A vielbein formalism for SHP general relativity

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Abstract. The 4+1 formalism in general relativity expresses the Einstein equations as a manifestly covariant initial value problem, resulting in a pair of first order evolution equations for the metric $\gamma_{\mu\nu}$ and intrinsic curvature $K_{\mu\nu}$ of *spacetime* geometry ($\mu,\nu = 0,1,2,3$). This approach extends the Stueckelberg-Horwitz-Piron (SHP) framework, a covariant approach to canonical particle mechanics and field theory employing a Lorentz scalar Hamiltonian K and an external chronological parameter τ. The SHP Hamiltonian generates τ-evolution of spacetime events $x^\mu(\tau)$ or $\psi(x,\tau)$ in an *a priori* unconstrained phase space; standard relativistic dynamics can be recovered *a posteriori* by imposing symmetries that express the usual mass shell constraint for individual particles and fields as conservation laws. As a guide to posing field equations for the evolving metric, we generalize the structure of SHP electrodynamics, with particular attention to O(3,1) covariance. Thus, the 4+1 method first defines a 5D pseudo-spacetime as a direct product of spacetime geometry and chronological evolution, poses 5D field equations whose symmetry must be broken to 4D, and then implements the implied 4+1 foliation to obtain evolution equations. In this paper, we sharpen and clarify the interpretation of this decomposition by introducing a fixed orthonormal *quintrad* frame and a 5D vielbein field that by construction respects the preferred 4+1 foliation. We show that for any diagonal metric, this procedure enables the evolution equation for the metric to be replaced by an evolution equation for the vielbein field itself, simplifying calculation of the spin connection and curvature.

1. Introduction

In general relativity (GR) the problem of time [1, 2] generally refers to the conflicting roles assigned to time in relativity: on the one hand, the chronological parameter required for posing equations of motion, and on the other hand, one of four spacetime coordinates, themselves dynamical quantities to be determined by equations of motion. This conflict became apparent to Stueckelberg [3, 4] in his work on classical and quantum electrodynamics. In seeking to interpret antiparticles as particles whose trajectory reverses time direction, he saw that neither coordinate time nor the proper time of the motion could serve as the parameter of chronological evolution. He was thus led to introduce an evolution parameter τ, independent of phase space and external to the spacetime manifold.

In the Stueckelberg framework, a particle worldline is the trajectory of an event $x^\mu(\tau)$ or $\psi(x,\tau)$ generated by a Lorentz scalar Hamiltonian K, establishing a canonical system familiar from nonrelativistic physics. In a suitable potential, the velocity component $\dot{x}^0 = dx^0/d\tau$ and energy $E = M\dot{x}^0$ may change sign, producing a trajectory observed in the laboratory as a particle/antiparticle interaction. Such pair processes are classically prohibited by the phase space relation $\dot{x}^\mu \dot{x}_\mu = -c^2$, which is here demoted in status from constraint to conservation law, applicable to systems governed by a Hamiltonian satisfying $\partial_\tau K = 0$. Clearly, if $\dot{x}^2$ can change sign, the proper time of the motion $ds = \sqrt{-\dot{x}^2}\,d\tau$ will not be a well-behaved evolution parameter.

Stueckelberg's work was extended by Piron and Horwitz [5] who constructed a relativistic canonical many-body theory [6–10], later generalized to a gauge theory of interacting spacetime events that recovers Maxwell electrodynamics in τ-equilibrium [11–14]. By including τ in the U(1) gauge function (but not the spacetime manifold), SHP electrodynamics requires five gauge potentials, whose interaction suggests a 5D symmetry such as O(3,2) or O(4,1). But to preserve the observed Lorentz invariance of spacetime, any higher symmetry must be broken to tensor and scalar representations of O(3,1) by choosing an appropriate structure for the matter kinetic term in the action. An analogy may be seen in classical acoustics, where the pressure wave equation appears to be invariant under Lorentz-like transformations, but no relativistic effects are expected as an observer approaches the speed of sound.

Extending the Stueckelberg-Horwitz-Piron (SHP) framework to curved spacetime, one obtains a classical and quantum theory of event evolution [15, 16], leading to a scalar event density $\rho(x,\tau)$ and energy-momentum tensor $T_{\mu\nu}(x,\tau)$. Following Wheeler's summary [17] of Einstein gravity as "spacetime tells matter how to move; matter tells spacetime how to curve," the τ-evolution of the mass/energy/momentum distribution associated with these events entails the τ-evolution of spacetime curvature [18–21], described by a local metric $\gamma_{\mu\nu}(x,\tau)$. To find appropriate field equations for this metric, we generalize the 3+1 formalism in geometrodynamics [22–25] to 4+1, regarding the spacetime manifold $\mathcal{M}$ as a 4D hypersurface embedded in a 5D pseudo-spacetime $\mathcal{M}_5$. Writing 5D field equations on $\mathcal{M}_5$, the 4+1 decomposition leads to a pair of manifestly covariant first order τ-evolution equations for the local metric $\gamma_{\mu\nu}(x,\tau)$ and an extrinsic curvature $K_{\mu\nu}(x,\tau)$, along with a set of propagating constraints [19].

In [20, 21] we studied the linearized SHP theory and showed that consistency with the phenomenology of weak gravitation requires that any 5D symmetry implied in the matter terms of the field equations must be explicitly broken to tensor and scalar representations of O(3,1), as previously seen in SHP electrodynamics. This strategic symmetry breaking also insures that the standard 4D Einstein equations are recovered in τ-equilibrium.

In this paper, we present a vielbein field theory approach to the 4+1 formalism, permitting us to construct the O(3,1) symmetric SHP field equations in a systematic way. In analogy to Einstein's quatrad method [26] we define a constant orthonormal quintrad basis for the pseudo-spacetime $\mathcal{M}_5$, so the inner products of basis vectors are the components of a 5D flat Minkowski pseudo-metric. The vielbein field transforms the constant basis to a local coordinate basis whose inner products provide a 5D local metric that induces the $\gamma_{\mu\nu}(x,\tau)$ of 4D spacetime by projection. By specifying a 5D vielbein field that respects the preferred foliation of $\mathcal{M}_5$, we find that the spacetime part (a vierbein field) contains the dynamic evolution, while the normal part propagates forward, enforcing the constraints. We may thus take the 4D vierbein field as the fundamental geometrical object, from which we obtain the spin connection, covariant derivative, and curvature. This foliation of the vielbein simplifies the formulation of evolution equations whose symmetry is limited to O(3,1), and their interpretation in terms of standard approaches to GR. We show that for any diagonal metric we may replace the evolution equation for the metric with an evolution equation for the vielbein field, simplifying calculation of the spin connection and curvature.

In Section 2 we briefly review the previous work in SHP GR and the considerations associated with constructing coordinate and vielbein frames. The foliation of $\mathcal{M}_5$ and the associated decomposition of the vielbein field is described in Section 3. In Section 4 we generalize the Einstein field equations to the 5D pseudo-spacetime and obtain the symmetry broken form, exhibiting tensor and scalar representations of standard O(3,1) covariance. Section 5 reviews the intrinsic and extrinsic geometry of spacetime, posed as an embedded 4D

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hypersurface of $\mathcal{M}_5$, and the resulting decomposition of the Riemann tensor. These pieces are assembled to write the evolution equations for the 4D spacetime metric and extrinsic curvature. We also obtain evolution equations for the vielbein field, permitting the formulation of initial value problems in the quintrad frame. An example is given in Section 6.

2. The pseudo-spacetime $\mathcal{M}_5$

We briefly review previous work generalizing SHP electrodynamics to general relativity. For additional details, see the references mentioned above.

2.1. SHP electrodynamics

In flat Minkowski spacetime, with $\eta_{\mu\nu} = \text{diag}\,(-1,1,1,1)$, the free particle Hamiltonian

$$K = \frac{1}{2M} p^\mu p_\mu \tag{1}$$

leads to the action

$$S = \int d\tau\, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu \tag{2}$$

made maximally U(1) gauge invariant [11,27] by introducing five gauge fields as

$$S_{\text{SHP}} = \int d\tau\, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\mu a_\mu(x,\tau) + \frac{e}{c} c_5 a_5(x,\tau) \tag{3}$$

$$= \int d\tau\, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\beta a_\beta(x,\tau) \tag{4}$$

where we partition the Greek indices as

$$\alpha,\beta,\gamma,\delta = 0,1,2,3,5 \qquad\qquad \lambda,\mu,\nu,\rho\ldots = 0,1,2,3 \tag{5}$$

and write $x^5 = c_5\tau$ in analogy to the notation $x^0 = ct$. Because $\dot{x}^\mu \dot{x}_\mu$, $\dot{x}^\mu a_\mu$, and a_5 are O(3,1) scalars, the action is 4D Lorentz invariant as required. But we emphasize that S_{SHP} enjoys the 5D gauge invariance $a_\alpha(x,\tau) \longrightarrow a_\alpha(x,\tau) + \partial_\alpha \Lambda(x,\tau)$. For a pure gauge potential $a_\alpha = \partial_\alpha \Lambda(x,\tau)$, the interaction term is seen to be a total τ-derivative. We may regard (4) as a standard 5D action with a symmetry-breaking matter term

$$S_{\text{5D}} = \int d\tau\, \frac{1}{2} M \dot{x}^\alpha \dot{x}_\alpha + \frac{e}{c} \dot{x}^\alpha a_\alpha \xrightarrow[\dot{x}^5 \equiv c_5]{} S_{\text{SHP}} = \int d\tau\, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\alpha a_\alpha \tag{6}$$

associated with the constraint $\dot{x}^5 = c_5$ that restricts the phase space to $(x^\mu, \dot{x}^\mu)$. In developing the field equations for the metric we will similarly break 5D symmetry when combining geometrical terms representing 5D gauge invariance with matter terms limited to 4D Lorentz symmetry.

Variation of the SHP action with respect to x^μ provides the Lorentz force [28]

$$M\ddot{x}_\mu = \frac{e}{c}\left(\dot{x}^\nu f_{\mu\nu} + c_5 f_{\mu 5}\right) = \frac{e}{c}\dot{x}^\beta f_{\mu\beta} \tag{7}$$

$$\frac{d}{d\tau}\left(-\frac{1}{2}M\dot{x}^\mu \dot{x}_\mu\right) = c_5 \frac{e}{c}\dot{x}^\beta f_{5\beta} = c_5 \frac{e}{c}\dot{x}^\mu f_{5\mu} \tag{8}$$

where the dynamics of the field strength

$$f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha \tag{9}$$

are determined by including the kinetic term

$$S_{\text{field}} = \int d\tau\, d^4x\, f^{\alpha\beta}(x,\tau) f_{\alpha\beta}(x,\tau) \tag{10}$$

in the total action, where $f_{\mu\nu}$ is the usual second rank tensor, while $f_{5\mu}$ is a vector field strength. Therefore, SHP electrodynamics differs in significant ways from Maxwell theory in 5D. In particular equation (8) permits mass exchange between particles and fields, setting the condition for non-conservation of proper time. Nevertheless, the total mass, energy, and momentum of particles and fields are conserved [28]. Compatibility of SHP electrodynamics with Maxwell theory requires $c_5 \ll c$ and we will neglect $(c_5/c)^2$ where appropriate.

Formally raising the five-index of $f_{\alpha\beta}$ in (10) suggests a 5D flat space metric

$$\eta_{\alpha\beta} = \text{diag}\,(-1,1,1,1,\sigma) \tag{11}$$

where $\sigma = \pm 1$. But regarding the kinetic term as

$$f^{\alpha\beta}(x,\tau) f_{\alpha\beta}(x,\tau) = f^{\mu\nu}(x,\tau) f_{\mu\nu}(x,\tau) + 2\sigma f^{\mu}_{5}(x,\tau) f_{\mu 5}(x,\tau) \tag{12}$$

we see that σ is simply the choice of sign for the vector-vector term.

2.2. *Curved spacetime*

In the parameterized SHP framework, we regard $x^\mu(\tau)$ as an irreversible physical event, occurring at time τ and spacetime coordinates x^μ. Then, $\mathcal{M}(\tau)$ represents the spacetime manifold of general relativity, a 4D block universe at τ. The scalar Hamiltonian K that generates evolution of the events of spacetime is thus said to evolve $\mathcal{M}(\tau)$ to an infinitesimally close 4D block universe $\mathcal{M}(\tau + d\tau)$. The structure of spacetime, including the past and future of coordinate time x^0, may change infinitesimally during the interval $d\tau$, and so the metric structure $\gamma_{\mu\nu}(x,\tau)$ of $\mathcal{M}(\tau)$ will be τ-dependent. A τ-independent 4D metric would thus have the character of an absolute background field, violating the goals of general relativity.

In [19] we introduced a pseudo-spacetime $\mathcal{M}_5$ as the image of an injective mapping

$$\Phi : \mathcal{M} \longrightarrow \mathcal{M}_5 = \mathcal{M} \times R \qquad\qquad X = \Phi(x,\tau) = (x, c_5\tau) \tag{13}$$

allowing us to characterize 4D spacetime $\mathcal{M}$ as a hypersurface embedded in $\mathcal{M}_5$ and borrow the mathematical tools of 3+1 geometrodynamics. The interval

$$dX = X_1 - X_2 = (x_1, c_5\tau_1) - (x_2, c_5\tau_2) \tag{14}$$

for $X \in \mathcal{M}_5$ refers to an event $x_1 \in \mathcal{M}(\tau_1)$ and an event $x_2 \in \mathcal{M}(\tau_2)$. This notion of 5D separation combines the *geometrical distance* δx^μ between arbitrary points in $\mathcal{M}(\tau)$ with the *dynamical distance* between events separated by the evolution $\mathcal{M}(\tau) \longrightarrow \mathcal{M}(\tau + \delta\tau)$. Taking the small variation $x_2 = x_1 + \delta x$ and $\tau_2 = \tau_1 + \delta\tau$, the 5D invariant interval between $X_1, X_2 \in \mathcal{M}_5$ becomes

$$\delta X^2 = \gamma_{\mu\nu}\left(\delta x^\mu + \frac{dx^\mu(\tau)}{d\tau}\delta\tau\right)\left(\delta x^\nu + \frac{dx^\nu(\tau)}{d\tau}\delta\tau\right) + \sigma c_5^2 \delta\tau^2 = g_{\alpha\beta}\,(x,\tau)\,\delta x^\alpha \delta x^\beta \tag{15}$$

referred to x_1 coordinates at $\tau = \tau_1$, where the spacetime metric $\gamma_{\mu\nu}$ must depend on x and τ in some manner to be determined. This interval suggests the free particle Lagrangian

$$L = \frac{1}{2} M g_{\alpha\beta}(x,\tau)\dot{x}^\alpha \dot{x}^\beta \tag{16}$$

where we must remove x^5 from the dynamical variables by asserting $\dot{x}^5 = c_5$. This constraint breaks the symmetry of the geodesic equations from

$$\ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \qquad \Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\beta\alpha}\right) \tag{17}$$

to

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \ddot{x}^\mu + \Gamma^\mu_{\lambda\sigma}\dot{x}^\lambda\dot{x}^\sigma + 2c_5\Gamma^\mu_{5\sigma}\dot{x}^\sigma + c_5^2\Gamma^\mu_{55} = 0 \qquad \ddot{x}^5 \equiv 0 \tag{18}$$

with $\Gamma^5_{\alpha\beta}$ playing no role in the particle dynamics.

As in 4D relativity [29] the 5D Ricci tensor is invariant under translations $x'^\alpha = x^\alpha + \Lambda^\alpha(x,\tau)$, leading to the Bianchi identity

$$\nabla_\alpha\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right) = 0 \qquad \nabla_\alpha X^\beta = \partial_\alpha X^\beta + X^\gamma\Gamma^\beta_{\gamma\alpha}\ . \tag{19}$$

By constructing a description of matter satisfying $\nabla_\beta T^{\alpha\beta} = 0$, we may write Einstein field equations in 5D, although we must break their 5D symmetry to O(3,1) at the interface between field terms expressed through $R_{\alpha\beta}$ and matter terms expressed through $T^{\alpha\beta}$.

2.3. *Mass-Energy-Momentum Tensor*

We define $n(x,\tau)$ to be the number of events per spacetime volume, so that

$$j^\alpha(x,\tau) = \rho(x,\tau)\dot{x}^\alpha(\tau) = Mn(x,\tau)\dot{x}^\alpha(\tau) \tag{20}$$

is the five-component event current. The continuity equation in flat space is

$$\partial_\alpha j^\alpha = \partial_\mu j^\mu + \partial_5 j^5 = \partial_\mu j^\mu + \frac{\partial\rho}{\partial\tau} = 0 \tag{21}$$

and is generalized for a local metric to

$$\nabla_\alpha j^\alpha = 0 \tag{22}$$

where again

$$\nabla_\alpha X^\beta = \frac{\partial X^\beta}{\partial x^\alpha} + X^\gamma\Gamma^\beta_{\gamma\alpha} \tag{23}$$

is the covariant derivative of a vector. Since j^5 is a scalar (n is scalar on physical grounds) for which the covariant derivative is the partial derivative, we have

$$\nabla_5 j^5 = \frac{\partial\rho}{\partial\tau} \tag{24}$$

and the continuity equation becomes

$$\frac{\partial\rho}{\partial\tau} + \nabla_\mu j^\mu = 0. \tag{25}$$

For non-interacting particles (non-thermodynamic dust under zero pressure), we write the mass-energy-momentum tensor [30] as

$$T^{\alpha\beta} = \rho\dot{x}^\alpha\dot{x}^\beta \longrightarrow \begin{cases} T^{\mu\nu} = \rho\dot{x}^\mu\dot{x}^\nu \\ T^{5\beta} = c_5 j^\beta \end{cases} \tag{26}$$

combining the 4D components $T^{\mu\nu}$ with the current density $T^{5\beta} = \dot{x}^5 \dot{x}^\beta \rho = c_5 j^\beta$. The conservation equation is

$$0 = \nabla_\beta T^{\alpha\beta} = \nabla_\beta \left(\rho \dot{x}^\alpha \dot{x}^\beta\right) = \dot{x}^\alpha \nabla_\beta \left(\rho \dot{x}^\beta\right) + \rho \dot{x}^\beta \nabla_\beta \dot{x}^\alpha = \dot{x}^\alpha \nabla_\beta j^\beta + \rho \dot{x}^\beta \nabla_\beta \dot{x}^\alpha \tag{27}$$

which vanishes by virtue of the continuity and geodesic equations

$$\nabla_\alpha j^\alpha = 0 \qquad\qquad \dot{x}^\beta \nabla_\beta \dot{x}^\alpha = \frac{D\dot{x}^\alpha}{D\tau} = 0\,. \tag{28}$$

Since the Bianchi relations are independent of dimension, the Einstein equations in 5D are

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta} \tag{29}$$

with Ricci tensor $R_{\alpha\beta}$ and Ricci scalar R obtained from $g_{\alpha\beta}$. The apparent 5D symmetry of (29) must be broken to 4+1, most conveniently achieved in a quintrad frame.

2.4. Quintrad frame

The standard basis vectors for the tangent space $\mathcal{T}\left(\mathcal{M}_5\right)$ in a coordinate (differential) frame are

$$\mathbf{g}_\alpha = \partial_\alpha = \frac{\partial}{\partial X^\alpha} \tag{30}$$

and the basis 1-forms for the dual space $\mathcal{T}^*\left(\mathcal{M}_5\right)$ are

$$\mathbf{g}^\alpha = \mathbf{d} X^\alpha \tag{31}$$

so that

$$\mathbf{g}^\alpha\left(\mathbf{g}_\beta\right) = \mathbf{g}^\alpha \cdot \mathbf{g}_\beta = \delta^\alpha_\beta \qquad\qquad \mathbf{g}_\alpha \cdot \mathbf{g}_\beta = g_{\alpha\beta} \qquad\qquad \mathbf{g}^\alpha \cdot \mathbf{g}^\beta = g^{\alpha\beta}\,. \tag{32}$$

Extending to 5D the vierbein formalism as presented in [26] and modifying the notation slightly, we define the quintrad frames $\{\mathbf{e}_a\}$ for $\mathcal{T}\left(\mathcal{M}_5\right)$ and $\{\mathbf{e}^a\}$ for $\mathcal{T}^*\left(\mathcal{M}_5\right)$ such that

$$\mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab} \qquad\qquad \mathbf{e}^a \cdot \mathbf{e}^b = \eta^{ab} \qquad\qquad \partial_a \mathbf{e}_b = \partial_a \mathbf{e}^b = 0 \tag{33}$$

where by convention the Latin letters

$$a, b, c, \cdots = 0, 1, 2, 3, 5 \qquad\qquad \eta_{ab} = \mathrm{diag}\left(-1, 1, 1, 1, \sigma\right) \tag{34}$$

indicate reference to the quintrad. The vielbein field provides the position-dependent components of the coordinate bases with respect to the quintrad as

$$\mathbf{g}_\alpha = E_\alpha{}^a\left(X\right) \mathbf{e}_a \qquad\qquad \mathbf{g}^\alpha = \bar{E}^\alpha{}_a\left(X\right) \mathbf{e}^a \tag{35}$$

invertible as

$$\mathbf{e}_a = e^\alpha{}_a\left(X\right) \mathbf{g}_\alpha \qquad\qquad \mathbf{e}^a = \bar{e}_\alpha{}^a\left(X\right) \mathbf{g}^\alpha \tag{36}$$

which for consistency requires

$$e^\alpha{}_a\left(X\right) E_\beta{}^a\left(X\right) = \delta^\alpha_\beta \qquad\qquad e^\alpha{}_a\left(X\right) E_\alpha{}^b\left(X\right) = \delta^b_a \tag{37}$$

with similar relations for $\bar{E}^{\alpha}{}_{a}$ and $\bar{e}_{\alpha}{}^{a}$. The duality relations impose the conditions

$$\begin{aligned}\delta^{a}_{a'} &= \mathbf{e}^{a}\cdot\mathbf{e}_{a'} = \bar{e}_{\alpha}{}^{a}\mathbf{g}^{\alpha}\cdot e^{\beta}{}_{a'}\mathbf{g}_{\beta} = \bar{e}_{\alpha}{}^{a}e^{\beta}{}_{a'}\delta^{\alpha}_{\beta} = \bar{e}_{\alpha}{}^{a}e^{\alpha}{}_{a'}\\ \delta^{\alpha}_{\alpha'} &= \mathbf{g}^{\alpha}\cdot\mathbf{g}_{\alpha'} = \bar{E}^{\alpha}{}_{a}\mathbf{e}^{a}\cdot E_{\alpha'}{}^{b}\mathbf{e}_{b} = \bar{E}^{\alpha}{}_{a}E_{\alpha'}{}^{b}\delta^{a}_{b} = \bar{E}^{\alpha}{}_{a}E_{\alpha'}{}^{a}\end{aligned} \tag{38}$$

which combined with (37) put the transformation equations into the form

$$\begin{aligned}\mathbf{g}_{\alpha} &= E_{\alpha}{}^{a}\mathbf{e}_{a} & \mathbf{e}_{a} &= e^{\alpha}{}_{a}\mathbf{g}_{\alpha}\\ \mathbf{g}^{\alpha} &= e^{\alpha}{}_{a}\mathbf{e}^{a} & \mathbf{e}^{a} &= E_{\alpha}{}^{a}\mathbf{g}^{\alpha}\end{aligned} \tag{39}$$

and so using (35) and (36) the metric is induced through

$$\begin{aligned}g_{\alpha\beta} &= \mathbf{g}_{\alpha}\cdot\mathbf{g}_{\beta} = \eta_{ab}\,E_{\alpha}{}^{a}E_{\beta}{}^{b} & g^{\alpha\beta} &= \mathbf{g}^{\alpha}\cdot\mathbf{g}^{\beta} = \eta^{ab}\,e^{\alpha}{}_{a}e^{\beta}{}_{b}\\ \eta_{ab} &= \mathbf{e}_{a}\cdot\mathbf{e}_{b} = g_{\alpha\beta}e^{\alpha}{}_{a}e^{\beta}{}_{b} & \eta^{ab} &= \mathbf{e}^{a}\cdot\mathbf{e}^{b} = g^{\alpha\beta}E_{\alpha}{}^{a}E_{\beta}{}^{b}\,.\end{aligned} \tag{40}$$

Since any vector can be written

$$\mathbf{V} = V^{\alpha}\mathbf{g}_{\alpha} = \left[V^{\alpha}E_{\alpha}{}^{a}(X)\right]\mathbf{e}_{a} = V^{a}\mathbf{e}_{a} \tag{41}$$

we may view

$$X^{a}_{b} = E_{\alpha}{}^{a}e_{a}{}^{\beta}X^{\alpha}_{\beta} \qquad\qquad X^{\alpha}_{\beta} = e^{\alpha}{}_{a}E_{\beta}{}^{b}X^{a}_{b} \tag{42}$$

as transformations between the coordinate frame and the vielbein frame.

For a tensor given with components in the vielbein frame, the covariant derivative is defined as

$$\nabla_{\alpha}X^{a}_{b} = \partial_{\alpha}X^{a}_{b} + \omega_{\alpha}{}^{a}{}_{c}X^{c}_{b} - \omega_{\alpha}{}^{c}{}_{b}X^{a}_{c} \tag{43}$$

where $\omega_{\alpha}{}^{a}{}_{c}$ is the spin connection. Writing the covariant derivative in a coordinate frame as

$$\nabla X = \left(\partial_{\alpha}X^{\beta} + \Gamma^{\beta}_{\alpha\gamma}X^{\gamma}\right)\mathbf{g}^{\alpha}\otimes\mathbf{g}_{\beta} \tag{44}$$

and transforming between frames, we are led to

$$\omega_{\alpha}{}^{b}{}_{a} = -e^{\beta}{}_{a}\left(\partial_{\alpha}E_{\beta}{}^{b}\right) + E_{\beta}{}^{b}e^{\gamma}{}_{a}\Gamma^{\beta}_{\alpha\gamma} \tag{45}$$

providing a relationship between the coordinate and spin connections. Acting with E^{a}_{δ} we find

$$\partial_{\alpha}E_{\delta}{}^{b} - \Gamma^{\gamma}_{\alpha\delta}E_{\gamma}{}^{b} + \omega_{\alpha}{}^{b}{}_{a}E_{\delta}{}^{a} = \nabla_{\alpha}E_{\delta}{}^{b} = 0 \tag{46}$$

expressing compatibility of the vielbein field. Using the known symmetries of the Christoffel connection

$$\Gamma^{\alpha}_{\gamma\beta} = \frac{1}{2}g^{\alpha\delta}\left(\partial_{\gamma}g_{\beta\delta} + \partial_{\beta}g_{\gamma\delta} - \partial_{\delta}g_{\beta\gamma}\right) = g^{\alpha\delta}\Gamma_{\gamma\beta\delta} \tag{47}$$

in (45) it is straightforward to demonstrate the antisymmetry of $(\omega_{\alpha})^{ab}$

$$(\omega_{\alpha})^{ba} = -(\omega_{\alpha})^{ab} \qquad \text{where} \qquad (\omega_{\alpha})^{ab} = \eta^{bb'}\omega_{\alpha}{}^{a}{}_{b'}\,. \tag{48}$$

It will prove convenient to transform the coordinate index α to a quintrad index as

$$(\omega_{c})^{ba} = e^{\alpha}{}_{c}(\omega_{\alpha})^{ba}\,. \tag{49}$$

Inserting the induced metric (40) into the coordinate connection (47) we are led to

$$\Gamma^{a}_{ba'} = \frac{1}{2}\left(e^{\beta}{}_{a'}\partial_b E_{\beta}{}^{a} + e^{\beta a}\partial_b E_{\beta a'} + e^{\beta}{}_{b}\partial_{a'} E_{\beta}{}^{a} + e^{\beta a}\partial_{a'} E_{\beta b} - e^{\beta}{}_{a'}\partial^{a} E_{\beta b} - e^{\beta}{}_{b}\partial^{a} E_{\beta a'}\right) \tag{50}$$

where

$$\Gamma^{a}_{ba'} = e^{\gamma}{}_{b} e^{\beta}{}_{a'} E_{\alpha}{}^{a}\Gamma^{\alpha}_{\gamma\beta} \tag{51}$$

and using (45) we express the spin connection in terms of the vielbein field as

$$(\omega_b)^{a}{}_{a'} = \frac{1}{2}\left(e^{\beta a}\partial_b E_{\beta a'} - e^{\beta}{}_{a'}\partial_b E_{\beta}{}^{a} + e^{\beta}{}_{b}\partial_{a'} E_{\beta}{}^{a} + e^{\beta a}\partial_{a'} E_{\beta b} - e^{\beta}{}_{a'}\partial^{a} E_{\beta b} - e^{\beta}{}_{b}\partial^{a} E_{\beta a'}\right) \tag{52}$$

from which we may derive the useful relation

$$(\omega_b)^{a}{}_{c} - (\omega_c)^{a}{}_{b} = e^{\beta}{}_{b}\partial_c E_{\beta}{}^{a} - e^{\beta}{}_{c}\partial_b E_{\beta}{}^{a} \,. \tag{53}$$

The 5D curvature for $\mathcal{M}_5$ in the quintrad frame is written

$$\left[\nabla_b, \nabla_a\right] X_c = X_d R^{d}_{cab} \tag{54}$$

where

$$R^{d}_{cab} = \partial_a \omega_b{}^{d}{}_{c} - \partial_b \omega_a{}^{d}{}_{c} + \omega_a{}^{d}{}_{c'}\omega_b{}^{c'}{}_{c} - \omega_b{}^{d}{}_{c'}\omega_a{}^{c'}{}_{c} \tag{55}$$

which in light of (52) contains only the vielbein field and its derivatives.

3. Foliation

3.1. Coordinate frame

By defining the scalar field $S(X) = X^5/c_5 = \tau$, the pseudo-spacetime $\mathcal{M}_5$ constructed in Section 2.2 admits the natural foliation defined by its level surfaces

$$\Sigma_{\tau_0} = \left\{X^{\alpha} \mid S\left(X\right) = X^5/c_5 = \tau_0\right\} \,. \tag{56}$$

The unit normal to the hypersurface Σ_{τ_0} was given in [19] as

$$n_{\alpha} = \sigma\frac{1}{\sqrt{|g^{55}|}}\,\partial_{\alpha} S(X) = \sigma\frac{1}{\sqrt{|g^{55}|}}\,\delta^{5}_{\alpha} \qquad\qquad g^{\alpha\beta} n_{\alpha} n_{\beta} = \sigma \tag{57}$$

where $S(X) =$ constant for $X \in \Sigma_{\tau_0}$ insures orthogonality. We similarly used this foliation to construct a coordinate frame $\{\mathbf{g}_{\alpha}\}$ for $\mathcal{T}\left(\mathcal{M}_5\right)$. For the hypersurface $\mathcal{T}\left(\Sigma_{\tau_0}\right) \subset \mathcal{T}\left(\mathcal{M}_5\right)$, we choose the four (5-component) vectors

$$\left(\mathbf{g}_{\mu}\right)^{\alpha} = \partial_{\mu}\Phi^{\alpha} = \left(\frac{\partial X^{\alpha}}{\partial x^{\mu}}\right)_{\tau_0} = \delta^{\alpha}_{\mu} \,. \tag{58}$$

We may choose as the fifth basis vector for $\mathcal{T}\left(\mathcal{M}_5\right)$ the linear combination

$$\mathbf{g}_5 = N^{\mu}\mathbf{g}_{\mu} + Nn \tag{59}$$

often called the ADM parameterization [25]. Here, the 4-vector N^{μ} generalizes the shift 3-vector in 3+1 formalisms and N is the lapse function with respect to τ. Designating

$$\gamma_{\mu\nu} = g_{\mu\nu} = \mathbf{g}_{\mu}\cdot\mathbf{g}_{\nu} \tag{60}$$

we find

$$g_{5\mu} = \mathbf{g}_\mu \cdot \mathbf{g}_5 = \gamma_{\mu\mu'} N^{\mu'} = N_\mu$$
$$g_{55} = \left(N^\mu \mathbf{g}_\mu + Nn\right) \cdot \left(N^{\mu'} \mathbf{g}_{\mu'} + Nn\right) = \gamma_{\mu\mu'} N^\mu N^{\mu'} + \sigma N^2 \; . \tag{61}$$

These expressions generalize the 3+1 ADM metric decomposition to

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix} \qquad g^{\alpha\beta} = \begin{bmatrix} \gamma^{\mu\nu} + \sigma \frac{1}{N^2} N^\mu N^\nu & -\sigma \frac{1}{N^2} N^\mu \\ -\sigma \frac{1}{N^2} N^\mu & \sigma \frac{1}{N^2} \end{bmatrix} \tag{62}$$

and put the unit normal into the form

$$n^\alpha = \frac{1}{N} \left(-N^\mu \mathbf{g}_\mu + \mathbf{g}_5\right)^\alpha = \frac{1}{N} \left(-N^\mu \delta^\alpha_\mu + \delta^\alpha_5\right)$$
$$n_\alpha = \sigma \frac{1}{\sqrt{|g^{55}|}} \delta^5_\alpha = \sigma N \delta^5_\alpha = \sigma N \left(\mathbf{g}^5\right)_\alpha \tag{63}$$

where the second expression is implicit in parameterization (59) through $n_\alpha = g_{\alpha\beta} n^\beta$.

3.2. *Quintrad frame*

We may perform the 4+1 decomposition through the vielbein field, without direct reference to the coordinate map Φ or the scalar field $S(X)$. We partition the quintrad indices, as we did for the coordinate indices, so that the index convention is now

$$\alpha, \beta, \gamma, \delta = 0,1,2,3,5 \qquad \lambda, \mu, \nu, \rho \ldots = 0,1,2,3$$
$$a, b, c, d, = 0,1,2,3,5 \qquad k, l, m, n, \ldots = 0,1,2,3 \tag{64}$$

where the five index with respect to the quintrad frame will be denoted $\bar{5}$ when necessary to avoid confusion. In this notation we expand the frame transformations as

$$\mathbf{g}_\mu = E_\mu{}^k \mathbf{e}_k + E_\mu{}^{\bar{5}} \mathbf{e}_5 \qquad \mathbf{e}_k = e^\mu{}_k \mathbf{g}_\mu + e^5{}_k \mathbf{g}_5$$
$$\mathbf{g}_5 = E_5{}^k \mathbf{e}_k + E_5{}^{\bar{5}} \mathbf{e}_5 \qquad \mathbf{e}_5 = e^\mu{}_{\bar{5}} \mathbf{g}_\mu + e^5{}_{\bar{5}} \mathbf{g}_5 \tag{65}$$

and

$$\mathbf{g}^\mu = e^\mu{}_k \mathbf{e}^k + e^\mu{}_{\bar{5}} \mathbf{e}^5 \qquad \mathbf{e}^k = E_\mu{}^k \mathbf{g}^\mu + E_5{}^k \mathbf{g}^5$$
$$\mathbf{g}^5 = e^5{}_k \mathbf{e}^k + e^5{}_{\bar{5}} \mathbf{e}^5 \qquad \mathbf{e}^5 = E_\mu{}^{\bar{5}} \mathbf{g}^\mu + E_5{}^{\bar{5}} \mathbf{g}^5 \tag{66}$$

for the dual basis, where once again the orthogonality relations (37) are required for consistency. In the quintrad we expect the spacetime hypersurface to be spanned by $\{\mathbf{e}_k\}$ and normal to $\mathbf{e}_5$, and so assign the unit normal vector and 1-form as

$$n = \mathbf{e}_5 \qquad \bar{n} = \sigma \mathbf{e}^5 \tag{67}$$

with normalization

$$n^2 = \mathbf{e}_5 \cdot \mathbf{e}_5 = \eta_{55} = \sigma \qquad \bar{n}^2 = \sigma^2 \mathbf{e}^5 \cdot \mathbf{e}^5 = \eta^{55} = \sigma \; . \tag{68}$$

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Combining this assignment with the general parameterization (59) provides

$$\mathbf{e}_5 = n = \frac{1}{N}\left(-N^\mu \mathbf{g}_\mu + \mathbf{g}_5\right) \qquad \mathbf{e}^5 = \sigma\bar{n} = \sigma^2 N\,\mathbf{g}^5 = N\,\mathbf{g}^5 \tag{69}$$

so that comparison with (65) and (66) determines four vielbein components

$$e^\mu{}_{\bar{5}} = -\frac{1}{N}N^\mu \qquad e^5{}_{\bar{5}} = \frac{1}{N} \qquad E_\mu{}^{\bar{5}} = 0 \qquad E_5{}^{\bar{5}} = N\,. \tag{70}$$

The orthogonality of the quintrad frame provides two additional conditions

$$\begin{aligned} 0 = \mathbf{e}_k \cdot \mathbf{e}_5 &= \left(e^\mu{}_k \mathbf{g}_\mu + e^5{}_k \mathbf{g}_5\right) \cdot \frac{1}{N}\left(-N^{\mu'}\mathbf{g}_{\mu'} + \mathbf{g}_5\right) \\ &= \frac{1}{N}\left(-e^\mu{}_k N_\mu + e^\mu{}_k g_{\mu 5} - e^5{}_k g_{\mu' 5} N^{\mu'} + e^5{}_k g_{55}\right) \end{aligned} \tag{71}$$

and

$$0 = \mathbf{e}^k \cdot \mathbf{e}^5 = \left(E_\mu{}^k \mathbf{g}^\mu + E_5{}^k \mathbf{g}^5\right) \cdot N\,\mathbf{g}^5 = N\left(E_\mu{}^k g^{\mu 5} + E_5{}^k g^{55}\right) \tag{72}$$

which combined with metric components from (62) provide the components

$$e^5{}_k = 0 \qquad E_5{}^k = E_\mu{}^k N^\mu\,. \tag{73}$$

Inserting $e^5{}_k = 0$ into (65) leads to

$$\mathbf{e}_k = e^\mu{}_k \mathbf{g}_\mu + e^5{}_k \mathbf{g}_5 = e^\mu{}_k\left(E_\mu{}^{k'}\mathbf{e}_{k'} + E_\mu{}^{\bar{5}}\mathbf{e}_5\right) \tag{74}$$

from which we conclude that

$$E_\mu{}^{\bar{5}} = 0 \qquad e^\mu{}_k E_\mu{}^{k'} = \delta_k^{k'}\,. \tag{75}$$

Finally, the transformations between coordinate and quintrad frames take the form

$$\begin{aligned} \mathbf{g}_\alpha &= E_\alpha{}^a \mathbf{e}_a = \delta_\alpha^\mu E_\mu{}^k \mathbf{e}_k + \delta_\alpha^5\left(E_\mu{}^k N^\mu \mathbf{e}_k + N\mathbf{e}_5\right) \\ \mathbf{e}_a &= e^\alpha{}_a \mathbf{g}_\alpha = \delta_a^k e^\mu{}_k \mathbf{g}_\mu + \delta_a^5 \frac{1}{N}\left(-N^\mu \mathbf{g}_\mu + \mathbf{g}_5\right) \end{aligned} \tag{76}$$

and we may summarize the vielbein field as

$$\begin{aligned} E_\alpha{}^a &= \delta_\alpha^\mu \delta_k^a E_\mu{}^k + \delta_\alpha^5\left(E_\mu{}^k N^\mu \delta_k^a + N\delta_5^a\right) \\ e^\alpha{}_a &= \delta_a^k \delta_\mu^\alpha e^\mu{}_k - \delta_a^5 \delta_\mu^\alpha \frac{1}{N} N^\mu + \delta_a^5 \delta_5^\alpha \frac{1}{N} \end{aligned} \tag{77}$$

which provides a quintrad basis for $\mathcal{T}\left(\mathcal{M}_5\right)$ with a 4+1 foliation built-in by construction. It was shown in [19] that the lapse and shift propagate with τ, enforcing the constraints on $g_{\alpha\beta}$ associated with the Bianchi relations, but are not subject to second order evolution equations. Since $e^\mu{}_k = E^\mu{}_k$ by orthogonality, the dynamical content of the Einstein equations is entirely contained in the spacetime vierbein $E_\mu{}^k$, where $\mu, k = 0, 1, 2, 3$.

3.3. Projection onto the hypersurface
The 4+1 decomposition of the Einstein equations is found by application of the projection operator P which acts on a vector as

$$V \in \mathcal{T}(\mathcal{M}_5) \longrightarrow V_\perp = P[V] \in \mathcal{T}(\Sigma_\tau) \subset \mathcal{T}(\mathcal{M}_5) \tag{78}$$

and in component form is

$$P[V] = \left(P^\alpha_{\alpha'} V^{\alpha'}\right) \mathbf{g}_\alpha = V^{\alpha'}\left(P^\alpha_{\alpha'}\mathbf{g}_\alpha\right) \qquad P[V] = \left(P^a_{a'} V^{a'}\right)\mathbf{e}_a = V^{a'}\left(P^a_{a'}\mathbf{e}_a\right) . \tag{79}$$

Using (63) for n_α, the components of P in the coordinate and quintrad frames are

$$\begin{aligned} P^\alpha_{\alpha'} &= \delta^\alpha_{\alpha'} - \sigma n^\alpha n_{\alpha'} = \delta^\alpha_{\alpha'} - \delta^5_{\alpha'}\left(\delta^\alpha_5 - N^\mu \delta^\alpha_\mu\right) \\ P^a_{a'} &= e_\alpha{}^a e^{\alpha'}{}_{a'} P^\alpha_{\alpha'} = \delta^a_{a'} - \delta^a_5 \delta^5_{a'} \end{aligned} \tag{80}$$

and we identify the expressions on the RHS as the component form of vectors

$$\begin{aligned} \mathbf{g}^\perp_{\alpha'} &= \mathbf{g}_{\alpha'} - \delta^5_{\alpha'}\left(\mathbf{g}_5 - N^\mu \mathbf{g}_\mu\right) = P^\alpha_{\alpha'}\mathbf{g}_\alpha \\ \mathbf{e}^\perp_{a'} &= \mathbf{e}_{a'} - \delta^5_{a'}\mathbf{e}_5 = P^a_{a'}\mathbf{e}_a . \end{aligned} \tag{81}$$

These can be rewritten

$$\mathbf{g}^\perp_\alpha = \mathbf{g}_\mu \delta^\mu_\alpha + N^\mu \mathbf{g}_\mu \delta^5_\alpha = \left(\delta^\mu_\alpha + N^\mu \delta^5_\alpha\right)\mathbf{g}_\mu \qquad \mathbf{e}^\perp_a = \delta^k_a \mathbf{e}_k \tag{82}$$

showing that $\mathbf{g}^\perp_\alpha$ and $\mathbf{e}^\perp_a$ lie in the hypersurface $\mathcal{T}(\Sigma_\tau)$, and that

$$\begin{aligned} V_\perp &= V^\alpha \mathbf{g}^\perp_\alpha &\longrightarrow\qquad V^\alpha_\perp &= \delta^\alpha_\mu\left(\delta^\mu_{\alpha'} + N^\mu \delta^5_{\alpha'}\right)V^{\alpha'} \\ V_\perp &= V^a \mathbf{e}^\perp_a &\longrightarrow\qquad V^a_\perp &= \delta^a_k \delta^k_{a'} V^{a'} . \end{aligned} \tag{83}$$

Because $V_\perp$ has four independent components we have the pull-back and push-forward relationships

$$v^\mu = \delta^\mu_\alpha V^\alpha_\perp \in \mathcal{T}(\mathcal{M}) \qquad V^\alpha_\perp = \delta^\alpha_\mu v^\mu \in \mathcal{T}(\Sigma_\tau) \tag{84}$$

where $v \in \mathcal{T}\left(\mathcal{M}(\tau)\right)$ is the spacetime four-vector homeomorphic to $V_\perp \in \mathcal{T}(\Sigma_\tau)$. More generally, writing

$$\begin{aligned} v^\mu &= \delta^\mu_\alpha V^\alpha_\perp = \delta^\mu_\alpha e^\alpha{}_a V^a_\perp = \delta^\mu_\alpha e^\alpha{}_{a'} P^{a'}_a V^a = \mathcal{E}^\mu_a V^a \\ v_\mu &= \delta^\alpha_\mu V^\perp_\alpha = \delta^\alpha_\mu E_\alpha{}^{a'} V^\perp_{a'} = \delta^\alpha_\mu E_\alpha{}^{a'} P^a_{a'} V^\perp_a = \mathcal{E}^a_\mu V^\perp_a \end{aligned} \tag{85}$$

and using (80) and (77) we define the composed pull-back operators

$$\mathcal{E}^\mu_a = \delta^\mu_\alpha e^\alpha{}_{a'} P^{a'}_a = \delta^k_a e^\mu{}_k \qquad \mathcal{E}^a_\mu = \delta^\alpha_\mu E_\alpha{}^{a'} P^a_{a'} = \delta^a_k E_\mu{}^k \tag{86}$$

which map a vector in a quintrad frame for $\mathcal{T}(\mathcal{M}_5)$ to a vector in a coordinate frame for $\mathcal{T}(\mathcal{M})$, using only the spacetime part of the vielbein field. Because the projector is idempotent, we see that

$$\mathcal{E}^\mu_{a'} P^{a'}_a = \mathcal{E}^\mu_a \qquad \mathcal{E}^{a'}_\mu P^a_{a'} = \mathcal{E}^a_\mu . \tag{87}$$

Defining the normal basis components

$$\mathbf{g}^\parallel_\alpha = \mathbf{g}_\alpha - \mathbf{g}^\perp_\alpha = \delta^5_\alpha\left(\mathbf{g}_5 - N^\mu \mathbf{g}_\mu\right) \qquad \mathbf{e}^\parallel_a = \mathbf{e}_a - \mathbf{e}^\perp_a = \delta^5_a \mathbf{e}_5 \tag{88}$$

we easily verify

$$\begin{aligned} \mathbf{g}^{\parallel}_{\alpha} \cdot \mathbf{g}^{\parallel}_{\beta} &= \sigma N^2 \delta^5_\alpha \delta^5_\beta \\ \mathbf{g}^{\perp}_{\alpha} \cdot \mathbf{g}^{\perp}_{\beta} &= \gamma_{\mu\mu'} \delta^\mu_\alpha \delta^{\mu'}_\beta + \delta^5_\alpha N_\mu \delta^\mu_\beta + \delta^5_\beta N_\mu \delta^\mu_\alpha + \gamma_{\mu\mu'} N^\mu N^{\mu'} \delta^5_\alpha \delta^5_\beta \\ \mathbf{g}^{\parallel}_{\alpha} \cdot \mathbf{g}^{\perp}_{\beta} &= 0 \end{aligned} \tag{89}$$

which decomposes the coordinate frame into tangent and normal components, and recovers the metric (62) through

$$g_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta = \mathbf{g}^{\parallel}_{\alpha} \cdot \mathbf{g}^{\parallel}_{\beta} + \mathbf{g}^{\perp}_{\alpha} \cdot \mathbf{g}^{\perp}_{\beta} \; . \tag{90}$$

Rearranging (80) we have the completeness relations

$$\delta^\alpha_\beta = P^\alpha_\beta + \sigma n^\alpha n_\beta \qquad\qquad g_{\alpha\beta} = P_{\alpha\beta} + \sigma n_\alpha n_\beta \tag{91}$$

which are useful in the 4+1 decomposition of objects defined on $\mathcal{M}_5$. On the Σ_τ hypersurface, the projection operator $P_{\alpha\beta}$ acts as the induced metric, as can be seen through

$$\gamma_{\mu\nu} = g_{\alpha\beta} \delta^\alpha_\mu \delta^\beta_\nu = \left(P_{\alpha\beta} + \sigma n_\alpha n_\beta \right) \delta^\alpha_\mu \delta^\beta_\nu = P_{\mu\nu} \tag{92}$$

where we used $n_\alpha \delta^\alpha_\mu = \sigma N \delta^5_\mu = 0$.

4. Field equations

Decomposition of the 4D Einstein equations into evolution equations requires an expression that isolates and projects the Ricci tensor into the hypersurface. In the 3+1 formalism, one starts from the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{93}$$

and takes the trace

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu} \tag{94}$$

using $g^{\mu\nu} g_{\mu\nu} = 4$ to obtain the trace-reversed form

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{95}$$

expressing the relationship of spacetime geometry on the LHS and the distribution of matter on the RHS. But in 5D we have $g^{\alpha\beta} g_{\alpha\beta} = 5 \neq 4$ leading to the trace-reversed form

$$R_{\alpha\beta} = \frac{8\pi G}{c^4} \left(T_{\alpha\beta} + \frac{\frac{1}{2} g_{\alpha\beta}}{1 - \frac{1}{2} g^{\gamma\delta} g_{\gamma\delta}} g^{\gamma'\delta'} T_{\gamma'\delta'} \right) = \frac{8\pi G}{c^4} \left(T_{\alpha\beta} - \frac{1}{3} g_{\alpha\beta} T \right) \; . \tag{96}$$

In [21] we studied the linearized SHP theory, and showed that this issue similarly appears when solving the wave equation for a weak gravitational perturbation. Unsurprisingly, consistency with standard phenomenology requires that we replace $\eta_{\alpha\beta} \longrightarrow \widehat{\eta}_{\alpha\beta} = (-1, 1, 1, 1, 0)$ as the flat background metric. This replacement recalls the observation in Section 2.1 that $\eta_{55} = \sigma$ can be understood as the choice of sign for certain field terms, but should play no role in matter terms.

Therefore, we similarly require the substitution $g_{\alpha\beta} \longrightarrow \widehat{g}_{\alpha\beta}$ satisfying $\widehat{g}^{\alpha\beta}\,\widehat{g}_{\alpha\beta} = 4$ among the matter terms, which permits us to recover the form

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}\widehat{g}_{\alpha\beta}\widehat{T}\right) \tag{97}$$

in 5D, where $\widehat{T} = \widehat{g}^{\alpha\beta}T_{\alpha\beta}$.

To obtain $\widehat{g}_{\alpha\beta}$ we write unbroken 5D Einstein equations in the quintrad frame

$$R_{ab} - \frac{1}{2}\eta_{ab}R = \frac{8\pi G}{c^4}T_{ab} \;\longrightarrow\; R_{ab} = \frac{8\pi G}{c^4}\left(T_{ab} + \frac{\frac{1}{2}\eta_{ab}}{1-\frac{1}{2}\eta^{cd}\eta_{cd}}\eta^{c'd'}T_{c'd'}\right) \tag{98}$$

where the metric η_{ab} is flat, while R_{ab} and T_{ab} are related to the coordinate frame tensors through the vielbein field. We break the 5D symmetry in the matter terms by replacing

$$\eta_{ab} \longrightarrow \widehat{\eta}_{ab} = (-1,1,1,1,0) = \delta_a^k\delta_b^l\,\eta_{kl} \tag{99}$$

on the RHS of (98), leaving the Ricci tensor R_{ab} unchanged. The SHP Einstein equations now take the form

$$R_{ab} = \frac{8\pi G}{c^4}\left(T_{ab} - \frac{1}{2}\widehat{\eta}_{ab}\widehat{T}\right) \tag{100}$$

where $\widehat{T} = \widehat{\eta}^{ab}T_{ab} = \eta^{kl}T_{kl}$. Using the vielbein field (77) we may transform (100) back to a coordinate frame, leading to

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}\widehat{g}_{\alpha\beta}\widehat{T}\right) \tag{101}$$

providing the symmetry-broken local metric as

$$\widehat{g}_{\alpha\beta} = E_\alpha{}^a E_\beta{}^b\,\widehat{\eta}_{ab} = g_{\alpha\beta} - \delta_\alpha^5\delta_\beta^5\sigma N^2 = g_{\alpha\beta} - \sigma n_\alpha n_\beta = P_{\alpha\beta}\,. \tag{102}$$

The breaking of 5D symmetry can thus be understood as replacing the metric in the matter terms of the field equation with the projector onto the 4D hypersurface. We recall from (92) that restricted to the Σ_τ hypersurface, the projection operator $P_{\alpha\beta}$ acts as the induced metric.

Using the completeness relation (91), we decompose the mass-energy-momentum tensor into

$$T_{ab} = T_{a'b'}\left(P_a^{a'} + \sigma n^{a'}n_a\right)\left(P_b^{b'} + \sigma n^{b'}n_b\right) = S_{ab} - 2\sigma n_a p_b + n_a n_b \kappa \tag{103}$$

where

$$S_{ab} = T_{a'b'}P_a^{a'}P_b^{b'} \qquad\qquad p_b = -n^{a'}P_b^{b'}T'_{a'b'} \qquad\qquad \kappa = n^{b'}n^{a'}T_{a'b'} \tag{104}$$

representing the 4D energy momentum tensor S_{ab}, a momentum vector p_b describing the flow of mass into spacetime, and a scalar mass density κ. The trace of T_{ab} is

$$T = \eta^{ab}T_{ab} = S + \sigma\kappa \tag{105}$$

but the symmetry broken trace is

$$\widehat{T} = \widehat{g}^{\alpha\beta}T_{\alpha\beta} = P^{\alpha\beta}T_{\alpha\beta} = P^{\alpha\beta}\left(S_{\alpha\beta} + 2\sigma n_\alpha p_\beta + n_\alpha n_\beta\kappa\right) = S \tag{106}$$

where we used $P^{\alpha\beta}n_\alpha = 0$. The SHP field equations now take the form

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}P_{\alpha\beta}S\right) \tag{107}$$

in a coordinate frame. In the unconstrained 8D phase space of SHP kinematics, mass is most readily understood through the independence of energy and 3-momentum in the dynamical quantity $-p^2 = -M^2\dot{x}^\mu \dot{x}_\mu$ associated with the motion of matter events. As in SHP electrodynamics, the dynamical evolution of an event in curved space may involve a variation in particle mass, with mass transferred to the gravitational field and transferred across spacetime through field momentum p_a and mass density κ.

5. Initial value problem

5.1. Review of 4+1 decomposition

In [19] we generalized the 3+1 formalism in geometrodynamics [24,25] to 4+1 to obtain a pair of first order evolution equations for the metric $\gamma_{\mu\nu}$ and intrinsic curvature $K_{\mu\nu}$ of spacetime geometry ($\mu, \nu = 0, 1, 2, 3$). Regarding the spacetime manifold as homeomorphic to the hypersurface Σ_τ embedded in the 5D pseudo-spacetime, the differential geometry of $\mathcal{M}$ is expressed in terms of projections of the corresponding structures for $\mathcal{M}_5$. By extension of 3+1 geometrodynamics, the initial value problem is found in the following steps:

(i) The covariant derivative D_α for $\mathcal{T}(\Sigma_\tau)$ is found by using $P_{\alpha\beta}$ to project the covariant derivative ∇_α for $\mathcal{M}_5$,
(ii) The extrinsic curvature $K_{\alpha\beta}$ is defined by projecting the covariant derivative of the unit normal n_α,
(iii) The projected curvature $\bar{R}^{\delta}_{\gamma\alpha\beta}$ on $\mathcal{T}(\Sigma_\tau)$ is defined through the non-commutation of projected covariant derivatives D_α and D_β,
(iv) Writing the explicit form of $P_{\alpha\beta}$ in the definition of $\bar{R}^{\delta}_{\gamma\alpha\beta}$ leads to the Gauss relation that decomposes the 5D curvature $R^{\delta}_{\gamma\alpha\beta}$ in terms of $\bar{R}^{\delta}_{\gamma\alpha\beta}$ and $K_{\alpha\beta}$,
(v) Projecting the 5D curvature $R^{\delta}_{\gamma\alpha\beta}$ on the unit normal n_α leads to the Codazzi relation providing a relationship between $K_{\alpha\beta}$ and p_α,
(vi) Lie derivatives of $P_{\alpha\beta}$ and $K_{\alpha\beta}$ along the unit normal — the direction of τ evolution — are combined with these ingredients, along with the trace-reversed field equation, to obtain τ-evolution equations for $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ and a pair of constraints on the initial conditions.

5.2. Intrinsic and extrinsic curvature in the quintrad frame

In Section 2.4 we wrote the covariant derivative (43) and curvature tensor (54) for $\mathcal{M}_5$ in a quintrad frame. Here we review some details of Steps (i) to (vi) where derivation in the quintrad frame helps clarify the content of the procedure.

The 4+1 projected derivative D_a is defined as

$$(DX)_{ab_1\cdots b_n} = P^{a'}_a P^{b'_1}_{b_1} \cdots P^{b'_n}_{b_n} \left(\nabla_{a'} X_{b'_1\cdots b'_n}\right) \tag{108}$$

and because

$$\nabla_a P_{bc} = \nabla_a \left(\eta_{bc} - \sigma n_b n_c\right) = -\sigma \left[\left(\nabla_a n_b\right) n_c + n_b \left(\nabla_a n_c\right)\right] \tag{109}$$

we have

$$(DP)_{abc} = P^{a'}_a P^{b'}_b P^{c'}_c \nabla_{a'} P_{b'c'} = 0 \,. \tag{110}$$

Since the $P_{aa'}$ acts as the metric on the hypersurface $\mathcal{T}(\Sigma_\tau)$, we see that D_a is the unique covariant derivative compatible with $P_{\alpha\beta}$ (and hence $\gamma_{\mu\nu}$). Pulling back the projected covariant derivative to $\mathcal{T}(\mathcal{M})$ we find

$$\begin{aligned} D_\mu v_\nu &= \mathcal{E}^a_\mu \mathcal{E}^b_\nu P^{a'}_a P^{b'}_b \left(\partial_a V_b - \omega_a{}^{a'}{}_b V_{a'}\right) = E_\mu{}^k E_\nu{}^l \left(\partial_k V_l - \omega_k{}^{l'}{}_l V_{l'} - \omega_k{}^{\bar{5}}{}_l V_{\bar{5}}\right) \\ &= \left(\partial_\mu v_\nu - \Gamma^\lambda_{\mu\nu} v_\lambda\right) - \omega_\mu{}^{\bar{5}}{}_\nu V_{\bar{5}} = \nabla^{(4)}_\mu v_\nu - \omega_\mu{}^{\bar{5}}{}_\nu V_{\bar{5}} \end{aligned} \tag{111}$$

where $\nabla_\mu^{(4)}$ contains only 4D components of the connection associated with a τ-independent spacetime. The additional $\omega_\mu{}^{\bar{5}}{}_\nu$ term suggests how D_μ retains information on the τ-evolution of the spacetime geometry. This point is sharpened by writing the extrinsic curvature

$$K_{ab} = -\left(Dn\right)_{ab} = -P_a^{a'} P_b^{b'} \nabla_{b'} n_{a'} = -P_b^{b'} \nabla_{b'} n_a \tag{112}$$

where the last equality follows from

$$0 = \nabla_{b'} \sigma = \nabla_{b'} n^2 = 2n^{a'} \left(\nabla_{b'} n_{a'}\right) \ . \tag{113}$$

Expanding the projector $P_b^{b'}$ we find

$$K_{ab} = -\nabla_b n_a + \sigma n_b \left(n^{b'} \nabla_{b'} n_a\right) = -\nabla_b n_a + \sigma n_b \left(-\frac{1}{N} D_a N\right) \tag{114}$$

which using $n_a = \sigma \delta_a^5$ and $n^a = \delta_5^a$ provides

$$\nabla_b n_a = \partial_b n_a - \omega_b{}^{a'}{}_a n_{a'} = -\sigma \omega_b{}^{\bar{5}}{}_a \qquad\qquad n^{b'} \nabla_{b'} n_b = -\omega_{\bar{5}}{}^{\bar{5}}{}_b \tag{115}$$

leading to

$$K_{ab} = \sigma \omega_b{}^{\bar{5}}{}_a - \sigma \delta_b^{\bar{5}} \omega_{\bar{5}}{}^{\bar{5}}{}_a = \sigma \delta_a^k \delta_b^l K_{kl} = \sigma \delta_a^k \delta_b^l \ \omega_l{}^{\bar{5}}{}_k \tag{116}$$

where $\omega_{\bar{5}}{}^{\bar{5}}{}_{\bar{5}} = 0$ by antisymmetry. Like the projected covariant derivative D_μ the extrinsic curvature $K_{\mu\nu}$ contains 5-components of the connection ω_{bc}^a not present in $\nabla_\mu^{(4)}$. Returning to the general expression (52) for ω_{bc}^a we may evaluate

$$\omega_{\bar{5}}{}^{\bar{5}}{}_b = \delta_b^k \ \omega_{\bar{5}}{}^{\bar{5}}{}_k = \delta_b^k \frac{1}{N} \partial_k N \tag{117}$$

in agreement with (114) and find the expression

$$K_{ab} = \frac{1}{2} \delta_a^k \delta_b^l \left[E_{\mu k} \left(\frac{1}{N} \partial_l N^\mu + \partial_5 e^\mu{}_l\right) + E_{\mu l} \left(\frac{1}{N} \partial_k N^\mu + \partial_5 e^\mu{}_k\right)\right] \tag{118}$$

for K_{ab} that depends on the lapse, shift, and spacetime part of the vielbein field.

The projected Riemann tensor $\bar{R}_{cab}^d$ is defined through the non-commutation of projected covariant derivatives

$$\left[D_b, D_a\right] X_c = X_d \bar{R}_{cab}^d \tag{119}$$

and describes the curvature of the hypersurface Σ_τ. Because D_μ contains projected 5-components of the connection, these components will also be present in the pull-back $\bar{R}_{\lambda\mu\nu}^\rho$ to $\mathcal{M}$ differing from the 4D curvature $R_{\lambda\mu\nu}^\rho$ associated with $\nabla_\mu^{(4)}$.

Decomposition of the 5D Riemann tensor R_{cab}^d into projected components $\bar{R}_{cab}^d$ and K_{ab} may be understood by contracting with the completeness relation (91)

$$R_{dab}^c = \left(P_a^{a'} + \sigma n_a n^{a'}\right) \left(P_b^{b'} + \sigma n_b n^{b'}\right) \left(P_{c'}^c + \sigma n^c n_{c'}\right) \left(P_d^{d'} + \sigma n_d n^{d'}\right) R_{d'a'b'}^{c'} \tag{120}$$

to obtain the sum of three projections

$$\left(P_a^{a'} P_b^{b'} P^c{}_{c'} P_d^{d'}\right) R^{c'}{}_{d'a'b'} \qquad \left(P_{c'}^c P_a^{a'} P_b^{b'}\right) n^d \ R_{da'b'}^{c'} \qquad \left(P_{aa'} P_b^{b'}\right) n^{c'} n^d \ R_{db'c'}^{a'} \tag{121}$$

where $n^{b'} n^{c'} n^d \; R^{a'}_{db'c'} = 0$ from the antisymmetry of R^d_{cab}. The first of (121) is found in Step (iv) above as the Gauss relation

$$R^{\mu}_{\nu\lambda\rho} = \bar{R}^{\mu}_{\nu\lambda\rho} - \sigma \left(K^{\mu}_{\lambda} K_{\rho\nu} - K^{\mu}_{\rho} K_{\lambda\nu} \right) \; . \tag{122}$$

In its original context, this shows that a curved surface embedded in a flat 3D space with $R^i_{jkl} = 0$ will derive its relevant curvature $\bar{R}^i_{jkl}$ entirely from the K_{ij} associated with the gradient of the normal to that surface.

In Step (v) above we write the definition (54) of R^c_{dab} while taking $X_d = n_d$ to obtain explicit expressions for the second and third of (121). Projecting the three remaining indices onto $\mathcal{T}\left(\Sigma_\tau\right)$ provides the Codazzi relation

$$\left(P^{a'}_a P^{b'}_b P^{c'}_c \right) n_d \; R^d_{c'a'b'} = D_b K_{ac} - D_a K_{bc} \tag{123}$$

while projecting twice onto $\mathcal{T}\left(\Sigma_\tau\right)$ and once onto $n^{c'}$ leads to

$$\left(P_{aa'} P^{b'}_b \right) n^{c'} n^d \; R^{a'}_{db'c'} = -K^c_a K_{cb} - \sigma \frac{1}{N} D_b D_a N + P^{a'}_a P^{b'}_b n^{c'} \nabla_{c'} K_{a'b'} \tag{124}$$

which we will combine with the trace-reversed field equation (107) to eliminate $R^{a'}_{db'c'}$.

5.3. Evolution equations in a coordinate frame

In Step (vi) above we construct an initial value problem for the τ-evolution of the metric and extrinsic curvature. This construction is naturally performed in a coordinate frame, because the metric in the quintrad frame is the constant η_{ab}. The τ-derivatives are found by evaluating the Lie derivatives of $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ in the direction of system evolution. Defining the normal evolution vector $m_\alpha = N n_\alpha$ we use the ADM parameterization (59) to write the Lie derivative along $\mathbf{g}_5$ as

$$\mathbf{g}_5 = N^\mu \mathbf{g}_\mu + N n = \mathbf{N} + m \quad \longrightarrow \quad \mathcal{L}_m = \mathcal{L}_{\mathbf{g}_5} - \mathcal{L}_{\mathbf{N}} \tag{125}$$

where the Lie derivative acts as

$$\mathcal{L}_B \, A_{\alpha\beta} = B^\gamma \nabla_\gamma A_{\alpha\beta} + A_{\gamma\beta} \nabla_\alpha B^\gamma + A_{\alpha\gamma} \nabla_\beta B^\gamma \; . \tag{126}$$

From the definition $(\mathbf{g}_5)^\gamma = \delta^\gamma_5$ the τ-derivative is found as

$$\mathcal{L}_{\mathbf{g}_5} A_{\alpha\beta} = \delta^\gamma_5 \partial_\gamma A_{\alpha\beta} + A_{\gamma\beta} \partial_\alpha \delta^\gamma_5 + A_{\alpha\gamma} \partial_\beta \delta^\gamma_5 = \partial_5 A_{\alpha\beta} = \frac{1}{c_5} \partial_\tau A_{\alpha\beta} \tag{127}$$

and the derivative along the normal evolution vector

$$\mathcal{L}_m \, A_{\alpha\beta} = m^\gamma \nabla_\gamma A_{\alpha\beta} + A_{\gamma\beta} \nabla_\alpha m^\gamma + A_{\alpha\gamma} \nabla_\beta m^\gamma \tag{128}$$

can be evaluated by rearranging (114) to write

$$\nabla_\alpha n_\beta = -K_{\alpha\beta} - n_\alpha \frac{1}{N} D_\beta N \;\longrightarrow\; \nabla_\beta m^\alpha = -N K^\alpha_\beta - n_\beta D^\alpha N + n^\alpha \nabla_\beta N \; . \tag{129}$$

Inserting (129) into (128) we easily evaluate

$$\mathcal{L}_m \, P_{\alpha\beta} = -2 N K_{\alpha\beta} \qquad\qquad \mathcal{L}_m \, P^\alpha_\beta = 0 \tag{130}$$

where again (92) permits the substitution $\gamma_{\alpha\beta} \to P_{\alpha\beta}$. For the extrinsic curvature we find

$$\mathcal{L}_m K_{\alpha\beta} = Nn^\gamma \nabla_\gamma K_{\alpha\beta} - 2NK_{\alpha\gamma}K^\gamma{}_\beta - K_{\alpha\gamma}D^\gamma Nn_\beta - K_{\beta\gamma}D^\gamma Nn_\alpha \tag{131}$$

which by projecting with $P^\alpha_{\alpha'} P^{\beta'}{}_\beta$ and recalling (122) and (124) becomes

$$P^{\alpha'}{}_\alpha P^{\beta'}{}_\beta R_{\alpha'\beta'} = \sigma \frac{1}{N} \mathcal{L}_m K_{\alpha\beta} + \sigma \frac{1}{N} D_\alpha D_\beta N + \bar{R}_{\alpha\beta} - \sigma K K_{\alpha\beta} + 2\sigma K^\delta_\alpha K_{\beta\delta} \,. \tag{132}$$

In light of (125) we see that equations (130) and (132) provide a pair of coupled evolution equations for $\gamma_{\mu\nu}$ and $K_{\mu\nu}$, of first order in ∂_τ, in which all terms except the 5D Ricci tensor $R_{\alpha\beta}$ are derived from the projected covariant derivative. These equations become an initial value problem for general relativity by replacing $R_{\alpha\beta}$ with the Einstein equations, which by projection on (107) take the form

$$P^{\alpha'}_\alpha P^{\beta'}_\beta R_{\alpha'\beta'} = P^{\alpha'}_\alpha P^{\beta'}_\beta \left(T_{\alpha'\beta'} - \frac{1}{2} P_{\alpha'\beta'} S \right) = S_{\alpha\beta} - \frac{1}{2} P_{\alpha\beta} S \,. \tag{133}$$

Pulling back to $\mathcal{M}$ and using (125) to expand $\mathcal{L}_m$ in (130) these expressions become

$$\frac{1}{c_5} \partial_\tau \gamma_{\mu\nu} = \mathcal{L}_\mathbf{N}\, \gamma_{\mu\nu} - 2NK_{\mu\nu} \tag{134}$$

$$\begin{aligned} \frac{1}{c_5} \partial_\tau K_{\mu\nu} &= -D_\mu D_\nu N + \mathcal{L}_\mathbf{N} K_{\mu\nu} \\ &\quad + N \left\{ -\sigma \bar{R}_{\mu\nu} + K K_{\mu\nu} - 2K^\lambda_\mu K_{\nu\lambda} + \sigma \frac{8\pi G}{c^4} \left(S_{\mu\nu} - \frac{1}{2} P_{\mu\nu} S \right) \right\} \end{aligned} \tag{135}$$

which differ from the expression found in [19] using unbroken field equations by the symmetry breaking replacement $g_{\mu\nu}\,(S + \sigma\kappa) \to P_{\mu\nu} S$.

To find the constraint equations, we first contract indices in the Gauss relation

$$R - 2\sigma R_{\alpha\beta} n^\alpha n^\beta = \bar{R} - \sigma \left(K^2 - K^{\alpha\beta} K_{\alpha\beta} \right) \tag{136}$$

and use the completeness relation to evaluate the trace of $R_{\alpha\beta}$

$$R = g^{\alpha\beta} R_{\alpha\beta} = P^{\alpha\beta} R_{\alpha\beta} + \sigma R_{\alpha\beta} n^\alpha n^\beta \tag{137}$$

to obtain

$$P^{\alpha\beta} R_{\alpha\beta} - \sigma R_{\alpha\beta} n^\alpha n^\beta = \bar{R} - \sigma \left(K^2 - K^{\alpha\beta} K_{\alpha\beta} \right) \,. \tag{138}$$

Projecting the SHP field equations onto the hypersurface and the unit normal provides

$$\begin{aligned} P^{\alpha\beta} R_{\alpha\beta} &= \frac{8\pi G}{c^4} P^{\alpha\beta} \left(T_{\alpha\beta} - \frac{1}{2} P_{\alpha\beta} S \right) = -\frac{8\pi G}{c^4} S \\ R_{\alpha\beta} n^\alpha n^\beta &= \frac{8\pi G}{c^4} \left(T_{\alpha\beta} - \frac{1}{2} P_{\alpha\beta} S \right) n^\alpha n^\beta = \frac{8\pi G}{c^4} \kappa \end{aligned} \tag{139}$$

from which we obtain the Hamiltonian constraint

$$\bar{R} - \sigma \left(K^2 - K^{\mu\nu} K_{\mu\nu} \right) = -\frac{8\pi G}{c^4} \left(S + \sigma\kappa \right) \,. \tag{140}$$

This expression differs from the expressions found in [19] using unbroken 5D field equations by the replacement

$$-\sigma\frac{16\pi G}{c^4}\kappa \;\longrightarrow\; -\frac{8\pi G}{c^4}\left(S+\sigma\kappa\right)\;. \tag{141}$$

Contracting the Codazzi relation (123) to

$$P^{\beta'}_{\ \ \beta}n^{\alpha}R_{\alpha\beta'} = D_\beta K - D_\alpha K^\alpha_\beta \tag{142}$$

and evaluating the mixed projection of the field equations

$$R_{\alpha\beta}n^{\alpha}P^{\beta'}_{\ \ \beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta'}-\frac{1}{2}P_{\alpha\beta'}S\right)n^{\alpha}P^{\beta'}_{\ \ \beta} = \frac{8\pi G}{c^4}T_{\alpha\beta'}n^{\alpha}P^{\beta'}_{\ \ \beta} = -\frac{8\pi G}{c^4}p_\beta \tag{143}$$

we find the momentum constraint

$$D_\mu K^\mu_\nu - D_\nu K = \frac{8\pi G}{c^4}p_\nu \tag{144}$$

in the same form found in [19]. Together, equations (134), (135), (140), and (144) decompose the SHP field equations into an initial value problem for the metric and extrinsic curvature.

5.4. Evolution of the vielbein field

In the quintrad the metric η_{ab} remains constant while the vielbein field that connects the quintrad to a coordinate frame evolves with τ. To obtain a pair of coupled evolution equations for $E_\alpha^{\ a}$ and K_{ab} we restrict our attention to line elements of the form

$$dX^2 = -g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + \sigma g_{55}(dx^5)^2 \tag{145}$$

in rectilinear coordinates and

$$dX^2 = -g_{00}dx_0^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}\sin^2\theta d\phi^2 + \sigma g_{55}dx^2 \tag{146}$$

in spherical coordinates. Following the prescription described in Section (2.4) we find the coordinate frames and vielbein fields to be

$$E_\alpha^{\ a} = \text{diag}\left(A_0, A_1, A_2, A_3, A_5\right) \tag{147}$$

in the rectilinear case, and

$$E_\alpha^{\ a} = \begin{bmatrix} A_0 & 0 & 0 & 0 & 0 \\ 0 & A_1\sin\theta\cos\phi & A_1\sin\theta\sin\phi & A_1\cos\theta & 0 \\ 0 & A_2\cos\theta\cos\phi & A_2\cos\theta\sin\phi & -A_2\sin\theta & 0 \\ 0 & -A_3\sin\theta\sin\phi & A_3\sin\theta\cos\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & A_5 \end{bmatrix} \tag{148}$$

in the spherical case, with $g_{\alpha\alpha} = (A_\alpha)^2$. We may write (148) as

$$E_\alpha^{\ a} = A_{(\alpha)}R_\alpha^{\ a} \tag{149}$$

where (α) indicates no summation and $R_\alpha^{\ 0} = \delta^0_\alpha$, $R_0^{\ 5} = \delta^5_\alpha$, and $R_\alpha^{\ a}$ is a standard 3D rotation matrix for $\alpha, a = 1,2,3$. We easily confirm that this vielbein also leads to the diagonal metric

$$\gamma_{\alpha\beta} = \eta_{ab}E_\alpha^{\ a}E_\beta^{\ b} = A_{(\alpha)}A_{(\beta)}\eta_{ab}R_\alpha^{\ a}R_\beta^{\ b} = A_{(\alpha)}A_{(\beta)}\eta_{\alpha\beta} = A^2_{(\alpha)}\eta_{\alpha\beta} \tag{150}$$

by virtue of its orthogonality.

For any diagonal metric we have $N^\mu = 0$, for which (118) simplifies to

$$K_{ab} = \frac{1}{2}\delta^k_a\delta^l_b \left(E_{\mu k}\partial_{\bar{5}}e^\mu{}_l + E_{\mu l}\partial_{\bar{5}}e^\mu{}_k\right) = -\frac{1}{2}\delta^k_a\delta^l_b \left(e^\mu{}_l\partial_{\bar{5}}E_{\mu k} + e^\mu{}_k\partial_{\bar{5}}E_{\mu l}\right) \tag{151}$$

where we used

$$e^\mu{}_l\partial_a E_{\mu k} + E_{\mu k}\partial_a e^\mu{}_l = \partial_a \left(e^\mu{}_l E_{\mu k}\right) = \partial_a \eta_{lk} = 0\,. \tag{152}$$

Writing the inverse vielbein field as

$$e^\alpha{}_b = \left[\left(E^{-1}\right)^T\right]^\alpha{}_b = \frac{1}{A_{(\alpha)}}R^\alpha{}_b \tag{153}$$

which we verify by

$$e^\alpha{}_b E_\alpha{}^a = \frac{1}{A_{(\alpha)}}R^\alpha{}_b A_{(\alpha)} R_\alpha{}^a = R^\alpha{}_b R_\alpha{}^a = \delta^a_b \tag{154}$$

we obtain

$$e^\alpha{}_b\partial_{\bar{5}}E_{\alpha a} = \eta_{aa'}e^\alpha{}_b\partial_{\bar{5}}E_\alpha{}^{a'} = \eta_{aa'}\frac{1}{A_{(\alpha)}}R^\alpha{}_b\partial_{\bar{5}}A_{(\alpha)}R_\alpha{}^{a'} = \frac{\partial_{\bar{5}}A_{(\alpha)}}{A_{(\alpha)}}\eta_{aa'}R^\alpha{}_b R_\alpha{}^{a'}\,. \tag{155}$$

Swapping the indices a and b we see that

$$e^\alpha{}_a\partial_{\bar{5}}E_{\alpha b} = \frac{\partial_{\bar{5}}A_{(\alpha)}}{A_{(\alpha)}}\eta_{ba'}R^\alpha{}_a R_\alpha{}^{a'} = \frac{\partial_{\bar{5}}A_{(\alpha)}}{A_{(\alpha)}}\eta_{ba'}R_\alpha{}^{a'}R^\alpha{}_a = \frac{\partial_{\bar{5}}A_{(\alpha)}}{A_{(\alpha)}}\eta_{aa''}R^\alpha{}_b R_\alpha{}^{a''} = e^\alpha{}_b\partial_{\bar{5}}E_{\alpha a} \tag{156}$$

from which

$$K_{kl} = -\frac{1}{2}\left(e^\mu{}_l\partial_{\bar{5}}E_{\mu k} + e^\mu{}_k\partial_{\bar{5}}E_{\mu l}\right) = -e^\mu{}_l\partial_{\bar{5}}E_{\mu k} \tag{157}$$

permitting us to rewrite (151) as

$$\partial_{\bar{5}}E_\mu{}^k = -E_\mu{}^l K^k_l\,. \tag{158}$$

Using (77) as the explicit form of the vielbein field we have

$$\partial_{\bar{5}} = e^\alpha{}_{\bar{5}}\partial_\alpha = \left[\delta^k_{\bar{5}}\delta^\alpha_\mu e^\mu{}_k - \delta^5_{\bar{5}}\frac{1}{N}\left(\delta^\alpha_\mu N^\mu - \delta^\alpha_5\right)\right]\partial_\alpha = \frac{1}{N}\partial_5 \tag{159}$$

where we used $\delta^k_{\bar{5}} = 0$ and $N^\mu = 0$. Expression (158) now becomes the evolution equation

$$\partial_5 E_\mu{}^k = -N E_\mu{}^l K^k_l \tag{160}$$

which we compare with the evolution equation for the metric (134)

$$\left(\frac{1}{c_5}\partial_\tau - \mathcal{L}_{\mathbf{N}}\right)\gamma_{\mu\nu} = -2NK_{\mu\nu} \quad\longrightarrow\quad \partial_5\gamma_{\mu\nu} = -2NK_{\mu\nu} \tag{161}$$

in the case that $N^\mu = 0$.

We now consider the coordinate evolution equation for $N^\mu = 0$

$$\partial_5 K_{\mu\nu} = -D_\mu D_\nu N + N\left\{-\sigma\bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K^\lambda_\mu K_{\nu\lambda} + \sigma\frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}S\right)\right\} \tag{162}$$

found as the pullback to $\mathcal{M}$ of an expression derived on the hypersurface Σ_τ. We rewrite this as a pullback from the quintrad using

$$\mathcal{E}^a_\mu = \mathcal{E}^\alpha_\mu E_\alpha{}^a = \delta^\alpha_\mu \left[\delta^{\mu'}_\alpha \delta^a_k E_{\mu'}{}^k + \delta^5_\alpha \left(E_{\mu'}{}^k N^{\mu'} \delta^a_k + N\delta^a_5\right)\right] = \delta^a_k E_\mu{}^k \tag{163}$$

where $N^\mu = 0$ and $\delta^5_\mu = 0$, to obtain

$$\partial_5 \left(\mathcal{E}^a_\mu \mathcal{E}^b_\nu K_{ab}\right) = -\mathcal{E}^a_\mu D_a \mathcal{E}^b_\nu D_b N + \mathcal{E}^a_\mu \mathcal{E}^b_\nu N \left\{-\sigma \bar{R}_{ab} + KK_{ab} - 2K^c_a K_{bc} + \sigma \frac{8\pi G}{c^4}\left(S_{ab} - \frac{1}{2}\eta_{ab} S\right)\right\} . \tag{164}$$

The push forward similarly reduces to

$$V^\perp_a = e^\alpha{}_a \delta^\mu_\alpha v_\mu = \left[\delta^k_a \delta^\alpha_{\mu'} e^{\mu'}{}_k - \delta^5_a \frac{1}{N}\left(\delta^\alpha_{\mu'} N^{\mu'} - \delta^\alpha_5\right)\right] \delta^\mu_\alpha v_\mu = \delta^k_a e^\mu{}_k v_\mu \tag{165}$$

and writing $\bar{\mathcal{E}}^\mu_a = \delta^k_a e^\mu{}_k$ we have

$$\bar{\mathcal{E}}^\mu_a \mathcal{E}^{a'}_\mu = \delta^k_a \delta^{a'}_{k'} e^\mu{}_k E_\mu{}^{k'} = \delta^k_a \delta^{a'}_k . \tag{166}$$

Applying $\bar{\mathcal{E}}^\mu_a$ to the evolution equation we are led to

$$\bar{\mathcal{E}}^\mu_k \bar{\mathcal{E}}^\nu_l \partial_5 \left(\mathcal{E}^a_\mu \mathcal{E}^b_\nu K_{ab}\right) = -D_k D_l N + N\left\{-\sigma \bar{R}_{kl} + KK_{kl} - 2K^j_k K_{lj} + \sigma \frac{8\pi G}{c^4}\left(S_{kl} - \frac{1}{2}\gamma_{kl} S\right)\right\} \tag{167}$$

where we used the compatibility relation (46) to write $D_a \mathcal{E}^b_\nu = 0$, and expand

$$K^c_k K_{lc} = \left(\eta^{cc'} \delta^{k'}_{c'} \delta^{l'}_k K_{k'l'}\right)\left(\delta^{k''}_l \delta^{l''}_c K_{k''l''}\right) = K^j_k K_{jl} . \tag{168}$$

On the LHS

$$\bar{\mathcal{E}}^\mu_k = \delta^{k'}_k e^\mu{}_{k'} = e^\mu{}_k \tag{169}$$

and

$$\mathcal{E}^a_\mu \mathcal{E}^b_\nu K_{ab} = \left(\delta^a_{k'} E_\mu{}^{k'}\right)\left(\delta^b_{l'} E_\nu{}^{l'}\right)\left(\delta^k_a \delta^l_b K_{kl}\right) = E_\mu{}^k E_\nu{}^l K_{kl} \tag{170}$$

so that

$$\bar{\mathcal{E}}^\mu_k \bar{\mathcal{E}}^\nu_l \partial_5 \left(\mathcal{E}^a_\mu \mathcal{E}^b_\nu K_{ab}\right) = e^\mu{}_k \left(\partial_5 E_\mu{}^{k'}\right) K_{k'l} + e^\nu{}_l \left(\partial_5 E_\nu{}^{l'}\right) K_{kl'} + \partial_5 K_{kl} \tag{171}$$

and using (160) to replace $\partial_5 E_\mu{}^k$ we find

$$\bar{\mathcal{E}}^\mu_k \bar{\mathcal{E}}^\nu_l \partial_5 \left(\mathcal{E}^a_\mu \mathcal{E}^b_\nu K_{ab}\right) = -2NK^{k'}_k K_{k'l} + \partial_5 K_{kl} . \tag{172}$$

Inserting this into (164) leads us finally to a slightly simplified evolution equation for the extrinsic curvature

$$\partial_5 K_{kl} = -D_k D_l N + N\left\{-\sigma \bar{R}_{kl} + KK_{kl} + \sigma \frac{8\pi G}{c^4}\left(S_{kl} - \frac{1}{2}\eta_{kl} S\right)\right\} . \tag{173}$$

We mention that the equilibrium condition for this system is found by taking $N = 1$ and $\partial_\tau E_\mu{}^k = 0$, which by (160) entails $K_{kl} = 0$. Now (173) becomes

$$0 = -\bar{R}_{kl} + \frac{8\pi G}{c^4}\left(S_{kl} - \frac{1}{2}\eta_{kl} S\right) \tag{174}$$

which we recognize as the trace-reversed form of the standard 4D Einstein field equations in the vielbein.

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6. Example — evolving Schwarzschild metric

In [19] we considered an event evolving with τ-varying squared mass $-p^\mu p_\mu = -(M\dot{x})^2$, and showed, using the weak field approximation, that it induces a τ-dependent metric. This metric, in turn, induces motion in a test particle whose mass varies similarly with τ, suggesting a transfer of mass carried by the gravitational field across spacetime. We also considered a generalized Schwarzschild solution with 5D metric

$$g_{\alpha\beta} = \text{diag}\left(-B, A, r^2, r^2\sin^2\theta, \sigma N\right) \tag{175}$$

with $B = B(r,\tau)$, $A = A(r,\tau)$, and $N = 1$. The spacetime components of the Christoffel connection and Ricci tensor are thus τ-dependent, but their functional form is unaffected, and so under boundary conditions

$$B(r,\tau) \xrightarrow[r\to\infty]{} 1 \qquad\qquad A(r,\tau) \xrightarrow[r\to\infty]{} 1 \tag{176}$$

for all τ, a solution satisfying $\bar{R}_{\mu\nu} = 0$ leads to

$$AB = 1 \qquad\qquad B = 1 + \frac{f(\tau)}{r} = 1 + \frac{2MG}{c^2 r}T(\tau) \tag{177}$$

which recovers the standard Schwarzschild solution for the function $T(\tau) = 1$. Solving the field equations to obtain $T(\tau)$ for given sources $S_{\mu\nu}$, p_μ, and κ will generally require numerical integration, but to gain a sense of the formalism we may calculate the sources for a given T. Using the unbroken 5D field equations in a coordinate frame we found

$$K_{\mu\nu} = -\frac{1}{2c_5}\partial_\tau\gamma_{\mu\nu} = -\frac{1}{c_5}\dot{T}(\tau)\,\frac{MG}{c^2 r}\,\text{diag}\left(1, \frac{1}{B^2}, 0, 0\right) \longrightarrow K = 0 \tag{178}$$

$$S_{00} = B^2 S_{11} = -\sigma\frac{c^2}{c_5^2}\frac{2M}{16\pi r}\ddot{T}(\tau) \qquad\qquad S_{22} = S_{33} = 0 \longrightarrow S = 0 \tag{179}$$

along with $\kappa \approx 0$.

In the quintrad frame, comparison of the metric (175) with (148) and (150) provides the vielbein field in the form

$$E_\alpha^{\ a} = \begin{bmatrix} \sqrt{B} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{A}\sin\theta\cos\phi & \sqrt{A}\sin\theta\sin\phi & \sqrt{A}\cos\theta & 0 \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta & 0 \\ 0 & -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{180}$$

where we have taken

$$A_0 = \sqrt{B} \qquad\qquad A_1 = \sqrt{A} = \frac{1}{\sqrt{B}} \qquad\qquad A_2 = A_3 = A_5 = 1\,. \tag{181}$$

leading to

$$\frac{\partial_a A_0}{A_0} = \frac{\partial_a B}{2B} \qquad\qquad \frac{\partial_a A_1}{A_1} = \frac{1}{2A}\partial_a A = -\frac{\partial_a B}{2B} \tag{182}$$

as in the standard Schwarzschild solution.

With $N = 1$, and thus $D_k D_l N = 0$, the evolution equations (160) and (173) are somewhat simplified. In (178) we found $K_{\mu\nu}$ by inverting the metric evolution equation, and here we similarly use (155) to invert the vielbein evolution equation (160) and write

$$K_{kl} = -e^{\mu}{}_k \partial_5 E_{\mu l} = -\frac{\partial_5 A_{(\mu)}}{A_{(\mu)}} \eta_{ll'} R^{\mu}{}_k R_{\mu}{}^{l'} = -\frac{\partial_5 B}{B} \left(\delta^0{}_k \eta_{l0} - \eta_{ll'} R^1{}_k R_1{}^{l'} \right) \tag{183}$$

from which we again find the vanishing trace as

$$K = \eta^{kl} K_{kl} = -\frac{\partial_5 B}{B} \eta^{kl} \left(\delta_0^0 - \delta_1^1 \right) = 0\ . \tag{184}$$

We notice that the coefficient $R^1{}_m$ is the m-component of the 3D unit vector

$$\hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{185}$$

so that

$$K_{kl} = \frac{\partial_5 B}{B} \left(\delta_k^0 \delta_l^0 + \hat{\mathbf{r}}_k \hat{\mathbf{r}}_l \right) \tag{186}$$

"points" in the kl-direction.

In the weak field approximation we neglected terms quadratic in $K_{\mu\nu}$ but here we readily evaluate

$$K^{kl} K_{kl} = \left(-\frac{\partial_5 A_{(\mu)}}{A_{(\mu)}} \eta^{kk'} R^{\mu}{}_{k'} R_{\mu}{}^{l} \right) \left(-\frac{\partial_5 A_{(\nu)}}{A_{(\nu)}} \eta_{ll'} R^{\nu}{}_k R_{\nu}{}^{l'} \right) = \frac{1}{2} \left(\frac{\partial_5 B}{B} \right)^2 . \tag{187}$$

Inserting (186) into the second evolution equation leads to nonlinear second order equations, as we expect to find from field equations for gravitation. Splitting (173) and the constraint equations into 3D space ($m, n = 1, 2, 3$) and 0-components, these become

$$-\partial_5 \left(\frac{\partial_5 B}{B} \right) = \sigma \frac{8\pi G}{c^4} S_{00} \qquad \hat{\mathbf{r}}_m \hat{\mathbf{r}}_n\, \partial_5 \left(\frac{\partial_5 B}{B} \right) = \sigma \frac{8\pi G}{c^4} S_{mn} \tag{188}$$

$$-\frac{1}{2} \left(\frac{\partial_5 B}{B} \right)^2 = \frac{8\pi G}{c^4} \kappa \tag{189}$$

$$-D_0 \left(\frac{\partial_5 B}{B} \right) = \frac{8\pi G}{c^4} p_0 \qquad D_m \left(\frac{\partial_5 B}{B} \hat{\mathbf{r}}^m \hat{\mathbf{r}}_n \right) = \frac{8\pi G}{c^4} p_n \tag{190}$$

from which we again find $S = \eta^{kl} S_{kl} = 0$. As for K_{kl}, we see that the spacetime energy momentum tensor S_{kl} "points" in the kl-direction. The previously neglected quadratic term (189) indicates that this solution requires a negative local mass density $\kappa < 0$. In [21] we showed that κ can be expressed as a cosmological term

$$\Lambda(x, \tau) = \sigma \frac{4\pi G}{c^4} \kappa \tag{191}$$

that is not constant, but scalar and independent of S_{kl}. For a weak field solution, it is seen that this term must be very small in magnitude.

7. Conclusion

As in SHP electrodynamics, a particle worldline is the trajectory over τ of the irreversible physical event $x^\mu(\tau)$, whose evolution is generated by a scalar Hamiltonian K. The spacetime manifold of general relativity is then $\mathcal{M}(\tau)$, a 4D block universe occurring at τ and evolving under K to an infinitesimally close 4D block universe $\mathcal{M}(\tau + d\tau)$. The local metric structure $\gamma_{\mu\nu}$ of $\mathcal{M}(\tau)$, including the coordinate past and future, must therefore be τ-dependent, and appropriate field equations must be determined to specify its evolution.

The field equations of standard general relativity (GR) express the reciprocal relationship between 4D geometry and the energy-momentum of matter present in spacetime, embodying O(3,1) covariance, general diffeomorphism invariance, and the translation invariance of the Ricci tensor that guarantees the Bianchi identity as a 4D gauge symmetry. The 3+1 formalism decomposes the field equations into a pair of coupled t-evolution equations for the geometry of 3D space, along with constraints among the initial conditions. As articulated by Wheeler [31], "A decade and more of work by Dirac, Bergmann, Schild, Pirani, Anderson, Higgs, Arnowitt, Deser, Misner, DeWitt, and others has taught us through many a hard knock that Einstein's geometrodynamics deals with the dynamics of geometry: of 3-geometry, not 4-geometry."

In SHP GR we proceed in the reverse direction, guided by SHP electrodynamics. Beginning with τ-evolution, we determine field equations possessing O(3,1) covariance and a 5D gauge symmetry associated with a Bianchi identity that obtains for the 5D Ricci tensor $R_{\alpha\beta}$ constructed on the pseudo-spacetime $\mathcal{M}_5$. This structure admits the natural foliation into 4D hyperspaces Σ_τ homeomorphic to $\mathcal{M}$, leading to the desired evolution equations for $\gamma_{\mu\nu}(x,\tau)$ and $K_{\mu\nu}(x,\tau)$ as (nearly) canonically conjugate fields [19].

The basic structure of the 4+1 formalism, generalizing the geometrical arguments of the 3+1 method to the SHP framework, was presented in [19]. In [20, 21] the linearized theory was explored, and it was shown that the evolution equations retain a 5D symmetry, possibly O(4,1) or O(3,2), that leads to deviation from standard gravitational phenomenology. As in SHP electrodynamics, the fields (geometry) may exhibit the higher symmetry in a vacuum, but the matter parts must be restricted to O(3,1). In this paper, we broke the symmetry at the interface between the field and matter parts of the field equations. To this end, we constructed the tangent space $\mathcal{T}(\mathcal{M}_5)$ and its foliation in a quintrad frame, for which the flat metric and its symmetry-broken form are easily chosen. Using the vielbein field, we transformed the symmetry-broken flat metric to its local form in a coordinate frame, leading to the desired coordinate frame field equations for the 5D metric $g_{\alpha\beta}$. Since the quintrad metric is constant and the τ-dependence is contained in the vielbein field, we obtained in an evolution equation for $E_\alpha{}^a$ that applies to any diagonal metric, and showed that it simplifies the evolution equation for $K_{\mu\nu}$. Thus, initial value problems for gravitational fields may be solved in the quintrad frame.

It follows from the 5D Bianchi identity [19–21] that the SHP field equations (107) split into

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}P_{\mu\nu}S\right) \qquad\qquad R_{5\alpha} = \frac{8\pi G}{c^4}T_{5\alpha} \tag{192}$$

where the spacetime components are ten unconstrained field equations corresponding to the τ-evolution equations for $\gamma_{\mu\nu}$ and $K_{\mu\nu}$, and the 5-components are constraints among the initial conditions that propagate forward with τ but do not evolve under second order differential equations. This decomposition was also seen [21] in the linearized SHP theory by expanding the Ricci tensor $R_{\alpha\beta}$ for a weak perturbation to the flat metric, so that equations (107) become a 5D wave equation. This wave equation similarly splits into spacetime components from which we find the linearized evolution equations, and 5-components that become the constraint equations. In this paper we saw that the vielbein field splits into a spacetime part that contains the dynamical content, and a 5-part that embodies the constraints. The natural separation of the field equations into unconstrained evolution equations and non-evolving constraints underlies the 4+1 formalism in SHP gravitation.

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