# Recursive properties of branching functions for affine Lie algebras

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**Abstract**

Using the recursion properties of the Costant-Heckman function we describe the reduction of the highest weight integrable modules of nontwisted affine Lie algebra with respect to a regular subalgebra.

## 1 Introduction

Quotienting affine Lie algebras is one of the key tools in constructing conformal field models and in particular in studying the string functions and fusion rules [1]. A general problem is to find the complete branching functions of the cosets. To solve this problem different kind of algorithms were proposed (see [2], [3], [4] and references therein).

We propose to apply the recursion properties of Costant-Heckman function to describe the reduction of the highest weight integrable modules of nontwisted affine Lie algebra with respect to its regular subalgebra.

The obtained recursion relations are formulated in terms of anomalous relative multiplicities. This technique provides a compact and effective algorithm to construct the branching functions.

## 2 Anomalous multiplicities.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be the affine Lie algebras with the regular injection $\mathfrak{b} \longrightarrow \mathfrak{a}$ and the corresponding finite-dimensional subalgebras $\mathfrak{a}_0 \longrightarrow \mathfrak{b}_0$. The following notations will be used:

$\Delta$ $(\Delta_{\mathfrak{b}})$– the root system; $\Delta^+$ $\left(\Delta_{\mathfrak{b}}^+\right)$– the positive root system;
$m(\alpha)$ $(m_{\mathfrak{b}}(\alpha))$ – the multiplicity of $\alpha \in \Delta$ ,$(\in \Delta_{\mathfrak{b}})$;
$\Delta_0$ , $(\Delta_{0\mathfrak{b}})$ – the finite root system of the subalgebra $\mathfrak{a}_0$ , $(\mathfrak{b}_0)$;
$L^{\mu}$ , $(L_{\mathfrak{b}}^{\nu})$ – the irreducible module of $\mathfrak{a}$ $(\mathfrak{b})$ with the highest weight $\mu$ $(\nu)$ ;
$W$ , $(W_{\mathfrak{b}})$– the Weil group;
$C$ , $(C_{\mathfrak{b}})$– the fundamental Weil chamber and $\overline{C}$ $\left(\overline{C}_{\mathfrak{b}}\right)$ – the corresponding closures;
$\rho$ , $(\rho_{\mathfrak{b}})$ – the Weil vector;
$P$ , $(P_{\mathfrak{b}})$ – the weight lattice;
$m_{\xi}^{(\mu)}$ , $\left(m_{\xi}^{(\nu)}\right)$ – the multiplicity of $\xi \in P$ $(\in P_{\mathfrak{b}})$ for the representation $L^{\mu}$ , $(L_{\mathfrak{b}}^{\nu})$;
$\mathcal{E}$ , $(\mathcal{E}_{\mathfrak{b}})$– the group algebra of the group $P$ $(P_{\mathfrak{b}})$ ;
$r$ , $(r_{\mathfrak{b}})$ – the rank of algebra $\mathfrak{a}$ $(\mathfrak{b})$ ;

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$\alpha_i$ , $(\alpha_{(\mathfrak{b})j})$ - the simple root for $\mathfrak{a}$ $(\mathfrak{b})$ , $i = 0, \ldots, r$ , $(j = 0, \ldots, r_{\mathfrak{b}})$ .

To describe the irreducible representations $L^{\mu}(\mathfrak{a})$ and $L^{\lambda}(\mathfrak{b})$ we shall use the so called anomalous multiplicities. Let $\xi \in P$ and $\mathfrak{b} \subset \mathfrak{a}$. The anomalous multiplicity $\widetilde{m}_{\xi}^{(\mu)}$ of $L^{\mu}$ is a function

$$\widetilde{m}_{\xi}^{(\mu)} = \det(w)\,\delta_{\xi, w(\mu+\rho)-\rho}. \tag{1}$$

Let $L^{\mu}(\mathfrak{a})$ be an irreducible integrable module with the highest weight $\mu$ and consider its reduction $L^{\mu}(\mathfrak{a})_{\downarrow\mathfrak{b}}$ with respect to the regular subalgebra $\mathfrak{b} \subset \mathfrak{a}$.

$$L^{\mu}(\mathfrak{a})_{\downarrow\mathfrak{b}} = \bigoplus_{\lambda \in P} n_{\lambda}^{(\mu)} L^{\lambda}(\mathfrak{b}), \tag{2}$$

($\lambda\chi \in \overline{C}_{\mathfrak{b}}$ are the coimages of $\chi \in \overline{C}_{\mathfrak{b}}$ in $P$) The set $M(L^{\mu})$ of relative highest weights $\lambda$ corresponding to the injection $\mathfrak{b} \longrightarrow \mathfrak{a}$ is formed,

$$M(L^{\mu}) = \left\{ \lambda \in P \mid n_{\lambda}^{(\mu)} > 0 \right\} \tag{3}$$

When $\xi$ is in $\mathrm{CoIm}\overline{C}_{\mathfrak{b}}$ the anomalous relative multiplicity $\widetilde{n}_{\xi}^{(\mu)}$,

$$\widetilde{n}_{\xi}^{(\mu)} = \sum_{\lambda \in M(L^{\mu})} \det\left(v^{\lambda}(\xi)\right) n_{\lambda}^{(\mu)} = \sum_{\lambda \in M(L^{\mu}), v \in W_{\mathfrak{b}}} \det(v)\, n_{\lambda}^{(\mu)} \delta_{\xi}^{v(\lambda+\rho_{\mathfrak{b}})-\rho_{\mathfrak{b}}}, \tag{4}$$

coincides with the relative multiplicity $n_{\xi}^{(\mu)}$ and thus defines (for the injection $\mathfrak{b} \longrightarrow \mathfrak{a}$ ) the branching rule in the form (2).

## 3 Generalized Kostant-Heckman formula

The formal characters in the relation

$$\mathrm{ch}L^{\mu}(\mathfrak{a}) = \sum_{\lambda \in P} n_{\lambda}^{(\mu)} \mathrm{ch}L^{\lambda}(\mathfrak{b}) \tag{5}$$

are [2]

$$\mathrm{ch}L^{\mu}(\mathfrak{a}) = \frac{\sum_{w \in W} \det(w) e^{w(\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^{+}} (1-e^{-\alpha})^{m(\alpha)}}, \qquad \mathrm{ch}L^{\lambda}(\mathfrak{b}) = \frac{\sum_{v \in W_{\mathfrak{b}}} \det(v) e^{v(\lambda+\rho_{\mathfrak{b}})-\rho_{\mathfrak{b}}}}{\prod_{\beta \in \Delta_{\mathfrak{b}}^{+}} (1-e^{-\beta})^{m_{\mathfrak{b}}(\beta)}}. \tag{6}$$

For regular injections, $\Delta_{\mathfrak{b}}^{+} \subset \Delta^{+}$, the branching rule can be rewritten,

$$\frac{\sum_{w \in W} \det(w) e^{w(\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^{+}} (1-e^{-\alpha})^{m(\alpha)-m_{\mathfrak{b}}(\alpha)}} = \sum_{\lambda \in P} n_{\lambda}^{(\mu)} \sum_{v \in W_{\mathfrak{b}}} \det(v) e^{v(\lambda+\rho_{\mathfrak{b}})-\rho_{\mathfrak{b}}} = \sum_{\xi \in P} \widetilde{n}_{\xi}^{(L)} e^{\xi}. \tag{7}$$

The coefficients $\widetilde{n}_{\xi}^{(L)}$ are the relative anomalous multiplicities for the representation $L^{\mu}(\mathfrak{a})$ (compare with (4)). In terms of Kostant-Heckman partition function $K_{\mathfrak{b}\subset\mathfrak{a}}$ related to the injection $\mathfrak{b} \longrightarrow \mathfrak{a}$ ,

$$\left( \prod_{\alpha \in \Delta^{+}} (1-e^{-\alpha})^{m(\alpha)-m_{\mathfrak{b}}(\alpha)} \right)^{-1} = \sum_{\xi'} K_{\mathfrak{b}\subset\mathfrak{a}}(\xi') e^{-\xi'}, \tag{8}$$

we have

$$\sum_{\xi'}\sum_{w\in W}\det(w)K_{\mathfrak{b}\subset\mathfrak{a}}(\xi')e^{w(\mu+\rho)-(\rho+\xi')}=\sum_{\xi\in P}\widetilde{n}_{\xi}^{(L)}e^{\xi} \tag{9}$$

and

$$\widetilde{n}_{\xi}^{(L)}=\sum_{W}\det(w)K_{\mathfrak{b}\subset\mathfrak{a}}(w(\mu+\rho)-(\rho+\xi)) \tag{10}$$

– the generalized Kostant-Heckman formula for affine Lie algebras.

## 4 Recursion relations for multiplicities $\widetilde{n}_{\xi}^{(L)}$

Consider the trivial representation $L^{0}(\mathfrak{a})$ with the highest weight $\mu=0$,

$$\widetilde{n}_{\varsigma}^{(0)} \;=\; \sum_{W}\det(w)K_{\mathfrak{b}\subset\mathfrak{a}}(w\rho-\rho-\varsigma)=\det\left(v^{0}(\varsigma)\right)n_{0}^{(0)}=\det\left(v^{0}(\varsigma)\right). \tag{11}$$

The recursion property for the function $K_{\mathfrak{b}\subset\mathfrak{a}}$ follows:

$$K_{\mathfrak{b}\subset\mathfrak{a}}(-\varsigma)=-\sum_{u\in W\setminus e}\det(u)K_{\mathfrak{b}\subset\mathfrak{a}}(-\varsigma+(u-1)\rho)+\det\left(v^{0}(\varsigma)\right). \tag{12}$$

Applying it to (10) we get

$$\begin{aligned}\widetilde{n}_{\xi}^{(L)} \;=\; & -\sum_{u\in W\setminus e}\det(u)\sum_{w\in W}\det(w)K_{\mathfrak{b}\subset\mathfrak{a}}\left(\begin{array}{c}w(\mu+\rho)-\\ -(\rho+(\xi+(1-u)\rho))\end{array}\right)+\\ & +\sum_{w\in W}\det(w)\det\left(v_{\mathfrak{b}}^{0}(\xi-w(\mu+\rho)+\rho)\right),\end{aligned} \tag{13}$$

and finally obtain the recursion relation for anomalous relative multiplicities:

$$\widetilde{n}_{\xi}^{(L)} \;=\; -\sum_{W\setminus e}\det(w)\widetilde{m}_{\xi+(1-w)\rho}^{(L)}+\sum_{W}\det(w)\det\left(v^{0}(\xi-w(\mu+\rho)+\rho)\right). \tag{14}$$

The recursion relation (14) is true for any regular subalgebra of $\mathfrak{a}$. In particular we can apply (14) to the injection in $\mathfrak{a}$ of its finite-dimensional subalgebra $\mathfrak{a}_0\longrightarrow\mathfrak{a}$. In that case the values of $v^0$ are formed by the finite subgroup $W_0\subset W$.

The relation (14) can be applied for both infinite- and finite-dimensional algebras. It can be considered as a generalization of recursion relations for classical Lie algebras obtained in [5].

## 5 Examples

Consider the branchings for the reduction $L^{\omega_0}\left(\widehat{sl(3)}\right)_{\downarrow\widehat{sl(2)\oplus u(1)}}$ and apply the recursion relations (14). They contain an additional information – the method provides not only the multiplicities but also the highest weights of subrepresentations. In our example the branchings (up to the grade $n = -5$) looks as follows (the highest weights are indicated in brackets by their $u(1)$-eigenvalue and the grade):

$$
\begin{array}{l}
L^{\omega_0}\left(\widehat{sl(3)}\right)_{\downarrow \widehat{sl(2)}} = L_b^{\omega_0}(2,-3) + L_b^{\omega_0}(2,-4) + 2L_b^{\omega_0}(2,-5) + \ldots \\
+L_b^{\omega_1}(1,-1) + L_b^{\omega_1}(1,-2) + 2L_b^{\omega_1}(1,-3) + 3L_b^{\omega_1}(1,-4) + 5L_b^{\omega_1}(1,-5) + \ldots \\
+L_b^{\omega_0}(0,0) + L_b^{\omega_0}(0,-1) + 2L_b^{\omega_0}(0,-2) + 3L_b^{\omega_0}(0,-3) + 5L_b^{\omega_0}(0,-4) + \ldots \\
+L_b^{\omega_1}(-1,-1) + L_b^{\omega_1}(-1,-2) + 2L_b^{\omega_1}(-1,-3) + 3L_b^{\omega_1}(-1,-4) + 5L_b^{\omega_1}(-1,-5) + \ldots \\
+L_b^{\omega_0}(-2,-3) + L_b^{\omega_0}(-2,-4) + 2L_b^{\omega_0}(-2,-5) + \ldots
\end{array}
\qquad (15)
$$

Notice that being reduced to the fixed $u(1)$-eigenvalue the branching functions here coincide with the inverse of the Euler function.

## 6 Conclusion

The recursion property of highest weight integrable representations is valid for any regular injection of affine Lie algebras.


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