denoted as $\vec{\mu}$, where ϵ is an infinitesimal gauge transformation. The hyperkähler quotient $\vec{\mu}^{-1}(0)/\mathcal{G}$ is clearly $\mathcal{M}_H$, and it is a theorem that this construction makes $\mathcal{M}_H$ into a hyperkähler manifold itself. The complex structures on $\mathcal{W}$ descend to complex structures on $\mathcal{M}_H$, which we continue to denote by I, J, K. In complex structure I, it is the moduli space of Higgs bundles.

How can we interpret the other complex structures? It turns out that any sufficiently nice (say, reductive) flat $G_{\mathbb{C}}$ connection ∇ (or $\mathcal{A}$), where $G_{\mathbb{C}}$ denotes the complexification of G, can be decomposed as

$$\nabla = D + \Phi + \Phi^* \quad (3.3.5)$$

where D (or A) is a *unitary* flat connection, and Φ a Higgs field, such that (A, Φ) solves the Hitchin equations. Thus (slightly glossing over stability considerations and precise details), the Hitchin moduli space is also the moduli space of flat $G_{\mathbb{C}}$ connections. The complex structures $J^{(\zeta)}, \zeta \in \mathbb{C}^*$ are all equivalent, and equal to the one arising from this description. For notation, in the future we parametrize these complex structures with a point $\zeta \in \mathbb{C}P^1$ so that the case $\zeta = 0, \infty$ corresponds to Higgs bundles, and $\zeta \in \mathbb{C}^*$ corresponds to flat connections. This correspondence is known as the ***nonabelian Hodge correspondence*** due to Hitchin and Simpson, and has been developed and extended in many ways since [29, 34, 35, 36].

3.3.2 Integrable systems

What makes the Hitchin moduli space so interesting is that, in addition to the properties above, it is (in the complex structure I) also an ***algebraically integrable system***, which is a complex integrable system with some extra conditions we will omit. In particular, we consider the map

$$p : \mathcal{M}_H \to \bigoplus_i H^0(C, K_C^{\otimes d_i}) := \mathbf{B} \tag{3.3.6}$$

where d_i are the degrees of the invariant polynomials on $\mathfrak{g}$, and $\mathbf{B}$ is called the ***Hitchin base***. Equipping $\mathcal{M}_H$ with the holomorphic symplectic form $\Omega_I = \omega_J + i\omega_K$ gives it the structure of a holomorphic symplectic manifold. Then p is a proper holomorphic map whose generic fibre is a Lagrangian abelian variety (in particular, a compact complex torus), making it into an algebraically integrable system.

As a concrete example, we can consider the case of $G = \mathrm{SU}(2)$, C compact with $g \geq 2$. In this case the Hitchin map $p : \mathcal{M}_H \to H^0(C, K_C^{\otimes 2})$ is just

$$(\mathcal{E}, \varphi) \mapsto \operatorname{tr} \varphi^2. \tag{3.3.7}$$

More generally one may choose a basis $\{\theta_i\}_{i=1\dots 3g-3}$ for the quadratic differentials and expand

$$p(E, \varphi) = \sum_{i=1}^{3g-3} H_i \theta_i \tag{3.3.8}$$

yielding $3g - 3$ holomorphic functions H_i, sometimes called ***Hitchin's Hamiltonians***, which Poisson commute, exhibiting the structure of an integrable system.

3.3.2.1 Singularities

So far, we have considered holomorphic Higgs bundles on compact Riemann surfaces. For our purposes, we will need to consider ***meromorphic Higgs bundles*** on punctured Riemann surfaces.

Given an effective divisor D, passing to meromorphic Higgs bundles requires a few modifications. The Higgs fields are now sections $\Phi \in H^0(C, \mathrm{End}(\mathcal{E}) \otimes K_C(D))$, the Hitchin base is now $\mathbf{B} = \bigoplus_i H^0(C, K_C(D)^{\otimes d_i})$. A more in-depth description of Higgs bundles with values in any line bundle can be found in original references such as [30, 31].

It is also possible to arrive at these spaces by considering the Hitchin equations with delta function sources [1, 37], though we do not take that point of view here.

3.4 Teichmüller space and Fenchel-Nielsen coordinates

3.4.1 Teichmüller space

The Teichmüller space is the geometric space arising from the study of all of the complex structures on a fixed topological surface S of genus g. By the uniformization theorem, this is equivalent (whenever $\chi(S) < 0$, as will always be the case for us) to studying hyperbolic metrics of constant curvature -1 on the surface. Usually, we will have a particular punctured Riemann surface C in mind, with underlying topological surface S. Then the ***Teichmüller space*** of C is the space of hyperbolic structures on S modulo equivalence by diffeomorphisms isotopic to the identity:

$$\mathcal{T}(S) = \{\text{hyperbolic structures on } S\}/\mathrm{Diff}_0(S) \tag{3.4.1}$$

The Teichmüller space can be given a topology either via coordinates, or more directly as follows. Let $U(X, \epsilon)$ be the set of $[X] \in \mathcal{T}(S)$ such that for every simple closed curve $\gamma \subset S$ with marking $\varphi : X \to S$, $|\log \ell_\gamma(X) - \log \ell_\gamma(C)| < \epsilon$. These form a basis for the topology, which coincides with the one coming from the Fenchel-Nielsen coordinates defined in the next section.

There is a crucial link between flat connections and Teichmüller space. In particular, there is an embedding of the Teichmüller space $\mathcal{T}(S)$ as a connected component of $\mathcal{M}_{\mathrm{dR}}(C, \mathrm{PSL}(2,\mathbb{R}))$ (for any C with underlying surface S). Since $\mathcal{M}_{\mathrm{dR}}(C, \mathrm{PSL}(2,\mathbb{R}))$ can be viewed as a real slice of $\mathcal{M}_{\mathrm{dR}}(C, \mathrm{PSL}(2,\mathbb{C}))$, we can thus make the link between the original "Fenchel-Nielsen" coordinates on the Teichmüller space and the restrictions of coordinates on the moduli space of flat rank 2 connections on C.

Kra's plumbing construction

Let us mention a procedure due to Kra [38] for gluing together three-holed spheres to construct Riemann surfaces. Fix q with $|q| < 1$. Take two copies D_1, D_2 of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and consider coordinates z_1, z_2 around 0. Let A_1, A_2 be annuli given by:

$$A_i = \left\{ z \in D_i \quad : \quad |q|^{\frac{1}{2}} < |z| < |q|^{-\frac{1}{2}} \right\} \tag{3.4.2}$$

Given two points $P_1 \in A_1$ and $P_2 \in A_2$, identify them if

$$z_1(P_1) z_2(P_2) = q$$

This gives a Riemann surface with transition function $z_i = q/z_j$ on the overlap. We can view q as a holomorphic coordinate on the Teichmüller space in a small punctured disk around 0, with 0 corresponding to the nodal curve in the boundary. One may iterate this process with more spheres or different values of q to construct Riemann surfaces of any topological type and any complex structure (at least in a neighbourhood of the nodal curve).

Fenchel-Nielsen coordinates

There is a well-known set of coordinates on the Teichmüller space which arises from a considering the surface as a collection of three-holed spheres. Suppose that C has genus g and n boundary components.

Definition. A ***pants decomposition*** of a Riemann surface C is a collection of mutually nonintersecting closed curves $P = \{c_1, \ldots c_{3g-3+n}\}$ whose complement is a disjoint union of pairs of pants.

A fundamental and well-known fact is the following:

Proposition 3.4.1. *The hyperbolic (and thus conformal) structure of a pair of pants S is uniquely determined by the lengths of its three boundary curves c_1, c_2, c_3. Conversely, for any $l_1, l_2, l_3 > 0$ there exists a pair of pants S with boundary curves c_1, c_2, c_3 of these lengths.*

So a pair of pants is determined entirely by the lengths of its boundary circles. This underlies the idea that we can (and should) study the hyperbolic (or complex) structure of Riemann surfaces by decomposing them into pairs of pants.

It can be shown that each pants curve is homotopic to a unique closed geodesic. The lengths of these geodesics are called the ***Fenchel-Nielsen length coordinates***, which evidently tell us something about the hyperbolic metric on C. Intuively, we can imagine then that knowing these numbers will help specify which point in the Teichmüller space C represents. This information is not enough however (as is obvious from the dimensions); it also matters how the pairs-of-pants are glued together.

The other half of the coordinates, called ***twist coordinates*** are given by measuring the deviation or "twisting" of geodesics from one pair of pants to another. These involve choosing another, complementary set of $3g-3+n$ curves. The details can be found in e.g. [39]. One consequence of the fact that the Fenchel Nielsen coordinates form a global coordinate system is

Theorem 3.4.2. *The Teichmüller space $\mathcal{T}(S)$ is homeomorphic to $(\mathbb{R}_+)^{3g-3+n} \times (\mathbb{R}_{\geq 0})^{3g-3+n}$. In particular, it is contractible.*

3.4.2 The Teichmüller component in $\mathcal{M}_{\mathrm{dR}}$

Let C be a point in $\mathcal{T}(S)$. Given its corresponding hyperbolic metric $h = e^{2\varphi}dzd\bar{z}$, we write $\nabla = \nabla' + \nabla''$

$$\nabla' = \partial + \begin{pmatrix} 0 & (\partial_z\varphi)^2 - \partial_z^2\varphi \\ 1 & 0 \end{pmatrix}, \qquad \nabla'' = \bar{\partial}. \tag{3.4.3}$$

which gives us a flat bundle on S. Conversely, given a flat $\mathrm{PSL}(2,\mathbb{C})$-bundle on S, the hyperbolic metric can be reconstructed from the solution to $\nabla s = 0$. As a result, there is a canonical connected component in $\mathcal{M}_{\mathrm{dR}}(C, \mathrm{PSL}_2(\mathbb{R})) \subset \mathcal{M}_{\mathrm{dR}}(C, \mathrm{PSL}_2(\mathbb{C}))$ identified with the Teichmüller space $\mathcal{T}(C)$. It is the set of all representations $\rho : \pi_1(C) \to \mathrm{PSL}_2(\mathbb{R})$ such that $\mathbb{H}/\rho$ is homeomorphic to C.

3.4.3 Complexified Fenchel-Nielsen coordinates

The classical Fenchel-Nielsen coordinates defined above were defined on the Teichmüller space. It turns out they can be found as the restriction of a set of holomorphic Darboux coordinates on the PSL_2 character variety. These coordinates were constructed by Tan [20] and Kourouniotis [21] and were studied by others as well [40, 39]. These are the coordinates found by Hollands-Neitzke [41] to arise as spectral coordinates, and which we will generalize in this thesis. Explicit formulas for them are written down later in §5.2.2.

3.5 Opers

A central object of our study will be a special subspace of the moduli space of flat connections, the space of so-called "opers" $\mathbf{L} \subset \mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K)$ (we leave C, $\mathcal{C}$ etc. implicit in the notation).

Let C be a compact Riemann surface. Throughout, we will fix a square root of the canonical bundle $K_C^{\frac{1}{2}}$ (also known as a spin structure or theta-characteristic). There are 2^{2g} possibilities for this choice (which is equivalent to choice of canonical lift of representations from $\mathrm{PSL}_2(\mathbb{C})$ to $\mathrm{SL}_2(\mathbb{C})$), and the connected components of $\mathbf{L}$ will labelled by this discrete data.

Definition. Let D be an effective divisor on C. A ***meromorphic*** SL_K***-oper*** $(\mathcal{E}, \{\mathcal{E}_i\}, \nabla)$ ***with poles bounded by*** D is a rank K vector bundle $\mathcal{E}$ over C, equipped with a flat meromorphic connection ∇ and a filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset ... \subset \mathcal{E}_{K-1} \subset \mathcal{E}_K = \mathcal{E}$ satisfying:

(i) $\nabla \mathcal{E}_i \subset \mathcal{E}_{i+1} \otimes \Omega^1_C(D)$

(ii) The induced maps $\widehat{\nabla} : \mathcal{E}_i/\mathcal{E}_{i-1} \to (\mathcal{E}_{i+1}/\mathcal{E}_i) \otimes \Omega^1_C(D)$ are isomorphisms.

(iii) $\mathcal{E}$ has trivial determinant (with fixed trivalization) and ∇ induces the trivial connection on it.

One may generalize this defintion to GL_K by dropping the third condition, or more generally to any algebraic group using principal bundles as in [42]. We will however be only concerned with SL_K-opers, in particular with $K = 2, 3$.

Not all bundles with connection admit an oper structure, but it can be shown that those which do admit a unique such structure[3]. Therefore if we fix the underlying bundle and the conjugacy classes we can think of the space of opers $\mathbf{L}$ as embedded into $\mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K)$.

The defining conditions imply that locally one may always choose a trivialization so that SL_K-opers can be put in the following form, uniquely:

[3] A proof in the compact (holomorphic) case can be found in [33], and with singularities in [43].

$$\partial_z + \begin{pmatrix} 0 & t_2 & \dots & t_K \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}. \tag{3.5.1}$$

It is thus clear that locally an oper is equivalent to an Kth order linear ODE, whose $(K-1)$th derivative term vanishes in the SL_K case. However, the relation between the coefficients t_i in different coordinate charts is not nice. There is an alternative "canonical form" one may give which results in nice transformation laws. First we need the notion of a differential operator between (line) bundles:

Definition. Let $\mathcal{E}, \mathcal{F}$ be holomorphic vector bundles over C. A linear map

$$\mathbf{D} : \mathcal{E} \to \mathcal{F} \tag{3.5.2}$$

is a ***k-th order (linear) differential operator*** if for any $s \in \mathcal{E}$ and $p \in C$, $j_k(s)(p) = 0$ implies $(\mathbf{D}s)(p) = 0$, where $j_k(s)$ denotes the k-jet of s. The sheaf of vector spaces of all such operators is denoted $\mathrm{Diff}_k(\mathcal{E}, \mathcal{F})$.

It is not hard to see that the map which sends $T \in \mathrm{Hom}(J^k(E), F)$ to $s \mapsto T_*(j_k(s))$ is an isomorphism of vector spaces $\mathrm{Hom}(J^k(E), F) \simeq \mathrm{Diff}_k(\mathcal{E}, \mathcal{F})$. The restriction

$$\sigma : \mathrm{Diff}_k(E, F) \to \mathrm{Hom}\,(K_X^{\otimes k} \otimes E, F) \tag{3.5.3}$$

is called the ***principal symbol*** of $\mathbf{D}$.

Now we introduce the notion of a "local system realized in a line bundle $\mathcal{L}$". By definition this is a local system $\mathbf{V}$ and an exact sequence

$$0 \longrightarrow \mathbf{V} \xrightarrow{\varphi} \mathcal{L} \xrightarrow{\mathbf{D}} \mathcal{L} \otimes K_C^K \longrightarrow 0$$

where $\mathbf{D}$ is a Kth order differential operator. Then we may observe that there is a correspondence between opers and local systems realized in $K_C^{(1-K)/2}$:

$$0 \longrightarrow \mathbf{V} \xrightarrow{\varphi} K_C^{(1-K)/2} \xrightarrow{\mathbf{D}} K_C^{(K+1)/2} \longrightarrow 0$$

To be precise, we may state the following ([44, 33]):

Theorem 3.5.1. *Let* $\mathbf{D} : K_C^{(1-K)/2} \to K_C^{(K+1)/2}$ *be a* $\mathbb{C}$*-linear differential operator, locally of the form*

$$\mathbf{D}y = y^{(K)} + Q_2 y^{(K-1)} + \dots + Q_K y \tag{3.5.4}$$

Then $\frac{12}{K(K^2-1)}Q_2$ *transforms as a projective connection, and for* $k \geq 3$*, there exist* w_k*, linear combinations of* Q_j *(*$j = 2, \ldots, k$*), and their derivatives, with coefficients polynomials in* Q_2*, such that* w_k *transform as k-differentials. Conversely, given one such operator and k-differentials* $w_k, k = 2, \ldots, n$*, these conditions uniquely determine the operator* $\mathbf{D}$*.*

The map from opers to differential operators in this theorem is explicitly given as follows. If ∇ is an oper and given y locally of the form $y(z)\,dz^{\otimes(1-K)/2}$, let $\{\phi_1 \ldots \phi_K\}$ be a set of linearly independent local flat sections of ∇. Then define

$$\mathbf{D}y := \det \begin{pmatrix} \phi_1 & \phi_2 & y \\ \vdots & \vdots & \vdots \\ \phi_1^{(K+1)} & \phi_K^{(K+1)} & y^{(K+1)} \end{pmatrix} \tag{3.5.5}$$

from which the transformation properties follow. The proof allows for an explicit calculation of w_k in terms of Q_k and vice-versa, which we write down for $k = 2, 3$ as these are the cases we will need:

$$Q_2 = w_2 \tag{3.5.6}$$

$$Q_3 = w_3 + \frac{K-2}{2} w_2' \tag{3.5.7}$$

In particular, the above theorem implies the following "oper transformation law" for the nontrivial component t_2 of an SL_2-oper, whenever y and w are local holomorphic coordinates with overlapping domains:

$$t(y) \mapsto (y'(w))^2 t(y(w)) + \frac{1}{2}\{y, w\} \tag{3.5.8}$$

where $\{y, w\}$ (also written $\mathcal{S}(y)$) is the ***Schwarzian derivative*** of y:

$$\{y, w\} := \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2.$$

Any collection of local expressions related in this way are in fact a known geometric object — this is precisely the defining condition for a ***projective connection*** on a Riemann surface. In particular, while it is not a quadratic differential, any two SL_2-opers differ by a quadratic differential. We can take a fixed projective connection P_0 and write any other one as $P = P_0 + \sum_{i=1}^{3g-3+n} H_i \theta_i$. The H_i are called ***accessory parameters***. Thus, the space of SL_2-opers is an affine space modelled on the quadratic differentials. From this result and the above theorem, it follows that (each connected component of) $\mathbf{L}$ is an affine space modelled on the Hitchin base

$\mathbf{B} = \bigoplus_{i=2}^{K} H^0(C, K_C^i)$. More generally, when singularities along a divisor D are present, we simply tensor with $\mathcal{O}_X(D)$ and restrict the residues at each puncture if desired.

In fact, it turns out that for an oper the bundle $\mathcal{E}$ is always just a jet bundle in disguise. If we let $\mathcal{L} := K_C^{\frac{1-K}{2}} = \mathcal{E}/\mathcal{E}_{K-1}$ be the top graded piece of the filtration, we will have, given an element $\varphi \in \mathbf{B}$ (and some additional choices, including a choice of principal sl_2 yielding a grading $\mathfrak{g} = \oplus V_i$) a canonical isomorphism F_φ:

$$\begin{array}{ccc} J^{K-1}(\mathcal{L}) & \xrightarrow{F_\varphi} & \mathcal{E} \\ & \searrow^{\text{pr}} & \downarrow{\scriptstyle \pi_{K-1}} \\ & & \mathcal{L} \end{array}$$

where pr is is the projection onto the 0-jet , and π_{K-1} is the quotient map.

The choices are as follows. Assume we have made a choice of Cartan subalgebra and positive roots $\{\alpha_i\}$ for $\mathfrak{g} = sl_K$, so we have a Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{3.5.9}$$

and $\mathfrak{n}_-$, $\mathfrak{n}_+$ are spanned by the so-called Cartan generators $\{f_i\}, \{e_i\}$, respectively. To see the isomorphism, we put the oper in the canonical local form

$$\partial_z + p_{-1} + \mathbf{v}(z) \tag{3.5.10}$$

where p_{-1} is the lower-triangular element of an sl_2-triple, and $\mathbf{v}(z) \in V^{\text{can}} = \oplus_{i=1}^{K-1} V_i^{\text{can}}$. Here V^{can} is the space of adp_1-invariants in $\mathfrak{n} = \mathfrak{n}_- \oplus \mathfrak{n}_+$, with V_i^{can} determined by the principal grading.

Let us write a choice of e_i and f_i out in the case of sl_3 with the usual decomposition into upper- and lower- triangular and diagonal subalgebras:

$$e_1 = E_{1,2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = E_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.5.11}$$

for $\mathfrak{n}_+$, and f_i their transposes. Then

$$p_{-1} = \sum_i f_i = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.5.12}$$

The grading, having chosen our principal sl_2, is

$$V_1^{\text{can}} = \text{span}\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \qquad V_2^{\text{can}} = \text{span}\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \tag{3.5.13}$$

The choices correspond to a basis element of each of these 1-dimensional V_i^{can} in the $K = 3$ case can be written as

$$\partial_z + \begin{pmatrix} 0 & \mu\varphi_2 & \nu\varphi_K \\ 1 & 0 & \mu\varphi_2 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.5.14}$$

so that we have some additional freedom beyond what the earlier theorem suggests – setting $\mu = 1/2, \nu = 1$ we recover the previous isomorphism. The only remaining non-canonicity is in the choice of principal sl_2 subalgebra.

It is instructive to note some of the different incarnations of projective connections.

Definition. A ***projective structure*** on C is an atlas of holomorphic charts for C whose transition functions consist entirely of restrictions of Möbius transformations.

Note that a projective connection and a projective structure are equivalent data, as follows: given a projective structure we simply take the projective connection to be ∂_z^2 in all coordinate charts. Since Möbius transformations are killed by the Schwarzian derivative, this is well-defined. Conversely, given a projective connection S and a point $p \in C$, consider the equation

$$\left(\partial_z^2 + \frac{1}{2}S\right)\phi = 0. \tag{3.5.15}$$

Let ϕ_1, ϕ_2 be linearly independent solutions in some small neighbourhood U. Then the coordinate $y := \phi_1/\phi_2$ is a chart in the atlas. The union of all such charts yields a projective structure.

Thus, once we have chosen a projective structure or projective connection, we may identify (the chosen connected component of) the space of opers with the Hitchin base, as the Schwarzian derivative term in the transformation law drops out, leaving us with an identification of t_2 with a quadratic differential. In particular, we may use the projective structure coming from Fuchsian uniformization – that is, if w is the uniformization coordinate, the corresponding projective connection is ∂_w^2.

Remark. For compact C, the variety $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}_2(\mathbb{C}))/\!/\mathrm{PSL}_2(\mathbb{C})$ splits into two irreducible connected components: those which lift to a representation of $\mathrm{SL}_2(\mathbb{C})$, and those which do not. The monodromy of projective structures always lift, and there are precisely 2^{2g} choices of such a lift. In fact, a theorem of Gallo, Kapovich, and Marden [45] identifies precisely the representations which arise as monodromies of projective structures: they are the "non-elementary" representations which lift to an SL_2 representation.

3.5.1 Generating function

It turns out that the opers (with fixed boundary monodromy) form a smooth, complex Lagrangian subvariety of $\mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K)$ with respect to the holomorphic symplectic structure Ω_J [43]. The half-dimensionality follows from the isomorphism with the Hitchin base described above, whereas the vanishing of the symplectic form follows from the infinite-dimensional Atiyah-Bott description and noting the integrand vanishes in the "oper gauge" (3.5.1).

In this thesis we will be interested in computing the generating function of the Lagrangian of opers. Let us explain what this means precisely:

Definition. Let C be a punctured curve, and $\mathcal{C}$ a collection of conjugacy classes at the punctures. Suppose we have holomorphic Darboux coordinates $\{\alpha_i, \beta_i\}$ on a neighbourhood $U \subset \mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K)$, and let $\mathbf{L}$ be the complex Lagrangian subvariety of opers. Suppose furthermore that all $\partial/\partial\beta_i$ are transverse to $\mathbf{L}$. A ***generating function for*** $\mathbf{L}$, relative to the the coordinate system $\{\alpha_i, \beta_i\}$, is a (unique up to additive constant) holomorphic function $W^{\mathrm{oper}} : \mathbf{L} \cap U \to \mathbb{C}$ such that if ∇ is an oper in the domain of W^{oper}, then

$$\beta_i(\nabla) = \frac{\partial W^{\mathrm{oper}}}{\partial \alpha_i}(\nabla) \tag{3.5.16}$$

for all i.

3.6 Monodromy of linear ODEs in the complex domain

3.6.1 Connections on $\mathbb{P}^1$

Consider vector bundles equipped with flat connections $(\mathcal{E}, \nabla)$ on $\mathbb{P}^1$; the parallel transport condition defines a system of ODEs, and if ∇ is an oper it gives a scalar equation whose monodromy is the holonomy of the connection. Conversely, given a linear ODE on $\mathbb{C}$ with coefficients whose singularities lie at at $z_1 \dots z_n$ we can consider its sheaf of solutions, which is locally free and thus corresponds to a vector bundle with flat connection over $\mathbb{C} \setminus \{z_1 \dots z_n\}$. The corresponding connection is locally just the ODE written in "companion matrix" form.

Because of this equivalence, we are interested in studying the monodromy of some particular ODEs that will be of importance to us. In particular, since all our bundles are holomorphic, we are interested in the equation

$$Lw := a_0(z)w^{(K)} + a_1(z)w^{(K-1)} + \dots + a_{K-1}(z)w' + a_K(z)w = 0 \tag{3.6.1}$$

defined on $\mathbb{P}^1 \setminus \{z_1 \dots z_n\}$ where a_i are all holomorphic and a_0 vanishes only at the $\{z_1 \dots z_n\}$, called the ***singular points*** or ***singularities*** of the equation. A singular point is called ***regular singular*** if the solutions grow at most polynomially in every sector around it, or irregular otherwise. An equation whose singularities are all regular is called ***Fuchsian*** (an important point is to note that this is true only for *equations*, and the word "Fuchsian" takes on different meaning for systems that are not equations). We can characterize the equations on $\mathbb{P}^1$ with this property with the following proposition [46].

Proposition 3.6.1. *Let $z \in \mathbb{P}^1$. Then*

1. *The system $\frac{dx}{dz} = Ax$ has finite singularities $z_1, \dots z_m$ which are regular singular if and only if*
$$A(z) = \sum \frac{A_j}{z - z_j}$$
2. *The equation $x^{(K)} + b_1(z)x^{(K-1)} + \dots b_K(z)x = 0$ has singular points $t_1, \dots, t_m$ that are regular singular if and only if*
$$b_j(z) = \frac{P_j(z)}{Q^j(z)}$$
where $Q(z) = (z - z_1) \dots (z - z_m)$ and P_j are polynomials of degree $\leq (m-1)j$.

Viewing the equation as a connection on $\mathbb{P}^1 \setminus \{z_1 \dots z_n\}$, the holonomy of the connection yields a representation of the fundamental group of C, $\rho : \pi_1(C) \to \mathrm{GL}_K(\mathbb{C})$ where we have chosen a basis for the fibre above the basepoint p. The representation is called the ***monodromy representation***, and its image is called the ***monodromy group*** of the equation.

A basic tool for both studying solutions of these equations, as well as their monodromy, is the ***indicial equation***, defined as follows. Write

$$L = r_n\theta^n + \ldots r_1\theta + r_0 \tag{3.6.2}$$

where we have defined the operator $\theta := z\frac{d}{dz}$. Then the indicial polynomial is $p(x) = r_n(0)x^n \ldots + r_1(0)x + r_0(0)$ and the ***indicial equation at the singularity 0*** is $p(x) = 0$. Its solutions are called the ***local exponents*** of the equation at the singularity 0. Their exponentials (after first multiplying by $2\pi i$), are the eigenvalues of the monodromy around 0. Similar definitions can be made at any singularity by Möbius transforming its position to 0.

The next important observation is that, as long as we are restricted to regular singularities, the monodromy of such equations can be built up from the monodromy of the functions z^λ and $\log z$ (e.g. [47]):

Theorem 3.6.2. *Let z_0 be a singular point of the equation*

$$a_0(z)w^{(n)} + a_1(z)w^{(n-1)} + \ldots + a_{n-1}(z)w' + a_n(z)w = 0 \tag{3.6.3}$$

Then the following are equivalent:

1. *The functions $b_k(z) := \frac{a_k(z)}{a_0(z)}$ have at worst a pole of order k at z_0.*

2. *The vector space of multivalued holomorphic functions in a sufficiently small punctured disk $\{0 < |z - z_0| < \delta\}$ which are solutions of (3.6.3), has dimension n and is generated by functions of the form*

$$(z - z_0)^\lambda (\log(z - z_0))^j f(z)$$

where $\lambda \in \mathbb{C}, j \in \mathbb{Z}, 0 \leq j \leq n-1$, and $f(z)$ is holomorphic in the disk $\{|z - z_0| < \delta\}$ and $f(z_0) \neq 0$.

In particular, if all the exponents modulo integers are distinct, then there are no logs in the solutions.

In general, we are interested in more than just the eigenvalues of the monodromy; we want to know the entire monodromy group. To do this explicitly requires computing the change of basis matrix between the solutions near singular points. This is in general highly nontrivial, but can be done in some special cases, as we will see next.

3.6.2 The hypergeometric equation

When $n = 2$ and we have exactly three singularities, at $(0, 1, \infty)$, we arrive at the study of the hypergeometric equation. Up to Möbius transformations, this is the most general Fuchsian ODE with three singularities, up to a minor subtlety (see below). The hypergeometric equation is written:

$$\left(z(z-1)\frac{d^2}{dz^2} + (\gamma - (\alpha+\beta+1)z)\frac{d}{dz} - \alpha\beta\right) f = 0 \tag{3.6.4}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ are parameters. It is easy to check that the exponents around $0, 1, \infty$ are $(0, \alpha), (0, \beta), (0, \gamma - \alpha - \beta)$ respectively. The solutions can be written in terms of Gauss's ***hypergeometric function***:

$${}_2F_1(\alpha, \beta, \gamma|z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \tag{3.6.5}$$

where the Pochhammer symbol is $(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\dots(a+k-1)$. In particular, a basis of solutions around 0 is given by

$$\left\{{}_2F_1(\alpha, \beta, \gamma|z), z^{1-\gamma}{}_2F_1(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma|z)\right\} \tag{3.6.6}$$

For the main applications in this thesis , we will also need to consider the third order analogue of this equation. This is the so-called ***generalized hypergeometric equation***:

$$[z(\theta-\alpha_1)\dots(\theta-\alpha_n) - (\theta-\beta_1+1)\dots(\theta-\beta_n+1)]\, f = 0 \tag{3.6.7}$$

Assuming β_i are distinct mod 1, which we always will (see below), the solutions around 0 can be given by

$$z^{1-\beta_i}{}_nF_{n-1}(1+\alpha_1-\beta_i, \dots, 1+\alpha_n-\beta_i\, ;\, 1+\beta_1-\beta_i, \overset{\vee}{\dots}, 1+\beta_n-\beta_i|z) \tag{3.6.8}$$

for $i = 1, \dots n$, where $^\vee$ denotes the omission of $1 + \beta_i - \beta_i$, and ${}_nF_{n-1}$ is the ***generalized hypergeometric function***

$${}_nF_{n-1}(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_{n-1}|z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_n)_k}{(\beta_1)_k \dots (\beta_{n-1})_k} \frac{z^k}{k!} \tag{3.6.9}$$

It is important to note that all three singularities are not equivalent for the generalized hypergeometric equation. In particular,

Proposition 3.6.3. *The generalized hypergeometric equation* (3.6.7) *has $n-1$ linearly independent holomorphic solutions in a neighbourhood around 1.*

So that in particular the monodromy around 1 only has one eigenvalue that isn't 1.

3.6.3 The Riemann equation

The hypergeometric equation is almost the most general second order meromorphic differential equation with three regular singular points (by a Möbius transformation, the singularities can always be assumed to be at $(0,1,\infty)$. However, one of the local exponents of the hypergeometric equation vanish at 0 and 1. The most general second order meromorphic linear equation with three regular singularities, however, may have arbitrary exponents $\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3$ satisfying the "Fuchs relation" $\sum(\sigma_i + \tau_i) = 1$. Fortunately, the solutions to such an equation are easily expressed in terms of the hypergeometric equation. Introduce the notation

$$P\begin{pmatrix} a_1 & a_2 & a_3 & \\ \sigma_1 & \sigma_2 & \sigma_3 & z \\ \tau_1 & \tau_2 & \tau_3 & \end{pmatrix}$$

denoting the set of solutions to the Riemann equation. In particular,

$$z^{\sigma_1}(z-1)^{\tau_1}F \tag{3.6.10}$$

is a solution of the Riemann equation with the desired exponents whenever F solves the corresponding hypergeometric equation. The same idea generalizes straightforwardly to higher order equations.

3.6.4 Monodromy of the generalized hypergeometric equation

It will be crucial for our calculations that we have explicit expressions for the monodromy of opers on the three-punctured sphere. For the generalized hypergeometric equation, these expressions are well-known [48, 49, 50]. We simply quote their results here.

Proposition 3.6.4 ([48], Cor 3.2.2)**.** *The monodromy of the generalized hypergeometric equation* (3.6.7) *is irreducible if and only if $\alpha_i - \beta_j$ are all non-integral for every i, j.*

We will always be interested in the irreducible case, and so when linear combinations of mass parameters are later identified with α_i, β_i, our results will hold away from the corresponding locus.

Proposition 3.6.5 ([49]). *An irreducible representation $\rho : \pi_1(\mathbb{C} \setminus \{0,1\}) \to \mathrm{GL}_K(\mathbb{C})$ is the monodromy of a hypergeometric equation if and only if the monodromy around $z = 1$ is a complex reflection matrix.*

Let us stick to the notation of the third order equation to avoid clutter. In the *Frobenius basis* of solutions around 0 given by

$$\left\{z^{1-\beta_1}F_1,\, z^{1-\beta_2}F_2,\, z^{1-\beta_3}F_3\right\}$$

where F_i are holomorphic, the monodromy around 0 is diagonal:

$$M_0 = \begin{pmatrix} e^{-2\pi i\beta_1} & 0 & 0 \\ 0 & e^{-2\pi i\beta_2} & 0 \\ 0 & 0 & e^{-2\pi i\beta_3} \end{pmatrix} \tag{3.6.11}$$

There is another basis, the Mellin-Barnes basis, in which it is possible to compute the monodromy around all three punctures explicitly. For details of the computation, see [50]. We quote the result:

Proposition 3.6.6 (Levelt). *Suppose α_k differ from β_l modulo 1 for all $1 \leq k, l \leq n$. The monodromy matrices in the Mellin-Barnes basis are*

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -Y_3 & -Y_2 & -Y_1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 + \frac{X_3 - Y_3}{Y_3} & \frac{X_2 - Y_2}{Y_3} & \frac{X_1 - Y_1}{Y_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_\infty = \begin{pmatrix} -\frac{X_2}{X_3} & -\frac{X_1}{X_3} & -\frac{X_0}{X_3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -X_3 & -X_2 & -X_1 \end{pmatrix}^{-1}$$

where $t^3 + Y_1t^2 + Y_2t + Y_3$ and $t^3 + X_1t^2 + X_2t + X_3$ are the polynomials with roots $e^{-2\pi i\beta_k}$ and $e^{-2\pi i\alpha_k}$, $k = 1, 2, 3$, respectively.

It turns out the change of basis between the two is explicitly known, e.g. [48, 50]. As a result, we have all the information we need to write the monodromy down explicitly. Setting our notation back to arbitrary n, we have:

Theorem 3.6.7 ([50],Theorem 2.8)**.** *Suppose $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are distinct modulo 1. Write*

$$C_i = \frac{\Gamma(\alpha_1 - \beta_i + 1) \dots \Gamma(\alpha_n - \beta_i + 1)}{\Gamma(\beta_1 - \beta_i + 1) \dots \Gamma(\beta_n - \beta_i + 1)},$$

and let $\tilde{c} = 2ie^{\pi i(\beta_1 - \alpha_1 \dots \beta_n - \alpha_n)}$. In the Frobenius basis at $z = 0$, the monodromy matrix around 1 is given by:

$$(M_1^{\mathcal{F}})_{kl} = \delta_{kl} + \tilde{c}\frac{C_l}{C_k} \cdot \prod_{m=1}^{n} \frac{\sin(\pi(\beta_l - \alpha_m))}{\sin(\pi(\beta_l - \beta_m))} \tag{3.6.12}$$

where $k, l = 1, 2, \dots, n$, and the term $\sin(\pi(\beta_l - \beta_l))$ denotes 1.

Chapter 4

Spectral networks and abelianization

With the preliminaries out of the way, we are ready to introduce the central tools of this thesis. The mathematical objects we will use to study $\mathcal{M}_{\mathrm{dR}}$ are known as spectral networks. These are certain collections of oriented paths on a punctured Riemann surface C equipped with some extra data. Spectral networks were introduced by Gaiotto, Moore, and Neitzke [1, 2, 3] as a geometric tool for counting BPS states in theories of class $\mathcal{S}$, though they also appeared previously in the literature on the "exact WKB method" as "Stokes graphs". For us, their primary use will be to construct certain nice coordinate systems on $\mathcal{M}_{\mathrm{dR}}$. In this chapter we summarize the relevant definitions for spectral networks and describe the *abelianization* construction for obtaining *spectral coordinates* which plays a crucial role through the rest of the thesis.

4.1 Spectral networks

We begin by defining a spectral network. More precisely, we will describe a class of networks known as "WKB spectral networks" which can be generated in a canonical way from a tuple of k-differentials. A more general definition, which abstracts the holomorphic data we will use into a purely topological notion, but which we will not need, can be found in [2, 51].

Consider a compact curve C equipped with an effective divisor D, and let the structure group (of the bundles we will study) be $G = \mathrm{SL}_K$ for some fixed integer $K \geq 2$. Fix some phase $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ and a tuple $\varphi = (\varphi_2, \ldots, \varphi_K) \in \mathbf{B}$. Write the corresponding spectral curve Σ in terms of a tuple $(\lambda_1, \ldots, \lambda_K)$ of meromorphic 1-differentials on C as

$$\Sigma = \left\{ \lambda \in T^*C \,:\, \lambda^K + \varphi_2 \,\lambda^{K-2} + \ldots + \varphi_K = \prod_{j=1}^{K} (\lambda - \lambda_j) = 0 \right\} \subset T^*C. \quad (4.1.1)$$

where we choose a collection of branch cuts for the projection and label the sheets away from them.

Define an ij ***trajectory of phase*** ϑ , for $i \neq j \in \{1, \ldots, K\}$, as an open path on C such that

$$(\lambda_i - \lambda_j)(v) \in e^{i\vartheta}\mathbb{R}, \quad (4.1.2)$$

for every nonzero tangent vector v to the path. In other words, an ij trajectory of phase φ is a curve $z(t)$ satisfying the differential equation

$$\big(x_i(z(t)) - x_j(z(t))\big)\, z'(t) = e^{i\vartheta}. \quad (4.1.3)$$

where x_i are the coordinate expressions for λ_i. The ***WKB spectral network*** $\mathcal{W}_\vartheta(\varphi)$ consists of a collection of such ij-trajectories on C, together with some labels, as follows.

Call any ij-trajectory that has an endpoint on a branch point of the covering $\Sigma \to C$ a ***wall***, and orient the wall so that it starts at the branch point. Any other ij-trajectory that has its endpoint at the intersection of previously defined walls is another wall. We orient this wall such that it starts at the intersection. The set of walls of the spectral network $\mathcal{W}_\vartheta(\varphi)$ is the union of all walls defined iteratively in this manner.

We ***label*** the walls as follows. The two sheets i and j of Σ over a wall correspond to the two differentials λ_i and λ_j. Given a positively oriented tangent vector v to the wall, the quantity $e^{-i\vartheta}(\lambda_i - \lambda_j)(v)$ is real. If it is positive we label the wall by the ordered pair ji, and if negative we label the wall by ij.

What do spectral networks look like? Apart from the case of $K = 2$, where it amounts to the classic subject of trajectories of quadratic differentials, little is known rigorously about the kinds of paths generated. Globally, the most straightforward way to examine the shape is to simply plot solutions, but we can make some local observations too. Generically, the spectral network in the neighbourhood of a simple branch point of the covering $\Sigma \to C$ looks as in Figure 4.1. In a neighbourhood of a simple intersection of walls the spectral network is illustrated in Figure 4.2. Generically, each wall ends at a puncture of C.

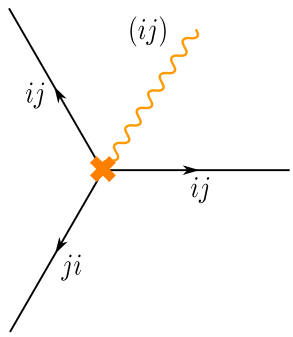


Figure 4.1: Configuration of walls around a simple branch point.

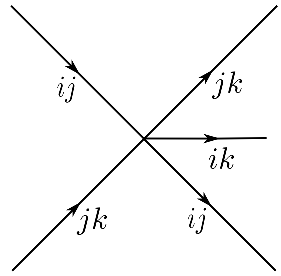


Figure 4.2: A wall with label *ik* is born at the intersection of two walls with label *ij* and *jk*.

We decorate a puncture with incoming walls as follows. Each root λ_i has a simple pole at the puncture with residue m_i. We decorate the puncture with an ordered tuple $i_1 \dots i_K$ such that $\mathrm{Re}(e^{-i\vartheta} m_{i_j}) > \mathrm{Re}(e^{-i\vartheta} m_{i_k})$ for each $j < k$. One then checks that the only walls which fall into the puncture are the ones whose labeling (read left-to-right) agrees with the cyclic ordering given by the decoration.

At special values for the differentials φ and the phase ϑ it might happen that two walls with labels ij and ji, with opposite orientations, overlap. This is illustrated in Figure 4.3. We say that the locus where the two walls overlap is a ***double wall***. If there is at least one double wall, the spectral network must be further equipped with a choice of a ***resolution***, which is either "British" or "American". We think of the resolution as telling us how the two constituents of a double wall are infinitesimally displaced from one another, and draw the walls as such.

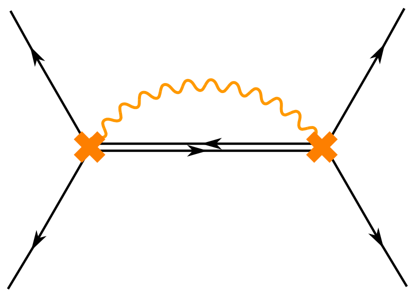


Figure 4.3: Local configuration with a double wall in the American resolution.

There is a natural notion of equivalence of spectral networks. Even though in the above we have fixed a complex structure on C and described a spectral network in terms of the tuple φ of differentials, we will later only be interested in the isotopy class of the spectral network on the topological surface underlying C. We thus define two spectral networks $\mathcal{W}$ and $\mathcal{W}'$ to be equivalent if one can be isotoped into the other.

4.1.1 Examples, $K = 2$

When $K = 2$, there can be no birthing and intersections (since that requires at least three distinct labels), so the possible spectral networks are somewhat constrained. It turns out they are the same thing as a well-known construction with quadratic differentials, arising as the so-called *critical graph* of an associated foliation.

Let φ_2 be a meromorphic quadratic differential on C, holomorphic away from the punctures z_l. Locally such a differential is of the form

$$\varphi_2 = u(z)(dz)^2 \tag{4.1.4}$$

It is well-known that given a phase ϑ, the differential φ_2 canonically determines a singular foliation $\mathcal{F}_\vartheta(\varphi_2)$ on C. By definition, its leaves are real curves on C such that, if v denotes a nonzero tangent vector to the curve,

$$e^{-2i\vartheta}\varphi_2(v^2) \in \mathbb{R}_+. \tag{4.1.5}$$

A generic φ_2 will yield a spectral network looking something like this:

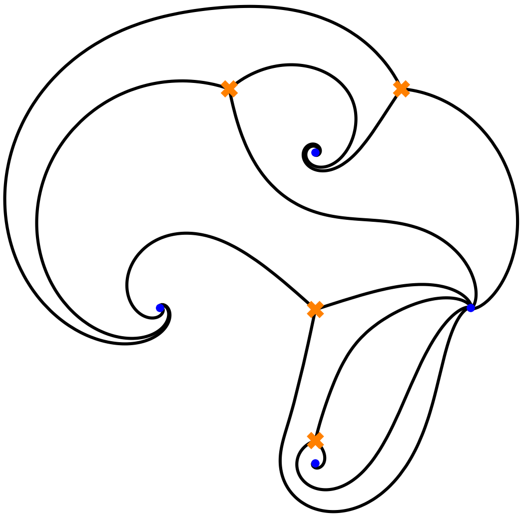


Figure 4.4: A Fock-Goncharov network. The punctures are at ± 1 and $\pm i$, depicted in blue.

All walls eventually fall into the punctures. These types of networks are known as "Fock-Goncharov", since they were shown [3] to induce the well-known Fock-Goncharov coordinates associated to (dual) ideal triangulations via their abelianization. On the other hand, one can find for special parameter values networks with double walls, which play an essential role in the original motivation for spectral networks, the counting of BPS states, and will be crucial to us. Furthermore, it can occur that there are no single walls at all, which leads to the so called "Fenchel-Nielsen" networks introduced by Hollands and Neitzke in [4]. The two main examples are:

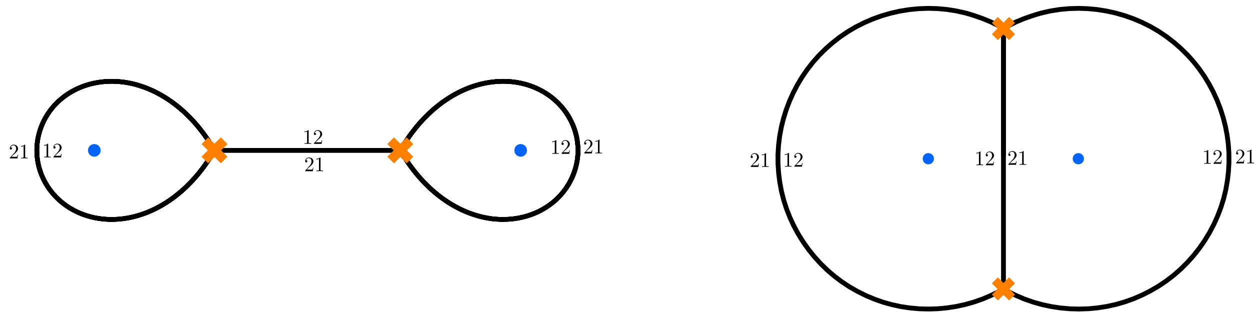


Figure 4.5: Fenchel-Nielsen networks on the three-punctured sphere. Left: "molecule I", Right: "molecule II". The blue dots are the punctures, and the orange crosses are branch points of the covering $\Sigma \to C$. All walls are double walls.

Using their terminology, we can say

Definition. [4] A $K = 2$ spectral network $\mathcal{W}$ is ***Fenchel-Nielsen*** if it consists of only double walls and respects some pants decomposition of C; that is, the restriction to every three-punctured sphere in the decomposition is itself a network of only double walls.

In particular, for such networks each wall both begins and ends on a branch point of the covering $\Sigma \to C$, and there are no incoming walls at any puncture. We will discuss the decoration at such punctures, as well as along the pants curves, later in this section.

Since by definition a Fenchel-Nielsen network respects a pants decomposition, we can glue it from Fenchel-Nielsen networks on the individual pairs of pants. In the next chapter, we will review the possible Fenchel-Nielsen networks on the three-punctured sphere for $K = 2$ and detail the gluing procedure.

4.1.2 Examples, $K > 2$

When $K = 3$ much more interesting behaviour can occur. We will simply give some examples here to give the reader some minimal intuition of what can happen. A slightly more systematic approach to a particular class of $K = 3$ networks will be seen in Chapter 7.

One relatively simple-looking $K = 3$ network with a joint phenomenon occurring only when $K > 2$ is depicted below:

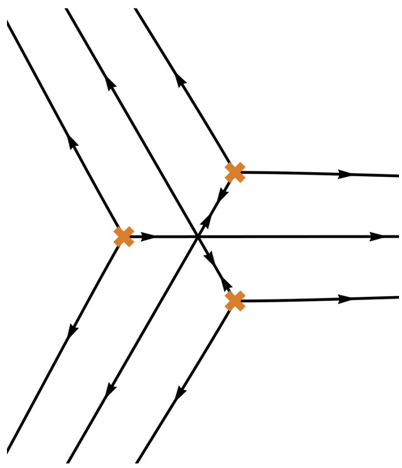


Figure 4.6: Behaviour possible when $K = 3$ but not when $K = 2$. Labels have been omitted.

In general, spectral networks can be much more complicated when $K > 2$. Apart from the additional sheets, the "birthing" in phenomenon Fig 4.1 is now possible. One such $K = 3$ network on $\mathbb{P}^1$ (with irregular singularity at ∞) that we will study later on is depicted below:

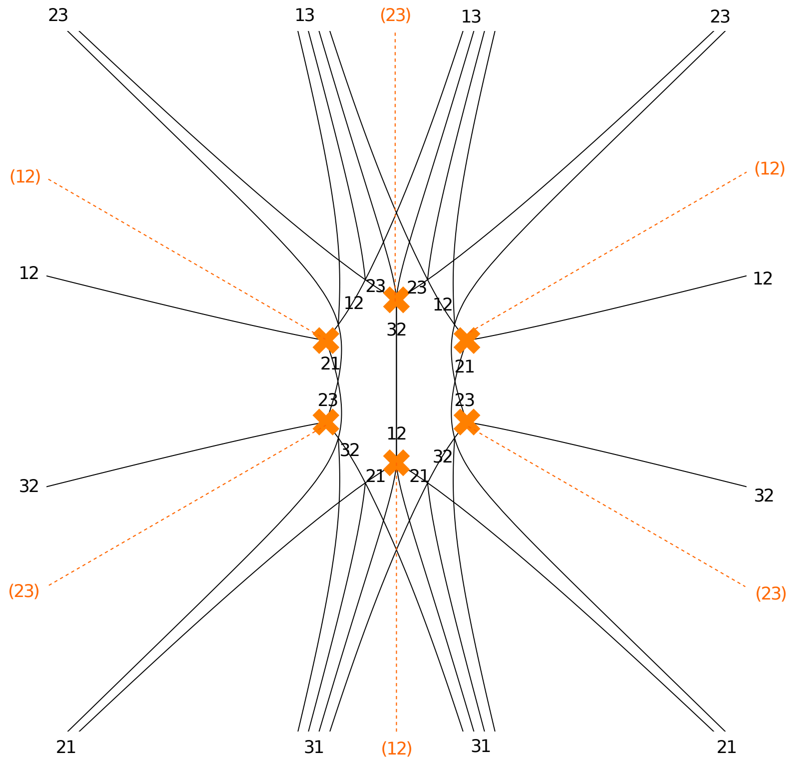


Figure 4.7: A "Grassmannian" network. The only singularity is at ∞.

On punctured spheres, choosing some generic parameter values, we can find networks that look like below:

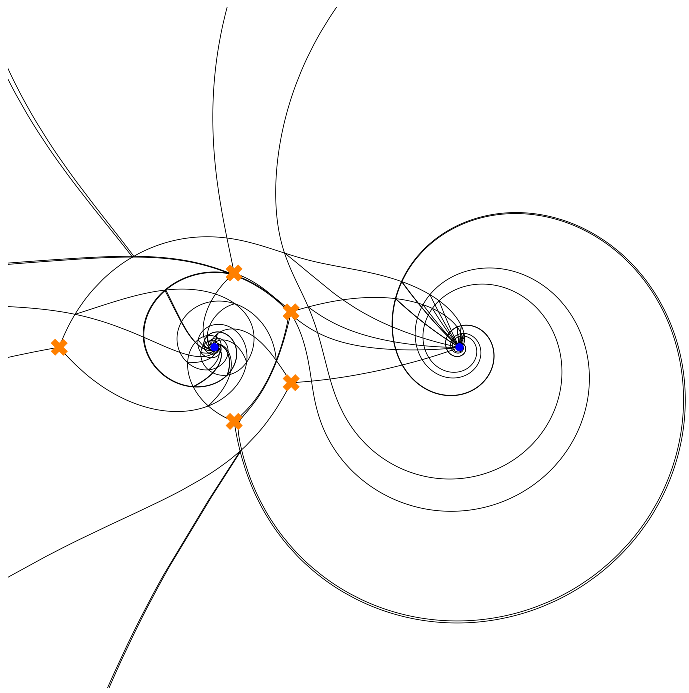


Figure 4.8: A $K = 3$ network on the three-punctured sphere. Punctures lie at $1, -1$, and ∞, and we have omitted labels and cuts.

These are the $K = 3$ analogue of the Fock-Goncharov networks above. These can be thought of as generalizing the notion of an ideal triangulation [3]. On the other hand, the case we will be most interested in is the opposite, in which all walls are double walls, so that we have networks like:

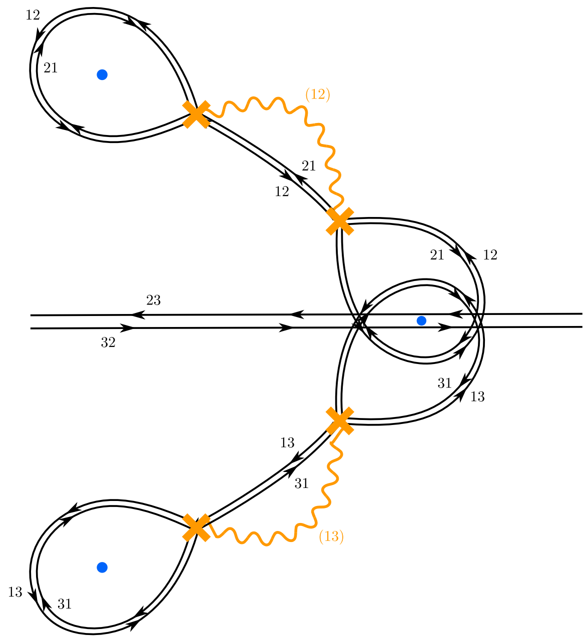


Figure 4.9: A "generalized Fenchel-Nielsen" network.

This particular network will be studied in much greater detail in the next chapter.

4.2 Abelianization and spectral coordinates

One of the most interesting applications of spectral networks is that they induce holomorphic Darboux coordinate systems on moduli spaces of flat connections, called spectral coordinates [2]. These are very special coordinate systems, subsuming a range of previously known examples, and will serve to produce new examples for us. In particular, in [3] it was found that for certain types of spectral networks the resulting coordinates are the same as coordinates introduced earlier by Fock and Goncharov. In [4] this was detailed in the special case of rank $K = 2$, and it was found that other types of spectral networks, namely the Fenchel-Nielsen networks, lead to (complexified) Fenchel-Nielsen length-twist coordinate systems. In this section we review the definitions and constructions, to be used in the following chapter to construct the higher length-twist coordinates.

In the following and throughout the sequel we replace all maximal punctures in a generalized Fenchel-Nielsen network by holes, to facilitate gluing.

4.2.1 Abelianization and nonabelianization

The key to the construction of spectral coordinate systems is the notion of "abelianization" and "nonabelianization" [4, 2]. Let C be a compact real oriented surface with some marked points, or ***singularities***, $C^\times = C \setminus \{s_1 \dots s_m\}$ (say, a punctured Riemann surface). Fix a branched covering $\pi : \Sigma \to C$ and a spectral network $\mathcal{W}$ "subordinate"[1] to this covering. Given a flat SL_K-connection ∇ on a complex vector bundle E over C, a $\mathcal{W}$-abelianization of ∇ is a way of putting ∇ in almost-diagonal form, by locally decomposing E as a sum of line bundles, which are preserved by ∇. Let C' and Σ' denote C and Σ respectively with the (preimages of) branch points removed.

Starting from a line bundle $(\mathcal{L}', \nabla^{\mathrm{ab}})$ over Σ, to nonabelianize, we begin with the crudest approximation to the bundle we want, $\pi_* \mathcal{L}'$, which takes $\mathcal{L}'$ to E restricted to $C' \setminus \mathcal{W}$ (this is the definition of E away from the branch points and the network). Going the other way, when we abelianize we aim to get as close to this situation as possible. Precisely, we may define $\mathcal{W}$-abelianization and $\mathcal{W}$-nonabelianization in terms of $\mathcal{W}$-pairs [4].

Definition. A $\mathcal{W}$***-pair*** $(E, \nabla, \iota, \mathcal{L}', \nabla^{\mathrm{ab}})$ for a network $\mathcal{W}$ subordinate to the branched covering $\pi : \Sigma \to C$ is the collection of data:

(i) A flat rank K bundle (E, ∇) over C

(ii) A flat rank 1 bundle $(\mathcal{L}', \nabla^{\mathrm{ab}})$ over Σ'

(iii) An isomorphism $\iota : E|_{C' \setminus \mathcal{W}} \to \pi_* \mathcal{L}'|_{C' \setminus \mathcal{W}}$

with the properties

(a) the isomorphism ι takes (the restrictions of) ∇ to $\pi_* \nabla^{\mathrm{ab}}$

[1] Here, we simply mean that $\mathcal{W}$ is a WKB network arising from a tuple $\varphi = (\varphi_2, \dots, \varphi_K)$ whose spectral curve is $\Sigma \to C$, where C is endowed with a complex structure. This definition makes sense verbatim in the context of general spectral networks [1, 4], in which case the word subordinate is defined without reference to a complex structure, but agrees with this definition when $\mathcal{W}$ is of WKB type.

(b) at each single wall $w \subset \mathcal{W}$, ι jumps by a map $\mathcal{S}_w = 1 + e_w \in \mathrm{End}(\pi'_* \mathcal{L}'|_{C' \setminus \mathcal{W}})$ where $e_w : \mathcal{L}'_i \to \mathcal{L}'_j$ if w carries the label ij, $e_w \neq 0$ (here by $\mathcal{L}'_i$ we mean the summand of $\pi_* \mathcal{L}'$ associated to sheet i; relative to diagonal local trivializations of $\pi_* \mathcal{L}$, this condition says $\mathcal{S}_w$ is upper or lower triangular). At each double wall $w'w$ ι jumps by a map $\mathcal{S}_{w'} \mathcal{S}_w$, with the ordering determined by the resolution.

We call two $\mathcal{W}$-pairs ***equivalent*** and write $(E_1, \nabla_1, \iota_1 \mathcal{L}'_1, \nabla^{\mathrm{ab}}_1,) \sim (E_2, \nabla_2, \iota_2, \mathcal{L}'_2, \nabla^{\mathrm{ab}}_2)$, if there exist bundle isomorphisms $\varphi : \mathcal{L}'_1 \to \mathcal{L}'_2$ and $\psi : E_1 \to E_2$ making the obvious diagrams commute. In particular, in this case we have equivalences $(\mathcal{L}'_1, \nabla^{\mathrm{ab}}_1) \sim (\mathcal{L}'_2, \nabla^{\mathrm{ab}}_2)$ and $(E_1, \nabla_1) \sim (E_2, \nabla_2)$ as flat bundles. Denote the moduli space of $\mathcal{W}$-pairs by $\mathcal{M}_{\mathrm{pair}}(\mathcal{W})$ (this depends on both π and $\mathcal{W}$), leaving the rank K implicit.

Definition. Given a flat SL_K-connection ∇ on a complex rank K bundle E over C, a ***$\mathcal{W}$-abelianization*** of ∇ is any extension of (E, ∇) to a $\mathcal{W}$-pair $(E, \nabla, \iota, \mathcal{L}', \nabla^{\mathrm{ab}})$.

Definition. Given an equivariant almost-flat GL_1-connection ∇^{ab} on a complex line bundle $\mathcal{L}'$ over Σ', a ***$\mathcal{W}$-nonabelianization*** of ∇^{ab} is any extension of $(\mathcal{L}', \nabla^{\mathrm{ab}})$ to a $\mathcal{W}$-pair $(E, \nabla, \iota, \mathcal{L}', \nabla^{\mathrm{ab}})$.

In fact, to $\mathcal{W}$-abelianize a flat SL_K-connection ∇, it is sufficient to define the flat GL_1-connection ∇^{ab} on $\mathcal{L}'$ restricted to $\Sigma' \backslash \pi^{-1}(\mathcal{W})$. Then ∇^{ab} automatically extends from $\Sigma' \backslash \pi^{-1}(\mathcal{W})$ to Σ':

Proposition 4.2.1. *Suppose $\mathcal{P} = (E, \nabla, \iota, \mathcal{L}', \nabla^{\mathrm{ab}})$ satisfies all the conditions for being a $\mathcal{W}$-pair, except that ∇^{ab} is a connection on the restriction to $\Sigma' \setminus \pi^{-1}(\mathcal{W})$. Then ∇^{ab} extends uniquely across the walls, extending $\mathcal{P}$ to a $\mathcal{W}$-pair.*

Proof. Straightforward extension of [4], section 5.1. □

4.2.2 Boundary

If C has boundary, it is useful to consider connections and $\mathcal{W}$-pairs with extra structure. We fix a marked point on each boundary component of C. Then, a $\mathcal{W}$-pair with boundary [4] consists of

- A $\mathcal{W}$-pair $(E, \nabla, \iota, \mathcal{L}', \nabla^{\mathrm{ab}})$,
- a basis of E_z for each marked point z,

- a basis of $\mathcal{L}'_{z_i}$ for each preimage $z_i \in \pi^{-1}(z)$ for each marked point z,
- a trivialization of the covering Σ over a neighbourhood of each marked point z,

such that ι maps the basis of E_z to the basis of $\pi_* \mathcal{L}'_z$ induced from those of $\mathcal{L}'_{z_i}$ and Σ.

Given two surfaces C_1, C_2 with boundary we can glue along a boundary component, in such a way that the marked points are identified. Suppose that we have a $\mathcal{W}_1$-pair with boundary on C_1 and a $\mathcal{W}_2$ pair on C_2, and that the monodromies around the glued component are the same (when written relative to the given trivializations at the marked points). Then using the trivializations we can glue the $\mathcal{W}_1$-pair to the $\mathcal{W}_2$-pair to obtain a $\mathcal{W}$-pair over the glued surface.

4.2.3 Equivariant $\mathrm{GL}(1)$ connections

The abelianization of a flat SL_K-connection ∇ amounts to choosing a basis $(s_1, \dots, s_K)$ of E at any point in $C \backslash \mathcal{W}$, with respect to which ∇ is diagonal, satisfying certain constraints ensuring the correct transition across walls. As a result, any GL_1 connection ∇^{ab} obtained by $\mathcal{W}$-abelianizing a flat SL_K-connection ∇ automatically carries some additional structure. We will capture this by saying that the GL_1 connection ∇^{ab} is equivariant on Σ [4]:

Definition. An ***equivariant line bundle over*** Σ (subordinate to the covering π) is a line bundle with connection $(\mathcal{L}', \nabla^{\mathrm{ab}})$ over Σ', equipped with a flat trivialization of $\det \pi_* \mathcal{L}'$. We say ∇^{ab} itself is an ***equivariant connection***.

We denote the moduli space of all equivariant line bundles on Σ up to equivalence by $\mathcal{M}_{\mathrm{eq}}(\Sigma, \mathrm{GL}_1)$ (note that this depends not just on Σ as a surface but also on the branching structure of π).

An equivariant connection cannot be extended to a flat connection over the whole of Σ, but we will sometimes say it is ***almost-flat*** on Σ, due to the next proposition.

Proposition 4.2.2. *Suppose all branch points of π are cyclic permutations of order r, and ∇^{ab} is an equivariant connection. Then $(\mathrm{Hol}_{\gamma_b^{(i)}} \nabla^{\mathrm{ab}})^r = 1$, where $\gamma_b^{(i)}$ is the ith lift of a small loop encircling a branch point $b \in C$.*

Suppose we are given a $\mathcal{W}$-pair $(E, \nabla, \iota, \mathcal{L}', \nabla^{\text{ab}})$ with ∇ a flat SL_K-connection. The isomorphism ι identifies the basis $(s_1, \ldots, s_K)$ with a basis $(\widetilde{\tau}_1, \ldots, \widetilde{\tau}_K)$ of the pushforward bundle $\pi_* \mathcal{L}'$, where $\widetilde{\tau}_j = \iota(s_j)$. The determinant line bundle $\det(\pi_* \mathcal{L}')$ is trivialized by the product

$$(\widetilde{\tau}_1 \wedge \ldots \wedge \widetilde{\tau}_K). \tag{4.2.1}$$

The triangular property of the jumps $\mathcal{S}_w$ shows that this trivialization extends over $\pi^{-1}(\mathcal{W})$. Furthermore, the trivialization is parallel with respect to the induced connection on the determinant line bundle $\det(\pi_* \mathcal{L}')$.

Equivariance just says that the parallel transport of a local frame $(\widetilde{\tau}_1, \ldots, \widetilde{\tau}_K)$ over a path in $C \backslash \mathcal{W}$ (not crossing a branch cut) is given by a diagonal matrix with determinant 1. It also implies that the holonomy of $\pi_* \nabla^{\text{ab}}$ around a simple branch point of type (ij) can be represented by the matrix whose only vanishing diagonal, and whose only nonvanishing off-diagonal, entries are

$$\begin{pmatrix} d_{ii} & d_{ij} \\ d_{ji} & d_{jj} \end{pmatrix} = \begin{pmatrix} 0 & d \\ -d^{-1} & 0 \end{pmatrix}. \tag{4.2.2}$$

This implies that the holonomy of ∇^{ab} around a simple branch point of type (ij) is diagonal with entry -1 corresponding to the two sheets interchanged, and 1 elsewhere . A connection ∇^{ab} with this property is called an almost-flat connection over Σ in [4].

The GL_1 connection ∇^{ab} furthermore carries additional structure at the punctures, characterized by the type of the puncture. In particular, since the monodromy of ∇ around a minimal puncture is a multiple of a reflection matrix, this implies that the monodromy of $\pi_* \nabla^{\text{ab}}$ around a minimal puncture is given by a diagonal matrix with $K - 1$ equal eigenvalues.

4.2.4 Framing

Let x_0 denote the chosen base point of the fundamental group π_1.

Definition. Let p be a maximal [2] puncture/boundary, and $\mathcal{E}, \nabla$ a flat bundle over the punctured surface C. A ***$\mathcal{W}$-framing of ∇ at p*** is a filtration of the fibre $\mathcal{E}_{x_0}$ by eigenspaces of the monodromy around p.

[2] The reason for only fixing a framing at the maximal punctures and maximal boundaries of C will become clear in §5.3, where we also discuss framings at other types of punctures and boundaries.

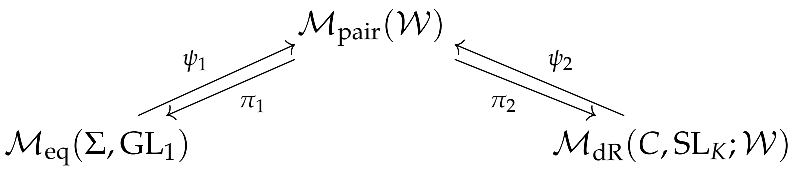
We define a $\mathcal{W}$-framed connection ∇ on C to be a flat SL_K connection on C together with a framing of ∇ at each maximal puncture and maximal boundary. The framing is just an ordered tuple of eigenlines $(l_{\alpha_1}, \ldots, l_{\alpha_K})$ of the monodromy M_+ (in the $+$ direction) around the maximal boundary or maximal puncture. As a genericity condition, we require furthermore that $l_{\alpha_i} \neq l_{\alpha_j}$ for $i \neq j$ and also that each of l_{α_i} for any puncture or boundary is distinct from each of l_{α_j} for any adjacent puncture or boundary (that is, a puncture or boundary belonging to the same pair of pants). Note that a $\mathcal{W}$-framing of ∇ exists only if all of the $M_\pm$ are diagonalizable.

4.2.5 Moduli spaces

Consider the following moduli spaces:

- $\mathcal{M}_{\mathrm{dR}}(C, \mathrm{SL}_K; \mathcal{W})$, the moduli space parametrizing flat $\mathcal{W}$-framed SL_K-connections over C, up to equivalence,
- $\mathcal{M}_{\mathrm{eq}}(\Sigma, \mathrm{GL}_1)$, the moduli space parametrizing equivariant GL_1-connections over Σ, up to equivalence,
- $\mathcal{M}_{\mathrm{pair}}(\mathcal{W})$, the moduli space parameterizing $\mathcal{W}$-pairs, up to equivalence.

The abelianization and nonabelianization constructions lead to the following diagram relating these spaces:



where π_1 and π_2 are the forgetful maps which map a $\mathcal{W}$-pair to the underlying equivariant GL_1-connection or $\mathcal{W}$-framed flat SL_K-connection respectively, whereas ψ_1 is the $\mathcal{W}$-nonabelianization map and ψ_2 the $\mathcal{W}$-abelianization map. From this description it is evident that $\pi_1 \circ \psi_1$ and $\pi_2 \circ \psi_2$ are the identity maps.

To avoid notational clutter we have not explicitly mentioned the restricted boundary monodromies in the above. Yet, all remains true if we consider flat $\mathcal{W}$-framed SL_K-connections with fixed conjugacy classes at the boundaries and punctures, and interpret their eigenvalues as the boundary monodromies for the equivariant GL_1 connections.

In [4] it was established that all of these mappings are bijections for $K = 2$ Fenchel-Nielsen networks $\mathcal{W}$. In particular, it was established that $\mathcal{W}$-abelianizations are in one-to-one correspondence with $\mathcal{W}$-framings for Fenchel-Nielsen networks $\mathcal{W}$, and that there is there is a unique nonabelianization for any equivariant GL_1-connection. In particular, this shows that the mapping $\Psi = \pi_1 \circ \psi_2$ is a bijection (in fact a diffeomorphism).

In the next section we will show that this result extends to $K = 3$ Fenchel-Nielsen networks $\mathcal{W}$ of length-twist type. We expect it to hold for any generalized Fenchel-Nielsen networks of length-twist type. $\mathcal{W}$-abelianizations for arbitrary generalized Fenchel-Nielsen networks (not of length-twist type) are more subtle, however, and are discussed in [52].

4.2.6 Spectral coordinates

Let Σ' denote Σ with the preimages of branch points removed. Given an equivariant GL_1 connection ∇^{ab} we can construct the holonomies

$$\mathcal{X}_\gamma = \mathrm{Hol}_\gamma(\nabla^{\mathrm{ab}}) \in \mathbb{C}^\times \tag{4.2.3}$$

where $\gamma \in H_1(\Sigma', \mathbb{Z})$. Together these form a coordinate system on the moduli space of equivariant GL_1 connections (because of the equivariance, we will really only need a sublattice of γ's). Through the abelianization map, these complex numbers also determine a coordinate system on the moduli space of $\mathcal{W}$-framed flat connections[3]. The resulting coordinates are called spectral coordinates.

Spectral coordinates have a number of good properties. First, they are multiplicative in the sense that

$$\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'} \tag{4.2.4}$$

for any two $\gamma, \gamma' \in H_1(\Sigma', \mathbb{Z})$.

Furthermore, they are "Darboux" coordinates with respect to the holomorphic Poisson structure on the moduli space of flat rank K connections over C (that is, holomorphic on the punctured curve or equivalently considered over the compact C with logarithmic singularities at the punctures):

$$\{\mathcal{X}_\gamma, \mathcal{X}_{\gamma'}\} = \langle \gamma, \gamma' \rangle \mathcal{X}_{\gamma+\gamma'}, \tag{4.2.5}$$

where $\langle .,. \rangle$ denotes the intersection pairing on $H_1(\Sigma', \mathbb{Z})$. In particular, for a symplectic basis $\{A_i, B_i\}$ of $H_1(\Sigma', \mathbb{Z})$, $\{\log \mathcal{X}_{A_i}, \log \mathcal{X}_{B_i}\}$ form Darboux coordinates in the usual sense. This fact will be essential to us in the coming chapters.

[3] We discuss framings at general regular punctures in §5.3

Chapter 5

Higher length-twist coordinates from abelianization

This chapter is based on the joint work [8] with L. Hollands, arXiv:1710.04438.

In this chapter we study the geometry of the moduli spaces of flat connections over a punctured Riemann surface C using the machinery of spectral networks and abelianization. Given class $\mathcal{S}$ data with only minimal or maximal punctures, we define and compute a generalization of (complexified) Fenchel-Nielsen coordinates on $\mathcal{M}_{\text{dR}}$. In the next chapter we utilize these coordinates to compute superpotentials associated to the class $\mathcal{S}$ theory.

In §5.1 we define the higher rank generalization of Fenchel-Nielsen networks and relate this to a generalized Strebel condition on the differentials. We use this to generate examples of generalized Fenchel-Nielsen networks on the four-punctured sphere. If the Riemann surface C is built out of three-punctured spheres with one minimal and two maximal punctures, by gluing the maximal punctures, there is an essentially unique (up to certain "moves") generalized Fenchel-Nielsen network. We call this a generalized Fenchel-Nielsen network of length-twist type.

The relevant moduli space $\mathcal{M}^{\mathcal{C}}_{\text{dR}}(C, \text{SL}_K)$ is the moduli space of flat connections on C with fixed conjugacy classes at each puncture. We require that each conjugacy class is semisimple, with K distinct eigenvalues for a maximal puncture and $K-1$ equal eigenvalues for a minimal puncture (or more generally, a partition of K eigenvalues corresponding to a puncture labeled by any Young diagram).

In §5.3 we show how to realize the higher rank length-twist coordinates as spectral coordinates through the abelianization method, focusing on our two main examples. In particular, we show that the abelianization and non-abelianization mappings are bijective. We then collect the resulting monodromy representations in terms of higher rank length-twist coordinates in §5.4, giving explicit formulas for the trace functions.

5.1 Generalized Fenchel-Nielsen networks

Fix a pants decomposition $P = (\alpha_1, \ldots \alpha_{3g-3+n})$ of the punctured curve C. In this section we define and study examples of a type of spectral network on C that respects this pants decomposition, called a generalized Fenchel-Nielsen network when $K > 2$. To be precise, recall that a network *respects the pants decomposition P* if the walls and pants curves are disjoint; in other words, the network is glued out of networks on three punctured spheres. We can then define:

Definition. We say that a $K > 2$ spectral network $\mathcal{W}_\vartheta(\varphi)$ is a ***generalized Fenchel-Nielsen*** network if it consists only of double walls and respects some pants decomposition of C.

In particular, for such networks each wall both begins and ends on a branch point of the covering $\Sigma \to C$, and there are no incoming walls at any puncture. We will discuss the decoration at such punctures, as well as along the pants curves, later in this section.

In [4] the case $K = 2$ was studied in detail and it was observed that the corresponding differential φ_2 satisfies the Strebel condition. In the following we will argue that for $K > 2$ there is a natural generalization of the Strebel condition which generates generalized Fenchel-Nielsen networks.

Since by definition a Fenchel-Nielsen network respects a pants decomposition, we can glue it from Fenchel-Nielsen networks on the individual pairs of pants. We analyze the possible Fenchel-Nielsen networks on the three-punctured sphere for $K = 2$ and $K = 3$ and detail the gluing procedure.

Even though in the above we have fixed a complex structure on C and described a spectral network in terms of the tuple φ of differentials, we will later only be interested in the isotopy class of the spectral network on the topological surface C. We thus define two spectral networks $\mathcal{W}$ and $\mathcal{W}'$ to be equivalent if one can be isotoped into the other.

5.1.1 $K = 2$

Let φ_2 be a meromorphic quadratic differential on C, holomorphic away from the punctures z_l. Locally such a differential is of the form

$$\varphi_2 = u(z)(dz)^2 \tag{5.1.1}$$

As we described in the previous chapter, given a phase ϑ, the differential φ_2 canonically determines a singular foliation $\mathcal{F}_\vartheta(\varphi_2)$ on C. The differential $e^{-2i\vartheta}\varphi_2$ is called ***Strebel*** if all leaves of the foliation $\mathcal{F}_\vartheta(\varphi_2)$ are either closed trajectories or saddle connections (i.e. trajectories that begin and end at a simple zero of φ_2).

Suppose that the singular foliation $\mathcal{F}_\vartheta(\varphi_2)$ respects a given pants decomposition of the surface C. That is, suppose that each pants curve α_k is homotopic to a closed trajectory of $\mathcal{F}_\vartheta(\varphi_2)$. Then the Strebel condition implies that the period of $\sqrt{-\varphi_2}$ around each pants curve α_k as well as around a small loop γ_l around each puncture z_l has phase ϑ, that is

$$e^{-i\vartheta} \oint_{\alpha_k} \sqrt{-\varphi_2} \in \mathbb{R} \quad \text{and} \quad e^{-i\vartheta} \oint_{\gamma_l} \sqrt{-\varphi_2} \in \mathbb{R}. \tag{5.1.2}$$

Conversely, given any pants decomposition $P = \{\alpha_1, \ldots, \alpha_{3g-3+n}\}$ consisting of simple closed curves of a punctured Riemann surface C and arbitrary $h_k > 0$, $k = 1, \ldots, 3g-3+n$ and $m_l > 0$, $l = 1, \ldots n$, there is a unique Strebel differential φ_2 whose foliation consists of punctured discs centered at the punctures and characteristic annuli homotopic to α_k, such that

$$\oint_{\alpha_k} \sqrt{-\varphi_2} = h_k \quad \text{and} \quad \oint_{\gamma_l} \sqrt{-\varphi_2} = m_l, \tag{5.1.3}$$

for a suitable choice of branch of the root $\sqrt{-\varphi_2}$ [53].

As explained in [4] a rank $K = 2$ spectral network $\mathcal{W}_\vartheta(\varphi_2)$ can be obtained from the critical locus of the singular foliation $\mathcal{F}_\vartheta(\varphi_2)$. The resulting network $\mathcal{W}_\vartheta(\varphi_2)$ is Fenchel-Nielsen if and only if the foliation respects a given pants decomposition of C, has no leaves ending on punctures and only compact leaves. This is equivalent to saying that $e^{-2i\vartheta}\varphi_2$ is a Strebel differential.

Example. Recall that any meromorphic quadratic differential φ_2 on the three-punctured sphere $\mathbb{P}_{0,1,\infty}$, with regular singularities and prescribed residues $-m_l^2$ can be written as

$$\varphi_2 = -\frac{m_\infty^2 z^2 - (m_\infty^2 + m_0^2 - m_1^2)z + m_0^2}{z^2(z-1)^2}\,(dz)^2. \tag{5.1.4}$$

The above differential is a Strebel differential if and only if all parameters m_l have the same phase $\vartheta - \frac{\pi}{2}$. Without loss of generality we can assume all m_l are real and $\vartheta = \frac{\pi}{2}$.

The isotopy class of the corresponding spectral network $\mathcal{W}_{\frac{\pi}{2}}(\varphi_2)$ depends on the precise values of the parameters m_0, m_1 and m_∞. The spectral network changes its isotopy class when one of the four hyperplanes defined by the equations

$$m_\infty = \pm m_0 \pm m_1 \tag{5.1.5}$$

in parameter space is crossed, which is when two branch-points of the covering $\Sigma \to C$ collide. The spectral networks on either side of such a hyperplane are related by a "flip move" (in the terminology of [54]), where two branch points approach each other, collide and then move away in perpendicular directions, as is illustrated in Figure 5.1.

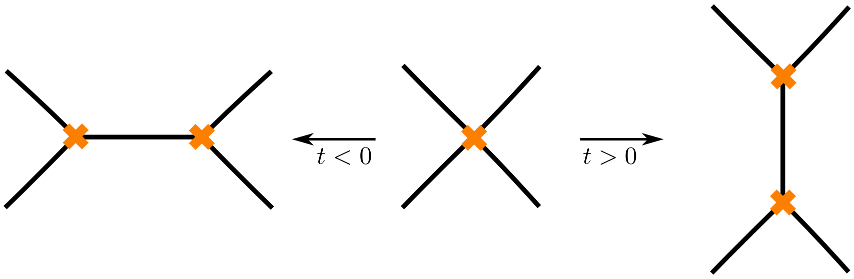


Figure 5.1: Flip move: when varying a real parameter t in φ from a small negative value to a small positive value two branch points come closer until they collide and then move away from each other in a perpendicular direction. All walls in this figure are double walls.

If we do not distinguish the three punctures on $\mathbb{P}_{0,1,\infty}$ there are only two inequivalent spectral networks, named "molecule I" and "molecule II", which are plotted in Figure 5.2 for $m_\infty = 1$ and $m_0 = m_1 = 0.45$ and $m_\infty = m_0 = m_1 = 1$, respectively. The illustrated molecules are related by varying the parameter $t = m_0 + m_1 - m_\infty$ from $t = -0.1$ to $t = 1$ (while keeping $m_\infty > -m_0 + m_1$).

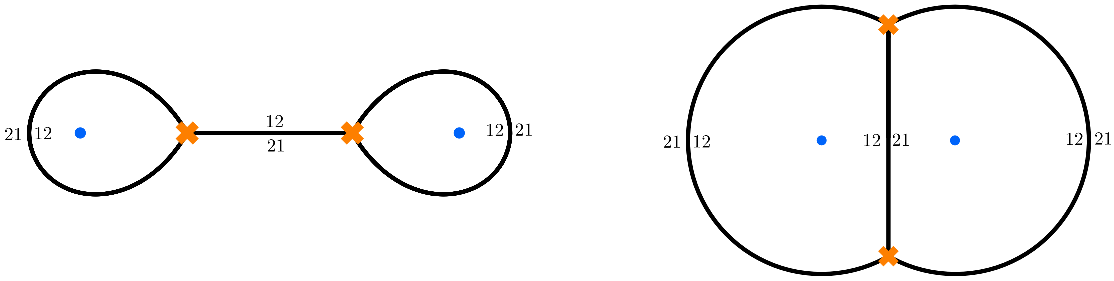


Figure 5.2: Fenchel-Nielsen networks on the three-punctured sphere. On the left is molecule I with $m_\infty = 1$ and $m_0 = m_1 = 0.45$. On the right is molecule II with $m_\infty = m_0 = m_1 = 1$. The blue dots are the punctures, and the orange crosses are branch points of the covering $\Sigma \to C$. All walls are double walls.

In applications we often need to study the two limits $\vartheta \to 0^{\pm}$. These correspond to the two "resolutions" of the network. In each of the two resolutions each double wall is split into two infinitesimally separated walls. The two resolutions of molecule I are shown in Figure 5.3. By drawing the branch cuts in this figure we have moreover fixed a local trivialization of the covering $\Sigma \to C$.

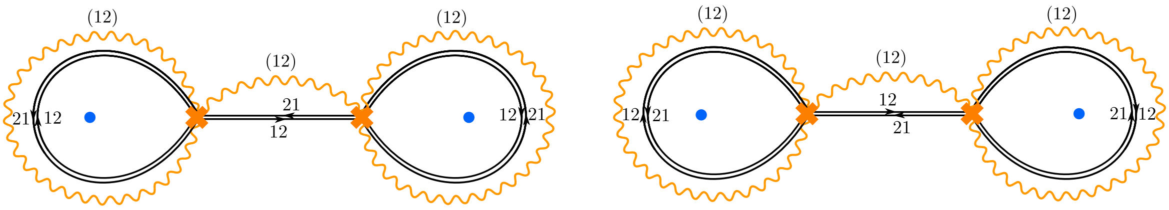


Figure 5.3: The two resolutions of molecule I: on the left the American resolution and on the right the British one. The wavy orange lines illustrate a choice of branch cuts of the covering $\Sigma \to C$.

A Fenchel-Nielsen network on a general Riemann surface C is defined with respect to a pants decomposition of C and can be constructed by gluing together molecules (in the same resolution) on the individual pairs of pants. The molecules are glued together along the boundaries of the pairs of pants, with marked points on the boundary identified, and inserting a circular branch cut around each pants curve.

Any puncture in a molecule is surrounded by a polygon of double walls. The decoration at a puncture is an assignment of an ordering of the sheets of the spectral curve Σ over the puncture to each direction around the puncture, compatible with the labelings of the double walls surrounding it, in such a way that reversing the direction reverses the ordering. In Figure 5.3 we have chosen the branch cuts such that the 12-walls run in the clockwise direction around each puncture. The decoration thus assigns the sheet ordering 12 to the clockwise orientation.

Similarly, any pants curve in a Fenchel-Nielsen network is surrounded on either side by a polygon of walls. We thus also associate a decoration to each pants curve. This is an assignment of an ordering of the sheets to each direction around the pants curve, compatible with the labelings of the double walls surrounding it, in such a way that reversing the direction reverses the ordering.

5.1.2 $K = 3$

We generalize the Strebel condition to $K > 2$ as follows.

Definition. A tuple of differentials $\varphi = (\varphi_2, \ldots, \varphi_K)$ is ***generalized Strebel*** (at phase ϑ) if there exists a symplectic basis for the compactified spectral cover $\overline{\Sigma}$, i.e. a choice of A- and B-cycles on $\overline{\Sigma}$, such that

$$e^{-i\vartheta} \oint_{A_k} \lambda \in \mathbb{R} \quad \text{and} \quad e^{-i\vartheta} \oint_{\widetilde{\gamma}_l} \lambda \in \mathbb{R}, \tag{5.1.6}$$

for each A-cycle A_k and each lift $\widetilde{\gamma}_l$ of a small loop around each puncture z_l to Σ, where λ is the tautological 1-form on Σ.

We say that the generalized Strebel tuple φ ***respects a pants decomposition*** P of C if the generalized Strebel condition (5.1.6) holds for a basis whose A-cycles are the lifts of each pants curve $\alpha \in P$ to Σ.

Recall that a spectral network $\mathcal{W}_\vartheta(\varphi)$ is called a generalized Fenchel-Nielsen network if it respects some pants decomposition and consists of only double walls. We use generalized Strebel tuples to generate our examples of Fenchel-Nielsen networks throughout the thesis, though we do not have a proof of why this must happen. Based on this, we propose

Conjecture 5.1.1. *Suppose P is a pants decomposition of C and φ is a generalized Strebel tuple at some phase ϑ which respects P. Then the WKB spectral network $\mathcal{W}_\vartheta(\varphi)$ is a generalized Fenchel-Nielsen network respecting P.*

(In [54, 55] a related class of networks, called BPS graphs, were given an interpretation in terms of BPS quivers. In the terminology of [4] they would be called generalized fully contracted Fenchel-Nielsen networks. In particular, they do not respect any pants decomposition.)

Example. The differentials φ_2, φ_3 on the three-punctured sphere $\mathbb{P}^1_{0,1,\infty}$ with two maximal and one minimal puncture were discussed in §2.2.3. Applying an automorphism of $\mathbb{P}^1$ to move the punctures to $z_a = 1$, $z_b = \omega$ and $z_c = \omega^2$, where ω is the third root of unity, these differentials can be written explicitly as

$$\varphi_2^{\text{bif}}(z) = \frac{-9m_a^2}{(1-z)(1-z^3)} + \frac{3(1-z)^2}{(1-z^3)^2}(m_{b,1}^2 + m_{b,1}m_{b,2} + m_{b,2}^2) \tag{5.1.7}$$

$$\varphi_3^{\text{bif}}(z) = \frac{9(1+z)m_a^3}{(1-z)(1-z^3)^2} - \frac{9(1-z)^2(1+z)}{(1-z^3)^3} m_{b,1}m_{b,2}(m_{b,1} + m_{b,2}), \tag{5.1.8}$$

where m_a is the single mass parameter at the minimal puncture at $z = 1$, and where we have set the mass parameters $m_{b,1}$ and $m_{b,2}$ at the maximal puncture at $z_b = \omega$ to be minus the ones at $z_c = \omega^2$.

The spectral network $(\varphi^{\text{bif}}, \vartheta)$ is a generalized Fenchel-Nielsen network if and only if all mass parameters m_a, $m_{b,1}$ and $m_{b,2}$ have the same phase $\vartheta - \frac{\pi}{2}$. This is precisely when the corresponding tuple φ^{bif} is generalized Strebel. Without loss of generality we can assume that the mass parameters are real and $\vartheta = \frac{\pi}{2}$.

Just as in the previous example, the different isotopy classes generated by φ^{bif} are classified by the connected components of the complement of the hyperplanes corresponding to the collision of two or more branch points of the covering $\Sigma \to C$. We refer to any of these isotopy classes as a $K = 3$ generalized Fenchel-Nielsen molecule with two maximal and one minimal puncture. Any two such molecules are related by a sequence of elementary local transformations, such as the flip move. Some molecules are shown in Figure 5.5.

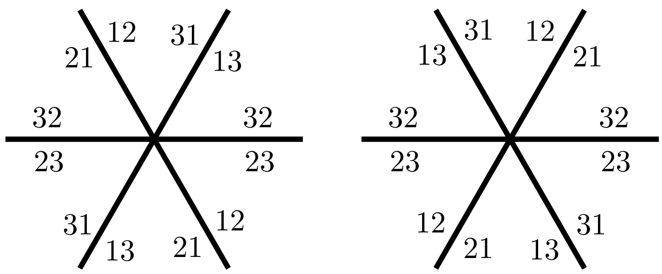


Figure 5.4: The two possible joints in which six double walls can intersect.

The generalized Fenchel-Nielsen molecules with two maximal and one minimal puncture share a number of features. They are built out of two (rank 2) Fenchel-Nielsen molecules, intersecting each other in (both of) the 6-joints illustrated in Figure 5.4. Maximal punctures are surrounded by a polygon of double walls, whereas minimal punctures lie on top of a double wall.

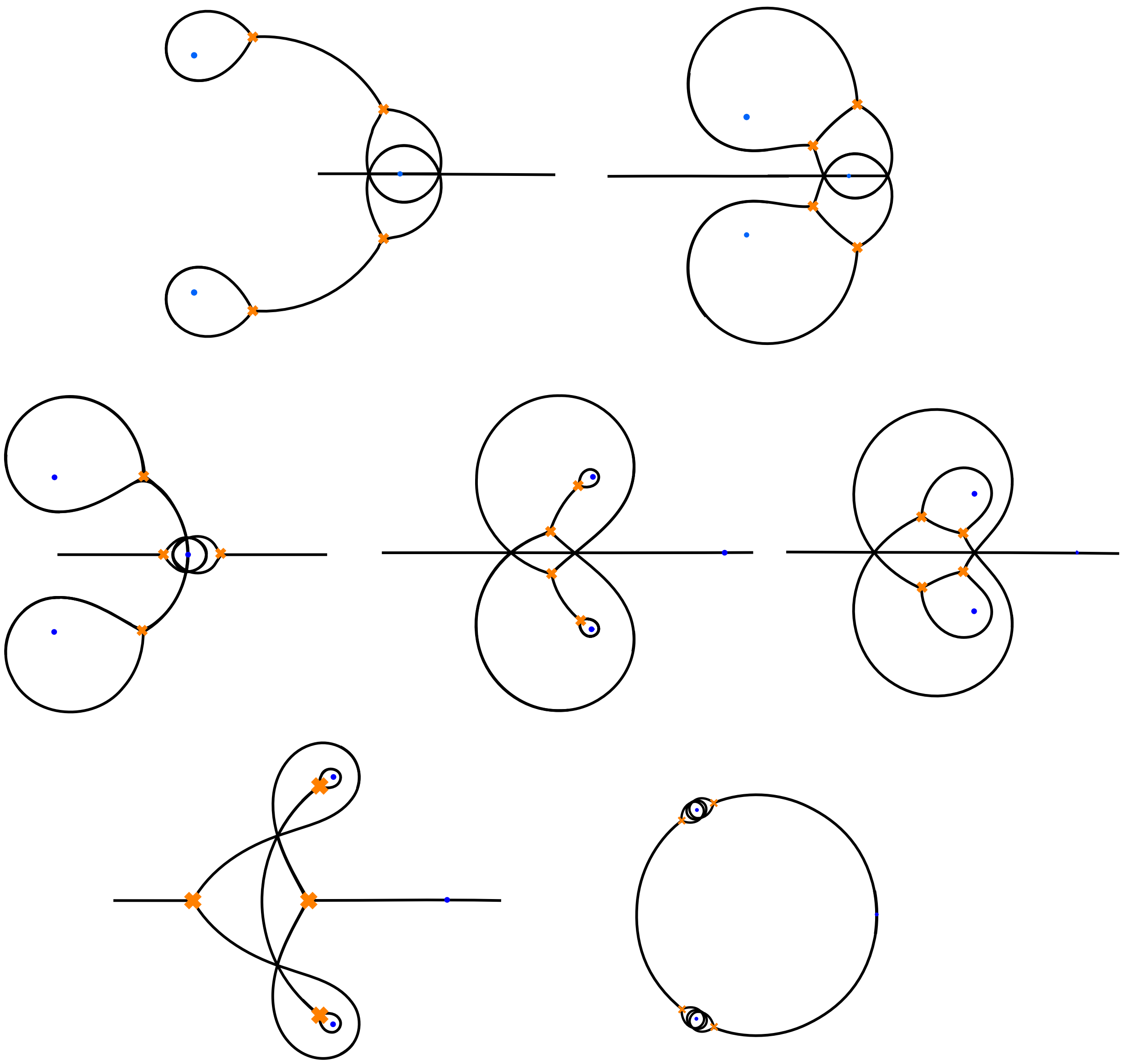


Figure 5.5: Examples of non-isotopic generalized Fenchel-Nielsen molecules with two maximal and one minimal puncture, symmetric about the horizontal. All walls are double walls.

Each $K = 3$ molecule comes with two resolutions, in which each double wall is split into two infinitesimally separated walls. For instance, the two resolutions of the molecule at the top-left in Figure 5.5 are illustrated in Figure 5.6. Note that a minimal puncture is in between two single opposite walls. In Figure 5.6 we have also chosen a local trivialization of the spectral cover Σ.

Each $K = 3$ molecule can be represented with several choices of wall labelings. For instance, for the $K = 3$ molecule in Figure 5.6 the wall labelings are completely determined if we fix the labels for the double wall surrounding the maximal puncture at $z = \omega$ as well as one of the two possible combinations of joints around the minimal puncture at $z = 1$. All different choices can be obtained from the representation in Figure 5.6 by introducing additional branch cuts around the punctures.

Each choice of wall labelings determines a decoration at the punctures and along the pants curves. As before, the decoration assigns an ordering of the sheets of the spectral curve Σ over the puncture or over the pants curve to each direction, in such a way that reversing the direction reverses the ordering. For instance, for the $K = 3$ molecule in Figure 5.6 the decoration at the maximal puncture at $z = \omega$ assigns the sheet ordering (123) to the clockwise direction and (321) to the counterclockwise direction, whereas the decoration at the maximal puncture at $z = \omega^2$ assigns the sheet ordering (321) to the clockwise direction and (123) to the counterclockwise direction. The decoration at the minimal puncture at $z = 1$ assigns the sheet ordering $(31;2)$ to the clockwise direction and $(13;2)$ to the counterclockwise direction, where 2 is the distinguished sheet that does not appear in the label of the double wall intersecting the minimal puncture.

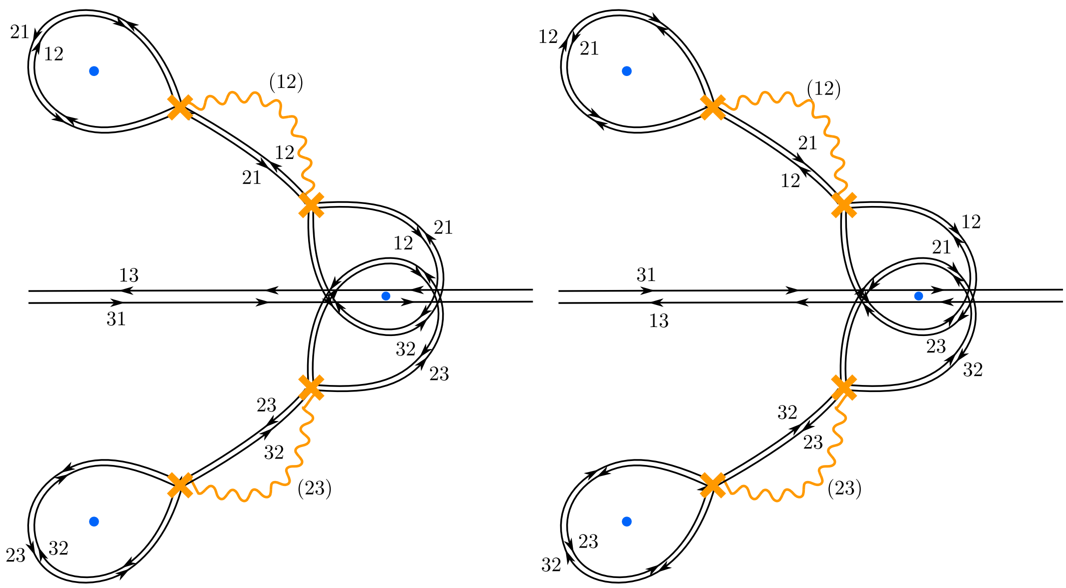


Figure 5.6: The two resolutions of the generalized Fenchel-Nielsen molecule at the top-left in Figure 5.5 together with a choice of local trivialization of the spectral cover Σ.

Example. Equations (2.2.14), (2.2.15) from Chapter 2 describe the 1-dimensional family of tuples (φ_2, φ_3) on the three-punctured sphere $\mathbb{P}^1_{0,1,\infty}$ with three maximal punctures. Each tuple defines a spectral cover Σ over C whose compactification has genus 1. This implies that the possible generalized Strebel tuples are labeled by a choice of A-cycle on $\overline{\Sigma}$. The generalized Strebel condition (5.1.6) fixes the parameters u and $m_{i,j}$ relative to the choice of the phase ϑ.

Generalized Fenchel-Nielsen networks on the three-punctured sphere $\mathbb{P}^1_{0,1,\infty}$ with three maximal punctures were classified in [41] in the special case when all parameters $m_{i,j}$ vanish. It was found that there is a single generalized Fenchel-Nielsen network at each phase $\vartheta_{[p,q]}$ with

$$\tan \vartheta_{[p,q]} = \frac{\sqrt{3}q}{q-2p}, \tag{5.1.9}$$

for any pair of coprime integers (p,q). Each generalized Fenchel-Nielsen network corresponds to a generalized Strebel differential φ_2 with

$$e^{-i\vartheta_{[p,q]}} \oint_{A_{p,q}} \lambda \in \mathbb{R}. \tag{5.1.10}$$

where $A_{p,q} = p\gamma_1 + q\gamma_2$, for a certain basis of 1-cycles γ_1 and γ_2 on Σ.

Generalized Fenchel-Nielsen networks on a punctured Riemann surface C are defined with respect to a pants decomposition of C and can thus be found by gluing together generalized Fenchel-Nielsen "molecules" on the individual pairs of pants. Allowed gluings require that not only the type of the punctures match, but also the decorations along the pants curves (possibly by inserting additional branch cuts).

In the following we restrict ourselves to Fenchel-Nielsen networks obtained from gluing Fenchel-Nielsen molecules with two maximal and one minimal puncture along maximal boundaries. We call this class of generalized Fenchel-Nielsen networks ***of length-twist type***. Figure 5.14 gives an example of such a length-twist type network on the four-punctured sphere (where we have replaced the two maximal punctures by boundaries).

5.2 Higher length-twist coordinates

Let ∇ be a flat SL_K-connection on a punctured curve $(C, \mathcal{D})$ with a fixed *semi-simple* conjugacy class

$$\mathcal{C}_l = \mathrm{diag}\{M_{l,1}, \ldots M_{l,K}\} \tag{5.2.1}$$

at each puncture with $M_{l,i} \in \mathbb{C}^\times$.

The partition of the K eigenvalues can be read off from the Young diagram assigned to the puncture: the height of each column in the Young diagram encodes the multiplicities of coincident eigenvalues. In particular, for generic[1] values of the eigenvalues, a conjugacy class at a minimal puncture is a scalar multiple of a complex reflection matrix.

We denote the (relative) moduli space of such flat connections by

$$\mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K), \tag{5.2.2}$$

where $\mathcal{C} = \{\mathcal{C}_l\}$ is the collection of conjugacy classes.

We will restrict ourselves to Riemann surfaces C that can be obtained by gluing spheres with two maximal and one minimal puncture, where gluing is permitted only along maximal boundaries[2]. Since a generic flat SL_K-connection on the sphere with two maximal and one minimal puncture is completely specified (up to equivalence) by the eigenvalues of the monodromy around the punctures, the moduli space of flat SL_K-connections on any such surface C is $(K-1)(6g-6+2n)$-dimensional, where $3g-3+n$ is the number of pants curves.

In this section we define a generalization of the standard Fenchel-Nielsen length-twist coordinates on the moduli space

$$\mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K; \mathcal{W})$$

of $\mathcal{W}$-framed flat SL_K connections, where C is built by gluing as above. In section 5.3 we show that these coordinates are realized as spectral coordinates through the abelianization method. The $\mathcal{W}$-framing will be crucial in picking out a canonical abelianization of ∇.

5.2.1 Higher length-twist coordinates

A flat SL_K connection ∇ on C with fixed boundary conjugacy classes is specified (up to equivalence) by $2K-2$ parameters at each pants curve α_i.

Half of this set of parameters, say $\ell_1, \ldots, \ell_{K-1}$, are simply the (logs of) eigenvalues of the monodromy M_i. The indexing of these parameters is determined by the decoration as well as the framing data. If the decoration at the boundary assigns

[1] $M_{l,1} = \ldots = M_{l,K-1}$ not equal to a K-th root of unity

[2] This constraint is natural from the physical point of view, where it corresponds to "gauging the flavor symmetry" associated to the punctures.

the sheet ordering $(i_1, \ldots, i_K)$ to the $+$ direction and the framing of ∇ at the boundary in the $+$ direction is given by the ordered tuple of eigenlines $(l_{\alpha_1}, \ldots, l_{\alpha_K})$, then we define

$$L_{i_j} = -e^{\pi i \, \ell_{i_j}} \tag{5.2.3}$$

as the eigenvalue corresponding to the eigenline l_{α_j}.[3]

The other half of the parameters, say $\tau_1, \ldots, \tau_{K-1}$, have a more indirect definition. One approach is in terms of their transformation under the following modification of the connection ∇. Suppose we cut the surface C into two pieces along a pants curve α.[4] We obtain two surfaces with boundary, say C_1 and C_2 carrying flat connections ∇_1 and ∇_2, as well as an isomorphism ι that relates ∇_1 to ∇_2. Let us now change ∇_1 by a gauge transformation κ that preserves the monodromy M around α, and then glue C back along the boundary α.

If the monodromy M_+ is diagonalized by the gauge transformation g, then the transformation κ can be written as

$$\kappa = g^{-1} \circ \operatorname{diag}\left(e^{\lambda_1}, \ldots, e^{\lambda_K}\right) \circ g \tag{5.2.4}$$

with $\sum_{i=1}^{K} \lambda_i = 0$. After gluing back we thus obtain a 1-parameter family of modified connections $\nabla(\lambda)$. This operation is sometimes called the (generalized) twist flow (see for instance [56] in the real-analytic setting, which builds on [57, 58, 59]).

Any choice of parameters $\tau_1, \ldots, \tau_{K-1}$ with the property that they change under the twist flow as

$$\tau_j \mapsto \tau_j + \frac{\lambda_j}{2} - \frac{\lambda_K}{2} \tag{5.2.5}$$

are called twist parameters. The twist parameters τ_i are thus only defined up to an additive function in the length parameters $\ell_1, \ldots, \ell_K$.

This definition of the length-twist coordinates $\ell_1, \ldots, \ell_{K-1}$ and $\tau_1, \ldots, \tau_{K-1}$ guarantees that they are Darboux coordinates on the moduli space of ($\mathcal{W}$-framed) flat SL_K connections. We refer to them as (complex) higher length and twist coordinates, respectively.[5] In §5.3 we will realize these coordinates explicitly as spectral coordinates associated to the generalized Fenchel-Nielsen network $\mathcal{W}$ of length-twist type, and obtain coordinate formulas for the trace functions on $\mathcal{M}_{\mathrm{dR}}$.

[3] A rationale for the slightly odd conventions is given in §6.3.

[4] Here we suppose that α is a separating loop, a similar discussion holds if it is nonseparating.

[5] This is a rather straightforward higher rank generalization of the definition of Fenchel-Nielsen length-twist coordinates in [4].

5.2.2 Fenchel-Nielsen twist coordinate

Let $K = 2$, so there is only one pair of coordinates (ℓ, τ) associated to each pants curve. The twist coordinate defined as above is only determined up to a canonical transformation $\tau' = \tau + f(\ell)$. However, certain other constructions yield a distinguished choice for the twist, given by the so-called complex Fenchel-Nielsen twist τ^{FN} [20, 21, 60, 61]. This twist parameter is identical to the NRS Darboux coordinate $\beta/2$ [7].

Example. On the four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$ fix the presentation of the fundamental group as illustrated in Figure 5.7, generated by the paths γ_0, γ, γ_1 and γ_∞ with the relation

$$\langle \gamma_0, \gamma, \gamma_1, \gamma_\infty | \, \gamma_0 \, \gamma \, \gamma_\infty \, \gamma_1 = 1 \rangle. \tag{5.2.6}$$

If the conjugacy class around the path γ_l is semisimple with eigenvalues M_l and M_l^{-1}, we have that the traces of the monodromy matrices $\mathbf{M}_\alpha = \mathbf{M}_{\gamma_0}\mathbf{M}_\gamma$ and $\mathbf{M}_\beta = \mathbf{M}_{\gamma_0}\mathbf{M}_{\gamma_\infty}$ are given by

$$\mathrm{Tr}\, \mathbf{M}_\alpha = L + L^{-1}, \tag{5.2.7}$$

$$\mathrm{Tr}\, \mathbf{M}_\beta = \sqrt{N(L)} \left(T + T^{-1} \right) + N_\circ(L), \tag{5.2.8}$$

where

$$\begin{aligned} L + L^{-1} &= -2\cos(\pi\ell), \\ T + T^{-1} &= -2\cosh(2\tau^{\mathrm{FN}}), \\ M_l + M_l^{-1} &= -2\cos(\pi m_l), \end{aligned} \tag{5.2.9}$$

and

$$\begin{aligned} N(L) &= \frac{c_{0q}(L) c_{1\infty}(L)}{\sin^4(\pi\ell)}, \\ c_{kl} &= \cos(\pi\ell)^2 + \cos(\pi m_k)^2 + \cos(\pi m_l)^2 + \cos(\pi\ell, \pi m_k, \pi m_l) - 4, \\ N_\circ(L) &= \frac{\cos(\pi\ell)\left(\cos(\pi m_0, \pi m_1) + \cos(\pi m, \pi m_\infty)\right) + \cos(\pi m, \pi m_1) + \cos(\pi m_0, \pi m_\infty)}{\frac{1}{2}\sin^2(\pi\ell)}, \end{aligned} \tag{5.2.10}$$

where we defined $\cos(x, y) = \cos(x)\cos(y)$. We realize the Fenchel-Nielsen length-twist coordinates ℓ and τ^{FN} as spectral coordinates in §5.4.2 by averaging over the two resolutions of a Fenchel-Nielsen network.

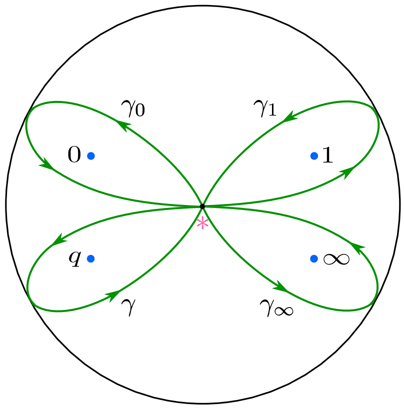


Figure 5.7: Generators for the fundamental group $\pi_1(C, *)$ of the four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$.

5.2.3 Higher length-twist coordinates as spectral coordinates

The coordinate system one obtains from abelianization depends in general on the isotopy class of the spectral network $\mathcal{W}$. The spectral coordinates for generalized Fenchel-Nielsen networks of length-twist type are higher length-twist coordinates, in the sense that they satisfy the twist flow property described in §5.2.1. The proof of this is a straightforward generalization of the argument in [4], where it was shown that the spectral coordinates corresponding to Fenchel-Nielsen networks are Fenchel-Nielsen length-twist coordinates.

Indeed, let us fix an annulus A, corresponding to a glued maximal boundary, and construct the corresponding basis of equivariant 1-cycles A_j and B_j, corresponding to a choice of A and B-cycles on the cover Σ.

Suppose that the decoration at the annulus A in the $+$ direction is $(i_1, \ldots i_K)$ and that the framing of ∇ in the $+$ direction is $(l_{\alpha_1}, \ldots, l_{\alpha_K})$. Fix a path $\wp$ going around the annulus A in the $+$ direction. Consider the lift $A_j \in H_1(\Sigma, \mathbb{Z})$ of $\wp$ onto sheet j. The spectral coordinate $\mathcal{X}_{A_j}$ is equal to the eigenvalue corresponding to the eigenline $l_{\alpha_{j'}}$ with $i_{j'} = j$, which according to §5.2.1 equals the higher length coordinate L_j.

Fix a 1-cycle B_j that crosses the j-th and K-th lift of the annulus A. Under the twist flow parametrized by $(\Lambda_1, \ldots, \Lambda_K)$ the section $(s_1, \ldots, s_K) \mapsto (\Lambda_1 s_1, \ldots, \Lambda_K s_K)$. This shows that the twist flow acts on $\mathcal{X}_{B_j}$ as

$$\mathcal{X}_{B_j} \mapsto (\Lambda_K)^{-1} \Lambda_j \, \mathcal{X}_{B_j},$$

which according to §5.2.1 implies that $\mathcal{X}_{B_j}$ is a higher twist coordinate.

The 1-cycles A_j and B_j satisfy

$$\langle A_j, B_k \rangle = \delta_{jk}$$

and thus indeed correspond to a choice of A and B-cycles on the cover Σ.

5.2.4 Representations

Instead of working directly with flat connections, we use the Riemann-Hilbert correspondence and work with the corresponding parallel transport maps. As in §6 of [4] we replace flat SL_K-connections of C by SL_K-representations of a groupoid $\mathcal{G}_C$ of paths on C and equivariant GL_1-connections by equivariant GL_1-representations of a groupoid $\mathcal{G}_{\Sigma'}$ of paths on Σ'. The objects of the groupoid $\mathcal{G}_C$ are basepoints on either side of a single wall in the spectral network $\mathcal{W}$, whereas the objects of the paths groupoid $\mathcal{G}_{\Sigma'}$ are lifts of these basepoints to the cover Σ'. Morphisms of the groupoid $\mathcal{G}_C$ (and $\mathcal{G}_{\Sigma'}$) are homotopy classes of oriented paths $\wp$ which begin and end at basepoints on C (and their lifts to Σ' respectively). Examples of such path groupoids are given in Figure 5.9 and 5.11. In figures, paths that do not cross any walls are coloured light-blue, whereas paths that connect the two basepoints attached to a single wall are coloured red.

5.3 Abelianization for higher length-twist networks

Let C be a punctured surface together with a pants decomposition into pairs of pants with two maximal and one minimal puncture. Furthermore, choose a generalized Fenchel-Nielsen network $\mathcal{W}$ of length-twist type on C respecting the pants decomposition. Our aim in this section is to show that the $\mathcal{W}$-nonabelianization mapping ψ_1 as well the $\mathcal{W}$-abelianization mapping ψ_2 both are bijections. We use the following strategy.

Fix a length-twist type network $\mathcal{W}$ and a $\mathcal{W}$-framed flat SL_K connection ∇. Suppose that we are given a $\mathcal{W}$-abelianization of ∇. That is, suppose that we are given local bases

$$(s_1^{\mathcal{R}}, \ldots, s_K^{\mathcal{R}})$$

on all domains $\mathcal{R}$ of $C \backslash \mathcal{W}$, and that the transformation $\mathcal{S}_w$ that relates the bases in adjacent domains, divided by a wall of type ij is of the form

$$\mathcal{S}_w = 1 + e_w, \tag{5.3.1}$$

where e_w lies in the 1-dimensional vector space $\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j)$.

The transformations e_w are not arbitrary, as they must satisfy some constraints. Encircling a (simple) branch point of the covering $\Sigma \to C$, due to almost-flatness, yields a constraint on their coefficients, as does encircling each joint of the spectral network $\mathcal{W}$ (which must give the identity). We show that these constraints admit a unique solution for the transformations $\mathcal{S}_w$, up to abelian gauge equivalence, if we demand the local bases around maximal punctures agree with the choice of $\mathcal{W}$-framing of ∇.

Recall that the $\mathcal{W}$-framing of ∇ is specified by a choice of framing at the maximal punctures and boundaries (see §4.2.4). Rather than requiring any similar data at the minimal puncture, we find that the constraints impose that the local basis at each minimal puncture may be expressed uniquely in terms of the transformations $\mathcal{S}_w$ and the framing data at any one of the maximal punctures or boundaries.

We conclude that there is a unique $\mathcal{W}$-abelianization of ∇ for every choice of $\mathcal{W}$-framing of ∇. But at the same time we deduce that there is a unique $\mathcal{W}$-nonabelianization of the corresponding equivariant $GL(1)$ connection ∇^{ab}. Furthermore, we find that all $\mathcal{W}$-abelianizations of ∇ are obtained in this way: if we have an abelianization of ∇, then the K lines $\iota^{-1}(\mathcal{L}_i)$ for $1 \leq i \leq K$ at each maximal puncture or maximal boundary must all be eigenlines of the monodromy of ∇ around that maximal puncture or maximal boundary. The only freedom is the choice which of these lines is which eigenline, i.e. the choice of framing. Hence we find that both mappings ψ_1 and ψ_2 (from §4.2.5) are bijections.

In the following we first spell out the details for a Fenchel-Nielsen and a generalized Fenchel-Nielsen molecule. We then use the gluing formalism to complete the argument for Fenchel-Nielsen and generalized Fenchel-Nielsen networks of length-twist type on any surface C.[6]

5.3.1 $K = 2$ molecule

Fix the Fenchel-Nielsen molecule $\mathcal{W}$ from Figure 5.8 on the three-holed sphere C. This was one of the examples from [4] (although we discuss the $\mathcal{W}$-abelianization in a slightly different way). Say that ∇ is a $\mathcal{W}$-framed flat SL(2) connection on C; the framing is a choice of eigenlines l_+ and l_- at each annulus A. Say that M is the

[6] To be precise, we show uniqueness only for networks glued from the molecule illustrated in Figure 5.10, but we expect it to hold for any network built from the $K = 3$ Fenchel-Nielsen molecules of length-twist type.

eigenvalue corresponding to the eigenline l_+ and M^{-1} the eigenvalue corresponding to the eigenline l_-.

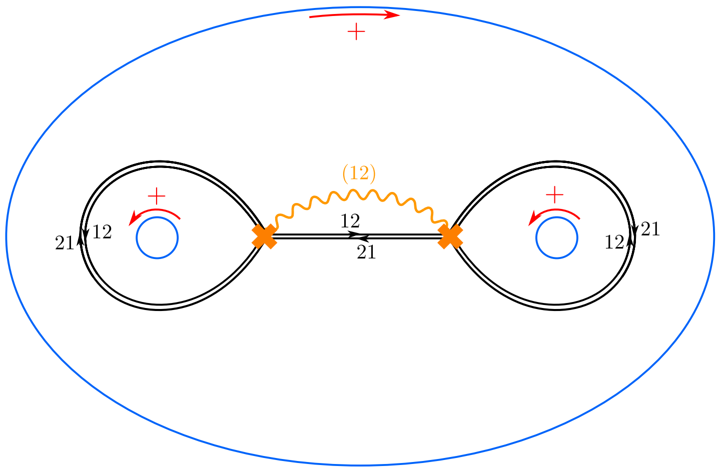


Figure 5.8: Length-twist network on the three-holed sphere together with a choice of $+$ direction around each hole.

Choose a trivialization of the covering $\pi : \Sigma \to C$, and suppose that ∇ admits a $\mathcal{W}$-abelianization. We require that the corresponding $\mathcal{W}$-abelianization singles out the basis of eigenlines $l_i = l_+$ and $l_j = l_-$ if the decoration in the $+$ direction is ij. We will now show that this uniquely determines the $\mathcal{W}$-abelianization.

The $\mathcal{W}$-abelianization corresponds to choosing a basis $(s_1, s_2) \in l_1 \oplus l_2$ in each annulus, such that the bases in adjacent domains are related by a transformation $\mathcal{S}_w$. Choose basepoints and generators of the path groupoids $\mathcal{G}_C$ and $\mathcal{G}_{\Sigma'}$ as in Figure 5.9. The section s_i changes by a constant when parallel transporting it along a light blue path that does not cross a branch cut. If the path does cross branch cuts it furthermore changes sheet accordingly. We encode the parallel transport of the basis (s_1, s_2) along light blue paths $\wp$ in matrices $D_\wp$ and along red paths w, connecting the red dots across walls, in matrices $\mathcal{S}_w$.

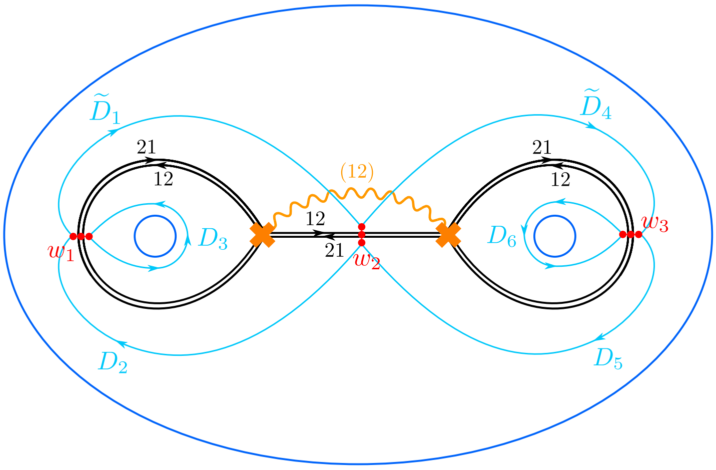


Figure 5.9: Length-twist network on the three-punctured sphere together with a choice of basepoints (the red dots) and a choice of paths (in light-blue and red). The red paths are admittedly barely visible in this figure, but connect the red dots across walls.

The matrices $D_\wp$ are not arbitrary, as they encode the parallel transport coefficients of the equivariant flat connection ∇^{ab}. When going around the boundaries we find the constraints

$$D_3 = \text{diag}(M_0, M_0^{-1}) \tag{5.3.2}$$
$$D_6 = \text{diag}(M_1, M_1^{-1}) \tag{5.3.3}$$
$$D_2 D_5 \widetilde{D}_4 \widetilde{D}_1 = \text{diag}(M_\infty^{-1}, M_\infty). \tag{5.3.4}$$

When traversing around the branch points we find the almost-flatness constraints

$$(D_3 D_2 \widetilde{D}_1)^2 = -1 \tag{5.3.5}$$
$$(D_6 D_5 \widetilde{D}_4)^2 = -1. \tag{5.3.6}$$

Together, these abelian flatness constraints determine the matrices $D_\wp$ up to transformations $G_z = \text{diag}(g_z, g_z^{-1})$ at the basepoints z that act on the matrices $D_\wp$ by

$$D_\wp \mapsto G_{f(\wp)} D_\wp \, G_{i(\wp)}^{-1}, \tag{5.3.7}$$

where $i(\wp)$ is the initial point of the path $\wp$ and $f(\wp)$ its end point. That is, up to abelian gauge transformations, $\mathcal{W}$-abelization determines a unique equivariant $\text{GL}(1)$ connection ∇^{ab} on the cover Σ.

It remains to check that there is a unique solution to the transformations $\mathcal{S}_w$ (up to an abelian gauge transformation). The transformations $\mathcal{S}_w$ are constrained by the requirement that following any contractible loop on the base C should result in the identity. Traversing around either branch point gives the constraints

$$D_2\,\mathcal{S}_{w_2}\,\widetilde{D}_1\,\mathcal{S}_{w_1}^{-1}\,D_3\,\mathcal{S}_{w_1} = 1 \tag{5.3.8}$$

$$\widetilde{D}_4\,\mathcal{S}_{w_2}^{-1}\,D_5\,\mathcal{S}_{w_3}^{-1}\,D_6\,\mathcal{S}_{w_3} = 1, \tag{5.3.9}$$

where

$$\mathcal{S}_{w_1} = \begin{pmatrix} 1 & 0 \\ \widetilde{c}_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \tag{5.3.10}$$

$$\mathcal{S}_{w_2} = \begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \widetilde{c}_2 & 1 \end{pmatrix} \tag{5.3.11}$$

$$\mathcal{S}_{w_3} = \begin{pmatrix} 1 & 0 \\ \widetilde{c}_3 & 1 \end{pmatrix}\begin{pmatrix} 1 & c_3 \\ 0 & 1 \end{pmatrix} \tag{5.3.12}$$

Indeed, this has a unique solution up to equivalence, with for instance,

$$c_1 = \frac{g_1^2\,(1 - M_0 M_1 M_\infty)\left(1 - \frac{M_0 M_\infty}{M_1}\right)}{(1 - M_\infty^2)} \qquad \widetilde{c}_1 = -\frac{1}{g_1^2(1 - M_0^2)}, \tag{5.3.13}$$

where g_1 is a coefficient of the abelian gauge transformation at basepoint 1. As explained in [4], the s-parameters c_z and $\widetilde{c}_z$ have an interpretation as abelian parallel transport along so-called "detour paths" which follow a wall back to its emanating branchpoint and return to a different preimage on Σ.

Note that that the unique solution to the branch point constraints crucially depends on the chosen framing at each of the annuli, but in a simple way: changing the framing at any one of the annuli A_l corresponds to replacing $M_l \mapsto M_l^{-1}$ in the expressions for the transformations $\mathcal{S}_w$.

5.3.2 $K = 3$ molecule

Fix the length-twist type network $\mathcal{W}$ from Figure 5.10 on the sphere C with two (maximal) holes and one minimal puncture, and suppose that ∇ is a $\mathcal{W}$-framed flat SL_3 connection on C. As before, the framing corresponds to an ordered tuple of three eigenlines around each boundary component. That is, an ordered tuple $(l_{0,\alpha}, l_{0,\beta}, l_{0,\gamma})$ at the top annulus and an ordered tuple $(l_{\infty,\alpha}, l_{\infty,\beta}, l_{\infty,\gamma})$ at the bottom annulus. We will show that there is a unique $\mathcal{W}$-abelianization of this ∇ that agrees with the framing.

One might ask why we did not introduce framings for minimal punctures. Suppose for the moment that we needed a "framing" at a minimal puncture, given by any ordered tuple $(l_{1,\alpha}, l_{1,\beta}; l_{1,\gamma})$ of eigenlines, where the eigenline $l_{1,\gamma}$ corresponds to the distinguished eigenvalue of the monodromy. It seems like this would introduce a continuous family of abelianizations, but in fact we find in the following that the abelianization constraints determine the tuple $(l_{1,\alpha}, l_{1,\beta}; l_{1,\gamma})$ uniquely in terms of the framings at the maximal punctures.

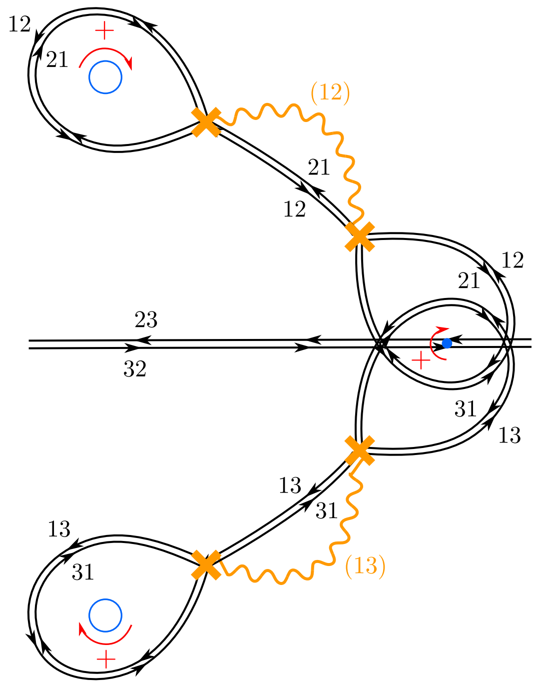


Figure 5.10: Rank 3 Fenchel-Nielsen molecule together with a + direction at each puncture and hole.

Choose a trivialization of the covering $\pi : \Sigma \to C$, and suppose that ∇ admits a $\mathcal{W}$-abelianization. Around each hole A_l we require that the corresponding $\mathcal{W}$-abelianization has eigenlines $l_{l,i} = l_{l,\alpha}$ and $l_{l,j} = l_{l,\beta}$ and $l_{l,k} = l_{l,\gamma}$ if the decoration in the + direction is (ijk). Around the minimal puncture we require that the corresponding $\mathcal{W}$-abelianization has eigenlines $l_{1,i} = l_{1,\alpha}$, $l_{1,j} = l_{1,\beta}$ and $l_{1,k} = l_{1,\gamma}$ if the decoration in the + direction is $(ij;k)$.

The $\mathcal{W}$-abelianization corresponds to the data of a basis $(s_1^{\mathcal{R}}, s_2^{\mathcal{R}}, s_3^{\mathcal{R}})$ on all domains $\mathcal{R}$ of $C \backslash \mathcal{W}$, such that the bases in adjacent domains are related by a transformation $\mathcal{S}_w$. Choose basepoints and generators of the path groupoids $\mathcal{G}_C$ and $\mathcal{G}_{\Sigma'}$ as in Figure 5.11. As before, we encode the parallel transport of ∇ in "abelian gauge" along light blue paths $\wp$ in matrices $D_\wp$ and along red paths w in matrices $\mathcal{S}_w$.

The matrices $D_\wp$ are not arbitrary, as they encode the parallel transport coefficients of the equivariant connection ∇^{ab} on the cover Σ'. For instance, when going around a loop encircling the top branch point twice we find the constraint

$$(\widetilde{D}_3 D_1 D_2)^2 = \mathrm{diag}(-1,-1,1). \tag{5.3.14}$$

The abelian holonomies in the + direction around the holes, labeled by 0 and ∞, and around the puncture labeled by 1 are given in terms of the monodromy eigenvalues as

$$\mathrm{Hol}_0 \nabla^{\mathrm{ab}} = \mathrm{diag}(M_{0,1}, M_{0,2}, (M_{0,1}M_{0,2})^{-1}), \tag{5.3.15}$$
$$\mathrm{Hol}_\infty \nabla^{\mathrm{ab}} = \mathrm{diag}(M_{\infty,1}, M_{\infty,2}, (M_{\infty,1}M_{\infty,2})^{-1}), \tag{5.3.16}$$
$$\mathrm{Hol}_1 \nabla^{\mathrm{ab}} = \mathrm{diag}((M_1)^{-2}, M_1, M_1). \tag{5.3.17}$$

Note that, whereas the framing at each annulus fixes the ambiguity of which eigenvalue corresponds to which sheet, for the puncture there is no such ambiguity. The abelian holonomy around a puncture must have coefficient M_1^{-2} for the distinguished sheet, and the coefficient M_1 for the two other sheets is the same.

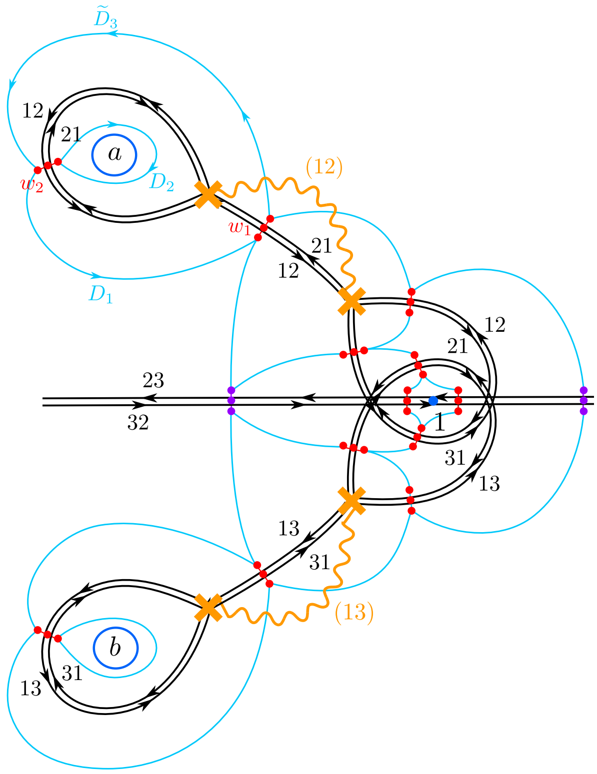


Figure 5.11: Rank 3 Fenchel-Nielsen molecule together with a choice of basepoints (the red and purple dots) and a choice of paths (in light blue and red). The purple basepoints should be identified. Even though we have only oriented and labeled a few paths to avoid cluttering of the picture, all paths are oriented and labeled.

Solving for all abelian flatness constraints one finds that the matrices $D_\wp$ are uniquely determined up to abelian gauge transformations. In other words, there is a unique equivariant $GL(1)$-connection ∇^{ab} on the cover Σ.

It remains to check that there is a unique solution to the transformations $\mathcal{S}_w$, which are constrained by nonabelian "branch point constraints" and "joint constraints". The former impose that ∇ has trivial monodromy around the branch points, and the latter that ∇ has trivial monodromy around the joints. For instance, encircling the top branch point gives the constraint (see Figure 5.11)

$$\widetilde{D}_3\, \mathcal{S}_{w_1} D_1\, \mathcal{S}_{w_2} D_2\, \mathcal{S}_{w_2}^{-1} = \mathbf{1}. \tag{5.3.18}$$

Furthermore, we need to enforce the boundary conditions at the punctures. For instance, going around the minimal puncture gives the constraint (see Figure 5.12)

$$\mathcal{S}_{w_{3,a}} D_4\, D_5\, \mathcal{S}_{w_4}^{-1}\, D_6\, D_7\, \mathcal{S}_{w_{3,b}} = \mathrm{Hol}_1 \nabla^{\mathrm{ab}}, \tag{5.3.19}$$

where $\mathcal{S}_{w_4} = \mathcal{S}_{w_{4,b}} \mathcal{S}_{w_{4,a}}$.

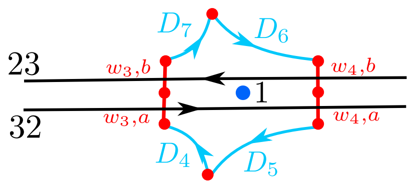


Figure 5.12: A close-up of Figure 5.11 near the minimal puncture labeled by 1. The red short paths w_3 and w_4 are split in half, labeled by the letters a and b.

Solving all these constraints shows that the matrices $\mathcal{S}_w$ have a canonical solution, which (just like for $K = 2$ abelianizations) have an interpretation as parallel transport along auxiliary paths.

The resulting expressions for the matrices $\mathcal{S}_w$ depend on the choice of framing at the (maximal) holes through the choice of ordering the eigenvalues in the abelian holonomy matrices $\mathrm{Hol}_0 \nabla^{\mathrm{ab}}$ and $\mathrm{Hol}_\infty \nabla^{\mathrm{ab}}$. In contrast, the particular choice of eigenlines $(l_{1,\alpha}, l_{1,\beta}; l_{1,\gamma})$ at the minimal puncture doesn't play any role in computing the $\mathcal{S}_w$.

Yet, the unique solution for the transformations $\mathcal{S}_w$ implies that the basis in any region $C \backslash \mathcal{W}$, and in particular near the minimal puncture, is uniquely determined in terms of the choice of eigenlines at the boundary components. That is, the abelianization of ∇ canonically determines the choice of eigenlines $(l_{1,\alpha}, l_{1,\beta}; l_{1,\gamma})$ at the minimal puncture. In particular, there is no framing ambiguity at the minimal puncture after all.

Note that this is consistent with the interpretation of the framing data in terms of the S matrices. Indeed, whereas a change in framing at an annulus A_l corresponds to a permutation of the mass parameters $M_{l,i}$, the permutation group of mass parameters at the minimal puncture is trivial. Generalizing this argument to any regular puncture, we expect that the framing ambiguity at a regular puncture with Young diagram Y is given by the group $S_{n_1} \times \ldots \times S_{n_k}$, where $n_1, \ldots, n_k$ counts columns of Y with the same height and S_n is the permutation group with n elements.

5.3.3 Gluing

Fix a length-twist type network $\mathcal{W}$ built out of two molecules. (The same argument can be extended to more molecules.) Say that ∇ is a $\mathcal{W}$-framed flat connection on C and suppose that ∇ admits a $\mathcal{W}$-abelianization.

Choose a pants cycle α relative to $\mathcal{W}$ and fix a marked point z_α on α. The monodromy ∇ along α is diagonal in the "abelian" gauge. Cut the surface C along α into two pair of pants C_1 and C_2. Say ∇_1 is the restriction of ∇ to C_1, and ∇_2 the restriction to C_2. ∇_1 and ∇_2 are both flat SL_K-connections with trivialization at the marked point z_α.

The $\mathcal{W}$-abelianization of ∇_1 (as well as ∇_2) is almost the same as described in the previous subsection. In particular, we still find the same unique solution to the S matrices $\mathcal{S}_w$. The only difference that we have to introduce an additional path $p_{\alpha,1}$ connecting the basepoint z_2 with z_α. The parallel transport matrix D_{α_1} along this path is diagonal and determined by ∇_1. This uniquely fixes the $\mathcal{W}$-abelianization on C_1 (and similarly on C_2).

If we glue back together the three-holed spheres C_1 and C_2, we can glue the two $\mathcal{W}$-abelianizations on C_1 and C_2 to obtain a unique $\mathcal{W}$-abelianization of ∇. Since we need to divide out by (diagonal) gauge transformations at the marked point z_α, the resulting equivariant GL_1 connection ∇^{ab} on C is characterized by its parallel transport along the lifts of the path $p_{\alpha,1} \circ p_{\alpha,2}^{-1}$ to Σ.

We conclude that the $\mathcal{W}$-framed connection ∇ admits a unique $\mathcal{W}$-abelianization and that the corresponding ∇^{ab} admits a unique $\mathcal{W}$-nonabelianization. Moreover, as before, different $\mathcal{W}$-abelianizations of ∇ (without the $\mathcal{W}$-framing) correspond to different $\mathcal{W}$-framings.

5.4 Monodromy representations in higher length-twist coordinates

In the previous section we have explicitly constructed $\mathcal{W}$-abelianizations as well as $\mathcal{W}$-nonabelianizations. With the resulting description of ∇ in terms of the parallel transport matrices D and transformations $\mathcal{S}_w$ it is a straightforward matter to write down the monodromy representation for ∇ in terms of the spectral coordinates $\mathcal{X}_\gamma$. In this section we summarize these monodromy representations in a few examples.

Recall that any length-twist type network carries a resolution, which can either be American or British. The spectral coordinates $\mathcal{X}_\gamma$ corresponding to either resolution are generalized Fenchel-Nielsen length-twist coordinates. The spectral length coordinates are the same in either resolution, while the spectral twist coordinates differ (corresponding to the ambiguity in the Fenchel-Nielsen twist coordinates).

In this section we will see that the NRS Darboux coordinates (i.e. the standard complex Fenchel-Nielsen length-twist coordinates) are only obtained by *averaging* over the two resolutions. More precisely, we define the average higher length-twist coordinates as[7]

$$L_i = \mathcal{X}^+_{A_i} = \mathcal{X}^-_{A_i} \tag{5.4.1}$$

$$T_i = \sqrt{\mathcal{X}^+_{B_i}\,\mathcal{X}^-_{B_i}}, \tag{5.4.2}$$

where A_i and B_i constitute a choice of A and B-cycles on the cover Σ, as defined in §4.2.6, and $+$ and $-$ refer to the American and the British resolution, respectively. Indeed, we find that the average length and twist agree with the standard length and twist of §5.2.2.

The only left-over ambiguity in the spectral coordinates is an ambiguity in defining the B-cycles on the cover Σ and a choice of (generalized) Fenchel-Nielsen length-twist network. Resultingly, we find that the higher length-twist coordinates are determined up to a multiplication by a simple monomial in the (exponentiated) mass parameters.

[7] We thank Andrew Neitzke for this suggestion.

5.4.1 Strategy

Let us spell out our strategy for computing the spectral coordinates of the length-twist network $\mathcal{W}$ on the four-holed sphere C illustrated on the left in Figure 5.13.

First we cut the four-holed sphere C into two three-holed spheres C_1 and C_2 along the pants cycle α. Say that C_1 is the upper and C_2 the lower three-holed sphere. Any flat SL_2-connection ∇ restricts to flat connections ∇_1 on C_1 and ∇_2 on C_2 with fixed trivialization at a marked point z_α at the boundary. The $\mathcal{W}$-abelianization of ∇_1 is outlined in §5.3.1. The $\mathcal{W}$-abelianization of ∇_2 is similar, but with opposite wall labels.

Then we can construct a monodromy representation for ∇_1 with base point z_{w_2} from the matrices

$$\mathbf{M}_0 = \mathcal{S}_{w_2} \widetilde{D}_1 D_2, \tag{5.4.3}$$

$$\mathbf{M}_1 = D_5 \widetilde{D}_4 \mathcal{S}_{w_2}^{-1} \tag{5.4.4}$$

$$\mathbf{M}_\alpha = \mathrm{diag}(M_\infty^{-1}, M_\infty) \tag{5.4.5}$$

with

$$\mathbf{M}_\alpha \cdot \mathbf{M}_1 \cdot \mathbf{M}_0 = 1. \tag{5.4.6}$$

Recall that the matrices $D_\wp$ encode the parallel transport coefficients along the paths $\wp$ illustrated in Figure 5.9.

Applying the same recipe to $\mathcal{W}_2$ yields a monodromy representation of ∇_2 on C_2, generated by the three matrices $M_{0'}$, $M_{1'}$ and $M_{\alpha'}$ with the constraint $M_{\alpha'} \cdot M_{1'} \cdot M_{0'} = \mathbf{1}$. We have that

$$\mathbf{M}_{\alpha'} = \mathrm{diag}(M_\infty, M_\infty^{-1}). \tag{5.4.7}$$

Now glue the three-holed spheres C_1 and C_2 along α together again, while introducing the matrix

$$\mathbf{P} = \mathrm{diag}(p, p^{-1}), \tag{5.4.8}$$

describing the parallel transport of ∇ along the annulus A (from basepoint w_2' to basepoint w_2). Then we can construct a monodromy representation for ∇ in terms of the matrices $\mathbf{M}_0$, $\mathbf{M}_1$, $\mathbf{M}_\alpha$, $\mathbf{M}_{0'}$, $\mathbf{M}_{1'}$, $\mathbf{M}_{\alpha'}$ and $\mathbf{P}$. For instance,

$$\mathbf{M}_\beta = \mathbf{M}_1 \cdot \mathbf{P} \cdot \mathbf{M}_{1'} \cdot \mathbf{P}^{-1}. \tag{5.4.9}$$

Since $\overline{\Sigma}$ is a torus, the monodromy representation depends on two spectral variables: the abelian monodromy along an A-cycle on $\overline{\Sigma}$, which can be expressed in terms of M_∞, and the abelian monodromy along a B-cycle on $\overline{\Sigma}$, which can be expressed in terms of p.

In the next section we give explicit expressions for invariants constructed from this monodromy representation.

5.4.2 $K = 2$, four-punctured sphere

Consider the $K = 2$ Fenchel-Nielsen network on the sphere $\mathbb{P}_{0,q,1,\infty}$ with four (maximal) punctures that is illustrated on the left in Figure 5.13 (where we have replaced all punctures by holes). We choose counter-clockwise abelian holonomies around the punctures and holes as

$$\mathbf{M}_l^{\mathrm{ab}} = \mathrm{diag}(M_l, \frac{1}{M_l}), \tag{5.4.10}$$

and two spectral coordinates $\mathcal{X}_A$ and $\mathcal{X}_B$ as the abelian holonomies along the 1-cycles A and B that are illustrated on the right in Figure 5.13. These 1-cycles form a symplectic basis of $H_1(\overline{\Sigma}, \mathbb{Z})$.

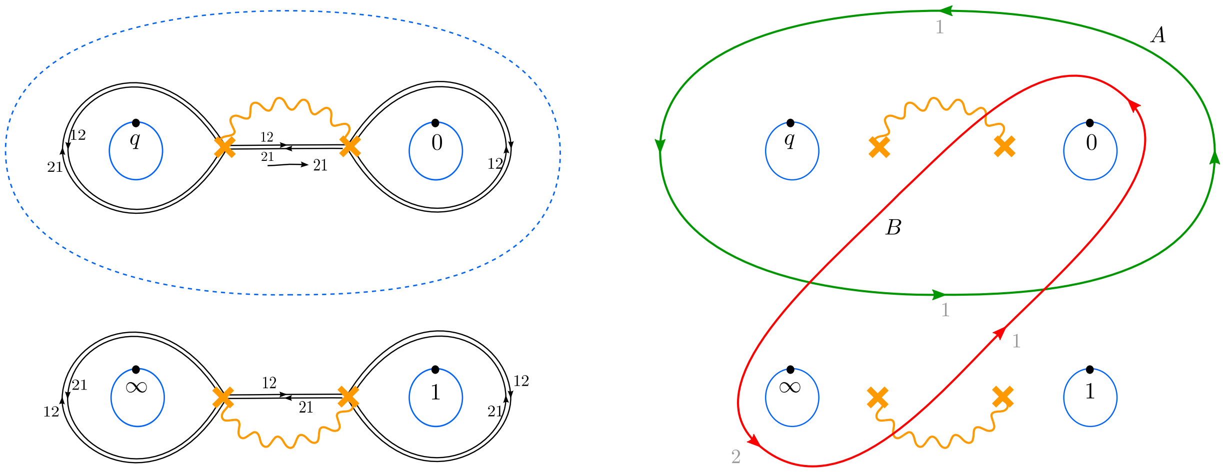


Figure 5.13: Left: Fenchel-Nielsen network on the four-holed sphere (in the British resolution). Right: basis of 1-cycles A and B on the compactified cover $\overline{\Sigma}$.

With the $\mathcal{W}$-abelianization construction the monodromy representation of a generic flat SL_2 connection ∇ can be expressed in terms of the spectral parameters $\mathcal{X}_A$, $\mathcal{X}_B$ and the mass parameters M_0, M, M_1, M_∞. Choose generators $\gamma_0, \gamma, \gamma_1$

and γ_∞ for the fundamental group of the four-punctured sphere as in Figure 5.7. The corresponding monodromy matrices $\mathbf{M}_{\gamma_0}$, $\mathbf{M}_{\gamma}$, $\mathbf{M}_{\gamma_1}$ and $\mathbf{M}_{\gamma_\infty}$ with

$$\mathbf{M}_{\gamma_0} \cdot \mathbf{M}_{\gamma} \cdot \mathbf{M}_{\gamma_\infty} \cdot \mathbf{M}_{\gamma_1} = 1 \tag{5.4.11}$$

whose conjugacy classes at the punctures are fixed such that

$$\mathrm{Tr}\, \mathbf{M}_{\gamma_l} = M_l + \frac{1}{M_l}. \tag{5.4.12}$$

Here we focus on the monodromies $\mathbf{M}_\alpha = \mathbf{M}_{\gamma_0}\mathbf{M}_{\gamma}$ and $\mathbf{M}_\beta = \mathbf{M}_{\gamma_0}\mathbf{M}_{\gamma_\infty}$ (although other monodromies are just as easy to compute).

In the British resolution, with spectral coordinates

$$L = \mathcal{X}_A^+ \tag{5.4.13}$$
$$T^+ = -\mathcal{X}_A^+ \mathcal{X}_B^+, \tag{5.4.14}$$

we find that

$$\mathrm{Tr}\, \mathbf{M}_\alpha = L + \frac{1}{L} \tag{5.4.15}$$
$$\mathrm{Tr}\, \mathbf{M}_\beta = N\, T^+ + N_\circ + \frac{1}{T^+}, \tag{5.4.16}$$

with

$$N(L) = \frac{(f_L^2 + f_0^2 + f^2 - f_L f_0 f - 4)(f_L^2 + f_1^2 + f_\infty^2 - f_L f_1 f_\infty - 4)}{(L - \frac{1}{L})^4} \tag{5.4.17}$$
$$N_\circ(L) = \frac{f_L(f_0\, f_1 + f\, f_\infty) - 2(f\, f_1 + f_0\, f_\infty)}{(L - \frac{1}{L})^2}, \tag{5.4.18}$$

and where $f_L = L + \frac{1}{L}$ and $f_l = M_l + \frac{1}{M_l}$.

On the other hand, in the American resolution, with spectral coordinates

$$L = \mathcal{X}_A^- \tag{5.4.19}$$
$$T^- = -\mathcal{X}_A^- \mathcal{X}_B^-, \tag{5.4.20}$$

we find that

$$\mathrm{Tr}\, \mathbf{M}_\alpha = L + \frac{1}{L} \tag{5.4.21}$$
$$\mathrm{Tr}\, \mathbf{M}_\beta = T^- + N_\circ + \frac{N}{T^-}. \tag{5.4.22}$$

Hence, in terms of the average spectral coordinates

$$L = \mathcal{X}_A \tag{5.4.23}$$

$$T = \mathcal{X}_A \sqrt{\mathcal{X}_B^+ \mathcal{X}_B^-}, \tag{5.4.24}$$

we have that

$$\operatorname{Tr} \mathbf{M}_\alpha = L + \frac{1}{L}, \tag{5.4.25}$$

$$\operatorname{Tr} \mathbf{M}_\beta = \sqrt{N(L)} \left(T + \frac{1}{T} \right) + N_\circ(L), \tag{5.4.26}$$

Later, it will be useful that N can be rewritten as

$$N(L) = \frac{\prod_{i,j \in \{\pm \frac{1}{2}\}} \left(L^{\frac{1}{2}} M_0^i M^j - L^{-\frac{1}{2}} M_0^{-i} M^{-j} \right) \left(L^{\frac{1}{2}} M_1^i M_\infty^j - L^{-\frac{1}{2}} M_1^{-i} M_\infty^{-j} \right)}{(L - \frac{1}{L})^4}. \tag{5.4.27}$$

Note that the monodromy invariants expressed in terms of the average length-twist coordinates agree with those in §5.2.2. This is our first new result. That is, the average length-twist coordinates L and T are the standard exponentiated complex Fenchel-Nielsen length-twist coordinates (which are equal to the NRS Darboux coordinates α and β), removing the ambiguity (up to constants arising from the choice of network and cycles, monomials in the $M_{l,j}$) otherwise present in the twist coordinate.

5.4.3 $K = 3$, sphere with two minimal and two maximal punctures

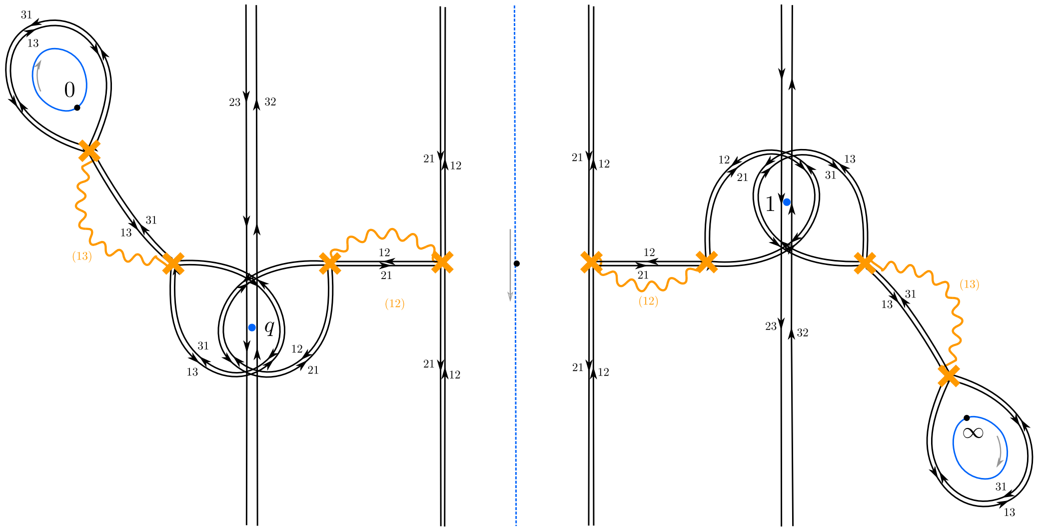


Figure 5.14: Fenchel-Nielsen network on an open region on the sphere $\mathbb{P}^1_{0,1,q,\infty}$ with two minimal and two maximal punctures (in the American resolution). The complete network on C is found by identifying the endpoints of all vertical double walls.

Next, consider the $K = 3$ Fenchel-Nielsen network on the sphere $\mathbb{P}_{0\underline{q}\underline{1}\infty}$ with two minimal and two maximal punctures that is illustrated in Figure 5.14. We choose counterclockwise abelian holonomies around the punctures and holes as

$$\mathbf{M}_0^{\text{ab}} = \text{diag}(M_{0,1}, M_{0,2}, \frac{1}{M_{0,1}M_{0,2}}) \tag{5.4.28}$$

$$\mathbf{M}^{\text{ab}} = \text{diag}(\frac{1}{M^2}, M, M) \tag{5.4.29}$$

$$\mathbf{M}_1^{\text{ab}} = \text{diag}(\frac{1}{M_1^2}, M_1, M_1) \tag{5.4.30}$$

$$\mathbf{M}_\infty^{\text{ab}} = \text{diag}(M_{\infty,1}, M_{\infty,2}, \frac{1}{M_{\infty,1}M_{\infty,2}}). \tag{5.4.31}$$

We choose four spectral coordinates $\mathcal{X}_{A_1}, \mathcal{X}_{A_2}, \mathcal{X}_{B_1}, \mathcal{X}_{B_2}$ as the abelian holonomies along the 1-cycles A_1, A_2, B_1, B_2, respectively, that are illustrated in Figure 5.15. These 1-cycles form a basis of 1-cycles on the compactified cover $\overline{\Sigma}$.

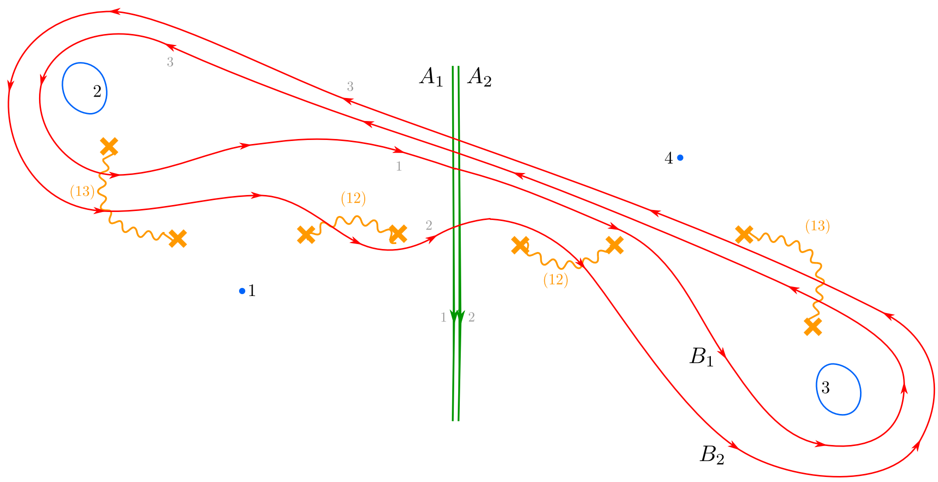


Figure 5.15: Illustrated in green and red are four 1-cycles A_1, A_2, B_1, B_2 which form a basis of 1-cycles on the compactified cover $\overline{\Sigma}$.

Nonabelianization with respect to the spectral network in Figure 5.14, in either the American or British resolution, yields a family of SL_3 flat connections, depending on the spectral parameters $\mathcal{X}_{A_1}$, $\mathcal{X}_{A_2}$, $\mathcal{X}_{B_1}$, $\mathcal{X}_{B_2}$ and the mass parameters $M_{0,1}, M_{0,2}, M, M_1, M_{\infty,1}$ and $M_{\infty,2}$. Their monodromy representations can be expressed in terms of $\mathbf{M}_{\gamma_0}$, $\mathbf{M}_{\gamma}$, $\mathbf{M}_{\gamma_1}$ and $\mathbf{M}_{\gamma_\infty}$ with

$$\mathbf{M}_{\gamma_0} \cdot \mathbf{M}_{\gamma} \cdot \mathbf{M}_{\gamma_\infty} \cdot \mathbf{M}_{\gamma_1} = 1 \tag{5.4.32}$$

whose conjugacy classes at the punctures are fixed such that

$$\mathrm{Tr}\, \mathbf{M}_{\gamma_0} = M_{0,1} + M_{0,2} + \frac{1}{M_{0,1}M_{0,2}} \tag{5.4.33}$$

$$\mathrm{Tr}\, \mathbf{M}_{\gamma} = \frac{1}{M^2} + 2M \tag{5.4.34}$$

$$\mathrm{Tr}\, \mathbf{M}_{\gamma_1} = \frac{1}{M_1^2} + 2M_1 \tag{5.4.35}$$

$$\mathrm{Tr}\, \mathbf{M}_{\gamma_\infty} = M_{\infty,1} + M_{\infty,2} + \frac{1}{M_{\infty,1}M_{\infty,2}}. \tag{5.4.36}$$

Here we focus on the monodromies $\mathbf{M}_\alpha = \mathbf{M}_{\gamma_0}\mathbf{M}_{\gamma}$ and $\mathbf{M}_\beta = \mathbf{M}_{\gamma_0}\mathbf{M}_{\gamma_\infty}$ (although other monodromies are just as easy to compute). In terms of the average generalized length and twist coordinates

$$L_1 = \mathcal{X}_{A_1} \qquad T_1 = \sqrt{M_{0,2}M_{\infty,2}}\sqrt{\mathcal{X}^+_{B_1}\mathcal{X}^-_{B_1}} \tag{5.4.37}$$

$$L_2 = \mathcal{X}_{A_2} \qquad T_2 = \sqrt{M_{0,2}M_{\infty,2}}\,\frac{\mathcal{X}_{A_1}}{\mathcal{X}_{A_2}}\sqrt{\mathcal{X}^+_{B_2}\mathcal{X}^-_{B_2}} \tag{5.4.38}$$

we find

$$\operatorname{Tr}\mathbf{M}_\alpha = L_1 + L_2 + L_3, \tag{5.4.39}$$

$$\operatorname{Tr}\mathbf{M}_\alpha^{-1} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}, \tag{5.4.40}$$

$$\begin{aligned}\operatorname{Tr}\mathbf{M}_\beta = {} & \underline{N}_\circ + \underline{N}(L_1, L_3)T_1 + \underline{N}(L_2, L_3)T_2 + \underline{N}(L_1, L_2)\frac{T_1}{T_2} \\ & + \underline{N}(L_2, L_1)\frac{T_2}{T_1} + \frac{\underline{N}(L_3, L_2)}{T_2} + \frac{\underline{N}(L_3, L_1)}{T_1},\end{aligned} \tag{5.4.41}$$

$$\begin{aligned}\operatorname{Tr}\mathbf{M}_\beta^{-1} = {} & \overline{N}_\circ + \overline{N}(L_1, L_3)T_1 + \overline{N}(L_2, L_3)T_2 + \overline{N}(L_1, L_2)\frac{T_1}{T_2} \\ & + \overline{N}(L_2, L_1)\frac{T_2}{T_1} + \frac{\overline{N}(L_3, L_2)}{T_2} + \frac{\overline{N}(L_3, L_1)}{T_1}.\end{aligned} \tag{5.4.42}$$

where we introduced $L_3 = 1/(L_1 L_2)$. Furthermore,

$$\underline{N}(L_k, L_l) = \frac{1}{\sqrt{MM_1}} \frac{N(L_k)N(L_l)}{N_\star(L_k, L_l)} \tag{5.4.43}$$

$$\overline{N}(L_k, L_l) = \sqrt{MM_1} \frac{N(L_k)N(L_l)}{N_\star(L_k, L_l)} \tag{5.4.44}$$

are symmetric in L_k and L_l, whose numerators are defined by

$$\begin{aligned} N(L_k) = {} & M^{-\frac{3}{4}} M_1^{-\frac{3}{4}} \sqrt{(L_k M M_{0,1} - 1)(L_k M M_{0,2} - 1)\left(\frac{L_k M}{M_{0,1} M_{0,2}} - 1\right)} \\ & \times \sqrt{\left(\frac{M_1 M_{\infty,1}}{L_k} - 1\right)\left(\frac{M_1 M_{\infty,2}}{L_k} - 1\right)\left(\frac{M_1}{L_k M_{\infty,1} M_{\infty,2}} - 1\right)} \end{aligned} \tag{5.4.45}$$

and whose denominators are defined as

$$N_\star(L_k, L_l) = L_k^{-\frac{5}{2}} L_l^{-\frac{5}{2}} (L_k - L_l)^2 (1 - L_k^2 L_l)(1 - L_k L_l^2). \tag{5.4.46}$$

Finally, we write the T_i-independent term (for $\operatorname{Tr}\mathbf{M}_\beta$ — the other is similar) as

$$\underline{N}_\circ = \frac{(\underline{N}_{\circ,1} + \underline{N}_{\circ,2} + \underline{N}_{\circ,3})}{M M_1 M_{0,1} M_{0,2} M_{\infty,1} M_{\infty,2}} \tag{5.4.47}$$

The formulas for the three terms are displayed on the following page.

$$\underline{N}_{\circ,1} = \frac{\left(-L_1L_2M^2\left(M_{0,1}^2M_{0,2}^2+M_{0,2}+M_{0,1}\right)+L_1^2L_2^2M\left(M_{0,2}M_{0,1}^2+M_{0,2}^2M_{0,1}+1\right)-\left(L_2L_1^2+L_2^2L_1-1\right)M_{0,1}M_{0,2}+M^3M_{0,1}M_{0,2}\right)\left(-\frac{M_1^2\left(M_{\infty,1}^2M_{\infty,2}^2+M_{\infty,2}+M_{\infty,1}\right)}{L_1L_2}+\frac{M_1\left(M_{\infty,2}M_{\infty,1}^2+M_{\infty,2}^2M_{\infty,1}+1\right)}{L_1^2L_2^2}-\left(\frac{1}{L_2^2L_1}+\frac{1}{L_2L_1^2}-1\right)M_{\infty,1}M_{\infty,2}+M_1^3M_{\infty,1}M_{\infty,2}\right)}{\left(-\frac{1}{L_2^2L_1}-\frac{1}{L_2L_1^2}+\frac{1}{L_2^3L_1^3}+1\right)\left(L_1^3L_2^3-L_1L_2^2-L_1^2L_2+1\right)} \tag{5.4.48}$$

$$\underline{N}_{\circ,2} = \frac{\left(L_1^2L_2M^3M_{0,1}M_{0,2}-L_1L_2M^2\left(M_{0,1}^2M_{0,2}^2+M_{0,2}+M_{0,1}\right)+L_2M\left(M_{0,2}M_{0,1}^2+M_{0,2}^2M_{0,1}+1\right)+\left(L_2L_1^2-L_2^2L_1-1\right)M_{0,1}M_{0,2}\right)\left(\frac{M_1^3M_{\infty,1}M_{\infty,2}}{L_1^2L_2}-\frac{M_1^2\left(M_{\infty,1}^2M_{\infty,2}^2+M_{\infty,2}+M_{\infty,1}\right)}{L_1L_2}+\frac{M_1\left(M_{\infty,2}M_{\infty,1}^2+M_{\infty,2}^2M_{\infty,1}+1\right)}{L_2}+\left(-\frac{1}{L_2^2L_1}+\frac{1}{L_2L_1^2}-1\right)M_{\infty,1}M_{\infty,2}\right)}{\left(\frac{1}{L_1}-\frac{1}{L_2}\right)\left(\frac{1}{L_1^2L_2}-1\right)\left(L_1-L_2\right)\left(L_1^2L_2-1\right)} \tag{5.4.49}$$

$$\underline{N}_{\circ,3} = \frac{\left(-L_1L_2^2M^3M_{0,1}M_{0,2}+L_1L_2M^2\left(M_{0,1}^2M_{0,2}^2+M_{0,2}+M_{0,1}\right)-L_1M\left(M_{0,2}M_{0,1}^2+M_{0,2}^2M_{0,1}+1\right)+\left(L_2L_1^2-L_2^2L_1+1\right)M_{0,1}M_{0,2}\right)\left(-\frac{M_1^3M_{\infty,1}M_{\infty,2}}{L_1L_2^2}+\frac{M_1^2\left(M_{\infty,1}^2M_{\infty,2}^2+M_{\infty,2}+M_{\infty,1}\right)}{L_1L_2}-\frac{M_1\left(M_{\infty,2}M_{\infty,1}^2+M_{\infty,2}^2M_{\infty,1}+1\right)}{L_1}+\left(-\frac{1}{L_2^2L_1}+\frac{1}{L_2L_1^2}+1\right)M_{\infty,1}M_{\infty,2}\right)}{\left(\frac{1}{L_1L_2^2}-1\right)\left(\frac{1}{L_1}-\frac{1}{L_2}\right)\left(L_1-L_2\right)\left(L_1L_2^2-1\right)} \tag{5.4.50}$$

Chapter 6

Twisted superpotentials from opers

This chapter is based on the joint work [8] with L. Hollands, arXiv:1710.04438.

In this chapter we study our main physical application of the higher length-twist coordinates developed in the previous chapter. Given the data of a class $\mathcal{S}$ theory we utilize these coordinates, following and generalizing a conjecture of Nekrasov-Rosly-Shatashvili, to compute the effective twisted superpotentials associated to the class $\mathcal{S}$ theory by examining how the Lagrangian subvariety of *opers* looks in them. These superpotentials can be interpreted as a two-dimensional shadow of the generating series for equivariant volumes of the instanton moduli spaces in four dimensions.

6.1 Introduction and summary

Let $(\mathfrak{g}, C, \mathcal{D})$ be a tuple of class S data — a complex (semi)simple Lie algebra $\mathfrak{g}$ and a punctured Riemann surface C with defects $\mathcal{D}$. We can obtain a four-dimensional $\mathcal{N} = 2$ quantum field theory T from this data via partially twisted compactification of the six-dimensional $(2,0)$ theory of type $\mathfrak{g}$ on C — this produces a theory of class $\mathcal{S}$. Throughout, we will assume $\mathfrak{g} = A_{K-1}$.

The low energy dynamics of T is described in terms of the Seiberg-Witten prepotential $\mathcal{F}_0(a; m, q)$, a holomorphic function of the Coulomb moduli $a = (a_1, \ldots, a_{K-1})$, the mass parameters $m = (m_1, \ldots, m_n)$ and the UV gauge couplings $q = (q_1, \ldots, q_{3g-3+n})$.

As we reviewed in §2.1.2, the prepotential $\mathcal{F}_0(a; m, q)$ is in general related to a classical algebraically integrable system [62]. It may be interpreted as a generating function of a Lagrangian submanifold $\mathbf{L}_0$ relating the Coulomb parameters a to the dual Coulomb parameters $a^D = \partial_a \mathcal{F}_0$. For theories of class $\mathcal{S}$ this integrable system is a Hitchin system associated to C [1, 10, 63].

Consider T as usual with spacetime

$$\mathbb{R}^4 = \mathbb{R}^2_{1,2} \oplus \mathbb{R}^2_{3,4} \tag{6.1.1}$$

and with the Ω-deformation with complex parameters ϵ_1 and ϵ_2, corresponding to the two isometries rotating the planes $\mathbb{R}^2_{1,2}$ and $\mathbb{R}^2_{3,4}$, respectively. The low energy dynamics of the resulting theory $T_{\epsilon_1,\epsilon_2}$ is described in terms of the ϵ_1,ϵ_2-deformed prepotential $\mathcal{F}(a;m,q,\epsilon_1,\epsilon_2)$. Then $\epsilon_1\epsilon_2\mathcal{F}(a;m,q,\epsilon_1,\epsilon_2)$ is analytic in ϵ_1, ϵ_2 near zero and becomes the prepotential $\mathcal{F}_0(a;m,q)$ in the limit $\epsilon_1,\epsilon_2 \to 0$ [5, 6, 14]. That is,

$$\mathcal{F}(a;m,q,\epsilon_1,\epsilon_2) = \frac{1}{\epsilon_1\epsilon_2}\mathcal{F}_0(a;m,q) + \ldots, \tag{6.1.2}$$

with ... denoting terms regular in ϵ_1 and ϵ_2.

6.1.1 Effective twisted superpotential

Recall the cigar theory of §2.4.2, where instead T is considered on $D \times \mathbb{R}^2$, where D is a cigar. We only turn on the Ω-deformation with parameter $\epsilon_1 = \epsilon$ corresponding to the rotation of the cigar, corresponding to the *Nekrasov-Shatashvili limit*. The resulting theory T_ϵ turns out to preserve a two-dimensional $\mathcal{N} = (2,2)$ super-Poincare invariance.

In [18] it is proposed that, at low energy, T_ϵ is described by a supersymmetric sigma model, whose fields in two dimensions are abelian gauge multiplets coupled to an effective twisted superpotential $\widetilde{W}^{\text{eff}}(a;m,q,\epsilon)$ for the twisted chiral fields in the abelian gauge multiplets. This effective twisted superpotential should be obtained from the four-dimensional partition function as

$$\widetilde{W}^{\text{eff}}(a;m,q,\epsilon) = \lim_{\epsilon_2\to 0} \epsilon_2 \log Z(a;m,q,\epsilon_1 = \epsilon,\epsilon_2), \tag{6.1.3}$$

where we denote the complex vevs of the twisted chiral fields by $a = (a_1,\ldots a_{K-1})$. Recall that mathematically, this limit interpreted term by term in q is our definition of $\widetilde{W}^{\text{eff}}$.

Since the (deformed) prepotential is given by

$$\mathcal{F}(a;m,q,\epsilon_1,\epsilon_2) = \log Z(a;m,q,\epsilon_1,\epsilon_2), \tag{6.1.4}$$

we have that

$$\widetilde{W}^{\text{eff}}(a;m,q,\epsilon) = \frac{1}{\epsilon}\mathcal{F}_0(a;m,q) + \ldots, \tag{6.1.5}$$

where ... are terms regular in ϵ, and $\mathcal{F}_0$ is the usual *Seiberg-Witten prepotential* of the undeformed theory.

Recall that the effective twisted superpotential $\widetilde{W}^{\text{eff}}(a;m,q,\epsilon)$ can be written

$$\widetilde{W}^{\text{eff}}(a;m,q,\epsilon) = \widetilde{W}^{\text{eff}}_{\text{clas}}(a;m,\epsilon)\log q + \widetilde{W}^{\text{eff}}_{1-\text{loop}}(a;m,\epsilon) + \widetilde{W}^{\text{eff}}_{\text{inst}}(a;m,\epsilon). \quad (6.1.6)$$

The first two terms — the "perturbative part" — consist of a classical contribution, proportional to $\log q$, and a 1-loop-term, which is independent of q. The instanton part is a (formal) series in powers of q of the form

$$\widetilde{W}^{\text{eff}}_{\text{inst}}(a;m,q,\epsilon) = \sum_{k=1}^{\infty} \widetilde{W}^{\text{eff}}_{k}(a;m,\epsilon)\, q^k. \quad (6.1.7)$$

For theories T with a known Lagrangian description, these terms have been computed explicitly, and the effective twisted superpotential $\widetilde{W}^{\text{eff}}(a;m,q,\epsilon)$ has a known expression.

Note that for the theories we are interested in, which are superconformal, the function $\widetilde{W}^{\text{eff}}(a;m,q,\epsilon)$ can simply be recovered from $\widetilde{W}^{\text{eff}}(a;m,q,1)$ by scaling the Coulomb and mass parameters with $\epsilon^{-\#}$, where # is their mass dimension. In the following we often leave out ϵ from the notation, knowing that we can simply reintroduce it by scaling the Coulomb and mass parameters.

Example. Let T_ϵ be the four-dimensional $\mathcal{N}=2$ superconformal "SU(2), $N_f = 4$" in the partial Ω-background with parameter ϵ.

The classical contribution to its effective twisted superpotential is simply

$$\widetilde{W}^{\text{eff}}_{\text{clas}}(a;m,\epsilon) = \frac{a^2}{4\epsilon}. \quad (6.1.8)$$

The 1-loop contribution $\exp \widetilde{W}^{\text{eff}}_{1-\text{loop}}$ may be computed as a product of determinants of differential operators. There is a certain freedom (though there are a number of natural choices) in its definition due to the regularization of divergences, which implies that it is only determined up to a phase [64, 65]. For a distinguished choice of phase $\exp \widetilde{W}^{\text{eff}}_{1-\text{loop}}$ may be identified with the square-root of the product of two Liouville three-point functions in the Nekrasov-Shatashvili (or $c \to \infty$) limit. In this "Liouville scheme" the 1-loop contribution is of the form

$$\widetilde{W}^{\text{eff}}_{1-\text{loop}}(a,m;\epsilon) = \widetilde{W}^{\text{eff}}_{\text{vector}}(a;\epsilon) + \widetilde{W}^{\text{eff}}_{\text{hyper}}(a;m,\epsilon) \quad (6.1.9)$$

with

$$\widetilde{W}^{\text{eff}}_{\text{vector}}(a;\epsilon) = -\frac{1}{2}\Upsilon(-a) - \frac{1}{2}\Upsilon(a) \tag{6.1.10}$$

$$\widetilde{W}^{\text{eff}}_{\text{hyper}}(a;m,\epsilon) = \frac{1}{2}\sum_{l=1}^{4}\Upsilon\left(\frac{\epsilon+a+m_l}{2}\right) + \frac{1}{2}\sum_{l=1}^{4}\Upsilon\left(\frac{\epsilon-a+m_l}{2}\right), \tag{6.1.11}$$

where

$$\Upsilon(x) = \int_{\frac{1}{2}}^{x} du \log\frac{\Gamma(u)}{\Gamma(1-u)}. \tag{6.1.12}$$

The instanton contributions may be written as a sum over Young tableaux [5, 6]. In particular, the 1-instanton contribution is given by

$$\widetilde{W}^{\text{eff}}_{1}(a;m,\epsilon) = \prod_{l=1}^{4}\frac{(a+m_l+\epsilon)}{16a(a+\epsilon)} + \prod_{l=1}^{4}\frac{(a-m_l-\epsilon)}{16a(a-\epsilon)}. \tag{6.1.13}$$

The effective twisted superpotential $\widetilde{W}^{\text{eff}}(a;m,q,\epsilon)$ turns out not only to characterize the low energy physics of the theory T_ϵ, but is part of a larger correpondence still under investigation. According to the philosophy of [18], it may also be identified with the Yang-Yang function governing the spectrum of a quantum integrable system. This quantum integrable system is the quantization of the classical algebraic integrable system describing the low energy effective theory of the four-dimensional $\mathcal{N}=2$ theory $\mathcal{T}$, with the deformation parameter ϵ playing the role of the complexified Planck constant. For theories of class $\mathcal{S}$ it is thus a quantization of a Hitchin system associated to C.

The corresponding moduli space of flat connections $\mathcal{M}_{\text{dR}}(C,\text{SL}_K)$ is holomorphic symplectic, and furthermore supports a distinguished complex Lagrangian subspace $\mathbf{L}$, the space of SL_K-*opers* on C [43]. The definition and some properties were given in 3.5, though we will be a little more concrete in this chapter since we will be interested in making some explicit computations. Recall that any SL_K oper can locally be written as a Kth order linear differential operator

$$\mathbf{D} = \partial_z^K + t_2(z)\,\partial_z^{K-2} + \ldots + t_K(z), \tag{6.1.14}$$

whose subleading ($(K-1)$th derivative) term vanishes. More precisely, we consider families of SL_K-valued ϵ-opers whose coefficients are dependent on the complex parameter ϵ. These may be obtained in the conformal limit $R,\zeta \to 0$, while $R/\zeta = \epsilon$ is kept fixed, of a certain family of flat connections coming from the non-abelian Hodge correspondence [66, 67].

Let us choose a Darboux coordinate system on $\mathcal{M}_{\mathrm{dR}}(C, \mathrm{SL}_K)$, say x_i, y_i with

$$\{x_i, y_j\} = \delta_{ij}. \tag{6.1.15}$$

Since the opers on C define a complex Lagrangian submanifold of $\mathcal{M}_{\mathrm{dR}}(C, \mathrm{SL}_K)$, we can guarantee they posses a *generating function* in this coordinate chart. This function W^{oper} is defined through the equation

$$y_i = \frac{\partial W^{\mathrm{oper}}(x, \epsilon)}{\partial x_i}. \tag{6.1.16}$$

and determined uniquely up to a constant in x.

6.1.2 Summary of results

We refine the methods to verify the NRS correspondence for $\mathrm{SU}(2)$ gauge theories and find the ingredients to extend the NRS correspondence to any superconformal theory of class $\mathcal{S}$. That is, using our generalization of the NRS Darboux coordinates, we describe the relevant spaces of opers, and compute the generating functions of these spaces of opers in some prototypical examples. Our main two examples are the superconformal $\mathrm{SU}(2)$ theory with four hypermultiplets and the superconformal $\mathrm{SU}(3)$ theory with six hypermultiplets. In these cases we calculate the generating function $W^{\mathrm{oper}}(a; m, q)$ as a perturbation in q and verify its agreement with the known superpotential $\widetilde{W}^{\mathrm{eff}}(a; m, q)$.

We find that the opers associated to a theory of class $\mathcal{S}$ with regular defects are described as certain Fuchsian differential operators with fixed semisimple conjugacy classes (with a certain "mass shift") at the punctures. Whereas for a surface C with only maximal punctures there are no further constraints, the space of opers on a surface C with other types of regular punctures is obtained by restricting the local exponents at the punctures, while keeping the conjugacy classes semisimple. This is analogous to the way that the space of differentials for a surface C with arbitrary regular punctures may be obtained from the space of differentials for the surface C with only maximal punctures, although the condition is different.

In particular, this implies that the locus of opers for the superconformal $\mathrm{SU}(2)$ theory coupled to four hypermultiplets is described by the family of Heun's opers, given by the differential equation (6.3.31), whereas the locus of opers for the superconformal $\mathrm{SU}(3)$ theory coupled to six hypermultiplets is described by the family of what we call "generalized Heun's opers", given by the differential equation (6.3.80). These families reduce in the limit $q \to 0$ to the hypergeometric and generalized hypergeometric oper, respectively, which we will see gives us a handle on explicitly describing their monodromy.

We describe how to calculate the monodromy representation explicitly for the family of (generalized) Heun's opers as a perturbation in the parameter q, and compute the result up to first order corrections in q. This is a generalization of the leading order computations of [68, 69], and a nontrivial extension of the work of [70] which computes the monodromy matrix around the punctures at $z = 0$ and $z = q$ in a perturbation series in q. The computation may be in principle generalized to any family of opers that depends on a small parameter.

We then calculate the generating function $W^{\text{oper}}(\ell; m, q)$ in the (generalized) NRS Darboux coordinates by comparing the monodromy representation for the opers to the monodromy representation in terms of the spectral coordinates. For $\mathrm{SU}(2)$, $N_f = 4$ theory we find

$$W^{\text{oper}}(a; m, q) = W^{\text{oper}}_{\text{clas}}(a; m) \log q + W^{\text{oper}}_{1-\text{loop}}(a; m) + W^{\text{oper}}_{1}(a; m)\, q + \mathcal{O}(q^2), \tag{6.1.17}$$

where the classical and the 1-loop contribution are computed in equation (6.5.16), and the 1-instanton contribution in equation (6.5.25). For the $\mathrm{SU}(3)$, $N_f = 6$ theory we find a similar expansion, where the classical and the 1-loop contribution are computed in equation (6.5.46).

We find that $W^{\text{oper}}_{1-\text{loop}}(a; m)$ in the $\mathrm{SU}(2)$ example equals the field theory expression (6.1.9). This computation is similar to and in agreement with the computation in [65]. Furthermore, we find that the 1-instanton correction $W^{\text{oper}}_{1}(a; m)$ is equal to (6.1.9), the four-dimensional 1-instanton correction in the Nekrasov-Shatashvili limit $\epsilon_2 \to 0$.

The interpretation of the generating function $W^{\text{oper}}(a, m, q)$ in the $\mathrm{SU}(3)$ example is similar. In particular, $\exp W^{\text{oper}}_{1-\text{loop}}(a; m)$ computes the square-root of the product of two Toda three-point functions with one semi-degenerate primary field in the Nekrasov-Shatashvili limit.

We conclude that our computation of the generating function of opers $W^{\text{oper}}(a; m, q)$, expressed in the generalized Nekrasov-Rosly-Shatashvili Darboux coordinates, indeed agrees with the known effective twisted superpotential $\widetilde{W}^{\text{eff}}(a; m, q)$ with the above "Liouville scheme" convention. Particularly interesting is that, while our computation of the generating function $W^{\text{oper}}(a; m, q)$ is a perturbation series in q, it is exact in ϵ.

Given an SL_2 ϵ-oper $\nabla^{\text{oper}}(\epsilon)$ there is another method to compute its monodromy representation, called the "exact WKB method" (see [71] for a good introduction). In the last section of the chapter we compare abelianization to the exact WKB method.

We argue that the monodromy representation for the oper $\nabla^{\text{oper}}(\epsilon)$ computed using the abelianization method is equal to its monodromy representation computed using the exact WKB method, when the spectral network is chosen to coincide with the Stokes graph, and with an appropriate choice of framing data. In this correspondence the so-called Voros symbols are identified with the spectral coordinates.

As a consequence it follows that the spectral coordinates X_γ, when evaluated on the ϵ-oper $\nabla^{\text{oper}}(\epsilon)$, have good WKB asymptotics in the limit $\epsilon \to 0$. In this limit X_γ is computed by what is sometimes called the quantum period $\Pi_\gamma(\epsilon)$ associated to $\nabla^{\text{oper}}(\epsilon)$. The asymptotic expansion in ϵ of the generating function $W^{\text{oper}}(\epsilon)$ may thus be simply found from the equation

$$\log \Pi_B = \frac{\partial W^{\text{oper}}(\Pi_A, \epsilon)}{\partial \log \Pi_A}. \tag{6.1.18}$$

This relates the Nekrasov-Rosly-Shatashvili correspondence to other approaches for computing the effective twisted superpotential [72].

We emphasize though that while the quantum periods are not particularly sensitive to the choice of Stokes graph, the exact resummed expressions are. In [73] it was found that there are "non-perturbative corrections" to the superpotential computed using quantum periods, but these corrections are ambiguous depending on the resummation process. In particular, the exact expression for the twisted effective superpotential $\widetilde{W}^{\text{eff}}(a, \epsilon)$ can only be found by applying the exact WKB method to the oper $\nabla^{\text{oper}}(\epsilon)$ where the phase of ϵ (and of other parameters) is chosen such that the corresponding Stokes graph is of Fenchel-Nielsen type. The results (6.1.17) then show that there are *no* non-perturbative corrections to $\widetilde{W}^{\text{eff}}(\epsilon)$, in agreement with [6].

This chapter is organized as follows.

We start in §6.2 by briefly recalling the geometric setup of §2.2.3 and writing down explicitly some families of differentials for various theories. In particular, we introduce our two main examples, the superconformal $SU(2)$ theory with $N_f = 4$ and the superconformal $SU(3)$ theory with $N_f = 6$.

Section 6.3 starts off with some explicit treatment of opers, in contrast with the more abstract treatment in §3.5, after which we introduce the relevant families of opers to our main examples. This is the family of Heun's opers for the superconformal SU(2) theory and the family of generalized Heun's opers for the superconformal SU(3) theory. We then define the moduli space of flat connections $\mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K)$ with fixed conjugacy classes $\mathcal{C}$ at the punctures associated to the theory of class S. In particular, we specify these conjugacy classes for the different kinds of punctures relevant to this chapter.

In §6.4 we detail the main calculation of the chapter, computing the monodromies of these opers in a perturbation series in the complex structure parameter q. While the computation of the instanton correction is based on the formal expansion of [68], perturbation theory for the lowest-order asymptotics can be justified by the results described in Appendix A (generalizing [69]), so that we have the lowest-order piece of a convergent expansion in q.

The final computations of the generating function of opers are contained in §6.5. Indeed, we find that in our two examples the generating function agrees with the effective twisted superpotential in an expansion in the parameter q, at the lowest order asymptotics for both SU(2) and SU(3) examples, and furthermore to 1-instanton level for SU(2).

In §6.6 we comment on the relation of the abelianization method with the exact WKB method and relate the NRS conjecture to other proposals for computing the effective twisted superpotential.

6.2 Class $\mathcal{S}$ geometry

Let us quickly review the geometry and notations described in §2.2.3. Fix a positive integer K and a tuple of class $\mathcal{S}$ data $(A_{K-1}, C, \mathcal{D})$. To each such choice corresponds a four-dimensional $\mathcal{N} = 2$ superconformal field theory $T = T_K[C, \mathcal{D}]$ of type A_{K-1} with regular defects $\mathcal{D}$.

Each puncture has an associated flavor symmetry attached to it. The flavour symmetry associated to a puncture with defect labelled by the Young diagram Y is

$$S[U(n_1) \times \cdots \times U(n_k)], \tag{6.2.1}$$

where $n_1, \ldots, n_k$ count columns of Y with the same height.

The Coulomb branch $\mathbf{B} = \mathbf{B}(T)$ of the theory T is equal to the corresponding Hitchin base, parametrized by tuples

$$(\varphi_2, \ldots, \varphi_K) \in \mathbf{B} = \bigoplus_{i=2}^{K} H^0(C, K_C(D)^{\otimes i}) \tag{6.2.2}$$

of k-differentials φ_k on C, with regular singularities of the appropriate pole structure at the punctures. Here $D = \sum_{i=1}^{n} 1 \cdot z_i$ denotes the divisor of punctures, and the residues are fixed at each puncture as specified by the data $\mathcal{D}$.

Thus, $\mathbf{B}$ is an affine space for the space of differentials with strictly lower order poles (possibly with restrictions as described in §2.2.3). Concretely, φ_k is given locally by

$$\varphi_k(z) = u_k(z) dz^{\otimes k} \tag{6.2.3}$$

where the function $u_k(z)$ has at most a pole of order k at each puncture.

To each tuple $(\varphi_2, \ldots, \varphi_K)$ we can associate the corresponding spectral curve $\Sigma \subset T^*C$, defined by the equation

$$\lambda^K + \lambda^{K-2}\varphi_2 + \ldots + \varphi_K = 0, \tag{6.2.4}$$

where λ is the tautological 1-form on T^*C, locally given by $\lambda = w dz$.

The spectral curve Σ is the Seiberg-Witten curve, and the restriction of λ to Σ is the Seiberg-Witten differential. The K residues of λ at each puncture z_l are fixed to be the mass parameters $m_{l,j}$.

In §2.2.3, we described some building blocks of class $\mathcal{S}$ theories, particularly for theories we will be interested in what follows. Our main examples in this chapter are the theories $T_K[C, \mathcal{D}]$ where C is the four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$, with $q \in \mathbb{C} \setminus \{0, 1\}$, and the rank is either $K = 2$ or $K = 3$. Let us recall these examples, and parameterize explicitly the points in their Coulomb branches.

6.2.1 $K = 2$

Recall when $K = 2$ there is only one possible regular defect, labeled by the Young diagram consisting of a single row with two boxes, and the mass parameters are generic, with $m_{l,2} = -m_{l,1}$. In the corresponding four-dimensional quantum field theory this defect corresponds to an SU(2) flavour symmetry group. Since there are no other possible diagrams, there is a single building block $T_2[\mathbb{P}^1_{0,1,\infty}]$.

Example. The theory $T_2[\mathbb{P}^1_{0,1,\infty}]$, describing a half-hypermultiplet in the trifundamental representation of $\mathrm{SU}(2)_0 \times \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_\infty$. Its Coulomb branch $\mathbf{B}$ is a single point corresponding to the quadratic differential

$$\varphi_2(z) = -\frac{m_\infty^2 z^2 - (m_\infty^2 + m_0^2 - m_1^2)z + m_0^2}{4z^2(z-1)^2}\, dz^2, \tag{6.2.5}$$

where the values of the mass parameters m_0, m_1 and m_∞ are fixed from the outset.

Recall that gauge fields are introduced by gluing three-punctured spheres, and the corresponding complex structure parameters q are identified with the gauge couplings $e^{2\pi i \tau}$. The limit $q \to 0$ corresponds to the weakly coupled description of the gauge theory at a cusp of the moduli space. For every pants cycle α there is a corresponding Coulomb parameter a_0, which is defined as the period integral $a_0 = \oint_A \lambda$ along a lift A of the pants cycle.

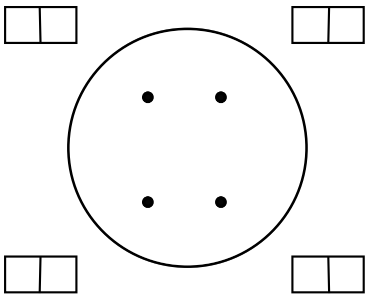


Figure 6.1: The UV curve for the theory $T_2[\mathbb{P}^1_{0,q,1,\infty}]$.

Example. The theory $T_2[\mathbb{P}^1_{0,q,1,\infty}]$ corresponds to the superconformal SU(2) gauge theory coupled to four hypermultiplets, see Figure 6.1. Its Coulomb branch $\mathbf{B}$ is the space of quadratic differentials with at most second order poles and fixed residues — it is 1-dimensional and parametrized by the family of quadratic differentials

$$\begin{aligned}\varphi_2(z) = -\Bigg(&\frac{m_0^2}{4z^2} + \frac{m^2}{4(z-q)^2} + \frac{m_1^2}{4(z-1)^2} \\ &+ \frac{m_0^2 + m^2 + m_1^2 - m_\infty^2}{4z(z-1)} - \frac{u}{z(z-q)(z-1)}\Bigg)(dz)^2,\end{aligned} \tag{6.2.6}$$

where the parameter u is free and the parameters m_0, m, m_1 and m_∞ are fixed from the outset.

The corresponding Seiberg-Witten curve Σ is (after compactifying) a genus one covering of $\mathbb{P}^1_{0,q,1,\infty}$ with four simple branch points. Let A be the lift of the 1-cycle α going counterclockwise around the punctures at $z = 0$ and $z = q$; then the Coulomb parameter $a_0 = a_0(u)$ is defined as the period integral $a_0 = \oint_A \lambda$.

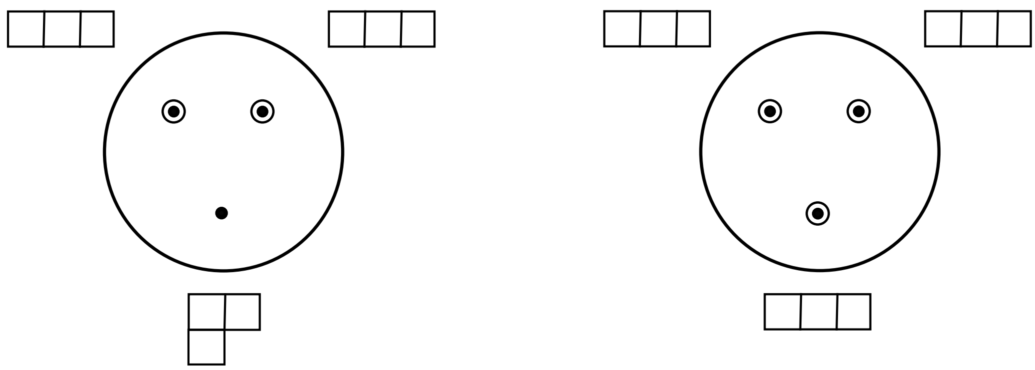


Figure 6.2: Left: the UV curve for the free bifundamental hypermultiplet $T_3[\mathbb{P}^1_{0,1,\infty}]$. Right: the UV curve for the non-Lagrangian E_6 theory $T_3[\mathbb{P}^1_{0,1,\infty}]$.

6.2.2 $K = 3$

Recall that for $K = 3$ there are two types of punctures, of "maximal" and "minimal". For a maximal puncture z_l the mass parameters $m_{l,i}$ are generic with $m_{l,1} \neq m_{l,2}$, whereas for a minimal puncture $m_{l,1} = m_{l,2}$.

As described in §2.2.3, a maximal puncture at $z = z_l$ turns into a minimal puncture if it satisfies two requirements:

(i) Two of the masses at the puncture should coincide:

$$m_{l,1} = m_{l,2} = m_l. \tag{6.2.7}$$

(ii) The discriminant of

$$\lambda^3 + (z - z_l)^2 \varphi_2 \, \lambda + (z - z_l)^3 \varphi_3 \tag{6.2.8}$$

should vanish up to order $(z - z_l)^2$.

We can use this condition to write down explicitly the differentials in the Coulomb branch in examples.

Example. The theory $T_3[\mathbb{P}^1_{0,1,\infty}]$ with two maximal and one minimal puncture, see on the left of Figure 6.2, corresponds to a free hypermultiplet in the bifundamental representation of $\mathrm{SU}(3)_0 \times \mathrm{SU}(3)_\infty$. We find its Coulomb vacua by applying the constraints (6.2.7) and (6.2.8) to the family of differentials for $T_3[\mathbb{P}^1_{0,1,\infty}]$ described in equation (2.2.14) and (2.2.15) at $z = 1$. Concretely, the latter constraint implies that

$$u = \left(\frac{m_1}{2}\right)^3 - d_0 - 2d_\infty + \frac{m_1}{2}(c_0 - c_\infty). \tag{6.2.9}$$

where d_i, c_i are symmetric polynomials in the mass parameters given in §2.2.3. Thus the Coulomb branch is a single point $\varphi^{\text{bif}} = (\varphi_2^{\text{bif}}, \varphi_3^{\text{bif}})$.

The resulting Seiberg-Witten curve Σ determines a 3-fold ramified covering of the UV curve $\mathbb{P}^1_{0,\underline{1},\infty}$ with four simple branch-points. It is therefore a punctured genus zero surface.

This is the $K = 3$ building block in the case that we will primarily be interested in (the E_6 theory is another possible choice, but we do not pursue it here). Once again, gauge fields are introduced by gluing three-punctured spheres at (maximal) punctures, so that this covers all cases in which we have n punctures, with $n-2$ of minimal type. The gauge coupling associated to the ith pants curve corresponds to the complex structure parameter q_i, where the gluing is performed according to the plumbing construction. The fundamental example we will be interested in is:

Example. The theory $T_3[\mathbb{P}^1_{0,\underline{q},\underline{1},\infty}]$ is the superconformal SU(3) gauge theory coupled to $N_f = 6$ hypermultiplets. It may be obtained by gluing two three-punctured spheres with two maximal and one minimal puncture. Its Coulomb branch $\mathbf{B}$ is parametrized by two parameters u_1 and u_2.

The explicit form of the differentials φ_2 and φ_3 can be obtained as before. First we write down the most general quadratic and cubic differential with regular poles at the punctures. Eight of the twelve parameters are fixed by writing the residues at each punctures in terms of the mass parameters. Two more parameters are fixed by additional requirements at both minimal punctures, analogous to equation (2.2.12) and (2.2.18). The resulting differentials can be written in the form

$$\varphi_2 = \frac{c_0}{z^2} + \frac{c}{(z-q)^2} + \frac{c_1}{(z-1)^2} + \frac{c_\infty - c_0 - c - c_1}{z(z-1)} + \frac{u_1}{z(z-q)(z-1)} \tag{6.2.10}$$

$$\varphi_3 = \frac{d_0}{z^3} + \frac{d}{(z-q)^3} + \frac{d_1}{(z-1)^3} + \frac{d_\infty - d_0 - d - d_1}{z(z-q)(z-1)} + \tag{6.2.11}$$

$$+ \frac{(1-q)(4c_0 - 3m^2 - 3m_1^2 - 4c_\infty)m_1}{8z(z-1)^2(z-q)} + \frac{u_2}{z^2(z-q)(z-1)} \tag{6.2.12}$$

$$- \frac{u_1}{z(z-1)^2(z-q)^2}\left(\frac{m_1}{2}(z-q) + \frac{m}{2}(z-1)\right) \tag{6.2.13}$$

where c, d are the symmetric polynomials in the mass parameters as in §2.2.3.

The resulting Seiberg-Witten curve Σ is a genus two (after compactifying) covering of $\mathbb{P}^1_{0,\underline{q},\underline{1},\infty}$ with eight simple branch points. The two Coulomb parameters a_0^1 and a_0^2 are defined as the period integrals $a_0^1 = \oint_{A^{(1)}} \lambda$ and $a_0^2 = \oint_{A^{(2)}} \lambda$ along two independent lifts of the pants cycle α to the Seiberg-Witten curve.

6.3 Opers from slass $\mathcal{S}$

The moduli space $\mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K)$ of flat SL_K connections has a distinguished complex Lagrangian submanifold

$$\mathbf{L} \subset \mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C, \mathrm{SL}_K) \tag{6.3.1}$$

of "SL_K-opers", known to physicists as the "brane of opers". These objects were first formalized in [42], and play an important role in the geometric Langlands program [43, 74]. They also appeared in a conjecture of Gaiotto [66] as the "conformal limit" of a certain canonical family of flat connections in the moduli space, which was recently proved in [67].

An SL_K oper is essentially a special kind of SL_K flat connection, which can locally be written in the form of a (single, scalar) differential equation

$$\mathbf{D}\, y(z) = y^{(K)}(z) + \sum_{i=2}^{K} t_k(z) y^{(K-i)}(z) = 0, \tag{6.3.2}$$

where globally $y(z)$ is not a function on C, but rather the local expression for a $(-\frac{K-1}{2})$-differential. The latter ensures that the differential equation is globally well-defined after specifying the transformation laws for the coefficients. So $\mathbf{D}$ is really a differential operator between line bundles

$$\mathbf{D} : K_C^{(1-K)/2} \to K_C^{(K+1)/2} \otimes \mathcal{O}(K \cdot D), \tag{6.3.3}$$

which in our SL_K case must have vanishing subprincipal symbol ($K-1$th order term in the local form). This definition is equivalent to the one given in other parts of thesis, and the most convenient for our purposes.

With the assumption we are acting on $K_C^{(1-K)/2}$, imposing that the differential equation (6.3.2) stays invariant under holomorphic coordinate transformations yields the transformation properties of the coefficients $t_k(z)$. As we spell out in detail below,

$$\frac{12}{K(K^2-1)}\, t_2 \tag{6.3.4}$$

transforms as a projective connection, whereas there exist linear combinations w_k of t_j ($j \leq K$) and its derivatives, such that the w_k transform as k-differentials. The SL_K flat connection is obtained from the oper equation (6.3.2) by writing it instead as a linear rank K differential equation.

We begin by characterizing the space of opers **L** associated to any type A theory $T_K[C,\mathcal{D}]$ of class $\mathcal{S}$ with regular defects. The fact that the defects are regular suggests that we should take the associated opers to be Fuchsian (i.e. have regular singularities at the punctures of C). Our recipe for obtaining **L** is similar to the recipe for obtaining the space of differentials associated to $T_K[C,\mathcal{D}]$, as explained in §2.2.3. That is, we first describe the space of opers on C with only maximal punctures, and then impose appropriate restrictions at the punctures to obtain the space of opers on C with minimal punctures (or more generally, any regular punctures) associated to $T_K[C,\mathcal{D}]$.

An important point is to note that the conjugacy classes are not simply the exponentiated mass parameters, but acquire a slight shift: the relevant space of opers associated to the theory $T_K[C,\mathcal{D}]$ sits inside $\mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C,\mathrm{SL}_K)$, where the $\mathcal{C}_l$ are such that

$$M_{l,i} = e^{2\pi i (m_{l,i} + \frac{K-1}{2})}. \tag{6.3.5}$$

This corresponds to the most symmetric choice of local exponents for an SL_K oper, and is necessary to ensure the desired equality between the generating function and the superpotential.[1] We will sometimes call these the ***shifted masses***

$$\mu_{l,i} := m_{l,i} + \tfrac{K-1}{2}\epsilon, \tag{6.3.6}$$

where we have reinserted ϵ. Let us stress, then:

Henceforth in this chapter, whenever $\mathcal{M}^{\mathcal{C}}_{\mathrm{dR}}(C,\mathrm{SL}_K)$ *is written, we are taking the* $\mathcal{C}_l$ *at punctures to be* $\mathcal{C}_l = [\mathrm{diag}(e^{2\pi i \mu_{l,1}},\ldots,e^{2\pi i \mu_{l,K}})]$*, having started with class* $\mathcal{S}$ *data* $(A_{K-1},C,\mathcal{D})$ *containing masses* $m_{l,i}$*.*

In particular, our characterization leads to a concrete description of the space of opers on any three-punctured sphere with regular punctures. These spaces may be seen as the building blocks for the space of opers. For instance, we find that opers on the three-punctured sphere with two maximal and one minimal puncture are characterized by the (generalized) hypergeometric equation. Furthermore, we find that the space of SL_K opers on the four-punctured surface $\mathbb{P}^1_{0,\underline{q},\underline{1},\infty}$ are characterized by the (generalized) Heun equation.

Although in the following we will only spell out the details for $K = 2$ and $K = 3$, it should be straightforward how to generalize the discussion to find the

[1] A related mass shift has been observed in the context of the AGT correspondence, see for instance [75].

space of SL_K opers associated to any building block, and more generally any theory $T_K[C, \mathcal{D}]$ of class $\mathcal{S}$ with regular punctures.

The locus of opers $\mathbf{L}$ may be interpreted as a quantization of the Coulomb moduli space $\mathbf{B}$ (or equivelantly, the spectral curves sitting above)in the sense of associating to each point a differential operator (see e.g. [76] for details). Indeed, the internal Coulomb parameters u as well as external mass parameters m carry mass dimensions. It is natural to introduce an additional parameter ϵ with mass dimension one such that all terms in the oper equation have the same mass dimension. In the semi-classical limit $\epsilon \to 0$ the family of opers then limits to the family of spectral curves over Coulomb moduli space $\mathbf{B}$.

6.3.1 SL_2 opers

An SL_2 oper is locally described by a scalar differential equation of the form

$$\mathbf{D}y = y''(z) + t_2(z)y(z) = 0, \tag{6.3.7}$$

where $y(z)$ is a $(-\frac{1}{2})$-differential on C. It is an instructive exercise to determine the transformation properties of the coefficient $t_2(z)$ directly, so we will consider what happens to the differential equation (6.3.7) under a holomorphic change of coordinates.

Under the holomorphic coordinate change $z \mapsto z(w)$ the $(-\frac{1}{2})$-differential $y(z)$ transforms into

$$\widetilde{y}(w) = y(z(w)) \left(\frac{dz}{dw} \right)^{-\frac{1}{2}}. \tag{6.3.8}$$

This implies that

$$\widetilde{y}''(w) = (z'(w))^{\frac{3}{2}} \left(y''(z(w)) - \frac{1}{2} \{w, z\}\, y(z(w)) \right). \tag{6.3.9}$$

where the brackets $\{\cdot, \cdot\}$ denote the Schwarzian derivative

$$\{w, z\} = \frac{w'''(z)}{w'(z)} - \frac{3}{2} \left(\frac{w''(z)}{w'(z)} \right)^2 = -\{z, w\}/z'(w)^2 \tag{6.3.10}$$

Under a holomorphic coordinate change $z \mapsto z(w)$ the differential equation (6.3.7) thus transforms into

$$\begin{aligned} 0 &= \widetilde{y}''(w) + \widetilde{t}_2(w)\, \widetilde{y}(w) \\ &= (z'(w))^{\frac{3}{2}} \left(y''(z(w)) + \frac{1}{2} \{w, z\}\, y(z(w)) + (z'(w))^{-2}\, \widetilde{t}_2(w)\, y(z(w)) \right), \end{aligned} \tag{6.3.11}$$

where we have not specified yet how the coefficient $t_2(z)$ transforms.

Now, demanding the differential equation (6.3.7) be invariant under the holomorphic coordinate change $z \mapsto z(w)$, we find that

$$\begin{aligned} \widetilde{t}_2(w) &= (z'(w))^2 \left(t_2(z(w)) - \frac{1}{2}\{w, z\} \right) \\ &= (z'(w))^2\, t_2(z(w)) + \frac{1}{2}\{z, w\} \end{aligned} \tag{6.3.12}$$

In other words, the coefficient t_2 should transform as a so-called projective connection on C.

Observe that the transformation properties (6.3.12) of the coefficient t_2 show that the difference between any two SL_2 opers is a quadratic differential on C. Thus the space of SL_2 opers $\mathbf{L}$ is an affine space modelled on the quadratic differentials.

SL_2 flat connection

The differential equation

$$\mathbf{D}y = y''(z) + t_2(z)y(z) = 0, \tag{6.3.13}$$

can be put in the form of an SL_2 flat connection

$$\nabla^{\mathrm{oper}} Y = dY + AY = \frac{dY(z)}{dz}dz + A_z\, dz\, Y(z) = 0 \tag{6.3.14}$$

where

$$Y(z) = \begin{pmatrix} -y'(z) \\ y(z) \end{pmatrix} \quad \text{and} \quad A_z = \begin{pmatrix} 0 & -t_2(z) \\ 1 & 0 \end{pmatrix}. \tag{6.3.15}$$

While under a change of variables $z \to z(w)$ we have that

$$\widetilde{y}(w) = y(z(w)) \left(\frac{dz}{dw} \right)^{-\frac{1}{2}} =: y(z(w))s(w), \tag{6.3.16}$$

the section Y transforms as

$$\widetilde{Y}(w) = \begin{pmatrix} s(w)^{-1} & -s'(w) \\ 0 & s(w) \end{pmatrix} Y(z(w)) =: U^{-1}(w)Y(z(w)), \tag{6.3.17}$$

and hence obeys

$$d\widetilde{Y} + \widetilde{A}\widetilde{Y} = \frac{d\widetilde{Y}(w)}{dw}dw + \widetilde{A}_w\, dw\, \widetilde{Y}(w) = 0, \tag{6.3.18}$$

with

$$\widetilde{A} = U^{-1}dU + U^{-1}AU = \widetilde{A}_w\, dw. \tag{6.3.19}$$

Since the new connection form is

$$\widetilde{A}_w = \begin{pmatrix} 0 & -\frac{t_2(z(w))}{s(w)^4} + \frac{s''(w)}{s(w)} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\widetilde{t}_2(w) \\ 1 & 0 \end{pmatrix}, \tag{6.3.20}$$

we find that the SL_2 oper $\mathbf{D}$ defined locally by equation (6.3.13) is equivalent to the SL_2 flat connection ∇^{oper} defined locally by equation (6.3.14).

More invariantly, the transformation property (6.3.17) says that Y transforms as a 1-jet, and says that we have converted the oper $\mathbf{D}$ into the flat connection ∇^{oper} on the rank 2 vector bundle $E = J^1(\mathcal{L})$ of 1-jets of sections of $\mathcal{L} = K_C^{-\frac{1}{2}}$.

Fuchsian SL_2 opers

The space $\mathbf{L}$ of SL_2 opers on any surface C with regular punctures consists of Fuchsian SL_2 opers on C, which are locally defined by a Fuchsian differential equation of order 2. We require that the local exponents of these SL_2 opers at each puncture z_l are given by[2]

$$\frac{1}{2} \pm \frac{m_l}{2}, \tag{6.3.21}$$

in terms of the mass parameters m_l. This implies that the Fuchsian SL_2 opers are SL_2 flat connections with a fixed semisimple conjugacy classes

$$\mathcal{C}_l = \text{diag}\left(-e^{-\pi i m_l}, -e^{\pi i m_l}\right). \tag{6.3.22}$$

at each puncture z_l. For reference, recall that we fixed the residues of the differentials $\sqrt{\varphi_2}$ in $\mathbf{B}$ at each puncture z_l to be $\pm\frac{m_l}{2}$.

Example. Locus of opers for $T_2[\mathbb{P}^1_{0,1,\infty}]$.

Recall that for a fixed choice of residues $\pm\frac{m_l}{2}$, the three-punctured sphere $\mathbb{P}^1_{0,1,\infty}$ admits the unique quadratic differential

$$\varphi_2(z) = -\frac{m_0^2}{4z^2} - \frac{m_1^2}{4(z-1)^2} - \frac{m_\infty^2 - m_0^2 - m_1^2}{4z(z-1)} \tag{6.3.23}$$

[2]Fuch's theorem necessitates that the exponents of an SL_2 oper add up to 1 at each puncture, hence (6.3.21) is the most symmetric choice.

with at most second-order poles at all punctures. The corresponding SL_2 oper is given by

$$\mathbf{D}\, y(z) = y''(z) + t_2(z)y(z) = 0, \tag{6.3.24}$$

with

$$t_2(z) = \frac{\delta_0}{z^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_0 - \delta_1}{z(z-1)}, \tag{6.3.25}$$

and

$$\delta_l = \frac{1-m_l^2}{4}. \tag{6.3.26}$$

This is equivalent (after a simple and standard transformation) to the classical Gauss' hypergeometric differential equation. Note that the local exponents of the SL_2 oper (6.3.24) are indeed given by $\frac{1}{2} \pm \frac{m_l}{2}$, and that $\epsilon^2\, t_2(z)$ reduces to $\varphi_2(z)$ in the semi-classical limit $\epsilon \to 0$ discussed in §6.3.3.

The hypergeometric oper (6.3.24) with vanishing masses $m_l = 0$ corresponds to a distinguished projective structure, namely the one induced by the Fuchsian uniformization of $\mathbb{P}^1_{0,1,\infty}$. Indeed, the three-punctured sphere $\mathbb{P}^1_{0,1,\infty}$ is uniformized by the modular lambda function

$$\begin{aligned} \lambda : \mathbb{H} &\to \mathbb{P}^1_{0,1,\infty} \\ w &\mapsto z = \lambda(w), \end{aligned} \tag{6.3.27}$$

invariant under the discrete group $\Gamma(2) \subset PSL_2(\mathbb{R})$. The uniformization oper $\mathbf{D}_{\text{unif}}$ on the three-punctured sphere is thus represented by the differential operator

$$\begin{aligned} \mathbf{D}_{\text{unif}} = \partial_w^2 &= \partial_z^2 + \frac{1}{2}\{w,z\} = \partial_z^2 - \frac{\{z,w\}}{2z'(w)^2} \\ &= \partial_z^2 + \frac{1-z+z^2}{4z^2(z-1)^2} = \partial_z^2 + \frac{1}{4z^2} + \frac{1}{4(z-1)^2} - \frac{1}{4z(z-1)}, \end{aligned} \tag{6.3.28}$$

where in the first line we have used the transformation law for projective connections. Both its local exponents are equal to $\frac{1}{2}$.

Note that the hypergeometric oper (6.3.24) itself is of the form

$$\mathbf{D}_{\text{unif}} + \varphi_2. \tag{6.3.29}$$

Example. Locus of opers for $T_2[\mathbb{P}^1_{0,q,1,\infty}]$.

The four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$ admits the 1-dimensional space of quadratic differentials

$$\varphi_2(z) = -\frac{m_0^2}{4z^2} - \frac{m^2}{4(z-q)^2} - \frac{m_1^2}{4(z-1)^2} - \frac{m_\infty^2 - m_0^2 - m^2 - m_1^2}{4z(z-1)} + \frac{u}{z(z-q)(z-1)} \tag{6.3.30}$$

with regular singularities at all punctures.

The corresponding 1-dimensional family of SL_2 opers are defined by the differential equation

$$\mathbf{D}\, y(z) = y''(z) + t_2(z) y(z) = 0, \tag{6.3.31}$$

with

$$t_2(z) = \frac{\delta_0}{z^2} + \frac{\delta}{(z-q)^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_0 - \delta - \delta_1}{z(z-1)} + \frac{H}{z(z-q)(z-1)}, \tag{6.3.32}$$

where H is a free complex parameter, the so-called accessory parameter, and

$$\delta_l = \frac{1 - m_l^2}{4}. \tag{6.3.33}$$

The differential equation (6.3.31) is known as Heun's differential equation. It is the most general Fuchsian equation of order 2 with four singularities.

As before we may write the Heun's opers in the form

$$\mathbf{D} = \mathbf{D}_0 + \varphi_2 \tag{6.3.34}$$

with respect to the base oper

$$\mathbf{D}_0 = \partial_z^2 + \frac{1}{4z^2} + \frac{1}{4(z-q)^2} + \frac{1}{4(z-1)^2} - \frac{1}{2z(z-1)} + \frac{\text{const}}{z(z-q)(z-1)}, \tag{6.3.35}$$

but unlike before we are not forced to fix the arbitrary constant.

In the limit $q \to 0$ the four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$ can be thought of as degenerating into two three-punctured spheres. In the same limit, the family of Heun's opers degenerates into a pair of hypergeometric opers.

More precisely, if we define ℓ through

$$H = \delta_\ell - \delta_0 - \delta + \mathcal{O}(q), \tag{6.3.36}$$

with $\delta_l = \frac{1-\ell^2}{4}$ in the limit $q \to 0$, the family of Heun's opers (6.3.31) has two interesting limits:

1. In the limit $q \to 0$ the family reduces to the hypergeometric oper (6.3.24) with parameters (ℓ, m_1, m_∞).
2. If we first map $z \mapsto qt$ and then take the limit $q \to 0$, the family reduces to the hypergeometric oper (6.3.24) with parameters (m_0, m, ℓ).

The definition of ℓ through equation (6.3.36) will be justified in §6.4, where it will be the eigenvalue of the monodromy around the pants curve enclosing 0 and q.

Using the AGT correspondence, the effective twisted superpotential for the superconformal SU(2) theory can be found as a series expansion of the accessory parameter H in q, through a generalized Matone relation [77, 78, 79, 80]. This is however *not* the route that we take here. Instead, we aim to find the effective twisted superpotential directly from the oper monodromies. (Our strategy might be useful though for establishing similar generalized Matone relations beyond SU(2) theories.)

6.3.2 SL_3 opers

An SL_3 oper is locally described by a differential equation of the form

$$\mathbf{D}y = y'''(z) + t_2(z)y'(z) + t_3(z)y(z) = 0, \tag{6.3.37}$$

where $y(z)$ is now the local expression for a section of K_C^{-1}, i.e. a (-1)-differential on C. Again, we work out the transformation properties of the coefficients $t_2(z)$ and $t_3(z)$, so let us consider again what happens to the differential equation (6.3.37) under a holomorphic change of coordinates.

Under a holomorphic coordinate change $z \mapsto z(w)$ the (-1)-differential $y(z)$ transforms as

$$\tilde{y}(w) = y(z(w)) \left(\frac{dz}{dw}\right)^{-1} =: s(w)\, y(z(w)). \tag{6.3.38}$$

This implies that

$$\tilde{y}'(w) = y'(z(w)) + s'(w)\, y(z(w)). \tag{6.3.39}$$

and

$$\tilde{y}'''(w) = \frac{y'''(z(w))}{s(w)^2} + \left(\frac{2s''(w)}{s(w)} - \frac{s'(w)^2}{s(w)^2}\right) y'(z(w)) + s'''(w)y(z(w)). \tag{6.3.40}$$

Under the holomorphic coordinate change $z \mapsto z(w)$ the differential equation (6.3.37) thus transforms into

$$0 = \widetilde{y}'''(w) + \widetilde{t}_2(w)\,\widetilde{y}'(w) + \widetilde{t}_3(w)\,\widetilde{y}(w) \tag{6.3.41}$$

$$= \frac{1}{s(w)^2}\Big(y'''(z(w)) + \Big(s(w)^2\,\widetilde{t}_2(w) + 2s(w)s''(w) - s'(w)^2\Big)\,y'(z(w)) + \tag{6.3.42}$$

$$+ \Big(s(w)^3\,\widetilde{t}_3(w) + s(w)^2 s'(w)\,\widetilde{t}_2(w) + s(w)^2 s'''(w)\Big)\,y(z(w))\Big), \tag{6.3.43}$$

where we have not specified yet how the coefficients $t_2(z)$ and $t_3(z)$ transform.

Now, since the differential equation (6.3.37) must be invariant under the holomorphic coordinate change $z \mapsto z(w)$, we find that

$$\widetilde{t}_2(w) = (s(w))^{-2}\left(t_2(z(w)) - 2s(w)s''(w) + s'(w)^2\right) \tag{6.3.44}$$

$$= (z'(w))^2\,t_2(z(w)) + 2\{z, w\}. \tag{6.3.45}$$

and

$$\widetilde{t}_3(w) = \frac{t_3(z(w))}{s(w)^3} - \frac{s'(w)}{s(w)}\,\widetilde{t}_2(w) - \frac{s'''(w)}{s(w)}. \tag{6.3.46}$$

Equation (6.3.44) says that the coefficient $t_2(z)/4$ transforms as a projective connection.

To find how $t_3(z)$ transforms, we read off from equation (6.3.44) that

$$\frac{1}{2}\partial_w \widetilde{t}_2(w) - \frac{1}{2}\frac{\partial_z t_2(z(w))}{s(w)^3} = -\frac{s'(w)}{s(w)}\,\widetilde{t}_2(w) - \frac{s'''(w)}{s(w)}. \tag{6.3.47}$$

Substituting this into equation (6.3.46) yields

$$\widetilde{t}_3(w) - \frac{1}{2}\partial_w \widetilde{t}_2(w) = (z'(w))^3\left(t_3(z(w)) - \frac{1}{2}\partial_z t_2(z(w))\right). \tag{6.3.48}$$

In other words, the combination $t_3(z) - \frac{1}{2}t_2'(z)$ transforms as a 3-differential.

SL_3 flat connection

Note that the differential equation

$$\mathbf{D}y = y'''(z) + t_2(z)y'(z) + t_3(z)y(z) = 0, \tag{6.3.49}$$

can be put in the form of an SL_3 flat connection

$$\nabla^{\mathrm{oper}} Y = dY + AY = \frac{dY(z)}{dz}dz + A_z\,dz\,Y(z) = 0 \tag{6.3.50}$$

where

$$Y(z) = \begin{pmatrix} y''(z) \\ -y'(z) \\ y(z) \end{pmatrix} \quad \text{and} \quad A_z = \begin{pmatrix} 0 & -t_2(z) & t_3(z) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6.3.51}$$

While under a change of variables $z \to z(w)$ we have that

$$\widetilde{y}(w) = y(z(w)) \left(\frac{dz}{dw}\right)^{-1} =: y(z(w))s(w), \tag{6.3.52}$$

the section Y transforms as

$$\widetilde{Y}(w) = \begin{pmatrix} s(w)^{-1} & -\frac{s'(w)}{s(w)} & s''(w) \\ 0 & 1 & -s'(w) \\ 0 & 0 & s(w) \end{pmatrix} Y(z(w)) =: U^{-1}(w)Y(z(w)), \tag{6.3.53}$$

and hence obeys

$$d\widetilde{Y} + \widetilde{A}\widetilde{Y} = \frac{d\widetilde{Y}(w)}{dw}dw + \widetilde{A}_w\, dw\, \widetilde{Y}(w) = 0, \tag{6.3.54}$$

with

$$\widetilde{A} = U^{-1}dU + U^{-1}AU = \widetilde{A}_w\, dw. \tag{6.3.55}$$

Since the new connection form is

$$\widetilde{A}_w = \begin{pmatrix} 0 & -\widetilde{t}_2(w) & \widetilde{t}_3(w) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{6.3.56}$$

with

$$\widetilde{t}_2(w) = \frac{t_2(z(w))}{s(w)^2} + \frac{s'(w)^2}{s(w)^2} - \frac{2s''(w)}{s(w)} \tag{6.3.57}$$

$$\widetilde{t}_3(w) = \frac{t_3(z(w))}{s(w)^3} - \frac{s'(w)t_2(z(w))}{s(w)^3} - \frac{s'(w)^3}{s(w)^3} + \frac{2s'(w)s''(w)}{s(w)^2} - \frac{s'''(w)}{s(w)} \tag{6.3.58}$$

in agreement with equations (6.3.44) and (6.3.48), we find that the SL_3 oper defined locally by equation (6.3.49) is equivalent to the SL_3 flat connection defined locally equation (6.3.50).

More invariantly, the transformation property (6.3.53) says that Y transforms as a 2-jet, and says that we have converted the oper $\mathbf{D}$ into the flat connection ∇^{oper} on the rank 3 vector bundle $\mathcal{E} = J^2(\mathcal{L})$ of 2-jets of sections of $\mathcal{L} = K_C^{-1}$.

Fuchsian SL_3 opers

The space $\mathbf{L}$ of SL_3 opers on a surface C with regular punctures consists of all Fuchsian SL_3 opers on C, which are locally given by a Fuchsian differential equation of order 3, with various possible restrictions at each puncture depending on the type of the puncture.

Maximal punctures

Suppose that the surface C has only maximal punctures. The corresponding space of SL_3 opers on C consists of all Fuchsian SL_3 opers on C, where we require that the local exponents at each such puncture z_l are given by[3]

$$1+\frac{m_{l,1}}{2},\ 1+\frac{m_{l,2}}{2},\ 1-\frac{m_{l,1}}{2}-\frac{m_{l,2}}{2}, \tag{6.3.59}$$

in terms of the mass parameters $m_{l,1}$ and $m_{l,2}$. This implies that the resulting SL_3 opers have fixed semisimple conjugacy class

$$\mathcal{C}_l = \mathrm{diag}\left(e^{\pi i m_{l,1}}, e^{\pi i m_{l,2}}, e^{-\pi i (m_{l,1}+m_{l,2})}\right), \tag{6.3.60}$$

at each maximal puncture z_l.

Example. Locus of opers for $T_3[\mathbb{P}^1_{0,1,\infty}]$.

As summarized in equations (2.2.14) and (2.2.15), the three-punctured sphere $\mathbb{P}^1_{0,1,\infty}$ with three maximal punctures admits a 1-dimensional family of differentials

$$\varphi_2 = \frac{c_\infty z^2 - (c_0 - c_1 + c_\infty)z + c_0}{z^2(z-1)^2}(dz)^2 \tag{6.3.61}$$

$$\varphi_3 = \frac{d_\infty z^3 + u z^2 + (d_0 + d_1 - d_\infty - u)z - d_0}{z^3(z-1)^3}(dz)^3. \tag{6.3.62}$$

The 1-dimensional family of Fuchsian SL_3 opers on $\mathbb{P}^1_{0,1,\infty}$ with three maximal punctures may be parametrized as

$$y'''(z) + t_2(z)y'(z) + t_3(z)y(z), \tag{6.3.63}$$

with coefficients

$$t_2 = \frac{(1+c_\infty)z^2 - (1+c_0-c_1+c_\infty)z + (1+c_0)}{z^2(z-1)^2} \tag{6.3.64}$$

$$t_3 = \frac{d_\infty z^3 + u z^2 + (d_0 + d_1 - d_\infty - u)z - d_0}{z^3(z-1)^3} + \frac{1}{2}t_2'. \tag{6.3.65}$$

[3]It is a simple exercise to check that the exponents of an SL_3 oper add up to 3 at each puncture, hence (6.3.59) is the most symmetric choice.

Its local exponents are given by

$$\begin{aligned} z=0: &\quad 1+\frac{m_{0,1}}{2}, 1+\frac{m_{0,2}}{2}, 1-\frac{m_{0,1}}{2}-\frac{m_{0,2}}{2} \\ z=1: &\quad 1+\frac{m_{1,1}}{2}, 1+\frac{m_{1,2}}{2}, 1-\frac{m_{1,1}}{2}-\frac{m_{1,2}}{2} \\ z=\infty: &\quad 1+\frac{m_{\infty,1}}{2}, 1+\frac{m_{\infty,2}}{2}, 1-\frac{m_{\infty,1}}{2}-\frac{m_{\infty,2}}{2}. \end{aligned} \tag{6.3.66}$$

The Fuchsian SL_3 oper (6.3.63) may be written in the form

$$y'''(z)+\left(4t_2^{\text{unif}}+\varphi_2(z)\right)y'(z)+\left(2\partial_z t_2^{\text{unif}}(z)+\frac{1}{2}\partial_z\varphi_2(z)+\varphi_3\right)y(z), \tag{6.3.67}$$

where

$$t_2^{\text{unif}}(z)=\frac{1-z+z^2}{z^2(z-1)^2} \tag{6.3.68}$$

is the coefficient of the Fuchsian uniformization oper (6.3.29). The Fuchsian SL_3 oper (6.3.63) thus again has an interpretation in terms of Fuchsian uniformization. In fact, in the limit $m_{l,i}\to 0$ it is equal to a lift of the Fuchsian uniformatization oper.

This is most easily seen by rewriting the SL_3 flat connection defined locally by equation (6.3.50) in the form[4]

$$\frac{d\widetilde{Y}(z)}{dz}dz+\begin{pmatrix} 0 & -\frac{1}{2}t_2(z) & t_3(z)-\frac{1}{2}\partial_z t_2(z) \\ 1 & 0 & -\frac{1}{2}t_2(z) \\ 0 & 1 & 0 \end{pmatrix} dz\,\widetilde{Y}(z)=0. \tag{6.3.69}$$

Indeed, in this form it is clear that the SL_3 oper defined locally by

$$y'''(z)+t_2(z)\,y'(z)+\frac{1}{2}\partial_z t_2(z)\,y(z)=0, \tag{6.3.70}$$

is the lift of the SL_2 oper defined locally by

$$y'(z)+\frac{1}{4}t_2(z)\,y(z)=0, \tag{6.3.71}$$

using the homomorphism $\rho: sl_2\to sl_3$ given by the spin 1 representation of sl_2.

The lift of the Fuchsian uniformization oper

$$\mathbf{D}^{(3)}_{\text{unif}}=\partial_z^3+4t_2^{\text{unif}}\,\partial_z+2\left(\partial_z t_2^{\text{unif}}\right) \tag{6.3.72}$$

has all three exponents equal to 1.

More generally, if the underlying surface C has complex structure moduli, such as for instance for $\mathbb{P}^1_{0,q,1,\infty}$, the SL_3 base oper $\mathbf{D}^{(3)}_0$ may be described as the lift (using the homomorphism $\rho: sl_2\to sl_3$ given by the spin 1 representation) of the SL_2 base oper $\mathbf{D}_0$, which in the example of $\mathbb{P}^1_{0,q,1,\infty}$ is written down in (6.3.35).

[4]This is the "canonical form" as in e.g. [42, 81], which makes the φ-action more obvious. This is also the form in which it plays a role in [67], in the scaling limit of Hitchin section.

Minimal punctures

Suppose that C has minimal punctures as well. We may obtain the space of Fuchsian SL_3 opers on C with minimal punctures from the locus of Fuchsian SL_3 opers on C with only maximal punctures, by simply enforcing the monodromy around the minimal punctures to be diagonal with two equal eigenvalues, i.e. a multiple of a reflection matrix.

This requires tuning two of the local exponents at each minimal puncture as well as tuning one internal parameter for each minimal puncture. These constraints can be expressed in terms of the differential equation (6.3.37) as follows:

(i) We set the mass parameters at the minimal puncture z_l equal to $m_l \pm 1$.

(ii) We require that if we multiply the differential equation

$$\widetilde{\mathbf{D}} y(z) = \mathbf{D} (z - z_l)^{\frac{1-m_l}{2}} y(z) = 0 \qquad (6.3.73)$$

by a factor $(z - z_l)^{\frac{1+m_l}{2}}$, such that the leading coefficient has an order 1 zero at $z = z_l$, the resulting differential equation has analytic coefficients at $z = z_l$.

This second condition implies that two of the solutions of the diffential equation $\widetilde{\mathbf{D}} y(z) = 0$ are holomorphic at $z = z_l$ (see for instance [49]). In return, that implies that the local monodromy of the SL_3 oper defined by the differential operator $\mathbf{D}$ around the puncture z_l is a multiple of a reflection matrix.

Example. Locus of opers for $T_3[\mathbb{P}^1_{0,1,\infty}]$.

Suppose that $z = 1$ is a minimal puncture. This imposes the constraints

$$\begin{aligned} m^{\mathrm{bif}}_{1,1} &= m_1 - 1 \\ m^{\mathrm{bif}}_{1,2} &= m_1 + 1, \end{aligned} \qquad (6.3.74)$$

as well as

$$u^{\mathrm{bif}} = \frac{1}{2^3} (m_1 - 1)\, m_1\, (m_1 + 1) - d_0 - 2 d_\infty + \frac{m_1}{2} (c_0 - c_\infty). \qquad (6.3.75)$$

on the family of Fuchsian SL_3 opers defined by the coefficients (6.3.64) and (6.3.65). This fixes the oper uniquely.

The resulting differential equation

$$\mathbf{D}^{\mathrm{bif}} y(z) = 0 \qquad (6.3.76)$$

can be written in the form of the generalized hypergeometric differential equation

$$\Big[z(\theta+\alpha_1)(\theta+\alpha_2)(\theta+\alpha_3)-(\theta+\beta_1-1)(\theta+\beta_2-1)(\theta+\beta_3-1)\Big]\widetilde{y}(z)=0 \tag{6.3.77}$$

where $\theta = z\partial_z$, with coefficients

$$\begin{aligned}
\alpha_1 &= \tfrac{1}{2}(-m_{\infty,1}+m_1+m_{0,3}-1+2\beta_3),\\
\alpha_2 &= \tfrac{1}{2}(-m_{\infty,2}+m_1+m_{0,3}-1+2\beta_3),\\
\alpha_3 &= \tfrac{1}{2}(-m_{\infty,3}+m_1+m_{0,3}-1+2\beta_3),\\
\beta_1 &= \tfrac{1}{2}(-m_{0,1}+m_{0,3}+2\beta_3),\\
\beta_2 &= \tfrac{1}{2}(-m_{0,2}+m_{0,3}+2\beta_3),
\end{aligned} \tag{6.3.78}$$

where $m_{0,3} = -m_{0,1} - m_{0,2}$, $m_{\infty,3} = -m_{\infty,1} - m_{\infty,2}$ and

$$\widetilde{y}(z) = z^{-\frac{\beta_1+\beta_2+\beta_3}{3}}(z-1)^{-\frac{\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2-\beta_3+3}{3}}y(z). \tag{6.3.79}$$

Finally, comparing the constraint (6.3.75) on the opers with the constraint (2.2.18) on the differentials, we notice a "quantum" difference. This implies that the generalized hypergeometric oper cannot be written in the form (6.3.67) for any choice of the coefficient t_2^{unif}.

Example. Locus of opers for $T_3[\mathbb{P}^1_{0,\underline{q},\underline{1},\infty}]$.

The space of SL_3 opers on the four-punctured sphere $\mathbb{P}^1_{0,\underline{q},\underline{1},\infty}$ with two maximal punctures at $z=0$ and $z=\infty$ and two minimal punctures at $z=q$ and $z=1$ is 2-dimensional. It may be obtained from the 4-dimensional family of Fuchsian SL_3 opers on the four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$ with four maximal punctures by imposing the conditions for a minimal puncture at $z=q$ and $z=1$.

The resulting family of opers may be written down as the differential equations

$$\mathbf{D}\,y(z) = y'''(z) + t_2(z)y'(z) + t_3(z)y(z) = 0, \tag{6.3.80}$$

with coefficients

$$t_2 = \frac{1+c_0}{z^2} + \frac{1+c}{(z-q)^2} + \frac{1+c_1}{(z-1)^2} + \frac{c_\infty - c_0 - c - c_1 - 2}{z(z-1)} + \frac{H_1}{z(z-q)(z-1)} \tag{6.3.81}$$

$$\begin{aligned}
t_3 = {} & \frac{d_0}{z^3} + \frac{d}{(z-q)^3} + \frac{d_1}{(z-1)^3} + \frac{d_\infty - d_0 - d - d_1}{z(z-q)(z-1)} + \\
& + \frac{(1-q)(4c_0 - 3m^2 - 3m_1^2 - 4c_\infty + 6)m_1}{8z(z-1)^2(z-q)} + \frac{H_2}{z^2(z-q)(z-1)} \\
& - \frac{H_1}{z(z-1)^2(z-q)^2}\left(\frac{m_1}{2}(z-q) + \frac{m}{2}(z-1)\right) + \frac{1}{2}t_2'.
\end{aligned} \tag{6.3.82}$$

We call this family (6.3.80) the family of ***generalized Heun's opers***. For any member of this family the monodromy around either minimal puncture is semisimple with two equal eigenvalues.

Note that the coefficients t_2 and $t_3 - \frac{1}{2}t_2'$ from equations (6.3.81) and (6.3.82) only differ with the differentials (6.2.10) and (6.2.11) in terms that have a smaller mass dimension. This difference goes to zero in the semi-classical limit $\epsilon \to 0$ discussed in §6.3.3.

In the limit $q \to 0$ the four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$ degenerates into two three-punctured spheres. In the same limit, the family of generalized Heun's opers degenerates into a pair of generalized hypergeometric opers.

If we assume that

$$H_1 = 1 + c_0 + c - c_\ell + \mathcal{O}(q) \tag{6.3.83}$$

$$H_2 = d_0 - d_\ell + \frac{m}{2}(c_0 - c_\ell) - \frac{1}{2^3}(m-1)m(m+1) + \mathcal{O}(q) \tag{6.3.84}$$

in the limit $q \to 0$, the family of generalized Heun's opers (6.3.80) has two interesting limits:

1. In the limit $q \to 0$ the family reduces to the generalized hypergeometric oper (6.3.77) with coefficients $(m_{0,i} \mapsto \ell_i)$, $m_{1,i}$ and $m_{\infty,i}$.

2. If we first map $z \mapsto qt$ and then take the limit $q \to 0$, the family reduces to the generalized hypergeometric oper (6.3.77) with coefficients $m_{0,i}$, $(m_{1,i} \mapsto m_i)$ and $(m_{\infty,i} \to -\ell_i)$.

The assumptions (6.3.83) will be justified in §6.4.

6.3.3 Semiclassical limit

It is natural to introduce an additional parameter ϵ with mass dimension 1 such that all terms in the Fuchsian differential equations have the same mass dimension. The corresponding locus of ϵ-opers $\mathbf{L}_\epsilon$ is a complex Lagrangian subspace of the moduli space of flat ϵ-connections. In the semiclassical limit $\epsilon \to 0$ the locus $\mathbf{L}_\epsilon$ limits to the space of quadratic (and higher if $K > 2$) differentials $\mathbf{B}$, or equivalently the spectral curves sitting above them.

In the following we often leave out the ϵ to avoid notational clutter, but at any stage it is a simple matter to reintroduce the ϵ-dependence.

6.4 Monodromy of opers

We now study the monodromy representation of the opers in the locus **L** in our main examples. For the superconformal SU(2), $N_f = 4$ theory this is Heun's differential equation (6.3.31), while for the superconformal SU(3), $N_f = 6$ theory this is its generalization (6.3.80) to $K = 3$. Since both are families of opers on a punctured sphere, there are no complications due to tricky coordinate transformations, and the monodromy representation is simply found as the fundamental system of solutions to the respective differential equations.

The relevant differential equations are too complicated for one to write down the monodromy representation explicitly in q. We use the fact that the underlying Riemann surface is the four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$ and write down the expressions in a series expansion in q, following and expanding arguments of [68, 69, 70].[5] We are helped by the fact that the leading contribution when $q \to 0$ is described by the (generalized) hypergeometric differential equation, whose monodromy has been explicitly computed [48, 50, 49].

The same method may be applied in principle to compute the monodromy representation of any family of opers of class $\mathcal{S}$ in a perturbation series in the complex structure parameters q, whenever exact expressions are known for the oper monodromies in the limit $q \to 0$.

6.4.1 Heun's differential equation

In this subsection we compute the monodromy representation of the Heun equation (6.3.31) in a perturbation series in q.

To compare to the monodromy representation (5.4.23) of any flat SL_2 connection on the four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$ in terms of average length-twist coordinates, we fix the monodromy $\mathbf{M}_\alpha^{\text{oper}}$ around the punctures $z = 1$ and $z = \infty$ such that it has trace

$$\operatorname{Tr} \mathbf{M}_\alpha^{\text{oper}} = -2\cos(\pi\ell), \tag{6.4.1}$$

with ℓ non-integer. The parameter ℓ will later play the role of the Coulomb parameter a.

[5] While finishing this work we noticed a seemingly related strategy for calculating the Painlevé VI tau-function for small q in [82].

As computed in [70], fixing the monodromy $\mathbf{M}_\alpha^{\text{oper}}$ in this way determines a series expansion of the accessory parameter H in q

$$H = \sum_{k=0}^{\infty} q^k H_k, \tag{6.4.2}$$

with for instance

$$H_0 = \delta_\ell - \delta_0 - \delta \tag{6.4.3}$$

and

$$H_1 = \frac{(\delta_\ell - \delta_0 + \delta)(\delta_\ell - \delta_\infty + \delta_1)}{2\delta_\ell} - H_0. \tag{6.4.4}$$

It remains to compute the monodromy $\mathbf{M}_\beta^{\text{oper}}$ of the Heun equation (6.3.31) around the punctures at $z = 0$ and $z = \infty$ in a perturbation series in q. Our strategy for this is as follows:

1. We define the rescaled Heun equation by substituting $z = qt$ in Heun's equation itself. We construct solutions $v_1(t)$ and $v_2(t)$ of the rescaled Heun equation in a neigbourhood of $t = 0$, in a perturbation series in q.

2. We analytically continue the solutions $v_1(t)$ and $v_2(t)$ to $t = \infty$ while keeping $z = qt$ finite, but very small. We re-organize the functions $w_1(z) = v_1(t/q)$ and $w_2(z) = v_2(t/q)$, which are solutions of the Heun equation itself, around $z = 0$ in a perturbation series in q.

3. We analytically continue the solutions $w_1(z)$ and $w_2(z)$ to $z = \infty$.

These three steps together determine the connection matrix $S_{\text{total}}(q)$ that relates the local solutions of Heun's differential equation near the puncture at $z = 0$ to the local solutions near the puncture at $z = \infty$. Say that M_0 and M_∞ are the local monodromies around $z = 0$ and $z = \infty$, respectively. Then the monodromy matrix of Heun's equation around the punctures $z = 0$ and $z = \infty$ is found as

$$\mathbf{M}_\beta^{\text{oper}} = M_0\, S_{\text{total}}(q)\, M_\infty\, \left(S_{\text{total}}(q)\right)^{-1}. \tag{6.4.5}$$

The computation is illustrated in Figure 6.3 and the result is summarized in equation (6.4.85).

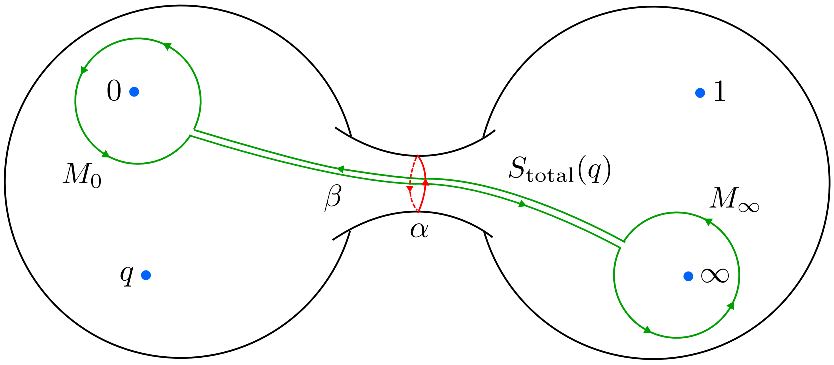


Figure 6.3: Decomposition of the cycle β on $\mathbb{P}^1_{0,q,1,\infty}$ into four paths corresponding to the computation of the monodromy matrix M_β on $\mathbb{P}^1_{0,q,1,\infty}$ as in equation (6.4.5).

Step 1: Perturbation of the rescaled Heun's differential equation

We first define the rescaled Heun's differential equation by substituting $z = qt$ in equation (6.3.31) and expand it in a perturbation series in q. We also expand its solutions $v(t,\ell)$ in q as

$$v(t,\ell) = \sum_{k=0}^{\infty} q^k v^{(k)}(t,\ell). \tag{6.4.6}$$

The leading contribution $v^{(0)}(t,\ell)$ is determined by the hypergeometric differential equation

$$\partial_t^2 v^{(0)}(t,\ell) + Q_0(t,\ell)\, v^{(0)}(t,\ell) = 0, \tag{6.4.7}$$

with

$$Q_0(t,\ell) = \frac{\Delta_0 - (\Delta_\ell - \Delta + \Delta_0)t + \Delta_\ell t^2}{t^2(1-t)^2}. \tag{6.4.8}$$

We find the two independent solutions

$$v_1^{(0)}(t,\ell) = t^{\frac{1-m_0}{2}}(1-t)^{\frac{1+m}{2}}\,{}_2F_1\left(\frac{1-\ell+m-m_0}{2}, \frac{1+\ell+m-m_0}{2}, 1-m_0, t\right) \tag{6.4.9}$$

$$v_2^{(0)}(t,\ell) = t^{\frac{1+m_0}{2}}(1-t)^{\frac{1+m}{2}}\,{}_2F_1\left(\frac{1-\ell+m+m_0}{2}, \frac{1+\ell+m+m_0}{2}, 1+m_0, t\right), \tag{6.4.10}$$

where ${}_2F_1\,(a,b,c,t)$ is the Gauss hypergeometric function

$${}_2F_1\,(a,b,c,t) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{t^k}{k!}, \tag{6.4.11}$$

with $(x)_k := x(x+1)\dots(x+k-1)$ the Pochhammer symbol.

The subleading contribution $v^{(1)}(t,\ell)$ is determined by the equation

$$\partial_t^2 v^{(1)}(t,\ell) + Q_0(t,\ell)\, v^{(1)}(t,\ell) + Q_1(t,\ell)\, v^{(0)}(t,\ell) = 0, \tag{6.4.12}$$

with

$$Q_1(t,\ell) = \frac{(\delta_\ell + \delta_1 - \delta_\infty)(\delta_\ell + \delta_1 - \delta_\infty - 2\delta_\ell t)}{2\,\delta_\ell\, t\,(1-t)}. \tag{6.4.13}$$

Its solutions can be found in two ways.

First, we may use general perturbation theory (following [69] and appendix A) to write the solution in the form

$$v_r^{(1)}(t,\ell) = S_{r1} v_1^{(0)}(t,\ell) + S_{r2} v_2^{(0)}(t,\ell) \tag{6.4.14}$$

with

$$S_{r1} = \frac{1}{m_0} \int_0^t v_2^{(0)}(s,\ell) Q_1(s,\ell) v_r^{(0)}(s,\ell) ds \tag{6.4.15}$$

$$S_{r2} = -\frac{1}{m_0} \int_0^t v_1^{(0)}(s,\ell) Q_1(s,\ell) v_r^{(0)}(s,\ell) ds. \tag{6.4.16}$$

In a perturbation series in t we find

$$S_{11} = -t\,(\sigma + \mathcal{O}(t)) \qquad S_{12} = -t^{1-m_0}\left(\frac{\sigma}{m_0 - 1} + \mathcal{O}(t)\right) \tag{6.4.17}$$

$$S_{21} = -t^{1+m_0}\left(\frac{\sigma}{m_0 + 1} + \mathcal{O}(t)\right) \qquad S_{22} = -t\,(\sigma + \mathcal{O}(t)) \tag{6.4.18}$$

with

$$\sigma = \frac{(\ell^2 + m_0^2 - m^2 - 1)(\ell^2 + m_1^2 - m_\infty^2 - 1)}{8 m_0 (\ell^2 - 1)}. \tag{6.4.19}$$

Note that there is the freedom of adding any multiple of $v_r^{(0)}(t,\ell)$ to $v_r^{(1)}(t,\ell)$. This only changes the boundary conditions of $v_r(t,\ell)$ at $t=0$. The choice made in equation (6.4.14) fixes

$$v_r^{(1)}(t=0,\ell) = 0. \tag{6.4.20}$$

This choice implies that the t-expansion of $v_1^{(1)}(t,\ell)$ starts off with a term proportional to $t^{\frac{3-m_0}{2}}$, and that the t-expansion of $v_2^{(1)}(t,\ell)$ starts off with a term proportional to $t^{\frac{3+m_0}{2}}$:

$$v_1^{(1)}(t,\ell) = -t^{\frac{3-m_0}{2}}\left(\frac{(\ell^2 + m_0^2 - m^2 - 1)(\ell^2 + m_1^2 - m_\infty^2 - 1)}{8(\ell^2-1)(m_0-1)} + \mathcal{O}(t)\right) \tag{6.4.21}$$

$$v_2^{(1)}(t,\ell) = t^{\frac{3+m_0}{2}}\left(\frac{(\ell^2 + m_0^2 - m^2 - 1)(\ell^2 + m_1^2 - m_\infty^2 - 1)}{8(\ell^2-1)(m_0+1)} + \mathcal{O}(t)\right) \tag{6.4.22}$$

Alternatively, we could make an ansatz of the form (following [68]):

$$\widetilde{v}_r^{(1)}(t,\ell) = A_{r,-1}^{(1)} v_r^{(0)}(t,\ell+2) + A_{r,0}^{(1)} v_r^{(0)}(t,\ell) + A_{r,1}^{(1)} v_r^{(0)}(t,\ell-2). \tag{6.4.23}$$

This ansatz confirms the value (6.4.4) for H_1 and fixes

$$A_{1,-1}^{(1)} = -\frac{(\ell \pm m - m_0 + 1)(\ell^2 + m_1^2 - m_\infty^2 - 1)}{16\ell(\ell+1)^2} \tag{6.4.24}$$

$$A_{1,1}^{(1)} = -\frac{(\ell \pm m + m_0 - 1)(\ell^2 + m_1^2 - m_\infty^2 - 1)}{16\ell(\ell-1)^2} \tag{6.4.25}$$

$$A_{2,-1}^{(1)} = -\frac{(\ell \pm m + m_0 + 1)(\ell^2 + m_1^2 - m_\infty^2 - 1)}{16\ell(\ell+1)^2} \tag{6.4.26}$$

$$A_{2,1}^{(1)} = +\frac{(\ell \pm m - m_0 - 1)(\ell^2 + m_1^2 - m_\infty^2 - 1)}{16\ell(\ell-1)^2}, \tag{6.4.27}$$

where $(\ell \pm b) = (\ell + b)(\ell - b)$.

The coefficient $A_{r,0}^{(1)}$ is left underdetermined, corresponding to the freedom of adding a multiple of $v_r^{(0)}(t,\ell)$ to $v_r^{(1)}(t,\ell)$. Comparing to equation (6.4.14) we choose

$$A_{r,0}^{(1)} = -A_{r,1}^{(1)} - A_{r,-1}^{(1)}, \tag{6.4.28}$$

to fix the boundary condition $v_r^{(1)}(t=0,\ell) = 0$. Indeed, we then find that

$$\widetilde{v}_r^{(1)}(t,\ell) = v_r^{(1)}(t,\ell). \tag{6.4.29}$$

The expansion of $v^{(1)}(t,\ell)$ in terms of hypergeometric functions as in equation (6.4.23) will be useful to analytically continue to $t = \infty$.

We can continue this perturbation to any order in q by expanding [68]

$$v_r^{(k)}(t,\ell) = \sum_{j=-k}^{k} A_{r,j}^{(k)} v_r^{(0)}(t,\ell-2j). \tag{6.4.30}$$

and find for instance that

$$A_{1,-2}^{(2)} = \frac{(\ell \pm m - m_0 + 3)(\ell \pm m - m_0 + 1)(\ell \pm m_1 \pm m_\infty + 1)}{512\ell(\ell+1)^2(\ell+2)^2(\ell+3)} + \frac{(\ell \pm m - m_0 + 3)(\ell \pm m - m_0 + 1)(m_1^2 - 1)}{128\ell(\ell+1)(\ell+2)(\ell+3)} \tag{6.4.31}$$

$$A_{1,2}^{(2)} = \frac{(\ell \pm m + m_0 - 3)(\ell \pm m + m_0 - 1)(\ell \pm m_1 \pm m_\infty - 1)}{512\ell(\ell-1)^2(\ell-2)^2(\ell-3)} + \frac{(\ell \pm m + m_0 - 3)(\ell \pm m + m_0 - 1)(m_1^2 - 1)}{128\ell(\ell-1)(\ell-2)(\ell-3)}. \tag{6.4.32}$$

Step 2: Solutions for $|z| < 1$

We have just seen that the solutions $v_r(t,\ell)$ of the rescaled Heun equation may be expanded in q as

$$\begin{aligned} v_r(t,\ell) =& v_r^{(0)}(t,\ell) \\ &+ q\left(A^{(1)}_{r,-1}\, v_r^{(0)}(t,\ell+2) + A^{(1)}_{r,0}\, v_r^{(0)}(t,\ell) + A^{(1)}_{r,1}\, v_r^{(0)}(t,\ell-2)\right) \\ &+ q^2 \sum_{k=-2}^{2} A^{(2)}_{r,k}\, v_r^{(0)}(t,\ell-2k) \\ &+ \mathcal{O}(q^3) \end{aligned} \tag{6.4.33}$$

Instead of considering the solutions $v_r(t,\ell)$ around $t=0$, we now want to analytically continue them to $|t| \gg 1$.

For $|t| \gg 1$ the hypergeometric functions $v_1^{(0)}(t)$ and $v_2^{(0)}(t)$ may be expanded as

$$\begin{aligned} v_r^{(0)}(t,\ell-2j) =& (-1)^{\frac{1-2m+m_0}{2}}(-t)^{\frac{1-\ell}{2}+j} B_t[\ell-2j]_{r1} \\ &\left(1 + \frac{(\ell-2j)^2+m^2-m_0^2-1}{4(\ell-2j+1)}\, t^{-1} + \mathcal{O}\left(t^{-2}\right)\right) \end{aligned} \tag{6.4.34}$$

$$\begin{aligned} &+ (-1)^{\frac{1-2m+m_0}{2}}(-t)^{\frac{1+\ell}{2}-j} B_t[\ell-2j]_{r2} \\ &\left(1 - \frac{(\ell-2j)^2+m^2-m_0^2-1}{4(\ell-2j-1)}\, t^{-1} + \mathcal{O}\left(t^{-2}\right)\right) \end{aligned} \tag{6.4.35}$$

with

$$B_t[\ell] = \begin{pmatrix} \frac{\Gamma[-\ell]\Gamma[1-m_0]}{\Gamma[\frac{1-\ell+m-m_0}{2}]\Gamma[\frac{1-\ell-m-m_0}{2}]} & \frac{\Gamma[\ell]\Gamma[1-m_0]}{\Gamma[\frac{1+\ell+m-m_0}{2}]\Gamma[\frac{1+\ell-m-m_0}{2}]} \\ \frac{\Gamma[-\ell]\Gamma[1+m_0]}{\Gamma[\frac{1-\ell+m+m_0}{2}]\Gamma[\frac{1-\ell-m+m_0}{2}]} & \frac{\Gamma[\ell]\Gamma[1+m_0]}{\Gamma[\frac{1+\ell+m+m_0}{2}]\Gamma[\frac{1+\ell-m+m_0}{2}]} \end{pmatrix}. \tag{6.4.36}$$

This implies that the solutions $v_r(t,\ell)$ have the expansion

$$\begin{aligned} v_r(t,\ell) =& (-1)^{\frac{1-2m+m_0}{2}}(-t)^{\frac{1-\ell}{2}}\Big(\sum_{l=0}^{\infty} B_t[\ell-2l]_{r1}\, A^{(l)}_{r,l}\, (-z)^{\mathcal{R}} + \mathcal{O}\left(t^{-1}\right)\Big) \\ &+(-1)^{\frac{1-2m+m_0}{2}}(-t)^{\frac{1+\ell}{2}}\Big(\sum_{l=0}^{\infty} B_t[\ell+2l]_{r2}\, A^{(l)}_{r,-l}\, (-z)^{\mathcal{R}} + \mathcal{O}\left(t^{-1}\right)\Big). \end{aligned} \tag{6.4.37}$$

for $|t| \gg 1$, yet $|z| = |qt| < 1$.

Leading order in q

Write the solutions $w_r(z,\ell) = v_r\left(\frac{z}{q},\ell\right)$ to the unrescaled Heun equation (6.3.31) in a q-expansion as

$$w_r(z,\ell) = \sum_k q^k\, w_r^{(k)}(z,\ell). \tag{6.4.38}$$

Equation (6.4.37) implies that the leading contribution $w_r^{(0)}(z,\ell)$ is given by

$$w_r^{(0)}(z,\ell) = (-1)^{\frac{1-2m+m_0}{2}}\left(-\frac{z}{q}\right)^{\frac{1-\ell}{2}} B_t[\ell]_{r1}\left(\sum_{l=0}^{\infty}\frac{B_t[\ell-2l]_{r1}\,A_{r,l}^{(l)}}{B_t[\ell]_{r1}}(-z)^{\mathcal{R}}\right) \tag{6.4.39}$$

$$+ (-1)^{\frac{1-2m+m_0}{2}}\left(-\frac{z}{q}\right)^{\frac{1+\ell}{2}} B_t[\ell]_{r2}\left(\sum_{l=0}^{\infty}\frac{B_t[\ell+2l]_{r2}\,A_{r,-l}^{(l)}}{B_t[\ell]_{r2}}(-z)^{\mathcal{R}}\right). \tag{6.4.40}$$

The coefficients in front of $(-z)$ are

$$\frac{B_t[\ell-2]_{r1}\,A_{r,1}^{(1)}}{B_t[\ell]_{r1}} = \frac{(\ell^2+m_1^2-m_\infty^2-1)}{4(\ell-1)} \tag{6.4.41}$$

$$\frac{B_t[\ell+2]_{r2}\,A_{r,-1}^{(1)}}{B_t[\ell]_{r2}} = -\frac{(\ell^2+m_1^2-m_\infty^2-1)}{4(\ell+1)}, \tag{6.4.42}$$

whereas the coefficients in front of $(-z)^2$ are

$$\frac{B_t[\ell-4]_{r1}\,A_{r,2}^{(2)}}{B_t[\ell]_{r1}} = \frac{(\ell-m_1\pm m_\infty-3)(\ell-m_1\pm m_\infty-1)}{32(\ell-1)(\ell-2)} + \frac{(m_1+1)(\ell-m_1\pm m_\infty-1)}{8(\ell-1)} + \frac{(m_1^2-1)}{8} \tag{6.4.43}$$

$$\frac{B_t[\ell+4]_{r2}\,A_{r,-2}^{(2)}}{B_t[\ell]_{r2}} = \frac{(\ell+m_1\pm m_\infty+3)(\ell+m_1\pm m_\infty+1)}{32(\ell+1)(\ell+2)} - \frac{(m_1+1)(\ell+m_1\pm m_\infty+1)}{8(\ell+1)} + \frac{(m_1^2-1)}{8}. \tag{6.4.44}$$

This suggests that $w_r^{(0)}(z,\ell)$ can be rewritten in the form

$$\begin{pmatrix} w_1^{(0)}(z,\ell) \\ w_2^{(0)}(z,\ell)\end{pmatrix} = B_t[\ell]\,T\begin{pmatrix} y_1^{(0)}(z,\ell) \\ y_2^{(0)}(z,\ell)\end{pmatrix} \tag{6.4.45}$$

with

$$T = \begin{pmatrix} q^{\frac{\ell-1}{2}} & 0 \\ 0 & q^{\frac{-\ell-1}{2}}\end{pmatrix}, \tag{6.4.46}$$

and

$$y_1^{(0)}(z,\ell) = (-1)^{\frac{1-2m+m_0}{2}}(1-z)^{\frac{1+m_1}{2}}(-z)^{\frac{1-\ell}{2}} \tag{6.4.47}$$
$$ {}_2F_1\left(\frac{1-\ell+m_1-m_\infty}{2}, \frac{1-\ell+m_1+m_\infty}{2}, 1-\ell, z\right)$$
$$y_2^{(0)}(z,\ell) = (-1)^{\frac{1-2m+m_0}{2}}(1-z)^{\frac{1+m_1}{2}}(-z)^{\frac{1+\ell}{2}} \tag{6.4.48}$$
$$ {}_2F_1\left(\frac{1+\ell+m_1-m_\infty}{2}, \frac{1+\ell+m_1+m_\infty}{2}, 1+\ell, z\right),$$

which is a basis of solutions to the unrescaled Heun equation (6.3.31) at $q=0$.

Indeed, since $w_r^{(0)}(z,\ell)$ is a solution to the unrescaled Heun equation at $q=0$, and since we have verified equation (6.4.45) up to order z^2, equation (6.4.45) must hold to any order.

subleading order in q

To find the subleading contribution $w_r^{(1)}$ in q we substitute the t^{-1}-expansion (6.4.34) of $v_r^{(0)}$ into the q-expansion (6.4.33) of v_r. The resulting expansion is

$$w_r^{(1)}(z,\ell) = (-1)^{\frac{1-2m+m_0}{2}}\left(-\frac{z}{q}\right)^{\frac{1-\ell}{2}} \sum_{m=-1}^{\infty} W_{1,m}^{(1)}\, z^m \tag{6.4.49}$$
$$+ \; (-1)^{\frac{1-2m+m_0}{2}}\left(-\frac{z}{q}\right)^{\frac{1+\ell}{2}} \sum_{m=-1}^{\infty} W_{2,m}^{(1)}\, z^m \tag{6.4.50}$$

with

$$W_{1,-1}^{(1)} = B_t[\ell]_{r1}\,\frac{\ell^2+m^2-m_0^2-1}{4(\ell+1)} \tag{6.4.51}$$
$$W_{1,0}^{(1)} = -\,B_t[\ell-2]_{r1}\,A_{r,1}^{(1)}\frac{(\ell-2)^2+m^2-m_0^2-1}{4(\ell-1)} + B_t[\ell]_{r1}\,A_{r,0}^{(1)} \tag{6.4.52}$$
$$W_{2,-1}^{(1)} = -B_t[\ell]_{r2}\,\frac{\ell^2+m^2-m_0^2-1}{4(\ell-1)} \tag{6.4.53}$$
$$W_{2,0}^{(1)} = B_t[\ell+2]_{r2}\,A_{r,-1}^{(1)}\frac{(\ell+2)^2+m^2-m_0^2-1}{4(\ell+1)} + \; B_t[\ell]_{r2}\,A_{r,0}^{(1)} \tag{6.4.54}$$

and so forth.

This expansion is consistent with the closed form

$$\begin{pmatrix} w_1^{(1)}(z,\ell) \\ w_2^{(1)}(z,\ell) \end{pmatrix} = B_t[\ell]\,T\begin{pmatrix} y_1^{(1)}(z,\ell) \\ y_2^{(1)}(z,\ell) \end{pmatrix}, \tag{6.4.55}$$

where

$$y_r^{(1)}(z,\ell) = C_{r,-1}^{(1)} y_r^{(0)}(z,\ell+2) + C_{r,0}^{(1)} y_r^{(0)}(z,\ell) + C_{r,1}^{(1)} y_r^{(0)}(z,\ell-2), \tag{6.4.56}$$

with coefficients

$$C_{1,-1}^{(1)} = \frac{1-\ell^2+m_0^2-m^2}{4(\ell+1)} \tag{6.4.57}$$

$$C_{1,0}^{(1)} = \frac{\ell}{4}\left(1+\frac{(m_1^2-m_\infty^2)(m_0^2-m^2)}{(\ell+1)^2(\ell-1)^2}\right) \tag{6.4.58}$$

$$-\frac{(m_1^2-m_\infty^2)(m_0^2-m^2)}{4(\ell+1)^2(\ell-1)^2} + \frac{(m_1^2-m_\infty^2)(1-m_0)}{4(\ell+1)(\ell-1)} - \frac{m_0}{4}$$

$$C_{1,1}^{(1)} = \frac{(\ell^2+m^2-m_0^2-1)(\ell\pm m_1\pm m_\infty-1)}{64\,\ell(\ell-1)^3(\ell-2)} \tag{6.4.59}$$

and $C_{2,k}^{(1)}(\ell) = C_{1,k}^{(1)}(-\ell)$.

Indeed, since $y_1^{(0)}(z,\ell)+qy_1^{(1)}(z,\ell)+\mathcal{O}(q^2)$ is a solution of the unrescaled Heun equation in a perturbation series in q, and since we can verify equation (6.4.55) up to second order in z, it must hold to any order in z.

Step 3: Solutions at $z=\infty$

Analytically continuing to $z=\infty$ gives

$$\begin{pmatrix} y_1^{(0)}(z,\ell) \\ y_2^{(0)}(z,\ell) \end{pmatrix} \approx B_z[\ell] \begin{pmatrix} (-1)^{\frac{1-2m+m_0}{2}}(-z)^{\frac{1-m_\infty}{2}} \\ (-1)^{\frac{1-2m+m_0}{2}}(-z)^{\frac{1+m_\infty}{2}} \end{pmatrix} \tag{6.4.60}$$

with

$$B_z[\ell] = \begin{pmatrix} \frac{\Gamma[1-\ell]\Gamma[-m_\infty]}{\Gamma[\frac{1-\ell-m_1-m_\infty}{2}]\Gamma[\frac{1-\ell+m_1-m_\infty}{2}]} & \frac{\Gamma[1-\ell]\Gamma[m_\infty]}{\Gamma[\frac{1-\ell-m_1+m_\infty}{2}]\Gamma[\frac{1-\ell+m_1+m_\infty}{2}]} \\ \frac{\Gamma[1+\ell]\Gamma[-m_\infty]}{\Gamma[\frac{1+\ell-m_1-m_\infty}{2}]\Gamma[\frac{1+\ell+m_1-m_\infty}{2}]} & \frac{\Gamma[1+\ell]\Gamma[m_\infty]}{\Gamma[\frac{1+\ell-m_1+m_\infty}{2}]\Gamma[\frac{1+\ell+m_1+m_\infty}{2}]} \end{pmatrix}. \tag{6.4.61}$$

This implies that

$$\begin{pmatrix} y_1^{(1)}(z,\ell) \\ y_2^{(1)}(z,\ell) \end{pmatrix} \approx S_z[\ell] \begin{pmatrix} (-1)^{\frac{1-2m+m_0}{2}}(-z)^{\frac{1-m_\infty}{2}} \\ (-1)^{\frac{1-2m+m_0}{2}}(-z)^{\frac{1+m_\infty}{2}} \end{pmatrix} \tag{6.4.62}$$

where

$$S_z[\ell]_{rs} = C_{r,-1}^{(1)}\,B_z[\ell+2]_{rs} + C_{r,0}^{(1)}\,B_z[\ell]_{rs} + C_{r,1}^{(1)}\,B_z[\ell-2]_{rs} \tag{6.4.63}$$

Hence

$$\begin{pmatrix} w_1(z,\ell) \\ w_2(z,\ell) \end{pmatrix} \approx S_{\text{total}}(q) \begin{pmatrix} (-1)^{\frac{1-2m+m_0}{2}} (-z)^{\frac{1-m_\infty}{2}} \\ (-1)^{\frac{1-2m+m_0}{2}} (-z)^{\frac{1+m_\infty}{2}} \end{pmatrix} \tag{6.4.64}$$

with

$$S_{\text{total}}[q] = B_t[\ell]\, T\, B_z[\ell] \left(1 + q\, B_z[\ell]^{-1} S_z[\ell] + \mathcal{O}(q^2)\right). \tag{6.4.65}$$

Step 4: Monodromy

Say that

$$M_0 = \begin{pmatrix} e^{\pi i(1-m_0)} & 0 \\ 0 & e^{\pi i(1+m_0)} \end{pmatrix} \tag{6.4.66}$$

and

$$M_\infty = \begin{pmatrix} e^{\pi i(1+m_\infty)} & 0 \\ 0 & e^{\pi i(1-m_\infty)} \end{pmatrix} \tag{6.4.67}$$

are the local monodromy matrices at zero and infinity, respectively. Then the monodromy matrix of Heun's differential equation around the punctures $z = 0$ and $z = \infty$ is given by

$$\mathbf{M}_\beta^{\text{oper}} = M_0\, S_{\text{total}}[q]\, M_\infty\, S_{\text{total}}[q]^{-1}. \tag{6.4.68}$$

We compute the inverse of $\mathbf{M}_\beta$ using that

$$B_z[\ell]^{-1} = B_t[-\ell]\Big|_{m=m_1, m_0=-m_\infty}. \tag{6.4.69}$$

and that

$$S_{\text{total}}[q]^{-1} = (1 - q\, B_z[\ell]^{-1}\, S_z[\ell])\, (B_t[\ell]\, T\, B_z[\ell])^{-1} + \mathcal{O}(q^2). \tag{6.4.70}$$

We then find

$$\mathbf{M}_\beta^{\text{oper}} = \mathbf{M}_\beta^{\text{oper},(0)} + q\, \mathbf{M}_\beta^{\text{oper},(1)} + \mathcal{O}(q^2) \tag{6.4.71}$$

with

$$\mathbf{M}_\beta^{\text{oper},(0)} = M_0\, B_t[\ell]\, T\, B_z[\ell]\, M_\infty\, B_z[\ell]^{-1}\, T^{-1}\, B_t[\ell]^{-1}. \tag{6.4.72}$$

and

$$\mathbf{M}_\beta^{\text{oper},(1)} = M_0\, B_t[\ell]\, T\, B_z[\ell] \left(B_z[\ell]^{-1}\, S_z[\ell]\, M_\infty - M_\infty\, B_z[\ell]^{-1}\, S_z[\ell]\right) \tag{6.4.73}$$

$$B_z[\ell]^{-1}\, T^{-1}\, B_t[\ell]^{-1}. \tag{6.4.74}$$

Leading order monodromy

In the limit $q \to 0$ the four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$ may be approximated by gluing two three-punctured spheres $\mathbb{P}^1_{0,1,\infty}$ using the plumbing construction. In the same limit Heun's differential equation (6.3.31) may be approximated by the two hypergeometric differential equations (6.3.24), one on each three-punctured sphere. It is well-known that $B_t[\ell]$ and $B_z[\ell]^{-1}$ are the connection matrices for these hypergeometric differential equations, respectively. Equation (6.4.72) shows that the leading order contribution in q to the monodromies of Heun's differential equation may simply be found from the monodromies of the hypergeometric differential equation by splicing in the gluing matrix T (see [68, 69] for an alternative proof, whose generalization is described in the appendix).

To leading order in q we calculate that

$$\mathrm{Tr}\, \mathbf{M}_{\beta}^{\mathrm{oper},0} = D_- \, q^{-\ell} + D_\circ + D_+ \, q^{\ell}, \tag{6.4.75}$$

where

$$D_- = -4\pi^2 \frac{\Gamma[1+\ell]^2 \Gamma[\ell]^2}{\Gamma\left[\frac{1}{2} + \frac{\ell \pm m_0 \pm m}{2}\right] \Gamma\left[\frac{1}{2} + \frac{\ell \pm m_1 \pm m_\infty}{2}\right]} \tag{6.4.76}$$

$$D_+ = -4\pi^2 \frac{\Gamma[1-\ell]^2 \Gamma[-\ell]^2}{\Gamma\left[\frac{1}{2} - \frac{\ell \pm m_0 \pm m}{2}\right] \Gamma\left[\frac{1}{2} - \frac{\ell \pm m_1 \pm m_\infty}{2}\right]} \tag{6.4.77}$$

and

$$D_\circ = \frac{\cos(\pi\ell, \pi m_0, \pi m_1) + \cos(\pi\ell, \pi m, \pi m_\infty) + \cos(\pi m_0, \pi m_\infty) + \cos(\pi m, \pi m_1)}{\frac{1}{2}\sin^2(\pi\ell)}, \tag{6.4.78}$$

where we defined

$$\cos(x_1, \ldots, x_n) = \cos(x_1) \cdots \cos(x_n). \tag{6.4.79}$$

subleading order monodromy

At subleading order in q we find

$$B_z[\ell]^{-1} \, S_z[\ell] \, M_\infty - M_\infty \, B_z[\ell]^{-1} \, S_z[\ell] = \begin{pmatrix} 0 & \delta M_+ \\ \delta M_- & 0 \end{pmatrix} \tag{6.4.80}$$

with

$$\delta M_+ = \frac{2i\pi^2 \left(C^{(1)}_{1,0} - C^{(1)}_{2,0} \right)}{\sin[\pi\ell]} \frac{\Gamma[1+m_\infty]}{\Gamma[1-m_\infty]} \frac{1}{\Gamma\left[\frac{1 \pm \ell \pm m_1 + m_\infty}{2}\right]} \tag{6.4.81}$$

and

$$\delta M_- = \frac{2i\pi^2\left(C^{(1)}_{2,0} - C^{(1)}_{1,0}\right)}{\sin[\pi\ell]} \frac{\Gamma[1-m_\infty]}{\Gamma[1+m_\infty]} \frac{1}{\Gamma\left[\frac{1\pm\ell\pm m_1+m_\infty}{2}\right]} \tag{6.4.82}$$

This leads to

$$\mathrm{Tr}\,\mathbf{M}^{\mathrm{oper},1}_\beta = D_-\left(C^{(1)}_{2,0} - C^{(1)}_{1,0}\right) q^{-\ell} + D_+\left(C^{(1)}_{1,0} - C^{(1)}_{2,0}\right) q^{\ell}. \tag{6.4.83}$$

with

$$C^{(1)}_{1,0} - C^{(1)}_{2,0} = \frac{\ell}{2}\left(1 + \frac{(m_1^2 - m_\infty^2)(m_0^2 - m^2)}{(\ell+1)^2(\ell-1)^2}\right). \tag{6.4.84}$$

The result

Up to order q we thus find that

$$\mathrm{Tr}\,\mathbf{M}^{\mathrm{oper}}_\beta = D_-\, q^{-\ell}\left(1 - c_1 q + \mathcal{O}(q^2)\right) + D_\circ + D_+\, q^{\ell}\left(1 + c_1 q + \mathcal{O}(q^2)\right), \tag{6.4.85}$$

with

$$D_- = -4\pi^2 \frac{\Gamma[1+\ell]^2\Gamma[\ell]^2}{\Gamma\left[\frac{1}{2} + \frac{\ell\pm m_0\pm m}{2}\right]\Gamma\left[\frac{1}{2} + \frac{\ell\pm m_1\pm m_\infty}{2}\right]} \tag{6.4.86}$$

$$D_+ = -4\pi^2 \frac{\Gamma[1-\ell]^2\Gamma[-\ell]^2}{\Gamma\left[\frac{1}{2} - \frac{\ell\pm m_0\pm m}{2}\right]\Gamma\left[\frac{1}{2} - \frac{\ell\pm m_1\pm m_\infty}{2}\right]}, \tag{6.4.87}$$

whereas

$$D_\circ = \frac{\cos(\pi m_0, \pi m_\infty) + \cos(\pi m, \pi m_1) + \cos(\pi\ell, \pi m_0, \pi m_1) + \cos(\pi\ell, \pi m, \pi m_\infty)}{\frac{1}{2}\sin^2(\pi\ell)}, \tag{6.4.88}$$

and

$$c_1 = \frac{\ell}{2}\left(1 + \frac{(m_1^2 - m_\infty^2)(m_0^2 - m^2)}{(\ell+1)^2(\ell-1)^2}\right). \tag{6.4.89}$$

Using the same techniques one can in principle compute the oper monodromies to any order in q.

We rewrite this result in terms of perturbative and instanton corrections to the effective twisted superpotential of the superconformal SU(2) theory coupled to four hypers in §6.5.1.

6.4.2 Generalized Heun's equation

The monodromies of the generalized Heun equation (6.3.80) may be computed perturbatively in an expansion in q in the same way,. Here we content ourselves with the leading contribution in q.

Again, we start with fixing the coefficients in the expansion

$$H_1 = \sum_{k=0}^{\infty} q^k H_{1,k} \tag{6.4.90}$$

$$H_2 = \sum_{k=0}^{\infty} q^k H_{2,k} \tag{6.4.91}$$

of the accessory parameters H_1 and H_2 in equation (6.3.80), by requiring that the monodromy $\mathbf{M}_\alpha^{\text{oper}}$ around the punctures $z = 1$ and $z = \infty$ has traces

$$\begin{aligned} \operatorname{Tr} \mathbf{M}_\alpha^{\text{oper}} &= e^{2\pi i\left(1+\frac{\ell_1}{2}\right)} + e^{2\pi i\left(1+\frac{\ell_2}{2}\right)} + e^{2\pi i\left(1-\frac{\ell_1}{2}-\frac{\ell_2}{2}\right)}, \\ \operatorname{Tr}\left(\mathbf{M}_\alpha^{\text{oper}}\right)^{-1} &= e^{2\pi i\left(1-\frac{\ell_1}{2}\right)} + e^{2\pi i\left(1-\frac{\ell_2}{2}\right)} + e^{2\pi i\left(1+\frac{\ell_1}{2}+\frac{\ell_2}{2}\right)}, \end{aligned} \tag{6.4.92}$$

for some fixed complex numbers ℓ_1 and ℓ_2. This determines the leading coefficients (6.3.83), i.e.

$$H_{1,0} = 1 + c_0 + c - c_\ell, \tag{6.4.93}$$

$$H_{2,0} = d_0 - d_\ell + \frac{m}{2}(c_0 - c_\ell) - \frac{(m-1)m(m+1)}{8}. \tag{6.4.94}$$

Recall that, on the one hand, the generalized Heun's equation (6.3.80) limits to the generalized hypergeometric oper (6.3.77) with coefficients $m_{0,i}$, $(m_{1,i} \mapsto m_i)$ and $(m_{\infty,i} \to \ell_i)$ if we first replace $z \mapsto qt$ and then take the limit $q \to 0$. A basis of three independent solutions of the limiting generalized hypergeometric differential equation at $t = 0$ is given by the generalized hypergeometric functions

$$\begin{aligned} v_r^{(0)}(t) &= t^{1+m_{0,r}} {}_3F_2(\alpha_{r,1}, \alpha_{r,2}, \alpha_{r,3}, \beta_{r,j}, \beta_{r,k}, t) \\ &= t^{1+m_{0,r}} \sum_{n=0}^{\infty} \frac{(\alpha_{r,1})_n (\alpha_{r,2})_n (\alpha_{r,3})_n}{(\beta_{r,j})_n (\beta_{r,k})_n} \frac{t^n}{n!}, \end{aligned} \tag{6.4.95}$$

with $j \neq r$ and $k \neq r$, and with coefficients

$$\alpha_{r,j} = \frac{1 + m + m_{0,r} - \ell_j}{2}, \quad \beta_{r,j} = 1 + \frac{m_{0,r} - m_{0,j}}{2}, \tag{6.4.96}$$

where $\ell_3 = -\ell_1 - \ell_2$ and $m_{0,3} = -m_{0,1} - m_{0,2}$.

The analytic continuation of the solutions $v_r^{(0)}(t)$ from $t = 0$ to $t = \infty$ is described by the connection matrix with coefficients

$$B_t[\ell]_{ij} = \prod_{k \neq i} \prod_{l \neq j} \frac{\Gamma[\frac{\ell_j - \ell_l}{2}]\Gamma[1 + \frac{m_{0,i} - m_{0,k}}{2}]}{\Gamma[\frac{1 - \ell_l + m + m_{0,i}}{2}]\Gamma[\frac{1 + \ell_j - m - m_{0,k}}{2}]}. \tag{6.4.97}$$

On the other hand, the generalized Heun's equation (6.3.80) limits to the generalized hypergeometric oper (6.3.77) with coefficients $(m_{0,i} \mapsto \ell_i)$, $m_{1,i}$ and $m_{\infty,i}$ if we just take the limit $q \to 0$.

The analytic continuation of its solutions $y_i^{(0)}(z)$ from $z = 0$ to $z = \infty$ is thus determined by the connection matrix

$$B_z[\ell]_{ij} = \prod_{k \neq i} \prod_{l \neq j} \frac{\Gamma[\frac{m_{\infty,j} - m_{\infty,l}}{2}]\Gamma[1 + \frac{\ell_i - \ell_k}{2}]}{\Gamma[\frac{1 - m_{\infty,l} + m_1 + \ell_i}{2}]\Gamma[\frac{1 + m_{\infty,j} - m_1 - \ell_k}{2}]}. \tag{6.4.98}$$

Similar to equation (6.4.69) the connection matrices $B_t[\ell]$ and $B_z[\ell]$ are related by

$$B_z[l]^{-1} = B_t[-l]\Big|_{m=m_1, m_0=-m_\infty}. \tag{6.4.99}$$

Going through the same steps as for the Heun's differential equation in the previous subsection, we find that the leading contribution to the monodromy matrix of the generalized Heun's equation around the punctures $z = 0$ and $z = \infty$ is computed by the expression

$$\mathbf{M}_\beta^{\text{oper},0} = M_0\, B_t[\ell]\, T\, B_z[\ell]\, M_\infty\, B_z[\ell]^{-1}\, T^{-1}\, B_t[\ell]^{-1}, \tag{6.4.100}$$

where now

$$T = \text{diag}\left(q^{\frac{-\ell_1}{2}}, q^{\frac{-\ell_2}{2}}, q^{-\frac{\ell_3}{2}}\right), \tag{6.4.101}$$

$$M_0 = \text{diag}\left(e^{2\pi i\left(1 - \frac{m_{0,1}}{2}\right)}, e^{2\pi i\left(1 - \frac{m_{0,2}}{2}\right)}, e^{2\pi i\left(1 - \frac{m_{0,3}}{2}\right)}\right), \tag{6.4.102}$$

$$M_\infty = \text{diag}\left(e^{2\pi i\left(1 + \frac{m_{\infty,1}}{2}\right)}, e^{2\pi i\left(1 + \frac{m_{\infty,2}}{2}\right)}, e^{2\pi i\left(1 + \frac{m_{\infty,3}}{2}\right)}\right). \tag{6.4.103}$$

We break the computation of $\mathbf{M}_\beta^{\text{oper},0}$ up in smaller pieces. We find that

$$\left(B_z[\ell]\, M_\infty\, B_z[\ell]^{-1}\right)_{ij} = e^{\frac{\pi i}{2}(2m_1 - 2 + \ell_i + \ell_j)}\Bigg(\delta_{i,j} + \tag{6.4.104}$$

$$2ie^{-\frac{3\pi i}{2}(m_1 - 1)} \frac{\prod_{k=1}^3 \cos\left(\frac{\pi(\ell_j + m_1 - m_{\infty,k})}{2}\right)}{\prod_{m \neq j} \sin\left(\frac{\pi(\ell_j - \ell_m)}{2}\right)} \frac{\prod_{k=1}^3 \Gamma\left[\frac{1 + \ell_j + m_1 - m_{\infty,k}}{2}\right] \prod_{l \neq i} \Gamma\left[1 + \frac{\ell_i - \ell_l}{2}\right]}{\prod_{k=1}^3 \Gamma\left[\frac{1 + \ell_i + m_1 - m_{\infty,k}}{2}\right] \prod_{m \neq j} \Gamma\left[1 + \frac{\ell_j - \ell_m}{2}\right]}\Bigg),$$

whereas

$$\left(B_t[\ell]^{-1}\, M_0\, B_t[\ell]\right)_{ij} = \left(B_z[-\ell]\, M_\infty\, B_z[-\ell]^{-1}\right)_{\{m_1\mapsto m, m_\infty\mapsto -m_0\},ij}. \tag{6.4.105}$$

Substituting these expressions into $\mathbf{M}_\beta^{\text{oper},0}$, we find that the leading order contribution to the traces is given by[6]

$$\begin{aligned}\operatorname{Tr}\mathbf{M}_\beta^{\text{oper},0} &= D_\circ + \underline{D}(\ell_1,\ell_3)\, q^{\frac{\ell_3-\ell_1}{2}} + \underline{D}(\ell_2,\ell_3)\, q^{\frac{\ell_3-\ell_2}{2}} + \underline{D}(\ell_1,\ell_2)\, q^{\frac{\ell_2-\ell_1}{2}} \\ &\quad + \underline{D}(\ell_2,\ell_1)\, q^{\frac{\ell_1-\ell_2}{2}} + \underline{D}(\ell_3,\ell_2)\, q^{\frac{\ell_2-\ell_3}{2}} + \underline{D}(\ell_3,\ell_1)\, q^{\frac{\ell_1-\ell_3}{2}}\end{aligned} \tag{6.4.106}$$

$$\begin{aligned}\operatorname{Tr}\left(\mathbf{M}_\beta^{(\text{oper},0}\right)^{-1} &= D_\circ + \overline{D}(\ell_1,\ell_3)\, q^{\frac{\ell_3-\ell_1}{2}} + \overline{D}(\ell_2,\ell_3)\, q^{\frac{\ell_3-\ell_2}{2}} + \overline{D}(\ell_1,\ell_2)\, q^{\frac{\ell_2-\ell_1}{2}} \\ &\quad + \overline{D}(\ell_2,\ell_1)\, q^{\frac{\ell_1-\ell_2}{2}} + \overline{D}(\ell_3,\ell_2)\, q^{\frac{\ell_2-\ell_3}{2}} + \overline{D}(\ell_3,\ell_1)\, q^{\frac{\ell_1-\ell_3}{2}}\end{aligned} \tag{6.4.107}$$

where

$$\underline{D}(\ell_k,\ell_l) = -4\pi^2 e^{-\pi i\left(\frac{m+m_1-2}{2}\right)}\frac{D_\star(\ell_k,\ell_l)}{D_\uparrow(\ell_k)D_\downarrow(\ell_l)}, \tag{6.4.108}$$

$$\overline{D}(\ell_k,\ell_l) = -4\pi^2 e^{\pi i\left(\frac{m+m_1-2}{2}\right)}\frac{D_\star(\ell_k,\ell_l)}{D_\uparrow(\ell_k)D_\downarrow(\ell_l)}, \tag{6.4.109}$$

and

$$\begin{aligned}D_\star(\ell_k,\ell_l) &= \Gamma\Big[1+\frac{\ell_k-\ell_l}{2}\Big]^2\Gamma\Big[\frac{\ell_k-\ell_l}{2}\Big]^2\Gamma\Big[1-\ell_l-\frac{\ell_k}{2}\Big] \\ &\quad\times\Gamma\Big[1+\ell_k+\frac{\ell_l}{2}\Big]\Gamma\Big[\ell_k+\frac{\ell_l}{2}\Big]\Gamma\Big[-\frac{\ell_k}{2}-\ell_l\Big],\end{aligned} \tag{6.4.110}$$

whereas

$$D_\uparrow(\ell_k) = \prod_{j=1}^{3}\Gamma\Big[\frac{1-m-m_{0,j}+\ell_k}{2}\Big]\Gamma\Big[\frac{1+m_1-m_{\infty,j}+\ell_k}{2}\Big] \tag{6.4.111}$$

and

$$D_\downarrow(\ell_k) = \prod_{j=1}^{3}\Gamma\Big[\frac{1+m+m_{0,j}-\ell_k}{2}\Big]\Gamma\Big[\frac{1-m_1+m_{\infty,j}-\ell_k}{2}\Big]. \tag{6.4.112}$$

We rewrite this result in terms of perturbative corrections to the effective twisted superpotential of the superconformal SU(3) theory coupled to six hypers in §6.5.2.

[6] The expression for $D_\circ$ is given by the same formulas as $N_\circ$ from the previous chapter, after identifying $e^{-\pi\ell_i}$ with L_i, M_l with $e^{2\pi I\mu_l}$, and $M_{l,j}$ with $e^{2\pi I\mu_{l,j}}$. It thus drops out in what follows, and we omit rewriting the unwieldy expression.

6.5 Generating function of opers

The locus of (framed) opers forms a complex Lagrangian subspace inside the moduli space of (framed) flat connections $\mathcal{M}_{\mathrm{dR}}^{\mathcal{C}}(C, \mathrm{SL}_K)$. Given any set of Darboux coordinates $\{x_i, y_i\}$ on $\mathcal{M}_{\mathrm{dR}}^{\mathcal{C}}(C, \mathrm{SL}_K)$ we can thus define a generating function $W^{\mathrm{oper}}(x, \epsilon)$ of the space of opers by the coupled set of equations

$$y_i = \frac{\partial W^{\mathrm{oper}}(x, \epsilon)}{\partial x_i}. \tag{6.5.1}$$

In this section we find the generating function of opers $W^{\mathrm{oper}}(x, \epsilon)$ in our two main examples, the superconformal SU(2) theory with four flavors and the superconformal SU(3) theory with six flavors, with respect to the length-twist coordinates L^i and T_i defined in Chapter 5. We do this by comparing the formulae for the oper monodromies in §6.4 to the formulae for the monodromies in terms of the length-twist coordinates L^i and T_i in §5.4.

Since the spectral twist coordinates T_i are only determined up multiplication by a simple monomial in the (exponentiated) mass parameters, due to the ambiguity in the choice of a Fenchel-Nielsen spectral network, the generating function $W^{\mathrm{oper}}(x, \epsilon)$ that we find in this section is determined up to a linear factor of the form mx.

6.5.1 Superconformal SU(2) theory with $N_f = 4$

Comparing the monodromy traces $\mathbf{M}_\alpha^{\mathrm{oper}}$ of the opers around the pants curve α to the monodromy traces $\mathbf{M}_\alpha$ in terms of the length-twist coordinates L and T, gives the identifications

$$M_l = -e^{\pi i m_l}, \tag{6.5.2}$$

$$L = -e^{\pi i \ell}. \tag{6.5.3}$$

These identifications in particular imply that the constant term $N_\circ$ in equation (5.4.26) agrees with the constant term $D_\circ$ in equation (6.4.85).

Next, we want to find the twist T as a function of the length L on the locus of opers.

Leading order contribution in q

Comparing the leading order contribution in q to the oper monodromy $\mathbf{M}_\beta^{\text{oper}}$, as computed in equation (6.4.75), to the monodromy $\mathbf{M}_\beta$ in terms of the length-twist coordinates L and T, as computed in equation (5.4.26), shows that up to leading order in q

$$T + \frac{1}{T} = \frac{D_+}{\sqrt{N}}\, q^\ell + \frac{D_-}{\sqrt{N}}\, q^{-\ell}, \tag{6.5.4}$$

where

$$N(\ell) = \frac{16 \cos\left(\frac{\pi\ell \pm \pi m_0 \pm \pi m}{2}\right) \cos\left(\frac{\pi\ell \pm \pi m_1 \pm \pi m_\infty}{2}\right)}{\sin(\pi\ell)^4}. \tag{6.5.5}$$

and $D_- = D_+|_{\ell=-\ell}$ with

$$D_+(\ell) = -4\pi^2 \frac{\Gamma[1-\ell]^2 \Gamma[-\ell]^2}{\Gamma\left[\frac{1}{2} - \frac{\ell \pm m_0 \pm m}{2}\right] \Gamma\left[\frac{1}{2} - \frac{\ell \pm m_1 \pm m_\infty}{2}\right]}. \tag{6.5.6}$$

Repeatedly using the identity

$$\cos\left(\frac{\pi x}{2}\right) \Gamma\left[\frac{1}{2} + \frac{x}{2}\right] \Gamma\left[\frac{1}{2} - \frac{x}{2}\right] = \pi, \tag{6.5.7}$$

we find that

$$\frac{D_+}{\sqrt{N}} = \sqrt{\frac{\Gamma[\frac{1}{2} + \frac{\ell \pm m_0 \pm m}{2}] \Gamma[\frac{1}{2} + \frac{\ell \pm m_1 \pm m_\infty}{2}]}{\Gamma[\frac{1}{2} - \frac{\ell \pm m_0 \pm m}{2}] \Gamma[\frac{1}{2} - \frac{\ell \pm m_1 \pm m_\infty}{2}]}} \frac{\Gamma[1-\ell]\Gamma[-\ell]}{\Gamma[1+\ell]\Gamma[\ell]}, \tag{6.5.8}$$

and hence that

$$\frac{D_-}{\sqrt{N}} = \frac{\sqrt{N}}{D_+} \tag{6.5.9}$$

This implies that equation (6.5.4) is solved by

$$T = \sqrt{\frac{\Gamma[\frac{1}{2} + \frac{\ell \pm m_0 \pm m}{2}] \Gamma[\frac{1}{2} + \frac{\ell \pm m_1 \pm m_\infty}{2}]}{\Gamma[\frac{1}{2} - \frac{\ell \pm m_0 \pm m}{2}] \Gamma[\frac{1}{2} - \frac{\ell \pm m_1 \pm m_\infty}{2}]}} \frac{\Gamma[1-\ell]\Gamma[-\ell]}{\Gamma[1+\ell]\Gamma[\ell]} q^\ell \tag{6.5.10}$$

up to leading order in q.

Classical and 1-loop contribution

Since the generating function of opers $W^{\text{oper}}(\ell, q)$ is defined by

$$\frac{1}{2}\log T = \frac{\partial W^{\text{oper}}(\ell, q)}{\partial \ell} \tag{6.5.11}$$

on the locus of opers, we find that

$$\begin{aligned}\frac{\partial W^{\text{oper}}(\ell, q)}{\partial \ell} &= \frac{\ell}{2}\log q + \frac{1}{4}\log\frac{\Gamma[\frac{1}{2}+\frac{\ell\pm m_0\pm m}{2}]}{\Gamma[\frac{1}{2}-\frac{\ell\pm m_0\pm m}{2}]} + \frac{1}{4}\log\frac{\Gamma[\frac{1}{2}+\frac{\ell\pm m_1\pm m_\infty}{2}]}{\Gamma[\frac{1}{2}-\frac{\ell\pm m_1\pm m_\infty}{2}]} \\ &\quad + \frac{1}{2}\log\frac{\Gamma[1-\ell]}{\Gamma[\ell]} + \frac{1}{2}\log\frac{\Gamma[-\ell]}{\Gamma[1+\ell]} + \mathcal{O}(q).\end{aligned} \tag{6.5.12}$$

To make contact with known formulae, we write the last equation in terms of the special function

$$\Upsilon(x) = \int_{\frac{1}{2}}^{x} du \log\frac{\Gamma(u)}{\Gamma(1-u)}, \tag{6.5.13}$$

which has the property that

$$\frac{\partial}{\partial x}\Upsilon(\beta+\gamma x) = \gamma\log\frac{\Gamma[\beta+\gamma x]}{\Gamma[1-\beta-\gamma x]} \tag{6.5.14}$$

as well as

$$\Upsilon[1-x] = \Upsilon[x]. \tag{6.5.15}$$

We thus find

$$W^{\text{oper}}(\ell, q) = W^{\text{oper}}_{\text{clas}}(\ell, \tau) + W^{\text{oper}}_{1-\text{loop}}(\ell) + \mathcal{O}(q) \tag{6.5.16}$$

with

$$W^{\text{oper}}_{\text{clas}}(\ell, \tau) = \frac{\ell^2}{4}\log q, \tag{6.5.17}$$

and

$$W^{\text{oper}}_{1-\text{loop}}(\ell) = W^{\text{oper}}_{\text{anti}-\text{hyp}}(\ell, m_0, m) + W^{\text{oper}}_{\text{vector}}(\ell) + W^{\text{oper}}_{\text{hyp}}(\ell, m_1, m_\infty) \tag{6.5.18}$$

with

$$W^{\text{oper}}_{\text{vector}}(\ell) = -\frac{1}{2}\Upsilon[-\ell] - \frac{1}{2}\Upsilon[\ell] \tag{6.5.19}$$

$$W^{\text{oper}}_{\text{anti}-\text{hyp}}(\ell, m_0, m) = \frac{1}{2}\Upsilon\Big[\frac{1}{2} + \frac{\ell\pm m_0\pm m}{2}\Big] \tag{6.5.20}$$

$$W^{\text{oper}}_{\text{hyp}}(\ell, m_1, m_\infty) = \frac{1}{2}\Upsilon\Big[\frac{1}{2} + \frac{\ell\pm m_1\pm m_\infty}{2}\Big], \tag{6.5.21}$$

up to an integration constant that is independent of ℓ.

If we identify the length coordinate ℓ with the Coulomb parameter a, and compare to the expression for the Nekrasov-Shatashvili effective twisted superpotential for the SU(2) gauge theory coupled to four hypermultiplets, given in equations (6.1.8), (6.1.9) and (6.1.13), we find that

$$W^{\text{oper}}_{\text{clas}}(a,\tau) = \widetilde{W}^{\text{eff}}_{\text{clas}}(a,\tau) \tag{6.5.22}$$

$$W^{\text{oper}}_{1-\text{loop}}(a) = \widetilde{W}^{\text{eff}}_{1-\text{loop}}(a). \tag{6.5.23}$$

In particular, $W^{\text{oper}}_{1-\text{loop}}(\ell)$ is equal to half the classical Liouville action on the nodal four-punctured sphere.

This computation is similar to and agrees with that in [65].

1-instanton correction

The 1-instanton correction $W^{\text{oper}}_1(\ell,q)$ in the generating function of opers,

$$W^{\text{oper}}(\ell,q) = W^{\text{oper}}_{\text{clas}}(\ell)\log q + W^{\text{oper}}_{1-\text{loop}}(\ell) + W^{\text{oper}}_1(\ell)\, q + \mathcal{O}(q^2), \tag{6.5.24}$$

is computed by the subleading order correction in q in equation (6.4.85) as

$$W^{\text{oper}}_1(\ell) = \frac{\ell^2}{8} + \frac{(m_0^2-m^2)(m_\infty^2-m_1^2)}{8(\ell+1)(\ell-1)} \tag{6.5.25}$$

$$\begin{aligned} &= \frac{(\ell \pm m_0 + m + 1)(\ell + m_1 \pm m_\infty + 1)}{16\ell(\ell+1)} \\ &\quad + \frac{(\ell \pm m_0 - m - 1)(\ell - m_1 \pm m_\infty - 1)}{16\ell(\ell-1)} \\ &\quad - \frac{1}{8}(m^2 - m_0^2 + m_1^2 - m_\infty^2 - 1) - \frac{1}{2}(1+m)(1+m_1), \end{aligned} \tag{6.5.26}$$

up to an integration constant that is independent of ℓ.

Comparing this to the 1-instanton contribution to the Nekrasov-Shatashvili effective twisted superpotential for the SU(2) theory with four hypermultiplets, given in equation (6.1.13), we conclude that

$$W^{\text{oper}}_1(a) = \widetilde{W}^{\text{eff}}_1(a), \tag{6.5.27}$$

after setting the integration constant. That is, $W^{\text{oper}}_1(a)$ computes the 1-instanton correction to the Nekrasov-Shatashvili effective twisted superpotential, up to a "spurious" factor that does not depend on the Coulomb parameter a.

So far we have hidden the dependence on ϵ, but let us now reintroduce this by scaling all parameters a and m_k as $a \mapsto \frac{a}{\epsilon}$ and $m_l \mapsto \frac{m_l}{\epsilon}$, respectively. It follows that the ϵ-expansion of $W_1^{\text{oper}}(a)$ is simply

$$W_1^{\text{oper}}(a,\epsilon) = \frac{1}{\epsilon^2}\frac{a^4 + (m_0^2 - m^2)(m_\infty^2 - m_1^2)}{4a^2} + \sum_{k=0}^{\infty} \epsilon^{2k} \frac{(m_0^2 - m^2)(m_\infty^2 - m_1^2)}{4a^{2k+4}}. \tag{6.5.28}$$

In particular, it does not have any odd powers in ϵ.

6.5.2 Superconformal SU(3) theory with $N_f = 6$

Comparing the monodromy traces $\mathbf{M}_\alpha^{\text{oper}}$ of the generalized Heun's opers around the pants curve α, given in equation (6.4.92), to the monodromy traces $\mathbf{M}_\alpha$ in terms of the higher length-twist coordinates L_1, L_2, T_1, T_2, given in equation (5.4.39), yields the identifications

$$L_1 = e^{\pi i \ell_1}, \quad L_2 = e^{\pi i \ell_2}. \tag{6.5.29}$$

Equating the eigenvalues of the local monodromies at the punctures, by comparing equations (5.4.28) to (6.3.60) for maximal punctures and (5.4.29) to (6.3.74) for minimal punctures, yields the identifications

$$M_{0,i} = e^{\pi i m_{0,i}}, \quad M = -e^{\pi i m}, \quad M_1 = -e^{\pi i m_1}, \quad M_{\infty,i} = e^{\pi i m_{\infty,i}}. \tag{6.5.30}$$

Next, we want to find the twists T_1 and T_2 as a function of the lengths L_1 and L_2 on the locus of generalized Heun's opers.

Leading order contribution

To leading order in q we need to equate equations (5.4.41) and (5.4.42), which capture the monodromy along the 1-cycle β on C in terms of the twist coordinates T_1 and T_2 as

$$\operatorname{Tr} \mathbf{M}_\beta = \underline{N}_\circ + \underline{N}(L_1, L_3)T_1 + \underline{N}(L_2, L_3)T_2 + \underline{N}(L_1, L_2)\frac{T_1}{T_2} + \underline{N}(L_2, L_1)\frac{T_2}{T_1} + \frac{\underline{N}(L_3, L_2)}{T_2} + \frac{\underline{N}(L_3, L_1)}{T_1}, \tag{6.5.31}$$

$$\operatorname{Tr} \mathbf{M}_\beta^{-1} = \overline{N}_\circ + \overline{N}(L_1, L_3)T_1 + \overline{N}(L_2, L_3)T_2 + \overline{N}(L_1, L_2)\frac{T_1}{T_2} + \overline{N}(L_2, L_1)\frac{T_2}{T_1} + \frac{\overline{N}(L_3, L_2)}{T_2} + \frac{\overline{N}(L_3, L_1)}{T_1}, \tag{6.5.32}$$

to equations (6.4.106) and (6.4.107), respectively, which capture the monodromy of the generalized Heun's equation to the leading order in q as

$$\begin{aligned}\operatorname{Tr}\mathbf{M}_\beta^{\text{oper},0} &= \underline{D}_\circ + \underline{D}(\ell_1,\ell_3)\,q^{\frac{\ell_3-\ell_1}{2}} + \underline{D}(\ell_2,\ell_3)\,q^{\frac{\ell_3-\ell_2}{2}} + \underline{D}(\ell_1,\ell_2)\,q^{\frac{\ell_2-\ell_1}{2}} \\ &\quad + \underline{D}(\ell_2,\ell_1)\,q^{\frac{\ell_1-\ell_2}{2}} + \underline{D}(\ell_3,\ell_2)\,q^{\frac{\ell_2-\ell_3}{2}} + \underline{D}(\ell_3,\ell_1)\,q^{\frac{\ell_1-\ell_3}{2}}\end{aligned} \tag{6.5.33}$$

$$\begin{aligned}\operatorname{Tr}\left(\mathbf{M}_\beta^{(\text{oper},0)}\right)^{-1} &= \overline{D}_\circ + \overline{D}(\ell_1,\ell_3)\,q^{\frac{\ell_3-\ell_1}{2}} + \overline{D}(\ell_2,\ell_3)\,q^{\frac{\ell_3-\ell_2}{2}} + \overline{D}(\ell_1,\ell_2)\,q^{\frac{\ell_2-\ell_1}{2}} \\ &\quad + \overline{D}(\ell_2,\ell_1)\,q^{\frac{\ell_1-\ell_2}{2}} + \overline{D}(\ell_3,\ell_2)\,q^{\frac{\ell_2-\ell_3}{2}} + \overline{D}(\ell_3,\ell_1)\,q^{\frac{\ell_1-\ell_3}{2}}\end{aligned} \tag{6.5.34}$$

and solve T_1 and T_2 as a function of L_1 and L_2.

With the identifications (6.5.29) and (6.5.30) we can check that $\underline{N}_\circ$ equals $\underline{D}_\circ$ and that $\overline{N}_\circ$ equals $\overline{D}_\circ$.

Furthermore, since

$$\frac{\overline{N}(L_k,L_l)}{\underline{N}(L_k,L_l)} = \frac{\overline{D}(\ell_k,\ell_l)}{\underline{D}(\ell_k,\ell_l)} = M_1 M_4, \tag{6.5.35}$$

it is sufficient to solve the equation

$$\operatorname{Tr}\mathbf{M}_\beta = \operatorname{Tr}\mathbf{M}_\beta^{\text{oper},0} \tag{6.5.36}$$

for T_1 and T_2.

By repeatedly using the identity (6.5.7) we can simplify the quotient

$$\frac{\underline{D}(\ell_k,\ell_l)}{\underline{N}(\ell_k,\ell_l)} = \widetilde{D}_\uparrow(\ell_k)\widetilde{D}_\downarrow(\ell_l)\widetilde{D}_*(\ell_k,\ell_l), \tag{6.5.37}$$

to a product of gamma functions.

Here,

$$\widetilde{D}_\uparrow(\ell_k) := \frac{8\pi^3 i}{D_\uparrow(\ell_k)N(\ell_k)} = \prod_{j=1}^{3}\sqrt{\frac{\Gamma\left[\frac{1+m+m_{0,j}-\ell_k}{2}\right]\Gamma\left[\frac{1-m_1+m_{\infty,j}-\ell_k)}{2}\right]}{\Gamma\left[\frac{1-m-m_{0,j}+\ell_k}{2}\right]\Gamma\left[\frac{1+m_1-m_{\infty,j}+\ell_k)}{2}\right]}}, \tag{6.5.38}$$

whereas

$$\widetilde{D}_\downarrow(\ell_k) = \frac{8\pi^3 i}{D_\downarrow(\ell_k)N_\diamond(\ell_k)} = \widetilde{D}_\uparrow(\ell_k)^{-1}. \tag{6.5.39}$$

Furthermore,

$$\widetilde{D}_\star(\ell_k,\ell_l) := \frac{D_\star(\ell_k,\ell_l)N_\star(\ell_k,\ell_l)}{16\pi^4} \tag{6.5.40}$$
$$= \frac{\Gamma\left[1+\frac{\ell_k-\ell_l}{2}\right]\Gamma\left[\frac{\ell_k-\ell_l}{2}\right]}{\Gamma\left[\frac{\ell_l-\ell_k}{2}\right]\Gamma\left[1+\frac{\ell_l-\ell_k}{2}\right]}\sqrt{\frac{\Gamma\left[1+\ell_k+\frac{\ell_l}{2}\right]\Gamma\left[\ell_k+\frac{\ell_l}{2}\right]\Gamma\left[1-\frac{\ell_k}{2}-\ell_l)\right]\Gamma\left[-\frac{\ell_k}{2}-\ell_l\right]}{\Gamma\left[-\ell_k-\frac{\ell_l}{2}\right]\Gamma\left[1-\ell_k-\frac{\ell_l}{2}\right]\Gamma\left[\frac{\ell_k}{2}+\ell_l)\right]\Gamma\left[1+\frac{\ell_k}{2}+\ell_l\right]}}.$$

It follows from the last four equations that

$$\frac{\underline{D}(\ell_k,\ell_l)}{\underline{N}(\ell_k,\ell_l)} = \frac{\underline{N}(\ell_l,\ell_k)}{\underline{D}(\ell_l,\ell_k)} \tag{6.5.41}$$

and also that

$$\frac{\underline{D}(\ell_1,\ell_3)}{\underline{N}(\ell_1,\ell_3)}\frac{\underline{N}(\ell_2,\ell_3)}{\underline{D}(\ell_2,\ell_3)} = \frac{\underline{D}(\ell_1,\ell_2)}{\underline{N}(\ell_1,\ell_2)}. \tag{6.5.42}$$

This implies that equation (6.5.36) is solved by the coupled system of equations

$$\begin{aligned} T_1 &= \frac{\underline{D}(\ell_3,\ell_1)}{\underline{N}(\ell_3,\ell_1)}q^{\frac{\ell_1-\ell_3}{2}} \\ T_2 &= \frac{\underline{D}(\ell_3,\ell_2)}{\underline{N}(\ell_3,\ell_2)}q^{\frac{\ell_2-\ell_3}{2}}. \end{aligned} \tag{6.5.43}$$

The generating function $W^{\text{oper}}(\ell_1,\ell_2)$ of the locus of generalized Heun's opers is defined as

$$\begin{aligned} \frac{1}{2}\log T_1 &= \partial_{\ell_1}W^{\text{oper}}(\ell_1,\ell_2) \\ \frac{1}{2}\log T_2 &= \partial_{\ell_2}W^{\text{oper}}(\ell_1,\ell_2), \end{aligned} \tag{6.5.44}$$

so that for instance

$$\begin{aligned} &\partial_{\ell_1}W^{\text{oper}}(\ell_1,\ell_2) = \\ &\frac{1}{4}\log\frac{\Gamma\left[1+\frac{\ell_3-\ell_1}{2}\right]^2\Gamma\left[\frac{\ell_3-\ell_1}{2}\right]^2}{\Gamma\left[\frac{\ell_1-\ell_3}{2}\right]^2\Gamma\left[1+\frac{\ell_1-\ell_3}{2}\right]^2}\frac{\Gamma\left[1+\frac{\ell_3-\ell_2}{2}\right]\Gamma\left[\frac{\ell_3-\ell_2}{2}\right]\Gamma\left[1+\frac{\ell_2-\ell_1}{2}\right]\Gamma\left[\frac{\ell_2-\ell_1}{2}\right]}{\Gamma\left[\frac{\ell_2-\ell_3}{2}\right]\Gamma\left[1+\frac{\ell_2-\ell_3}{2}\right]\Gamma\left[\frac{\ell_1-\ell_2}{2}\right]\Gamma\left[1+\frac{\ell_1-\ell_2}{2}\right]} \\ &\frac{1}{4}\sum_{j=1}^{3}\log\frac{\Gamma\left[\frac{1-m-m_{0,j}+\ell_1}{2}\right]\Gamma\left[\frac{1+m+m_{0,j}-\ell_3}{2}\right]\Gamma\left[\frac{1+m_1-m_{\infty,j}+\ell_1)}{2}\right]\Gamma\left[\frac{1-m_1+m_{\infty,j}-\ell_3)}{2}\right]}{\Gamma\left[\frac{1+m+m_{0,j}-\ell_1}{2}\right]\Gamma\left[\frac{1-m-m_{0,j}+\ell_3}{2}\right]\Gamma\left[\frac{1-m_1+m_{\infty,j}-\ell_1)}{2}\right]\Gamma\left[\frac{1+m_1-m_{\infty,j}+\ell_3)}{2}\right]}. \end{aligned} \tag{6.5.45}$$

To leading order in q, in terms of the function $\Upsilon(x)$ defined in equation (6.5.13), we thus find that

$$W^{\text{oper}}(\ell_1,\ell_2) = \frac{\ell_1^2+\ell_2^2+\ell_1\ell_2}{2}\log q + W^{\text{oper}}_{\text{anti-hyp}}(\ell_1,\ell_2,m,m_{0,1},m_{0,2}) \\ + W^{\text{oper}}_{\text{vector}}(\ell_1,\ell_2) + W^{\text{oper}}_{\text{hyp}}(\ell_1,\ell_2,m_1,m_{\infty,1},m_{\infty,2}) + O(q) \tag{6.5.46}$$

where

$$W^{\text{oper}}_{\text{vector}}(\ell_1,\ell_2) = -\frac{1}{2}\sum_{j=1}^{3}\Upsilon\left[\frac{\ell_j-\ell_{j+1}}{2}\right] - \frac{1}{2}\sum_{j=1}^{3}\Upsilon\left[\frac{\ell_{j+1}-\ell_j}{2}\right] \tag{6.5.47}$$

$$W^{\text{oper}}_{\text{anti-hyp}}(\ell_1,\ell_2,m,m_{0,1},m_{0,2}) = \frac{1}{2}\sum_{j,k=1}^{3}\Upsilon\left[\frac{1-m-m_{0,k}+\ell_j}{2}\right] \tag{6.5.48}$$

$$W^{\text{oper}}_{\text{hyp}}(\ell_1,\ell_2,m_1,m_{\infty,1},m_{\infty,2}) = \frac{1}{2}\sum_{j,k=1}^{3}\Upsilon\left[\frac{1+m_1-m_{\infty,k}+\ell_j}{2}\right], \tag{6.5.49}$$

up to an integration constant that is independent in ℓ_1 and ℓ_2.

If we identify the length coordinates ℓ_i with the Coulomb parameters a_i, the above expressions agree with the classical and 1-loop contributions to the Nekrasov-Shatashvili effective twisted superpotential $\widetilde{W}^{\text{eff}}$ for the SU(3) gauge theory coupled to six hypermultiplets. Precisely, we find the 1-loop contribution in agreement with the "Liouville scheme" regularization method as explained earlier.

Instanton contributions to W^{oper} may be obtained by computing the monodromies of the generalized Heun equation up to a higher order in q, following the strategy of §6.4.1. We leave this for future work.

6.6 WKB asymptotics

In this section we remark on the relation of our approach, exact in ϵ but expanding in q, to another approach used in the literature where one expands in ϵ instead.

Given an ϵ-oper $\nabla_\epsilon^{\text{oper}}$ there is yet another method to compute its monodromy representation, called the "exact WKB method" [83, 71, 84]. We will review this approach in §6.6.1, following [71].

In §6.6.2 we compare the monodromy representation for the oper $\nabla_\epsilon^{\text{oper}}$ obtained from the exact WKB method to that obtained from the abelianization mapping. We conclude $\nabla_\epsilon^{\text{oper}}$ is abelianized by the Borel sums (defined below) in the direction $\vartheta = \arg\epsilon$ of its WKB solutions.

As a consequence, this implies that the spectral coordinate $\mathcal{X}_\gamma(\nabla_\epsilon^{\text{oper}})$ has an asymptotic expansion in the limit $\epsilon \to 0$ given by

$$\mathcal{X}_\gamma(\nabla_\epsilon^{\text{oper}}) \sim \exp\left(\oint S_{\text{odd}}(\epsilon)dz\right), \tag{6.6.1}$$

where $S_{\text{odd}}(\epsilon)$ is a solution to the Riccati equation (6.6.7). These WKB-asymptotics relate the Nekrasov-Rosly-Shatashvili correspondence to the approach of computing the ϵ-asymptotics of the effective twisted superpotential $\widetilde{W}^{\text{eff}}(a, q, \epsilon)$ using quantum periods (pioneered in [72] for the pure $SU(2)$ gauge theory).

In §6.6.3 we conclude that while the ϵ-asymptotics of the effective twisted superpotential may be found by computing quantum periods, the analytic result is found by computing the Borel sums of the quantum periods in a critical direction ϑ_0 corresponding to a Fenchel-Nielsen network.

Whereas we restrict ourselves to $K = 2$ in this section, a similar discussion holds for higher rank.

6.6.1 Monodromy representation from exact WKB

We start off with a brief review of the exact WKB method, following [71, 84]. We rephrase some of the previously introduced notions in the language commonly used in the exact WKB literature.

Let ϵ be a small complex parameter with phase ϑ. Fix an SL_2 ϵ-oper $\nabla_\epsilon^{\text{oper}}$ on C locally given by the differential operator

$$\mathbf{D}(\epsilon) = \epsilon^2\partial_z^2 - Q(z, \epsilon), \tag{6.6.2}$$

where $Q(z, \epsilon) = \sum_{j=0}^{N} Q_j(z)\epsilon^j$ is a polynomial in ϵ with coefficients $Q_j(z)$ that are meromorphic on C, satisfying conditions outlined in [84]. The principal part $Q_0(z)$ of $Q(z, \epsilon)$ defines a meromorphic quadratic differential $\varphi_2 = Q_0(z)(dz)^2$ on C.

The zeroes and poles of φ_2 on C are called turning points and singular points, respectively. Stokes curves are paths on C emanating from the turning points such that

$$e^{-i\vartheta}\sqrt{\varphi_2}(v) \in \mathbb{R} \tag{6.6.3}$$

for every nonzero tangent vector v to the path. We orient the Stokes curves such that the real part of $e^{-i\vartheta}\int^z \sqrt{\varphi_2}$ increases along the trajectory in the positive direction. We assign signs $+$ and $-$ to the singular poles so that the trajectories with positive directions flow from $-$ to $+$.

Stokes curves oriented away from turning points are called dominant, while those oriented towards turning points are called recessive. The Stokes curves, the turning and the singular points form a graph on C which is called the Stokes graph $\mathcal{G}_\vartheta(\varphi_2)$.

The WKB ansatz for the solutions to the differential equation

$$\left(\epsilon^2\partial_z^2 - Q(z,\epsilon)\right)\psi(z) = 0 \tag{6.6.4}$$

is by definition the series

$$\psi_\pm(z) = \frac{1}{\sqrt{S_{\text{odd}}}}\exp\left(\pm\int_{z_0}^{z} S_{\text{odd}}\,dz\right), \tag{6.6.5}$$

with base-point z_0. Here

$$S_{\text{odd}} = \sum_{n=0}^{\infty}\epsilon^{2n-1}S_{2n-1} \tag{6.6.6}$$

is the odd part of the formal solution $S = \sum_{k=-1}^{\infty}\epsilon^k S_k$ of the Riccati equation

$$S'(z) + S^2 = \epsilon^{-2}Q_0(z). \tag{6.6.7}$$

Note that $S_{-1} = \sqrt{Q_0}$.

Now, recall that a formal power series $f(\epsilon) = \sum_{k=0}^{\infty} f_k\epsilon^k$ is said to be *Borel summable in the direction* ϑ if

$$f_B(y) = \sum_{k=1}^{\infty} f_k\frac{y^{k-1}}{(k-1)!} \tag{6.6.8}$$

converges in a neighbourhood of $y = 0$, can be analytically continued to some connected open Ω containing the ray $e^{i\vartheta}\mathbb{R}_{\geq 0}$, and satisfies the bound

$$|f_B(y)| \leq C_1e^{C_2|y|} \tag{6.6.9}$$

on Ω. The *Borel sum in the direction* ϑ of f is then

$$\mathcal{S}_\vartheta[f](\epsilon) = f_0 + \int_0^{e^{-i\vartheta}\infty} e^{-\frac{1}{\epsilon}y}f_B(y)dy, \tag{6.6.10}$$

and this is an analytic function of ϵ whose asymptotic expansion in a sufficiently small sector $S_\vartheta = \{\epsilon \in \mathbb{C} \mid |\arg_\epsilon| < \frac{\pi}{2}, |\epsilon| < \text{const}\}$ is given by the original formal series f.

Suppose that the differential φ_2 is generic, such that there are no saddle trajectories. Then the WKB solutions $\psi_\pm$ are Borel resummable in the direction ϑ in each connected region of $C\backslash\mathcal{G}_\vartheta(\varphi_2)$ [85]. The Borel sums of $\psi_\pm$ give analytic solutions to the differential equation (6.6.4).

Any solution $\psi_{b,\pm}^{\mathcal{R}}$ obtained upon Borel resummation in a region $\mathcal{R}$ can be analytically continued into a neighbouring region $\mathcal{R}'$. It is related to the solution $\psi_{b,\pm}^{\mathcal{R}'}$ obtained upon Borel resummation in the region $\mathcal{R}'$ by a so-called connection formula.

Say that we cross a dominant Stokes line clockwise with regard to the turning point b that it emanates from. Then the Borel sums $\psi_{b,\pm}^{\mathcal{R}}$ and $\psi_{b,\pm}^{\mathcal{R}'}$ of

$$\psi_\pm = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int_b^z S_{\text{odd}}\, dz \right), \tag{6.6.11}$$

on either side of the Stokes line are related by the transformation

$$\begin{aligned} \psi_{b,-}^{\mathcal{R}} &= \psi_{b,-}^{\mathcal{R}'} \\ \psi_{b,+}^{\mathcal{R}} &= \psi_{b,+}^{\mathcal{R}'} + i\psi_{b,-}^{\mathcal{R}'}, \end{aligned} \tag{6.6.12}$$

Indeed, $\psi_{b,+}^{\mathcal{R}}$ is dominant in this region, and hence is allowed to pick up a recessive contribution without changing the WKB asymptotics.

Say that we cross a recessive Stokes line clockwise with regard to the turning point b that it emanates from. Then the Borel sums $\psi_{b,\pm}^{\mathcal{R}}$ and $\psi_{b,\pm}^{\mathcal{R}'}$ on either side of the Stokes line are related by the transformation

$$\begin{aligned} \psi_{b,+}^{\mathcal{R}} &= \psi_{b,+}^{\mathcal{R}'} \\ \psi_{b,-}^{\mathcal{R}} &= \psi_{b,-}^{\mathcal{R}'} + i\psi_{b,+}^{\mathcal{R}'}, \end{aligned} \tag{6.6.13}$$

Indeed, in this situation $\psi_{b,-}^{\mathcal{R}}$ is dominant, and hence is allowed to pick up a recessive contribution without changing the WKB asymptotics.

With the above data we can produce the monodromy representation for the SL_2 oper $\nabla_\epsilon^{\text{oper}}$. Suppose that we want to compute the monodromy matrix along a path C_0 with begining and end point at z_0 with respect to the Borel sums of the WKB solutions $\psi_\pm$. Label the Stokes regions that the path C_0 crosses as U_l (with l increasing) and say that $\psi_\pm^l$ is the Borel sum in Stokes region U_l. Then we can determine the monodromy matrix along C_0 by computing the basis transformation that relates $\psi_\pm^{l+1}$ to $\psi_\pm^l$.

Let b be the turning point that the Stokes line emanates from. The connection formulae tell us how to relate the (local) Borel sums $\psi_{b,\pm}^{\mathcal{R}}$ in the neighbouring regions U_l across Stokes lines. The transformation is of the form

$$(\psi_{b,+}^{\mathcal{R}}, \psi_{b,-}^{\mathcal{R}}) = (\psi_{b,+}^{l+1}, \psi_{b,-}^{l+1})\, V^{l,l+1}, \tag{6.6.14}$$

where the matrix $V^{l,l+1}$ is determined by equation (6.6.12) or (6.6.13). It depends on the type of the Stokes line and the direction of crossing.

We would like to know the transformation in terms of the Borel sums $\psi_\pm^{\mathcal{R}}$, which are defined with respect to the base-point z_0. Now $\psi_\pm^{\mathcal{R}}$ differs from $\psi_{b,\pm}^{\mathcal{R}}$ by the transformation

$$(\psi_+^{\mathcal{R}}, \psi_-^{\mathcal{R}}) = (\psi_{b,+}^{\mathcal{R}}, \psi_{b,-}^{\mathcal{R}})\, D_{z_0}^b, \tag{6.6.15}$$

where $D_{z_0}^b$ is the Borel sum of the matrix

$$\begin{pmatrix} \exp\left(+\int_{z_0}^b S_{\rm odd}\, dz\right) & 0 \\ 0 & \exp\left(-\int_{z_0}^b S_{\rm odd}\, dz\right) \end{pmatrix}. \tag{6.6.16}$$

(Note that since the integrals do not depend on the position z, the Borel summed $D_{z_0}^b$ does not depend on the Stokes region.)

Hence we find that $\psi_\pm^{l+1}$ and $\psi_\pm^{\mathcal{R}}$ are related by the transformation

$$(\psi_+^{\mathcal{R}}, \psi_-^{\mathcal{R}}) = (\psi_+^{l+1}, \psi_-^{l+1})\, \widetilde{V}_{z_0}^{l,l+1}, \tag{6.6.17}$$

with connection matrix

$$\widetilde{V}_{z_0}^{l,l+1} = (D_{z_0}^b)^{-1}\, V^{l,l+1}\, D_{z_0}^b. \tag{6.6.18}$$

The monodromy matrix along the path C_0 is then found by multiplying all connection matrices $\widetilde{V}_{z_0}^{l,l+1}$ along it.

If the differential equation (6.6.4) is Fuchsian, the resulting monodromy representation may be expressed in terms of the characteristic exponents at the regular singular points and the Borel sums (in the direction ϑ) of the contour integrals

$$\exp\left(V_\gamma\right) := \exp\left(\oint_\gamma S_{\rm odd}\, dz\right) \tag{6.6.19}$$

along 1-cycles γ on the covering Σ. The exponent V_γ is called the ***Voros symbol*** for the cycle γ.

So far we have kept the phase $\vartheta = \arg \epsilon$ fixed. The connection formulae (6.6.14) describe the analytic continuation of the Borel sums $\psi_{b,\pm}$ of the WKB solutions in the z-plane. Let us now consider what happens if we vary the phase ϑ. We make the dependence on ϑ explicit in the notation by writing $\psi_{b,\pm}^{\vartheta}$.

Suppose a Stokes line crosses a point $z \in C$ at a critical phase ϑ_0. Then the Borel sums $\psi_{b,\pm}^{\vartheta_0-\delta}$ and $\psi_{b,\pm}^{\vartheta_0+\delta}$ are not equal, but related by connection formulae similar to (6.6.14) in a neighbourhood of the point z, for small enough δ. This is the so-called "Stokes phenomenon". The Borel sums $\psi^{\vartheta}_{b,\pm}$ do have the same asymptotic expansion, given by $\psi_{b,\pm}$, in the whole sector $\{\epsilon \in \mathbb{C} \mid |\theta - \vartheta_0| < \pi/2, |\epsilon| \ll 1\}$.

The Borel sums of the Voros symbols V_γ are affected by the Stokes phenomenon as well. The Voros symbol V_γ is Borel summable (in the direction ϑ) if the cycle γ does not intersect with a saddle trajectory of the Stokes graph $\mathcal{G}_\vartheta(\varphi_2)$. This Borel summability is broken if a saddle trajectory appears, say at the phase ϑ_0. The Borel sums of the Voros symbol V_γ in the directions $\vartheta_0 \pm \delta$ are related by "jump formulae", for sufficiently small δ (see for instance [84] for explicit expressions). Of course, both Borel sums do have the same asymptotic expansion in the limit $|\epsilon| \to 0$, given by the Voros symbol V_γ itself.

6.6.2 Relating exact WKB to abelianization

The above procedure of finding the monodromy representation using the exact WKB method is very similar to finding the monodromy representation of a flat SL_2 connection ∇ using the abelianization mapping. In fact, in this section we show that the resulting monodromy representations are equivalent on the locus of opers.[7]

Fix an ϵ-oper $\nabla^{\text{oper}}_\epsilon$ locally given by (6.6.2), with fixed phase $\vartheta = \arg \epsilon$. It is easy to see that the corresponding Stokes graph $\mathcal{G}_\vartheta(\varphi_2)$ and spectral network $\mathcal{W}_\vartheta(\varphi_2)$ are equivalent notions. Indeed, the Stokes curves of the Stokes graph $\mathcal{G}_\vartheta(\varphi_2)$ have the same definition as the walls of the spectral network $\mathcal{W}_\vartheta(\varphi_2)$. Furthermore, the labels of the walls determine the orientations of the Stokes curves and vice versa. In particular, notice that the labels of the walls in the spectral network are chosen in such a way that the recessive (or small) section s_i of the flat connection ∇^{oper} stays invariant across a wall.

Let us remind ourselves how we compute the monodromy along the path C_0 for any flat SL_2 connection ∇ using the abelianization method with respect to the spectral network $\mathcal{W}_\vartheta(\varphi_2)$ (see §5.3, or for some more detail §6 and §7 of [4]). Suppose the flat SL_2 connection ∇ is abelianized with respect to $\mathcal{W}_\vartheta(\varphi_2)$ by the equivariant connection ∇^{ab}. To find the monodromy of ∇ along C_0 we cut the path C_0 into a collection of smaller paths $\wp$ that do not cross any walls nor branch-cuts of

[7] While finalizing this work we heard about an alternative argument from A. Neitzke, proved by N. Nikolaev [86].

$\mathcal{W}_\vartheta(\varphi_2)$. The monodromy along C_0 is then given by the product of abelian parallel transport matrices $D_\wp$ over all paths $\wp$, where we splice in a branch cut matrix when crossing a branch cut and a unipotent matrix $\mathcal{S}_w$ when crossing a wall.

As is shown in [4], and reviewed in §5.3, the abelianization mapping is unique for any $K = 2$ Fock-Goncharov or (resolution of a) $K = 2$ Fenchel-Nielsen spectral network. Furthermore, the unipotent matrices $\mathcal{S}_w$ are of a rather special form. They have 1's on the diagonal, and the only nonzero off-diagonal component of $\mathcal{S}_w$ can be written as the abelian parallel transport of ∇^{ab} along an auxiliary (or detour) path that starts at a lift of the basepoint w, follows the wall in the opposite orientation, circles around the branch point b, and returns to the other lift of the basepoint w.

More precisely, the previous description is valid if we choose the branch cut matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.6.20}$$

as is conventional. If instead we choose the branch cut matrix to be the quasi-permutation matrix

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{6.6.21}$$

the nonzero off-diagonal component of $\mathcal{S}_w$ is multiplied by an additional factor $\pm i$.

Let us now decompose the connection matrix $\widetilde{V}_{z_0}^{l,l+1}$ from equation (6.6.18) as

$$\widetilde{V}_{z_0}^{l,l+1} = (D_{z_0}^w)^{-1}\, \mathcal{S}_w\, D_{z_0}^w. \tag{6.6.22}$$

Then the matrix

$$\mathcal{S}_w = (D_w^b)^{-1}\, V^{l,l+1}\, D_w^b. \tag{6.6.23}$$

has as only nonzero off-diagonal component the Borel sum of

$$\exp\left(\pm \int_w^b S_{\text{odd}}\, dz\right) (\pm i) \exp\left(\mp \int_b^w S_{\text{odd}}\, dz\right), \tag{6.6.24}$$

where two signs in the exponentials are opposite and depend on the orientation of the Stokes curve, whereas the sign in front of the factor i depends on the direction of crossing the Stokes curve.

It follows that the monodromy representation for the oper $\nabla_\epsilon^{\text{oper}}$ obtained using the exact WKB method can be brought in the form of a monodromy representation obtained using the abelianization mapping. In fact, it shows that $\nabla_\epsilon^{\text{oper}}$ is abelianized in each region l by the Borel sums $\psi_\pm^{\mathcal{R}}$ (in the direction ϑ) of the WKB solutions $\psi_\pm$ of the differential equation (6.6.4).

Indeed, with respect to this basis the abelian parallel transport matrix is the Borel sum of the matrix

$$D_\wp = \begin{pmatrix} \exp\left(+\int_\wp S_{\text{odd}}\, dz\right) & 0 \\ 0 & \exp\left(-\int_\wp S_{\text{odd}}\, dz\right) \end{pmatrix}, \tag{6.6.25}$$

while the branch cut matrix is of the non-conventional form (6.6.21), due to the square-root $\sqrt{S_{\text{odd}}}$ in the denominator of the definition of $\psi_\pm$, and the nonzero off-diagonal component of the unipotent matrix $\mathcal{S}_w$ is the Borel sum of the expression (6.6.24).

Thus we conclude the monodromies obtained from the exact WKB method are equivalent to the monodromies obtained through the abelianization mapping on the locus of ϵ-opers.

In particular, this relation shows that the spectral coordinates $\log \mathcal{X}_\gamma(\nabla_\epsilon^{\text{oper}})$ are equal to the Borel sums (in the direction ϑ) of the Voros periods V_γ. As an immediate consequence it follows that the spectral coordinates $\mathcal{X}_\gamma(\nabla_{\epsilon'}^{\text{oper}})$ have the WKB asymptotics

$$\mathcal{X}_\gamma(\nabla_{\epsilon'}^{\text{oper}}) \sim \exp\left(\oint_\gamma S_{\text{odd}}(\epsilon')\, dz\right) \tag{6.6.26}$$

in the limit $\epsilon' \to 0$ with $|\arg \epsilon' - \vartheta| < \pi/2$. (This was already shown by a different argument in [1].) We emphasize that while the spectral coordinates $\mathcal{X}_\gamma$ are thus rather sensitive to the choice of the phase ϑ, their WKB asymptotics are not. For example, the spectral coordinates for the two resolutions of a Fenchel-Nielsen network generated by a Strebel differential differ, but their WKB asymptotics in the limit $\epsilon \to 0$ agree.

The right-hand side of equation (6.6.26) is also known as a quantum period

$$\Pi_\gamma(\epsilon) = \exp\left(\oint_\gamma S_{\text{odd}}(\epsilon)\, dz\right). \tag{6.6.27}$$

6.6.3 Quantum periods and non-perturbative corrections

In the main part of this chapter we computed the generating function of the space of ϵ-opers on the four-punctured sphere $\mathbb{P}^1_{0,q,1,\infty}$, with respect to the complexified length-twist coordinates (ℓ, τ), by computing the monodromy representations and comparing with the formula in terms of the coordinates. The complexified length-twist coordinates were realized as spectral coordinates

$$L = -\exp(\pi i \ell) \qquad (6.6.28)$$
$$T = -\exp(2\tau)$$

by abelianizing with respect to a Fenchel-Nielsen network $\mathcal{W}$. The discussion in §6.6.2 indicates an alternative way of computing this generating function.

Fix the phase ϑ_0 and the mass parameters m_l such that $e^{-2i\vartheta_0}\varphi_2$ is a Strebel differential, generating a Fenchel-Nielsen network isotopic to $\mathcal{W}$. This is certainly possible in the weakly coupling limit $q \to 0$ where

$$u = \frac{a_0^2}{2} + \mathcal{O}(q), \qquad (6.6.29)$$

with $a_0 = \oint_A \sqrt{\varphi_2}$.

According to the discussion in §6.6.2, the length-twist coordinates $(\ell, \tau^{\pm})$ restricted to the space of ϵ-opers may be computed as the Borel sums of the Voros symbols V_A and V_B, respectively, in the direction $\arg \epsilon = \vartheta_0 \pm \delta$ for sufficiently small δ. Let us denote these Borel sums as $V_A^{\pm}$ and $V_B^{\pm}$, respectively.

The generating function of ϵ-opers is found by inverting the relation

$$\frac{\ell}{2} = V_A^{+}(H, q, \epsilon) = V_A^{-}(H, q, \epsilon), \qquad (6.6.30)$$

where H denotes the accessory parameter, and substituting the result into the expression for $V_B^{\pm}(H, q, \epsilon)$, to find

$$\frac{\partial W^{\text{oper}}(\ell, q, \epsilon)}{\partial \ell} = \frac{V_B^{+}(\ell, q, \epsilon)}{4} + \frac{V_B^{-}(\ell, q, \epsilon)}{4}. \qquad (6.6.31)$$

By construction, this generating function agrees analytically with the generating function of opers as computed in §6.5 (after reintroducing the ϵ-dependence in the latter).

The ϵ-asymptotics of the generating function $W^{\text{oper}}(\ell, q, \epsilon)$ are simply obtained by computing the Voros symbols V_A and V_B in an ϵ-expansion as quantum periods. This relates the Nekrasov-Rosly-Shatashvili correspondence to the approach of computing the ϵ-asymptotics of the NS superpotential $\widetilde{W}^{\text{eff}}(a, q, \epsilon)$ using quantum periods [72].

The exact NS superpotential $\widetilde{W}^{\text{eff}}(a, q, \epsilon)$ is found by Borel resumming its asymptotic expansion in a critical direction ϑ_0 corresponding to a Fenchel-Nielsen network. In particular, we find that in this critical direction the NS superpotential does ***not*** acquire any non-perturbative corrections.

Chapter 7

Uniqueness of abelianization for Grassmannian networks

In Chapter 5 we showed that the higher length-twist networks $\mathcal{W}$ induce unique abelianizations of flat connections ∇ once equipped with an appropriate notion of framing at the punctures and/or annuli. In this chapter we consider another class of networks $\mathcal{W}$ equipped with appropriate framing data and consider whether there is a unique $\mathcal{W}$-abelianization compatible with this data.

The relevant moduli spaces will be essentially Grassmannians, and the corresponding spectral coordinates are expected to be cluster coordinates in some sense — we leave a detailed study of this structure to future study and here focus on the more basic problem of existence and uniqueness of abelianization in this setting.

We will be interested in using a spectral network $\mathcal{W}$ to study certain meromorphic flat connections on $\mathbb{P}^1$ with an irregular singularity at infinity, and no singularities elsewhere. In particular, the monodromy is trivial. This means that only the Stokes geometry remains, and the connections are characterized by their Stokes data. The main question we would like to answer, motivated below, is

Given an abelianization "at infinity", to what extent does it extend uniquely, up to equivalence, to an abelianization over the whole surface?

By "at infinity" here we simply mean that the isomorphism ι is specified on the componenets of $C \setminus \mathcal{W}$ adjacent to ∞, but not given to us completely.

In this chapter we study this problem and give some partial results that the author has obtained thus far, focusing on examples. We hope to develop these results further and continue studying the many questions which follow.

7.1 Motivation and setup

It is important to establish the basic properties of various classes of spectral networks such as their geometry, the properties their of abelianizations, etc. On the other hand, the particular class of networks we will study here appears "in real life" in a conjecture of A. Neitzke describing an application of the GMN integral operators [1] to compute certain invariants describing the asymptotics of harmonic maps

$$g : \mathbb{C} \to \mathrm{SO}(3)\backslash\mathrm{SL}(3, \mathbb{R}) \tag{7.1.1}$$

It is thus important to ensure that the spectral coordinates always exist are well-defined in this problem.

Furthermore, it turns out the moduli spaces in this context are essentially Grassmannians $\mathrm{Gr}(3, n+3)$. The spectral coordinates are expected to be cluster variables in the corresponding cluster structure on the homogeneous coordinate ring, with topological changes in the network corresponding to mutations. We expect the abelianization should be unique, at least when $n \leq 5$, which corresponds precisely to the Grassmannians whose known cluster structure [87] contains finitely many clusters.

7.1.1 Geometric setup

Fix some integer $n \geq 2$, and let $D_i = (n-3+i) \cdot \infty$, and let $\mathbf{B} = \oplus_{i=2}^{3} H^0(C, K_C^{\otimes i}(D_i))$ denote the corresponding Hitchin base with "wild" singularities at ∞ and no singularities elsewhere.

Fix a point $\varphi = (\varphi_2, \varphi_3) \in \mathbf{B}$, and write $\varphi_i = P_i(z) dz^{\otimes i}$ for the i-differential so that P_2, P_3 are polynomials with $\deg P_2(z) < n$ and $\deg P_3(z) = n$. We will sometimes set $\varphi_2 = 0$ (this is motivated by the desire for the corresponding wild Higgs bundle moduli space to be hyperkähler, see [88]). From this data and for each choice of angle ϑ, we can produce a WKB spectral network $\mathcal{W}$, which we will call the "Grassmannian networks".

We will study meromorphic connections on the trivial rank 3 bundle $\mathcal{E}$ over $\mathbb{P}^1$, with connection matrix $A(z)$, holomorphic everywhere except ∞. We assume that the characteristic polynomial of the leading term $A_0(z)$ in the expansion around ∞ is

$$\lambda^3 + z^n = 0. \tag{7.1.2}$$

Let λ_i denote the eigenvalues of $A_0(z)$. It will be useful to define

Definition. Let ∇ be a meromorphic connection as above. The ***Stokes rays of type*** ij of the connection are given by

$$\int^z \lambda_i - \lambda_j > 0, \tag{7.1.3}$$

and the ***anti-Stokes rays of type*** ij are given by

$$e^{-i\pi/2} \int^z \lambda_i - \lambda_j > 0. \tag{7.1.4}$$

An example of the kind of network we will study is Figure 7.1 below. In this figure Stokes and anti-Stokes rays are the asymptotes of the walls going out to infinity. They divide the plane into $2(n+3)$ equal sectors.

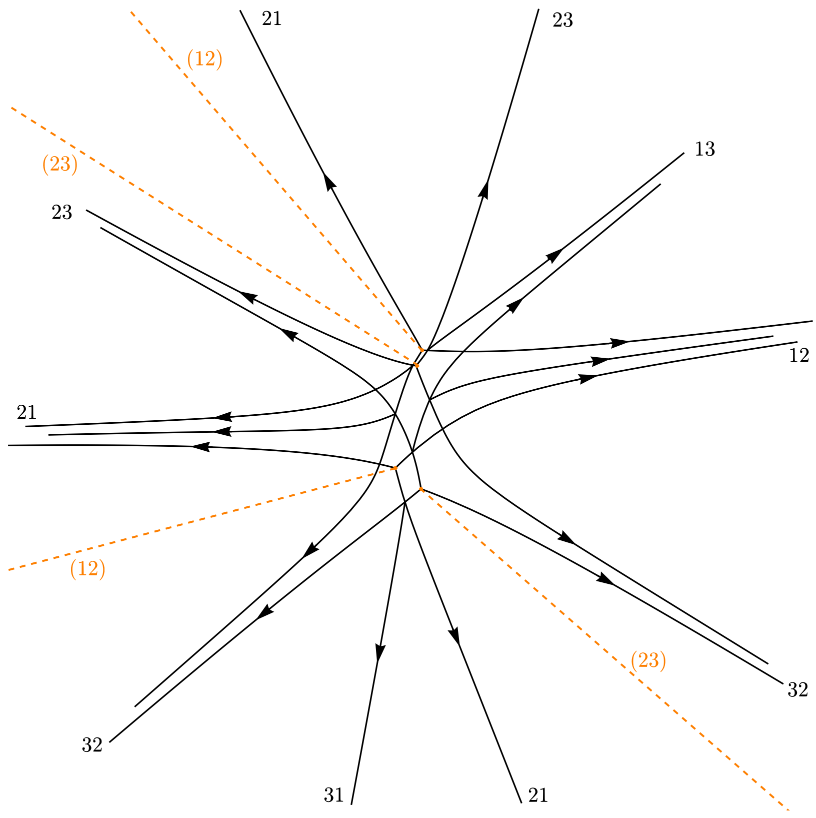


Figure 7.1: A network with $n = 3$, with walls asymptoting to (anti)-Stokes rays.

We will use $\mathcal{W}$ to study connections ∇ on $\mathbb{P}^1$ with irregular singularity at infinity, and no singularities elsewhere. Furthermore, we will fix the (anti-)Stokes ray structure up to rotation, but not the Stokes data itself. In particular, we will denote by $\mathcal{A}_{3,n+3}$ the space of all connections with the characteristic polynomial (7.1.2) and whose Stokes rays are of the same type as the network. The detailed behaviour of the network is not easy to predict, but the asymptotic structure is quite simple: if n is the degree of the polynomial, we have $n+3$ Stokes and anti-Stokes rays each, and all nondegenerate walls of the network asymptote to these rays, up to rotation. We will call the collection of walls asymptoting to a given ray a ***cable***.

7.1.2 Flags and framings

Generically, away from the Stokes and anti-Stokes lines, there is a canonical ordering of the eigenvalues by "dominance":

$$\mathrm{Re}\lambda_1 \gg \mathrm{Re}\lambda_2 \gg \mathrm{Re}\lambda_3 \tag{7.1.5}$$

and thus eigenlines, yielding a flag $F^{\mathcal{R}}$ associated to each sector. Intuitively, the flag orders the (generically distinct) eigenlines according to the decay rate of the corresponding formal solutions. A ***framing*** of ∇ is simply a flag in each sector, so that we have a *canonical* framing associated to each $\nabla \in \mathcal{M}_{\mathrm{dR}}$. Thus, we don't need to consider $\mathcal{W}$-framed connections at all.

These eigenlines form a line decomposition $\mathcal{E} = \bigoplus_{i=1}^{3} L_i$. Furthermore, when crossing a Stokes line only two lines are interchanged, meaning if we choose arbitrary elements (s_1, s_2, s_3) in adjacent regions $\mathcal{R}_1, \mathcal{R}_2$ separated by a wall of type (ij), the transition matrices are of the "abelianization form" (5.3.1). In particular, given two bases $(s_1^{\mathcal{R}}, s_2^{\mathcal{R}}, s_3^{\mathcal{R}})$, $(s_1^{\mathcal{R}'}, s_2^{\mathcal{R}'}, s_3^{\mathcal{R}'})$ associated to adjacent regions $\mathcal{R}, \mathcal{R}'$,

$$\mathcal{S}_c = \mathrm{diag} + \mathrm{const} * E_{ji} = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \tag{7.1.6}$$

This is the structure we want to extend:

We will call such a structure (that is, a flag of line decompositions in each of the $2(n+3)$ Stokes regions satisfying the transition property (7.1.6)) an ***abelianization of*** ∇ ***near infinity***.

7.1.3 Grassmannians and cluster structures

It turns out that the moduli spaces of the connections we are studying are essentially Grassmannians (up to a certain quotient). Roughly speaking, the canonical flags produce $n+3$ distinct lines in the 3-dimensional $\mathcal{E}$, which can dually be identified as 3-planes in $n+3$ space. We avoid giving the details here for brevity, since our concern is primarily in motivating interest in these networks and making the link to cluster algebras.

Recall the ***Grassmannian of k-planes in*** $\mathbb{C}^n$ $\mathrm{Gr}(k,n)$ is the set of k-planes up to change of basis. It is well-known that $\mathrm{Gr}(k,n)$ is a smooth, compact, projective variety. Recall the *Plücker embedding*

$$\mathrm{Gr}(k,n) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{C}^n) \tag{7.1.7}$$

is given by sending $(v_1, \ldots, v_k) \mapsto v_1 \wedge \ldots \wedge v_k$. The ***Plücker coordinates*** x_I associated to an ordered k-tuple $I = \{i_1, \ldots, i_k\}$ is the homogeneous coordinate in $\wedge^k \mathbb{C}$ associated to I.

The homogeneous coordinate rings $\mathbb{C}[\mathrm{Gr}(k,n)]$ for are known to possess cluster structures [87]. The structure is simplest in the case of $\mathrm{Gr}(2,n)$. In that case one draws an n-gon labelled clockwise from 1 to n. To each diagonal one associates the Plücker coordinate x_{ij}. The clusters are the Plücker coordinates associated to the edges of a triangulation, and the Plücker relations correspond to flips of a triangulation. This specifies the cluster structure on $\mathbb{C}[\mathrm{Gr}(2,n)]$. For $k = 3$, it is more complicated in general, but at least for $\mathrm{Gr}(3,5)$ the $k = 2$ case suffices since $\mathrm{Gr}(k,n) \simeq \mathrm{Gr}(n-k,n)$.

We will in fact focus our attention exclusively on the case $k = 3$. The reason for this is that, apart from the relatively simple case of $k = 2$, a theorem of Scott [87] shows that the *only* Grassmannians (in the non-redundant range $2 < k \leq n$) whose cluster algebra is of *finite type*[1] are precisely the cases $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, $\mathrm{Gr}(3,8)$. There is nothing stopping us from considering other values of k and n too, but for these reasons we focus our attention on rank 3 connections here.

Each network $\mathcal{W}$ produces a collection of spectral coordinates $\mathcal{X}_\gamma \in \mathbb{C}[\mathrm{Gr}(3, n+3]$. It is known in some examples that these coordinates always coincide exactly with a cluster from the known cluster structure. Thus, one expects there should be a canonical way to identify isotopy classes of spectral networks with the clusters of the Grassmannians. We do not investigate the cluster structure here explicitly, but it is one of our main motivations for considering this problem.

7.2 Abelianization

Recall from Chapter 4 the notion of $\mathcal{W}$-pairs and abelianization. All flat bundles below are assumed to be SL_K.

A $\mathcal{W}$-pair $(E, \nabla, \iota, \mathcal{L}', \nabla^{\mathrm{ab}})$ for a network $\mathcal{W}$ subordinate to the branched covering $\pi : \Sigma \to C$ is the collection of data:

[1] that is, possessing finitely many clusters

(i) A flat rank K bundle (E, ∇) over C

(ii) A flat rank 1 bundle $(\mathcal{L}', \nabla^{\mathrm{ab}})$ over Σ'

(iii) An isomorphism $\iota : E|_{C' \setminus \mathcal{W}} \to \pi_* \mathcal{L}'|_{C' \setminus \mathcal{W}}$

such that

(a) the isomorphism ι takes (the restrictions of) ∇ to $\pi_* \nabla^{\mathrm{ab}}$

(b) at each single wall $w \subset \mathcal{W}$, ι jumps by a map $\mathcal{S}_w = 1 + e_w \in \mathrm{End}(\pi'_* \mathcal{L}'|_{C' \setminus \mathcal{W}})$ where $e_w : \mathcal{L}'_i \to \mathcal{L}'_j$ if w carries the label ij. At each double wall $w'w$ ι jumps by a map $\mathcal{S}_{w'} \mathcal{S}_w$, with the ordering determined by the resolution.

Then an abelianization of some $(\mathcal{E}, \nabla)$ is simply the remaining data $(\iota, \mathcal{L}', \nabla^{\mathrm{ab}})$ needed to form a $\mathcal{W}$-pair.

Given some ∇, we would like to abelianize it. As usual, this can depend on various data attached to punctures/annuli, but in our case there is a particular canonical choice of this "framing" data given by the eigenflags ordered by the dominance of the eigenvalue. The appropriate notion of framing is given here by the eigenflags attached to each asymptotic region. Equipped with this data, we can ask the question,

Given an abelianization "at infinity", to what extent does it extend uniquely, up to equivalence, to an abelianization over the whole surface?

To answer this question we should think about what it boils down to. The most nontrivial part of an abelianization is the isomorphism ι. One may think of it as a collection of local trivializations of $\mathcal{E}$ given over each component $\mathcal{R}$ of the complement of the network. This amounts to a collection of K sections of $\mathcal{E}$ given over each component. Condition (b) then tells us what the change of basis matrix should be between frames over cells with a common wall as a boundary. Since in our case the flat sections are globally defined, we have a global trivialization and we can identify the fibres of $\mathcal{E}$ with a single vector *space* $\mathcal{E}$ (abusing notation) and have no bundles etc. to worry about. So we can work concretely by thinking of abelianization as a collection of bases $\{s_1^{\mathcal{R}}, s_2^{\mathcal{R}}, s_3^{\mathcal{R}}\}$ for $\mathcal{E}$, one assigned to each $\mathcal{R}$, with this transition property. Then the question is about producing unique bases satisfying the abelianization condition on the whole of $C \setminus \mathcal{W}$.

At first glance, it does not look like we are done yet. We have produced ι, and have $(\mathcal{E}, \nabla)$, but need to find $(\mathcal{L}', \nabla^{\mathrm{ab}})$ to complete the $\mathcal{W}$-pair. However, we can take $\mathcal{L}'$ to be the trivial bundle (as it must be trivializable) and use ι to define $\nabla^{ab} := \iota_* \nabla$ which is well-defined away from the (preimages of the) walls. Then by Proposition 4.2.1, ∇^{ab} extends uniquely across the walls, and $(\mathcal{E}, \nabla, \iota, \mathcal{L}', \nabla^{ab})$ is the unique $\mathcal{W}$-pair as desired.

For the purpose of stating our solution, we will henceforth say ∇ is ***generic*** if no two flags associated to adjacent regions near infinity are equal — that is, if no cable matrix is strictly diagonal.

With this in mind, we provide a solution to the above question first in some examples for the case of $n = 2, 3$, corresponding to $\mathrm{Gr}(3,5)$ and $\mathrm{Gr}(3,6)$, below, before turning to the question of how to generalize these somewhat ad-hoc methods to the whole class of networks of interest.

7.3 Particular examples

7.3.1 Gr(3,5) network

The simplest of all the networks involved is the one related to $\mathrm{Gr}(3,5)$:

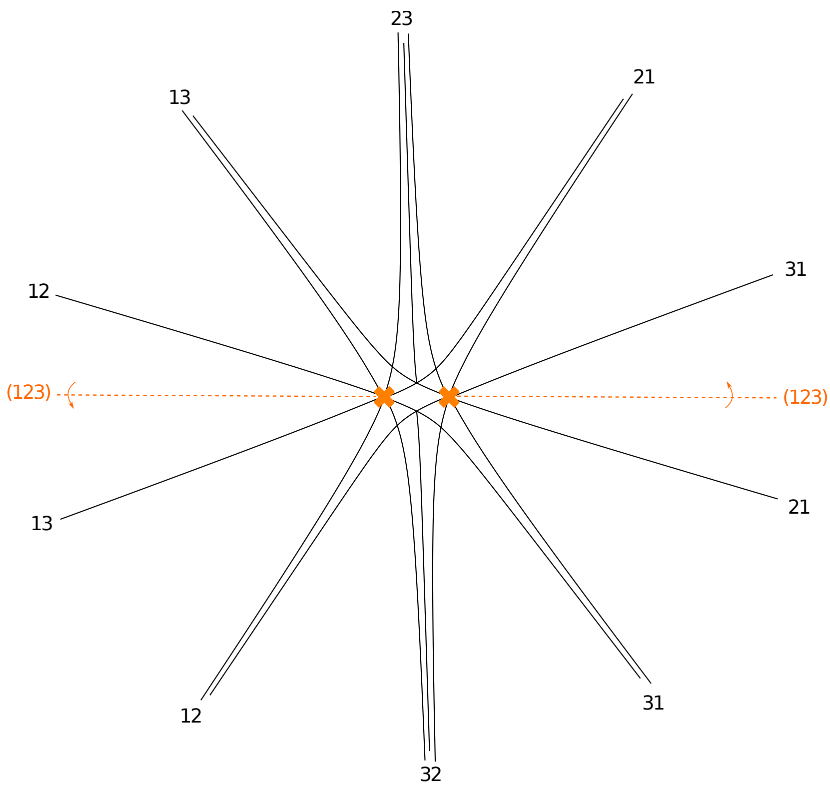


Figure 7.2: An $n = 2$ network.

Our problem is to determine whether the abelianization at infinity arising from the asymptotics of the connection is sufficient to determine a bona fide $\mathcal{W}$-abelianization. Given a generic connection ∇, a $\mathcal{W}$–abelianization at infinity amounts to the following. On each of the ten "big" domains $\mathcal{R}$, we have a collection of bases $(s_1^{\mathcal{R}}, s_2^{\mathcal{R}}, s_3^{\mathcal{R}})$ of the vector space $\mathcal{E}$ of global flat sections of ∇ and the matrix that goes $(s_1^{\mathcal{R}'}, s_2^{\mathcal{R}'}, s_3^{\mathcal{R}'}) = (s_1^{\mathcal{R}}, s_2^{\mathcal{R}}, s_3^{\mathcal{R}}) \cdot \mathcal{S}_c$ between adjacent domains via a cable c should be of the form (say, for a cable of type $(ij) = (31)$)

$$\mathcal{S}_c = \text{diag} + \text{const} * E_{ji} = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \tag{7.3.1}$$

where E_{ij} is the elementary matrix with 1 in the (i, j)th position. We may rescale any of the bases without changing the $\mathcal{W}$–abelianization up to equivalence. So for any wall (again say of type (31)), one can bring it into the "nice" form

$$\mathcal{S}_c = \mathbf{1} + s_c \cdot E_{ji} = \begin{pmatrix} 1 & 0 & s_c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.3.2}$$

To extend this to a true $\mathcal{W}$–abelianization, we need a basis on all of the domains, with each jump $\mathcal{S}_w$ at the walls (not just the cables) with the above property. Let us consider here the uniqueness of such an extension.

Proposition 7.3.1. *Let $\mathcal{W}$ be the network in Figure 7.2, and let $\nabla \in \mathcal{A}_{2,5}$ be generic. Given a $\mathcal{W}$–abelianization at infinity of ∇, there is a unique $\mathcal{W}$-abelianization which restricts to it.*

Proof. Suppose there is an abelianization, so that we are given bases on all of the domains, each basis depicted by a dot in Figure 7.3. Consider the one in the center, $(s_1^{\mathcal{R}}, s_2^{\mathcal{R}}, s_3^{\mathcal{R}})$, where the grey dot is.

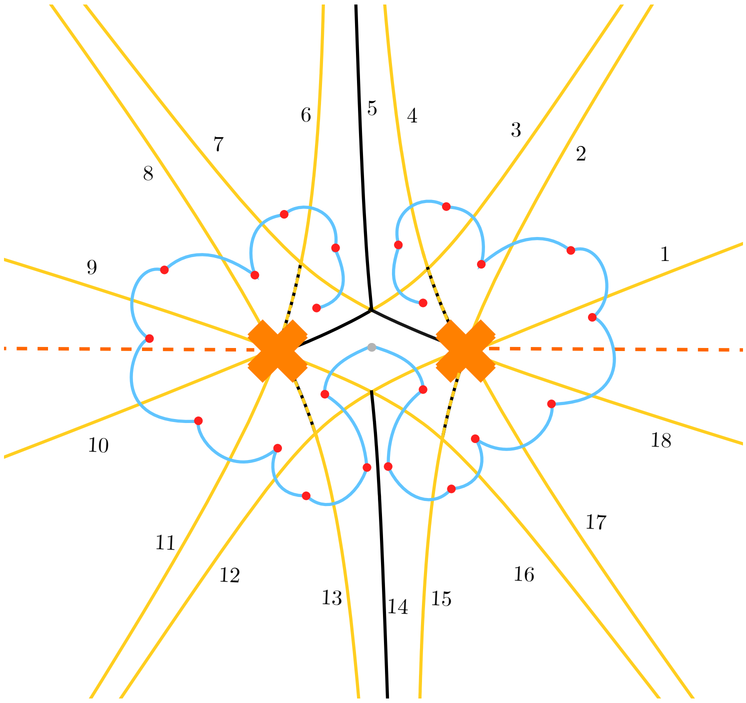


Figure 7.3: Making as many walls as possible nice.

Follow the blue paths in either direction, each time entering a new domain and rescaling the basis in this domain to trivialize (put into the nice form) the wall just crossed. The walls which are now in the “nice” form are shown in yellow. Now consider the joints at which no birthing takes place. We know 3 of of the 4 walls are nice, but it follows from triviality around a loop that a) the fourth wall is also nice (shown in dotted-yellow), and b) the incoming and outgoing walls of the same type actually have the same S-matrix.

So now all yellow and dotted-yellow walls are nice, and the distinct walls we wish to solve for with are actually those depicted in colour (which are nice), plus a, b, c, d (which are not necessarily nice) in Figure 7.4:

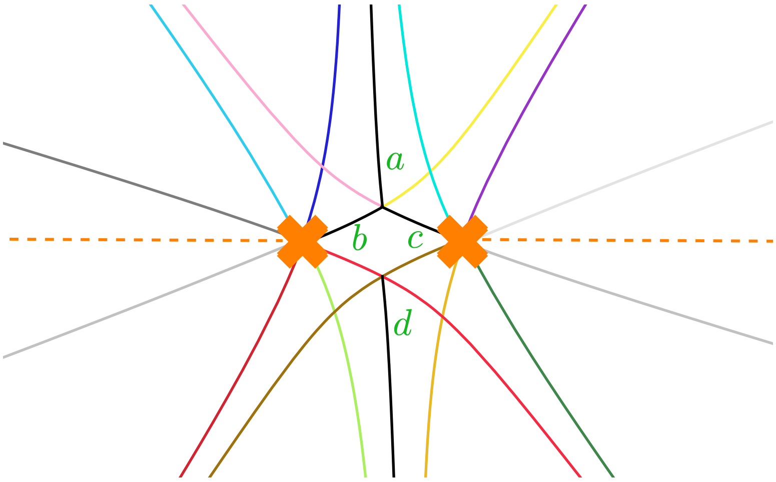


Figure 7.4: All walls are nice except those in black.

(the single-wall-cables on the sides were part of the initial data, so we don't need to solve for them).

Now we can impose the condition that loops not encircling the origin should give the identity. First we check triviality around each branch point. Since every emanating wall except b, c are nice, they just have one nontrivial entry, and the condition forces each corresponding s_w to be certain very simple ratios of $s_{1,9,10,18}$ corresponding to the black walls, uniquely. The result for b and c is

$$\mathcal{S}_b = \begin{pmatrix} -\frac{s_9}{s_{10}} & \frac{1}{s_{10}} & 0 \\ 0 & \frac{1}{s_9} & 0 \\ 0 & 0 & -s_{10} \end{pmatrix}, \qquad \mathcal{S}_c = \begin{pmatrix} -\frac{s_1}{s_{18}} & 0 & 0 \\ 0 & -s_{18} & 0 \\ \frac{1}{s_{18}} & 0 & \frac{1}{s_1} \end{pmatrix}$$

Triviality at the lower joint implies $\mathcal{S}_d$ is nice and its nontrivial entry is the product $s_d = s_{14} = -s_{12}s_{16} = -(s_{10}s_{18})^{-1}$.

Finally, we are left with the upper joint, which should satisfy

$$\mathcal{S}_c\mathcal{S}_3\mathcal{S}_a\mathcal{S}_7\mathcal{S}_b = \mathbf{1}_3 \tag{7.3.3}$$

which forces

$$\mathcal{S}_a = \begin{pmatrix} \frac{s_{10}s_{18}}{s_1 s_9} & 0 & 0 \\ 0 & -\frac{s_9}{s_{18}} & 0 \\ 0 & 1 & -\frac{s_1}{s_{10}} \end{pmatrix}$$

In particular, there is only one possible solution. □

7.3.2 Gr(3,6) network

A similar but more complicated argument can be made for the network:

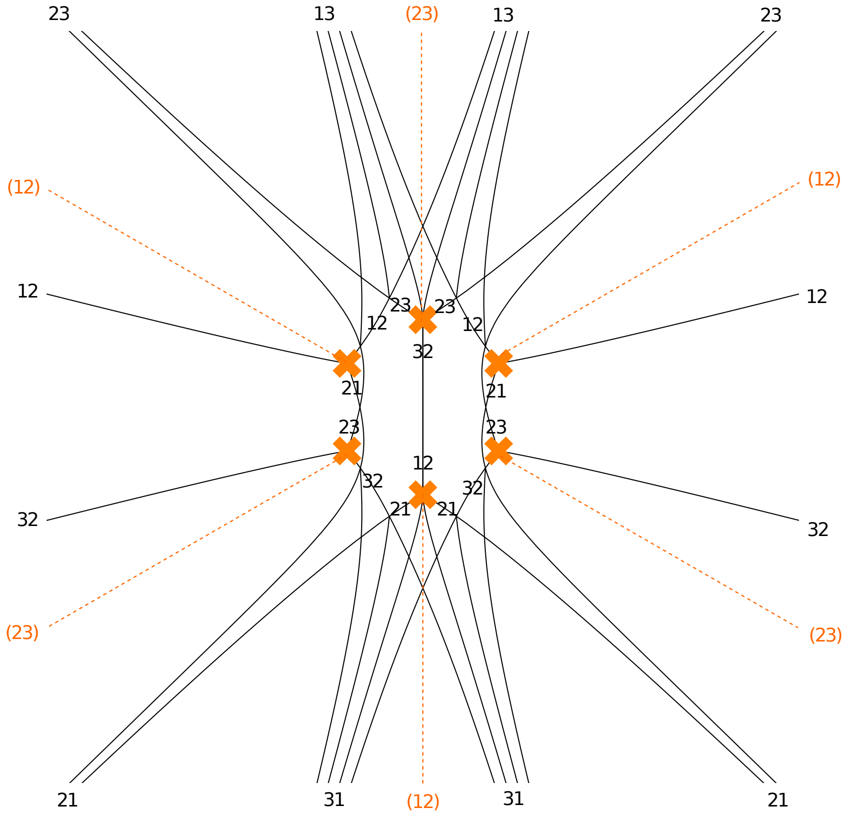


Figure 7.5: A particularly symmetric looking $n = 3$ network.

Proposition 7.3.2. *Let $\mathcal{W}$ be the network of Figure 7.5, and let $\nabla \in \mathcal{A}_{3,6}$ be generic. Given a $\mathcal{W}$–abelianization at infinity of ∇, there is a unique $\mathcal{W}$-abelianization which restricts to it.*

We omit the proof, but it is similar to the previous case with more steps.

7.3.3 Gr(3,9) network

Here is one example of a Gr(3,9) network to illustrate the more complicated behaviour when $n > 5$, though we do not attempt to solve it:

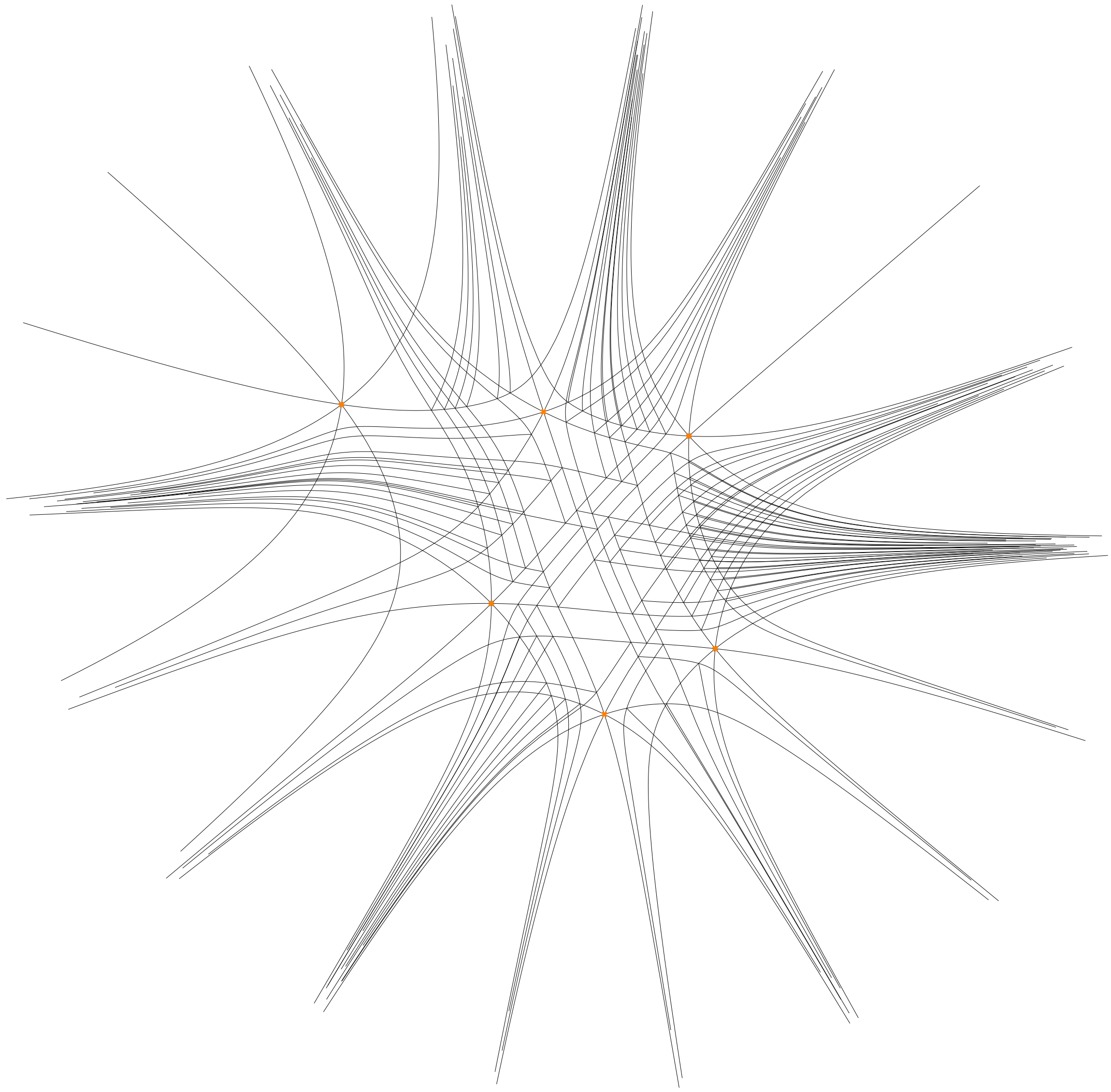


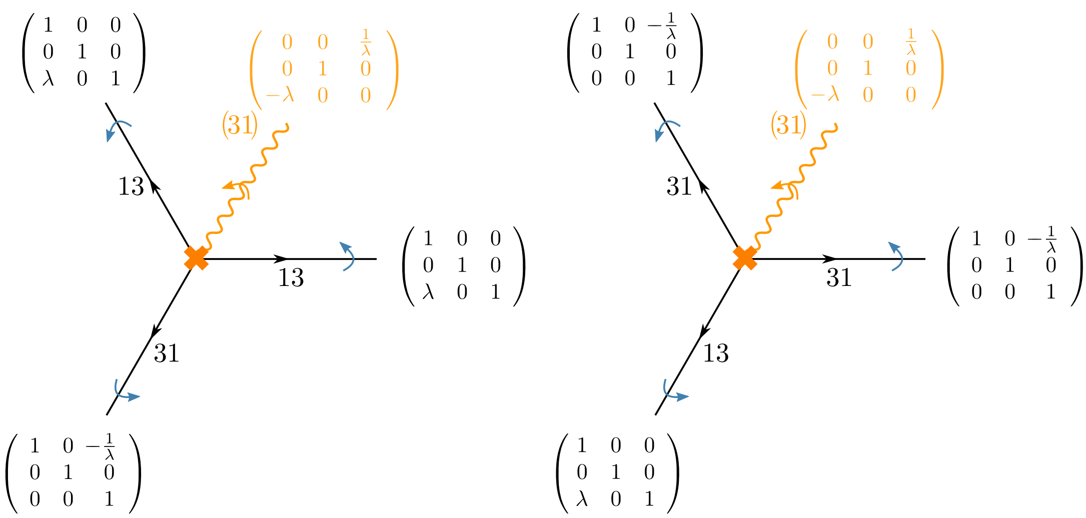
Figure 7.6: A network with $n = 6$ (we omit labels and orientations).

It would be interesting to find an example explicitly demonstrating the appearance of either infinitely many walls or infinitely many network topologies upon varying the phase, both of which are expected to occur.

7.4 A Cleaner Proof

We can do away with the issue of niceness by making the following observation. Consider the connection ∇^{ab}, but restricted to the complement of branch cuts going off to infinity, preferably not intersecting a wall. Then this $\nabla^{\mathrm{ab}}_{\mathrm{cut}}$ is necessarily the trivial connection, and we can find a gauge in which all its parallel transport matrices are simply the identity. This ensures that all the S-matrices are nice, at the expense of turning the permutation matrix at the branch cut into a ″quasi-permutation″ matrix which has nontrivial parameters. The uniqueness problem becomes a question of solving for the S-walls in terms of these parameters.

The constraints determining the isomorphism ι, via the matrices S_w, are now



which has a unique solution. Thus we see that to each branch point we can associate one parameter λ and assume all the walls emanating are nice, at the cost of introducing a so-called "monodromy cut", replacing the permutation matrix with a quasi-permutation matrix.

In principle one can resolve each network into one with simple branch points by perturbing φ, but it is sometimes preferred to deal explicitly with networks whose branch points have cyclic monodromy. We can also compute the local model for this behaviour, which looks as follows:

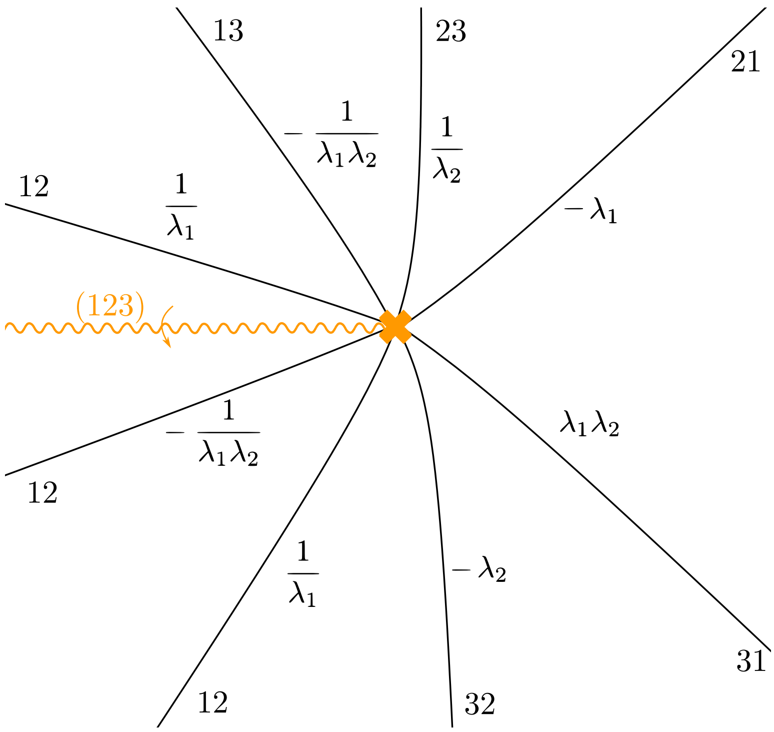
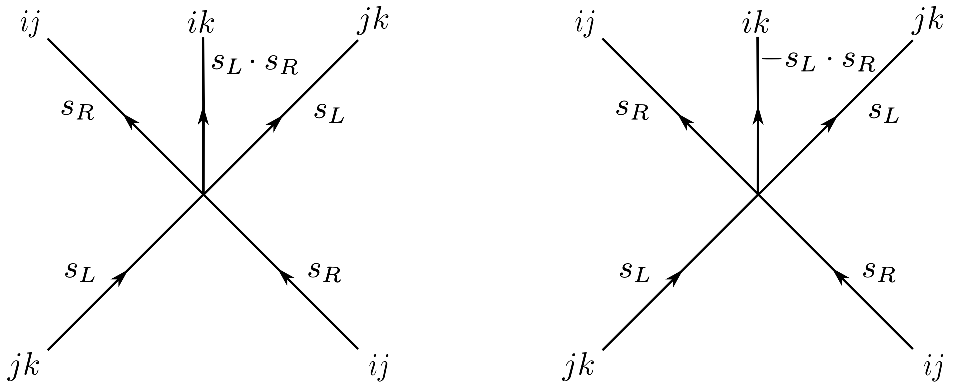


Figure 7.7: The local behaviour around a 3-cyclic branch point.

Next, we write down the constraint around a joint — when all walls are nice, it is constrained so that the birthed wall has s-parameter equal to (± 1 times, depending on the labeling) the product of the s-parameters of the incoming walls, depicted below:



$(s_L, s_R) \longmapsto (s_L, s_L \cdot s_R, s_R)$ $\qquad$ $(s_L, s_R) \longmapsto (s_L, -s_L \cdot s_R, s_R)$

Finally, we check the constraint whenever a wall intersects a branch cut (we have picked some labels, but any other situation is related by relabelling):

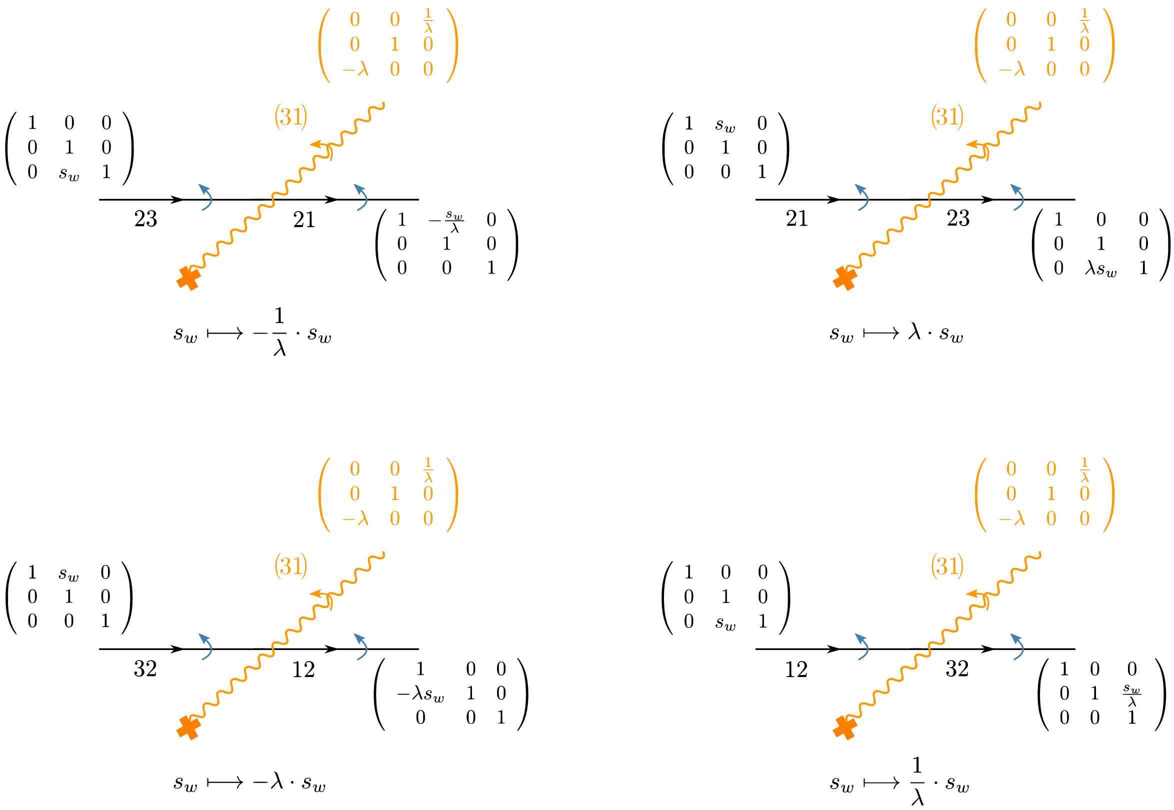


Figure 7.8: The various configurations crossing a branch cut.

In particular, we have again an explicit expression for the monomial attached to a wall even after it crosses a branch cut.

As a result of the above local computations, we see that all walls can be taken to be nice, and the nontrivial entry for every S_w depends on a set of parameters $\{\lambda_i\}$, one associated to each branch point. Thus, the (Laurent) monomials attached to the walls near infinity are fully determined in terms of the parameters associated to the branch points, which coincide with the off-diagonal terms of certain S-matrices. Furthermore, we can very easily determine the expressions by starting at a branch point b_i, where all emanating S_w and P_{b_i} have an explicit form in terms of λ_i, following the walls and modifying the associated (Laurent) monomial whenever a branch cut is crossed, or multiplying two monomials at the birth from a joint. This is clearly a well-defined procedure that yields a single monomial in the λ_i's associated to each wall near infinity.

Now, the data we are given is that of constants c_i associated to every cable, and the constraints imply the sum of terms associated to the walls of a cable must equal c_i. Clearing denominators on these $p_i - c_i$, we call the ideal $I_{\mathcal{W}}$ generated by the resulting polynomials the ***abelianization ideal*** $I_{\mathcal{W}}$ associated to $\mathcal{W}$.

The question thus becomes to show the existence and uniqueness of a solution to the set of polynomial equations given by $I_{\mathcal{W}}$ has a unique solution. In other words, that the corresponding affine variety is a point.

7.4.1 Gr(3,5) examples

Let us first go back to our original network which we previously solved, Figure 7.2. Applying the new approach using our rules, we can obtain the terms associated to each cable:

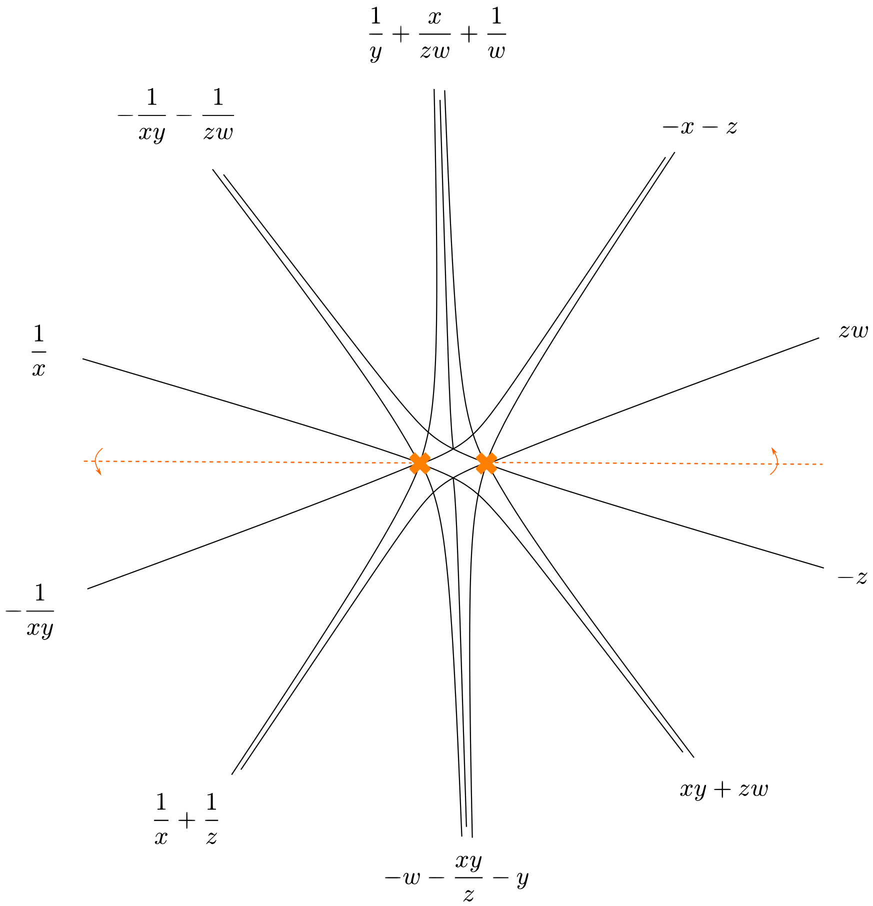


Figure 7.9: Monomials attached to each wall at infinity.

The abelianization ideal $I_{\mathcal{W}}$ is:

$$I_{\mathcal{W}} = \langle 1-c_1x, -c_2y-wy-1, -c_3y+wxy+wyz+1, \tag{7.4.1}$$

$$-c_4z-xz+1, 1-c_5wz, 1-c_6z, -c_7w-wy-1, \tag{7.4.2}$$

$$-c_8w+wxy-wyz+1, -c_9x-xz+1, -c_{10}xy-1\rangle \tag{7.4.3}$$

where c_i is the s-parameter of the corresponding cable transformation. Accounting for the constraints between the c_i arising from the triviality of traversing a large circle, one can readily verify the solution is unique and furthermore must exist for generic (as defined above) initial data:

$$x = \frac{1}{c_1}, \quad y = \frac{3}{c_3}, \quad z = \frac{1}{c_1}, \quad w = \frac{3}{c_3} \tag{7.4.4}$$

Another example, this time with $P_2 \neq 0$, is the network of Figure 7.10 below. We can depict our process of deducing the monomial attached to a wall visually by letting a solid colour denote an s-parameter of λ_i and a dotted line of the same colour denote $-\frac{1}{\lambda_i}$, and drawing an additional colour on the wall when necessary due to birthing or crossing a branch cut. Then we have:

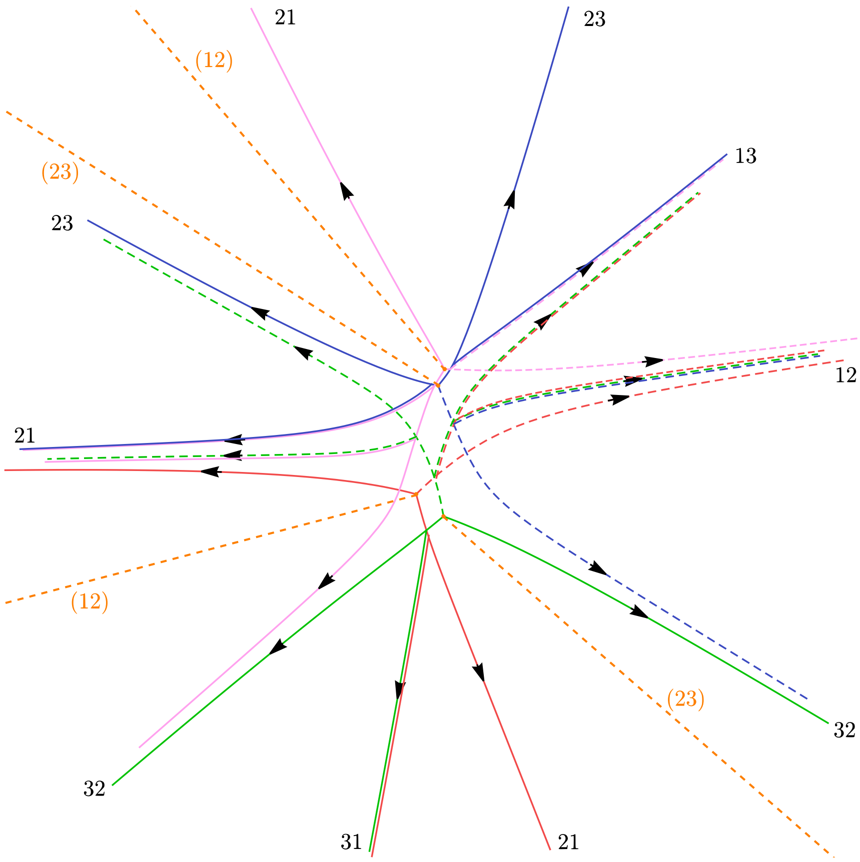


Figure 7.10: Another example using the rules to evaluate the cable terms, $P_2 \neq 0$.

This time, the abelianization ideal is:

$$I_{\mathcal{W}} = \langle -c_1 v - 1, y - c_2, -c_3 + vy + xz, -c_4 y + vy + xy - xz, \tag{7.4.5}$$

$$-c_5 yz - y - z, -c_6 - x, 1 - c_7 xz, -c_8 vyz - vy - xz, \tag{7.4.6}$$

$$-c_9 vxy - vy - xy - xz, -c_{10} + y + z \rangle \tag{7.4.7}$$

One can again verify readily that there exists a unique solution to these equations whenever c_i come from an abelianization at infinity of a generic connection.

7.4.2 Gr(3,6) examples

For $n = 3$ we find again that existence and uniqueness holds in examples, such as the one depicted below.

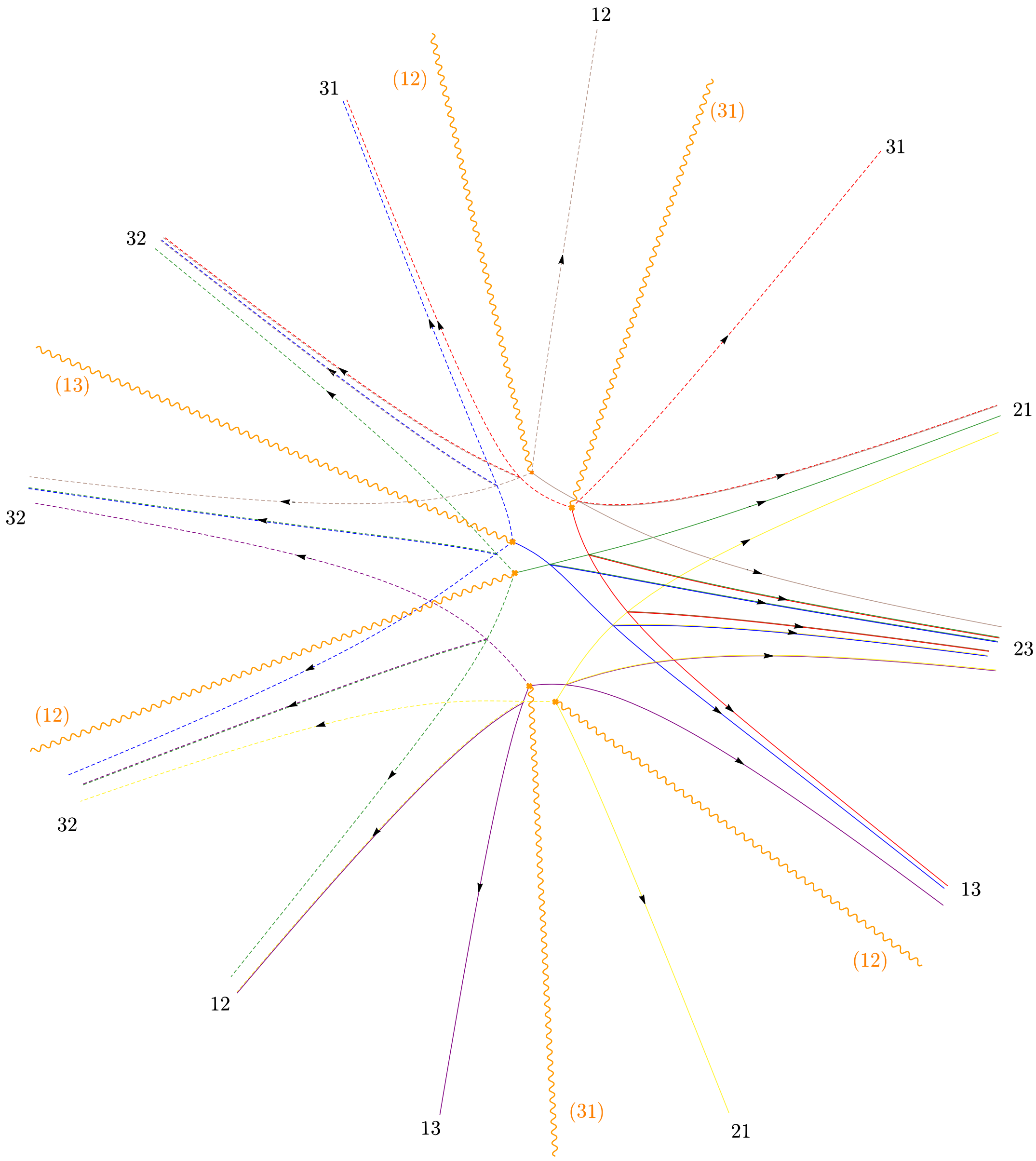


Figure 7.11: Computing the polynomials for a randomly chosen $n = 3$ network.

Existence and uniqueness of solutions has held true in every example the author checked. In principle for $n \leq 5$ this can be proved by brute force since there are expected to be a finite number of network topologies, but it would be much more satisfying to find a more conceptual proof.

7.5 Towards a general proof

As we have seen in examples above, it is fairly straightforward, if tedious, to check whether uniqueness holds for a given network. However, it is not so easy to see why this should always be the case, or to check it individually for the large (perhaps infinite) number of isotopy classes of networks that can be generated in situations of interest.

Since we are dealing with solutions to systems polynomial equations, the theory of Gröbner bases seems like a natural weapon of choice to try to get a better hold on the problem. To that end, we provide here a lightning review of only the elements of this theory that we need.

7.5.1 Gröbner bases

We end up with numerous examples of systems of polynomial equations in n_b variables, and we would like to conclude that the solutions are unique in each case. The most natural systematic way to approach this question seems to be via the theory of Gröbner bases. Given a polynomial ideal $I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{C}[x_1, \ldots, x_N]$, a Gröbner basis is essentially a "normal form" for writing this ideal – that is, a particularly nice collection of polynomials $(g_1, \ldots, g_m)$ such that $I = \langle g_1, \ldots g_m \rangle$. The fundamental theorem of Buchberger is that these always exist, and are unique once a particular monomial ordering is fixed.

The idea is that from the point of view of algebraic geometry, we can study a variety with ideal I by studying an equivalent but simpler generating set $(g_1, \ldots g_m)$. The crucial point is that it turns out one can read off many of the properties of the variety, often by inspection!

In particular, the Gröbner basis (once computed) gives us an easy method to check if the number of solutions is finite and how many there are. To state this, we need to we fix a monomial ordering so that we can unambiguously say what the *initial term* of a given polynomial f is. Then we define

Definition. The ideal generated by the initial terms of all the f_i is called the ***initial ideal*** of I. A monomial $h \in \mathbb{C}[x_1, \ldots, x_N]$ is called ***standard*** if it is not contained in the initial ideal.

Then we have:

Proposition 7.5.1. *The variety $V(I)$ is a finite set if and only if the number of standard monomials n_{std} is finite. In this case the number of solutions, counted with multiplicity, is given by n_{std}.*

Using this criterion, following the Buchberger algorithm explicitly ought to show us exactly how the uniqueness properties arise in practice, i.e. what starting property of the polynomial ends up propagating throughout the calculation to give us the uniqueness.

In order to understand this, we will need examples to understand the pattern, especially for higher values of n. The rules we have worked out in the previous subsection provide us with a simple algorithm for producing the polynomial equations associated to a given network. We are thus able to easily generate a large number of examples, whose properties will be studied in the future.

7.5.2 Future considerations

In summary, we have recast our original problem as follows

Problem. *What can be said about the ideals $I_{\mathcal{W}}$, how do they depend on φ, and how are these properties felt by the Buchberger algorithm?*

An answer to this question will most likely require some understanding of the dependence of geometry of the networks on our initial data, so we are led to consider:

Problem. *What can we prove about the behaviour of the rays and the shape of the spectral network by looking at the differentials φ?*

If we are able to establish in full generality the basic properties we have studied here, there are several questions that naturally follow. Can we compute the spectral coordinates explicitly? Do spectral coordinates always yield cluster variables? Roughly speaking, each network should correspond to a cluster — can we determine in general which cluster a given network's spectral coordinates belong to? The computations we have made along the way in this chapter should allow us to go beyond the question of existence and uniqueness and actually examine these questions in practice.

We hope to explore many of these questions and others in the future.

Appendix A

Perturbation theory for the generalized Heun equation

In this appendix we sketch the argument of [69] and its generalization to higher rank equations. This result is crucial in establishing rigorously the lowest-order asymptotics of the monodromy traces in Chapter 6. While the result is effectively present in [69], it is not entirely clear there what the general statement should be. Thus, we give a sketch following that article with some more general notation that may serve as a reference.

A.1 Statement

The result will hold for equations with regular singularities

$$z^{r_1}(z+q)^{r_2}\,\partial_z^K w + P_1(z,q)\partial_z^{K-1} w + \ldots + P_K(z,q)w = 0 \qquad (*)$$

where $P_i(z,q)$ are rational functions with singularities outside some small disc containing 0 and q, and for which the limiting equation as $q \to 0$:

$$z^{r_1+r_2}\,\partial_z^K w + P_1(z,0)\partial_z^{K-1} w + \ldots + P_K(z,0)w = 0 \qquad (**)$$

also has regular singularities (that is, for which all $P_i(z,0)$ are divisible by $z^{r_1+r_2-K}$ — we can check that this is true in our situation). For simplicity, take $r_1 = K-1$, $r_2 = 1$.

The main idea is to rescale the equation $(*)$ by $z = q\,t$ and prove the validity of variation of parameters.

Throughout, we assume z_0 is a "basepoint" lying outside the closed disc $0 < |q| < q_0$, and that z_0 is a regular point of $(*)$ and $(**)$. We will usually assume it is at $z_0 = \frac{1}{2}$.

Applying the rescaling, we arrive at the rescaled versions of $(*)$, $(**)$:

$$t^{K-1}(t+1)\partial_t^K v(t) + \frac{P_1(qt,q)}{q^{K-1}}\partial_t^{K-1} v(t) \ldots + P_K(qt,q)v(t) \tag{$\star$}$$

$$t^{K-1}(t+1)\partial_t^K u(t) + p_1(t)\partial_t u(t)^{K-1} \ldots + p_K(t)u(t) \tag{$\star\star$}$$

where

$$p_i(t) := \lim_{q \to 0} \frac{P_i(qt,q)}{q^{K-i}} \tag{A.1.1}$$

is required to exist.

Exponents will be denoted by the convention that the exponents of the equation $(*)$ at the point p are denoted by $\nu_k^{(p)}$. It is not too hard to check that $(*)$,$(**)$,$(\star)$, and $(\star\star)$ all have the same exponents *except* that when the singularities coalesce in $(**)$, the new exponents at 0 are modified– call them $\mu_k^{(0)}$– and the ones at $-q$ disappear (obviously), and for $(\star\star)$ likewise but at ∞ instead. Furthermore,

$$\mu_k^{(0)} = -\mu_k^{(\infty)} \tag{A.1.2}$$

so we drop the superscripts and just define $\mu_k := \mu_k^{(0)}$. In summary, we have

$$\begin{array}{ccc} & \text{unscaled} & \text{scaled} \\ q \neq 0 & \nu_k^{(p)} & \nu_k^{(p)} \\ q = 0 & \mu_k, p = 0 & -\mu_k, p = \infty \end{array} \tag{A.1.3}$$

The main statements are (with some notations introduced momentarily):

Proposition A.1.1. *To lowest order, we have*

$$\mathbf{T}_0(q) \sim \mathbf{V}_0(q)\, \mathbf{\Gamma}\, \mathbf{U}^{-1}(q)\, \mathbf{T}_l \tag{A.1.4}$$

where $\mathbf{\Gamma}$ *is the connection matrix going from* ∞ *to* 0 *for the hypergeometric equation with exponents* $\mu_k, \nu_k^{(0)}$ *at* $\infty, 0$.

Theorem A.1.2. *Suppose* $\operatorname{Re} q > 0$ *and that the coefficient functions* $P_i(z,q)$ *satisfy* $z^{m_1+m_2-K} \mid P(z,0)$. *Then the connection matrices have the asymptotic behaviour:*

$$\mathbf{T}_\infty(q)\mathbf{T}_0^{-1}(q) = \mathbf{U}(q)\left(\mathbf{\Gamma} + q \log q\, O(1)\right)\mathbf{V}_0^{-1}(q) \tag{A.1.5}$$

where $\mathbf{\Gamma}$ *(which depends on* $\mu_k^{(0)} = \alpha_k/2\pi i + (K-1)/2$*) is the connection matrix from 0 to 1 for the limiting equation* ($\star\star$)*,* $O(1)$ *means a matrix with components which are* $O(1)$*, and*

$$\mathbf{U}(q) = \text{diag}\,\left(q^{\mu_1}, \ldots, q^{\mu_k}\right), \qquad \mathbf{V}_0(q) = \text{diag}\,\left(q^{\nu_1^{(0)}}, \ldots, q^{\nu_k^{(0)}}\right) \tag{A.1.6}$$

Theorem A.1.3. *We have*

$$\mathbf{T}_0(q)\mathbf{T}_s^{-1}(q) = \mathbf{V}_0(q)\Big(\mathbf{M}\big(\mathbb{I} + q\,\mathbf{Q}(q)\big)\Big)\mathbf{V}_s^{-1}(q) \tag{A.1.7}$$

where $\mathbf{V}_p(q) = \text{diag}(q^{\nu_1^{(s)}}, \ldots q^{\nu_K^{(s)}})$*, and* $\mathbf{Q}(q) = O(1)$*. Furthermore, the correction* $\mathbf{Q}(q)$ *is a power series in q converges in some* $0 < |q| < q_0$ *and so may be determined iteratively.*

Thus the monodromy around, say, the A-cycle encircling 0 and $-q$ is (denoting $\mathbf{S} := \mathbf{T}_0\mathbf{T}_s^{-1}$)

$$\mathbf{M}_A = \text{diag}(e^{2\pi i\nu_1^{(s)}}, \ldots, e^{2\pi i\nu_K^{(s)}})\mathbf{S}^{-1}(q)\,\text{diag}(e^{2\pi i\nu_1^{(0)}}, \ldots, e^{2\pi i\nu_K^{(0)}})\mathbf{S}(q) \tag{A.1.8}$$

recalling that $\mathbf{M}$ depends on α. Then we can impose that $\text{tr}(\mathbf{M})_A = -2\cosh(\alpha)$ to get conditions on $\mathbf{Q}(q)$, from which we can then determine $\mathbf{M}_B$.

Conventions, normalizations, etc.

Let the solutions of ($*$) be normalized as

$$w_k^{(0)} = z^{\nu_k^{(0)}}\,T_k(z,q) \tag{A.1.9}$$

$$w_k^{(q)} = (z+q)^{\nu_k^{(s)}}\,\widetilde{T}_k(z,q) \tag{A.1.10}$$

where T_k (resp. $\widetilde{T}_k$) is holomorphic at 0 (resp. $-q$) and satisfy $T_k(0,q) = \widetilde{T}_k(0,q) = 1$

We will always assume that z_0 is a regular point of ($*$) and therefore ($**$) as well, and that the only singularities within the disc $|z| < |z_0|$ are 0 and $-q$. Then we define solutions at z_0 by:

$$w_1^{(z_0)} = 1 + (z-z_0)\,r_1(z,q) \tag{A.1.11}$$

$$w_2^{(z_0)} = (z-z_0) + (z-z_0)^2\,r_2(z,q) \tag{A.1.12}$$

$$\vdots \tag{A.1.13}$$

$$w_K^{(z_0)} = \frac{(z-z_0)}{(K-1)!}^{K-1} + (z-z_0)^K\,r_K(z,q) \tag{A.1.14}$$

for $r_k(z,q)$ some functions holomorphic in a neighbourhood of $z = z_0$. This ensures that $w_i^{(z_0)}$ satisfy the initial conditions $\partial_z^j w_i(z_0) = \delta_{i,j}$ for $0 \le j < K$.

Assume, furthermore, that $y_k(z)$ are solutions of $(**)$ with the same normalization at z_0. Then we also define solutions in a neighbourhood of 0:

$$y_k^{(0)} = z^{\mu_k} X_k(z), \qquad k = 1, 2, \ldots, K \tag{A.1.15}$$

where $X_k(z)$ are holomorphic in a neighbourhood of 0 with $X_k(0) = 1$. *The exponents are assumed to be nonresonant* i.e. no two μ_k differ by an integer—as is the case for us .

In summary, notationally we have that $w^{(s)}$ denotes local solutions around a singular point s of the perturbed equation (to avoid confusion, the solution at $z = -q$ will be denoted with superscript (s), for the "singularity" that we are varying), and $w^{(z_0)}$ denotes local (holomorphic) solutions around the (regular) base-point $z_0 = \frac{1}{2}$.

Local solutions are related by the connection matrices

$$\begin{pmatrix} w_1^{(s)}(z,q) \\ \vdots \\ w_K^{(s)}(z,q) \end{pmatrix} = \mathbf{T}_s(q) \begin{pmatrix} w_1^{(z_0)}(z,q) \\ \vdots \\ w_K^{(z_0)}(z,q) \end{pmatrix} \tag{A.1.16}$$

$$\begin{pmatrix} w_1^{(0)}(z,q) \\ \vdots \\ w_K^{(0)}(z,q) \end{pmatrix} = \mathbf{T}_0(q) \begin{pmatrix} w_1^{(z_0)}(z,q) \\ \vdots \\ w_K^{(z_0)}(z,q) \end{pmatrix} \tag{A.1.17}$$

where the connection matrices $\mathbf{T}_{0,s}(q)$ and $\mathbf{T}_l$ are defined by picking a contour for analytic continuation. In particular, we pick

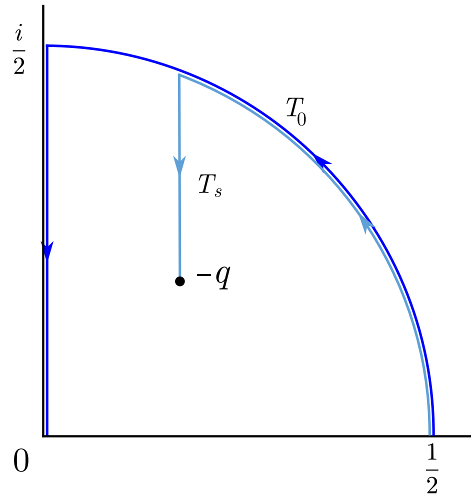


Figure A.1: Contours defining T_0, T_s.

for T_0 and T_s. Finally, $\mathbf{T}_l$ is defined as the connection matrix from z_0 to 0 for the equation (∗∗):

$$\begin{pmatrix} y_1^{(0)}(z) \\ \vdots \\ y_K^{(0)}(z) \end{pmatrix} = \mathbf{T}_l \begin{pmatrix} y_1^{(z_0)}(z) \\ \vdots \\ y_K^{(z_0)}(z) \end{pmatrix} \tag{A.1.18}$$

Near a regular point

Around a regular point z_0, we have the usual theorem describing the behaviour of the solutions:

Proposition 1. *Let $w_k(z,q), y_k(z)$ be solutions of eq and eq, respectively. In some fixed neighbourhood of $z = z_0$,*

$$w_k^{(z_0)}(z,q) = y_k^{(z_0)}(z)[1 + q\, d_k(z,q)] \tag{A.1.19}$$

where $d_k(z,q)$ $k = 1,\ldots,n$ functions holomorphic and bounded in U

A.2 Rescaled equation

The crucial strategy for the perturbation theory is to rescale the variable as $z = qt$, so that the equation (∗), (∗∗) become (⋆), (⋆):

$$t^{K-1}(t+1)\partial_t^K v(t) + \frac{P_1(qt,q)}{q^{K-1}}\partial_t^{K-1} v(t) \ldots + P_K(qt,q)v(t) \tag{$\star$}$$

and the rescaled limiting equation $(**)$ becomes

$$t^{K-1}(t+1)\partial_t^K u(t) + p_1(t)\partial_t u(t)^{K-1} \ldots + p_K(t)u(t) \tag{$\star\star$}$$

where

$$p_i(t) := \lim_{q\to 0} \frac{P_i(qt,q)}{q^{K-i}} \tag{A.2.1}$$

In particular we need this limit to exist.

The equations $(*)$ and $(\star)$ have the same exponents, since they are just rescalings of one another. However, $(**)$ and $(\star\star)$ have different exponents (since the singularities don't collide in the limit $(\star\star)$. We will always denote by $\mu_i^{(0)}$ the exponents of the *rescaled* limiting equation $(\star\star)$, and likewise $\mu_i^{(\infty)}$ at infinity.

Since the singularities at $0, -q$ are regular, the Frobenius method assures us that there exist bases of linearly independent local solutions of the form

$$v_k^{(0)} = t^{\nu_k^{(0)}} f_k(t,q) \tag{A.2.2}$$

$$v_k^{(s)} = t^{\nu_k^{(0)}} g_k(t,q) \tag{A.2.3}$$

where f_i, g_i are holomorphic in a neighbourhood of $t = 0$, $t = -1$, respectively, and we may single out the normalization $f_i(0,q) = g_i(0,q) = 1$ for $i = 1,2,3$.

It is easy to see we can recover the original solutions of $(*)$ from the scaled solutions of $(\star)$ via

$$v_k^{(0)}(z/q,q) = q^{-\nu_k^{(0)}} w_k^{(0)}(z,q) \tag{A.2.4}$$

$$v_k^{(s)}(z/q,q) = q^{-\nu_k^{(s)}} w_k^{(s)}(z,q) \tag{A.2.5}$$

Define the functions $u_k^{(0)}$ to be the solutions to $(\star\star)$ guaranteed locally by Frobenius satisfying

$$u_k^{(0)} = t^{\mu_k^{(0)}} \Phi_k(t) \tag{A.2.6}$$

where $\Phi_i(t)$ are holomorphic on $\mathbb{D}_1$ and $\Phi_i(0) = 1$ for $i = 1, \ldots, n$. Likewise define $u_k^{\infty}(t) = t^{\mu_k^{(\infty)}}(1 + x_k(t)/t)$.

We proceed by making an ansatz that the local solution v should be of the form

$$v(t,q) = \sum_{i=1}^{K} C_i(t,q)u_i(t) \tag{A.2.7}$$

and impose further the conditions that

$$\sum_{i=1}^{K} \dot{C}_i(t,q)\partial_t^j u_i = 0 \tag{A.2.8}$$

for all $j = 0, 1, \ldots, K-1$, where again $\dot{}$ denotes derivative with respect to t. This condition ensures that

$$\partial_t^k v(t,q) = \sum_{i=1}^{K} C_i \partial_t^k u_i \tag{A.2.9}$$

for all but the higher derivative $\partial_t^K v$.

With these definitions we find that after imposing (1.9) and using the fact that u_i satisfy $(\star\star)$, we have the remaining equation

$$t^{K-1}(t+1)\sum_{i=1}^{K} \dot{C}_i \partial_t^{K-1} u_i = -\left(\sum_{j=1}^{K} h_j(t,q)\left(\sum_{i=1}^{K} C_i \partial_t^{K-j} u_i\right)\right) \tag{A.2.10}$$

where $h_i(t,q) := P_i(qt,q)/q^{K-i} - p_i(t)$ i.e. $P_i(qt,q)/q^{K-i} = p_i(t) + h_i(t,q)$, h_i are the perturbations.

For each i, we will do the following (but for notational purposes we will always assume $i = 1$). Define m_i by:

$$C_1(t,q) = 1 + qm_1(t,q), \quad C_k(t,q) = qm_k(t,q), \qquad k \neq 1 \tag{A.2.11}$$

We can then rewrite (1.9) and (1.10) together as

$$\begin{pmatrix} u_1 & \ldots & u_K \\ \vdots & & \vdots \\ \partial_t^{K-2} u_1 & \ldots & \partial_t^{K-2} u_K \\ \partial_t^{K-1} u_1 & \ldots & \partial_t^{K-1} u_K \end{pmatrix} \begin{pmatrix} \dot{m}_1 \\ \vdots \\ \dot{m}_K \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{F}u_1 \end{pmatrix} + \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ \mathbf{F}u_1 & \ldots & \mathbf{F}u_K \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_K \end{pmatrix} \tag{A.2.12}$$

where we have defined

$$(\mathbf{F}u)(t) := -\frac{1}{q}\frac{1}{t^{K-1}(t+1)}\Big(h_1(t,q)\partial_t^{K-1}u(t) + \ldots + h_{K-1}(t,q)\partial_t u(t) + h_K(t,q)u(t)\Big) \tag{A.2.13}$$

Or as an inhomogeneous linear sytem for m_i:

$$\begin{pmatrix} \dot{m}_1 \\ \vdots \\ \dot{m}_K \end{pmatrix} = \begin{pmatrix} R_1 \\ \vdots \\ R_K \end{pmatrix} + \mathbf{M}\begin{pmatrix} m_1 \\ \vdots \\ m_K \end{pmatrix} \tag{A.2.14}$$

with [1]

$$\mathbf{M} = \mathbf{J}^{-1}\mathbf{F} \tag{A.2.15}$$

$$\mathbf{R} = \mathbf{J}^{-1}\mathbf{F}_1 \tag{A.2.16}$$

Notation: $\mathrm{W}(t)$ is the Wronskian of the solutions $(u_1, \ldots u_K)$, i.e. $\det \mathbf{J}$. Note by Abel's identity, since our particular equation has $P_{K-1}(z,q)$ vanishing, $\mathrm{W}(t)$ is a nonzero constant (with no q dependence either) for any choice of linearly independent solutions.

To solve this system of differential equations for m, we convert them into a system of integral equations:

$$\begin{pmatrix} m_1 \\ \vdots \\ m_K \end{pmatrix} = \begin{pmatrix} \int_0^t R_1 \\ \vdots \\ \int_0^t R_K \end{pmatrix} + \int_0^t \mathbf{M} \begin{pmatrix} m_1 \\ \vdots \\ m_K \end{pmatrix} \tag{A.2.17}$$

Here one should keep in mind that $\mathbf{M}(q)$ depends on q, though by assumption it goes to 0 as $q \to 0$.

A.3 Existence

First we show that our ansatz works for the solutions near $z = 0$. Let $\mathbb{D}_r$ here and throughout denote the closed disk of radius r. We will start off by looking for solutions on $\mathbb{D}_{2/3}$ which are holomorphic and bounded.

Theorem A.3.1. *Suppose* $\mathrm{Re}\, \nu_k^{(0)} < 0$, $k = 1, \ldots K$. *Then there exist functions* $\lambda_k(t,q)$, *holomorphic and bounded in* $\mathbb{D}_{2/3}$, *such that*

$$v_k^{(0)}(t,q) = u_k^{(0)} \left(1 + q\, \lambda_k(t,q)\right) \tag{A.3.1}$$

for $k = 1, \ldots, n$

Proof. Follow the same method as [69] making the substitutions above. Sketch as follows: Let $L = [\mathrm{Re}\nu_k^{(0)}]$.

$$m_1(t) = l_1, m_i(t) = t^{\nu^{(0)}} \left(\sum_{j=0}^{L-1} c_{j,i} t^j + t^L l_i \right) \tag{A.3.2}$$

[1] Note: this is the transpose of what is used in [70]

Consider our equations on the space

$$X = \left\{(l_1(t), \ldots, l_K(t), \{c_{i,j}\}) : l_i \text{ are holomorphic and bounded on } \mathbb{D}_{2/3},\ c_{i,j} \in \mathbb{C}\right\}. \tag{A.3.3}$$

X is a Banach space when equipped with the sup-norm. Our integral equations are now mapping $T : X \to X$. Choose q small enough to make this a contraction mapping and the result follows from the Banach fixed point theorem. □

Remark. In particular, the proof shows that for sufficiently small q, the solutions w actually converge in q. We can use this to actually calculate the corrections as a convergent power series in q, though it may not be that the integrals are tractable. Menotti [70] uses a trick to do this for the first order correction when computing the A-cycle monodromy.

Continuing, we can essentially follow the same proofs as [69] and end up with the final formulas given at the beginning.

A.4 Rank 2 worked out

We use the fact that the formulas above hold even when the basepoint is the singularity $z = 1$. So we have

$$T_{1,0}(q) = V \cdot \Gamma \cdot U^{-1} \cdot T_{l,s} \tag{A.4.1}$$

where $U = \mathrm{diag}(q^{2\pi\nu}, q^{-2\pi\nu})$.

Letting

$$D_{pt} := \begin{pmatrix} e^{2\pi i \nu_1^{(pt)}} & 0 \\ 0 & e^{-2\pi i \nu_1^{(pt)}} \end{pmatrix} \tag{A.4.2}$$

the trace around the B-cycle is thus

$$\operatorname{tr} \mathbf{M}_B = \operatorname{tr}\left(D_{(1)}\, T_l^{-1}\, \mathrm{U}\, \Gamma^{-1}\, V_{-1}\, D_{(0)}\, V\, \Gamma\, U^{-1}\, T_l\right) \tag{A.4.3}$$

$$= \operatorname{tr}\left(\left(T_l D_1 T_l^{-1}\right) \cdot \mathrm{U} \cdot \left(\Gamma^{-1}\, D_0\, \Gamma\right) \cdot \mathrm{U}^{-1}\right) \tag{A.4.4}$$

but recall that T_l is simply the change of basis from solutions around 1 to solutions around 0 of the limiting, unrescaled equation ($**$). So the structure is clear: take the monodromy of the first pants (the “z-pants”) in a basis at 0 and go around 1, use the transition matrix U to cross the pants curve, and then take the monodromy of the second pants (the “t-pants”). This justifies the procedure employed by Nekrasov-Rosly-Shatashvili heuristically, and (what amounts to the same) that of Teschner-Vartanov [65], which uses CFT.

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