On the soft limit of tree-level string amplitudes

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We study the soft behavior of string scattering amplitudes at three level with massless and massive external insertions, relying on different techniques to compute 4-points amplitudes respectively with open or closed strings.

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1. General introduction

The study of the soft limit of scattering amplitudes is a well established topic in QFT since the pioneering works of Low[1] in quantum electrodynamics, and Weinberg[2], Gross and Jackiw[3] in Einstein gravity. Recently, this subject has raised renewed attention due to the discovery[4] of its connection with the B²MS large diffeomorphisms invariance of general relativity for asymptotically flat spaces[5–12]. Soft theorems have been derived for a wide class of QFTs and strings from the examination of scattering amplitudes at tree and loop level[13–22]. Concretely, if we consider color-ordered gluon amplitudes, the soft theorem relates the $n+1$ amplitude $\mathcal{A}_{n+1}(k_1, ..., k_n; k_{n+1})$ to the n-points one $\mathcal{A}_n(k_1, ..., k_n)$ through a differential operator depending on the momentum and the polarization of the $(n+1)$-th external leg, when $k_{n+1} = \delta \hat{k}_{n+1}$ and $\delta \to 0$; in formulae

$$\mathcal{A}_{n+1}(k_1, ..., k_n; k_{n+1}) = \left(\frac{a_{n+1}k_1}{k_{n+1}k_1} - \frac{a_{n+1}k_n}{k_{n+1}k_n} + \frac{f_{n+1}:J_1}{k_{n+1}k_1} - \frac{f_{n+1}:J_n}{k_{n+1}k_n} \right) \mathcal{A}_n(k_1, ..., k_n),$$
(1)

up to terms of order $\mathcal{O}(\delta^0)$. In Eq. (1) we introduced the field-strength $f_{\mu\nu} = a_\mu k_\nu - a_\nu k_\mu$, and the angular-momentum operator $J_{\mu\nu}$. A stronger statement holds for gravity amplitudes: the soft behavior is "universal" up to terms of order $\mathcal{O}(\delta)$, instead of order $\mathcal{O}(\delta^0)$ as for the Yang-Mills case

$$\mathcal{M}_{n+1}(k_1, ..., k_n; k_{n+1}) =$$
$$\sum_{i=1}^{n} \left(\frac{k_i h_{n+1} k_i}{k_i k_{n+1}} + \frac{k_i h_{n+1} J_i k_{n+1}}{k_i k_{n+1}} + \frac{k_{n+1} J_i h_{n+1} J_i k_{n+1}}{2 k_i k_{n+1}} \right) \mathcal{M}_n(k_1, ..., k_n). \quad (2)$$

For open superstring and bosonic string theories, Eq. (1) describes the soft behavior of massless string amplitudes, notwithstanding the presence of an interaction term $\alpha' F^3$ in the bosonic case, that in principle could spoil the validity of Eq. (1). Closed superstring amplitudes satisfy Eq. (2), while closed bosonic strings don't because at order $\mathcal{O}(\delta)$ there is a mixing between the graviton and the dilation, due to the interaction term $\alpha' \varphi R^2$.

2. Soft theorems in QFT and strings

In this section we sketch in some detail the proof of Eq. (1) for color-ordered gluon amplitudes in QFT – see Ref. [13] for a more complete discussion, and compare this approach to the one relying on the OPE expansion presented in Ref. [14].

2.1. *Gluon amplitudes in QFT*

The singular soft behavior of the tree level color ordered amplitude $\mathcal{A}_{n+1}(k_1, ..., k_n; k_s)$, when k_s becomes soft, arises from the propagators in the channels $k_1 + k_s$ and $k_n + k_s$, in formulae

$$\mathcal{A}_{n+1}(k_1, ..., k_n; k_s) =$$
$$\frac{V_\mu(k_s, k_1)}{(k_s+k_1)^2}\mathcal{A}_n^\mu(k_1+k_s, ..., k_n) + \frac{V_\mu(k_n, k_s)}{(k_s+k_n)^2}\mathcal{A}_n^\mu(k_1, ..., k_n+k_s) + R_{n+1}(k_1, ..., k_n; k_s). \tag{3}$$

The Yang-Mills vertex for incoming momenta k_s, k_i, k_I, with $k_I = -k_i - k_s$, can be written as

$$V_\mu(k_s, k_i) = V_{\mu,i}^{(0)} + V_{\mu,i}^{(1)} = 2a_i^\mu a_s k_i + 2k_s^\mu a_s a_i - 2a_s^\mu a_i k_s, \tag{4}$$

$$V_{\mu,i}^{(0)} = V_\mu(0, k_i) = 2a_i^\mu a_s k_i \qquad \text{and} \qquad V_{\mu,i}^{(1)} = k_s\frac{\partial}{\partial k_s}V_\mu(0, k_i) = 2k_s^\mu a_s a_i - 2a_s^\mu a_i k_s. \tag{5}$$

The expansion of the n-point amplitude $\mathcal{A}_n$ around k_i yields

$$\mathcal{A}_n^\mu(..., k_s+k_i, ...) = \mathcal{A}_{n,i}^{\mu(0)} + \mathcal{A}_{n,i}^{\mu(1)} + ... = \mathcal{A}_n^\mu(..., k_i, ...) + k_s\frac{\partial}{\partial \hat{k}_i}\mathcal{A}_n^\mu(..., \hat{k}_i, ...)|_{\hat{k}_i=k_i} + ..., \tag{6}$$

Combining it with Eq. (4) produces the full soft expansion of Eq. (3). Imposing gauge invariance order by order in the soft parameter δ, we can determine the leading term of the unknown contribution $R_{n+1}(k_1, ..., k_n; k_s)$. At leading order

$$\frac{V_{\mu,1}^{(0)}}{2k_s k_1}\mathcal{A}_{n,1}^{\mu(0)} - \frac{V_{\mu,n}^{(0)}}{2k_s k_n}\mathcal{A}_{n,n}^{\mu(0)} = \left(\frac{a_s k_1}{k_s k_1} - \frac{a_s k_n}{k_s k_n}\right)\mathcal{A}_n(k_1, ..., k_n) \overset{a_s \to k_s}{\longrightarrow} 0. \tag{7}$$

At sub-leading order, it is possible to recognize a manifest gauge invariant contribution

$$\frac{V_{\mu,1}^{(1)}}{2k_s k_1}\mathcal{A}_{n,1}^{\mu(0)} - \frac{V_{\mu,n}^{(1)}}{2k_s k_n}\mathcal{A}_{n,n}^{\mu(0)} = \frac{f_s^{\mu\nu}a_{1[\nu}\partial/\partial a_1^{\mu]}}{2k_s k_1}\mathcal{A}_n^{(0)} - \frac{f_s^{\mu\nu}a_{n[\nu}\partial/\partial a_n^{\mu]}}{2k_s k_n}\mathcal{A}_n^{(0)} \overset{a_s \to k_s}{\longrightarrow} 0, \tag{8}$$

which reconstructs the action of the spin part of the angular momentum, and a non gauge invariant part

$$\frac{V_{\mu,1}^{(0)}}{2k_s k_1}\mathcal{A}_{n,1}^{\mu(1)} - \frac{V_{\mu,n}^{(0)}}{2k_s k_n}\mathcal{A}_{n,n}^{\mu(1)} + a_s\widetilde{R}_{n+1}(k_1, ..., k_n; k_s=0)$$
$$\overset{a_s \to k_s}{\longrightarrow} k_s\frac{\partial}{\partial k_1}\mathcal{A}_n - k_s\frac{\partial}{\partial k_n}\mathcal{A}_n + k_s\widetilde{R}_{n+1}(k_s = 0) = 0. \tag{9}$$

Solving Eq. (9) for $R_{n+1}(k_s=0)$ allows us to recover the action of the orbital part of the angular-momentum.

2.2. *Gluon amplitudes in string theory*

The above analysis, performed for a QFT, can be reproduced as well by the OPE of vertex operators in string theory. To compute and $n+1$ superstring gluon amplitude, we need two operators in the $q=-1$ super-ghost picture and $n-2$ vertices with $q=0$. Let us suppose the soft vertex operator to be integrated and in the $q=0$ super-ghost picture, and the adjacent operators to be in the $q=-1$ picture. The leading order in k_s, when z_s approaches z_{s+1}, is given by

$$\int^{z_{s+1}1} dz_s\, a_s i\partial X\, e^{ik_s X}(z_s)\, a_{s+1}\psi e^{-\varphi} e^{ik_{s+1}X}(z_{s+1})\ldots \sim$$
$$\int^{z_{s+1}1} dz_s\, a_s k_{s+1}(z_s-z_{s+1})^{k_s k_{s+1}}\ldots \approx \frac{a_s k_{s+1}}{k_s k_{s+1}}\ldots \qquad (10)$$

Taking the sub-leading order in the soft expansion of Eq. (10), it happens that BRST invariance is responsible for reproducing the action of the orbital part of the angular momentum, as well as gauge invariance does in Eq. (9)[14]. The OPE of the manifestly BRST invariant sub-leading term $\frac{1}{2}\psi^\mu f_{\mu\nu}\psi^\nu e^{ikX}$ reproduces the action of the spin part of the soft operator. The above analysis is remarkable because can be easily generalized when the OPE is performed between the soft operator and a massive higher-spin string belonging to the leading Regge trajectory.

2.3. *Soft behavior of disk scattering amplitudes with massive states*

In the open bosonic string, at the first massive level, we have only one BRST-invariant state given by the vertex operator

$$V_H = H_{\mu\nu} i\partial X^\mu i\partial X^\nu\, e^{ipX} \qquad (11)$$

with $p^\mu H_{\mu\nu}=0$, symmetric and traceless. Conversely, in the open superstring we have two different states interpolating in the $q=-1$ picture by

$$V_H = H_{\mu\nu} i\partial X^\mu \psi^\nu e^{-\varphi} e^{ipX} \quad \text{and} \quad V_C = C_{\mu\nu\rho}\psi^\mu\psi^\nu\psi^\rho e^{-\varphi} e^{ipX}, \qquad (12)$$

with $p^\mu C_{\mu\nu\rho}=0$ and completely antisymmetric. In Ref.[24] we have shown that string disk-amplitudes with the insertions of the aforementioned vertex operators satisfy Eq. (1). Following the steps outlined in Sec. 2.2, it is straightforward to show that also amplitudes with the insertion of massive higher-spin strings like

$$V_{H^s} = H_{\mu_1\ldots\mu_s} i\partial X^{\mu_1}\ldots i\partial X^{\mu_{s-1}}\psi^{\mu_s} e^{-\varphi} e^{ipX} \qquad (13)$$

satisfy the soft theorem.

3. Massive string amplitudes from Yang-Mills

In Ref. 23 it is shown that open string disk-amplitudes can be expressed in terms of Yang-Mills tree-amplitudes covered by suitable generalized hypergeometric functions

$$\mathcal{A}_n^{ST}(1,...,n)=\sum_{\sigma\in S_{n-3}} F(1,[2_\sigma,...,(n-2)_\sigma],n-1,n)\mathcal{A}_n^{YM}(1,[2_\sigma,...,(n-2)_\sigma],n-1,n).$$

(14)

For $n=5$ Eq. (14) yields

$$\mathcal{A}_5^{ST}(12345)=F(1[23]45)\mathcal{A}_5^{YM}(1[23]45)+F(1[32]45)\mathcal{A}_5^{YM}(1[32]45).$$

(15)

As we suggested in Ref 24, it is possible to get 4-point amplitudes with massive states taking the residue at the massive pole in some 2-particle channels of the 5-points amplitude with massless strings

$$\lim_{s_{12}\to -1}(s_{12}+1)\mathcal{A}_n^{ST}(1,2,...,n)=\sum_{S'}\mathcal{A}_3^{ST}(1,2,S')\otimes\mathcal{A}_{n-1}(S',3,...,n).$$

(16)

3.1. *Dimensional reduction at $D=4$*

Before showing an explicit example, let us briefly review the content of the first massive level of the superstring theory in the bosonic sector. The first massive super-multiplet in $D=10$ yields a long multiplet of $\mathcal{N}=4$ in $D=4$

$$\{H_{MN},C_{MNP}\}\quad\rightarrow\quad\{H_{\mu\nu},8\psi_\mu,27Z_\mu,48\chi,42\varphi\}.$$

(17)

The bosonic degrees of freedom are related to the 10-dimensional vertex operators as follows[24]

$$H_{\mu\nu}\leftarrow H_{\mu\nu}\qquad 27\,Z_\mu\leftarrow 6\,H_{\mu,i},15\,C_{\mu,ij},6\,C_{\mu\nu,i}\qquad 42\,\varphi\leftarrow 21\,H_{ij},20\,C_{ijk},C_{\mu\nu\rho}.$$

(18)

Among these states, only three of them are R-symmetry singlets in $D=4$: the spin-2 $H_{\mu\nu}$, the pseudo-scalar $C_0=\varepsilon_{\mu\nu\rho\sigma}p^\sigma C^{\mu\nu\rho}/M$ and the scalar H_0 coming from the fact that the BRST-invariant string state H_{MN} is traceless only in $D=10$, therefore after dimensional reduction we are left with the partial trace $H_{ij}=H_0\delta_{ij}/2$.

3.2. *Massive string amplitudes in $D=4$*

In $D=4$ there are only MHV or anti-MHV 5-points gluon amplitudes. For example, fixing the helicity of the gluons, Eq. (15) yields

$$\mathcal{A}_5^{ST}(1^-2^-3^+4^+5^+)=\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}F(1[23]45)+\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 45\rangle\langle 51\rangle}F(1[32]45).$$

(19)

Taking the residue at the pole $s_{12}+1$, we get

$$\lim_{s_{12}\to -1}(s_{12}+1)\mathcal{A}_5^{ST}(1^-2^-3^+4^+5^+)=\mathcal{B}(1-\alpha's,1-\alpha't)\times$$

$$\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}[s_{23}s_{35}+(s_{34}+s_{35})s_{24}]-\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 45\rangle\langle 51\rangle}s_{13}s_{24}.$$

(20)

It is remarkable to notice that a totally symmetric and traceless state, as $H_{\mu\nu}$, or any other spin$-s$ state $S_{\mu_1 \ldots \mu_s}$ couples only to vector bosons with opposite helicity, let's say $f_1^{\pm}$ and $f_2^{\mp}$. Conversely, a scalar state couples only to vector bosons with the same helicity $f_1^{\pm}$ and $f_2^{\pm}$. This can be easily understood looking at the properties of the 't Hooft symbols which form a basis for the selfdual or anti-selfdual 4×4 matrices

$$(f_1^+)_{\mu\nu}(f_2^-)^{\nu\rho} \propto \eta_{\mu\nu}^a \bar{\eta}^{b\nu\rho} = S_{\mu}^{ab\rho} \tag{21}$$

$$(f_1^+)_{\mu\nu}(f_2^+)^{\nu\rho} \propto \eta_{\mu\nu}^a \eta^{b\nu\rho} = \delta^{ab}\delta_{\mu}^{\rho} + \varepsilon^{abc}\eta^c{}_{\mu}{}^{\rho}. \tag{22}$$

This observation allows us to disentangle the contribution due to the polarization H^{++} or H^{--} from those due to the propagation of H_0 or C_0 [24]

$$\mathcal{A}_5^{ST}(1^-2^-3^+4^+5^+) \xrightarrow{s_{12}\to-1} \frac{\langle 12 \rangle^2}{M} \times \mathcal{B}(1-\alpha's, 1-\alpha't)\frac{M[35]}{\langle 34 \rangle\langle 45 \rangle} \qquad C_0/H_0 \tag{23}$$

$$\mathcal{A}_5^{ST}(1^+2^+3^-4^-5^+) \xrightarrow{s_{12}\to-1} \frac{[12]^2}{M} \times \mathcal{B}(1-\alpha's, 1-\alpha't)\frac{\langle 34 \rangle^3[35]}{M^3\langle 45 \rangle} \qquad C_0/H_0 \tag{24}$$

$$\mathcal{A}_5^{ST}(1^+2^-3^-4^+5^+) \xrightarrow{s_{12}\to-1} M \times \mathcal{B}(1-\alpha's, 1-\alpha't)\frac{\langle 13 \rangle^4[35]}{M\langle 12 \rangle^2\langle 34 \rangle\langle 45 \rangle} \qquad H^{++}/H^{--}. \tag{25}$$

Using $SO(3)$ little group transformations that leave unchanged the momentum of the massive particle $p = u\bar{u}+v\bar{v}$, it is straightforward to get the expression of the amplitude for all the other polarizations of the tensor H. Defining

$$L_x \quad : \quad u' = \frac{1}{\sqrt{2}}(u+v) \qquad v' = \frac{1}{\sqrt{2}}(-u+v) \tag{26}$$

$$L_y \quad : \quad u' = \frac{1}{\sqrt{2}}(u+iv) \qquad v' = \frac{1}{\sqrt{2}}(iu+v), \tag{27}$$

and noticing that $H^{++}+H^{--} \xrightarrow{L_x-L_y} H^0$, $H^{++}-H^{--} \xrightarrow{L_x} (H^- - H^+)/2$, and $H^{++}-H^{--} \xrightarrow{L_y} i(H^- + H^+)/2$, we get the full amplitude rotating the polarizations using linear combinations of the operators in Eq. (27)

$$\sum_h c_h \mathcal{A}(1^-2^+3^+H^h) = \mathcal{B}(1-\alpha's, 1-\alpha't)$$
$$\times \frac{[13]\langle 14 \rangle^2\langle 15 \rangle^2}{M\langle 12 \rangle\langle 23 \rangle\langle 45 \rangle^2}\left(c_{++}\frac{\langle 14 \rangle^2}{\langle 15 \rangle^2} - 4c_{+0}\frac{\langle 14 \rangle^2}{\langle 15 \rangle^2} + 6c_{00}\frac{\langle 14 \rangle^2}{\langle 15 \rangle^2} - 4c_{0-}\frac{\langle 14 \rangle^2}{\langle 15 \rangle^2} + c_{--}\frac{\langle 14 \rangle^2}{\langle 15 \rangle^2}\right). \tag{28}$$

4. Closed superstring amplitudes

We conclude with some few comments about the soft behavior of closed string amplitudes with gravitons and massive states studied in detail in Ref. 25. Using

the amplitudes computed in Ref.[24], and the KLT formula[26]

$$\mathcal{M}_4(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{K}_4 + \mathcal{L}_4 + \mathcal{U}_4) = \sin\left(\pi \frac{\alpha' t}{4}\right) \mathcal{A}_{4L}(1, 2, 3, H_4 + C_4) \otimes \mathcal{A}_{4R}(1, 3, 2, H_4 + C_4).$$
(29)

We checked the correct soft behavior up to the sub-sub-leading order in $D = 4$ using the spinor-helicity formalism for amplitudes involving gravitons, dilatons, and the massive states $\mathcal{K}$ and $\mathcal{H}$

$$\mathcal{M}_4(1^{-2}, 2^{+2}, 3^{+2}, \mathcal{K}_4^{+4}) \quad \mathcal{M}_4(1^0, 2^0, 3^{+2}, \mathcal{K}_4^{+4}) \quad \mathcal{M}_4(1^0, 2^{+2}, 3^{+2}, \mathcal{H}_4^{+2}), \quad (30)$$

when the graviton with momentum k_3 becomes soft. While for bosonic closed string amplitudes we have found a discrepancy at sub-sub-leading order due to α' dependent terms in agreement with Ref. 27.

Acknowledgments

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