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Antisymmetric tensor fields in 4D: actions, symmetries and first order Duffin–Kemmer–Petiau formulations

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Abstract

Analyzing the representations of the Lorentz group, we give a systematic count and construction of all the possible Lagrangians describing an antisymmetric rank two tensor field. The count yields two scalars, which provide the ingredients of the gauge invariant Kalb–Ramond model, equivalent to the sigma model and familiar from super gravity and string theory, and of the conformally invariant Avdeev–Chizhov model, which describes self-dual tensors. The count also includes a pseudoscalar with symmetric coupling, and an additional pseudoscalar, an antisymmetrized form of the Avdeev–Chizhov Lagrangian. This quantity was first derived in an $SU(2/1)$ superalgebraic model of the weak interactions. It is also conformally invariant, and naturally implements the Landau CP symmetry. Then, by extending the DKP ten component formalism, we recover the model Lagrangians as first order systems. To complete the analysis, we classify all local Lorentz invariant potentials (mass terms and quartic couplings) for charged antisymmetric tensor fields coupled to a Yang-Mills field.

Keywords: antisymmetric tensor fields, conformal symmetry, Kemmer matrices, Lorentz invariant, DKP equation, first order formulation

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1. Introduction and main results

The theoretical project of constructing invariant actions for 'higher spin' relativistic fields can plausibly be claimed to have originated with the Klein-Kaluza models, in curved space, but was subsequently shaped by Dirac's formulation for spin-$\frac{1}{2}$ and generalizations [1]. Famous early examples include the Proca spin-1 equation [2], and, for example the treatment by Fierz and Pauli of spin-$\frac{3}{2}$ and spin-2, [3], but the topic has spawned an enormous and still expanding literature. The successful validation of the elementary particle standard model has led to the quantum field-theoretical focus on 'fundamental' entities , as currently understood to be quarks and leptons (spin-$\frac{1}{2}$ Fermions), together with vector exchange particles (spin-1 gauge Bosons), being augmented by an additional, apparently fundamental, spin-0 entity, the Higgs Boson, with deep implications for the origin of mass, and cosmology. While a full understanding of the physics of the symmetry breaking phenomena is incomplete, it is prudent, on a theoretical level, to consider possible alternative presentations of, or potential extensions to, the Bosonic sector.

In this vein we reconsider the case of local relativistic antisymmetric tensor fields (rank 2 tensors in 4 dimensional Minkowski space). In the guise of higher spin gauge models [4], such degrees of freedom are a known concomitant of various string theories and dualities (see for example [5]), and indeed have been much studied in topological quantum field theory, in applications to geometric invariant theory [6]. Skew symmetric tensor gauge fields have also been invoked in connection with the quantum chromodynamics (QCD) $U(1)$ problem [7]. Moreover, a different formulation, of conformally invariant antisymmetric tensor fields, proposed and developed some time ago [8] has also been explored as a possible accompaniment of phenomenological scalar degrees of freedom in the standard model symmetry breaking Higgs sector [9] (for technical aspects see also [10, 11]).

In section 2 below we present a systematic count of all local invariant combinations (one dimensional representations) which are quadratic in a generic antisymmetric tensor and in derivatives, thereby enumerating all admissible linearly independent kinetic terms for the free fields. The tensor analysis is based on the analysis of representations of the special Lorentz group $SO(3,1)$, but also on group character theory in the full Lorentz group $O(3,1)$. The result of this counting is that in symmetric coupling, as to be expected, there are just two linearly independent Lorentz invariant scalar candidate terms, which reproduce (in appropriate linear combinations) the Lagrangians for the known gauge invariant two-form potential model, and conformally invariant antisymmetric tensor model, referred to above. Furthermore, the count of one dimensional representations includes two additional Lorentz pseudo-scalar terms: one symmetrically coupled, and one antisymmetrically coupled. The antisymmetrically coupled pseudo-scalar, which is also conformally invariant, has also been exploited, as a Lagrangian model, in our recent work [12], in which 'chiral Bosons' (see [11]) appear as part of the $SU(2/1)$ superalgebraic extension of the electroweak gauge sector. (See section 6 for further discussion).

In section 3, we flesh out the previous classification by giving free Lagrangians with the desired properties. From the literature, these include the two symmetric scalar Lagrangians: the Kalb-Ramond gauge invariant model $\mathscr{L}^{KR}$ [5] and the Avdeev-Chizhov self-dual tensor model $\mathscr{L}^{AC}$ [8], and thirdly the antisymmetric self-dual pseudo-scalar Lagrangian $\mathscr{L}^{CP}$ from [12]. The remaining, fourth case is a trivial, symmetric pseudo-scalar Lagrangian, describing a collection of 6 independent fields.

Amongst the contributions to the early work on relativistic wave equations should also be included the papers of Duffin, Kemmer and Petiau (DKP) [13–15]. In particular, the seminal paper of Kemmer presented a general first order equation, involving the Kemmer β matrices, a weaker algebraic structure than that of the Dirac matrices. Two different cases, with five- and ten-component wave functions, provided (at least for free fields) for formulations equivalent to the Klein–Gordon (massive complex scalar) spin-0 equation and the Proca (massive complex vector) spin-1 equation.

In section 4 and appendix B below, we present an extension of the ten component DKP system. Specifically, we show that there are two 'twisted' variants of the Duffin-Kemmer-Petiau local action providing first order equations, which after elimination of auxiliary fields, precisely reproduce the two known physical models identified above in the count for the symmetric coupling (the two-form gauge potential and the conformally invariant antisymmetric tensor field, respectively). For antisymmetric field coupling, we demonstrate a third variant, which recovers the pseudo-scalar term identified above. As an illustration of the method, the standard Proca system for a massive vector field is given in first order DKP form in the appendix C.1. The corresponding details for the two form gauge potential and the two antisymmetric tensor field cases are given in appendix C.2. The conformal invariance of these two models is discussed in appendix D.

Moving from kinetic terms to couplings, for completeness we also provide in section 5 a count of Lorentz invariant local interactions, ('mass' terms and quartic potentials), for physically relevant cases of a complex scalar and complex antisymmetric tensor, invariant under global internal $U(n) \cong SU(n) \times U(1)$, in the fundamental n-dimensional representation.

In section 6, we summarize our main results and discuss the implications of these models for physics.

2. Lorentz invariant tensor polynomials: kinetic energy

Relativistic fields in Minkowski space are labeled as representations of the Lorentz group by their associated spins. For the special Lorentz group $SO(3,1)$, we exploit the isomorphism with the direct product $sl(2) \times sl(2)$, and denote irreducible representations by pairs (j_1, j_2) with integer and half-integer spins $j_1, j_2 = 0, \frac{1}{2}, 1, \cdots$ and dimension $(2j_1+1)(2j_2+1)$. Thus a Lorentz vector is the representation $(\frac{1}{2}, \frac{1}{2})$, while $(1,1)$ represents a traceless symmetric tensor (such as the gravitational metric in some gauges). Finally, an antisymmetric tensor, of dimension 6, is a reducible representation $(1,0)+(0,1)$ (see below).

For enumerating candidate kinetic terms we take local invariants quadratic in derivatives and fields, of the form

$$\mathscr{L} = \mathtt{C}^{\mu\nu\alpha\beta\gamma\delta} \partial_\mu T_{\alpha\beta} \partial_\nu T_{\gamma\delta} + \text{h.c.}, \tag{1}$$

where[3] $T_{\mu\nu} = -T_{\nu\mu}$ and $\mathtt{C}^{\mu\nu\alpha\beta\gamma\delta}$ is any numerical tensor made from (monomials in) the invariant tensors $\eta_{\kappa\lambda}$ and $\varepsilon_{\rho\sigma\tau\upsilon}$. With this in mind we have the derivative terms contributing a symmetric tensor (of dimension 10) with the two T terms providing additional factors in the overall tensor product, written symbolically as a product of irreducible representation labels (or group characters),

$$((0,0)+(1,1)) \cdot ((1,0)+(0,1))^2 . \tag{2}$$

Thus far we have suppressed (global, or eventually local) internal symmetry transformations, which would require appending to the fields an additional internal suffix $^{\mathsf{a},\mathsf{b}}, \cdots,$. Here we

[3] We use standard notation for Minkowski space with coordinates $x^\mu = (x^0, x^i) = (ct, \underline{x})$, metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and Levi-Civita tensor $1 = \varepsilon_{0123} = -\varepsilon^{0123}$.

Table 1. Count of symmetric and antisymmetrically coupled $SO(3,1)$ and $O(3,1)$ invariant kinetic terms.

	Symmetric		Antisymmetric	
$SO(3,1)$	3		1	
$O(3,1)$	scalar 2	pseudo-scalar 1	scalar 0	pseudo-scalar 1

keep this implicit, but to accommodate later extensions (see sections 3 and 5 below) we separately take account of which invariants arise from symmetric, or antisymmetric, couplings (corresponding to orthogonal or symplectic representations of internal symmetry groups, respectively)[4].

A simple count (appendix A.1) shows that there are four invariants in total: three symmetric, and one antisymmetric.

We now augment the analysis, by identifying the above invariants under the full Lorentz group $O(3,1)$. At this level the machinery of tensor representations is somewhat more intricate than for $SO(3,1)$, and relevant details are provided in appendix A.1 below, based on formal group character manipulation.

Table 1 shows how the $SO(3,1)$ count is refined under $O(3,1)$. Clearly, under restriction the latter count (of one dimensional representations) must reduce to that for the special Lorentz group. In symmetric coupling we find two scalars and one pseudo-scalar, while the single $SO(3,1)$ invariant in antisymmetric coupling is a Lorentz pseudo-scalar.

Capitalizing on previous knowledge, we present below for each of these cases, a corresponding Lagrangian, written in the standard covariant way. In this context, the interest of the present section is the proof that in four dimensions our enumeration of these Lagrangians is complete.

3. Lorentz invariant tensor polynomials: free Lagrangians

In the previous section we established that in four dimensional Minkowski space there are precisely four linearly independent local terms which are suitable ingredients for kinetic terms of antisymmetric tensor Lagrangians.

It is straightforward to construct, in standard relativistic tensor notation (compare (1)) an explicit basis of local monomials which, given the count, must provide the ingredients for any physical models. Four such candidates, $\mathscr{L}^{\Box}$, $\mathscr{L}^{\nabla}$, ${}^*\mathscr{L}^{\Box}$,and ${}^*\mathscr{L}^{\nabla}$ are listed in table 2, with properties in accord with table 1 above.

We now review and discuss various Lagrangians incorporating these terms, and which (as mentioned in the introduction) which have been considered as models for the physics of antisymmetric tensor fields. Note that the third invariant in table 2, the pseudoscalar ${}^*\mathscr{L}^{\Box} = {}^*T\Box T$, does not appear to have been used in model building, and will not be discussed further. The symmetrically coupled, scalar, contributions are incorporated in the Kalb-Ramond and Avdeev-Chizhov models [4, 5, 8],

$$\mathscr{L}^{KR} = -\mathscr{L}^{\Box} - 2\mathscr{L}^{\nabla} = (\partial^{\rho}T_{\mu\nu})(\partial_{\rho}T^{\mu\nu}) - 2(\partial^{\rho}T_{\rho\mu})\eta^{\mu\nu}(\partial^{\sigma}T_{\sigma\nu})\,; \tag{3}$$

[4] The role of the Hodge dual ${}^*T$ in the projection of a tensor T into self-dual and anti self-dual parts $(1,0)$ and $(0,1)$ is explained below in relation to explicit local forms of the invariants.

Table 2. Explicit local candidates $\mathscr{L}^{\Box}$, $\mathscr{L}^{\nabla}$, ${}^*\mathscr{L}^{\Box}$, and ${}^*\mathscr{L}^{\nabla}$ for $SO(3,1)$ and scalar (S) and pseudoscalar (P) $O(3,1)$ invariant kinetic terms. The notation ${}^*T$ denotes the Hodge dual (see text). The first three rows are for symmetric coupling, and the last row (vanishing if $T'=T$) is for antisymmetric coupling. For discussion purposes, an informal representation of each candidate term is also listed.

S	$\mathscr{L}^{\Box}$	$T_{\mu\nu}\partial^\rho\partial_\rho T^{\mu\nu}$	$(T\Box T)$
S	$\mathscr{L}^{\nabla}$	$(\partial^\rho T_{\rho\mu})\eta^{\mu\nu}(\partial^\sigma T_{\sigma\nu})$	$(\partial\cdot T)(\partial\cdot T)$
P	${}^*\mathscr{L}^{\Box}$	${}^*T_{\mu\nu}\partial^\rho\partial_\rho T^{\mu\nu}$	$({}^*T\Box T)$
P	${}^*\mathscr{L}^{\nabla}$	$(\partial^{\rho\,*}T_{\rho\mu})\eta^{\mu\nu}(\partial^\sigma T'_{\sigma\nu})$	$(\partial\cdot{}^*T)(\partial\cdot T')$

$$\mathscr{L}^{AC} = -\mathscr{L}^{\Box} - 4\mathscr{L}^{\nabla} = (\partial^\rho T_{\mu\nu})(\partial_\rho T^{\mu\nu}) - 4(\partial^\rho T_{\rho\mu})\eta^{\mu\nu}(\partial^\sigma T_{\sigma\nu})\,. \tag{4}$$

Note that $\mathscr{L}^{AC}$ can be re-written [8] in terms of the self-dual and anti self-dual projections of a complex antisymmetric tensor $Z_{\mu\nu}$,

$$Z := T + \mathrm{i}^*T, \qquad \overline{Z} := T - \mathrm{i}^*T,$$

where ${}^*T_{\mu\nu} := \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}T^{\rho\sigma}$ is the Hodge dual such that ${}^{**}T = -T$ in Minkowski space, with ${}^*Z = -\mathrm{i}Z$, ${}^*\overline{Z} = +\mathrm{i}\overline{Z}$. With these definitions we find [8]

$$\mathscr{L}^{AC} = \partial^\rho \overline{Z}_{\rho\mu}\eta^{\mu\nu}\partial^\sigma Z_{\sigma\nu}\,. \tag{5}$$

The Kalb-Ramond model does not admit an analogous rearrangement, but instead can be re-written

$$\mathscr{L}^{KR} = \tfrac{1}{6}\delta^{\lambda\mu\nu}{}_{\rho\sigma\tau}\partial_\lambda T_{\mu\nu}\partial^\rho T^{\sigma\tau} \tag{6}$$

in terms of the totally antisymmetric determinant invariant[5] $\delta^{\lambda\mu\nu}{}_{\rho\sigma\tau} = \delta^\lambda{}_\rho\delta^\mu{}_\sigma\delta^\nu{}_\tau \pm\cdots$.

Rather than expanding ${}^*\mathscr{L}^{\nabla} = (\partial\cdot\overline{Z})(\partial\cdot Z)$ in terms of projections as above, consider now the corresponding expansion starting with two distinct fields Z, Z' and projections T, T' and ${}^*T$, ${}^*T'$. This reproduces the analogue of $\mathscr{L}^{AC}$ (symmetrized in T, T'), but also returns a cross term, which is the antisymmetric pseudoscalar listed in table 2 above,

$${}^*\mathscr{L}^{\nabla} := \mathrm{i}\,(\partial^{\rho\,*}T_{\rho\mu})\,\eta^{\mu\nu}\,(\partial^\sigma T'_{\sigma\nu})\,. \tag{7}$$

Endowed with a multiplet of complex fields $Z^{\mathtt{a}}$, $a = 1, 2, \cdots$, belonging to a representation of some internal symmetry group, with antisymmetric quadratic invariant $\kappa_{\mathtt{ab}} = -\kappa_{\mathtt{ba}}$, this becomes the model of antisymmetric tensors first identified in [12, 16, 17] via Fermion loop corrections in a superalgebra-enhanced standard model, augmented by antisymmetric tensor fields in the gauge sector[6]:

$$\mathscr{L}^{CP} := \mathrm{i}\left(\partial^{\rho\,*}T^{\mathtt{a}}_{\rho\mu}\right)\eta^{\mu\nu}\left(\partial^\sigma T^{\mathtt{b}}_{\sigma\nu}\right)\kappa_{\mathtt{ab}}\,. \tag{8}$$

[5] We have $\delta^{\lambda\mu\nu}{}_{\rho\sigma\tau} = \varepsilon^{\kappa\lambda\mu\nu}\varepsilon_{\kappa\rho\sigma\tau}$; recall from (1) that scalar and pseudoscalar Lagrangian densities must be even or odd in $\varepsilon_{\alpha\beta\gamma\delta}$, respectively. That this form is also $GL(4)$ covariant is evident from the character analysis (appendix A.1 and equation (17)). A general discussion of the role of the structure group $GL(d)$ in higher dimensions is given in [18].

[6] This coupling applies in particular when the fields are allocated to a real reducible representation comprised of a conjugate pair, (as in section 5) where an antisymmetric bilinear form is always available.

Importantly, the models $\mathscr{L}^{KR}$, $\mathscr{L}^{AC}$ and $\mathscr{L}^{CP}$ which we have identified are distinguished by their symmetries. Evidently, $\mathscr{L}^{KR}$ is gauge invariant[7], while $\mathscr{L}^{AC}$ (containing a different admixture of $\mathscr{L}^{\square}$ and $\mathscr{L}^{\nabla}$) is not. Vice versa, as noted in the original literature, $\mathscr{L}^{AC}$ is invariant under the conformal group acting in four-dimensional Minkowski space, whereas $\mathscr{L}^{KR}$ is not.

For further details of the physics of the identified models, we refer to the literature cited [4, 5, 8–11]. In particular we note that quantization of the models $\mathscr{L}^{KR}$ and $\mathscr{L}^{AC}$ entails one degree of freedom (two for complex fields). In the case of the model $\mathscr{L}^{CP}$, while the propagator appears to be in agreement with fermion couplings to one loop, full analysis of consistency and unitarity are still under investigation (see [12] and section 6 for concluding remarks).

Beyond four dimensions, we note that $\mathscr{L}^{KR}$, $\mathscr{L}^{AC}$ (and also $\mathscr{L}^{CP}$) are specific instances of hierarchies of models of generalized antisymmetric tensors, of various ranks p, and space-time dimension D. For example, $\mathscr{L}^{KR}$ itself is a generalization to rank 2 of Maxwell electromagnetism, which is well known to be conformally invariant [19] as well as being gauge invariant; however for antisymmetric rank 2, conformal invariance holds in $D=6$ rather than $D=4$. For a comprehensive discussion we refer the reader to [20], where it is also proven that the generalization of $\mathscr{L}^{CP}$, for self-dual fields, is conformally invariant in any dimension $D=2p$ in Minkowski space. Further details are beyond the scope of this paper; however, given the significance of the results, in appendix D we provide for completeness, a derivation of conformal invariance of $\mathscr{L}^{CP}$ and $\mathscr{L}^{AC}$ within the four dimensional first order formulation.

4. Ten component DKP formulation for antisymmetric tensor fields in 4D

It is well known that the wave equations associated with the Dirac algebra

$$\{\gamma_\mu,\gamma_\nu\}=2\eta_{\mu\nu}\,,$$

together with those for the Duffin–Kemmer algebra

$$\beta_\mu\beta_\rho\beta_\nu+\beta_\nu\beta_\rho\beta_\mu=\eta_{\mu\rho}\beta_\nu+\eta_{\nu\rho}\beta_\mu\,,$$

provide the only single-mass instances of the general class of $SO(5)$-related Bhabha wave equations. The Dirac equation of course pertains to Fermionic fields of spin-$\frac{1}{2}$, while (as reviewed above), the DKP equation either to Bosonic spin 0 (for 5 component wave functions) or spin 1 (for ten component wave-functions, in the standard picture)[8]. Both cases have been much studied as alternatives to the complex scalar Klein–Gordon model and the complex vector Proca model, both in phenomenological applications, and also in relation to their equivalence to these standard systems, especially in the interacting case or in curved backgrounds (for an overview see [21], for classical solutions see for example [22, 23]. A detailed analysis has been presented in [24, 25], and for investigation at the second quantized level see [26]).

[7] In the language of exterior forms, the action corresponding to the Lagrangian density $\mathscr{L}^{KR}$ is proportional to $\int {}^*\mathrm{d}T\wedge\mathrm{d}T$, which is manifestly gauge invariant under $T\to T+\mathrm{d}X$.

[8] These cases correspond to the irreducible five- and ten- dimensional representations of the Kemmer algebra and, together with the trivial one-dimensional representation, saturate the dimension of the adjoint representation $126=1^2+5^2+10^2$ in accordance with Wedderburn's theorem.

An explicit representation of the Kemmer β-matrices in the 10×10 case is conveniently written in $3+3+3+1$ block form as follows[9]:

$$\beta^0 = \left[\begin{array}{ccc|c} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array}\right], \quad \beta^i = \left[\begin{array}{ccc|c} \cdot & \cdot & \cdot & e_i \\ \cdot & \cdot & \ell_i & \cdot \\ \cdot & \ell_i & \cdot & \cdot \\ \hline -f_i & \cdot & \cdot & \cdot \end{array}\right]. \tag{9}$$

The three triplet plus singlet partitioning reflects the reducible Lorentz structure of the DKP wave-function at the level of the rotation group: namely, antisymmetric tensor (dimension 6) plus four-vector. The parity transformation, such that for the ten-component wave-function $\Phi'(ct, -\underline{x}) = -\eta\Phi(ct, \underline{x})$, leads to the standard parity matrix [14]

$$\eta := 2\left(\beta^0\right)^2 - 1 = \mathrm{diag}\left(1, -1, 1, -1\right) \tag{10}$$

and in the usual way to the conjugate wave-function $\overline{\Phi} := \Phi^\dagger \eta$.

In appendix C.1 we demonstrate for completeness the well-known standard DKP derivation of the Proca action for a (free) massive complex spin-1 field. Kemmer [14] pointed out that the roles of the antisymmetric tensor (field strength) and vector in the 10 component DKP wave-function could be reversed (now with an axial vector as field strength tensor). In this vein we proceed to develop a corresponding extended DKP formalism as follows. Firstly note that the pseudo-scalar object $\beta_5 := \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}[\beta_\mu, \beta_\nu][\beta_\rho, \beta_\sigma] \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\beta_\mu\beta_\nu\beta_\rho\beta_\sigma$ is the analogue for the Kemmer algebra of the pseudo-scalar $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ in the Dirac case, and takes the form[10]

$$\beta_5 = \left[\begin{array}{ccc|c} \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array}\right]. \tag{11}$$

We now assume that in an extension of the DKP equation, the Kemmer matrix β^μ may be replaced by an analogue taken from the enveloping algebra. Here we investigate the choice

$$\breve{\beta}^\mu := [\beta_5, \beta^\mu], \tag{12}$$

from which

$$\breve{\beta}^0 = \left[\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array}\right], \quad \breve{\beta}^i = \left[\begin{array}{ccc|c} \cdot & \cdot & \ell_i & \cdot \\ \cdot & \cdot & \cdot & -e_i \\ \ell_i & \cdot & \cdot & \cdot \\ \hline \cdot & f_i & \cdot & \cdot \end{array}\right]. \tag{13}$$

For this twisted system the appropriate parity matrix is

$$\eta' := -\left(1 + 2(\beta_5)^2\right)\eta, \tag{14}$$

[9] Here e_i, $i = 1, 2, 3$ are standard three-component unit column vectors, $f_i = {}^\top e_i$ the corresponding row vectors, and $\ell_i := \frac{1}{2}\varepsilon_{ijk}\ell_{jk}$ elementary 3×3 rotation generators. (For details see appendix B).

[10] $\beta_5 \equiv 0$ in the five dimensional representation.

Table 3. Extended DKP models (see appendix C.2 for derivations). In each case the DKP action is given along with the type of ten component wave-function (complex, real, self dual, i.e. $\beta_5\Phi^\wedge = \pm \mathrm{i}\Phi^\wedge$). The reduced $\overline{G}^\mu G_\mu$ forms after elimination of variables for the two-form gauge, antisymmetric tensor and antisymmetric tensor in antisymmetric coupling are explicitly compared with the standard tensor expressions in the concluding discussion section 6.

	$\mathcal{L}$ (DKP form)	Φ	$\mathcal{L}$ ($\overline{G}^\mu G_\mu$ form)
$\mathcal{L}^{KR}$	$\overline{\overline{\Phi}}\mathrm{i}\breve{\beta}^\mu\partial_\mu\Phi + m\overline{\overline{\Phi}}(1+\beta_5^2)\Phi$	r.(or c.)	$G_\mu = \varepsilon_{\mu\nu\rho\sigma}\partial^\nu Z^{\rho\sigma}$
$\mathcal{L}^{AC}$	$\overline{\Phi}\breve{\beta}^\mu\frac{1}{2}\overleftrightarrow{\partial_\mu}\Phi + m\overline{\Phi}\Phi$	c.s.d.	$G_\mu = \partial^\rho Z_{\rho\mu}$
$\mathcal{L}^{CP}$	$\frac{1}{2}\left(\Phi^{c\,\mathtt{a}}\beta^\mu\partial_\mu\Phi^{\mathtt{b}} + \mathrm{i}m\overline{\Phi}^{\mathtt{a}}\Phi^{\mathtt{b}}\right)\kappa_{\mathtt{ab}}\big) + \text{h.c.}$	c.s.d	$G^{\mathtt{a}}_\mu := \partial^\rho Z^{\mathtt{a}}_{\rho\mu}$

with modified conjugate wave-function $\overline{\overline{\Phi}} := \Phi^\dagger\eta'$, enabling current bilinears to be constructed in the usual way[11]. Just as in the case of Dirac spinors, it is possible to work with projected wave-functions. For example $\Phi^\wedge := (-\beta_5^2)\Phi$ is the restriction to the upper (antisymmetric tensor) components, from which we can further project out self-dual and anti self-dual components as $\pm\mathrm{i}$ eigenvectors of the chirality matrix β_5 in correspondence with the constraint ${}^*Z_{\mu\nu} = \pm\mathrm{i}Z_{\mu\nu}$ on the Hodge dual ${}^*Z_{\mu\nu} := \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}Z^{\rho\sigma}$ of the antisymmetric tensor.

It remains to present the variants of the extended DKP formalism which recover the known models for antisymmetric tensor fields (the *KR* two-form gauge field and the *AC* antisymmetric tensor field in symmetric coupling, as well as the *CP* antisymmetric coupling case). In table 3 we list in each case the candidate action in terms of the DKP wave-function, and the corresponding action after elimination of auxiliary fields, which recovers the known model in question. Details of the calculations are given in appendix C.2, following the method of appendix C.1.

5. Lorentz invariant tensor polynomials: potential energy

Given the possibility of antisymmetric tensor fields providing ingredients (together with scalar fields) of an extended symmetry breaking sector, we supplement our count and identification of admissible kinetic terms (for free fields), as in sections 2 and 4, with a systematic count of invariant potential terms (both self-coupling interaction and mass terms, where applicable). Invariants are again identified by group theoretical techniques where fields are represented by appropriate group characters, and local monomials in fields are classified by character manipulations (with powers in fields counted as symmetrized products).

Here we work at the $SO(3,1)$ level with complex self-dual antisymmetric tensors , which are allowed to carry distinct representations of an internal symmetry, taken to be $SU(n)\times U(1) \cong U(n)$, in the fundamental representation, which includes the standard model for $n=2$, and the Abelian case for $n=1$. We also include a complex scalar as proxy for the physical Higgs field. The complex self-dual antisymmetric tensor field is now represented as

$$(1,0)\square + (0,1)\bar{\square},$$

where the Young diagram stands for a representation of $SU(n)$ with accompanying $U(1)$ eigenvalue the diagram weight ±1 for the fundamental and contragradient, $\square$ and $\bar{\square}$, respectively. To

[11] For example, $\overline{\overline{\Phi}}\breve{\beta}^\mu\Phi$ is a Lorentz four-vector.

Table 4. Number of Lorentz invariants for contributions to potential, from monomials of the form $(\text{scalar})^p \cdot (\text{antisymmetric tensor})^q$.

p, q	2,0	1,1	0,2	4,0	2,2	0,4
count	1	0	0	1	2	1

include the scalar content we append

$$(0,0)\square + (0,0)\bar{\square},$$

and seek Lorentz invariants amongst monomials in the complex scalar and tensor at degrees p, q up to quadratic and quartic contributions (where cross terms with odd powers are disallowed, as the $U(1)$ charge, viz. $(\pm 3) + (\pm 1)$ is non-vanishing). The resulting count (table 4) lists admissible mass terms for the complex scalar and antisymmetric tensor, as well as quartic interaction potentials in each, and a possible quartic term with mixed powers (a bilinear scalar-tensor mixing being excluded). See appendix A.2 for details. The count includes, as expected a single complex scalar mass term and its square, the quartic scalar potential, while forbidding a mass term for the complex self-dual antisymmetric tensor. The quartic pure tensor, and quartic mixed quadratic scalar-tensor, invariants arise from combinations of the symmetrized contributions of one component of the tensor, say $(1,0)\cdot\square$, with the corresponding opposite weight counterpart(s), in the expansions of either the tensor, or the scalar, respectively[12]. For further details see appendix A.2.

6. Conclusions

In this work we have provided a robust count of all possible invariant local Lagrangian densities for antisymmetric tensor fields, including both kinetic and potential energy terms, and these have been correlated with different physical models.

While it is straightforward (section 3) to construct invariants using standard tensor notation for such relativistic fields, we have been at pains in section 2 to provide a realization-free identification of the number of admissible linearly independent contributions and their correspondence with known models, which we have also reproduced in section 4 via first order DKP formulations involving an extension to the Kemmer algebra which we believe to be new. The universality of our 'existence' count guarantees that, in any alternative presentations such as generalized Bargmann-Wigner, Schwinger, multispinor or other methods (see [29, 30]), only and precisely the same local Lagrangian densities must enter. At the same time, the first order DKP formulation, in which the Kemmer matrix β_5 plays a central role, analogous to that of γ_5 in the Dirac equation and spin-$\frac{1}{2}$ context, deepens the analogy between 'chiral Fermions' and 'chiral Bosons'.

For further details of the known models identified, including degree of freedom count, quantization and renormalizability, we refer to the literature cited. However, one of the most interesting results of the present paper concerns the identification of the conformally invariant self-dual tensor Lagrangian $\mathscr{L}^{CP}$. This antisymmetric tensor model has been exploited in

[12] Using the complex vector $\underline{C} := \underline{S} \pm \mathrm{i}\underline{R}$ (for one choice of sign) and its conjugate [27, 28], (compare equation (22)), local forms for the tensor-scalar and tensor quartic invariants are $\underline{C}^a \cdot \underline{C}^b H^*_a H^*_b + c.c.$ and $\underline{C}^a \cdot \underline{C}^b \underline{C}^*_a \cdot \underline{C}^*_b$, respectively. The abstract count establishes unequivocally that these are the only candidates. Quartic interactions for antisymmetric tensors have also been discussed in [9–11].

our recent work [12], in which such 'chiral Bosons' (see [9]) appear as part of an $SU(2/1)$ superalgebraic extension of the electroweak gauge sector[13]. In [12] it was noticed that if a complex self-dual antisymmetric tensor is coupled to Fermions using the odd matrix generators of $SU(2/1)$ in the appropriate representations, then the Fermion loop counterterms of the quantum field theory induce a new type of tensor propagator, corresponding precisely to that of $\mathscr{L}^{CP}$. Moreover, this Lagrangian, being pseudoscalar and antisymmetric in the internal charge space, naturally implements the Landau CP symmetry of the weak interactions. From the enumeration of Lorentz invariants done in this study, this model indeed occupies the niche allowed by the additional pseudoscalar term, as distinct from the known scalars (tables 1 and 2 above). This surprising discovery was part of the motivation for the present work.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Lorentz invariant tensor polynomials

A.1. Kinetic energy

With the notation $[\cdots]^{\underline{\otimes}(2)}, [\cdots]^{\underline{\otimes}(1^2)}$ to represent the symmetric or antisymmetric symmetrized quadratic tensor power, our task is to count $SO(3,1)$ invariants in the expansion of equation (2), namely

$$((0,0)+(1,1))\cdot[(1,0)+(0,1)]^{\underline{\otimes}(2)} + ((0,0)+(1,1))\cdot[(1,0)+(0,1)]^{\underline{\otimes}(1^2)}. \tag{15}$$

Using the standard rules for symmetrization of rotation group representations, viz. $(1)^{\underline{\otimes}(2)} = (0)+(2)$, $(1)^{\underline{\otimes}(1^2)} = (1)$, and the fact that the symmetrization or antisymmetrization of a sum of two parts contains a single copy of the cross term, we require $SO(3,1)$ invariants in

$$((0,0)+(1,1))\cdot[(0,0)+(2,0)+(0,0)+(0,2)+(1,1)] \quad \text{and}$$
$$((0,0)+(1,1))\cdot[(1,0)+(0,1)+(1,1)],$$

[13] In 1979, Ne'eman [31] and Fairlie [32] independently proposed embedding the electroweak Lie algebra of the standard model, $SU(2)\times U(1)$, into the simple Lie-Kac superalgebra $SU(2/1)$ (of which it is the even Lie subalgebra). Indeed, the weak isospin and electric charge content of not only the leptons $((\nu_L, e_L)/e_R)$, but also of the colored quarks $(u_R/(u_L,d_L)/d_R)$, graded by their left/right chirality, match the lowest triplet and quartet irreducible representations of $SU(2/1)$ [33, 34]. Also, we recently presented general results on superalgebra representations of real and indecomposable type [35–37], consolidating the original observation [38] that in the case of $SU(2/1)$, the observed three generations of Fermions, associated to the electron, the muon and the tau families, can be naturally accommodated.

and by matching we find three invariants in symmetric, and one in antisymmetric coupling, respectively:

$$\begin{array}{c|cc} SO(3,1) & \underbrace{((0,0)+(1,1))\cdot[(1,0)+(0,1)]^{\underline{\otimes}(2)}}_{3} & \underbrace{((0,0)+(1,1))\cdot[(1,0)+(0,1)]^{\underline{\otimes}(1^2)}}_{1} \\ \text{count} & & \end{array} \tag{16}$$

A short cut to the needed classification of terms under the full Lorentz group is provided by working initially with tensors under the larger structure group $GL(4)$, and restricting to $O(3,1)$ in the final step. In this case the kinetic term is now an irreducible symmetric second rank tensor, and the antisymmetric tensor field simply an irreducible antisymmetric second rank tensor, represented as $\ydiagram{2}$ and $\ydiagram{1,1}$, respectively[14]. The counting needed is thus to identify $O(3,1)$ invariants in the reduction of the $GL(4)$ characters

$$\ydiagram{2}\cdot\ydiagram{1,1}^{\underline{\otimes}(2)} \quad \text{and} \quad \ydiagram{2}\cdot\ydiagram{1,1}^{\underline{\otimes}(1^2)},$$

respectively. Given that

$$\ydiagram{1,1}^{\underline{\otimes}(2)} = \ydiagram{1,1,1,1} + \ydiagram{2,2}, \quad \text{and} \quad \ydiagram{1,1}^{\underline{\otimes}(1^2)} = \ydiagram{2,1,1},$$

we look for orthogonal group[15] invariants in the general linear group products[16]

$$\begin{aligned} &\ydiagram{2}.\ydiagram{2,2} + \ydiagram{2}.\ydiagram{1,1,1,1} = \ydiagram{4,2} + \ydiagram{3,2,1} + \ydiagram{2,2,2} + \ydiagram{3,1,1,1} \\ &\text{and} \quad \ydiagram{2}.\ydiagram{2,1,1} = \ydiagram{4,1,1} + \ydiagram{3,2,1} + \ydiagram{3,1,1,1} + \ydiagram{2,2,1,1}. \end{aligned}$$

Reduction to the orthogonal group requires diagram manipulation equivalent to removing all possible contractions with the metric tensor. In symmetric coupling we have firstly

$$\begin{aligned} &\ydiagram{4,2} \Rightarrow \ydiagram{4,2} + \ydiagram{4} + \ydiagram{3,1} + \ydiagram{2,2} + 2.\ydiagram{2} + \bullet, \\ &(126 = 42 + 25 + 30 + 10 + 18 + 1); \\ &\ydiagram{2,2,2} \Rightarrow \ydiagram{2,2,2} + \ydiagram{2,2} + \ydiagram{2} + \bullet \Rightarrow -\ydiagram{2,2} + \ydiagram{2,2} + \ydiagram{2} + \bullet, \\ &(10 = 9 + 1), \end{aligned}$$

where '$\bullet$' stands for the scalar (one dimensional) representation, and dimension checks have been included. Note that the second expansion contains a non-standard orthogonal group diagram (with more than two rows) which modifies to the negative of an irreducible character[17].

14 The computation is aided visually by use of Young diagram notation for tensor manipulation (see for example [39]).

15 We use combinatorial character manipulation methods with $O(4)$ diagrams being a proxy for (finite dimensional) tensor reps. We have $\ydiagram{1} \cong \underline{4}$, $\ydiagram{1,1} \cong \underline{6}$. Under $O(3,1) \downarrow SO(3,1)$, two-rowed diagrams reduce to inequivalent irreducible parts, viz. $\ydiagram{1,1} \cong \ydiagram{1,1}_+ + \ydiagram{1,1}_-$ (self-dual and anti self-dual tensors, respectively). For $\ydiagram{3,2} \cong \ydiagram{3,2}_+ + \ydiagram{3,2}_-$ etc with row lengths $[\ell_1, \ell_2]$, the rep is irreducible if $\ell_2 = 0$ and reduces to $SO(3,1)$ $(\frac{1}{2}\ell_1, \frac{1}{2}\ell_1)$, and for $\ell_1 \geqslant \ell_2 > 0$ the decomposition is $(\frac{1}{2}(\ell_1+\ell_2), \frac{1}{2}(\ell_1-\ell_2)) + (\frac{1}{2}(\ell_1-\ell_2), \frac{1}{2}(\ell_1+\ell_2))$.

16 Diagrams with five or more rows have been removed because the corresponding $GL(4)$ characters are zero.

17 If there are $2+h$ rows the modification is by removal of boundary strip of length $2h$, starting with the offending rows, with accompanying sign reversal, or change to an associated representation ('*'), or removal if an improper diagram results. For notation and methods see [40, 41].

From the above diagrams it is of note that one $GL(4)$ term occurs in both symmetric and antisymmetric coupling, corresponding to two distinct (linearly independent) couplings. The orthogonal reduction follows

$$\ydiagram{3,1,1,1} \Rightarrow \ydiagram{3,1,1,1} + \ydiagram{1,1,1,1} + \ydiagram{2,1,1} \Rightarrow \quad \bullet^* + \ydiagram{2}^*$$
$$(10 = 1 + 9)\,,$$

where the modification rules have been applied (with removal of a vanishing character). This term therefore supplies the pseudo-scalar candidate $\bullet^*$ in both coupling symmetries.

Verifying that the remaining $GL(4)$ terms do not contain further scalars or pseudoscalars completes the accounting of kinetic terms admissible under the full Lorentz group, with the results (cf equation (16) and table 1):

$GL(4)$	$\ydiagram{2} \cdot \ydiagram{2}^{\underline{\otimes}(2)}$	$\ydiagram{2} \cdot \ydiagram{1,1}^{\underline{\otimes}(1^2)}$
$O(3,1)$ scalar	2	0
$O(3,1)$ pseudo-scalar	1	1

(17)

A.2. Potential energy

The count of admissible local invariant interaction terms for the complex scalar and self-dual antisymmetric tensor fields (section 5) by group methods requires working with expansions of symmetrized powers of the characters representing them. Here we provide more details for the quartic couplings (the scalar quartic is the square of the quadratic mass term). As already noted, the the requirement to have only monomials with net vanishing $U(1)$ charge considerably restricts the count. For example at quartic degree, the only possible contributions from the antisymmetric tensor field come from the nine terms in

$$\begin{aligned}[(1,0)\cdot\ydiagram{1}]^{\underline{\otimes}(2)} \cdot [(1,0)\cdot\overline{\ydiagram{1}}]^{\underline{\otimes}(2)} &= [(0+2,0)\cdot\ydiagram{2} + (1,0)\cdot\ydiagram{1,1}] \cdot \left[(0,0+2)\cdot\overline{\ydiagram{2}} + (0,1)\cdot\overline{\ydiagram{1,1}}\right] \\ &= (0+2,0+2)\cdot\ydiagram{2}\cdot\overline{\ydiagram{2}} + (0+2,1)\cdot\ydiagram{2}\cdot\overline{\ydiagram{1,1}} + (1,0+2)\cdot\overline{\ydiagram{2}}\cdot\ydiagram{1,1} + (1,1)\ydiagram{1,1}\cdot\overline{\ydiagram{1,1}}.\end{aligned}$$

wherein the $(0,0)\cdot\ydiagram{2}\cdot\overline{\ydiagram{2}}$ term contains a unique one dimensional representation[18]. On the other hand for the mixed quadratic scalar-tensor term we have

$$[\ydiagram{2} + \ydiagram{1}\cdot\overline{\ydiagram{1}} + \overline{\ydiagram{2}}] \cdot [(0+2,0)\cdot\ydiagram{2} + (1,1)\cdot\ydiagram{1}\cdot\overline{\ydiagram{1}} + (0,0+2)\overline{\ydiagram{2}}]$$

wherein two invariants will come from the expansions of

$$\ydiagram{2}\cdot(0,0)\cdot\overline{\ydiagram{2}} \quad \text{and} \quad \overline{\ydiagram{2}}\cdot(0,0)\cdot\ydiagram{2}.$$

Note that the present count (table 4) holds for internal symmetry $U(n) \cong SU(n) \times U(1)$ which includes the standard model $SU(2) \times U(1)$ and also the Abelian case of complex fields.

[18] Compound diagrams provide a succinct way to handle contragradient representations [41].

Appendix B. The DKP algebra

For completeness, we give explicit representations of the Kemmer β-matrices for (in 1+3+1 block form) the 5×5:

$$\beta^0 = \left[\begin{array}{c|c|c} \cdot & \cdot & 1 \\ \hline \cdot & \cdot & \cdot \\ \hline 1 & \cdot & \cdot \end{array}\right], \quad \beta^i = \left[\begin{array}{c|c|c} \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & e_i \\ \hline \cdot & -f_i & \cdot \end{array}\right];$$

and (in 3+3+3+1 block form) 10×10:

$$\beta^0 = \left[\begin{array}{ccc|c} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array}\right], \quad \beta^i = \left[\begin{array}{ccc|c} \cdot & \cdot & \cdot & \mathtt{a}e_i \\ \cdot & \cdot & \mathtt{b}\ell_i & \cdot \\ \cdot & \mathtt{b}'\ell_i & \cdot & \cdot \\ \hline \mathtt{a}'f_i & \cdot & \cdot & \cdot \end{array}\right];$$

representations. Here e_i, $i=1,2,3$ are standard 3-component unit column vectors, $f_i = {}^{\top}e_i$ the corresponding row vectors, and $\ell_i := \frac{1}{2}\varepsilon_{ijk}\ell_{jk}$ elementary 3×3 rotation generators[19]. Examination of the form of the $SO(3,1)$ rotation generators $[\beta_i,\beta_j]$, the boost generators $[\beta_0,\beta_i]$, as well as algebra identities such as $\{\beta^j,(\beta^i)^2\} = -\beta^j$, $\{(\beta^i)^2,(\beta^0)\} = -\beta^0$ yields $\mathtt{aa}' = -1$, $\mathtt{bb}' = +1$. Hereafter we choose $\mathtt{a} = \mathtt{b} = +1$.

For the pseudo-scalar $\beta_5 := \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\beta_\mu\beta_\nu\beta_\rho\beta_\sigma$ direct evaluation gives after normalization

$$\beta_5 = \left[\begin{array}{ccc|c} \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array}\right],$$

from which we derive the 10-component twisted $\check{\beta}_\mu = [\beta_5,\beta_\mu]$ matrices,

$$\check{\beta}^0 = \left[\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array}\right]; \quad \check{\beta}^i = \left[\begin{array}{ccc|c} \cdot & \cdot & \ell_i & \cdot \\ \cdot & \cdot & \cdot & -e_i \\ \ell_i & \cdot & \cdot & \cdot \\ \hline \cdot & f_i & \cdot & \cdot \end{array}\right].$$

The parity operators are defined

$$\begin{aligned} \eta &:= 2\beta_0^2 - 1 = \mathrm{diag}\,(1,-1,1,-1)\,, \\ \eta' &:= -\left(1+2\beta_5^2\right)\eta = \mathrm{diag}\,(1,-1,-1,1)\,, \\ \text{with}\quad \eta\beta^{\mu\top}\eta &= +\beta^\mu = -\eta'\beta^{\mu\top}\eta'\,, \quad \eta'\check{\beta}^{\mu\top}\eta' = +\check{\beta}^\mu = -\eta\check{\beta}^{\mu\top}\eta\,, \end{aligned} \tag{18}$$

from which we define $\overline{\Phi} := \Phi^\dagger\eta$ and $\overline{\overline{\Phi}} := \Phi^\dagger\eta'$. Sundry properties which we shall use are

$$\begin{aligned} \left\{\beta_5^2,\beta^\mu\right\} &= -\beta^\mu\,, \quad \left\{\beta_5^2,\check{\beta}^\mu\right\} = -\check{\beta}^\mu\,; \\ \beta_\mu\beta^\mu &= 3+\beta_5^2\,, \quad \check{\beta}_\mu\beta^\mu = 2\beta_5^2\,, \\ (\beta_5)^3 &= -\beta_5\,, \quad \beta_\mu\beta^\sigma\beta^\mu = 2\beta^\sigma\,. \end{aligned} \tag{19}$$

[19] Where $\ell_{jk} := e_{jk} - e_{kj} = e_j\otimes f_k - e_k\otimes f_j$, with commutation relations $[\ell_i,\ell_j] = -\varepsilon_{ijk}\ell_k$ and matrix elements $(\ell_i)_{jk} = \varepsilon_{ijk}$, and for example $\ell_i e_i = 0$, $(\ell_i)^2 = e_{ii} - 1$.

Appendix C. Extended DKP formalism

C.1. Proca massive vector field

As a case study, and to illustrate the reduction of first order DKP actions to standard forms, we reproduce here the standard treatment for the derivation of the Proca (complex massive spin-1) system. In an obvious notation we have

$$\overline{\Phi}\left(\mathrm{i}\beta^{\mu}\partial_{\mu}\Phi\right) - m\overline{\Phi}\Phi = \left(\underline{\overline{E}}, -\underline{\overline{B}}, \underline{\overline{A}}, -A_0\right)\left[\begin{array}{ccc|c} -m & \cdot & \partial_0 & \underline{\nabla} \\ \cdot & -m & \underline{\ell}\cdot\underline{\nabla} & \cdot \\ \partial_0 & \underline{\ell}\cdot\underline{\nabla} & -m & \cdot \\ \hline -\underline{\nabla}\cdot & \cdot & \cdot & -m \end{array}\right]\left(\begin{array}{c} \underline{E} \\ \underline{B} \\ \underline{A} \\ A_0 \end{array}\right).$$

$$= \underline{\overline{E}}\cdot\left(\mathrm{i}\partial_0\underline{A} + \mathrm{i}\underline{\nabla}A_0\right) + \underline{E}\cdot\left(-\mathrm{i}\partial_0\underline{\overline{A}} - \mathrm{i}\underline{\nabla}\overline{A}_0\right) + \underline{\overline{B}}\cdot\left(\mathrm{i}\underline{\nabla}\times\underline{A}\right) + \underline{B}\cdot\left(-\mathrm{i}\underline{\nabla}\times\underline{\overline{A}}\right) +$$
$$- m\left(\underline{\overline{E}}\cdot\underline{E} - \underline{\overline{B}}\cdot\underline{B}\right) - m\left(\underline{\overline{A}}\cdot\underline{A} - \overline{A}_0A_0\right)$$

The expansion in three-vector calculus involving $\underline{E}, \underline{B}$ and $\underline{A}$ (with scalar A_0) and complex conjugates follows using matrix forms such as $(\underline{\ell}\cdot\underline{\nabla}) \Rightarrow -\underline{\nabla}\times$ and use of partial integration. After eliminating auxiliary fields $\underline{E}, \underline{B}$ using their equations of motion we find

$$\overline{\Phi}\left(\mathrm{i}\beta^{\mu}\partial_{\mu}\Phi\right) - m\overline{\Phi}\Phi \Rightarrow +\frac{1}{m}\left(\partial_0\underline{\overline{A}} + \underline{\nabla}\overline{A}_0\right)\cdot\left(\partial_0\underline{A} + \underline{\nabla}A_0\right) - \frac{1}{m}\left(\underline{\nabla}\times\underline{\overline{A}}\right)$$
$$\cdot\left(\underline{\nabla}\times\underline{A}\right) + m\left(\overline{A}_0A_0 - \underline{\overline{A}}\cdot\underline{A}\right)$$

which after rescaling $A_0 \to -\sqrt{m}A_0$, $\underline{A} \to \sqrt{m}\underline{A}$ accords with the free action for a complex vector field of mass m,

$$\mathscr{L}_{Proca} = -\tfrac{1}{2}\overline{F}^{\mu\nu}F_{\mu\nu} + m^2\overline{A}^{\mu}A_{\mu}$$

with field strength $F_{\mu\nu} = \partial_{\mu}A_{\mu} - \partial_{\nu}A_{\mu}$ and the standard identification of electric and magnetic fields[20] $\underline{E}, \underline{B}$.

C.2. Two-form gauge potentials and antisymmetric tensor fields

In the sequel we adopt the following notation for fields in the ten component wave-function of the modified DKP systems:

$$\Phi = {}^{\top}\left(\underline{S}, \underline{R}, \underline{G}, G_0\right), \quad \overline{\Phi} = \left(\underline{\overline{S}}, -\underline{\overline{R}}, \underline{\overline{G}}, -\overline{G}_0\right), \quad \overline{\overline{\Phi}} = \left(\underline{\overline{S}}, -\underline{\overline{R}}, -\underline{\overline{G}}, +\overline{G}_0\right), \tag{20}$$

together with $\Phi^c := {}^{\top}\Phi\eta$, in which we identify the three-vector $\underline{S} = (Z_{01}, Z_{02}, Z_{03})$, and pseudo-vectors $\underline{R} = (Z_{23}, Z_{31}, Z_{12})$ as putative 'electric' and 'magnetic' parts of an antisymmetric tensor $Z_{\mu\nu}$, the Hodge dual of whose three-form curl (in the twisted case) becomes the axial vector field strength $G_{\mu} \cong (G_0, \underline{G})$ consisting of a pseudo-vector together with a pseudoscalars. The allocations in (20) to the components of $Z_{\mu\nu}$ in order to reassemble the DKP-derived actions as kinetic terms in local relativistic invariants in standard tensor notation, follow after field rescaling.

[20] Elimination of auxiliary fields amounts to changing the normalization of the generating function by a functional Gaussian. From the above expansion, it is evident that elimination of $A_0, \underline{A}$ instead of $\underline{E}, \underline{B}$ would lead to a 'dual' model equivalent to the Proca theory, at least at the non-interacting level.

Case 1: two-form gauge field [4, 5]:
Following the above discussion we consider

$$\begin{aligned}&\overline{\overline{\Phi}}\mathrm{i}\breve{\beta}^{\mu}\partial_{\mu}\Phi+m\overline{\overline{\Phi}}\left(1+\beta_5^2\right)\Phi\\&\quad=\overline{\underline{G}}\cdot(\mathrm{i}\partial_0\underline{R}+\mathrm{i}\underline{\nabla}\times\underline{S})+\overline{G}_0\left(-\mathrm{i}\underline{\nabla}\cdot\underline{R}\right)+\underline{G}\cdot\left(-\mathrm{i}\partial_0\overline{\underline{R}}-\mathrm{i}\underline{\nabla}\times\overline{\underline{S}}\right)\\&\qquad+G_0\left(+\mathrm{i}\underline{\nabla}\cdot\overline{\underline{R}}\right)+m\left(\overline{G}_0G_0-\overline{\underline{G}}\dot{\underline{G}}\right)\\&\quad\Rightarrow\frac{1}{m}\left(\left(\partial_0\overline{\underline{R}}+\underline{\nabla}\times\overline{\underline{S}}\right)\cdot\left(\partial_0\underline{R}+\underline{\nabla}\times\underline{S}\right)-\left(\underline{\nabla}\cdot\overline{\underline{R}}\right)\left(\underline{\nabla}\cdot\underline{R}\right)\right)\end{aligned}$$

so finally with $\underline{R},\underline{S}\to-\sqrt{m}\underline{R},\sqrt{m}\underline{S}$,

$$\overline{\overline{\Phi}}\mathrm{i}\breve{\beta}^{\mu}\partial_{\mu}\Phi+m\overline{\overline{\Phi}}\left(1+\beta_5^2\right)\Phi\Rightarrow\left(\left(-\underline{\nabla}\cdot\overline{\underline{R}}\right)\left(-\underline{\nabla}\cdot\underline{R}\right)-\left(-\partial_0\overline{\underline{R}}+\underline{\nabla}\times\overline{\underline{S}}\right)\cdot\left(-\partial_0\underline{R}+\underline{\nabla}\times\underline{S}\right)\right)$$

Here the four-vector components of Φ have again been eliminated using their equations of motion. The action is proportional to the (complex) Lorentz invariant length, when re-expressed in terms of derivatives of $\underline{S},\underline{R}$. We find

$$\begin{aligned}&\tfrac{1}{4}\mathscr{L}^{KR}=\overline{\overline{\Phi}}\mathrm{i}\breve{\beta}^{\mu}\partial_{\mu}\Phi-m\overline{\overline{\Phi}}\left(1+\beta_5^2\right)\Phi\Rightarrow\overline{G}^{\mu}G_{\mu}\\&\text{where}\quad G_{\mu}:=\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^{\nu}Z^{\rho\sigma}\\&\text{with}\qquad G_0=-\underline{\nabla}\cdot\underline{R},\quad\underline{G}=-\partial_0\underline{R}+\underline{\nabla}\times\underline{S}.\end{aligned}\tag{21}$$

□

As pointed out in section 2, under the special Lorentz group, an antisymmetric tensor field $Z_{\mu\nu}$ can be projected into two irreducible, so-called self-dual and anti self-dual components, using the Hodge dual

$$^{*}Z_{\mu\nu}:=\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}Z^{\rho\sigma}=\pm\mathrm{i}Z_{\mu\nu}\,.$$

Individual terms in local actions are no longer separately fixed under parity, but the contributions of the (anti) self-dual components to the total count of invariants can easily be inferred from equations (16) and (17) above. For imposing duality restrictions on the DKP ten-component wave-function, it is evident that the pseudo-scalar β_5 and Hodge dual projection (both of whose eigenvalues are $\pm$i) play analogous roles, and indeed

$$^{*}Z_{\mu\nu}:=\pm\mathrm{i}Z_{\mu\nu}\quad\Leftrightarrow\beta_5\Phi^{\wedge}=\pm\mathrm{i}\Phi^{\wedge}\tag{22}$$

where $\Phi^{\wedge}$ is the invariant projection via $\mathbb{P}^{\wedge}:=-\beta_5^2$ on to the upper (antisymmetric tensor) components[21] of Φ. In terms of the components in equation (20) we find correspondingly $\underline{R}=\pm\mathrm{i}\underline{S}$.

Note that, for a self-dual or anti self-dual wave function with $\beta_5(\beta_5^2)\Phi=\pm\mathrm{i}\beta_5^2\Phi$, we have[22]

$$\begin{aligned}\overline{\Phi}\beta_5\Phi&=-\Phi^{\dagger}\eta\beta_5\left(\beta_5^2\right)\Phi=\mp\mathrm{i}\Phi^{\dagger}\eta\left(\beta_5^2\right)\Phi\\&\equiv+\Phi^{\dagger}\left(\beta_5\right)^2\beta_5\eta\Phi=\pm\mathrm{i}\Phi^{\dagger}\eta\left(\beta_5\right)^2\Phi\,,\end{aligned}$$

[21] With $\mathbb{P}^{\vee}:=1-\mathbb{P}^{\wedge}=1+\beta_5^2$ the lower projection.
[22] Note $\beta_5^3=-\beta_5$ and $\{\eta,\beta_5\}=0$.

whence $\overline{\Phi}\beta_5\Phi = 0 = \overline{\Phi}\beta_5^2\Phi$, or equivalently, $\overline{\Phi}\Phi \equiv \overline{\Phi}(1+\beta_5^2)\Phi$, expressing the fact that only the projection onto the vector part is present, as is evident from the following explicit calculations.

Case 2: antisymmetric tensor field [8, 9]:
Take for example a complex anti self-dual DKP wave-function ($^*Z_{\mu\nu} = -\mathrm{i}Z_{\mu\nu},\ \beta_5\Phi^\wedge = -\mathrm{i}\Phi^\wedge$) with

$$\Phi = {}^\top\left(\underline{S}, -i\,\underline{S}, +\underline{G}, G_0\right), \quad \overline{\Phi} = \Phi^\dagger\eta = \left(\underline{\overline{S}}, -\mathrm{i}\underline{\overline{S}}, +\underline{\overline{G}}, -\overline{G}_0\right).$$

Consider the action

$$\mathscr{L}^{AC} := \overline{\Phi}\check{\beta}^\mu \tfrac{1}{2}\overleftrightarrow{\partial_\mu}\Phi + m\overline{\Phi}\Phi\,. \tag{23}$$

We have (see appendix C.1 for details)

$$\begin{aligned}\mathscr{L}^{AC} = \tfrac{1}{2}\Big(&\underline{\overline{S}}\cdot(-\underline{\nabla}\times\underline{G}) + \left(-\mathrm{i}\underline{\overline{S}}\right)\cdot(-\partial_0\underline{G}+\underline{\nabla}G_0) + \underline{\overline{G}}\cdot(+\mathrm{i}\partial_0\underline{S}-\underline{\nabla}\times\underline{S}) - \overline{G}_0\,(\mathrm{i}\underline{\nabla}\cdot\underline{S})\\ &+\underline{S}\cdot\left(-\underline{\nabla}\times\underline{\overline{G}}\right) + (+\mathrm{i}\underline{S})\cdot\left(-\partial_0\underline{\overline{G}}+\underline{\nabla}\overline{G}_0\right) + \underline{G}\cdot\left(-\mathrm{i}\partial_0\underline{\overline{S}}-\underline{\nabla}\times\underline{\overline{S}}\right) - G_0\left(-\mathrm{i}\underline{\nabla}\cdot\underline{\overline{S}}\right)\Big)\\ &+m\left(\underline{\overline{G}}\cdot\underline{G}-\overline{G}_0G_0\right)\\ \Rightarrow\ &\frac{1}{m}\left(\left(\underline{\nabla}\cdot\underline{\overline{S}}\right)\left(\underline{\nabla}\cdot\underline{S}\right) - \left(\partial_0\underline{\overline{S}}-\mathrm{i}\underline{\nabla}\times\underline{\overline{S}}\right)\cdot\left(\partial_0\underline{S}+\mathrm{i}\underline{\nabla}\times\underline{S}\right)\right)\end{aligned}$$

after elimination. With the re-scaling $\underline{S}\to\sqrt{m}\underline{S}$ we have

$$\begin{aligned}&\mathscr{L}^{AC} = \overline{\Phi}\check{\beta}^\mu\tfrac{1}{2}\overleftrightarrow{\partial_\mu}\Phi + m\overline{\Phi}\Phi \Rightarrow \partial^\rho\overline{Z}_{\rho\mu}\partial_\sigma Z^{\sigma\mu} = \overline{G}^\mu G_\mu\,,\\ \text{where}\quad &G_\mu := \partial^\rho Z_{\rho\mu}\,,\\ \text{with}\quad &G_0 = \underline{\nabla}\cdot\underline{G},\quad \underline{G} = \partial_0\underline{S}+\mathrm{i}\underline{\nabla}\times\underline{S}.\end{aligned} \tag{24}$$

in accord with the (complex) Lorentz invariant length of the gradient of the (anti) self-dual complex antisymmetric tensor field [10, 42]. □

Case 3: antisymmetric tensor field (antisymmetric coupling) [12, 18]:
The final invariant coupling identified in section 2 above arises from a scenario in which there is an internal symmetry (possibly local) conferring an antisymmetric bilinear invariant. Note that in the standard DKP Proca formalism a kinetic term such as $\Phi^\top\eta\beta^\mu\partial_\mu\Phi$ becomes a total derivative in view of the symmetry of $\eta\beta^\mu$; in the complex self-conjugate form $\Phi^\dagger\eta\mathrm{i}\beta^\mu\partial_\mu\Phi \equiv \overline{\Phi}\mathrm{i}\beta^\mu\partial_\mu\Phi$, this is of course averted. Appending an additional internal index $\Phi^{\mathtt{a}}$, and antisymmetric bilinear invariant $\kappa_{\mathtt{ab}} = -\kappa_{\mathtt{ba}}$ however, again with an anti self-dual DKP wave-function, permits the alternative action[23] in terms of $\Phi^c := \Phi^\top\eta$,

$$\mathscr{L}^{CP} := \Phi^{c\mathtt{a}}\left(\beta^\mu\partial_\mu\Phi^{\mathtt{b}}\kappa_{\mathtt{ab}}\right) + \mathrm{h.c.} - \mathrm{i}m\overline{\Phi}^{\mathtt{a}}\Phi^{\mathtt{b}}\kappa_{\mathtt{ab}}\,. \tag{25}$$

[23] The possible combination $\eta'\check{\beta}^\mu$ also has the correct ingredients for a variant $\Phi^\top\eta'\check{\beta}^\mu\partial_\mu\Phi$.

Expanding components in terms of three-vector notation as above,

$$\begin{aligned}\mathscr{L}^{CP} :=& 2\underline{G}^{\mathtt{a}}\cdot\left(\partial_0\underline{S}^{\mathtt{b}}+\mathrm{i}\underline{\nabla}\times\underline{S}^{\mathtt{b}}\right)\kappa_{\mathtt{ab}}+2G_0^{\mathtt{a}}\left(\underline{\nabla}\cdot\underline{S}^{\mathtt{b}}\right)\kappa_{\mathtt{ab}}+2\underline{\overline{G}}^{\mathtt{a}}\cdot\left(\partial_0\underline{\overline{S}}^{\mathtt{b}}-\mathrm{i}\underline{\nabla}\times\underline{\overline{S}}^{\mathtt{b}}\right)\kappa_{\mathtt{ab}}\\ &+2\overline{G}_0^{\mathtt{a}}\left(\underline{\nabla}\cdot\underline{\overline{S}}^{\mathtt{b}}\right)\kappa_{\mathtt{ab}}+\mathrm{i}m\left(\overline{G}_0^{\mathtt{a}}G_0^{\mathtt{b}}-\underline{\overline{G}}^{\mathtt{a}}\cdot\underline{G}^{\mathtt{b}}\right)\kappa_{\mathtt{ab}}\\ &\Rightarrow-\frac{4}{\mathrm{i}m}\left(\underline{\nabla}\cdot\underline{\overline{S}}^{\mathtt{a}}\right)\left(\underline{\nabla}\cdot\underline{S}^{\mathtt{b}}\right)\kappa_{\mathtt{ab}}+\frac{4}{\mathrm{i}m}\left(\partial_0\underline{\overline{S}}^{\mathtt{a}}-\mathrm{i}\underline{\nabla}\times\underline{\overline{S}}^{\mathtt{a}}\right)\cdot\left(\partial_0\underline{S}^{\mathtt{b}}+\mathrm{i}\underline{\nabla}\times\underline{S}^{\mathtt{b}}\right)\kappa_{\mathtt{ab}},\end{aligned}$$

again after elimination. With the field redefinition $\underline{S}\rightarrow\frac{1}{2}\mathrm{i}\sqrt{m}\underline{S}$, we have finally

$$\begin{aligned}\mathscr{L}^{CP} :=& \Phi^{c\mathtt{a}}\left(\beta^\mu\partial_\mu\Phi^{\mathtt{b}}\kappa_{\mathtt{ab}}\right)+\text{h.c.}-\mathrm{i}m\overline{\Phi}^{\mathtt{a}}\Phi^{\mathtt{b}}\kappa_{\mathtt{ab}}\Rightarrow\mathrm{i}\partial^\rho\overline{Z}^{\mathtt{a}}_{\rho\mu}\partial_\sigma Z^{\mathtt{b}\sigma\mu}\kappa_{\mathtt{ab}}\\ =& \mathrm{i}\overline{G}^{\mathtt{a}\mu}G^b_\mu\kappa_{\mathtt{ab}},\quad\text{where}\quad G^{\mathtt{a}}_\mu:=\partial^\rho Z^{\mathtt{a}}_{\rho\mu},\quad\text{with}\\ G_0^{\mathtt{a}}=&\underline{\nabla}\cdot\underline{S},\quad\underline{G}^{\mathtt{a}}=\partial_0\underline{S}^{\mathtt{a}}+\mathrm{i}\underline{\nabla}\times\underline{S}^{\mathtt{a}}.\end{aligned}\tag{26}$$

□

Appendix D. Conformal invariance

To complete the discussion of symmetry aspects of the physical antisymmetric tensor models, reviewed in section 3 above, we here provide a demonstration of the conformal invariance both of the antisymmetrically coupled pseudoscalar $\mathscr{L}^{CP}$ (equation (8)) here using its first order four dimensional DKP formulation (table 3 and equation (25)). A similar calculation obtains for the symmetrically coupled scalar $\mathscr{L}^{AC}$ (table 3 and equations (5), (24)).

We adapt to the present case the analysis of Jackiw and Pi [43], which gave a convenient criterion for verifying conformal invariance in a scale-invariant system. In the first order formulations however it is necessary to adjust for the fact that the scaling behaviour of the 10 component DKP wave function (table 2) is no longer a diagonal multiplier, but must be taken as $\mathtt{D}:=\mathrm{diag}(1,1,2,2)$ in 3+3+3+1 block form[24]. Given that $\beta_5^2=\mathrm{diag}(-1,-1,0,0)$, other Lorentz invariants can be written similarly as

$$\beta_\mu\beta^\mu=\mathrm{diag}(2,2,3,3)=\left(3+\beta_5^2\right),\quad\mathtt{D}=\left(2+\beta_5^2\right),$$

with the projectors $\mathsf{Id}=(1+\beta_5^2)+(-\beta_5^2)$.

Checking the scale invariance of the first order Lagrangian (25), we have[25]

$$\begin{aligned}\delta\left(\overline{\Phi}\left(1+\beta_5^2\right)\Phi\right)&==\overline{\Phi}\left\{\mathtt{D},\left(1+\beta_5^2\right)\right\}\Phi=\overline{\Phi}\left\{\left(2+\beta_5^2\right),\left(1+\beta_5^2\right)\right\}\Phi=4\left(\overline{\Phi}\left(1+\beta_5^2\right)\Phi\right),\\ \delta\left(\Phi^c\beta^\mu\partial_\mu\Phi\right)&=\left(\Phi^c\beta^\mu\partial_\mu\Phi\right)+\left(\Phi^c\left\{\beta^\mu,\left(2+\beta_5^2\right)\right\}\partial_\mu\Phi\right)\\ &=5\left(\Phi^c\beta^\mu\partial_\mu\Phi\right)+\left(\Phi^c\left\{\beta^\mu,\beta_5^2\right\}\partial_\mu\Phi\right)=4\left(\Phi^c\beta^\mu\partial_\mu\Phi\right).\end{aligned}$$

The criterion of Jackiw and Pi [43] for conformal invariance is that the field virial,

$$J^\sigma=\frac{\delta\mathscr{L}}{\delta\partial^\mu\varphi}\left(\mathtt{D}\eta^{\mu\sigma}-\Sigma^{\mu\sigma}\right)\varphi,$$

[24] In the Kemmer presentation of the Proca massive vector field we would have $\mathtt{D}':=\mathrm{diag}(2,2,1,1)$.

[25] Recall that $\overline{\Phi}\Phi=\overline{\Phi}(1+\beta_5^2)\Phi$ for (anti) self-dual wave functions, $\beta_5^4=-\beta_5^2$, and that $\{\beta_5^2,\beta^\sigma\}=-\beta^\sigma$.

is a divergence, $J^\sigma = \partial_\mu W^{\mu\sigma}$. Here the sum is over all fields φ of scaling dimension $\mathtt{D}$, and $\Sigma^{\mu\sigma}$ are the Lorentz group generators applied to the field multiplet φ.

We assume here that the method of [43] simply goes through, now with the scaling dimension being the operator $\mathtt{D}$, and also with use of the reducible representation of the Lorentz group generators, provided by the Kemmer 10×10 matrices. Proceeding with the functional differentiation of (25) we have

$$J^\sigma = \Phi^c \beta_\mu \left(\mathtt{D}\eta^{\mu\sigma} + [\beta^\mu, \beta^\sigma]\right) \Phi$$

where
$$\begin{aligned}\beta_\mu \left(\mathtt{D}\eta^{\mu\sigma} + [\beta^\mu, \beta^\sigma]\right) &= \beta^\sigma \left(2 + \beta_5^2\right) + \beta_\mu \beta^\mu \beta^\sigma - \beta_\mu \beta^\sigma \beta^\mu \\ &= \beta^\sigma \left(2 + \beta_5^2\right) + \left(3 + \beta_5^2\right) \beta^\sigma - 2\beta^\sigma \equiv 3\beta^\sigma + \left\{\beta_5^2, \beta^\sigma\right\} = 2\beta^\sigma\end{aligned}$$

using $\{\beta_5^2, \beta^\sigma\} = -\beta^\sigma$. Hence with antisymmetric coupling

$$J^\sigma = 2\kappa_{\mathtt{ab}} \left(\Phi^{\mathtt{a}}\right)^\top \eta\beta^\sigma \Phi^{\mathtt{b}} + h.c. \equiv 0 + 0$$

because $(\eta\beta^\sigma)$ is symmetric and $\kappa_{\mathtt{ab}}$ is antisymmetric.

A similar computation applies to the (symmetric) Avdeev–Chizhov Lagrangian (24) for scaling invariance, as $\{\beta_5^2, \check{\beta}^\sigma\} = -\check{\beta}^\sigma$. A detailed expansion shows that the field virial is now proportional to $\overline{\Phi}\check{\beta}^\sigma \Phi + h.c.$, which again vanishes, but now due to the *antisymmetry* of $(\eta\check{\beta}^\sigma)$.

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References

[1] Dirac P A M 1936 Relativistic wave equations *Proc. R. Soc.* A **155** 447–59
[2] Proca A L 1936 Sur la théorie ondulatoire des électrons positifs et négatifs *J. Phys. Radium* **7** 347–53
[3] Fierz M and Ernst Pauli W 1939 On relativistic wave equations for particles of arbitrary spin in an electromagnetic field *Proc. R. Soc.* A **173** 211–32
[4] Ogievetsky V I and Polubarinov I V 1967 The notoph and its possible interactions *Yad. Fiz.* **4** 216–23
[5] Kalb M and Ramond P 1974 Classical direct interstring action *Phys. Rev.* D **9** 2273–84
[6] Blau M and Thompson G 1991 Topological gauge theories of antisymmetric tensor fields *Ann. Phys., NY* **205** 130–72
[7] Hata H, Kugo T and Ohta N 1981 Skew symmetric tensor gauge field theory dynamically realized in QCD $U(1)$ channel *Nucl. Phys.* B **178** 527
[8] Avdeev L V and Chizhov M V 1994 Antisymmetric tensor matter fields. An Abelian model *Phys. Lett.* B **321** 212–18
[9] Chizhov M V 2011 Theory and phenomenology of spin-1 chiral particles *Phys. Part. Nuc.* **42** 93–183
[10] Lemes V, Renan R and Sorella S P 1995 ϕ_4^4-theory for antisymmetric tensor matter fields in Minkowski space-time *Phys. Lett.* B **352** 37–42
[11] Wetterich C 2008 Quantization of chiral antisymmetric tensor fields *Int. J. Mod. Phys.* A **23** 1545–79
[12] Thierry-Mieg J and Jarvis P D 2021 SU(2/1) superchiral self-duality: a new quantum, algebraic and geometric paradigm to describe the electroweak interactions *J. High Energy Phys.* JHEP04(2021)001
[13] Duffin R J 1938 On the characteristic matrices of covariant systems *Phys. Rev.* **54** 1114
[14] Kemmer N 1939 The particle aspect of meson theory *Proc. R. Soc.* A **173** 91–116

[15] Petiau G 1936 *Thesis* University of Paris (Published in: Acad. Roy. de Belg., Classe sci., Mem in 8o 16, no. 2 (1936))
[16] Thierry-Mieg J 2020 Scalar anomaly cancellation reveals the hidden superalgebraic structure of the quantum chiral SU(2/1) model of leptons and quarks *J. High Energy Phys.* JHEP10(2020)167
[17] Thierry-Mieg J 2021 Chirality, a new key for the definition of the connection and curvature of a Lie-Kac superalgebra *J. High Energy Phys.* JHEP01(2021)111
[18] Bekaert X and Boulanger N 2007 Tensor gauge fields in arbitrary representations of $GL(d)$. II. Quadratic actions *Commun. Math. Phys.* **271** 723–73
[19] Bateman H 1910 The transformation of the electrodynamical equations *Proc. London Math. Soc.* **2** 223–64
[20] Thierry-Mieg J and Jarvis P D 2023 Conformal invariance of antisymmetric tensor field theories in any even dimension *J. Phys. A.* submitted (arXiv:2311.01701v2 [hep-th])
[21] Krajcik R A and Nieto M M 1977 Historical development of the Bhabha first-order relativistic wave equations for arbitrary spin *Am. J. Phys.* **45** 818–22
[22] Nedjadi Y and Barrett R C 1993 On the properties of the Duffin-Kemmer-Petiau equation *J. Phys. G: Nucl. Part. Phys.* **19** 87–98
[23] Nedjadi Y and Barrett R C 1994 Solution of the central field problem for a Duffin–Kemmer–Petiau vector boson *J. Math. Phys.* **35** 4517–33
[24] Kruglov S I 2010 Kalb–Ramond fields in the Petiau-Duffin-Kemmer formalism and scale invariance *Mod. Phys. Lett.* A **25** 2745–51
[25] Kruglov S I 2011 Field theory of massive and massless vector particles in the Duffin-Kemmer-Petiau formalism *Int. J. Mod. Phys.* A **26** 2487–501
[26] Beltran J, Pimentel B M and Soto D E 2020 The causal approach proof for the equivalence of $SDKP_4$ and $SQED_4$ at tree-level *Mod. Phys.* A **35** 2050042
[27] Aste A 2012 Complex representation theory of the electromagnetic field *J. Geom. Symmetry Phys.* **28** 47–58
[28] Silberstein L 1907 Elektromagnetische Grundgleichungen in bivektorieller Behandlung *Ann. Phys., Lpz.* **327** 579–86
[29] Weinberg S 1964 Feynman rules for any spin *Phys. Rev.* **133** B1318
[30] Weinberg S 1964 Feynman rules for any spin. II. Massless particles *Phys. Rev.* **134** B882
[31] Ne'eman Y 1979 Irreducible gauge theory of a consolidated Salam-Weinberg model *Phys. Lett.* B **81** 190–4
[32] Fairlie D B 1979 Higgs fields and the determination of the Weinberg angle *Phys. Lett.* B **82** 97–100
[33] Dondi P H and Jarvis P D 1979 A supersymmetric Weinberg-Salam model *Phys. Lett.* B **84** 75–78
[34] Ne'eman Y and Thierry-Mieg J 1980 Geometrical gauge theory of ghost and Goldstone fields and of ghost symmetries *Proc. Natl Acad. Sci.* **77** 720–3
[35] Thierry-Mieg J and Jarvis P D 2023 New Hermitian conjugations and real forms in $sl(m/n)$ and $osp(2/2n)$ superalgebras modelling quark and lepton families Preprint (in preparation)
[36] Jarvis P D and Thierry-Mieg J 2022 Indecomposable doubling for representations of the type I Lie superalgebras $sl(m/n)$ and $osp(2/2n)$ *J. Phys. A: Math. Theor.* **55** 475206
[37] Thierry-Mieg J, Jarvis P D and Germoni J (with an appendix by Gorelik M) 2023 Construction of matryoshka nested indecomposable N-replications of Kac-modules of quasi-reductive Lie superalgebras, including the $sl(m/n)$ and $osp(2/2n)$ series (arXiv:2207.06538v4 [math.RT])
[38] Coquereaux R 1991 Elementary fermions and $su(2|1)$ representations *Phys. Lett.* B **261** 449–58
[39] Fulling S A, King R C, Wybourne B G and Cummins C J 1992 Normal forms for tensor polynomials. I. The Riemann tensor *Class. Quantum Grav.* **9** 1151
[40] King R C and Wybourne B G 1982 Some noteworthy spin plethysms *J. Phys. A: Math. Gen.* **15** 1137
[41] Black G R E, King R C and Wybourne B G 1983 Kronecker products for compact semisimple Lie groups *J. Phys. A: Math. Gen.* **16** 1555
[42] Lemes V, Renan R and Sorella S P 1995 New algebraic renormalization of antisymmetric tensor matter fields *Phys. Lett.* B **344** 158–63
[43] Jackiw R and Pi S-Y 2011 Tutorial on scale and conformal symmetries in diverse dimensions *J. Phys. A: Math. Theor.* **44** 223001