Journal of Physics: Conference Series **287** (2011) 012018 doi:10.1088/1742-6596/287/1/012018

# The partition function of a ferromagnet up to three loops

**C P Hofmann**

Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo 340, Colima, Colima 28045, México.

E-mail: `christoph@ucol.mx`

**Abstract.** The low-temperature behavior of ferromagnets with a spontaneously broken symmetry O(3) → O(2) is analyzed within the perspective of effective Lagrangians. The leading coefficients of the low-temperature expansion for the partition function are calculated up to three loops and the manifestation of the spin-wave interaction in this series is discussed. The effective field theory method has the virtue of being completely systematic and model-independent.

## 1. Motivation

The question of how the low-temperature expansion of the partition function of a ferromagnet is affected by the spin-wave interaction, has a long history. After a series of erroneous attempts, Dyson, in his pioneering paper on the thermodynamic behavior of an ideal ferromagnet [1], showed that the spin-wave interaction in the free energy density starts manifesting itself only at order $T^5$. Here we go beyond Dyson's analysis and explicitly calculate the effect of the spin-wave interaction beyond $T^5$ in the free energy density of a ferromagnet. To the best of our knowledge, this is the first time that the general structure of this power series and the explicit calculation of the corresponding coefficients is given. Here, we just present some preliminary results - a detailed exposition of the evaluation and, in particular, an extensive discussion of the low-temperature series for the spontaneous magnetization of an ideal ferromagnet, will be given elsewhere [2].

In the present work, we will make use of an approach which has the virtue of being completely systematic and model-independent: the method of effective Lagrangians. Within the effective Lagrangian framework, the structure of the low-temperature expansion of the partition function for an O(3) ferromagnet was analyzed in Ref. [3] up to order $T^5$ and Dyson's series was reproduced in a straightforward manner. In the effective language, this corresponds to including Feynman diagrams for the partition function up to two loops. Here we consider Feynman diagrams up to three-loop order in the perturbative expansion of the partition function. As it turns out, in the free energy density of a ferromagnet, the next-to-leading interaction term already sets in at order $T^{\frac{11}{2}}$ – remarkably, this term is completely determined by the two low-energy coupling constants of the leading-order effective Lagrangian $\mathcal{L}^2_{eff}$ and does not involve any higher-order effective constants from $\mathcal{L}^4_{eff}$ related to the anisotropies of the cubic lattice.

## 2. Effective field theory evaluation

The effective Lagrangian method relies on an analysis of the symmetry properties of the underlying theory, i.e., the Heisenberg model in our case. In particular, it can universally be applied to systems with a spontaneously broken symmetry and is formulated in terms of Goldstone boson fields which are the relevant degrees of freedom at low energies [4]. Microscopic details of the system, such as the structure of the lattice, are taken into account through a few low-energy coupling constants in the effective Lagrangian. Symmetry does not fix the actual numerical values of these couplings – these have to be determined experimentally or in a numerical simulation of the underlying model. Symmetry, however, does unambiguously determine the derivative structure of the terms in the effective Lagrangian.

Whereas the Heisenberg model is invariant under global O(3) spin rotations, the ground state of the ferromagnet is invariant under the subgroup O(2) only. We then have one type of spin-wave excitation – or one magnon particle – in the low-energy spectrum of the ferromagnet which obeys a quadratic dispersion relation.

The leading-order effective Lagrangian (see Ref. [5]) is of order $p^2$ and takes the form

$$\mathcal{L}^2_{eff} = \Sigma \frac{\epsilon_{ab} \partial_0 U^a U^b}{1+U^3} + \Sigma \mu H U^3 - \frac{1}{2} F^2 \partial_r U^i \partial_r U^i \, . \tag{1}$$

The two real components of the magnon field, $U^a (a = 1, 2)$ are the first two components of the three-dimensional unit vector $U^i = (U^a, U^3)$, which transforms with the vector representation of the rotation group. The quantity $H$ is the third component of the external magnetic field $\vec{H} = (0, 0, H), H > 0$. While the structure of the above terms is unambiguously determined by the symmetries of the underlying theory, at this order, we have two a priori unknown low-energy constants: the spontaneous magnetization $\Sigma$ and the constant $F$. The above Lagrangian implies a quadratic dispersion relation

$$\omega(\vec{k}) = \gamma \vec{k}^2 + \mathcal{O}(|\vec{k}|^4) \, , \quad \gamma \equiv \frac{F^2}{\Sigma} \, , \tag{2}$$

characteristic of ferromagnetic magnons. It is important to note that – in the power counting scheme – one temporal derivative is on the same footing as two spatial derivatives – in the derivative expansion, two powers of momentum thus count as only one power of energy or temperature: $k^2 \propto \omega, T$.

The next-to-leading order Lagrangian is of order $p^4$ and takes the form [3]

$$\mathcal{L}^4_{eff} = l_1 (\partial_r U^i \partial_r U^i)^2 + l_2 (\partial_r U^i \partial_s U^i)^2 + l_3 \Delta U^i \Delta U^i \, . \tag{3}$$

It involves the three effective coupling constants $l_1, l_2$ and $l_3$. Higher order pieces which are also relevant for our calculation are

$$\mathcal{L}^6_{eff} = c_1 U^i \Delta^3 U^i \, , \qquad \mathcal{L}^8_{eff} = d_1 U^i \Delta^4 U^i \, . \tag{4}$$

We conclude this section with a remark concerning effects induced by the anisotropy of the lattice. For a cubic lattice, as shown in [6], the anisotropies start manifesting themselves only at the four-derivative level: indeed, the pieces $\mathcal{L}^4_{eff}$, $\mathcal{L}^6_{eff}$ and $\mathcal{L}^8_{eff}$ contain additional terms – not displayed in Eqs.(3) and (4) – which are not invariant under space rotations, such as

$$\sum_{s=1,2,3} \partial_s \partial_s U^i \, \partial_s \partial_s U^i \, . \tag{5}$$

In the present analysis, however, we do not take care of these extra terms and assume space rotation symmetry up to order $p^8$. The conclusions of the present paper regarding the manifestation of the spin-wave interaction in the partition function are not affected by this idealization.

In finite-temperature field theory the partition function is represented as a Euclidean functional integral

$$\mathrm{Tr}\,[\exp(-\mathcal{H}/T)] \;=\; \int [dU]\;\exp\Big(-\int_{\mathcal{T}} d^4x\,\mathcal{L}_{eff}\Big)\,. \tag{6}$$

The integration is performed over all field configurations which are periodic in the Euclidean time direction $U(\vec{x}, x_4+\beta) = U(\vec{x}, x_4)$, with $\beta \equiv 1/T$. The periodicity condition imposed on the magnon fields also reflects itself in the thermal propagator

$$G(x) \;=\; \sum_{n=-\infty}^{\infty} \Delta(\vec{x}, x_4 + n\beta)\,, \tag{7}$$

where $\Delta(x)$ is the Euclidean propagator at zero temperature,

$$\Delta(x) \;=\; \int \frac{dk_4 d^3k}{(2\pi)^4}\,\frac{e^{i\vec{k}\vec{x}-ik_4x_4}}{\gamma\vec{k}^2 - ik_4 + \mu H}\,. \tag{8}$$

At finite temperature the derivative expansion in powers of the momenta amounts to an expansion in powers of the temperature.

## 3. Results

While a detailed account of the perturbative evaluation can be found in Ref. [2], here we just present some of the main results. The free energy density of the O(3) ferromagnet up to order $p^{11}$ is given by

$$\begin{aligned} z = \; & - \; \Sigma\mu H \; - \; \frac{1}{8\pi^{\frac{3}{2}}\gamma^{\frac{3}{2}}}\,T^{\frac{5}{2}}\sum_{n=1}^{\infty}\frac{e^{-\mu Hn\beta}}{n^{\frac{5}{2}}} \; - \; \frac{15\,l_3}{16\pi^{\frac{3}{2}}\Sigma\gamma^{\frac{7}{2}}}\,T^{\frac{7}{2}}\sum_{n=1}^{\infty}\frac{e^{-\mu Hn\beta}}{n^{\frac{7}{2}}} \\ & - \; \frac{105}{32\pi^{\frac{3}{2}}\Sigma\gamma^{\frac{9}{2}}}\left(\frac{9l_3^2}{2\gamma\Sigma} - c_1\right)T^{\frac{9}{2}}\sum_{n=1}^{\infty}\frac{e^{-\mu Hn\beta}}{n^{\frac{9}{2}}} \\ & - \; \frac{3(8l_1+6l_2+5l_3)}{128\pi^3\Sigma^2\gamma^5}\,T^5\left\{\sum_{n=1}^{\infty}\frac{e^{-\mu Hn\beta}}{n^{\frac{5}{2}}}\right\}^2 \\ & - \; \frac{945\,d_1}{64\pi^{\frac{3}{2}}\Sigma\gamma^{\frac{11}{2}}}\,T^{\frac{11}{2}}\sum_{n=1}^{\infty}\frac{e^{-\mu Hn\beta}}{n^{\frac{11}{2}}} \; + \; \frac{10395\,l_3c_1}{64\pi^{\frac{3}{2}}\Sigma^2\gamma^{\frac{13}{2}}}\,T^{\frac{11}{2}}\sum_{n=1}^{\infty}\frac{e^{-\mu Hn\beta}}{n^{\frac{11}{2}}} \\ & - \; \frac{\Sigma^{\frac{5}{2}}}{2F^9}\,\bar{j}\,T^{\frac{11}{2}} + \mathcal{O}(T^6)\,. \end{aligned} \tag{9}$$

The first term is temperature-independent. Contributions which involve half integer powers of the temperature – $T^{\frac{5}{2}}, T^{\frac{7}{2}}, T^{\frac{9}{2}}$ and the first two terms of order $T^{\frac{11}{2}}$ – arise from one-loop graphs and are thus all related to the free energy density of noninteracting magnons. Remarkably, up to order $p^{10}$ there is only one term in the above series – the order $T^5$ contribution, originating from a two-loop graph – which is due to the spin-wave interaction and represents the Dyson term. Note that there is no term of order $T^4$ in the above series of the free energy density.

The last term of order $T^{\frac{11}{2}}$ is related to a three-loop graph: The dimensionless function $\bar{j}$,

$$\bar{j} = \bar{j}(\sigma)\,, \quad \sigma = \frac{\mu H}{T}\,, \tag{10}$$

in the limit $\sigma \to 0$, may be parametrized by

$$\bar{j}(\sigma) = j_1 + \mathcal{O}(\sigma)\,, \tag{11}$$

where the coefficient $j_1$ is pure number given by

$$j_1 = 0.0000091\,. \tag{12}$$

With the above results, the low-temperature series for the pressure takes the form

$$P \;=\; h_0 T^{\frac{5}{2}} \;+\; h_1 T^{\frac{7}{2}} \;+\; h_2 T^{\frac{9}{2}} \;+\; h_3 T^5 \;+\; h_4 T^{\frac{11}{2}} + \mathcal{O}(T^6)\,, \tag{13}$$

with coefficients $h_i$ given by

$$\begin{aligned}
h_0 &= \frac{1}{8\pi^{\frac{3}{2}}\gamma^{\frac{3}{2}}} \sum_{n=1}^{\infty} \frac{e^{-\mu H n\beta}}{n^{\frac{5}{2}}}\,, \\
h_1 &= \frac{15\, l_3}{16\pi^{\frac{3}{2}}\Sigma\gamma^{\frac{7}{2}}} \sum_{n=1}^{\infty} \frac{e^{-\mu H n\beta}}{n^{\frac{7}{2}}}\,, \\
h_2 &= \frac{105}{32\pi^{\frac{3}{2}}\Sigma\gamma^{\frac{9}{2}}} \left( \frac{9 l_3^2}{2\gamma\Sigma} - c_1 \right) \sum_{n=1}^{\infty} \frac{e^{-\mu H n\beta}}{n^{\frac{9}{2}}}\,, \\
h_3 &= \frac{3(8l_1 + 6l_2 + 5l_3)}{128\pi^3\Sigma^2\gamma^5} \left( \sum_{n=1}^{\infty} \frac{e^{-\mu H n\beta}}{n^{\frac{5}{2}}} \right)^2\,, \\
h_4 &= \frac{945}{64\pi^{\frac{3}{2}}\Sigma\gamma^{\frac{11}{2}}} \left( d_1 - 11 l_3 c_1 F^{-2} \right) \sum_{n=1}^{\infty} \frac{e^{-\mu H n\beta}}{n^{\frac{11}{2}}} + \frac{\Sigma^{\frac{5}{2}}}{2F^9}\,\bar{j}\,.
\end{aligned} \tag{14}$$

In the limit $\sigma = \frac{\mu H}{T} \to 0$, these coefficients become temperature independent and the sums reduce to Riemann zeta functions,

$$\begin{aligned}
\tilde{h}_0 &= \frac{1}{8\pi^{\frac{3}{2}}\gamma^{\frac{3}{2}}}\, \zeta(\tfrac{5}{2})\,, \\
\tilde{h}_1 &= \frac{15\, l_3}{16\pi^{\frac{3}{2}}\Sigma\gamma^{\frac{7}{2}}}\, \zeta(\tfrac{7}{2})\,, \\
\tilde{h}_2 &= \frac{105}{32\pi^{\frac{3}{2}}\Sigma\gamma^{\frac{9}{2}}} \left( \frac{9 l_3^2}{2\gamma\Sigma} - c_1 \right) \zeta(\tfrac{9}{2})\,, \\
\tilde{h}_3 &= \frac{3(8l_1 + 6l_2 + 5l_3)}{128\pi^3\Sigma^2\gamma^5}\, \zeta^2(\tfrac{5}{2})\,, \\
\tilde{h}_4 &= \frac{945}{64\pi^{\frac{3}{2}}\Sigma\gamma^{\frac{11}{2}}} \left( d_1 - 11 l_3 c_1 F^{-2} \right) \zeta(\tfrac{11}{2}) + \frac{\Sigma^{\frac{5}{2}}}{2F^9}\, j_1\,.
\end{aligned} \tag{15}$$

Of particular interest are the last two terms involving the coefficients $h_3$ and $h_4$, as they contain the spin-wave interaction part. The contribution displaying five powers of the temperature is the famous Dyson term, while the interaction contribution contained in $T^{\frac{11}{2}}$ – the last term in the coefficient $h_4$ – is our main new result. Note that all other contributions to the pressure originate from one-loop graphs – hence, those graphs describe noninteracting magnons and merely modify the dispersion relation.

Up to order $T^5$, we thus reproduce Dyson's series. In the effective Lagrangian framework, the famous interaction term of order $T^5$ in the free energy density originates from a two-loop graph with an insertion from the next-to-leading order Lagrangian $\mathcal{L}^4_{eff}$. Remarkably, the next-to-leading term resulting from the spin-wave interaction already sets in at order $T^{\frac{11}{2}}$. Note again that the coefficient of this three-loop interaction term, contained in $h_4$, does not involve any higher-order low-energy coupling constants. It only involves $\Sigma$ and $F$, as well as the quantity $\bar{j}$, which is a dimensionless function determined by the symmetries of the underlying Heisenberg model.

## 4. Conclusions

The effective Lagrangian method, which is based on symmetry considerations, has the virtue of being completely systematic and model-independent. It allowed us to go beyond the results of Dyson in a rigorous way and thereby solve the long-standing question of how the spin-wave interaction manifests itself in the low-temperature expansion of the partition function of an ideal ferromagnet. In particular, we have identified the correct temperature-power beyond the Dyson term due to the spin-wave interaction: it is of order $T^{\frac{11}{2}}$ in the free energy density and does not depend on the anisotropies of the cubic lattice.

A detailed exposition of the perturbative evaluation of the partition function within the effective theory framework will be presented elsewhere [2]. In the same article, more results will be provided. In particular, a detailed account of the low-temperature series for the spontaneous magnetization of an ideal ferromagnet and its comparison with the condensed matter literature will be given there.

## Acknowledgments

Support by CONACYT grant No. 50744-F is gratefully acknowledged.

## References

[1] Dyson F J 1956 *Phys. Rev.* **102** 1217; **102** 1230
[2] Hofmann C P 2011 to be published
[3] Hofmann C P 2002 *Phys. Rev.* B **65** 094430
[4] Leutwyler H 1994 *Ann. Phys.* **235** 165
[5] Leutwyler H 1994 *Phys. Rev.* D **49** 3033
[6] Hasenfratz P and Niedermayer F 1993 *Z. Phys. B: Condens. Matter* **92** 91