ELLIPTIC AND HEUN FUNCTIONS IN SPATIALLY-FLAT FRIEDMANN-ROBERTSON-WALKER COSMOLOGIES

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Abstract. In the framework of the celebrated perfect-fluid approach to the $k = 0$ – Friedmann-Robertson-Walker models, with decoupled mixed sources, the case of radiation and cosmological dust has been investigated. We have obtained the algebraic equations which provide the scale factor and therefore computed the essential cosmological parameters. Finally, the quantum cosmological approach based on this minisuperspace and its heuristic correspondence with the classical regime have been received a particular consideration.

Key words: FRW-cosmology, Heun functions, non-interacting fluids.

1. INTRODUCTION

Nowadays, some modern trends in cosmology are revealed by the consistent investigations focusing on mixed matter sources, most of them elaborated in the framework of Einstein's theory. According to observations, we are provided with the conclusion that, within the present epoch, the main contribution to the energy density is reserved to the cosmological constant, the most trivial and appealing candidate to dark energy. It proves that it is the dark energy component the one which drives the expansion of the universe at an accelerated rate. The role of the cosmological constant in the evolution of the Universe is confirmed by the large scale experimental observations on the distribution of galaxies and clusters and by WMAP measurements on the fluctuations detected in CMB radiation [1].

It is well-known that the expansion rate is intimately related to the types of energy that the Universe contains. As the universe undergoes a continuous evolution, with progressive phases in which new forms of matter are generated, with various middle stages that affect the behavior of the universe as a whole, theoretical studies on cosmologies with a mixed composition of matter seem legitimate and very appealing. Also, one may get a more accurate description of the Universe in its present state, where the transitions between various epochs might be revealed by the way these matter sources intermingle to generate new physically interesting dynamics [2].

Furthermore, the isotropy and homogeneity which characterizes the large scale universe are consistent with the fluid-type *Friedmann-Robertson-Walker* (FRW) evolving cosmological models [3] which are locally isotropic and spatially homogeneous [4]. The detection of the CMB radiation generated within a very early and hectic phase as a label of the relic radiation comes to support this idea. According to the data collected by the COBE mission, it appears that the CMB radiation is very smooth to at least one part in 10^5 [5], possessing a black body spectrum whose temperature isotropy constitutes a direct probe in favor of the physical constraint of homogeneity and isotropy of the Universe at its megagalactic scale. Observations on temperature variations in CMB indicate that after leaving the radiation era, the geometry of the Universe is described in terms of a spatially-flat FRW universe [6]. We specify that the large-scale homogeneity as a characteristic of the observable universe founds a powerful explanation within the inflation scenario [7].

Our paper is following a previous investigation [8], where we developed a mixed matter cosmology with non-interacting ideal fluids namely, stiff matter and dust. In the first stage of our present study, within the fluid formalism, we consider the universe composed of cosmological dust and radiation. In the second stage, we add one more component, namely the cosmological constant in order to study its effects on the dynamics of the considered model. We specify that the species of fluid involved are treated as non-interacting matter sources. Historically, models with decoupled matter and radiation start with the ones elaborated by Lemaitre [9], Stabell [10] and McIntosh [11, 12], followed by the ones developed in [13, 14]. These focused not only on flat models ($k = 0$), with a non-linear time-dependent equation of state parameter, but also on models with other types of geometries ($k = \pm 1$).

At the basis of our choice for the togetherness of the radiation and dust species lies the phenomenology of the influence of radiation pressure on the cosmic dust mostly at the level of galaxies. For a long period way back in the past, this issue has been an acute and debatable one [15]. Among the studies on cosmologies with decoupled dust and radiation sources, we mention the theoretical investigative models [16, 17].

Within the frame of quantum minisuperspace theory, in [18], the authors are analyzing the FRW universe with non-interacting fluids. Following an analytical procedure, the paper shows that, for a combined dust-radiation source, the dust component can be created as a quantum effect. Moreover, when quantum effects come into play, the radiation also possesses an exotic character, leading to the formation of bounces.

With reference to the (small) positive cosmological constant we are considering here, this is in agreement with recent measurements of two independent groups, High-Z Supernova Team and the Supernova Cosmological Project [19, 20, 21].

Our approach is inspired by the work of Chavanis [17] which we find inspiring for allowing a deeper and more elaborate analysis that unveils intricacies behind the model' dynamics.

In what it concerns the quantum cosmological treatment developed in the final part of our work, this can be compared to similar recent investigations [22, 23, 24], where the solutions to the Wheeler–DeWitt proved to be expressed in terms of Heun functions.

2. DYNAMICS OF THE MODEL WITHIN THE CONTEXT OF FLUIDIC FORMALISM

The line element in comoving spherical coordinates for the quadridimensional homogeneous and isotropic flat space with constant local curvature is given by the $k = 0 -$ *Friedmann-Robertson-Walker* (FRW) metric

$$\mathrm{d}s^2 = a^2(t)\left[\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right] - \mathrm{d}t^2, \tag{1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$. The physical quantity $a(t): \mathbb{R} \to \mathbb{R}_+$, $a(t) = a_0 e^{f(t)}$ defines the scale factor, a function of the global time coordinate namely the cosmic time denoted here by t. We introduce the tetradic pseudo-orthonormal frame $e_{a\left(a=\overline{1,4}\right)}$, whose dual base is

$$\omega^1 = a\mathrm{d}r, \omega^2 = ar\mathrm{d}\theta, \ \omega^3 = ar\sin\theta\mathrm{d}\varphi, \ \omega^4 = \mathrm{d}t,$$

so that $\mathrm{d}s^2 = \eta_{ab}\omega^a\omega^b$, with $\eta_{ab} = \mathrm{diag}\left[1,\ 1,\ 1,\ -1\right]$.

Within the framework of an uniform perfect fluid of isotropic pressure with the energy-momentum tensor components given by $T_{\alpha\alpha} = p$, $T_{44} = \rho$, the Einstein's system of equations, $G_{ab} + \eta_{ab}\Lambda = \kappa T_{ab}$, admits the explicit representation

$$\begin{aligned} &2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \Lambda = -\kappa p; \\ &3\frac{\dot{a}^2}{a^2} - \Lambda = \kappa\rho, \end{aligned} \tag{2}$$

where $\kappa = 8\pi G / c^4$ and Λ is the positive cosmological constant. The second relation in (2), *i.e.*

$$H^2 = \frac{\kappa}{3}\rho + \frac{\Lambda}{3}, \tag{3}$$

where H is the Hubble function defined as $H = \dot{a}/a$, is known as the Friedmann's equation for $k = 0$ – FRW Universe. This will be of principal use in the present study as it correlates two of the most essential cosmological parameters responsible for the dynamics of the Universe: the total energy density ρ of the perfect fluid and the Hubble expansion parameter H.

Our cosmological model is containing a combined matter source made of non-miscible radiation and dust matter with the total energy density given by

$$\rho = \rho_{01}\left(\frac{a_0}{a}\right)^3 + \rho_{02}\left(\frac{a_0}{a}\right)^4 = \frac{3}{\kappa}\left(\frac{\beta}{a^3} + \frac{\gamma}{a^4}\right), \tag{4}$$

where we have performed the following notations:

$$\beta = \frac{\kappa}{3}\rho_{01}a_0^3, \quad \gamma = \frac{\kappa}{3}\rho_{02}a_0^4$$

and the zero index corresponds to the present day values. Thus, the general Friedmann equation (3) becomes equivalent to the differential equation

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \sqrt{\frac{\beta}{a} + \frac{\gamma}{a^2}}\,, \tag{5}$$

whose integration, with the origin condition $a_i(t = t_i = 0) = 0$, leads to the following cubic equation for the scale function

$$a^3 - 3\frac{\gamma}{\beta}a^2 = \frac{9}{4}\beta t^2 - 6\frac{\gamma^{3/2}}{\beta}t\,. \tag{6}$$

To be mentioned that our result can be put into correspondence with the one obtained in [17] for the same choice of Universe composition.

Before proceeding to an analysis of the solutions of this equation, we will focus on deducing the expression of the other essential cosmological quantities.

From (5), one is able to compute the Hubble parameter as depending on the scale function as

$$H(a) = \sqrt{\frac{\beta}{a^3} + \frac{\gamma}{a^4}}\,, \tag{7}$$

and, inserting this result into the Einstein equation, one can deduce the evolution of the pressure with respect to the scale factor as described by the relation

$$p(a) = \frac{\gamma}{\kappa a^4}\,. \tag{8}$$

The last formula together with (4) allows the finding of a non-linear dependence of the cosmological energy density on the fluid pressure:

$$\rho(p) = 3\left[\beta\left(\kappa\gamma^3\right)^{-1/4} p^{3/4} + p\right]. \tag{9}$$

This result is the inverse function of the *Equation of State* (EoS) of our present model, $p(\rho)$, the latter possessing an intricate mathematical representation.

From the model' time dependent EoS, $p = w(t)\rho$, one may compute the effective EoS parameter

$$w(a) = \frac{\gamma}{3\left(\beta a + \gamma\right)},$$

which results to decrease with the ratio $\frac{\beta}{\gamma}$ of the dust species over the radiation component. Also, as the scale function approaches the primordial cosmological singularity, $a \to 0$, the EoS parameter goes to a constant, $w \to \frac{1}{3}$, meaning an early universe dominated by the radiative component. Obviously, the same w asymptotic tendency appears if $\frac{\beta}{\gamma} \to 0$ (a dominance of the radiative component over the dust species). In the last years, time-variation of the EoS parameter has been under ardent investigations and discussions, for instance, within the context of the models with viscous fluids [25] or quintessence models with scalar fields [26, 27]. Techniques based on data for recovering the physical quantity $w(t)$ have been discussed in [28].

Last but not least, we compute the parameter

$$q(a) = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{2\gamma + \beta a}{2\left(\beta a + \gamma\right)} > 0, \tag{10}$$

pointing out a decelerating Universe (*i.e.*, $q(t) > 0$).

Now, we will return to the equation (6) and by calling down the theory of third degree equation [29], we are going to discuss the nature of roots of the cubic equation in terms of the model' parameters. Hence, the discriminant for the equation (6) is

$$\Delta = -27Q^2 - 108\frac{\gamma^3}{\beta^3}Q. \tag{11}$$

where we have introduced the notation

$$Q \equiv \frac{9}{4}\beta t^2 - 6\frac{\gamma^{3/2}}{\beta}t = \frac{9}{4}\beta t\left(t - t_*\right), \text{ with } t_* = \frac{8}{3}\frac{\gamma^{3/2}}{\beta^2}. \tag{12}$$

As we are going to see in the followings, depending on the signs of both the discriminant (11) and the expression in (12), different situations distinguish.

i. $\Delta < 0$ and $Q > 0$.

In this case, the equation (6) possesses one real and two complex conjugate roots. Because the scale factor has to be real, the corresponding solution is given by

$$\begin{aligned} a(t) \equiv a(Q) = \frac{\gamma}{\beta} + \frac{2^{1/3}\gamma^2}{\beta}\left(\beta^3 Q + 2\gamma^3 - \beta^3 Q\sqrt{1+\frac{4\gamma^3}{\beta^3 Q}}\right)^{-1/3} + \\ + \frac{2^{-1/3}}{\beta}\left(\beta^3 Q + 2\gamma^3 - \beta^3 Q\sqrt{1+\frac{4\gamma^3}{\beta^3 Q}}\right)^{1/3} \end{aligned} \tag{13}$$

with Q representing the time dependent function defined in (12).

In the very late phases of the Universe evolution, compatible with small values of the ratio $\frac{4\gamma^3}{\beta^3 Q}$, *i.e.* high values of Q, one can perform a series expansion of the square root so that the scale factor admits the asymptotic representation

$$a_{t\to\infty} \approx \frac{\gamma}{\beta} + Q^{1/3} + \frac{\gamma}{\beta}Q^{-1/3} \approx Q^{1/3}$$

Recalling (12), for large values of the time variable, *i.e.* $t \gg t_*$, we have the behavior $Q \approx \frac{9}{4}\beta t^2$ from where we find that the radius of the universe increases algebraically as $a \sim t^{2/3}$. Thus, as expected, in late times, the radiative component does not manifest.

ii. $\Delta < 0$ and $Q < 0$ ($t < t_*$).

Even though, in this case, the solution is the same as in the previous one, it proves that this situation cannot take place as the two inequalities cannot be valid simultaneously. Thus, this case is not a physically acceptable one.

iii. $\Delta = 0$ and $Q < 0$ ($t < t_*$).

For this combination, it results that $Q = -\frac{4\gamma^3}{\beta^3}$ which transforms our cubic equation into the following configuration

$$a^3 - 3\frac{\gamma}{\beta}a^2 + 4\frac{\gamma^3}{\beta^3} = 0 .$$

The above expression admits a static solution $a_1 = a_2 = \frac{2\gamma}{\beta}$ and the root $a_3 = \frac{\gamma^2}{\beta^2}Q + 3\frac{\gamma}{\beta}$. If we ask for a_3 to be a positive physical quantity, we have to deal with a time limited universe characterized by the parametric interval

$$t \in \left(0,\ \frac{1}{4}t_*\right) \cup \left(\frac{3}{4}t_*,\ t_*\right) \quad i.e. \quad t \in \left(0,\ \frac{2\gamma\sqrt{\gamma}}{3\beta^2}\right) \cup \left(\frac{2\gamma\sqrt{\gamma}}{\beta^2},\ \frac{8\gamma\sqrt{\gamma}}{3\beta^2}\right).$$

3. THE EFFECT OF THE COSMOLOGICAL CONSTANT IN THE CLASSICAL AND QUANTUM EVOLUTION OF THE FRW UNIVERSE

In this section, we dedicate ourselves to the study of the effects that a cosmological constant can generate, if inserted in the model we already discussed. Thus, the Hubble parameter in terms of the scale function is given by

$$H(a) = \sqrt{\frac{\beta}{a^3} + \frac{\gamma}{a^4} + \lambda}\ , \tag{14}$$

with $\lambda = \frac{\Lambda}{3}$. In the corresponding Friedmann equation,

$$\dot{a}^2 = \frac{\beta}{a} + \frac{\gamma}{a^2} + a^2\lambda\ , \tag{15}$$

one may notice the contribution brought by the cosmological constant to the Hubble function, while the cosmological energy density remains the same.

After performing the variable separation, the differential equation (15) becomes

$$\frac{ada}{\sqrt{\lambda\left(\frac{\beta}{\lambda}a+\frac{\gamma}{\lambda}+a^4\right)}}=\mathrm{d}t.$$

and, by introducing the substitutions

$$p=\frac{\beta}{\lambda},\ q=\frac{\gamma}{\lambda} \tag{16}$$

it transforms into

$$\frac{ada}{\sqrt{\lambda}\sqrt{a^4+pa+q}}=\mathrm{d}t. \tag{17}$$

If it is to consider that $a_{i\left(i=\overline{1,4}\right)}$ are defining the four roots of the quartic polynomial a^4+pa+q, then the differential equation (17) can be rewritten as

$$\frac{ada}{\sqrt{\lambda}\sqrt{(a-a_1)(a-a_2)(a-a_3)(a-a_4)}}=\mathrm{d}t\ , \tag{18}$$

leading, by integration, to the following algebraic transcendental relation

$$\frac{2}{\sqrt{\lambda}\sqrt{(a_1-a_4)(a_2-a_3)}}\left\{a_1 EllipticF\left[Z,\ \varepsilon\right]+\left(a_2-a_1\right)EllipticPi\left[\varepsilon',\ Z,\ \varepsilon\right]\right\}=t\ , \tag{19}$$

with

$$Z=\mathrm{Arcsin}\left[\sqrt{\frac{(a-a_2)(a_1-a_4)}{(a-a_1)(a_2-a_4)}}\right],\varepsilon=\frac{(a_1-a_3)(a_2-a_4)}{(a_2-a_3)(a_1-a_4)},\varepsilon'=\frac{a_2-a_4}{a_1-a_4}. \tag{20}$$

Asymptotic representations for the elliptic functions allow for the transcendental equation (19) to reduce to a new, more simplified and algebraically convenient transcendental form:

$$\frac{2Z}{\sqrt{\lambda}\sqrt{(a_1-a_4)(a_2-a_3)}}\left\{a_2+\frac{a_2(\varepsilon+2\varepsilon')-2\varepsilon' a_1}{6}Z^2\right\}=t\ . \tag{21}$$

An appropriate manner of tackling these transcendental equations are given by the numerical procedures. For more details into the behaviour of these elliptic functions, their asymptotic approximations, series expansions, inequalities, one can consult the papers [30, 31] and references within.

We add the remark that the four roots $a_{i\left(i=\overline{1,4}\right)}$ are satisfying a set of mathematical relations in terms of the model parameters known as the Viete's relations.

In what follows, we will discuss the nature of the roots $a_{i\left(i=\overline{1,4}\right)}$ of the polynomial a^4+pa+q, in the framework of the quartic equation theoretical background. Thence, the discriminant

$$\Delta = 256q^3 - 27p^4, \tag{22}$$

in terms of our notations in (16), is equivalent with the algebraic expression

$$\Delta = \frac{1}{\lambda^3}\left(256\gamma^3 - 27\frac{\beta^4}{\lambda}\right). \tag{23}$$

To be mentioned that the sign of the discriminant (23) has a significant influence on the nature of the roots $a_{i\left(i=\overline{1,4}\right)}$. The analysis over this aspect can be refined through considering the sign of the polynomials below [32]

$$P=0,\quad Q=8p=8\frac{\beta}{\lambda},\quad D=64q=64\frac{\gamma}{\lambda},\quad \Delta_0=12q=12\frac{\gamma}{\lambda}. \tag{24}$$

Thus, depending on the signs combination for the expressions defined in (24) together with the sign of the discriminant (23), one distinguishes a number of situations that determine the nature of the roots $a_{i\left(i=\overline{1,4}\right)}$ and their algebraic formulas. We will proceed to expose the relevant situations for our polynomial under investigations.

i. $\Delta<0$

In this case, the quartic equation $a^4+pa+q=0$ possesses two real and two complex roots, namely

$$a_{1,2}=-S\pm\sqrt{-S^2+\frac{p}{4S}}\in\mathbb{R},\ a_{3,4}=S\pm i\sqrt{S^2+\frac{p}{4S}}\in\mathbb{C}, \tag{25}$$

where

$$S=\frac{1}{2\sqrt{3Q'}}\sqrt{Q'^2+12q}\text{ , with } Q'=\sqrt[3]{\frac{\Delta_1+\sqrt{\Delta_1^2-4\Delta_0^3}}{2}}\ \text{ and } \Delta_1=27p^2=27\frac{\beta^2}{\lambda^2},$$

$$\Delta_0=12q=12\frac{\gamma}{\lambda}.$$

Because the scale factor is a real quantity, particularly, we will be interested in the real roots.

By virtue of identity $\Delta_1^2 - 4\Delta_0^3 = -27\Delta = 27^2 \frac{\beta^4}{\lambda^4} - 6912\frac{\gamma^3}{\lambda^3}$, we have that

$$Q' = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}} = \sqrt[3]{\frac{\Delta_1 + \sqrt{-27\Delta}}{2}} > 0,$$

where the discriminant Δ in terms of the model parameters is given by (23).
In these considerations, the real solutions are described by the algebraic expression

$$a_{1,2} = -\frac{1}{2\sqrt{3Q'\lambda}}\sqrt{\lambda Q'^2 + 12\gamma} \pm \frac{1}{2}\sqrt{-\frac{1}{3Q'\lambda}\left(\lambda Q'^2 + 12\gamma\right) + \frac{2\beta\sqrt{3Q'}}{\sqrt{\lambda}\sqrt{\lambda Q'^2 + 12\gamma}}}. \quad (26)$$

In addition, by calling down the inequality $\Delta < 0$, we find that the parameters of our model are subjected to the constraint:

$$\gamma^3\lambda < 0{,}1\beta^4. \quad (27)$$

ii. $\Delta > 0$. As for our model $P = 0$, we find out that this situation is not reflected in theory.

iii. $\Delta = 0$.

In this case, by considering the condition $D > 0$, one can identify one real double root and two complex ones. In the same time, we have the relation $\Delta_1^2 - 4\Delta_0^3 = 0$ which leads to the following relation between the model's parameters

$$\frac{\beta^4}{\gamma^3\lambda} = \frac{256}{27} \simeq 9.4, \quad (28)$$

The real solutions are described by the same algebraic construction in (26), with $Q' = \frac{3p^{2/3}}{2^{1/3}} = \frac{3}{2^{1/3}}\left(\frac{\beta}{\lambda}\right)^{2/3}$ and therefore one has the following set of real solutions:

$$a_1 = a_2 = -\left(\frac{q}{3}\right)^{1/4} = -\left(\frac{\gamma}{3\lambda}\right)^{1/4}. \quad (29)$$

Before we finish this section, let us derive, from the first Einstein equation (2), the evolution of pressure with respect to the scale function:

$$p(a) = \frac{\gamma}{\kappa a^4} - \frac{3\lambda}{\kappa}. \tag{30}$$

In the same manner, one can deduce the dependence of the energy density on the scale factor,

$$\rho(a) = \frac{3}{\kappa}\left(\frac{\beta}{a^3} + \frac{\gamma}{a^4}\right), \tag{31}$$

so that we are able to determine the following effective pressure dependence on the energy density, where a scale factor 'interference' is to be noticed,

$$p(\rho,\, a) = \rho - \frac{1}{\kappa}\left(\frac{3\beta}{a^3} + \frac{2\gamma}{a^4} + 3\lambda\right). \tag{32}$$

For the quantum analysis, to which the final part of our work is dedicated, let us write the Friedmann equation (15), as the Hamiltonian-constraint

$$H = \dot{a}^2 + V(a) = 0\,\cdot$$

This contains the non-positive potential

$$V(a) = -\left[\frac{\beta}{a} + \frac{\gamma}{a^2} + \lambda a^2\right], \tag{33}$$

which tends to $-\infty$ for both $a \to 0$ and $a \to \infty$.

In view of the theory developed in [23], the *Wheeler–DeWitt* (WDW) equation [33, 34]

$$H\Psi = 0 \tag{34}$$

with the effective Hamiltonian

$$H = \frac{a_0}{4} p^2 + \frac{1}{a_0} V(a) \tag{35}$$

and the momentum operator

$$p \to \hat{p} = -i\frac{\partial}{\partial a},$$

gets the explicit form

$$\frac{d^2\Psi}{d\zeta^2}+\left[\frac{\bar{\beta}}{\zeta}+\frac{\bar{\gamma}}{\zeta^2}+\bar{\lambda}\zeta^2\right]\Psi=0 . \tag{36}$$

Here, we have introduced the characteristic length a_0, so that the variable and the parameters get the dimensionless expressions $\zeta = a/a_0$ and

$$\bar{\beta} \equiv \frac{4\beta}{a_0},\ \bar{\gamma} \equiv \frac{4\gamma}{a_0^2},\ \bar{\lambda} = 4\lambda a_0^2 .$$

Up to a normalization constant, the solution to the differential equation (36) is written in terms of the Heun biconfluent function of variable [35]

$$x=\frac{\sqrt{2}}{2}(i-1)\bar{\lambda}^{1/4}\zeta=(i-1)\sqrt{\frac{\lambda^{1/2}}{a_0}}a , \tag{37}$$

as

$$\Psi=\sqrt{\zeta}\zeta^{\pm i\sqrt{\bar{\gamma}}}\exp\left[\frac{i}{2}\sqrt{\bar{\lambda}}\zeta^2\right]HeunB\left[\pm 2i\sqrt{\bar{\gamma}},\ 0,\ 0,\ \frac{\sqrt{2}(1+i)\bar{\beta}}{\bar{\lambda}^{1/4}};\ x\right]. \tag{38}$$

The absolute value of the amplitude (38) represented in Fig. 1, as a function of x. This is starting from zero and is increasing to the maximum value for the Universe with $a=\sqrt{a_0/\sqrt{\lambda}}$. Then, it has an oscillating behavior, with periodic decreasing maxima corresponding to high probabilities of those Universes.

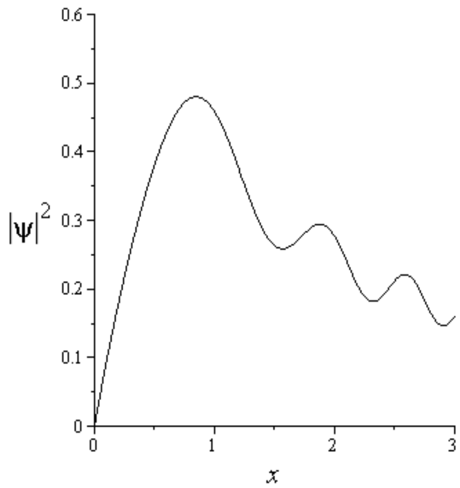

Fig. 1 – The absolute value of the function (38) with respect to the variable x defined in (37).

4. CONCLUSIONS

For the spatially-flat FRW Universe filled with a mixture of cosmic dust and radiation, the Friedmann equation is analytically integrable, leading to the cubic equation (6) for the scale function. As one is expecting, the radiative epoch characterizes the early times, while, at late times, it was identified the dust-like behavior of the universe.

By adding a positive cosmological constant, its effects on the model dynamics have been pointed out. In this respect, it was found a non-trivial dependence of the pressure on the energy density and scale factor. The non-linear, polytropic equation (32) is the algebraic sum of a standard linear EoS ($p \cong \rho$) and a non-linear term depending on the model's parameters. With respect to the linear dependence, we outline that within this level of approximation (the ultra-relativistic limit), it might be viewed as a stiff matter EoS. This has been brought into evidence by the Zel'dovich pioneering model of the Universe composed, in early stages, of a cold gas of baryons interacting through a meson field [36]. A similar non-linear EoS has been obtained in [8], for a mixed cosmology with stiff fluid, cosmic dust and a cosmological constant, in the context of fluid dynamics with viscous effects.

Finally, within a quantum analysis, the wave function of the Universe is expressed in terms of the intricate Heun functions in their biconfluent form. The absolute value of the wave function is characterized by a zero initial value for $a = 0$, indicating a smooth departure from the vacuum state; from this singularity the wave function is starting to increase until it achieves the most prominent maximum, at $a = \sqrt{a_0 / \sqrt{\lambda}}$. As the scale function evolves towards higher values, the most probable Universes possess successive decreasing maxima, alternating with non-vanishing minima.

As an ultimate remark, it is worth noticing that the present work can be expanded in several directions, as for example by embedding the $k = 0-$ FRW branes in a higher dimensional space [37].

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