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Analogy between gravitational and gauge field Wilson line operators

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Abstract The gauge field Wilson line operator is an important concept in the gauge field theory and can be derived from the nonlocal interaction. A similar analysis can be carried out for the nonlocal interaction in the curved space-time to obtain the gravitational Wilson line operator explicitly. In this paper we first review the connection between the nonlocal interaction and the gauge field Wilson line operator, then discuss the gravitational Wilson line operator. Finally, we prove the gauge invariance of the energy–momentum tensor to explore the effect of the Wilson line operator.

1 Introduction

In the field theory, the gauge invariance is an important condition because it is directly associated with the conservation of currents and charges. Generally, under local gauge transformation, the theory recovers gauge invariance by introducing the corresponding gauge field. However, in the bilocal operator or nonlocal interactions, the gauge invariance can not be fulfilled as in the conventional way. Since the nonlocal interactions give rise two phase factors at two different space-time points. In this circumstance, these phase factors can not counteract with that of the gauge field. This can be seen more concretely from the nonlocal interactions. In nonlocal field theory, the main assumption is that fields interact with each other at different space-time points. Thus under the gauge transformation, the nonlocal interaction is not gauge invariant. Therefore it is necessary to introduce a new operator, which can neutralize these phase factors [1–4]. Such an operator is usually called Wilson line operator. The main property of the Wilson line operator is that under the gauge transformation it generates two phase factors at different space-time points too. According to the gauge transformation behavior of the Wilson line operator, it can be parameterized in terms of the path-ordered exponential of the gauge field by solving the parallel transport equation of the gauge field [5–7].

As for the theory in the curved space-time, the corresponding gauge field is assumed to be gravity. By analogy to gauge theory, in the curved space-time the gauge invariance can be preserved if gravity is included. Similarly, the gauge invariance of the nonlocal interaction in the curved space-time can be ensured by including the Gravitational Wilson line operator. As mentioned above, in a straightforward manner, one is able to obtain the gravitational Wilson line operator by solving the parallel transport equation for the Christoffel symbol, which is the gauge connection in the curved-pace time [8–12]. However, such parametrization of the gravitational Wilson line operator is not consistent with the single copy of gauge theory [8]. In Refs. [10,13,14], it has been proposed that the gravitational Wilson line operator can be defined in terms of the phase factor of the heavy relativising particle in the curved space-time. Although the theory with such gravitational Wilson line operator satisfies gauge invariance, the intrinsic connection between nonlocality and gravitational Wilson line operators is not yet clear. Therefore, it is necessary to systematically investigate the connection between nonlocality and Wilson line operator, especially, unify the method in which the gauge and gravitational Wilson line operators can be derived in an analogous way.

This paper organized as follows: in Sects. 2 and 3, we review the main framework of nonlocal regularization and the derivation of the corresponding gauge field Wilson line operator from the nonlocal field operator. In Sect. 4, we will derive the gravitational Wilson line operator from the nonlocal interaction in the curved space-time. In Sect. 5, as an application, we explicitly verify the conservation of nonlocal energy–momentum tensor and illustrate the effects of the gravitational Wilson line operator. In the final section, we will draw a short conclusion.

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2 Nonlocal regularization

One of the main challenges in quantum field theory (QFT) is that ultraviolet (UV) divergences arise when one considers the loop contributions to physical observables. This can be illustrated more clearly through the loop calculations of specific interactions. For example, let us consider the interaction between the complex scalar field $\phi(x)$ and the electromagnetic field $A_\mu(x)$, which is described by the Lagrangian density,

$$\mathcal{L}^{\rm local} = [D_\mu\phi(x)]^* D^\mu\phi(x) - m_\phi^2\phi^*(x)\phi(x) - \frac{\lambda}{4}[\phi^*(x)\phi(x)]^2, \quad (1)$$

where the electromagnetic covariant derivative is defined as $D_\mu = \partial_\mu - igA_\mu(x)$, λ is the coupling constant. It is obvious that the local Lagrangian $\mathcal{L}^{\rm local}$ is gauge invariant under the $U(1)$ gauge transformation $\phi'(x) = U(x)\phi(x)$. However, the point is that when we calculate the loop corrections to the coupling constant λ, the loop contribution is UV divergent. This problem stems from the ill-defined nature of the product of two local field operators at the same space-time point. More specifically, local fields interact with one another at the same space-time point [15]. Traditionally, to address this issue, various regularization methods have been developed, such as dimensional and Pauli–Villars regularization. On the other hand, from the perspective of single-point interactions, one can eliminate the ultraviolet divergence by positioning the fields at different space-time points [16–21]. This procedure is called delocalization of field operators. After the delocalization, one can obtain a new field operator, which is called nonlocal field operator. The terminology for 'nonlocality' refers to scenario in which the original local field operator $\phi(x)$ has been shifted to a new space-time point $\phi(x+a)$. Furthermore, the nonlocal field operator $\phi(x+a)$ hides the information about the infinite-order derivatives of the local field operator since nonlocal field operator $\phi(x+a)$ can also be written as $\phi(x+a) = e^{a.\partial}\phi(x)$. From this aspect, nonlocal regularization is also referred to as infinite-order derivatives regularization [22–30].

Compared to other conventional regularization schemes, nonlocal regularization can be systematically encoded into a modified interaction Lagrangian. Generally, this modified Lagrangian can be obtained by replacing the local field operator $\phi(x)$ in the local interaction Lagrangian with the nonlocal operator $\phi(x+a)$. However, it is evident that if a theory includes the nonlocal scalar field operator $\phi(x+a)$, then it will lose its original gauge invariance. To preserve gauge invariance in this case, one must invoke the gauge field Wilson line operator. A straightforward method to recover gauge invariance is to replace the nonlocal field operator $\phi(x+a)$ with the modified operator $\Phi(x, a, A) = W(x, a, A)\phi(x+a)$, where $W(x, a, A)$ symbolically represents the gauge field Wilson line operator and A is the gauge field. According to the gauge transformation of the local field operator, it can be shown that the Wilson line operator transforms as $W'(x, a, A) = U(x)W(x, a, A)U^\dagger(x+a)$. Thus, the nonlocal operator $\Phi(x, a, A)$ generates a local phase factor similar to its local counterpart, i.e., $\Phi(x, a, A)' = U(x)\Phi(x, a, A)$. Consequently, the theory that incorporates both the nonlocal field operator and the Wilson line operator remains gauge invariant. Summarizing this, the nonlocal Lagrangian is constructed as,

$$\begin{aligned}\mathcal{L}^{\rm nonlocal} = {} & [D_\mu\phi(x)]^* D^\mu\phi(x) - m_\phi^2\phi^*(x)\phi(x) \\ & -\frac{\lambda}{4}\int d^4a F(a)\int d^4b F(b)[W^*(x, b, A) \\ & \times\phi^*(x+b)W(x, a, A)\phi(x+a)]^2, \end{aligned} \quad (2)$$

where $F(a)$ is the nonlocal regulator function, which includes an additional cutoff parameter and satisfies certain conditions. More details can be found in the Refs. [19–21]. Furthermore, in the local limit where $F(a) = \delta^4(a)$, the nonlocal Lagrangian $\mathcal{L}^{\rm nonlocal}$ reduces to the local one. It is clear that the modified Lagrangian $\mathcal{L}^{\rm nonlocal}$ still preserves gauge invariance under $U(1)$ transformations. More importantly, the loop contributions are ultraviolet (UV) finite. This conclusion can be intuitively understood from loop calculations. Specifically, in perturbative loop calculations, the Wilson line operator $W(x, a, A)$ generates additional Feynman diagrams, ensuring that the total loop contribution remains gauge invariant and UV finite [4,31,32]. However, the regularization of loop contributions depends on the Wilson line operator and the shape of the nonlocal regulator. Therefore, it is essential to define the explicit expression of the Wilson line operator.

3 Gauge field Wilson line operator

Conventionally, the Wilson line operator is usually given by the solution of the parallel transport equation of the corresponding gauge field $A_\mu(x)$. To unify the method that will be used in the next section, in this work, we will present another method to parameterize the gauge field Wilson line operator. In such a method, the Wilson line operator for gauge field or gravity can be straightforwardly obtained from the nonlocality of the field.

Now we proceed to discuss the connection between nonlocality and gauge field Wilson line operators. Let us start with a nonlocal scalar field operator $\phi(x+a)$. Indeed, it can be rewritten as $\phi(x+a) = e^{a.\partial}\phi(x)$. However, its right side does not manifestly transform under the local gauge transformation. To cure this issue, we replace the partial derivative with the covariant derivative of the local scalar field. Then the modified field operator $\tilde{\Phi}(x, a, A) = e^{a.(\partial - igA)}\phi(x)$, which exactly transforms as $\tilde{\Phi}'(x, a, A) = U(x)\Phi(x + a, A)$. On the other hand, the nonlocal field operator $\tilde{\Phi}(x, a, A)$ also can be transformed into the form $\tilde{\Phi}(x, a, A) = e^{a.(\partial - igA)}e^{-a.\partial}\phi(x + a)$. Comparing these two nonlocal operators $\Phi(x, a, A)$ and $\tilde{\Phi}(x, a, A)$, one can find that the gauge field Wilson line operator can be defined as,

$$W(x, a, A) = e^{a.(\partial - igA)}e^{-a.\partial}. \tag{3}$$

Applying the Baker–Campbell Hausdorff formula, Eq. (3) is simplified as,

$$\begin{aligned} W(x, a, A) &= \text{Exp}\left[-ig\sum_{n=1}\frac{(a.\partial)^{n-1}}{n!}a.A(x)\right] \\ &= \mathcal{P}\text{Exp}\left[-ig\int_x^{x+a} dz^\mu A_\mu(z)\right], \end{aligned} \tag{4}$$

where $\mathcal{P}$ is the path ordering operator. Obviously, Eq. (4) is identical to the conventional expression of gauge field Wilson line operator, which can be found in QFT textbooks and other references [6,7].

4 Gravitational Wilson line operator

An action for the matter field in the curved space-time is obtained by introducing the metric and vielbein fields which contract with the modified covariant derivative and vector field. However, the local scalar field is unchanged and the same as the flat space-time. As for the nonlocal interaction in the curved space-time, it can constructed by transforming the nonlocal field operator into curved space-time following the similar prescription stated above. However, it should be noted that a nonlocal scalar field operator does not obey such a simple rule. Since the nonlocal scalar field operator in the flat space-time $\phi(x + a)$ is rewritten in the form $\phi(x + a) = e^{a.\partial}\phi(x)$. This indicates that the nonlocal field operator hides higher order derivatives of local field operator with respect to the space-time. Therefore in the curved space-time, the nonlocal scalar field operator can be written as,

$$\phi(x + a) = e^{e^\mu_b a^b \partial_\mu}\phi(x), \tag{5}$$

where the e^μ_b represents vielbein field. It is necessary to mention that the infinitesimal translation vector a^b is a constant vector. Therefore the curved space-time version of operator $a.\partial$ can be obtained by the replacement $e^\mu_b a^b \partial_\mu$[33]. It will be more clear in the following section that in the weak gravitational field limit, the vielbein field can be expanded as $e^\mu_b = \delta^\mu_a - \frac{\kappa}{2}\eta_{a\lambda}h^{\lambda\mu} + \mathcal{O}(\kappa^2)$, where κ is defined as $\kappa^2 = 32\pi G$ [34–36]. Substituting the expansion into the Eq. (5), one obtains $\phi(x + a) = e^{a.\partial - \frac{\kappa}{2}h^{\mu\nu}a_\mu\partial_\nu}\phi(x)$. Analogy to Eq. (3), the exponential operator can be defined as the gravitational Wilson line operator,

$$W(x, a, h) = e^{a.\partial - \frac{\kappa}{2}h^{\mu\nu}a_\mu\partial_\nu}e^{-a.\partial}. \tag{6}$$

By analogy to Eq. (4), further simplification gives rise to an explicit parametrization for the gravitational Wilson line as,

$$W(x, a, h) = \mathcal{P}\text{Exp}[-\frac{\kappa}{4}\int_x^{x+a} h^{\mu\nu}(z)dz_{\{\mu}\partial_{\nu\}}], \tag{7}$$

where the symmetrization notation $a_{\{\mu}b_{\nu\}}$ is defined as $a_{\{\mu}b_{\nu\}} = a_\mu b_\nu + a_\nu b_\mu$. Comparing the Eq. (7) to the gauge Wilson line operator, the gravitational Wilson line operator contains partial derivatives of local scalar field. If the partial derivative is replaced with the four momentum of the local scalar field, the Eq. (7) then can be rewritten as $W(x, a, h) = \text{Exp}[i\frac{\kappa}{2}m\int_0^\tau dt v^\mu v^\nu h_{\mu\nu}(x + vt)]$, where k^ν denotes the four momentum of delocalized particle and is defined $k^\mu = mv^\mu$, v^ν is the four velocity and defined as $v^\mu = \frac{a^\mu}{\tau}$. Unsurprisingly, this expression is identical to the massive particle phase factor result given in [10,13,14,37].

5 Gauge invariance of pseudo-scalar interaction

5.1 Nonlocal action in the curved space-time

In this section, as a simple example, we will explore the gauge invariancy of nonlocal matter field action in the curved space-time with the gravitational Wilson line operator. Analogy to electrometric conserved current, the gravitational gauge invariancy results in the conservation of energy–momentum tensor even at the loop level. Therefore, we will investigate the gauge invariance of the pion loop corrections to the proton energy–momentum tensor. On the other hand, it is known that the Yukawa-type interaction between the nucleon and pion fields is governed by pseudoscalar Lagrangian density, which is given by $\mathcal{L}_{\text{int}} = -ig\bar{N}(x)\gamma^5\boldsymbol{\tau}.\boldsymbol{\pi}(x)N(x)$, where $N(x)$ and $\boldsymbol{\tau}.\boldsymbol{\pi}(x)$ represent the nucleon and pion fields, and τ_i are the Pauli matrices, g is the effective coupling constant [6,38,39]. Following the prescription developed in [40,41], it is easy to construct an effective action in the curved space-time for the nucleon and pion kinetic term as well as for the PS interaction as,

$$\begin{aligned} S^{\text{local}}_{\text{PS}} = &\int d^4x\sqrt{-g(x)}\Big[\frac{i}{2}\bar{N}(x)e^\mu_a(x)\gamma^a\nabla_\mu N(x) \\ &-\frac{i}{2}\nabla_\mu\bar{N}(x)e^\mu_a(x)\gamma^a N(x) - m\bar{N}(x)N(x) \\ &+\delta m\bar{N}(x)N(x)\Big] + \frac{1}{2}\int d^4x\sqrt{-g(x)} \end{aligned}$$

$$\times\left[\frac{g^{\mu\nu}(x)}{2}\partial_{\{\mu}\boldsymbol{\pi}(x).\partial_{\nu\}}\boldsymbol{\pi}(x) - m_\pi^2\boldsymbol{\pi}(x).\boldsymbol{\pi}(x) + \delta m_\pi^2\boldsymbol{\pi}(x).\boldsymbol{\pi}(x)\right] - ig\int d^4x\sqrt{-g(x)} \times\bar{N}(x)\gamma^5\boldsymbol{\tau}.\boldsymbol{\pi}(x)N(x), \quad (8)$$

where $g^{\mu\nu}(x)$ and $e_a^\mu(x)$ are the metric and vielbein field, which represent the gravitational background of the matter field, $\sqrt{-g(x)}$ is the determinant of the metric field $g^{\mu\nu}(x)$, δm and δm_π^2 are the mass counterterms of nucleons and pions [42–44]. It is worthwhile to mention that we have included in pion and nucleon counterterms since they have a nonzero contribution to the energy–momentum tensor and violate the gauge invariancy. This is contrary to the electrometric case, in which the counterterm fulfills the gauge invariance independently. Therefore electrometric gauge invariance can proved for the loop contribution and counterterm respectively. On the other hand, as mentioned above the interaction between the nonlocal particles can be described by the nonlocal Lagrangian. In this work, we treat the pion field as a nonlocal field. Correspondingly, the nonlocal version of the Eq. (8) can be constructed as,

$$\begin{aligned}S_{\rm PS}^{\rm nonlocal} = &\int d^4x\sqrt{-g(x)}\Big[\frac{i}{2}\bar{N}(x)e_a^\mu(x)\gamma^a\nabla_\mu N(x) \\ &-\frac{i}{2}\nabla_\mu\bar{N}(x)e_a^\mu(x)\gamma^a N(x) - m\bar{N}(x)N(x) \\ &+\delta m\bar{N}(x)N(x)\Big] + \frac{1}{2}\int d^4x\sqrt{-g(x)} \\ &\times\left[\frac{g^{\mu\nu}(x)}{2}\partial_{\{\mu}\boldsymbol{\pi}(x).\partial_{\nu\}}\boldsymbol{\pi}(x) - m_\pi^2\boldsymbol{\pi}(x).\boldsymbol{\pi}(x) \right. \\ &\left.+\delta m_\pi^2\boldsymbol{\pi}(x).\boldsymbol{\pi}(x)\right] - ig\int d^4x \\ &\times\int d^4aF(a)\sqrt{-g(x)}\bar{N}(x)\gamma^5 \\ &\times W(x,a,h)\boldsymbol{\tau}.\boldsymbol{\pi}(x+a)N(x), \end{aligned} \quad (9)$$

where $W(x, a, h)$ is the gravitational Wilson line operator. In fact, the metric and vielbein fields mix the flat space-time background with curved one. More concretely, we can not distinguish the flat space-time contribution from Eq. (9). Alternatively, it is worth to separate metric and vielbein into flat and curved space-time components. In the weak field limit, $g^{\mu\nu}$ and e_a^μ can be expanded around the flat space-time background as [34–36],

$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\nu} - \kappa h^{\mu\nu} + \mathcal{O}(\kappa^2), \\ \sqrt{-g} &= 1 + \frac{1}{2}\kappa h + \mathcal{O}(\kappa^2), \\ e_a^\mu &= \delta_a^\mu - \frac{\kappa}{2}\eta_{a\lambda}h^{\lambda\mu} + \mathcal{O}(\kappa^2), \end{aligned} \quad (10)$$

where h is defined as $h = \eta_{\mu\nu}h^{\mu\nu}$. Substituting Eq. (10) into Eq. (9), we can find that the first terms of Eq. (10) give rise to PS action in the Minkowskian space-time. As for the second terms, they generate terms proportional to h and $h^{\mu\nu}$, which represent the true curved space-time background. Furtherly, the energy–momentum tensor is defined as,

$$T_{\mu\nu} = -\frac{2}{\kappa}\frac{\delta S}{\delta h^{\mu\nu}}. \quad (11)$$

Adopting this definition, from action in Eq. (9), we obtain the energy–momentum tensor for nucleon -pion PS interaction as,

$$\begin{aligned}T_{\mu\nu}(x) = &\frac{i}{4}\big[\bar{N}(x)\gamma_{\{\mu}\partial_{\nu\}}N(x) - \partial_{\{\mu}\bar{N}(x)\gamma_{\nu\}}N(x)\big] \\ &-\eta_{\mu\nu}\big[\frac{i}{2}\bar{N}(x)\gamma^\alpha\partial_\alpha N(x) - \frac{i}{2}\partial_\alpha\bar{N}(x)\gamma^\alpha N(x) \\ &-m\bar{N}(x)N(x) + \delta m\bar{N}(x)N(x)\big] \\ &+\frac{1}{2}\partial_{\{\mu}\boldsymbol{\pi}(x).\partial_{\nu\}}\boldsymbol{\pi}(x) - \frac{\eta_{\mu\nu}}{2}\big[\partial^\alpha\boldsymbol{\pi}(x).\partial_\alpha\boldsymbol{\pi}(x) \\ &-m_\pi^2\boldsymbol{\pi}(x).\boldsymbol{\pi}(x) + \delta m_\pi^2\boldsymbol{\pi}(x).\boldsymbol{\pi}(x)\big] - ig \\ &\times\int d^4aF(a)\big[-\eta_{\mu\nu}\bar{N}(x)\gamma_5\boldsymbol{\tau}.\boldsymbol{\pi}(x+a)N(x) \\ &+\frac{1}{2}\int_0^1 dt\bar{N}(x)\gamma_5\boldsymbol{\tau}.\boldsymbol{\pi}(x+a-at)N(x)a_{\{\mu}\partial_{\nu\}}\big], \end{aligned} \quad (12)$$

where the last term represents the additional gauge link energy–momentum tensor current. Actually, this term tends to zero in the local limit $F(a) \to \delta^4(a)$. Using the energy–momentum tensor currents and nonlocal action, one can calculate the pion one loop corrections to the proton self energy and the energy–momentum tensor. On the other hand, the loop corrections to the energy–momentum can be parameterized in terms of gravitational form factors as [45–47],

$$\begin{aligned}\langle p'|T_{\mu\nu}|p\rangle = \bar{u}(p', m)\Bigg[&A(t)\frac{\gamma_{\{\mu}P_{\nu\}}}{2} + iB(t)\frac{P_{\{\mu}\sigma_{\nu\}\alpha}q^\alpha}{4m} \\ &+ D(t)\frac{q_\mu q_\nu - \eta_{\mu\nu}q^2}{4m} + m\bar{c}(t)\eta_{\mu\nu}\Bigg]u(p, m), \end{aligned} \quad (13)$$

where $u(p, m_N)$ is the proton spinor, m_N is the physical mass of the proton, kinematic variables P and q are defined as $P = (p+p')/2, q = p'-p$, and four momentum transfer $t = q^2 = -Q^2$, $\eta_{\mu\nu}$ is the flat space-time metric, four Lorentz invariant scalar functions $A(t)$, $B(t)$, $D(t)$ and $\bar{c}(t)$ are called gravitational form factors of the proton. From Eq. (13) it can be seen that the first three terms satisfy gauge invariance independently for each loop diagram except the last $\bar{c}(t)$ term. If the total energy–momentum tensor is conserved $q_\mu T^{\mu\nu} = 0$, then the total $\bar{c}(t)$ from the all loop contributions vanishes

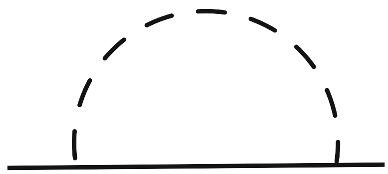

Fig. 1 Pion one loop corrections to the proton self-energy, where the solid and dashed lines represent pion and nucleon fields, respectively

i.e $\sum_i \bar{c}_i(t) = 0$. To prove this, in the subsequent analysis, we first calculate the pion one loop correction to the energy–momentum tensor and then prove the gauge invariance.

Collecting energy–momentum tensor currents and nucleon-pion nonlocal interactions, one can compose possible one loop Feynman diagrams as shown in Figs. 1 and 2.

From Fig. 1a, one can write down pion one loop contribution to the proton self energy as,

$$\Sigma(\not p, m) = -i\, 3\, g^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_5(\not p - \not k + m)\gamma_5}{D_\pi(k) D_N(p-k)} \tilde{F}^2(k), \tag{14}$$

where $\tilde{F}(k)$ is Fourier transformation of the regulator function $F(a)$ and defined as $\tilde{F}(k) = \int d^4a F(a) e^{-ik.a}$, $D_\pi(k)$ and $D_N(k)$ are defined as $D_\pi(k) = k^2 - m_\pi^2 + i\epsilon, D_N(k) = k^2 - m^2 + i\epsilon$. Using the self-energy, the mass counterterm δm is related to the self-energy via $\delta m = \Sigma(\not p = m, m)$. For the tree diagram Fig. 2(a), one has energy–momentum tensor vertex operator as,

$$\Gamma^{\mu\nu}_{\text{fig.2.a}} = \bar{u}(p') \left[\frac{P_{\{\mu}\gamma_{\nu\}}}{2} - \eta^{\mu\nu}(\not P - m) \right] u(p), \tag{15}$$

where P is defined as $P = \frac{1}{2}(p' + p)$. Similarly, the energy–momentum tensor vertex operator for the gravitational counterterm diagram Fig. 2b reads,

$$\Gamma^{\mu\nu}_{\text{fig.2.b}} = -\eta^{\mu\nu} \delta m \bar{u}(p') u(p). \tag{16}$$

In the same way, one can obtain the energy–momentum tensor operator for the nucleon rainbow diagram Fig. 2c,

$$\begin{aligned}\Gamma^{\mu\nu}_{\text{fig2.c}} = -3\, i\, g^2 \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} \gamma^5 (\not p' - \not k - m) \\ \times \left\{ \frac{1}{2}(P-k)^{\{\mu}\gamma^{\nu\}} - \eta^{\mu\nu}(\not P - \not k - m) \right\} \\ \times \frac{(\not p - \not k + m)\gamma^5}{D_N(p-k) D_N(p'-k) D_\pi(k)} \tilde{F}^2(k) u(p).\end{aligned} \tag{17}$$

Similarly, tensor vertex operator for the leading order pion rainbow diagram Fig. 2d is obtained as,

$$\begin{aligned}\Gamma^{\mu\nu}_{\text{fig2.d}} = -3\, i\, g^2\, \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} \left[k'^{\{\mu} k^{\nu\}} - \left(k.k' - m_\pi^2\right)\eta^{\mu\nu} \right] \\ \times \frac{\gamma_5(\not p - \not k + m)\gamma_5}{D_\pi(k) D_\pi(k') D_N(p-k)} u(p) \tilde{F}(k) \tilde{F}(k').\end{aligned} \tag{18}$$

For the energy–momentum tensor current arises from the Kroll–Ruderman (KR) type diagram Fig. 2e, one has,

$$\begin{aligned}\Gamma^{\mu\nu}_{\text{fig2.e}} = -3\, i\, g^2 \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{\eta^{\mu\nu}\gamma_5(\not p - \not k + m)\gamma_5}{D_\pi(k) D_N(p-k)} \right. \\ \left. + \frac{\eta^{\mu\nu}\gamma_5(\not p' - \not k + m)\gamma_5}{D_\pi(k) D_N(p'-k)} \right\} \tilde{F}^2(k) u(p).\end{aligned} \tag{19}$$

In the nonlocal regularization, vertex generated from the gravitational Wilson line operator give rise to an additional KR-like diagram Fig. 2f and the corresponding vertex operator can be obtained as,

$$\begin{aligned}\Gamma^{\mu\nu}_{\text{fig2.f}} = 3\, i\, g^2 \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{\gamma_5(\not p - \not k + m)\gamma_5}{D_\pi(k) D_N(p-k)} \right. \\ \times \left[\frac{1}{2} k^{\{\mu} \frac{\partial}{\partial k^{\nu\}}} \int_0^1 dt\, \tilde{F}(k + qt) \right] + \frac{\gamma_5(\not p' - \not k + m)\gamma_5}{D_\pi(k) D_N(p'-k)} \\ \left. \times \left[\frac{1}{2} k^{\{\mu} \frac{\partial}{\partial k^{\nu\}}} \int_0^1 dt\, \tilde{F}(k - qt) \right] \right\} \tilde{F}(k) u(p),\end{aligned} \tag{20}$$

where t is a dimensionless parameter. It is obvious that loop contribution nucleon rainbow, pion rainbow and KR diagrams in the local limit $\tilde{F}(k) = 1$ recover to the local counterpart except the additional gauge link diagram, which vanishes due to the derivative of the regulator $\tilde{F}(k)$.

5.2 Gauge invariance

As mentioned in the Sect. 1, like electromagnetic current, the total energy–momentum tensor currents from the tree and loop contributions must be conserved. This directly indicates that the total $\bar{c}(t)$ gravitational form factor vanishes if we combine the $\bar{c}(t)$ from each loop contribution. In this way, we can prove the conservation of current straightforwardly without explicitly calculating the gravitational form factors $A(t)$, $B(t)$, $D(t)$ and $\bar{c}(t)$. To do this, we first calculate the identity $q_\mu \Gamma^{\mu\nu}(p, q)$ for each Feynman diagram then combine all of them. Since as shown in Eq. (13), Lorentz structure of gravitational form factors $A(t)$, $B(t)$ and $D(t)$ satisfy on-shell condition except the $\bar{c}(t)$. Thus one can extract the $\bar{c}(t)$ from the identity according to $q_\mu \Gamma^{\mu\nu}(p, q) = q^\nu \bar{c}(t)$. Multiplying q_μ to energy–momentum tensor vertex operator $\Gamma^{\mu\nu}(p, q)$ for the tree diagram (Fig. 2a), from Eq. (15), we easily get the identity,

$$q_\nu \Gamma^{\mu\nu}_{\text{fig.2.a}} = 0. \tag{21}$$

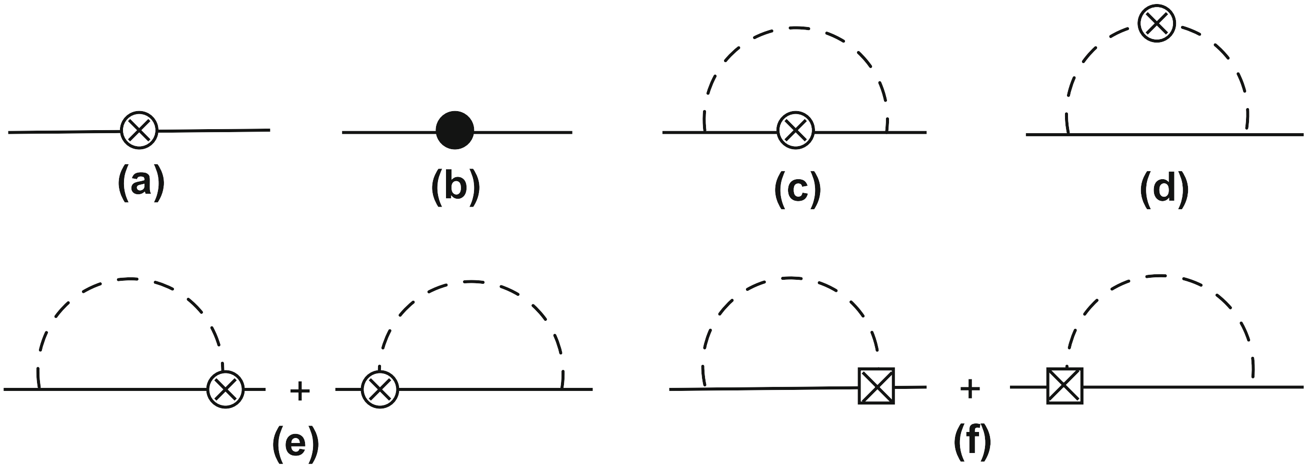

Fig. 2 Pion one loop corrections to the proton gravitational form factors. The crossed circle denotes the insertion of the energy–momentum tensor currents, the crossed square represents the additional gauge link energy–momentum tensor currents arising from the gravitational Wilson line operator, and the solid dot represents the gravitational counterterm

This implies that contribution from the tree diagram is gauge invariant without including the other diagrams. As for the gravitational counterterm diagram (Fig. 2b), the gauge invariancy identity reads,

$$q_\nu \Gamma^{\mu\nu}_{\text{fig.2.b}} = -q^\mu \bar{u}(p')\Sigma(\slashed{p} = m, m)u(p), \tag{22}$$

where the leading order pion contribution to the proton self energy $\Sigma(\slashed{p}, m)$ is defined in Eq. (14). Simplifying Eq. (17) yields the gauge invariance identity for the nucleon rainbow diagram (Fig. 2c) as,

$$\begin{aligned} q_\nu \Gamma^{\mu\nu}_{\text{fig2.c}} = & -3\, i\, g^2 \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} \gamma^5 \left[\frac{(-\frac{q^\mu}{2})(\slashed{p} - \slashed{k} + m)}{2D_N(p-k)} \right. \\ & + \frac{(-\frac{3q^\mu}{2})(\slashed{p}' - \slashed{k} - m)}{2D_N(p'-k)} + \frac{(\slashed{p}' - \slashed{k} + m)k^\mu}{D_N(p'-k)} \\ & \left. - \frac{k^\mu(\slashed{p} - \slashed{k} + m)}{D_N(p-k)} \right] \frac{\gamma^5}{D_\pi(k)} \tilde{F}^2(k) u(p), \end{aligned} \tag{23}$$

where we have used identity $k.q = \frac{1}{2}[D_N(p-k) - D_N(p'-k)]$, and replace $\slashed{q}$ with $\slashed{q} = (\slashed{p}' - \slashed{k} - m) - (\slashed{p} - \slashed{k} - m)$. Therefore the numerator becomes $(\slashed{p} - \slashed{k} - m)(\slashed{p} - \slashed{k} + m) = D_N(p-k), (\slashed{p}' - \slashed{k} - m)(\slashed{p}' - \slashed{k} + m) = D_N(p'-k)$. Similarly, from Eq. (18), we obtain the identity for the pion rainbow diagram (Fig. 2d) as,

$$\begin{aligned} q_\nu \Gamma^{\mu\nu}_{\text{fig2.d}} = & -i3g^2 \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} \left[k^\mu \frac{\gamma_5(\slashed{p} - \slashed{k} + m)\gamma_5}{D_N(p-k)D_\pi(k)} \right. \\ & \times F(k)F(k') - \frac{k^\mu \gamma_5(\slashed{p}' - \slashed{k} + m)\gamma_5}{D_N(p'-k)D_\pi(k)} \\ & \left. \times F(k-q)F(k) \right] u(p), \end{aligned} \tag{24}$$

where we have used the identity $2\,k.q + q^2 = D_\pi(k') - D_\pi(k)$, and $D_\pi(k')$ is transformed into $D_\pi(k)$ by the replacement $k' \to k$. In the same way, from Eq. (19), we obtain gauge invariance identity for the KR diagram (Fig. 2e) as,

$$\begin{aligned} q_\nu \Gamma^{\mu\nu}_{\text{fig2.e}} = & -3\, i\, g^2 \bar{u}(p') q^\mu \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{\gamma_5(\slashed{p} - \slashed{k} + m)\gamma_5}{D_\pi(k)D_N(p-k)} \right. \\ & \left. + \frac{\gamma_5(\slashed{p}' - \slashed{k} + m)\gamma_5}{D_\pi(k)D_N(p'-k)} \right\} \tilde{F}^2(k)u(p). \end{aligned} \tag{25}$$

Finally, the identity for the additional gauge link diagram is given by,

$$\begin{aligned} q_\nu \Gamma^{\mu\nu}_{\text{fig2.f}} = & 3ig^2 \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} \left\{ k_\mu \frac{\gamma_5(\slashed{p} - \slashed{k} + m)\gamma_5}{D_\pi(k)D_N(p-k)} \right. \\ & \times [\tilde{F}(k') - \tilde{F}(k)] - k_\mu \frac{\gamma_5(\slashed{p}' - \slashed{k} + m)\gamma_5}{D_\pi(k)D_N(p'-k)} \\ & \left. [\tilde{F}(k-q) - \tilde{F}(k)] \right\} \tilde{F}(k)u(p). \end{aligned} \tag{26}$$

Combining the expressions in Eqs. (22, 23, 24, 25, 26), the result reads,

$$q_\nu \Gamma^{\mu\nu}_{\text{fig.2.b}} + q_\nu \Gamma^{\mu\nu}_{\text{fig2.c}} + q_\nu \Gamma^{\mu\nu}_{\text{fig2.d}} + q_\nu \Gamma^{\mu\nu}_{\text{fig2.e}} + q_\nu \Gamma^{\mu\nu}_{\text{fig2.f}} = 0. \tag{27}$$

Equation (27) shows that the total $\bar{c}(t)$ form factor from the counterterm and loop contributions vanishes with the help of the gravitational Wilson line operator. In addition, in the local limit $\tilde{F}(k) = 1$ the additional gauge link contribution of Eq. (26) tends to zero so that local $\bar{c}(t)$ from factor from

Eqs. (22, 23, 24, 25) is also equal to zero. In summary, both in the local and nonlocal frameworks the energy–momentum tensor is conserved.

6 Conclusion

The Wilson line operator is an important topic in the gauge field theory and gravity. It plays an important role as one considers the gauge invariance of the nonlocal interactions in the flat and curved space-time. In such a model, the nonlocal field operator was introduced to renormalize the UV divergence of the loop contribution. However, the price for this is that the gauge invariance of the theory is violated. To recover the gauge invariancy, it is necessary to include both the gauge field and the gravitational Wilson line operator. Consequently, loop contributions to physical observables are gauge invariant and UV finite.

In this paper, we developed a novel method in which the gravitational Wilson line operator can be derived from the nonlocal interactions. Our analysis shows that the nonlocal result for the gravitational Wilson line operator is identical to previous result. More importantly, by virtue of the gravitational Wilson line operator total $\bar{c}(t)$ gravitational form factor is zero. This implies that the energy–momentum tensor is conserved even in the non-local framework.

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