DICE-2024 IOP Publishing
Journal of Physics: Conference Series **3017** (2025) 012011 doi:10.1088/1742-6596/3017/1/012011

Some remarks on the geometrization of physics and the topological theories of quantum fields

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Abstract. This article aims at examining the role of geometrical and topological concepts in the recent developments of theoretical physics, especially in gauge theory and string theory, as well as at showing the great significance of these concepts for a deeper understanding of the dynamics of physics and the mathematical structure of spacetime. Contemporary developments in theoretical physics suggest that another "revolution", after those of general relativity and quantum mechanics, may be in progress, through which a new kind of geometry may enter physics, and spacetime itself can be reinterpreted as an approximate, derived concept. One of the main purposes of this paper is to show the deep significance of spacetime geometry and topology in influencing the laws which are supposed to govern the behavior of matter, and further to support the thesis that matter itself might be built from geometry, in the sense that particles of matter as well as the other forces of nature emerges in a similar way that gravity emerges from geometry. This article is an attempt to show that it is very unsatisfactory try to develop physics on a rigorous basis to give formal justification of the experimental conclusions; this is the traditional (or "instrumental") role of mathematical physics since, in fact, the Newton's time. Contrary to this conception, we try to understand the deeper meaning of the physics-mathematics connection. Rather than to view mathematics as a tool to establish physical theories, or physics as a way of pointing to mathematical truths, we can try to dig more deeply into the relationship between them. This investigation may not only have theoretical and philosophical interest, but it could lead to a better understanding of and even to new insights into the geometrical and topological structures acting at the frontier between mathematics and physics.

1. A new era in the relationship between geometry and physics: topology as guiding principle

Beginning in the 1970's, it was recognized that mathematically gauge theory is essentially one branch of differential geometry that uses the new concept of "fiber spaces" with "connections". This notion is central in the understanding of the relation between mathematical structures and physical theories, and it directly links geometry and physics to the point that it can be said that the two are coextensive.

Indeed, consider the mathematical concept of a space with a connection and its curvature (see Manin, 1981). Let $f: M \to N$ be a map between spaces M, N, where M, say, represents a model of spacetime, and at each point p of M there is localized a physical system with the space of internal states $f^{-1}(p)$. A connection on a geometrical object is a rule permitting the transport of the system along the curves in M. In other words, if we know part of the world-lines and the initial internal state of a system in M, then, thanks to the corresponding displacement determined by the connection, we can know the future states of the system. According to recent physical theories, a gravitational field is a connection in the space of internal degrees of freedom of a gyroscope; the connection allows us to follow the evolution of the gyroscope in spacetime. An electromagnetic field is also a connection in the space of internal degrees of freedom of a quantum electron; the connection allows us to follow the evolution of the electron in spacetime. A Yang-Mills field is yet a connection, in the space of internal degrees of freedom of a quark.

This geometrical image seems now to be the most universal mathematical model of an ideal universe with a small number of basic interactions. The state of matter in spacetime, at each point and each moment, is described by a section of an appropriate fiber space $N \rightarrow M$. A field is described by a connection on this fiber space. Matter acts on the connection by imposing restrictions on its curvature, and the connection acts on matter by forcing it to propagate by "parallel displacement" along world-lines. The famous equations of Einstein, Maxwell-Dirac and Yang-Mills are exactly the embodiment of this idea. The geometrical concept of connection has thus become an essential element of physics.

One can see that each physical entity relates to a geometrical or global differential concept. For example, field strength is identified with the curvature of the connection; the action integral is but a global measure of curvature. Certain topological and algebraic invariants in the theory of characteristic classes have been seen to be most appropriate to describe the charge of the particle in the sense of Yang-Mills. More generally, we can establish a direct *correspondence* from the concepts of gauge field theory to those of the differential geometry- and topology of fiber spaces. But how can we precisely understand the nature of such a correspondence? Inspired by an idea already proposed by Weyl in another manner (see Weyl, 1918), we suggest the thesis that, essentially, *physics is but geometry in act*. This implies not only that geometry yields mathematical abstract concepts like manifolds, groups, curvature, connections, bundles, but also that it is, in a way, physically rooted, because it is integral part of the properties of physical entities and the features of phenomena (see Boi, 1992).

One could go so far as to claim that there must be a geometrical structure, continuous or discrete according to the theory and the class of phenomena considered, underlying any given physical variety of phenomena. Or maybe a topological structure which would encompass at the same time the continuous and discrete characters of space and of nature into a more general mathematical scheme. To convince oneself of this, it suffices to remember that some principles of geometrical symmetry (or, equivalently, some groups), can be transformed into dynamical principles that are in turn responsible for changes in the phenomena. Should we then affirm: "*In the beginning was the symmetry or the group...*"? However, this concept is not just abstract, for, in fact, the mathematical properties related to any group have simultaneously an explanatory power and a capacity to generate a world of forces, interactions, and energy..., so that the mathematical understanding of this world cannot be separate of the understanding of reality itself. Indeed, at a deeper level, one is increasingly led to believe that symmetry may, in a hidden sense, determine almost everything. Moreover, in view of all this it is not unreasonable to look on topology, like symmetry, as some kind of underlying or unifying principle which helps us to understand natural phenomena at the microscopic as well as at the macroscopic levels (see Atiyah, 1979).

2. The structure of fiber bundles and the topological significance of physical theories

Let us now return to the concept of fiber bundle or fiber space. That notion, being global in character, arose in topology. At first it was an attempt to find new examples of manifolds. Fiber spaces are locally, but not globally, product spaces. The presence of such a distinction is a sophisticated mathematical fact. The development of fiber spaces must wait until invariants are found to distinguish the fiberings or even to show that globally there are nontrivial ones. The first such invariants are the characteristic classes introduced by H. Whitney and by E. Stiefel in 1935. Topology, however, forgets the algebraic structure, and in applications vector bundles, with the linear structure intact, are more useful.

A vector bundle π: $E \rightarrow M$ over a manifold M is, roughly speaking, a family of vector spaces parametrized by M such that it is locally a product. The vector space $E_X = \pi^{-1}(x)$ corresponding to $x \in M$ is called the fiber at x. Examples are the tangent bundle M and all tensor bundles associated to it. A more trivial bundle is the product bundle $M' \times V$, where V is a fixed vector space and (x, V), $x \in M$, is the fiber at x. A vector bundle is called *real* or *complex* according as the fiber is a real or complex vector space. Its dimension is the dimension of the fibers. It is important that the linear structure on the fibers has a meaning so that the general linear group $GL(n, \mathbf{R})$ plays a fundamental role in matching the fibers; it is called the *structure group*. A real (respectively complex) vector bundle is called *Riemannian* (respectively *Hermitian*) if the fibers are provided with inner products. In this case the structure group is reduced to $O(n)$ (respectively $U(n)$), n being the dimension of the fibers; the bundle is then called an $O(n)$ (respectively $U(n)$-bundle). Similarly, we have the notion of an $SU(n)$-bundle. A section of the bundle E is an attachment, in a continuous and smooth manner, to every point $x \in M$, a point of the fiber E_X. In other words, it is a continuous mapping σ: $M \rightarrow E$ such that the composition $\pi \circ s$ is the identity. This notion is a natural generalization of a vector-valued function and of a tangent vector field. To differentiate σ we need a so-called *connection* in E.

The latter allows the definition of the covariant derivative $D_X\sigma$ (X being a vector field in M), which is a new section of E. Covariant differentiation is generally not commutative; that is, $D_X \circ D_Y \neq D_Y \circ D_X$ for two vector fields X, Y in M. The measure of the noncommutativity gives the curvature of the connection; this is an analytic version of the geometric concept of non-holonomy introduced by Elie Cartan. According to him, it is important to regard the curvature as a matrix-valued exterior quadratic differential form. Its trace is a closed 2-form. More generally, the sum of all its principal minors of order k is a closed $2k$-form. It is called a *characteristic class*. By the de Rham theory the characteristic form of degree $2k$ determines a cohomology class of dimension $2k$, to be called a *characteristic class*. Whereas the characteristic forms depend on the connection, the characteristic class depend only on the bundle. They are the simplest invariants of the bundle. In some sense, it must be an act of nature that the nontriviality of a vector bundle is recognized through the need of a covariant differentiation and that its noncommutativity accounts for the first global invariants. This introduction of the characteristic classes gives emphasis on its local character, and the characteristic forms contain more information than the classes. When M is a compact oriented manifold, a characteristic class of the top dimension (that is, of dimension equal to that of M) gives by integration a characteristic number. When it is an integer, it is called a *topological quantum number*.

These differential-geometric notions have been found to be the likely mathematical basis of a unified field theory (see Atiyah 1979; Yang 1983). Weyl's gauge theory deals with a circle bundle or a $U(1)$-bundle, that is, a complex Hermitian bundle of dimension one. In studying the isotopic spin Yang-Mills used what is essentially a connection in a $SU(2)$-bundle. It is the first instance of a non-Abelian gauge theory. From the connection the "action" can be defined. A connection in an $SU(2)$-bundle at which the action takes the minimum is called an *instanton* (see Witten, 1994). Its curvature has a simple expression and is called *self-dual*. An instanton is thus a self-dual solution of the Yang-Mills equation. When the space R^4 is compactified into the four-dimensional sphere S^4, the $SU(2)$-bundles are determined up to an isomorphism by a topological quantum number k, which is an integer. It has been proved that over S^4 the moduli (or parameter space) for the set of connections with self-dual curvature on the $SU(2)$-bundle with given $k > 0$ is a smooth manifold of dimension $8k - 3$. In physical terms this is the dimension of the space of instantons with topological quantum number $k > 0$. Instantons can claim a relation to Einstein through the following result. The group $SO(4)$ is locally isomorphic to $SU(2) \times SU(2)$, so that a Riemannian metric on a four-dimensional manifold M gives rise through projection to connections in the $SU(2)$-bundles. M is an Einstein manifold if and only if these connections are self-dual or anti-dual.

The notion of fiber bundle generalizes that of a Cartesian product on a manifold (see Trautman 1980). Two examples from physics and geometry will clarify the need for such a generalization.

(i) In *Aristotelian physics* both space and time are absolute, every event being defined by an instant of time and a location in space. This is equivalent to saying that spacetime E is a Cartesian product $T \times S$, where T is the time axis and S is the three-dimensional space.

(ii) In *Galilean physics* time remains absolute, but space is relative. This can be described by saying that there is a *projection* $\pi : E \to T$, i.e., a surjective (onto) map π that associates to any event $p \in E$ the corresponding instant of time $t = \pi(p) \in T$. The set (line) T is called the *base space* and the set $\pi^{-1}(t)$ of all events simultaneous with p is called the *fiber* over t. Each fiber is isomorphic to the Euclidean three-dimensional space $\mathbf{R}^3$, which is therefore called the *typical fiber*. The total space E of this bundle may be trivialized, i.e., represented as the Cartesian product $T \times \mathbf{R}^3$. Any such trivialization (map) $h : E \to T \times \mathbf{R}^3$ is of the form $h(p) = (\pi(p), \mathbf{r}(p))$, where $\mathbf{r}(p) = (x(p), y(p), z(p))$ are the space coordinates of the event p relative to an inertial observer. One can say that Galilean spacetime E is the total space of a fiber bundle which is *trivial*, i.e., isomorphic to the product bundle $T \times \mathbf{R}^3$, without there being a natural isomorphism between these bundles.

(iii) Consider now the two-dimensional sphere S^2 with a preferred orientation. Define a "dyad" as a pair of unit orthogonal vectors tangent to S^2 at a point. Let P be the set of all dyads whose orientation agrees with that of S^2. One can make P into the total space of a bundle in such a way that $\pi : P \to S^2$ is the map sending a dyad into the point at which its vectors are attached to S^2. If $e = (e_1, e_2)$ is a dyad at $x \in S^2$, then so is the pair (e'_1, e'_2), where

$$e'_2 = -e_1 \sin\varphi + e_2 \cos\varphi \qquad e'_1 = e_1 \cos\varphi + e_2 \sin\varphi, \tag{2.1}$$

and all dyads at x may be obtained in this manner from (e_1, e_2). Therefore, $SO(2)$ is the typical fiber of the bundle $\pi : P \to S^2$. Equation (2.1) defines an *action* of the (*structure*) *group* $SO(2)$ on P. The bundle $\pi : P \to S^2$ is a simple example of a *principal bundle*. Moreover, this bundle is nontrivial in the following sense: there is no diffeomorphism $k : S^2 \times SO(2) \to P$ such that $\pi \circ k(x, a) = x$. Indeed, if such a k existed, then $\sigma : S^2 \to P$, defined by $s(x) = k(x, a_0)$, would determine a smooth field of unit vectors on S^2. By the "no combing of S^{2n}" theorem of Brouwer, such a field σ does not exist. In general, if $\pi : E \to M$ is a bundle and N is an open subset of M, then a smooth map $\sigma : N \to P$, such that $\pi \circ \sigma = \mathrm{id}_N$, is called a (local) *section of* π.

If $N = M$, then s is a *global section*. For a principal bundle, the existence of a global section is equivalent to its triviality. Incidentally, the bundle of dyads occurs in the description of a magnetic pole of unit strength. The nontrivial nature of the bundle $\pi : P \to S^2$, shows up in the occurrence of a "string singularity" in the expression for the vector potential of the magnetic pole.

The last remark leads to what is probably the most important domain of applications of fiber bundles in theoretical physics: *infinitesimal connections on principal bundles provide good geometrical models of classical gauge fields*. Of striking interest are the analogies between electromagnetism and gravitation. For instance, the discussion on the significance of the electromagnetic potentials become clear when electromagnetism is interpreted as an (infinitesimal) connection in the space of phases. Namely, the experiments proposed by Aharonov and Bohm (1959) have a very simple analog in elementary differential geometry: the surface of a cone is locally flat, but a vector undergoing parallel transport along a loop enclosing the vertex does not return to its original position. Similarly, the phase of a wave function of a charged particle undergoes parallel transport determined by the potential. The region with the magnetic field is analogous to the vertex of the cone. Electromagnetism potentials should not be slighted, but considered for what they are: *the coefficients of a connection*.

A heuristic approach to the notion of a connection on a principal bundle shows how this concept is related to the physicist's view of gauge potentials (see Regge, 1992): Let $\pi : P \to M$ be a principal bundle with structure group G. The result of action of $a \in G$ on $p \in P$ is another point $pa \in P$, lying in the same fiber as p, $\pi(pa) = \pi(p)$. A local section $\sigma : N \to P$ defines a diffeomorphism $k : N \times G \to \pi^{-1}(N)$ by $k(x, a) = s(x)a = p$. The section σ being fixed for the moment, we may identify $\sigma(x)$ with (x, ε) and $\sigma(x)a$ with $(x, a) = (x, \varepsilon)a$, where ε is the unit element of G. An infinitesimal connection on P defines parallel displacement of elements of P. If $dx = (dx^\mu)$ is a small displacement at $x = \pi(p) \in N$, then the parallel transport of (x, ε) along dx results in $(x + dx, \varepsilon - A)$, where $A = A_\mu dx^\mu$ is a 1-form on N, with values in the Lie algebra $\mathcal{G}$ of G. Parallel transport should commute with the action of G : (x, a) displaced along dx becomes $(x + dx, a - Aa)$. If $\sigma' : N' \to P$ is another section, then there is a map $U : N \times N' \to G$ such that

$$s'(x) = s(x)\, U(x) \tag{2.2}$$

for $x \in N \times N'$. The section σ' leads to the diffeomorphism $k' : N' \times G \to \pi^{-1}(N')$, $k'(x, a) = s'(x)a = s(x)U(x)a$, and

$$k'(x, a) = k(x, Ua)$$
$$k'(x + dx, a) = k(x + dx, (U + dU)a)$$

Relative to k', parallel transport is described by a 1-form $A' = A'_\mu dx^\mu$. By parallel transport, the point $k'(x, \varepsilon)$ becomes $k'(x + dx, \varepsilon - A')$, which is the same as $k(x + dx, (U + dU)(\varepsilon - A'))$. On the other hand, $k'(x, \varepsilon) = k(x, U)$ is parallel to $k(x + dx, U - AU)$. Since parallel displacement in P

should not depend on the choice of section (gauge), $(U + dU)(\varepsilon - A') = U - AU$. This leads to the transformation law:

$$A' = U^{-1}(dU + AU) \tag{2.3}$$

of the potential under gauge transformations of the second kind. It follows from (2.3) that the $\mathcal{G}$-valued 1-form:

$$\omega = a^{-1}(da + Aa) \tag{2.4}$$

is independent of the section. The form ω has a simple geometric interpretation: $\varepsilon + \omega$ is the element of G that moves the point (x, a) into the point (x, a) $(\varepsilon + \omega) = (x, a + da + Aa)$ parallel to $(x + dx, a + da)$. The section-independent 1-form ω on P is called the *connection form*; it is the gauge- independent counterpart of the potential A. Relation (2.3) contains, as special cases, the transformation laws of the coefficients of a linear connection (Christoffel symbols, Ricci rotation coefficients), of the electromagnetic potentials, and of non-Abelian gauge potentials of the Yang- Mills type. The advantage of the connection form ω, defined on P, over the potential A, defined on $N \in$ □M, results from the following considerations: the connection form ω is defined independently of any section, whereas A refers to a (local) section of the bundle. Therefore, for a nontrivial bundle, the potentials are defined only locally, whereas the connection form ω is defined globally, all over P.

An interesting application of the bundle approach to gauge fields is the construction of Riemannian geometries of the Kaluza-Klein type. If there is a connection form ω on P, $g = g_{\mu\nu}dx^{\mu}dx^{\nu}$ is a metric tensor on M and h is a bi-invariant metric on G, the one can define a metric tensor γ on P by the formula

$$\gamma\,(u, v) = \gamma(T\pi(u), T\pi(v)) + \text{const}\, h(\omega(u), \omega(v)) \tag{2.5}$$

where u and v are vectors tangent to P, and $T\pi : TP \to TM$ is the projection of such vectors on M, induced by π. The metric γ is invariant under the action of G on P. For $G = SO(2)$ it coincides with the metric considered in five-dimensional, “unified’ theories of gravitation and electromagnetism.

3. Einstein General Relativity and Generalized Theories of Gravitation

Relativistic theories of gravitation—such as Einstein’s theory of general relativity—may also be considered as gauge theories. The bundle P consists in this case of orthonormal linear frames (tetrads, *Vierbeine*) of the spacetime manifold M and G is the Lorentz group. Alternatively, one can take P to be the bundle of orthonormal affine frames, in which case G is the inhomogeneous Lorentz group[1]. There are, however, important differences between Einstein’s theory and gauge

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[1] An important point is that Lorentz group leads to the study of representation theory of non-compact Lie groups. Recall that the classical Maxwell’s equations in vacuum for the electromagnetic field are invariant under the finite dimensional Lorentz group of linear isometries of $R^{3,1}$. One of the principles of the special theory of relativity is the invariance of the field equations under the non-compact Lorentz group or inhomogeneous Poincaré group. The Lorentzian structure plays a fundamental role in relativity. It has also interesting geometrical interpretations. Indeed, this happens when we study the geometry of spheres in space. All the contact transformations in space

theories such as electrodynamics or the Yang-Mills theory. First, the bundle of frames is soldered to the base M whereas in other gauge theories the bundle is rather loosely connected to M. The soldering results in appearance, in theories of gravitation, of *torsion*, in addition to *curvature*, which occurs in gauge theory. (Torsion is zero in Riemannian geometry, but being zero is different from not existing at all[2].) Moreover, the form of Einstein's equations of gravitation is different from the "generic" form of the field equations assumed in gauge theories. The latter are derived from Lagrangians quadratic in curvature, whereas the former are based on a linear Lagrangian. The possibility of constructing such a linear Lagrangian is also related to the existence of the soldering form on P.

In the past, there was much research and discussion on whether and in what sense gravitation is a gauge theory. Recently, this problem has been considered in connection with the program of constructing a "supersymmetric" theory of gravitation. In classical relativity, the following questions have been raised and given diverse answers by different authors:

1. What is the gauge group of gravitation?
2. What are the corresponding gauge potentials; what is the status of the metric tensor?
3. Can the form of the field equations be derived from arguments of gauge invariance?

The theoretical physicist R. Utiyama was the first to say that gravitation may be looked upon as a gauge theory; he identified its potentials with the coefficients of the Riemannian connection on spacetime (see Utiyama, 1956). Using gauge arguments, Dennis W. Sciama argued in favor of an asymmetric connection as the basis of gravitation and showed that spin may be the source of torsion. Independently, on the ground of heuristic considerations invoking a gauge group with translations (in addition to Lorentz transformations), Tom Kibble derived the full set of field equations of gravitation with spin and torsion; the Sciama-Kibble theory was later recognized as being essentially equivalent to Elie Cartan's theory of 1923. Cartan proposed to modify the Einstein theory of gravitation by allowing space-time to have torsion and relating it to the density of intrinsic angular momentum of a continuous medium. The idea of connecting torsion to spin has known new developments in the 1970th until recent years (see Kibble, 1979; and Trautman, 1976).

The need to propose alternative or more general gravitational theories stem from a dichotomy in theoretical physics. Strong, electromagnetic, and weak interactions find their successful description within the framework of relativistic quantum field theory in flat Minkowski space-time. These quantum fields reside in space-time but are separate from it. Gravitation, according to Einstein, deforms Minkowski space and enters in the dynamic

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carrying spheres to spheres form a 15-parameter Lie-group, and these that carry planes to planes a 10-paramet subgroup. Besides, the application of Lorentz group lead to the study of representation theory of non-compact Lie groups. These are representations which have to take place in Hilbert space because, for compact groups, the irreducible representations are finite-dimensional, but non-compact groups require infinite dimensions, and it was physicists (Weyl, Wigner) who first realized it.

[2] A definitive treatment of affine connections, together with a generalization to connections with torsion, was given by Elie Cartan in his fundamental paper, "Sur les variétés à connexion affine et la théorie de la relativité générale," (1923-24). The paper is more than a theory of affine connections. Its idea can be easily generalized to give a theory of connections in a fiber bundle with any Lie group, for whose treatment the Ricci calculus is no longer adequate. The paper shows, among other things, how Einstein's theory is a direct generalization of Newton's theory when the latter is expressed in an intrinsic form, that is, in general coordinates. Specifically, the following contributions could be singled out: 1) the introduction of the equations of structure and the interpretation of the so-called Bianchi identities as the result of exterior differentiation of the equations of structure; 2) the recognition of curvature as a tensor-valued exterior quadratic differential form.

Riemannian geometry of space-time. One branch of fundamental physics is highly successful in a flat and rigid spacetime, but gravitation requires a non-flat and dynamic spacetime.

(i) The geometrical independence of the metric g and linear connection Γ leads to the idea of treating these quantities as independent variable in the sense of a principle of least action. If g and Γ are assumed to be compatible, then the freedom in the choice of Γ reduces to that of the torsion tensor Q.

(ii) According to relativistic quantum theory, the Poincaré—or the inhomogeneous Lorentz group—is physically more significant than the Lorentz group itself. The Poincaré group has two fundamental invariants: mass and spin. The first among them is related to translations and to energy-momentum. In Einstein's theory, the density of energy-momentum is source of curvature whereas spin has no such direct dynamical significance. In a sense, Einstein-Cartan theory restores—to some extent—the symmetry between mass and spin. It introduces also an unexpected "duality": via Noether's theorem, energy-momentum is generated by translations whereas Einstein's equation relates it to curvature, which is responsible for rotations of vectors undergoing parallel transport. Conversely, spin is generated by rotations, but torsion induces translations in the tangent space to a manifold ("Cartan displacement"). This duality can be traced to the fact that the Einstein-Cartan Lagrangian is linear in curvature.

(iii) There is an interesting analogy between the description of magnetic moments in electrodynamics and spin in the theory of gravitation. In a phenomenological description of electromagnetism, the external magnetic field produced by a ferromagnet may be obtained in at least three ways: by considering a surface current equivalent to the actual distribution of microscopic currents and magnetic moments, by replacing the latter by a volume distribution of "Ampère currents", or, finally, by introducing a smooth field of the magnetization vector. In the Einstein theory, there are analogues for the first two descriptions, whereas the Einstein-Cartan theory provides the third.

The Einstein-Cartan theory assumes, as a model of spacetime, a four-dimensional manifold with a linear connection Γ compatible with a metric tensor g. The gravitational part of the Lagrangian, $\sqrt{-gR}$, if formed from the curvature tensor of Γ. The left-hand sides of the field equations are obtained by varying this Lagrangian with respect to g and Q. Variation with respect to g may be replaced by that relative to the field of frames. The sources of gravitational field are described by expressions resulting either from phenomenology or by varying an action integral obtained by applying the principle of minimal gravitational coupling to a special-relativistic Lagrangian. The Einstein-Cartan equations are:

$$R_{\square\nu} - 1/2g_{\square\nu}R = 8\pi G/c^4\ T_{\square\nu} \tag{3.1}$$

$$Q^{\theta}{}_{\square\nu\square} - \delta^{\theta}{}_{\square}Q^{\sigma}{}_{\sigma\nu} - \delta^{\theta}{}_{\nu}Q^{\sigma}{}_{\square\sigma} = 8\pi G/c^3\ s^{\theta}{}_{\square\nu} \tag{3.2}$$

The Cartan equation (3.2) is trivial in the sense that if the spin density vanishes, $\sigma^{\theta}{}_{\square\nu} = 0$, then so does torsion, $Q^{\theta}{}_{\square\nu} = 0$. Quite independently of this, torsion is topologically trivial: any linear connection can be deformed into a connection without torsion. The Einstein-Cartan theory may be physically relevant only when the density of energy is of the same order of magnitude as the spin density squared. For matter consisting of particles of mass m and spin $h/2$, this will occur at densities of order m^2c^4/Gh^2.

Chen Ning Yang pointed out that Einstein's theory is different from other gauge theories in being based on a Lagrangian that is linear, rather than quadratic, in curvature. He proposed considering a theory of gravitation based on Riemannian geometry and a Lagrangian of the form:

$$ {}^*\Omega^{\mu}{}_{\nu} \wedge \Omega^{\nu}{}_{\mu} \tag{3.3}$$

(the dual ${}^*\Omega_\nu{}^\mu$ of the curvature form $\Omega_\nu{}^\mu$ (where $\Omega_\nu{}^\mu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$) and the conformally invariant Lagrangian density). The source-free equations of this theory, $\nabla_\mu R_{\nu\rho} = \nabla_\nu R_{\mu\rho}$, appear to be too weak, e.g., they admit as a solution the de Sitter universe with an arbitrary radius of curvature. There is a modification of Yang's theory based on a metric connection with torsion and two sets of field equations, as in the Einstein–Cartan theory. It is clear, from the diversity of results and views, that there is no unique "gauge theory of gravitation." This is because gravitation is a "rich" theory from the geometrical point of view: it contains several invariants which may be used to build the kinetic part of the gravitational Lagrangian. The correspondence principle of relativistic gravity to the Newtonian theory suggest—but probably does not require—a Lagrangian linear in curvature, whereas the analogy with electrodynamics leads to the idea of a quadratic Lagrangian.

4. The Geometry of Gauge Theories

A gauge theory is any physical theory of a dynamical variable which, at the classical level, may be identified with a connection on a principal bundle. The structure group G of the bundle P is the group of gauge transformations of the first kind; the group $\mathcal{G}$ of gauge transformations of the second kind may be identified with a subgroup of the group Aut P of all automorphisms of P. In this sense, gravitation is a gauge theory: the basic gauge field is a linear connection ω. In addition to ω, there is a metric tensor g which plays the role of a Higgs field. The most important difference between gravitation and other gauge theories is due to the soldering of the bundle of the frames LM to the base manifold M. The bundle LM is constructed in a natural and unique way from M, whereas a noncontractible M may be the base of inequivalent bundles with the same structure group. For example, LS^2 reduced to $SO(2)$ is isomorphic to $SO(3)$, but there is a denumerable set of inequivalent $SO(2)$ bundles over S^2, corresponding to the different elements of $\pi_1(SO(2)) = \mathbf{Z}$. The soldering form θ leads to torsion which has no analog in nongravitational theories. Moreover, it affects the group $\mathcal{G}$, which now consists of the automorphisms of LM preserving θ. This group contains no vertical automorphism other than the identity; it is isomorphic to the group Diff M of all diffeomorphisms of M. In a gauge theory of the Yang-Mills type over Minkowski spacetime, the group $\mathcal{G}$ is isomorphic to the semidirect product of the Poincaré group by the group $\mathcal{G}_0$ of vertical automorphisms of P. In other words, in the theory of gravitation, the group $\mathcal{G}_0$ of "pure gauge" transformations reduces to the identity; all elements of $\mathcal{G}$ correspond to diffeomorphisms of M. What is the structure group G of the gravitational principal bundle? Since spacetime M is four dimensional, if $P = LM$ then $G = GL(4, \mathbf{R})$. But one can equally well take for P the bundle AM of affine frames: in this case G if the affine group. There is a simple correspondence between affine and linear connections which makes it immaterial whether one works with LM or AM. If one assumes—as usually one does— that ω and γ are compatible, then the structure group of LM or AM can be restricted to the Lorentz or the Poincaré group, respectively. It is also possible to take, as the underlying bundle for a theory of gravitation, another bundle attached in a natural manner to spacetime, such as the bundle of projective frames or the first extension of LM. The corresponding structure group are natural extensions of $GL(4, \mathbf{R})$, $O(1, 3)$ or the Poincaré group.

The importance of gauge theories in modern theoretical physics is well-known. Yang and Mill's new gauge theory should especially serve as a model for the study of strong interactions, including the quantum effects on them. The main feature of this gauge theory is the use of a non-Abelian Lie group, the simplest of the non-commutative continuous groups, as its invariance group. This mathematical property of the symmetry group gives a very rich structure to the theory, whose field equations are more general than Maxwell's. This already illustrates the fundamental role of both geometrical and internal symmetries in physical problems which can be handled by gauge theories. Already in Weyl's theory, in addition to the position variables of spacetime, there is an internal space parameter on which the phase group acts. The field identified with the particle's wave function can therefore be seen as associating to each point of spacetime a point of the internal space, or an angle (of rotation) in the case of electromagnetism. A gauge requires that the coordinates of spacetime be combined with the parameters of the internal space. Weyl's theory satisfies the "principle of local invariance": i.e., the field equations are invariant under a gauge shift.

5. Reflection upon some *aporetic* questions: how geometrical objects may yield physical effects? About the topology of the universe

It would be very fruitful to study these questions, both epistemologically and mathematically, in order to clarify the relation between geometrical concepts and physical entities that has become increasingly close during the last century and the current. From the general context we have sketched above, we would like to mention the following problems, and at the same time to raise some crucial issues in theoretical physics.

(1) In which way are the contents of certain recent physical theories, like gauge theories and topological quantum field theories, determined by the mathematical concepts underlying these theories? Up to what extent the geometrical structures underlying physical theories determine the physical behavior of phenomena? Putted differently, are physical "phenomena" essentially different from geometrical "structures" and what about the reasons?

In this regard, let us remark here that a connection, which is a well-defined geometrical object, is more primitive than the curvature. Therefore, we should consider the gauge potential to be more primitive that the gauge field. In fact, in the case of electromagnetism we can show experimentally that the field can be identically zero, but physical effects can still be detected: this is because the parallel transport need not be trivial if the region of space is not simply-connected. The vanishing of curvature only gives information about the parallel transport round very small, closed paths. Physically, the parallel transport is generally described in terms of a non-integrable phase factor. The property of non-integrability refers locally to the existence of a non-vanishing field, whereas large scale non-integrability is of a topological nature and may arise even if the field is zero.

Classically, the concept of potential was introduced as a mathematical device to simplify the field equations, and the arbitrary nature of the gauge characteristic in the choice of potential indicated that the potential did not really have a physical meaning. But geometrically, one can in fact show that such an interpretation is not satisfactory. The connection is a geometrical object and so the potential should be considered as having a physical nature. It is the choice of gauge describing the potential which has no physical meaning, and this corresponds to the fact that the geometrical fiber space where the connection sits has no (natural) horizontal sections.

(2) A more general problem concerns the relation between purely mathematical geometry and physical geometry. According to an idea going back to Riemann and Clifford and next developed by T. Levi-Civita, E. Cartan, H. Weyl and A. Einstein, physical concepts cannot be dissociated from geometrical ones, and conversely. Some remarks about the general relativity theory can help to understand what we mean by that. In this theory, the gravitational field is seen as the effect of a geometric distortion, a curvature or warping of spacetime. In this theory, as is well- known, freely falling bodies are not treated as subject to gravitational forces but are instead regarded as following the straightest possible path (a geodesic) in an underlying curved spacetime. In Newton's theory of gravitation, the Earth's orbit curves around the Sun because the Sun's gravity forces it to depart from its natural straight-line motion. In Einstein's theory there is non-gravitational forces as such. The Sun produces a warping of spacetime in its vicinity and the Earth travels freely along a geodesic in this curved spacetime. Gravity is treated as a geometrical effect precisely because it is universal; it affects all test objects in the same way. Thus, even light will follow a curved path in a gravitational field.

On a large scale, the distribution of galaxies throughout the universe will depend on the geometry of space. The fact that there might be a regular curvature of space on a cosmological scale raises the interesting question of the *topology of the universe*. So long as space is flat, it must be either infinite in extent, or else possess some sort of boundary. But if space is curved there are other possibilities. Think of the situation with a two-dimensional sheet. A curved sheet could be closed into a sphere, for example, or a torus. It is possible to envisage a three-dimensional version of a closed spherical surface, called a hypersphere. If the universe had the topology of a hypersphere, it would possess a finite volume, but there would be no boundary or edge to space. It is not known what topology space actually possesses, but the issue is crucial to the superstring theory.

One of the basic assumptions in modern cosmology, the *Cosmological Principle*, is that on large-scale average our universe is spatially homogeneous and isotropic. The apparent isotropy on large scales is normally explained because of spatial homogeneity, which in turn is understood as a natural result of an "inflationary" period of the early universe. An alternative approach to explaining the apparent homogeneity is to assume an expanding universe with small and finite space sections with a non-trivial topology, the "small universe" model. From the theoretical point of view, it is possible to have quantum creation of the universe, where fluctuations prevail, with a multiply connected topology. From the observational side, this model has been used to explain the "observed" periodicity in the distribution of quasars and galaxies.

(3) Another point of view (which is complementary to the idea in (2)) that has been recently developed by geometers and mathematical physicists, maintains that physical events in an n-dimensional space (n being arbitrary, but preferably large or infinite) chosen as the mathematical model for these events, are but a set of geometrical forms ruled by abstract mathematical objects. At the same time the physical phenomena somehow should embody these abstract mathematical models and evolve according to them. This theoretical assumption was advocated in the last years notably by M. Gromov, A. Connes and R. Penrose. In fact, one could say that space is not at all fixed or of a predetermined nature, but rather that it could be imagined as say, a topological frame or architecture, a sort of regulating principle, from which arise the essential rules governing the unfolding of physical phenomena.

6. The concept of space in quantum field theories

The theoretical question of explain the kind of relationships which could exist between the "mathematical" and the "ontological" nature of physical theories is by no means easy to solve. At any rate, it seems difficult to entirely accept the radical "platonic" vista which assumes that physical phenomena are but nothing than abstract idealizations arising from purely "objective" (that is, "existing in se") mathematical laws. Nevertheless, it is clear that recent physical theories (relativistic and quantum fields theories) do not share the same ontology with Newtonian physics. Indeed, the last had a definite ontology: the world consists of massive particles situated in Euclidean space. In that sense, the nature of space plays a fundamental role. In the mathematical development of Newtonian mechanics, however, the role of space is not so clear. There is little fundamental difference between the description of two particles moving in $\mathbf{R}^3$ and that of a single particle moving in $\mathbf{R}^6$, not between a pivoted rigid body and a point moving on the group-manifold $SO(3)$.

In quantum mechanics the idea of space is even more elusive, for there seems to be no ontology at all, and, whatever wave-functions are, they are certainly not functions defined in space. Still, for about seventy years we have known that elementary particles must be described not by quantum mechanics but by quantum field theory, and in field theory the role of space is quite different. Although it is an important fact that quantum field theory cannot be reconciled with general relativity, one could emphasize that the two theories have a virtual feature in common, for in both the points of space play a central and objective dynamical role. In quantum field theory two electrons are not described by a wave-function on $\mathbf{R}^6$; instead, they constitute a state of a field in $\mathbf{R}^3$ which is excited in the neighborhood of two points. Thus, the points of space *index* the observables in the theory.

The mathematics of quantum field theory is an attempt to describe the nature of space, but it proposes to look at space in a completely different way. Like quantum field theory, Penrose's twistor theory is a radical attempt to get rid of space as a primary concept. The Connes's program of non-commutative geometry amounts to a huge generalization of the classical notion of a manifold. Finally, string theory proposed a scheme for making space as an approximation to some more general kind of structure.

From a more philosophical point of view, one can say that it should be interesting to develop a theory that classifies the ways of actualizing and deploying abstract mathematical structures in real physical phenomena. In other words, how can one conceive that a set of abstract mathematical forms could, by a sequence of transformations of its basic elements, generates some changes in the properties of microscopic and macroscopic bodies? Such a theory would possibly let us understand the processes underlying the constitution of physical phenomena and the criterion for their individuation with respect to ideal geometrical "beings". It is worthwhile noting that the most abstract mathematical structures and concepts corresponds to geometrical operations and diagrammatic patterns which are often very essential, and that the generation of new space dimensions and structures relates to changes in the physical state of phenomena. For example, we know that the qualitative properties of certain physical (dynamical) system are sensitive to the dimension of the space, and that the geometrical and topological structure of the space puts constraints on the evolution of the system.

It is here worth mentioning an outstanding example. In 1984 the Britain physicist Michael Berry showed that the adiabatic evolution of energy eigenfunctions, with respect to a time-dependent quantum Hamiltonian $H(t)$, contains a phase of deeply geometrical origin in addition to the familiar dynamical phase:

$$\exp - 1/h \int E(t)dt. \tag{6.1}$$

The additional phase approaches a finite, non-zero limit as the Hamiltonian is taken more and more slowly around a closed path in its parameter space. The geometric phase $\gamma(C)$ (where C is a closed circuit on a sphere) measures the *anholonomy* of a physical (classical or quantum) system. Anholonomy is a geometrical phenomenon in which nonintegrability causes some variable to fail to return to their original values when others, which drive them, are altered round a cycle. The simplest anholonomy is in the parallel transport of vectors, two examples being the change in the direction of swing of a Foucault pendulum after one rotation of the earth, and the change in the direction of linear polarization of light along a twisting ray or coiled optical fiber whose direction is altered in a cycle. *Adiabaticity* is slow change and therefore denotes phenomena at the border between dynamics and statics. Adiabatic change provides the simplest way to make quantum parallel transport happen. The variables which are cycled are parameters in the Hamiltonian of a system. If the cycling is slow, the adiabatic theorem guarantees that the system returns to its original state. But it usually acquires a nontrivial phase, a manifestation of anholonomy.

Moreover, some mathematical ideas can provide a deep and powerful connection between, on the one hand, the geometrical symmetries of space, and on the other, the dynamical behavior of material bodies. In fact, forbidding the absence of spontaneous changes in motion amounts to a statement of the laws of conservation of momentum and regular momentum. The translation symmetry of space leads directly to momentum conservation for particles, whereas the rotational symmetry implies angular momentum conservation. In addition to this, the conservation of energy can be shown to follow from the translation symmetry of time. Thus, the most fundamental and comprehensive laws of physics are seen to follow from the basic fact that empty space and time are featureless. It well illustrates the power of symmetry in ordering the natural world. An interesting question now arises. Do all the forces of nature necessarily respect the geometrical symmetries of space and time? Certainly, Maxwell's electromagnetic theory as well as Einstein's general relativity theory incorporates all the symmetries we have just mentioned. What about the discrete (quantic) geometrical symmetries? How can the laws of physics be tested for them?

7. Continuous and discrete in mathematics and physics

Another important issue concerns the discrepancy between the continuous and the discrete in relation to the nature of space-time and the essential features of the physical world. How do we explain a world of phenomena that is essentially continuous macroscopically (on which are based the classical and relativistic theories of spacetime) but discontinuous microscopically at the quantum scale? Seemingly these two conceptions of nature are fundamentally different. On the one hand, there is the spacetime invariance of relativistic phenomena (homogeneity of space plus symmetry of time), where the mathematical model underlying the description of these phenomena is that of a compact differentiable pseudo-Riemannian manifold (M^4, g) of four dimensions endowed with a metric tensor g. Each event is localized at a point $p \in M$, and its evolution in time is represented by a (geodesic) curve γin M. In this theory, it seems impossible to separate the postulate of spacetime localization and the law of gravitation from the geometric structure of spacetime. On the other hand, it is precisely the axiom of spacetime continuous localization (and description) that got lost in quantum mechanics. It is specifically contradicted by the Bohr principle of complementarity and the Heisenberg uncertainty relations. The last states the impossibility of knowing simultaneously the exact position and velocity of an electron, and which are based on a model in which the electron jumps quickly from one orbit to another, radiating all energy thus liberated in the form of a global package, a *quantum* of light. These jumps give rise mathematically to discrete “quantum” numbers (integers). So mathematically, Hilbert space and operator theory symmetry, through group theory, established itself as a guiding principle and algebra took over. Hilbert space and operator algebra theory is the realm of discrete mathematics.

Should this prevent us from discovering a deeper, yet unknown level of theory and experience, where the discrete and the continuous character of the laws of physics are but special cases according with each other in the framework of a new unitary mathematical theory? The theory of supergravity, developed mathematically in the l970's, which generalizes a theory of gravitation conceived by Weyl in l923 and another by Kaluza- Klein about the same time, as well as the more recent superstring theory, gives some hope (only in theory, actually) of unifying the laws of physics. In fact, at the basis of this last theory there is a new symmetry called *supersymmetry* that acts even on a global level. It links the two large classes of elementary particles, the fermions (such as the electron, the proton, and the neutron) and the bosons (such as the photon), which, as is well-known, have very different properties. Since supersymmetry extends from the global to the local level, it leads to a theory which includes gravity, and which suggests the possibility of unifying it with the other forces.

In this new perspective, it would be very interesting to study particularly the relation between the topological structure of certain (local and global) groups acting on a certain family of non-smooth (quasi-conformal or symplectic?) manifolds and the corresponding kinds of physical symmetries and symmetries breaking. In fact, the study of the gauge theory invariants seems intimately related to the problem of constructing diffeomorphisms between 4-manifolds, or finding embedded surfaces of a given genus, which would complement the obstructions and invariants which have been found.

8. Invariants of four-manifold and quantum field theory

There is at present no doubt that some mathematical concepts of fiber bundle theory have become an established part of mathematical physics because fiber bundles provide a natural and very deep framework for discussing the concepts of relativity and invariance, describing gravitation and other gauge fields, and giving a geometrical interpretation to quantization and the canonical formalism of particles and fields. Fiber bundles provide the language which is needed for dealing with local problems of differential geometry and field theory. They are necessary to understand and solve global, topological problems, such as those arising in connection with magnetic monopoles and instantons. For example, to understand the properties of Donaldson invariants of four manifolds, E. Witten presented a new approach to using physics to illuminate Donaldson theory. He suggested that, instead of computing the Donaldson invariants by counting $SU(2)$ instanton solutions, one can obtain the same invariants by counting the solutions of the dual equations, which involve $U(1)$ gauge fields and monopoles. From a physical point of view the dual description via monopoles and abelian gauge fields should be simpler than the microscopic $SU(2)$ description since in the renormalization group sense it arises by “integrating out the irrelevant degrees of freedom.”

The new monopole equations and the topological invariants of four-manifolds introduced by Witten involve two entities, a $U(1)$ connection and a “spinor” field. Thus, a main pre-requisite for their study is a knowledge of spinors on 4-manifolds. More precisely, the most relevant notion is that of $Spin^c$ structure. Recall that if X is an oriented, closed Riemannian four-manifold, a spin structure on X is a lift of the structure group of the tangent bundle from $SO(4)$ to its double cover $Spin(4)$. The exceptional isomorphism $Spin(4) = SU(2) \times SU(2)$ means that this can be given a more concrete description in terms of vector bundles. Giving a spin structure is the same as giving a pair of complex 2-plane bundles $S^+, S^- \rightarrow X$, each with structure group $SU(2)$ and related to the tangent bundle by a structure map $c : TX \rightarrow Hom(S^+, S^-)$. Now the map $e \wedge f \rightarrow c(e)^* c(f) - c(f)^* c(e)$ induces a map ρ from the self-dual 2-forms Λ^+ to $Hom(S^+, S^-)$, which corresponds to the standard isomorphism between the Lie algebras of $SU(2)$ and $SO(3)$.

The map c is the symbol of the Dirac operator $D : \Gamma(S^+) \rightarrow \Gamma(S^-)$, and one of the most fruitful calculations in differential geometry leads to the Lichnerowicz-Weitzenbock formula for the Dirac operator:

$$D^*D\psi = \nabla^*\nabla\psi + 1/4R\psi. \tag{8.1}$$

Here ∇ is the covariant derivative on spinors, induced by Levi-Civita connection, and R is the scalar curvature, which acts in equation (8.1) by scalar multiplication at each point. If we have an additional auxiliary bundle $E \rightarrow X$, with a Hermitian metric and connection, we may consider spinors with values in E—sections of $S \otimes E$. The Dirac operator on these coupled spinors satisfies:

$$D^*D\psi = \nabla^*\nabla\psi + 1/4R\psi - F^+{}_E\,(\psi) \tag{8.2}$$

where $F^+{}_E$ is self-dual part $1/2(F_E + {*F_E})$ of the curvature E. Here the self-dual forms act on spinors in the way described above. Now a spin structure may not exist globally—the Stiefel-Whitney class $w_2(X) \in H^2(X; \mathbf{Z}/2)$ is the obstruction—but a variant, a $Spin^c$ structure, always does. A $Spin^c$ structure is given by a pair of vector bundles $W^\pm$ over X with an isomorphism $\Lambda^2 W^+ = \Lambda^2 W^- = L$ say, such that locally $W^\pm = S^\pm \otimes L^{1/2}$, where $L^{1/2}$ is a local square root of L: $L^{1/2} \otimes L^{1/2} = L$. An old result of Hirzebruch and Hopf assures the existence of $Spin^c$ structures on any oriented, closed four-manifold; up to an action of the finite group $H^1(X; \mathbf{Z}/2)$ they are classified by the lifts of $w^2(X)$ to $H^2(X; \mathbf{Z})$, the first Chern class of the line bundle L. A connection on L gives a Dirac operator $D : G(W^+) \otimes G(W^-)$, which is locally just the same as the Dirac operator on $L^{1/2}$-valued spinors. In particular, we get the Lichnerowicz formula:

$$D^*D\psi = \nabla^*\nabla\psi + 1/4R\psi - 1/2F^+{}_L(\psi), \tag{8.3}$$

where the factor of 1/2 comes from the square root of L. Note that $Hom(W^+, W^+) \sim Hom(S^+, S^+)$. Now, the Seiberg-Witten equations for a four-manifold X with $Spin^c$ structure $W^\pm$ are equations for a pair (A, y) where:

1. A is a unitary connection on $L = \Lambda^2 W^\pm$,
2. ψ is a section of W^+.

If ξ and η are in W^+, we write $\xi\eta^*$ for the endomorphism $\theta \rightarrow \langle\theta, \eta\rangle\xi$ of W^+. The trace-free part of this endomorphism lies in the image of the map ρ, and we write $\tau(\xi, \eta)$ for the corresponding element of $\Lambda^+ \otimes \mathbf{C}$. So τ is a sesquilinear map $\tau : W^+ \times W^+ \rightarrow \otimes \mathbf{C}$. The Seiberg-Witten equations are:

$$D_A\psi = 0, \tag{8.4}$$

$$F^+_A = -\,\tau(\psi, \psi). \tag{8.5}$$

The sign of the quadratic form $\tau(\psi, \psi)$ is crucial and underpins the whole theory.

Witten showed that, in general, the number of solutions of a system of equations weighted by the sign of the determinant of the operator analogous to T (an elliptic operator $T : \Lambda^1 \otimes (S^+ \otimes L) \to \Lambda^0 \otimes \Lambda^{2,+} \otimes (S^- \otimes L)$ is defined by $T = s^* \otimes t$) is always a topological invariant if a suitable compactness holds. If one has a gauge invariant system of equations, and one wishes to count gauge orbits of solutions up to gauge transformations, then one requires (i) compactness; and (ii) free action of the gauge group on the space of solutions. By contrast with Donaldson theory, according to which for $SU(2)$ instantons, compactness fails precisely because an instanton can shrink to zero size, the monopole equations are scale invariant but they have no non-constant L^2 solutions on flat $\mathbf{R}^4$.

The general problem behind the above result is that of find topological invariant defined by solutions of partial differential equations. In differential topology one is familiar with many contexts in which the solutions of an equation $f(x) = y$ is, at the level of homology, unchanged by continuous variations of parameters. For example, f might be a map $f : P \to Q$ between oriented manifolds, then the homology class in $H_*(P)$ of $f^{-1}(y)$, for generic y in Q, is a homotopy invariant of f—just the Poincaré dual of the pull-back of the fundamental cohomology class of Q. Or f might be a section of an oriented vector bundle $V \to P$, and $y = 0$, so the solutions are the zero set of the section which, assuming transversality, give a submanifold representing the Poincaré dual of the Euler class of V. Now if we have a family of partial differential equations, depending on continuous parameters, we may hope to find similar invariants from the homology class of the solution space. This can be developed abstractly in the framework of differential topology in certain manifolds. The key points one need to establish to find invariants analogous to the finite dimensional case are:

1. The maps involved should be *Fredholm* maps, which in practice means that the linearization of the equations about a solution should be represented by linear *elliptic* differential equations, say over a compact manifold. The index of the linearized equation gives the “expected dimension” of the solutions space.
2. One need to establish the *compactness* of the space of solutions, or some weaker analogue of this.
3. One need to establish orientability, analogous to the finite dimensional case; otherwise, one only gets invariants modulo 2. This can be set up in terms of the index theory of families of operators. In the cases arising from gauge theory, the equations are invariant under the action of the gauge group of bundle automorphisms, and one studies spaces of solutions modulo this action.
4. One must not encounter reducible solutions in generic 1-parameter families of equations.

Now one can show that the essential features of Seiberg-Witten equations listed above, define differential-topological invariants of the underlying four-manifold. Indeed, the theory is significantly simpler than for the Donaldson instanton equations. To check the Fredholm property we can ignore the quadratic term $\tau(\psi, \psi)$ since this does not affect the symbol (leading term) of the linearization. At the level of the symbol, the linearization is given by the sum of the linearization of the $U(1)$ instanton equation, which modulo gauge is represented by the operator $d^* + d^+$ acting on ordinary forms, and the Dirac operator D_A. Regarding compactness, unlike the instanton case, the Seiberg-Witten moduli spaces are compact, without qualification. This follow from *a priori* estimates on the solutions. These can be obtained from energy estimates, using integration by parts as in the previous section, or, more directly by the maximum principle applied to second-order equations. The remaining issues are reducibles and orientations. If a non-trivial gauge transformation $g \in \mathrm{Aut}(L)$ fixes a pair (A, ψ), then ψ must be zero, and $g \in U(1)$ a constant scalar. Thus the only reducible Seiberg-Witten solutions are the self-dual $U(1)$ connections, and these do not occur in generic r-dimensional families of metrics on X, so long as $b^+(X) > r$. Thus, if $b^+ > 1$, reducibles do not interfere with the definition of invariants. Considering orientations: an orientation of the moduli space is furnished by an orientation of the "determinant line" of the relevant index bundle over the space C^* of all irreducible pairs (A, ψ) modulo gauge transformation.

The most straightforward application of the Seiberg-Witten invariants is to distinguish differentiable four-manifolds within the same homeomorphism type. Myriads of examples could be given: the simplest being to show that connected sums $X_{p,q}$, say, of p copies of $\mathbf{CP}^2$ and q copies of $\mathbf{CP}^2$, $q > 1$, for which the Seiberg-Witten invariants vanish, are not diffeomorphic to Kähler surfaces (or any other manifolds with non-zero Seiberg-Witten invariants). The Seiberg-Witten equations have led to astounding advances in four-manifold theory. To some extent they may well have brought the study of the gauge theory invariants to a fairly complete form, resolving many of the main problems that drove research in this area in the last ten years. Perhaps the most exciting challenge for is to come to grips with the quantum field theory ideas which led to these new advances. —in parallel with other important developments such as mirror symmetry, 3-manifold invariants, conformal field theory—and to understand in a rigorous way the intricate structures discovered by Seiberg and Witten. At the same time there are notable questions which are left open at present. One is the question of whether all simply-connected manifolds are of simple type. A more wide- ranging problem is to understand the structure of the invariants of families of 4-manifolds, and the relation between the instanton and Seiberg-Witten theories, for manifolds with $b^+ = 1$. By considering an r-dimensional family of equations, of either kind, one should get invariants which are roughly speaking cohomology classes in $H^r(B\ \mathrm{Diff}\ (X))$ where $B\ \mathrm{Diff}\ (X)$ is the classifying space of the diffeomorphism group of a 4-manifold X. Then the same issues which complicate the story for ordinary invariants when $b^+ = 1$ should arise, for any X, once $r \geq b^+ - 1$. In another direction one may consider 4-manifolds which are not smooth. The instanton theory can be extended to the class of *quasi-conformal* 4-manifolds, where the coordinate change maps are only quasi-conformal, not necessarily smooth.

To see the relation of these results to quantum field theory, one must recall the analysis of $N = 2$ supersymmetric Yang-Mills theory. To begin, we work on flat $\mathbf{R}^4$. It has long been known that this theory has a family of quantum vacuum states parametrized by a complex variable u, which corresponds to the four-dimensional class in Donaldson theory. For $u \to \infty$, the gauge group is spontaneously broken down to the maximal torus, the effective coupling is small, and everything can be computed using asymptotic freedom. For small u, the effective coupling is strong. Classically, at $u = 0$, the full $SU(2)$ gauge symmetry is restored. But the classical approximation is not valid near $u = 0$. Quantum mechanically, the u plane turns out to parametrize a family of elliptic curves, in fact, the modular curve of the group $\Gamma(2)$. The family can be described by the equation:

$$y^2 = (x^2 - \Lambda^4)\,(x - u), \tag{8.6}$$

where Λ is the analog of a parameter that often goes by the same name in the theory of strong interactions. (The fact that $\Lambda \neq 0$ means that the quantum theory does not have the conformal invariance of the classical theory.) The curve (8.6) is smooth for generic u, but degenerates to a rational curve for $u = \Lambda^2$, –□Λ^2, or ∞. Near each degeneration, the theory becomes weakly coupled, and everything is calculable, if the right variables are used. At $u = \infty$, the weak coupling is (by asymptotic freedom) in terms of the original field variables. Near $u = \pm\,\Lambda^2$, a magnetic monopole becomes massless; the light degrees of freedom are the monopole or dyon and a dual photon or $U(1)$ gauge boson. In terms of the dyon and dual photon, the theory is weakly coupled and controllable near $u = \pm\,\Lambda^2$.

9. Remarks on space-time continuum and its relationship with the new physical theories.

A major problem in contemporary physics is the understanding of the structure of space and time. Recent research seems to tend toward a very different conception from everyday experience. The various most current relevant theories (of quantum gravity, supergravity, superstring, noncommutative geometry) have in common a tendency to unify spacetime and dynamics, geometrical structures, and interactions of physical theories. General relativity and quantum mechanics have already led to a profound revision of our conception of space and time, especially at the cosmological and quantum levels. The first is based on a model of spacetime which is a continuum of 4 dimensions with a finite metric. Quantum mechanics, on the other hand, admits that the space at the subatomic level of particles has a discrete structure and his metric presents infinite terms.

The attempts to develop a relativistic (local) quantum theory, that is a quantum field theory, were made to find a continuous structure for spacetime that was finer than the model of general relativity. This development of concepts of space and time characterizes the quantum theory of gravity. We can now highlight some new ideas partly responsible for the importance of quantum fields theories:

(1) Since quantum fields theories deal with spacetime distributions, field theories pertain directly to the concept of the spacetime continuum.

(2) The renormalization program in quantum fields theories, which must be seen as a constituent part of them, concerns the definition of the theory at short distances—i.e., his very finite spacetime structure. The results of the renormalization program suggest an intimate connection between the short distance structure of spacetime and interaction, which is a fundamental concept in quantum field theory.

Modern physics is dominated by the paradigm of local, relativistic, quantum field theory which represents thus the implementation of a new theoretical scheme. These theories possess an extraordinary potential for the unification of the fundamental phenomena, since:

(i) They provide a unique object, the quantum field, to account both for those phenomena which in some well-defined classical limit have a particle appearance as well as for those which in the same limit have field (wave) appearance.

(ii) They substitute action at distance by local interaction and solve in this way the old "forces/bodies" dualism. All fields contribute here with the same right—although taking various roles. So, interaction between matter fields (e.g., electrons) is transmitted by the gauge fields (e.g., photons) and vice versa. Since particles creation and annihilation is a fundamental feature of these theories, the kinematical concept of particle loses its primordial character, while the role of the interaction (dynamics) becomes dominant. We achieve in this way, among other things, a consistent treatment of the “transmutation” phenomena—but also a new picture of the vacuum.

(iii) They helped recognize a fundamental constructive principle—gauge symmetry and broken symmetry—which may help develop the mathematical scheme for a unified view of the various phenomena (electromagnetic, weak, strong, gravitation). In general, the treatment and understanding of symmetries is well promoted by the quantum field theoretical formalism.

(iv) They introduced a construction—the renormalization group (in the sense of Kenneth Wilson)— which allows one to follow quantum phenomena to smaller and smaller scales and trace back in this way the emergence of the diversity of phenomena as induced by phase transitions from a unified, fundamental interaction (Grand Unified Theory). Thereby, and in connection with symmetries, an intimate connection between dynamics and the spacetime continuum was suggested, which then led to several new ideas about the structure of the latter, but which may lead to far reaching conceptual consequences in future theoretical developments.

One can further outline some new ideas relating to the structure of spacetime in the most recent physical theories, to start with general relativity.

(1) *The geometric structure of spacetime gives rise to the dynamics of this same spacetime*, and remarkably of the gravitational field. The other physical fields, electromagnetic and the field of matter, however, remain outside this theory. But some physical theories developed in the last four decades shows quite compellingly that in fact even the electromagnetism and the other physical forces (the nuclear—strong and weak—interactions of elementary particles) emerges as dynamical effects from the geometrical and topological structure of spacetime. Conversely, the spacetime itself must be henceforth thought, in some way, as a derived, variable concept which can be subject to the quantum fluctuations of the different fields of matter. Moreover, in string theory the auxiliary two-dimensional field theory (which is needed to describe a vibrating string) plays in a way a more fundamental role than spacetime, and spacetime exists only to the extent that it can be reconstructed from the two-dimensional field theory. So, string theory, if correct, entails radical change in our concepts of spacetime and a new way of thinking the interactions between geometry and physics.

(2) *The physical symmetries dictate the different interactions between forces and between particles*. This is a very general principle, and it is the crucial idea at the heart of quantum field theories. Actually, all the natural phenomena seem to be founded upon such principle. Symmetry, Lie’s groups, and gauge invariance are now recognized, through theoretical and experimental developments, to play essential role in determining the basic forces of nature. Furthermore, and this is very exciting, while great successes have been achieved in these developments, we are still far from a grand synthesis. It is reasonable to believe this is because the full meaning of the word symmetry is not yet understood, and key additional concepts are still missing.

(3) *Gauge invariance has been recognized as a universal physical principle governing the fundamental forces and interactions between particles and matter*. All physical theories known so far, can be formulated by using this principle. Moreover, there is the hope that all physical theories might be related to each other by a common gauge group of symmetries.

There were two stated motivations that lay behind the historical discovery by Yang-Mills (1954) of the first non-Abelian gauge theory. First, they wanted to find *a principle* that would enable him to select a theory and determine the interactions. The principle was that of a local symmetry. The second motivation was simply to generalize the local gauge invariance of electrodynamics of the non-Abelian symmetry of isotopic-spin. Isotopic-spin was the first symmetry that was evident in the strong interactions, which was introduced by Werner Heisenberg and Eugene Wigner. The novel idea—that the isotopic-spin connection, and therefore the potential, acts like the *SU*(2) symmetry group—is the most important result of the Yang-Mills theory. This concept lies at the heart of local gauge theory. It shows explicitly how the gauge symmetry group is built into the dynamics of the interaction between particles and field[lxxiv]. In the late 1960s, Yang attempts to give a new formulation of gauge fields, through the approach of nonintegrable phase factors. The remarkable thing here is that the physical contents of gauge theory and the mathematical formalism in post-Riemannian geometry, which is based on the concept of connection and fiber space with connection, are strictly similar.

(4) The principal problem of theoretical physics today, and perhaps of science in general, *is that of arriving at a unifying theory of all fundamental physical forces*. More specifically, it is a matter of unifying the various quantum fields theories with the theory of gravity. We know that quantum mechanics and general theory of relativity are mutually incompatible. General relativity fails to comply with the quantum laws that govern the behavior of elementary particles, whereas on the opposite scale, black holes are challenging the very foundations of quantum mechanics.

It seems more and more justified to believe that superstring achieve remarkable progress in the search for a theory of all fundamental interactions in nature. One of the most important features of string theories is the unification of gauge couplings. There are notably two reasons why this is a particularly compelling feature to study. On the one hand, the unification of gauge coupling—like the appearance of gravity or of gauge symmetry in the first place—is a feature intrinsic to string theory. On the other hand, viewing the situation from an experimental perspective, the unification of gauge couplings is arguably the highest-energy phenomenon that any extrapolation from low energy data can uncover; in this sense it sits at what is believed to be the frontier between our low-energy $SU(3) \times SU(2) \times U(1)$ world, and whatever may lie beyond. Thus, the unification of gauge couplings provides a fertile meeting-ground where string theory can be tested against the results of low-energy experimentation.

However, the problem is a theorists' problem par excellence. Experiment provides little guide, and the inconsistency between the theory of gravity and quantum mechanics is an important problem that clearly illustrates the intermingling of philosophical, mathematical, and physical thought. In fact, to unify general relativity with quantum field theory, it seems necessary to conceive a new mathematical framework, which will generalize Riemannian geometry and therefore our present conception of space and spacetime.

Contrary to the conception (which goes back to Isaac Newton and even more to Michael Faraday) that try to develop physics on a rigorous basis to give formal justification to the experimental conclusions, we should attempt to understand the deeper meaning of the physics-mathematics connection. Rather than view mathematics as a tool to establish physical theories, or physics as a way of pointing to mathematical truths, we can try to dig more deeply into the relation between them. This investigation may not only have philosophical or theoretical interest, but it could lead to better understanding and even to new insights in the geometric and topological structures to be found at the frontier between mathematics and physics.

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