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The Sobolev Wavefront Set of the Causal Propagator in Finite Regularity

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Abstract. Given a globally hyperbolic spacetime $M = \mathbb{R} \times \Sigma$ of dimension four and regularity C^τ, we estimate the Sobolev wavefront set of the causal propagator K_G of the Klein–Gordon operator. In the smooth case, the propagator satisfies $WF'(K_G) = C$, where $C \subset T^*(M \times M)$ consists of those points $(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta})$ such that $\tilde{\xi}, \tilde{\eta}$ are cotangent to a null geodesic γ at $\tilde{x}$ resp. $\tilde{y}$ and parallel transports of each other along γ. We show that for $\tau > 2$,

$$WF'^{-2+\tau-\epsilon}(K_G) \subset C$$

for every $\epsilon > 0$. Furthermore, in regularity $C^{\tau+2}$ with $\tau > 2$,

$$C \subset WF'^{-\frac{1}{2}}(K_G) \subset WF'^{\tau-\epsilon}(K_G) \subset C$$

holds for $0 < \epsilon < \tau + \frac{1}{2}$. In the ultrastatic case with Σ compact, we show $WF'^{-\frac{3}{2}+\tau-\epsilon}(K_G) \subset C$ for $\epsilon > 0$ and $\tau > 2$ and $WF'^{-\frac{3}{2}+\tau-\epsilon}(K_G) = C$ for $\tau > 3$ and $\epsilon < \tau - 3$. Moreover, we show that the global regularity of the propagator K_G is $H^{-\frac{1}{2}-\epsilon}_{loc}(M \times M)$ as in the smooth case.

Contents

1. Introduction

The quantisation of the scalar field forms part of the basis for the subject of Algebraic Quantum Field Theory. While the main mathematical framework for the smooth setting was initiated more than 20 years ago, see e.g. [23,28,32,39,50], ongoing research continues to develop new techniques, particularly in connection with microlocal analysis [29,37,38], the importance of Hadamard states [20,27,40,55,58], locality and covariance [12,26,44], perturbation theory [11,14,34], Dirac fields [15,24,31,33] and gauge theory [7,13].

Moreover, it is now possible to approach certain mathematical questions related to quantum fields propagating in spacetimes of finite regularity. This is motivated by the deep foundational work on causality theory [8,18,42,46] and advances in our understanding of nonlinear hyperbolic equations [17,19,41], which were needed as a first step towards a full understanding of Einstein's equations as a well-posed Cauchy problem, which requires solutions that go beyond the smooth ones. Additionally, there are several astrophysical models of phenomena such as neutron stars, self-gravitating fluids and gravitational collapse that are not smooth [2,16,47].

The quantisation proceeds in two steps. First, one constructs an algebra of observables, then one represents this algebra on a Hilbert space of physical states.

A common candidate for such physical quantum states, ω, are quasifree states that satisfy the microlocal spectrum condition.

To state it, it is useful to introduce the sets

$$C = \left\{(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}) \in T^*(M\times M)\backslash 0; g^{ab}(\tilde{x})\tilde{\xi}_a\tilde{\xi}_b = g^{ab}(\tilde{y})\tilde{\eta}_a\tilde{\eta}_b = 0, (\tilde{x},\tilde{\xi}) \sim (\tilde{y},\tilde{\eta})\right\}$$
$$C^+ = \left\{(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}) \in C; \tilde{\xi}^0 \geq 0, \tilde{\eta}^0 \geq 0\right\}, \tag{1.1}$$

where $(\tilde{x},\tilde{\xi}) \sim (\tilde{y},\tilde{\eta})$ means that there is a null geodesic γ joining $\tilde{x}$ and $\tilde{y}$ such that $\tilde{\xi},\tilde{\eta}$ are cotangent to the null geodesic γ at $\tilde{x}$ resp. $\tilde{y}$ and parallel transports of each other.

Using the above sets, one can define the microlocal spectrum condition as follows:

Definition 1.1. A quasifree state ω_H on the algebra of observables satisfies the microlocal spectrum condition if its two-point function $\omega_H^{(2)}$ is a distribution in $\mathcal{D}'(M \times M)$ and satisfies the following wavefront set condition

$$WF'(\omega_H^{(2)}) = C^+,$$

where $WF'(\omega_H^{(2)}) := \{(\tilde{x}, \tilde{\xi}; \tilde{y}, -\tilde{\eta}) \in T^*(M \times M); (\tilde{x}, \tilde{\xi}; \tilde{y}, \tilde{\eta}) \in WF(\omega_H^{(2)})\}$.

These states, called Hadamard states, have been constructed in the smooth setting. They encompass both ground and KMS states [29,37]. Moreover, they are particularly well suited for point-splitting renormalisation, a technique used for calculating key physical quantities like the renormalised energy-momentum tensor [63,64].

A central goal now is the construction of suitable quantum states in non-smooth scenarios following the techniques in [29,38], which requires a thorough knowledge of the wavefront set of the causal propagator. This is the question we address in this article. To be precise, we characterise the wavefront set of the causal propagator of the Klein–Gordon operator in non-smooth globally hyperbolic spacetimes. The causal propagator is constructed using the inverses associated with the Cauchy problem, which makes it a classical propagator. It is worth noting that there exist other bisolutions such as the two-point functions described above, which are non-classical (see [22] for further details on this convention).

The microlocal analysis of the propagators of the wave equation and its parametrices in low-regularity spacetimes introduces several technical challenges due to the lack of a complete theory of Fourier integral operators with non-smooth symbols and amplitudes. However, progress has been made using the paradifferential calculus introduced by Bony [10] (see also [6,45,61]). In addition, Szeftel has constructed a parametrix which requires only control over the L^2 curvature of the metric in order to prove the L^2-curvature conjecture related to Einstein's field equations [41,59]. Moreover, Tataru [60] has constructed parametrices of the wave equations in low regularity for metrics with $C^{1,1}$ coefficients as a preliminary step to show suitable Strichartz estimates and analyse nonlinear PDE's using phase space transforms. In addition, his results allowed even lower regularity at the expense of showing weaker results. Finally, we mention Smith's construction of parametrices for the $C^{1,1}$ case using wave packets [56] (see [65] for a parametrix construction using Gaussians). The contribution of our paper is establishing the microlocal singular structure of the causal propagator when the regularity of the spacetime is finite. The main theorems we prove are:

Theorem (Theorem 5.1). *Let (M, g) be a C^τ globally hyperbolic spacetime with $\tau > 2$ and K_G the causal propagator of the Klein–Gordon operator P. Then,*

$$WF'^{-2+\tau-\epsilon}(K_G) \subset C$$

for every $\epsilon > 0$, C as in Eq.(1.1), and

Theorem (Theorem 5.2). *For a $C^{\tau+2}$ globally hyperbolic spacetime with $\tau > 2$,*

$$C \subset WF'^{-\frac{1}{2}}(K_G) \subset WF'^{\tau-\epsilon}(K_G) \subset C,$$

and hence equality, holds for $0 < \epsilon < \tau + \frac{1}{2}$.

In the ultrastatic case, sharper results are available. For completeness, we state these in the Appendix, see Lemmas 6.5 and 6.7, Theorems 6.9 and 6.11.

1.1. The Smooth Setting

Consider a pair (M, g), where M is a smooth manifold and g is a smooth Lorentzian metric. The Klein–Gordon operator P on (M, g) is given by

$$P := g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + m^2\phi = (\Box_g + m^2)\phi \tag{1.2}$$

where $g^{\mu\nu}$ is the inverse metric tensor, ∇_μ is the covariant derivative and m is a positive real number.

The starting point is the notion of advanced and retarded Green operators in this situation.

Definition 1.2. Let M be a time-oriented connected Lorentzian manifold and let P be the Klein–Gordon operator. An *advanced Green operator* G^+ is a linear map $G^+ : \mathcal{D}(M) \to C^\infty(M)$ such that

1. $P \circ G^+ = \mathrm{id}_{\mathcal{D}(M)}$
2. $G^+ \circ P|_{\mathcal{D}(M)} = \mathrm{id}_{\mathcal{D}(M)}$
3. $\mathrm{supp}(G^+\phi) \subset J^+(\mathrm{supp}(\phi))$ for all $\phi \in \mathcal{D}(M)$.

A *retarded Green operator* G^- satisfies (1) and (2), but (3) is replaced by the condition $\mathrm{supp}(G^-\phi) \subset J^-(\mathrm{supp}(\phi))$ for all $\phi \in \mathcal{D}(M)$.

In [5, Corollary 3.4.3], it is shown that these exist and are unique on a globally hyperbolic manifold.

The advanced and retarded Green operators are then used to define the causal propagator

$$G := G^+ - G^-$$

which maps $\mathcal{D}(M)$ to $C^\infty_{\mathrm{sc}}(M)$, the space of spatially compact maps, i.e. the smooth maps ϕ such that there exists a compact subset $K \subset M$ with $\mathrm{supp}(\phi) \subset J(K)$. If M is globally hyperbolic, then one has the following *exact sequence* [5, Theorem 3.4.7]:

$$0 \longrightarrow \mathcal{D}(M) \xrightarrow{P} \mathcal{D}(M) \xrightarrow{G} C^\infty_{\mathrm{sc}}(M) \xrightarrow{P} C^\infty_{\mathrm{sc}}(M),$$

Since G is a continuous linear operator, the Schwartz Kernel Theorem implies that there exists one and only one distribution $K_G \in \mathcal{D}'(M \times M)$ such that

$$K_G(u \otimes v) = \langle G(v), u\rangle, \quad u, v \in \mathcal{D}(M). \tag{1.3}$$

It follows from Duistermaat and Hörmander's characterisation using Fourier integral operators that the kernel K_G satisfies

$$WF'(K_G) = C. \tag{1.4}$$

More explicitly, they showed that $K_G \in I^{-\frac{3}{2}}(M \times M, C')$, where $I^\mu(X, \Lambda)$ denotes the space of Lagrangian distributions of order μ over the manifold X associated to the Lagrangian submanifold Λ. In this case $\Lambda = C' = \{(\tilde{x}, \tilde{\xi}; \tilde{y}, -\tilde{\eta}); (\tilde{x}, \tilde{\xi}; \tilde{y}, \tilde{\eta}) \in C\}$, see [25, Theorem 6.5.3]. Using [25, Theorem 5.4.1, Theorem 6.5.3], one obtains that in four dimensions, K_G belongs to the Sobolev space $H_{loc}^{-\frac{1}{2}-\epsilon}(M \times M)$ for any $\epsilon > 0$. For details on the Sobolev spaces mentioned, see Sect. 6.1 and [36, Appendix B].

2. The Non-Smooth Setting

Next we will consider the case, where g is a non-smooth metric. We will specify the precise regularity in each section.

The definition of the Green operators in the non-smooth setting will require us to choose suitable spaces of functions based on Sobolev spaces as domain and range. We let

$$\begin{aligned} V_0 &= \{\phi \in H^2_{comp}(M); P\phi \in H^1_{comp}(M)\} \\ V_{sc} &= \{\phi \in H^2_{loc}(M); P\phi \in H^1_{loc}(M) \\ &\quad \text{and } \operatorname{supp}(\phi) \subset J(K), \text{ where } K \text{ is a compact subset of } M\}. \end{aligned} \tag{2.1}$$

Definition 2.1. An *advanced Green operator* for the Klein–Gordon operator P is a linear map

$$G^+ : H^1_{\text{comp}}(M) \to H^2_{loc}(M)$$

satisfying the properties

1. $PG^+ = \text{id}_{H^1_{comp}(M)}$,
2. $G^+ P|_{V_0} = \text{id}_{V_0}$,
3. $\operatorname{supp}(G^+(f)) \subset J^+(\operatorname{supp}(f))$ for all $f \in H^1_{comp}(M)$,

A *retarded Green operator* G^- is defined correspondingly.

It is shown in [36, Theorem 5.8] that these operators exist and are unique on Lorentzian manifolds that satisfy the condition of generalised hyperbolicity. This condition is satisfied in particular for $C^{1,1}$ globally hyperbolic spacetimes. Moreover, one obtains a short exact sequence for the low-regularity causal propagator, $G := G^+ - G^-$, similar to that in the smooth case

$$0 \longrightarrow V_0 \xrightarrow{P} H^1_{comp}(M) \xrightarrow{G} V_{\text{sc}} \xrightarrow{P} H^1_{loc}(M).$$

3. Pseudodifferential Operators with Non-Smooth Symbols

3.1. Symbol Classes

Let $\{\psi_j; j = 0, 1, \ldots\}$ be a Littlewood-Paley partition of unity on $\mathbb{R}^n$, i.e. a partition of unity $1 = \sum_{j=0}^{\infty} \psi_j$, where $\psi_0 \equiv 1$ for $|\xi| \le 1$ and $\psi_0 \equiv 0$ for $|\xi| \ge 2$ and $\psi_j(\xi) = \psi_0(2^j\xi) - \psi_0(2^{1-j}\xi)$. The support of ψ_j, $j \ge 1$, then lies in an annulus around the origin of interior radius 2^j and exterior radius 2^{1+j}.

Definition 3.1. (a) For $\tau \in (0, \infty)$, the Hölder space $C^\tau(\mathbb{R}^n)$ is the set of all functions f with

$$\|f\|_{C^\tau} := \sum_{|\alpha| \le [\tau]} \|\partial_x^\alpha f\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\alpha| = [\tau]} \sup_{x \ne y} \frac{|\partial_x^\alpha f(x) - \partial_x^\alpha f(y)|}{|x - y|^{\tau - [\tau]}} < \infty. \quad (3.1)$$

(b) For $\tau \in \mathbb{R}$, the Zygmund space $C_*^\tau(\mathbb{R}^n)$ consists of all functions f with

$$\|f\|_{C_*^\tau} = \sup_j 2^{j\tau} \|\psi_j(D) f\|_{L^\infty} < \infty. \quad (3.2)$$

Here, $\psi_j(D)$ is the Fourier multiplier with symbol ψ_j, i.e. $\psi_j(D)u = \mathcal{F}^{-1}\psi_j \mathcal{F}u$, where $(\mathcal{F}u)(\xi) = (2\pi)^{-n/2} \int e^{-ix\xi} u(x)\, d^n x$ is the Fourier transform.

We have the following relations: $C^\tau = C_*^\tau$ if $\tau \notin \mathbb{N}$, and $C^\tau \subset C_*^\tau$ if $\tau \in \mathbb{N}$.

We next introduce symbol classes of finite Hölder or Zygmund regularity, following Taylor [61]. We use the notation $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}^n$.

Definition 3.2. (a) Let $0 \le \delta < 1$. A symbol $p(x, \xi)$ belongs to $C_*^\tau S_{1,\delta}^m := C_*^\tau S_{1,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n)$ if

$$\|D_\xi^\alpha p(\cdot, \xi)\|_{C_*^\tau} \le C_\alpha \langle \xi \rangle^{m - |\alpha| + \tau\delta} \text{ and } |D_\xi^\alpha p(x, \xi)| \le C_\alpha \langle \xi \rangle^{m - |\alpha|}.$$

(b) We obtain the symbol class $C^\tau S_{1,\delta}^m := C^\tau S_{1,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n)$ for $\tau > 0$ by requiring that

$$\|D_\xi^\alpha p(\cdot, \xi)\|_{C^s} \le C_\alpha \langle \xi \rangle^{m - |\alpha| + s\delta}, \quad 0 \le s \le \tau.$$

(c) A symbol $p(x, \xi)$ is in $C^\tau S_{cl}^m$ provided $p(x, \xi) \in C^\tau S_{1,0}^m$ and $p(x, \xi)$ has a classical expansion

$$p(x, \xi) \sim \sum_{j \ge 0} p_{m-j}(x, \xi)$$

in terms p_{m-j} homogeneous of degree $m - j$ in ξ for $|\xi| \ge 1$, in the sense that the difference between $p(x, \xi)$ and the sum over $0 \le j < N$ belongs to $C^\tau S_{1,0}^{m-N}$.

The pseudodifferential operator $p(x, D_x)$ with the symbol $p(x, \xi) \in C^\tau S_{1,\delta}^m$ is given by

$$(p(x, D_x)u)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi)(\mathcal{F}u)(\xi) d^n \xi, \quad u \in \mathcal{S}(\mathbb{R}^n). \quad (3.3)$$

It extends to continuous maps

$$p(x, D_x) : H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n), \quad -\tau(1-\delta) < s < \tau. \tag{3.4}$$

While it is possible to extend the theory of pseudodifferential operators with non-smooth symbols to manifolds (see [1]), due to the local nature of our results it is a key point of this article that we can work entirely on $\mathbb{R}^n$.

3.2. Symbol Smoothing

Given $p(x,\xi) \in C^\tau S^m_{1,\gamma}$ and $\delta \in (\gamma, 1)$ let

$$p^\#(x,\xi) = \sum_{j=0}^{\infty} J_{\epsilon_j} p(x,\xi)\psi_j(\xi). \tag{3.5}$$

Here, J_ϵ is the smoothing operator given by $(J_\epsilon f)(x) = (\phi(\epsilon D)f)(x)$ with $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi(\xi) = 1$ for $|\xi| \leq 1$, and we take $\epsilon_j = 2^{-j(\delta-\gamma)}$.

Letting $p^b(x,\xi) = p(x,\xi) - p^\#(x,\xi)$, we obtain the decomposition

$$p(x,\xi) = p^\#(x,\xi) + p^b(x,\xi), \tag{3.6}$$

where $p^\#(x,\xi) \in S^m_{1,\delta}$ and $p^b(x,\xi) \in C^\tau S^{m-\tau(\delta-\gamma)}_{1,\delta}$.

The symbol estimates for $p^\#$ are a consequence of the estimate

$$\|\partial_x^\beta J_\epsilon f\|_{L^\infty} \leq \begin{cases} C\|f\|_{C^\tau} & |\beta| \leq \tau \\ C\epsilon^{-(|\beta|-\tau)}\|f\|_{C^\tau} & |\beta| > \tau, \end{cases}$$

and that $\epsilon_j = 2^{-j(\delta-\gamma)}$. For details, see Proposition 1.3 E and Equation (1.3.21) in [61].

3.3. Microlocal Sobolev Regularity

Let $p \in C^\tau S^m_{\rho,\delta}$, $\tau > 0$, with $\delta < \rho$. Suppose that there is a conic neighbourhood Γ of (x_0, ξ_0) and constants $c, C > 0$ such that $|p(x,\xi)| \geq c|\xi|^m$ for $(x,\xi) \in \Gamma$, $|\xi| \geq C$. Then, (x_0,ξ_0) is called *non-characteristic*. If p has a homogeneous principal symbol p_m, the condition is equivalent to $p_m(x_0,\xi_0) \neq 0$. The complement of the set of non-characteristic points is the set of characteristic points denoted by $\mathrm{Char}(p)$.

A distribution u is *microlocally in* H^s at $(x_0,\xi_0) \in T^*M\backslash 0$ if there exists a conic neighbourhood Γ_0 of ξ_0 and a smooth function $\varphi \in C_0^\infty(M)$ with $\varphi(x_0) \neq 0$ such that

$$\int_{\Gamma_0} \langle\xi\rangle^{2s} |\mathcal{F}(\varphi u)(\xi)|^2 d^n\xi < \infty.$$

Otherwise we say that (x_0,ξ_0) lies in the H^s-wavefront set $WF^s(u)$.

If u is microlocally in H^s in an open conic subset $\Gamma \subset T^*M\backslash 0$, we write $u \in H^s_{mcl}(\Gamma)$.

3.4. Propagation of Singularities for Bisolutions of the Klein–Gordon Operator

A globally hyperbolic spacetime is of the form $\mathbb{R}\times\Sigma$, where Σ is not assumed to be compact, and we will write local coordinates in the form

$$\tilde{x}=(t,x),\tilde{y}=(s,y) \tag{3.7}$$

and the associated covariables as

$$\tilde{\xi}=(\xi^0,\xi),\tilde{\eta}=(\eta^0,\eta). \tag{3.8}$$

On the product $(\mathbb{R}\times\Sigma)\times(\mathbb{R}\times\Sigma)$, we use $(\mathbf{x},\boldsymbol{\xi})$ with

$$\mathbf{x}=(\tilde{x},\tilde{y}),\boldsymbol{\xi}=(\tilde{\xi},\tilde{\eta}). \tag{3.9}$$

In the sequel, we shall apply the Klein–Gordon operator also to functions and distributions on $M\times M$. Using the coordinates in Eqs. (3.7), (3.8) and (3.9), we distinguish the cases, where P acts on the first set of variables (t,x) or on the second set (s,y), and write $P_{(t,x)}$ and $P_{(s,y)}$, respectively. Explicitly,

$$P_{(t,x)}(\mathbf{x},D_{\mathbf{x}})=P_{(t,x)}(\tilde{x},D_{\tilde{x}},\tilde{y},D_{\tilde{y}})=(\Box_{g(\tilde{x})}+m^2)\otimes I$$
$$P_{(s,y)}(\mathbf{x},D_{\mathbf{x}})=P_{(s,y)}(\tilde{x},D_{\tilde{x}},\tilde{y},D_{\tilde{y}})=I\otimes(\Box_{g(\tilde{y})}+m^2)$$

In particular,

$$\begin{aligned}\mathrm{Char}(P_{(t,x)})&=\mathrm{Char}(P)\times T^*M\cup\{(\mathbf{x},\boldsymbol{\xi})\in T^*(M\times M)\backslash 0,\tilde{\xi}=0\}\\ \mathrm{Char}(P_{(s,y)})&=T^*M\times\mathrm{Char}(P)\cup\{(\mathbf{x},\boldsymbol{\xi})\in T^*(M\times M)\backslash 0,\tilde{\eta}=0\}.\end{aligned} \tag{3.10}$$

Theorem 3.3. *Let the metric g be of class C^τ, $\tau>1$, $0\le\sigma<\tau-1$ and $v\in H^{2+\sigma-\tau+\epsilon}_{loc}(M\times M)$ for some $\epsilon>0$ with $P_{(t,x)}(\mathbf{x},D_{\mathbf{x}})v=0$. Then,*

$$WF^{\sigma+2}(v)\subset\mathrm{Char}(P_{(t,x)}).$$

Proof. Being interested in the wavefront set of v near a point $\mathbf{x}$, we multiply v by a function $\varphi\in\mathcal{D}(M\times M)$ with $\varphi\equiv 1$ near $\mathbf{x}$ and consider φv. So we can assume that v has support in a small neighbourhood of $\mathbf{x}$ contained in a single coordinate patch and consider v as an element of $H^{2+\sigma-\tau+\epsilon}(\mathbb{R}^4\times\mathbb{R}^4)$. In order to distinguish points $(\mathbf{x},\boldsymbol{\xi})=(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta})$ from their representation in local coordinates, we will write the latter in the form $(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})=(\underline{\tilde{x}},\underline{\tilde{\xi}},\underline{\tilde{y}},\underline{\tilde{\eta}})$. In this local setting, $P_{(t,x)}(\underline{\mathbf{x}},D_{\underline{\mathbf{x}}})$ is given by the symbol

$$P_{(t,x)}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})=P_{(t,x)}(\underline{\tilde{x}},\underline{\tilde{\xi}},\underline{\tilde{y}},\underline{\tilde{\eta}})=\underbrace{g^{\mu\nu}(\underline{\tilde{x}})\underline{\xi_\mu\xi_\nu}}_{p_2(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})}+\underbrace{ig^{\mu\nu}(\underline{x})\Gamma^\rho_{\mu\nu}(\underline{x})\underline{\xi}_\rho}_{p_1(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})}+\underbrace{m^2}_{p_0(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})}. \tag{3.11}$$

The symbol smoothing (Eq. (3.6)) on p_2,p_1 gives a decomposition

$$p_2(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})=p_2^{\#}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})+p_2^{b}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})$$
$$p_1(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})=p_1^{\#}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})+p_1^{b}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})$$

$$\begin{aligned}P_{(t,x)}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) &= (p_2^{\#}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) + p_1^{\#}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})) + p_2^b(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) + p_1^b(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) + p_0(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})\\ &= q^{\#}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) + p_2^b(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) + p_1^b(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}),\end{aligned}$$

where

$$q^{\#}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) = (p_2^{\#}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) + p_1^{\#}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) + p_0(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})) \in S^2_{1,\delta}(\mathbb{R}^8 \times \mathbb{R}^8), \tag{3.12}$$

$$p_2^b(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) \in C^\tau S^{2-\tau\delta}_{1,\delta}(\mathbb{R}^8 \times \mathbb{R}^8) \quad p_1^b(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) \in C^{\tau-1} S^{1-(\tau-1)\delta}_{1,\delta}(\mathbb{R}^8 \times \mathbb{R}^8). \tag{3.13}$$

Taking $0 \le \delta < 1$ so close to 1 that $2 - \tau\delta < 2 - \tau + \epsilon$ we have $v \in H^{2+\sigma-\tau\delta}(\mathbb{R}^4 \times \mathbb{R}^4)$ (notice this implies $v \in H^{1+\sigma-(\tau-1)\delta}(\mathbb{R}^4 \times \mathbb{R}^4)$), and we have

$$q^{\#}(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})v = -(p_2^b(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}}) + p_1^b(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}}))v = f, \tag{3.14}$$

where $f \in H^\sigma(\mathbb{R}^4 \times \mathbb{R}^4)$, since $p_2^b(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})v \in H^\sigma(\mathbb{R}^4 \times \mathbb{R}^4)$ and $p_1^b(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})v \in H^{\sigma+1-\delta}(\mathbb{R}^4 \times \mathbb{R}^4)$.

Now if $(\underline{\tilde{x}_0}, \underline{\tilde{\xi}_0}, \underline{\tilde{y}_0}, \underline{\tilde{\eta}_0}) \notin \mathrm{Char}(P_{(t,x)})$, there are $C, c > 0$ such that

$$|P_{(t,x)}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})| \ge c|\underline{\boldsymbol{\xi}}|^2 \text{ for } |\underline{\boldsymbol{\xi}}| \ge C$$

in a conical neighbourhood Γ that contains $(\underline{\tilde{x}_0}, \underline{\tilde{\xi}_0}, \underline{\tilde{y}_0}, \underline{\tilde{\eta}_0})$.

Since $p_2^b(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) \in C^\tau S^{2-\tau\delta}_{1,\delta}$ and $p_1^b(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}}) \in C^{\tau-1} S^{1-(\tau-1)\delta}_{1,\delta}$, there exists a $\tilde{C} > 0$ such that

$$\begin{aligned}|q^{\#}(\underline{\mathbf{x}},\underline{\boldsymbol{\xi}})| &\ge C(1+|\underline{\boldsymbol{\xi}}|^2) - (1+|\underline{\boldsymbol{\xi}}|^2)^{\frac{2-\tau\delta}{2}} - (1+|\underline{\boldsymbol{\xi}}|^2)^{\frac{1-(\tau-1)\delta}{2}}\\ &\ge \tilde{C}(1+|\underline{\boldsymbol{\xi}}|^2) \text{ for large } |\underline{\boldsymbol{\xi}}|.\end{aligned}$$

Therefore, $(\underline{\tilde{x}_0}, \underline{\tilde{\xi}_0}, \underline{\tilde{y}_0}, \underline{\tilde{\eta}_0}) \notin \mathrm{Char}(q^{\#})$.

Since $q^{\#} \in S^2_{1,\delta}$ and $(\underline{\tilde{x}_0}, \underline{\tilde{\xi}_0}, \underline{\tilde{y}_0}, \underline{\tilde{\eta}_0}) \notin \mathrm{Char}(q^{\#})$, there is a microlocal parametrix with symbol $\tilde{q} \in S^{-2}_{1,\delta}(\mathbb{R}^8 \times \mathbb{R}^8)$ such that

$$v + r(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})v = \tilde{q}(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})q^{\#}(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})v = \tilde{q}(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})f,$$

where $(\underline{\tilde{x}_0}, \underline{\tilde{\xi}_0}, \underline{\tilde{y}_0}, \underline{\tilde{\eta}_0}) \notin WF(r(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})v)$ and $\tilde{q}(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})f \in H^{\sigma+2}(\mathbb{R}^4 \times \mathbb{R}^4)$ which shows that $(\underline{\tilde{x}_0}, \underline{\tilde{\xi}_0}, \underline{\tilde{y}_0}, \underline{\tilde{\eta}_0}) \notin WF^{\sigma+2}(\tilde{q}(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})f)$. Since

$$WF^{\sigma+2}(v) \subset WF^{\sigma+2}(\tilde{q}(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})f) \cup WF(r(\underline{\mathbf{x}}, D_{\underline{\mathbf{x}}})v), \tag{3.15}$$

we see that $(\underline{\tilde{x}_0}, \underline{\tilde{\xi}_0}, \underline{\tilde{y}_0}, \underline{\tilde{\eta}_0}) \notin WF^{\sigma+2}(v)$.

By definition of the wavefront set, this means that $(\mathbf{x}_0, \boldsymbol{\xi}_0)$ is not in the wavefront set of v, considered as a distribution on $M \times M$. □

Remark 3.4. In the proof presented above, we showed that the microlocal results are local estimates, which can be done within a chart in the cotangent bundle $T^*\mathbb{R}^8$. To streamline the discussion and avoid frequently alternating between the notation of the chart and the manifold, we will forego this distinction in Section 5. However, it is important to bear in mind that the proofs in that section are analogous to the one detailed above, involving localization within a chart.

Remark 3.5. Applying the symbol smoothing directly to $P_{(t,x)} \in C^{\tau-1}S^2_{1,0}$ would leave us with $P^b_{(t,x)} \in C^{\tau-1}S^{2-(\tau-1)\delta}_{1,\delta}$. The advantage of the decomposition in Theorem 3.3 with $p^b_1 \in C^{\tau-1}S^{1-(\tau-1)\delta}_{1,\delta}$ and $p^b_2 \in C^{\tau}S^{2-\tau\delta}_{1,\delta}$ is that the associated operators map a given $u \in H^{2+s-\tau\delta}$ to H^s and $H^{s+1-\delta}$, respectively, for $-(1-\delta)(\tau-1) < s < \tau-1$, so that the sum is in H^s instead of $H^{s-\delta}$.

The theorem, below, will be crucial for our main result. Proofs can be found in [61, Proposition 6.1.D] or [62, Proposition 11.4]. In [62, p.215], Taylor points out that Zygmund regularity C^2_* for the metric suffices.

Theorem 3.6. *Let $u \in \mathcal{D}'(M \times M)$ solve $P_{(t,x)}u = f$. Let γ be an integral curve of the Hamiltonian vector field H_{p_2} with p_2 as in Eq. (3.11). If for some $s \in \mathbb{R}$, we have $f \in H^s_{mcl}(\Gamma)$ and $P^b_{(t,x)}u \in H^s_{mcl}(\Gamma)$, where $\gamma \subset \Gamma$ with Γ a conical neighbourhood and $u \in H^{s+1}_{mcl}(\gamma(0))$, then $u \in H^{s+1}_{mcl}(\gamma)$.*

Remark 3.7. If $u \in H^{2+s-\tau\delta}_{comp}$, then $P^b_{(t,x)}u \in H^s$, see Remark (3.5). Moreover, using the divergence structure of the operator one can show that, if $u \in H^{1+s-\tau\delta}_{comp}, f \in H^{s-1}, u \in H^s_{mcl}(\gamma(0))$, then $u \in H^s_{mcl}(\gamma)$ for $-2(1-\delta) < s \leq 2$; see [62, p.210] for details.

Remark 3.8. Notice that the $s \in \mathbb{R}$ is constrained by the microlocal regularity of $P^b_{(t,x)}u$ and not only that of f. In fact, one can use the stronger hypothesis that $u \in H^{s-\tau\delta}_{comp}(U)$ for a suitable domain U, regularity τ and $\delta \in (0,1)$ in order to guarantee that $P^b_{(t,x)}u \in H^s(U) \subset H^s_{mcl}(\Gamma)$

4. Support and Global Regularity of K_G

The following two lemmas contain the main results of this section. The first lemma shows that only causally connected points belong to the support of K_G. The second lemma establishes that $K_G \in H^{-1-\epsilon}_{loc}(M \times M)$.

Lemma 4.1. *Let $(\tilde{x}, \tilde{y}) \in M \times M$ be such that $\tilde{x}$ and $\tilde{y}$ are not causally related, i.e. $\tilde{x} \notin J(\tilde{y})$. Then, $(\tilde{x}, \tilde{y}) \notin \operatorname{supp}(K_G)$.*

Proof. Since the support of K_G is the complement of the largest open set where K_G vanishes, it is enough to show that there are open neighbourhoods V of $\tilde{x}$ and U of $\tilde{y}$ such that K_G vanishes in $W = V \times U$.

We construct the sets V and U as follows: For globally hyperbolic spacetimes, there exist a time function and a foliation by Cauchy surfaces, i.e. $M = \mathbb{R} \times \Sigma$, see [8, Theorem 1.1], [53, Theorem 5.9]. Let $\tilde{x} \in \{t\} \times \Sigma$ and $\tilde{y} \in \{s\} \times \Sigma$. Without loss of generality, we assume $t \leq s$. Since M is globally hyperbolic, $J(\tilde{y}) \cap (\{t\} \times \Sigma)$ is compact and by hypothesis does not contain $\tilde{x}$. Therefore, there exists a neighbourhood $\tilde{V}$ of $\tilde{x}$ in $\{t\} \times \Sigma$ such that $\overline{\tilde{V}} \cap (J(\tilde{y}) \cap (\{t\} \times \Sigma)) = \emptyset$. By symmetry, $\tilde{y} \notin J(\overline{\tilde{V}}) \cap (\{s\} \times \Sigma) = \overline{J(\tilde{V})} \cap (\{s\} \times \Sigma)$, and we thus also find a neighbourhood $\tilde{U}$ of $\tilde{y}$ in $\{s\} \times \Sigma$ such that $\tilde{U} \cap J(\tilde{V}) \cap (\{s\} \times \Sigma) = \emptyset$.

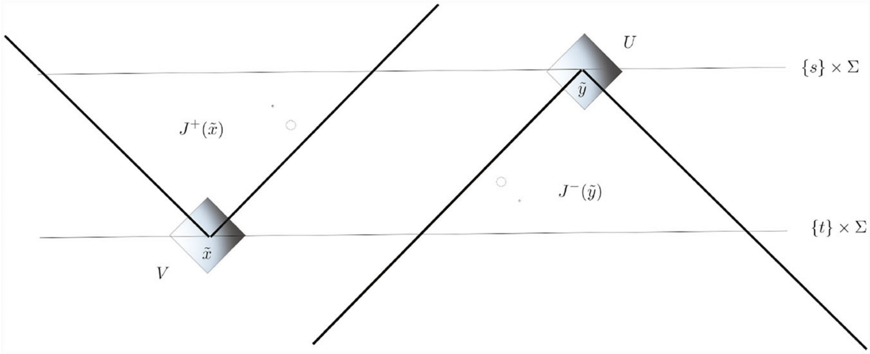

FIGURE 1. $U \cap J(V) = \emptyset$ and $V \cap J(U) = \emptyset$

Now we consider the total domain of dependence of both sets, i.e. $D(\tilde{U})$ and $D(\tilde{V})$.[1] Notice that $J(D(\tilde{V})) \cap D(\tilde{U}) = \emptyset$ and $J(D(\tilde{U})) \cap D(\tilde{V}) = \emptyset$. Otherwise, we could construct a causal curve between $\tilde{U}$ and $\tilde{V}$. We define $V := \mathrm{Int}D(\tilde{V})$ and $U := \mathrm{Int}D(\tilde{U})$, see Fig. 1.

Now we show that K_G vanishes in $W = V \times U$: Choose smooth functions ψ and ϕ with $\mathrm{supp}(\psi) \subset V$ and $\mathrm{supp}(\phi) \subset U$. Then,

$$\begin{aligned} K_G(\psi \otimes \phi) = \langle G(\psi), \phi \rangle &= \int_M G(\psi)\phi\sqrt{g}dx \\ &= \int_{J(\mathrm{supp}(\psi))\cap \mathrm{supp}(\phi)} G(\psi)\phi\sqrt{g}dx \\ &= \int_{J(V)\cap U} G(\psi)\phi\sqrt{g}dx = 0. \end{aligned}$$

□

Remark 4.2. Notice that a totally analogous proof shows that $(\tilde{x}, \tilde{y}) \notin \mathrm{supp}(K_{G^\pm})$ if $\tilde{x} \notin J^\pm(\tilde{y})$.

Regarding the global regularity of the causal propagator for $C^{1,1}$ globally hyperbolic spacetimes, we find a slightly weaker result compared to the smooth case. Nevertheless, in the ultrastatic setting we show that the same regularity as in the smooth setting holds (Lemma 6.11).

Lemma 4.3. *Let (M, g) be a $C^{1,1}$-globally hyperbolic spacetime. Then $K_G \in H^{-1-\epsilon}_{loc}(M \times M)$ for every $\epsilon > 0$.*

Proof. We have to show that, given $\psi_1, \psi_2 \in \mathcal{D}(M)$, the Schwartz kernel of the product $\psi_2 G \psi_1$ is in $H^{-1-\varepsilon}(M)$ for every $\varepsilon > 0$. Since the proof is local, we may assume (using possibly disconnected coordinate charts) that ψ_1 and ψ_2 have their support in the same coordinate neighbourhood for M. We will

[1]Given a subset S of M, the domain of dependence of S is the set of all points p in M such that every inextendible causal curve through p intersects S.

therefore work in $\mathbb{R}^4$, using the notation ψ_1, ψ_2 and G also for the representations in local coordinates. In order to distinguish the standard variables and covariables on $\mathbb{R}^4$ from those chosen for M we shall denote them by $\underline{x}$, $\underline{\xi}$, etc. Moreover, we choose $\psi_3, \psi_4 \in \mathcal{D}(\mathbb{R}^4)$ supported in the same coordinate chart, satisfying $\psi_3\psi_2 = \psi_2$ and $\psi_4\psi_3 = \psi_3$. Finally, we denote by Λ^s, $s \in \mathbb{R}$, the pseudodifferential operator of order s with symbol $(1 + |\underline{\xi}|^2)^{s/2}$ on $\mathbb{R}^4$.

We have

$$\psi_2 G\psi_1 = \Lambda^{1+\varepsilon}\Lambda^{-1-\varepsilon}\psi_3\psi_2 G\psi_1 = \Lambda^{1+\varepsilon}(\psi_4 + (1 - \psi_4))\Lambda^{-1-\varepsilon}\psi_3\psi_2 G\psi_1. \tag{4.1}$$

The operator $\psi_4\Lambda^{-1-\varepsilon}\psi_3\psi_2 G\psi_1$ maps $H^1(\mathbb{R}^4)$ to $H^{3+\varepsilon}_{comp}(\mathbb{R}^4)$ and therefore is a Hilbert–Schmidt operator. Hence, it has an integral kernel in $L^2(\mathbb{R}^4 \times \mathbb{R}^4)$. The operator $(1 - \psi_4)\Lambda^{-1-\varepsilon}\psi_3$ is obviously smoothing, since $1 - \psi_4$ and ψ_3 have disjoint support. Hence, it maps $H^2(\mathbb{R}^4)$ to $H^\infty(\mathbb{R}^4) = \bigcap_s H^s(\mathbb{R}^4)$. But more is true: In the identity

$$\underline{x}_j(1 - \psi_4)\Lambda^{-1-\varepsilon}\psi_3 = (1 - \psi_4)\Lambda^{-1-\varepsilon}\underline{x}_j\psi_3 + (1 - \psi_4)[\underline{x}_j, \Lambda^{-1-\varepsilon}]\psi_3,$$

both operators on the right hand side map $H^2(\mathbb{R}^4)$ to $H^\infty(\mathbb{R}^4)$ (recall that $[\underline{x}_j, \Lambda^{-1-\varepsilon}]$ has the symbol $D_{\underline{\xi}_j}(1 + |\underline{\xi}|^2)^{-1-\varepsilon})$. Iterating this identity, we find that $(1 + |\underline{x}|^{2N})(1 - \psi_4)\Lambda^{-1-\varepsilon}\psi_3 \in \mathcal{B}(H^2(\mathbb{R}^4), H^\infty(\mathbb{R}^4))$ for every $N \in \mathbb{N}$. Hence, $(1 - \psi_4)\Lambda^{-1-\varepsilon}\psi_3$ maps $H^2(\mathbb{R}^4)$ to $\mathcal{S}(\mathbb{R}^4)$.

Therefore, it also has an integral kernel in $L^2(\mathbb{R}^4 \times \mathbb{R}^4)$. Denote for the moment the L^2-integral kernel of $\Lambda^{-1-\varepsilon}\psi_3\psi_2 G\psi_1$ by $k_A = k_A(\underline{x}, \underline{y})$. Then, the kernel $k = k(\underline{x}, \underline{y})$ of $\psi_2 G\psi_1$ is given by

$$\Lambda^{1+\varepsilon}_{(\underline{x})} k_A(\underline{x}, \underline{y}).$$

Here, the notation $\Lambda^{1+\varepsilon}_{(x)}$ indicates that we view $\Lambda^{1+\varepsilon}$ as an operator on $\mathbb{R}^4 \times \mathbb{R}^4$ that acts only with respect to the first copy of $\mathbb{R}^4$. In this sense, it is a pseudodifferential operator with symbol in the Hörmander class $S^{1+\varepsilon}_{0,0}$ and thus maps $L^2(\mathbb{R}^4 \times \mathbb{R}^4)$ to $H^{-1-\varepsilon}(\mathbb{R}^4 \times \mathbb{R}^4)$. This shows the assertion. □

Remark 4.4. Notice that since only the mapping properties of G were used we have also that $K_{G^+}, K_{G^-} \in H^{-1-\epsilon}_{loc}(M \times M)$.

5. Proof of the Main Theorems

A globally hyperbolic spacetime is given by a family of Riemannian metrics $\{h_t\}_{t\in\mathbb{R}}$ on Σ and a function $\beta(x, t) > 0$ such that the spacetime metric (M, g), where $M = \mathbb{R} \times \Sigma$, is given by

$$ds^2 = \beta^2(t, x)dt^2 - h_t, \tag{5.1}$$

see [9, Theorem 1.1]. We will assume that the regularity of the spacetime metric g is C^τ.

In this section, we will prove the following results:

Theorem 5.1. *Let (M, g) be a C^τ globally hyperbolic spacetime with $\tau > 2$ and K_G the causal propagator of the Klein–Gordon operator P. Then,*

$$WF'^{-2+\tau-\epsilon}(K_G) \subset C$$

for every $\epsilon > 0$, C as in Eq. (1.1).

Theorem 5.2. *For a $C^{\tau+2}$ globally hyperbolic spacetime with $\tau > 2$,*

$$C \subset WF'^{-\frac{1}{2}}(K_G) \subset WF'^{\tau-\epsilon}(K_G) \subset C$$

holds for $0 < \epsilon < \tau + \frac{1}{2}$.

Remark 5.3. In the non-smooth case, we cannot expect $G(f) \in C^\infty(M)$ even if $f \in \mathcal{D}(M)$ as a consequence of the fact that $G(f)$ solves the homogeneous Cauchy problem. We know from [38, Proposition B.8] that for $f \in \mathcal{D}(M)$,

$$WF^s(G(f)) \subset \{(\tilde{x}, \tilde{\xi}) \in T^*M; (\tilde{x}, \tilde{\xi}, \tilde{y}, 0) \in WF^s(K_G) \text{ for some } y \in M\}.$$

Therefore, $WF'^s(K_G)$ might contain points that are not in C.

Remark 5.4. Since K_G is antisymmetric, we have that for $\rho(\tilde{x}, \tilde{y}) = (\tilde{y}, \tilde{x})$, $\rho^* K_G = -K_G$. This implies that if $(\tilde{x}, \tilde{\xi}, \tilde{y}, 0) \in WF^s(K_G)$ for some $y \in M$, then $(\tilde{y}, 0, \tilde{x}, \tilde{\xi}) \in WF^s(K_G)$ for some $y \in M$.

5.1. Proof of Theorem 5.1

Let $u \in H^{1+s-\tau\delta}_{comp}(M \times M)$ satisfy $P_{(t,x)}(\mathbf{x}, D_\mathbf{x})u = 0$.

Then also

$$\partial_\nu \left(\sqrt{|g|} g^{\mu\nu} \partial_\mu u \right) = 0.$$

Using the decomposition $\sqrt{|g|} g^{\mu\nu} \partial_\mu = (\sqrt{|g|} g^{\mu\nu} \partial_\mu)^\# + (\sqrt{|g|} g^{\mu\nu} \partial_\mu)^b$, we obtain

$$P_{(t,x)}(\mathbf{x}, D_\mathbf{x})u = \frac{1}{\sqrt{|g|}} \partial_\nu \left((\sqrt{|g|} g^{\mu\nu} \partial_\mu)^\# u + (\sqrt{|g|} g^{\mu\nu} \partial_\mu)^b u \right). \tag{5.2}$$

We state the behaviour outside the characteristic in this setting.

Lemma 5.5. *For $\tau > 2$ and any $\tilde{\epsilon} > 0$,*

$$WF^{-1-\tilde{\epsilon}+\tau}(K_G) \subset \operatorname{Char}(P_{(t,x)}) \cap \operatorname{Char}(P_{(s,y)}). \tag{5.3}$$

Proof. As the statement is microlocal, we can work in local coordinates in $T^*(\mathbb{R}^4 \times \mathbb{R}^4)$ and consider φK_G for $\varphi \in \mathcal{D}(\mathbb{R}^4 \times \mathbb{R}^4)$ with $\varphi = 1$ near $\mathbf{x}_0$.

Let $(\mathbf{x}_0, \boldsymbol{\xi}_0) = (\tilde{x}_0, \tilde{\xi}_0, \tilde{y}_0, \tilde{\eta}_0) \notin \operatorname{Char}(P_{(t,x)})$[2] Then, $0 < \sqrt{|g(\tilde{x})|}$ and $|g^{\mu\nu}(\tilde{x})\sqrt{|g(\tilde{x})|}\xi_\mu \xi_\nu| \geq C|\boldsymbol{\xi}|^2$ for suitable $C > 0$ in a conic neighbourhood of $(\mathbf{x}_0, \boldsymbol{\xi}_0)$.

In particular, $(\mathbf{x}_0, \boldsymbol{\xi}_0) \notin \operatorname{Char}(\partial_\nu(\sqrt{|g|} g^{\mu\nu} \partial_\mu)^\#)$, so there exists a microlocal parametrix $\tilde{q} \in S^{-2}_{1,\delta}$ such that

$$\tilde{q} \partial_\nu (\sqrt{|g|} g^{\mu\nu} \partial_\mu)^\# = I + r, \tag{5.4}$$

[2]Underscores to differentiate between the manifold points and points in $\mathbb{R}^8$ will be omitted. See Remark 3.4.

where $r(\mathbf{x}, D_{\mathbf{x}})$ is microlocally smoothing near $(\mathbf{x}_0, \boldsymbol{\xi}_0)$.

Since $P_{(t,x)}(\mathbf{x}, D_{\mathbf{x}})K_G = 0$, we have near $\mathbf{x}_0$

$$0 = \partial_\nu(\sqrt{|g|}g^{\mu\nu}\partial_\mu)K_G \tag{5.5}$$

$$= \partial_\nu(\sqrt{|g|}g^{\mu\nu}\partial_\mu)^{\#}\varphi K_G + \partial_\nu(\sqrt{|g|}g^{\mu\nu}\partial_\mu)^b\varphi K_G, \tag{5.6}$$

Since $(\sqrt{|g|}g^{\mu\nu}\xi_\mu)^b \in C^\tau S^{1-\tau\delta}_{1,\delta}$ for every $0 \le \delta < 1$, we obtain a bounded map

$$\partial_\nu(\sqrt{|g|}g^{\mu\nu}\partial_\mu)^b : H^{s+1-\tau\delta}(\mathbb{R}^4 \times \mathbb{R}^4) \to H^{s-1}(\mathbb{R}^4 \times \mathbb{R}^4), \tag{5.7}$$

$-(1-\delta)\tau < s < \tau\delta$.

Since $K_G \in H^{-1-\epsilon}_{loc}(M \times M)$ for every $\epsilon > 0$ by Lemma 4.3, we can choose δ such that $s = -2 + \tau\delta - \epsilon > 0$ so that by Eq. (5.5), we have locally

$$\partial_\nu(\sqrt{|g|}g^{\mu\nu}\partial_\mu)^{\#}\varphi K_G = -\partial_\nu(\sqrt{|g|}g^{\mu\nu}\partial_\mu)^b\varphi K_G \in H^{-3+\tau\delta-\epsilon}(\mathbb{R}^4 \times \mathbb{R}^4). \tag{5.8}$$

Applying the microlocal parametrix $\tilde{q}$, we obtain

$$\tilde{q}\partial_\nu(\sqrt{|g|}g^{\mu\nu}\partial_\mu)^{\#}\varphi K_G \in H^{-1+\tau\delta-\epsilon}(\mathbb{R}^4 \times \mathbb{R}^4). \tag{5.9}$$

By Eq. (5.4), Eq. (5.9) equals

$$(I + r(\mathbf{x}, D_{\mathbf{x}}))\varphi K_G. \tag{5.10}$$

Hence, $K_G \in H^{-1+\tau\delta-\epsilon}(M \times M)$ microlocally near $(\mathbf{x}_0, \boldsymbol{\xi}_0)$, so that $(\mathbf{x}_0, \boldsymbol{\xi}_0) \notin WF^{-1+\tau\delta-\epsilon}(K_G)$ for any $\epsilon > 0, 0 \le \delta < 1$. Choosing δ appropriately, we find that for every $\tilde{\epsilon} > 0$

$$WF^{-1-\tilde{\epsilon}+\tau}(K_G) \subset \mathrm{Char}(P_{(t,x)}). \tag{5.11}$$

Arguing analogously for $P_{(s,y)}$, we can see that

$$WF^{-1+\tau-\tilde{\epsilon}}(K_G) \subset \mathrm{Char}(P_{(t,x)}) \cap \mathrm{Char}(P_{(s,y)}). \tag{5.12}$$

□

Notice that

$$\mathrm{Char}(P_{(t,x)}) \cap \mathrm{Char}(P_{(s,y)}) = (\mathrm{Char}(P) \times \mathrm{Char}(P)) \cup \mathcal{A} \cup \mathcal{B},$$

where $\mathcal{A} := \{(\tilde{x}, 0, \tilde{y}, \tilde{\eta}) \in T^*(M \times M) : (\tilde{y}, \tilde{\eta}) \in \mathrm{Char}(P)\}$ and $\mathcal{B} := \{(\tilde{x}, \tilde{\xi}, \tilde{y}, 0) \in T^*(M \times M) : (\tilde{x}, \tilde{\xi}) \in \mathrm{Char}(P)\}$.

We will show now that the sets $\mathcal{A}$ and $\mathcal{B}$ do not belong to $WF^{-2+\tau-\tilde{\epsilon}}(K_G)$. Nevertheless, for higher wavefront sets, that may not be the case, see Remarks 5.3 and 5.4.

In order to show the result, we will need the following lemma.

Lemma 5.6. *$(\tilde{x}, \tilde{\xi}, \tilde{x}, \mu\tilde{\xi}) \notin WF^{-2+\tau-\tilde{\epsilon}}(K_{G^\pm})$ for $\mu \ne -1$.*

Proof. Consider a point $(\tilde{y}, \tilde{\eta}) \neq (\tilde{x}, \tilde{\xi})$ on the null bicharacteristic $\gamma(\tilde{x}, \tilde{\xi})$, with $\tilde{y} \in J^-(\tilde{x})$. Since $PG^+ = I$, it holds

$$K_I = K_{PG^+} = P_{(t,x)} K_{G^+} \tag{5.13}$$

with wavefront set the conormal to the diagonal. As $\mu \neq -1$, $(\tilde{x}, \tilde{\xi}, \tilde{x}, \mu\tilde{\xi})$ is not part of it, and neither are the points of the set $\gamma(\tilde{x}, \tilde{\xi}) \times \{(\tilde{x}, \mu\tilde{\xi})\}$. Hence, there exists an open conic neighbourhood W of the set of all $(\tilde{z}, \tilde{\zeta}, \tilde{x}, \mu\tilde{\xi}) \in T^*(M \times M)$, where $(\tilde{z}, \tilde{\zeta})$ lies on $\gamma(\tilde{x}, \tilde{\xi})$ between $(\tilde{x}, \tilde{\xi})$ and $(\tilde{y}, \tilde{\eta})$, that does not intersect $WF(K_I)$. We can assume that the base point projection $\Pi(W)$ is relatively compact. We choose $\varphi \in \mathcal{D}(M \times M)$ with $\varphi = 1$ on ΠW. Then,

$$\emptyset = WF(K_I) \cap W = WF(P_{(t,x)} K_{G^+}) \cap W. \tag{5.14}$$

Moreover, $P^{\#}_{(t,x)}(\varphi K_{G^+}) = P_{(t,x)}(\varphi K_{G^+}) - P^b_{(t,x)}(\varphi K_{G^+})$.

According to Remark 4.4, $K_{G^+} \in H^{-1-\epsilon}_{loc}(M \times M)$ for every $\epsilon > 0$, therefore $P^b_{(t,x)}(\varphi K_{G^+}) \in H^{-3-\epsilon+\tau}$. We now apply Theorem 3.6 with $u = \varphi K_{G^+}$, $s = -3-\tilde{\epsilon}+\tau$, $\Gamma = W$, $f = P_{(t,x)} K_{G^+} \in H^{\infty}_{mcl}(W)$, $P^b_{(t,x)}(\varphi K_{G^+}) \in H^s$. We have $\varphi K_{G^+} \in H^{\infty}_{mcl}$ near $(\tilde{y}, \tilde{\eta}, \tilde{x}, \mu\tilde{\xi})$, since $(\tilde{y}, \tilde{x})$ is not in the support of K_{G^+}. Hence, Theorem 3.6 implies that $K_{G^+} \in H^{-2-\epsilon+\tau}_{mcl}$ also in a conic neighbourhood of $(\tilde{x}, \tilde{\xi}, \tilde{x}, \mu\tilde{\xi})$, as this point lies on the integral curve of the Hamiltonian vector field for the principal symbol of $P_{(t,x)}$. Hence, $(\tilde{x}, \tilde{\xi}, \tilde{x}, \mu\tilde{\xi}) \notin WF^{-2+\tau-\epsilon}(K_{G^+})$. In an analogous way, we see that $(\tilde{x}, \tilde{\xi}, \tilde{x}, \mu\tilde{\xi}) \notin WF^{-2+\tau-\epsilon}(K_{G^-})$ by considering a point $(\tilde{y}, \tilde{\eta})$ on $\gamma(\tilde{x}, \tilde{\xi})$ with $\tilde{y} \in J^+(\tilde{x})$. □

Remark 5.7. Notice that the fact that the wavefront set of K_I is the conormal to the diagonal does not allow one to repeat the same argument in the case $(\tilde{x}, \tilde{\xi}, \tilde{x}, -\tilde{\xi}) \in WF^s(K_{G^+})$.

Remark 5.8. A similar argument holds for the case $(\tilde{x}, \lambda\tilde{\xi}, \tilde{x}, \tilde{\xi}) \notin WF^{-2+\tau-\tilde{\epsilon}}(K_{G^\pm})$ by using $P_{(s,y)}$.

Lemma 5.9. *For $\tau > 2$ and any $\tilde{\epsilon} > 0$,*

$$WF^{-2+\tau-\tilde{\epsilon}}(K_G) \subset \mathrm{Char}(P) \times \mathrm{Char}(P). \tag{5.15}$$

Proof. Using Lemma 5.5, we just need to show that there are no points from the sets $\mathcal{A}$ or $\mathcal{B}$. Let $(\tilde{x}, \tilde{\xi}, \tilde{y}, 0) \in \mathcal{B} \cap WF^{-2+\tau-\tilde{\epsilon}}(K_G)$ then by Theorem 3.6, we have that $(\gamma(\tilde{x}, \tilde{\xi}), \tilde{y}, 0) \in WF^{-2+\tau-\tilde{\epsilon}}(K_G)$. Now $\tilde{y} = (s_1, y_1)$ for some $s_1 \in \mathbb{R}, y_1 \in \Sigma$. By global hyperbolicity, $\gamma(\tilde{x}, \tilde{\xi})$ intersects $\{s_1\} \times \Sigma$ in exactly one point with the covector $\chi \neq 0$. Since causally separated points are not in $\mathrm{supp}(K_G)$, the point of intersection has to be (s_1, y_1). Hence, $(s_1, y_1, \chi, s_1, y_1, 0) \in WF^{-2+\tau-\tilde{\epsilon}}(K_G) \subset (WF^{-2+\tau-\tilde{\epsilon}}(K_{G^+}) \cup WF^{-2+\tau-\tilde{\epsilon}}(K_{G^-}))$. This is a contradiction to Lemma 5.6. A similar argument holds for points in $\mathcal{A}$. □

Remark 5.10. The existence of symmetries allows one to show that the Sobolev wavefront set in Lemma 5.5 is already disjoint from the sets $\mathcal{A}$ and $\mathcal{B}$. For example, if M is stationary, K_G is of the form $K_G(t - s, x, y)$. Therefore, one has the additional equation $(\partial_t + \partial_s)K_G = 0$, that implies $WF^l(K_G) \subset \mathrm{Char}(\partial_t+\partial_s)$ for $l \in \mathbb{R}$. Moreover, $\mathrm{Char}(\partial_t+\partial_s)\cap\mathcal{A} = \emptyset$ and $\mathrm{Char}(\partial_t+\partial_s)\cap\mathcal{B} = \emptyset$. A similar argument holds in the case of a sufficiently spatially symmetric spacetime, e.g. cosmological space of the form $ds^2 = a(t)(-dt^2 + dx^2 + dy^2 + dz^2)$. In this case, K_G is of the form $K_G(t, s, x_1 - x_2, y_1 - y_2, z_1 - z_2)$ due to the spatial invariance.

Now we establish that points above the diagonal are of a specific form.

Lemma 5.11. *If* $(\tilde{x}, \tilde{\xi}, \tilde{x}, \tilde{\eta}) \in WF^{-2+\tau-\tilde{\epsilon}}(K_G)$ *for* $\tau > 2$*, and some* $\tilde{\epsilon} > 0$*, then* $\tilde{\eta} = -\tilde{\xi}$.

Proof. Suppose $\tilde{\eta}$ and $\tilde{\xi}$ are linearly independent, i.e. $\tilde{\eta} \neq \mu\tilde{\xi}$ for $\mu \in \mathbb{R}$. By Lemma (5.9) $(\tilde{x}, \tilde{\xi}, \tilde{x}, \tilde{\eta}) \in \mathrm{Char}(P)\times\mathrm{Char}(P)$. Now we choose a Cauchy hypersurface $\Sigma_{t_0} = \{t_0\} \times \Sigma$ such that the null geodesic with initial data $(\tilde{x}, \tilde{\xi})$ and the null geodesic with initial data $(\tilde{x}, \tilde{\eta})$ intersect it. These points of intersections are unique by global hyperbolicity. Moreover, using the condition $\tilde{\eta} \neq \mu\tilde{\xi}$, we can choose Σ_{t_0} such that these points are distinct. We denote these points by $(t_0, x_0), (t_0, y_0)$. Furthermore, they are not causally related. Now $K_G \in H^{-1-\epsilon}_{loc}(M \times m)$ so $\partial_\nu(\sqrt{|g|}g^{\mu\nu}\partial_\mu)^b K_G \in H^{-3-\epsilon+\tau\delta}(\mathbb{R}^4 \times \mathbb{R}^4)$ and therefore if $(\tilde{x}, \tilde{\xi}, \tilde{y}, -\tilde{\eta}) \in WF^{-2+\tau-\epsilon}(K_G)$ then $(\gamma(\tilde{x}, \tilde{\xi}), \gamma(\tilde{y}, -\tilde{\eta})) \in WF^{-2-\epsilon+\tau}(K_G)$ where $\gamma(\tilde{x}, \tilde{\xi})$ is the null bicharacteristic with initial data $(\tilde{x}, \tilde{\xi})$ and $\gamma(\tilde{x}, \tilde{\eta})$ is the null bicharacteristic with initial data $(\tilde{x}, \tilde{\eta})$.

In particular $(t_0, x_0, t_0, y_0) \in \Pi(WF^{-\frac{1}{2}-\epsilon+\tau}(K_G))$, where Π is the projection from $T^*(M\times M)$ to $M\times M$. However, this is a contradiction to Proposition 4.1, since $(t_0, x_0, t_0, y_0) \notin \mathrm{supp}(K_G)$. Therefore, $\tilde{\eta} = \mu\tilde{\xi}$.

Now as a consequence of the fact that $K_G = K_{G^+}+K_{G^-}$ and $WF^s(K_G) \subset WF^s(K_{G^+}) \cup WF^s(K_{G^-})$ for all s, Lemma 5.6 implies that $\mu = -1$. □

Proof of Theorem 5.1. Let $(\tilde{x}, \tilde{\xi}, \tilde{y}, -\tilde{\eta}) \in WF^{-2+\tau-\epsilon}(K_G)$. The propagation of singularities result (Theorem 3.6) implies that $(\gamma(\tilde{x}, \tilde{\xi}), \gamma(\tilde{y}, -\tilde{\eta})) \in WF^{-2-\epsilon+\tau}(K_G)$, where $\gamma(\tilde{x}, \tilde{\xi})$ is the null bicharacteristic with initial data $(\tilde{x}, \tilde{\xi})$ and $\gamma(\tilde{y}, -\tilde{\eta})$ is the null bicharacteristic with initial data $(\tilde{y}, -\tilde{\eta})$.

Now we choose a Cauchy surface $\Sigma_{t_1} = \{t_1\} \times \Sigma$ and suppose that $(t_1, x_1, \tilde{\xi}_1, t_1, x_2, \tilde{\xi}_2) \in (\gamma(\tilde{x}, \tilde{\xi}), \gamma(\tilde{y}, -\tilde{\eta})) \cap (\Sigma^2_{t_1})$. By Lemmas 4.1 and 5.9, $(t_1, x_1, \tilde{\xi}_1), (t_1, x_2, \tilde{\xi}_2) \in \mathrm{Char}(P)$, $x_1 = x_2$, and $\tilde{\xi}_2 = -\tilde{\xi}_1$.

Next we define a curve $\tilde{\gamma} : (-\infty, \infty) \to M$ as follows. First, we shift the parametrization λ in the definition of the null bicharacteristics so that

$$\gamma(\tilde{x}, \tilde{\xi})(t_1) = (t_1, x_1, \tilde{\xi}_1), \quad \gamma(\tilde{y}, -\tilde{\eta})(t_1) = (t_1, x_1, -\tilde{\xi}_1).$$

Then, we denote by $\Pi : T^*M \to M$ the canonical projection and define two curves in M by

$$\gamma_1(\lambda) := \Pi(\gamma(\tilde{x}, \tilde{\xi})(\lambda)), \quad \gamma_2(\lambda) := \Pi(\gamma(\tilde{y}, -\tilde{\eta})(\lambda)).$$

Notice that we have $\gamma_1(t_1) = (t_1, x_1), \dot{\gamma_1}(t_1) = g^{-1}(\tilde{\xi}_1, \cdot)$ and $\gamma_2(t_1) = (t_1, x_1), \dot{\gamma_2}(t_1) = g^{-1}(-\tilde{\xi}_1, \cdot)$. Moreover, we can assume that $\tilde{x} = \gamma_1(a)$ and $\tilde{y} = \gamma_2(b)$ for suitable $a, b \in \mathbb{R}$ with $a < t_1 < b$.

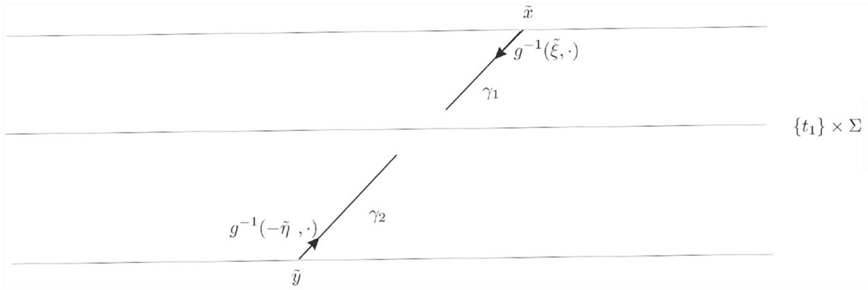

FIGURE 2. γ_1 is a null geodesic that satisfies $\gamma(a) = \tilde{x}, \dot{\gamma}_1(a) = g^{-1}(\xi, \dot{)}$ and γ_2 is a null geodesic that satisfies $\gamma(b) = \tilde{y}, \dot{\gamma}_2(b) = g^{-1}(-\eta, \cdot)$

Finally, let

$$\tilde{\gamma}(\lambda) = \begin{cases} \gamma_1(\lambda) & \lambda \in (-\infty, t_1] \\ -\gamma_2(\lambda) & \lambda \in (t_1, \infty) \end{cases} \tag{5.16}$$

where $-\gamma_2$ denotes the curve with opposite orientation.

Then, $\tilde{\gamma}(a) = \tilde{x}$, $\tilde{\gamma}(b) = \tilde{y}$; moreover, $g(\cdot, \dot{\tilde{\gamma}})|_{T_{\tilde{x}}M} = \tilde{\xi}$, $g(\cdot, \dot{\tilde{\gamma}})|_{T_{\tilde{y}}M} = \tilde{\eta}$, and therefore, $\tilde{\gamma}$ is a null geodesic between $\tilde{x}$ and $\tilde{y}$ with cotangent vectors $\tilde{\xi}$ at $\tilde{x}$ and $\tilde{\eta}$ at $\tilde{y}$, i.e. $(\tilde{x}, \tilde{\xi}, \tilde{y}, -\tilde{\eta}) \in C' := \{(\tilde{x}, \tilde{\xi}, \tilde{y}, -\tilde{\eta}); (\tilde{x}, \xi; \tilde{y}, \tilde{\eta}) \in C\}$, see Fig. 2.

This shows

$$WF^{-2-\epsilon+\tau}(K_G) \subset C' \tag{5.17}$$

or, equivalently $WF'^{-2-\epsilon+\tau}(K_G) \subset C$. □

5.2. Proof of Theorem 5.2

Now we show that C is contained in $WF'^{-\frac{1}{2}}(K_G)$.

Lemma 5.12. *Let P be the Klein–Gordon operator with $g \in C^{\tau+2}, \tau > 2$. Then, $C \subset WF'^{-\frac{1}{2}}(K_G)$*

Proof. Using Proposition C.1 of [28], see also [48], there exists an interpolating spacetime of regularity C^τ, $(\bar{M}, \bar{g})$, which satisfies the following conditions: There exist times t_1 and t_2 such that for $t < t_1$, $(\bar{M}, \bar{g})$ is isometric to a neighbourhood of a Cauchy surface $\tilde{\Sigma}$ of a smooth, globally hyperbolic spacetime (M_s, g_s). Furthermore, for $t > t_2$, $(\bar{M}, \bar{g})$ is isometric to a neighbourhood of a Cauchy surface Σ of the non-smooth spacetime (M, g).

Now if $K_{\bar{G}}$ is the causal propagator associated to $(\bar{M}, \bar{g})$, its restriction to $t < t_1$, denoted $K_{\bar{G}}|_{t<t_1}$, corresponds to the smooth causal propagator [5, Proposition 3.5.1] and therefore

$$WF'(K_{\bar{G}}|_{t<t_1}) = \bar{C} \cap T^*(\{(t,x) \in \bar{M}; t < t_1\} \times \{(t,x) \in \bar{M}; t < t_1\}),$$

where $\bar{C}$ denotes the canonical relationship associated to $\bar{g}$.

Let $(\tilde{x}, \tilde{\xi}, \tilde{x}, -\tilde{\xi}) \in \bar{C}'$ in the non-smooth region, i.e. $\tilde{x} = (t_3, x)$ with $t_3 > t_2$.

By global hyperbolicity, the base point projections of the null bicharacteristics $\gamma(\tilde{x}, \tilde{\xi})$ and $\gamma(\tilde{x}, -\tilde{\xi})$ intersect the hypersurface $t = t_0 < t_1$ at one unique point denoted w. Moreover, as a consequence of being in $\bar{C}'$, we have $(w, \chi, w, -\chi) = (\gamma(\tilde{x}, \tilde{\xi}) \times \gamma(\tilde{x}, -\tilde{\xi})) \cap (\tilde{\Sigma}_{t_0} \times \tilde{\Sigma}_{t_0})$.

Since we are in the smooth part, smooth theory implies, in particular, that $(w, \chi, w, -\chi) \in WF^s(K_{\bar{G}}|_{t<t_1})$ for $-\frac{1}{2} \leq s$ by combining [25, Theorem 6.5.3] and [38, Proposition B.10]. Now, an application of Theorem 3.6 gives $(\tilde{x}, \tilde{\xi}, \tilde{x}, -\tilde{\xi}) \in WF^s(K_{\bar{G}})$ for $-\frac{1}{2} \leq s$.

Furthermore, by [36, Theorem 5.10, Theorem 5.8], the restriction of $K_{\bar{G}}$ to $t > t_2$, denoted $K_{\bar{G}}|_{t>t_2}$, in a neighbourhood of Σ_{t_3} is the same as the restriction of the non-smooth causal propagator, K_G, associated to (M, g). Hence, $(\tilde{x}, \tilde{\xi}, \tilde{x}, -\tilde{\xi}) \in WF^s(K_G)$.

Another application of Theorem 3.6 using the null bicharacteristics from (M, g) gives $C' \subset WF^{-\frac{1}{2}}(K_G)$, i.e. $C \subset WF'^{-\frac{1}{2}}(K_G)$. □

Proof of Theorem 5.2. The combination of Lemma 5.12 and Theorem 5.1 gives the result. □

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

6. Appendix

6.1. Sobolev Spaces

$H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the set of all tempered distributions u on $\mathbb{R}^n$ whose Fourier transforms $\mathcal{F}u$ are regular distributions satisfying

$$\|u\|^2_{H^s(\mathbb{R}^n)} := \int \langle \xi \rangle^{2s} |\mathcal{F}u(\xi)|^2 \, d^n\xi < \infty.$$

Let (M, g) be a (possibly) non-compact Riemannian manifold which is geodesically complete. The Laplace–Beltrami operator $-\Delta_g$ is essentially self-adjoint if the regularity of the metric is C^τ for $\tau \geq 2$, [57, Theorem 2.4]. For lower regularity, see Appendix 6.2. By $H^s(M)$, we denote the completion of $\mathcal{D}(M)$ with respect to the norm

$$\|u\|_{H^s(M)} := \|(I - \Delta_g)^{s/2} u\|_{L^2(M)}.$$

If M is compact, $H^s(M)$ is independent of the metric.

For an open subset U of M, we define the local Sobolev spaces:

$$H^s_{loc}(U) := \{u \in \mathcal{D}'(M); \varphi u \in H^s(M) \text{ for all } \varphi \in \mathcal{D}(U)\}.$$

and

$$H^s_{comp}(U) := \{u \in \mathcal{D}'(M); u \in H^s(M) \text{ and } \operatorname{supp}(u) \subset U \text{ is compact}\}.$$

Notice that given a manifold M, the spaces $H^s_{loc}(U)$ and $H^s_{comp}(U)$ are independent of the Riemannian metric used to define the Sobolev spaces $H^s(M)$.

For a compact n-dimensional manifold Σ, we can also define Sobolev spaces on $\mathbb{R} \times \Sigma$ relying on local coordinates. Namely, suppose $\{U_j : j \in J\}$ is an open cover of Σ by coordinate charts and $\{\varphi_j : j \in J\}$ is a subordinate partition of unity. Given a function u on $\mathbb{R} \times \Sigma$, we say that $u \in \tilde{H}^s(\mathbb{R} \times \Sigma)$, provided that, using local coordinates on Σ, $\varphi_j(x)u(t,x) \in H^s(\mathbb{R} \times \mathbb{R}^n)$ for $j = 1, \ldots, J$ (more formally: For the coordinate map $\kappa_j : U_j \to \mathbb{R}^n$, we have $(id \times \kappa_{j*})(\varphi_j u) \in H^s(\mathbb{R} \times \Sigma)$). For integer k, this is equivalent to asking that, for all multi-indices α with $|\alpha| \leq k$, we have $\partial^\alpha_{t,x} u \in L^2(\mathbb{R} \times \mathbb{R}^n)$ in local coordinates. Moreover, $\mathbb{R} \times \Sigma$ is a manifold of bounded geometry and the Sobolev spaces introduced in this setting coincide with the spaces $\tilde{H}^s$, see e.g. Theorem 3.9 in [30].

Lemma 6.1. *Let $g = dt^2 + h_{ij}dx^i dx^j$ be an ultrastatic metric of regularity C^τ on $\mathbb{R} \times \Sigma$ with $\tau > 1$. Then,*

$$H^s(\mathbb{R} \times \Sigma) = \tilde{H}^s(\mathbb{R} \times \Sigma), \quad 0 \leq s \leq 2,$$

i.e. the two Hilbert spaces coincide up to equivalent norms.

Proof. The assertion is obvious for $s = 0$, when $\tilde{H}^0(\mathbb{R} \times \Sigma) = L^2(\mathbb{R} \times \Sigma) = H^0(\mathbb{R} \times \Sigma)$. We have

$$H^s(\mathbb{R} \times \Sigma) = \mathscr{D}((I - \Delta_g)^{s/2}) = [L^2(\mathbb{R} \times \Sigma), \mathscr{D}(I - \Delta_g)]_{s/2},$$

where the first equality holds by definition and the second is [3, Section I.2.9] for complex interpolation.

In view of the interpolation property for the standard Sobolev spaces, it is sufficient to show the assertion for $s = 2$. Assuming that $\tau > 1$, the operator $I - \Delta_g$ is strongly elliptic with coefficients in $C^{\tau-1}$. By elliptic regularity, its maximal domain is $\tilde{H}^2(\mathbb{R} \times \Sigma)$. This is a well-known fact, although a reference seems to be hard to find. In order to see it we first note that, by Lax–Milgram's theorem, every $u \in L^2(\mathbb{R} \times \Sigma)$ with $\Delta_g u \in L^2(\mathbb{R} \times \Sigma)$ belongs to $H^1(\mathbb{R} \times \Sigma)$. Symbol smoothing as in Remark 3.5 then shows that u even belongs to $\tilde{H}^2(\mathbb{R} \times \Sigma)$. Hence, the maximal domain is a subset of $\tilde{H}^2(\mathbb{R} \times \Sigma)$.

The minimal domain is also $\tilde{H}^2(\mathbb{R} \times \Sigma)$, since $\mathcal{D}(\mathbb{R} \times \Sigma)$ is dense in $\tilde{H}^2(\mathbb{R} \times \Sigma)$. Hence, $\mathscr{D}(I - \Delta_g) = \tilde{H}^2(\mathbb{R} \times \Sigma)$. □

Remark 6.2. An analogous construction can be performed for $\mathbb{R}^2 \times \Sigma^2$ and the analogue of Lemma 6.1 holds.

6.2. Essential Self-Adjointness of the Laplace–Beltrami Operator

Theorem 6.3. *Let (Σ, h) be a smooth compact n-dimensional manifold equipped with a Riemannian metric of regularity $C^1(\Sigma)$. Then, the Laplace–Beltrami operator Δ_h is essentially self-adjoint.*

We follow Strichartz's article [57] that uses the following criterion [51, Theorem X.1].

Theorem 6.4. *Let A be any closed negative-definite symmetric, densely defined operator on a Hilbert space H. Then, $A = A^*$ if and only if there are no eigenvectors with positive eigenvalue in the domain of A^*.*

Now we will state the following helpful result

Proposition 6.5. *Let u be an $L^2(\Sigma)$ function that satisfies $\Delta u = \lambda u$ for some $\lambda > 0$. Then, u is identically zero.*

Proof. Let u be a weak solution which by elliptic regularity satisfies $u \in H^2(\Sigma)$.

Hence,

$$\lambda\,(u, u)_{L^2(\Sigma)} = (\Delta u, u)_{L^2(\Sigma)} = -\,(du, du)_{L^2(\Sigma)} \tag{6.1}$$

Now $\lambda > 0$ so we have $u = 0$. □

Proof of Theorem 6.3. By direct computation, Δ_h is negative-definite and symmetric. That it is densely defined follows from the density of $\mathcal{D}(M)$ in $L^2(M)$ for continuous metrics (see [4, Proposition 7] for even rougher cases). The application of Theorem 6.4 taking into account Proposition 6.5 gives the result. □

For the non-compact case, one could follow the construction in Strichartz's article. However, suitable modifications are required under the regularity of Theorem 6.3. For example, one would have to use an integral distance as in [21, Theorem 5.11] to show the desired properties of the approximations to unity. Then, one would need to verify that the elliptic regularity results hold in that situation as well.

Since we are only interested in the case of $M = \mathbb{R}^2 \times \Sigma^2$ and the operator $2mI - \partial_{tt} - \partial_{ss} - \Delta_{h_x} - \Delta_{h_y}$ under $C^{1,1}$ regularity assumptions, we will proceed in a different manner.

Lemma 6.6. *The operator $2mI - \partial_{tt} - \partial_{ss} - \Delta_{h_x} - \Delta_{h_y}$, where h_x, h_y are Riemannian metrics of regularity $C^{1,1}$, is essentially self-adjoint with domain $H^2(\mathbb{R}^2 \times \Sigma^2)$.*

Proof. By [52, Lemma 2.1], we obtain that $-\partial_{tt} - \partial_{ss} - \Delta_{h_x} - \Delta_{h_y}$ is essentially self-adjoint in $L^2(\mathbb{R}^2 \times \Sigma^2)$ with domain $\mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{R}) \otimes C^\infty(\Sigma^2)$. Since $\mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{R}) \otimes C^\infty(\Sigma^2)$ is dense in $H^2(\mathbb{R}^2 \times \Sigma^2)$ which carries the graph norm of $-\partial_{tt} - \partial_{ss} - \Delta_{h_x} - \Delta_{h_y}$, we obtain that the closure of the domain is $H^2(\mathbb{R}^2 \times \Sigma^2)$. Now $-\partial_{tt} - \partial_{ss} - \Delta_{h_x} - \Delta_{h_y}$ and $2mI$ commute and are self-adjoint. By [49, Lemma 4.16.1], $2mI - \partial_{tt} - \partial_{ss} - \Delta_{h_x} - \Delta_{h_y}$ is self-adjoint with domain $H^2(\mathbb{R}^2 \times \Sigma^2)$. □

6.3. An Equivalent Sobolev Norm

The main results of this section are the following proposition and Corollary 6.10.

Proposition 6.7. *Let Σ be a compact manifold and $\{\phi_j \otimes \phi_k; j, k = 1, 2, \ldots\}$ be an orthonormal basis of $L^2(\Sigma) \otimes_H L^2(\Sigma)$ associated to the eigenfunctions $\{\phi_j\}$ of the operator $mI - \Delta_h$, $m > 0$. Writing $u \in L^2((\mathbb{R} \times \Sigma) \times (\mathbb{R} \times \Sigma)) \cong L^2(\mathbb{R}^2 \times \Sigma^2) \cong L^2(\mathbb{R}^2) \otimes_H L^2(\Sigma) \otimes_H L^2(\Sigma)$ in the form*

$$u(t, s, x, y) = \sum_{j,k} u_{jk}(t, s)\phi_j(x)\phi_k(y) \quad \text{with } u_{jk} = \langle u, \phi_j \otimes \phi_k \rangle \in L^2(\mathbb{R}^2), \tag{6.2}$$

we obtain the following alternative description of the Sobolev spaces: For $0 \leq s \leq 2$

$$H^s(\mathbb{R}^2 \times \Sigma^2) = \left\{ u \in \mathcal{S}'(\mathbb{R}^2 \times \Sigma^2); \sum_{j,k} \int_{\mathbb{R}^2} (\xi_0^2 + \eta_0^2 + \lambda_j^2 + \lambda_k^2)^s |(\mathcal{F}u_{jk})(\xi_0, \eta_0)|^2 d\xi_0 d\eta_0 < \infty \right\}.$$

Here, $\mathcal{S}'(\mathbb{R}^2 \times \Sigma^2)$ is the dual space to $\mathcal{S}(\mathbb{R}^2 \times \Sigma^2) := \mathcal{S}(\mathbb{R}^2)\hat{\otimes}_\pi C^\infty(\Sigma^2)$. First we show the result in the particular case $s = 2$:

Lemma 6.8.

$$H^2(\mathbb{R}^2 \times \Sigma^2) = \left\{ u \in \mathcal{S}'(\mathbb{R}^2 \times \Sigma^2); \sum_{j,k} \int_{\mathbb{R}^2} (|\xi_0|^2 \right.$$

$$+|\eta_0|^2 + \lambda_j^2 + \lambda_k^2)^2 |(\mathcal{F}u_{jk})(\xi_0, \eta_0)|^2 d\xi_0 d\eta_0 < \infty \Bigg\}.$$

Proof. By definition (see Appendix 6.1)

$$\begin{aligned} H^2(\mathbb{R}^2 \times \Sigma^2) \ &= \{u \in L^2(\mathbb{R}^2 \times \Sigma^2); (I - \partial_{tt} - \partial_{ss} - \Delta_{h_x} - \Delta_{h_y})u \\ &\quad \in L^2(\mathbb{R}^2 \times \Sigma^2)\}, \end{aligned} \tag{6.3}$$

where we have equipped $\Sigma \times \Sigma$ with the product metric $\tilde{h}$ induced by the metric h on each of the components so that $\Delta_{\tilde{h}} = \partial_{tt} + \partial_{ss} + \Delta_{h_x} + \Delta_{h_y}$, where Δ_{h_x} and Δ_{h_y} are the Laplacians for the metric h on the first and second components of $\Sigma \times \Sigma$, respectively. Since $m > 0$ we may (at the expense of obtaining an equivalent norm) replace I in Eq. (6.3) by $2mI$. Writing $u \in H^2(\mathbb{R}^2 \times \Sigma^2) \subset L^2(\mathbb{R}^2 \times \Sigma^2)$ in the form (6.2) and using the orthonormality of the set $\{\phi_l\}_l$ in $L^2(\Sigma)$, we obtain

$$\begin{aligned} &\|u\|^2_{H^2(\mathbb{R}^2\times\Sigma^2)} \\ &\quad := \int_{\mathbb{R}^2} \int_{\Sigma\times\Sigma} |(2mI - \partial_{tt} - \partial_{ss} - \Delta_{h_x} - \Delta_{h_y})u|^2 \sqrt{h(x)}\sqrt{h(y)} dt ds dx dy \\ &\quad = \int_{\mathbb{R}^2} \sum_{j,k} | - (\partial_{tt} + \partial_{ss})u_{jk}(t,s) + \lambda_j^2 u_{jk}(t,s) + \lambda_k^2 u_{jk}(t,s)|^2 dt ds. \end{aligned}$$

Applying the Fourier transform in (s, t), Plancherel's theorem shows that

$$\|u\|^2_{H^2(\mathbb{R}^2\times\Sigma^2)} = \sum_{j,k} \int_{\mathbb{R}^2} \left(\xi_0^2 + \eta_0^2 + \lambda_j^2 + \lambda_k^2\right)^2 |(\mathcal{F}u_{jk})(\xi_0, \eta_0)|^2 d\xi_0 d\eta_0$$

which proves the result. □

Before proving the main proposition, we state the following result found in Amann [3, I.(2.9.8)].

Theorem 6.9. *Let A be a nonnegative self-adjoint operator. Then, we have the following relation for the domains of the powers of A:*

$$\mathscr{D}(A^{(1-\theta)\alpha+\theta\beta}) = [\mathscr{D}(A^\alpha), \mathscr{D}(A^\beta)]_\theta$$

for $0 \le \operatorname{Re}\alpha < \operatorname{Re}\beta$ and $0 < \theta < 1$. Here $[\cdot,\cdot]_\theta$ denotes complex interpolation.

Proof of Proposition 6.7. Since $(\mathbb{R} \times \Sigma) \times (\mathbb{R} \times \Sigma)$ is a complete manifold, the operator $2mI - \Delta_{\tilde{h}}$ is positive and self-adjoint (see Appendix 6.2). Using Theorem 6.9, we obtain for $0 < \theta < 1$

$$\begin{aligned} H^{2\theta}(\mathbb{R}^2 \times \Sigma^2) &= \mathscr{D}((2mI - \Delta_{\tilde{h}})^\theta) \\ &= \{u \in \mathcal{S}'(\mathbb{R}^2 \times \Sigma^2); (2mI - \Delta_{\tilde{h}})^\theta u \in L^2(\mathbb{R}^2 \times \Sigma^2)\}. \end{aligned}$$

Since $2mI - \Delta_{\tilde{h}} = 2mI - \partial_{tt} - \partial_{ss} - \Delta_{h_x} - \Delta_{h_y}$ can be written as a multiplication operator in the form

$$(2mI - \Delta_{\tilde{h}})u$$

$$= \sum_{j,k} \phi_j(x)\phi_k(y) \int_{\mathbb{R}^2} e^{i(\xi_0 t+\eta_0 s)} \left(\xi_0^2 + \eta_0^2 + \lambda_j^2 + \lambda_k^2\right) (\mathcal{F}u_{jk})(\xi_0, \eta_0) d\xi_0 d\eta_0,$$

we infer from the orthonormality of the ϕ_j that $(2mI - \Delta_{\tilde{h}})^\theta u$ in $L^2(\mathbb{R}^2 \times \Sigma^2)$, if and only if

$$\Big\| \sum_{j,k} \phi_j(x)\phi_k(y) \int_{\mathbb{R}^2} e^{i(\xi_0 t+\eta_0 s)} \left(\xi_0^2 + \eta_0^2 + \lambda_j^2 + \lambda_k^2\right)^\theta (\mathcal{F}u_{jk})(\xi_0, \eta_0) d\xi_0 d\eta_0 \Big\|^2_{L^2(\mathbb{R}^2\times\Sigma^2)}$$
$$= \sum_{j,k} \int_{\mathbb{R}^2} \left(\xi_0^2 + \eta_0^2 + \lambda_j^2 + \lambda_k^2\right)^{2\theta} |(\mathcal{F}u_{jk})(\xi_0, \eta_0)|^2 d\xi_0 d\eta_0 < \infty.$$

This establishes the required equivalence. □

Corollary 6.10. *For $-1 \le \theta \le 0$, we obtain by L^2-duality that*

$$H^{2\theta}(\mathbb{R}^2 \times \Sigma^2) = (H^{-2\theta}(\mathbb{R}^2 \times \Sigma^2))'$$
$$= \left\{ u \in \mathcal{S}'(\mathbb{R}^2 \times \Sigma^2); \sum_{j,k} \int_{\mathbb{R}^2} (|\xi_0|^2 + |\eta_0|^2 + \lambda_j^2 + \lambda_k^2)^{2\theta} |\mathcal{F}u_{jk}(\xi_0, \eta_0)|^2 d\xi_0 d\eta_0 < \infty \right\},$$

with $u_{j,k} = \langle u, \phi_j \otimes \phi_k \rangle \in \mathcal{S}'(\mathbb{R}^2)$.

6.4. The Ultrastatic Case

In this case, we consider a Lorentzian metric g on $M = \mathbb{R} \times \Sigma$ with Σ compact of the form

$$ds^2 = dt^2 - h_{ij}(x) dx^i dx^j$$

where $h_{ij}(x)$ are the components of a time independent Riemannian metric of Hölder regularity C^τ (when $\tau \in \mathbb{N}$ we will consider the Zygmund spaces C_*^τ, introduced in Definition 3.1).

The Klein–Gordon operator P on M is

$$P\phi = \partial_{tt}\phi - \Delta_h \phi + m^2 \phi \tag{6.4}$$

with $\Delta_h \phi = \frac{1}{\sqrt{h}} \partial_{x^i}(h^{ij}\sqrt{h}\partial_{x^j}\phi)$ and $m > 0$.

The causal propagator G is given by $-\dfrac{\sin(A^{\frac{1}{2}}(t-s))}{A^{\frac{1}{2}}}$ where $A := -\Delta_h + m^2$ is self-adjoint on $L^2(\Sigma)$, see Appendix 6.2.

Moreover, the spectrum of A is a discrete set of positive eigenvalues which we denote by $\{\lambda_j^2; j = 1, 2, \ldots\}$, listed according to their (finite) multiplicity. The associated set $\{\phi_j\}_{j\in\mathbb{N}}$ of normalised real eigenfunctions is an orthonormal basis of $L^2(\Sigma)$, see [43, Theorem 5.8]. For $u, v \in \mathcal{D}(M)$, we have $G(v) \in \mathcal{D}'(M)$ given by

$$\langle G(v), u \rangle$$

$$:= \int_\Sigma \int_{-\infty}^{\infty} \left(-\frac{\sin(A^{\frac{1}{2}}(t-s))}{A^{\frac{1}{2}}} v \right)(t,x) u(t,x) \sqrt{h(x)} dx dt$$

$$= -\int_M \left(\int_{-\infty}^{\infty} \sum_j \lambda_j^{-1} \sin(\lambda_j (t-s)) \phi_j(x) \int_\Sigma \phi_j(y) v(s,y) \sqrt{h(y)} dy ds \right)$$

$$u(t,x)\sqrt{h(x)} dx dt. \tag{6.5}$$

Using that $\langle G(v), u\rangle = \langle K_G, v \otimes u\rangle$ gives the singular integral kernel representation

$$K_G(t,x;s,y) = -\sum_j \lambda_j^{-1} \sin(\lambda_j(t-s))\phi_j(x)\phi_j(y). \tag{6.6}$$

6.4.1. Global Regularity. Now we show in Lemma 6.11 that, in ultrastatic spacetimes, the global regularity of the causal propagator is the same as in the smooth case

Lemma 6.11. *$K_G \in H_{loc}^{-\frac{1}{2}-\epsilon}(M \times M)$ for every $\epsilon > 0$.*

Proof. This follows from Corollary 6.10 similar to the computation in [54, Theorem 4.10]. □

It will be useful to consider the following bidistribution, K_A that satisfies $\partial_t K_A = K_G$.

Corollary 6.12. *Let $K_A \in \mathcal{D}'(M \times M)$ be the bidistribution given by*

$$K_A(u \otimes v) := \int_M \left(\int_{-\infty}^{\infty} \sum_j \lambda_j^{-2} \cos(\lambda_j(t-s))\phi_j(x) \int_\Sigma \phi_j(y) v(s,y)\sqrt{h(y)} dy ds \right)$$

$$u(t,x)\sqrt{h(x)} dx dt,$$

Then,

$$K_A \in H_{loc}^{\frac{1}{2}-\epsilon}(M \times M) \text{ for every } \epsilon > 0. \tag{6.7}$$

Proof. This follows from Proposition 6.7 similar to the computation in [54, Corollary 4.11]. □

6.4.2. Wavefront Set Estimates. Now we show some helpful lemmas in order to prove Theorems 6.15 and 6.17 which are the main results of the section.

First, we establish the microlocal regularity of K_G outside the set $\mathrm{Char}(P) \times \mathrm{Char}(P)$.

In the following proofs, we use the distribution K_A, because a direct application of Theorem 3.3 for K_G is not possible, since for δ close to 1, the above σ cannot take the value $-\frac{1}{2}$.

Lemma 6.13. *For $\tau > 2$ and any $\tilde{\epsilon} > 0$,*

$$WF^{-\frac{1}{2}-\tilde{\epsilon}+\tau}(K_G) \subset \mathrm{Char}(P) \times \mathrm{Char}(P). \tag{6.8}$$

Proof. This is an application of Theorem 3.3, the observation that K_A satisfies $(\partial_t + \partial_s)K_A = 0$ and $WF^{-\frac{1}{2}-\tilde{\epsilon}+\tau}(K_G) \subset WF^{\frac{1}{2}-\tilde{\epsilon}+\tau}(K_A)$. The proof is along the lines [54, Lemma 4.13] □

Now we establish that points above the diagonal are of a specific form.

Lemma 6.14. *If $(\tilde{x}, \tilde{\xi}, \tilde{x}, \tilde{\eta}) \in WF^{-\frac{3}{2}-\tilde{\epsilon}+\tau}(K_G)$ for $\tau > 2$ and some $\tilde{\epsilon} > 0$, then $\tilde{\eta} = -\tilde{\xi}$.*

Proof. This is a consequence of Theorem 3.7 combined with the support properties of K_G. The proof is along the lines of that for [54, Lemma 4.16] □

Now we state one of the main results:

Theorem 6.15. *Let (M, g) be a C^τ ultrastatic spacetime with $\tau > 2$ and K_G the causal propagator. Then, $WF'^{-\frac{3}{2}-\epsilon+\tau}(K_G) \subset C$ for every $\epsilon > 0$ and C as in Eq. (1.1).*

Proof. Let $(\tilde{x}, \tilde{\xi}, \tilde{y}, -\tilde{\eta}) \in WF^{-\frac{3}{2}-\epsilon+\tau}(K_G)$. The propagation of singularities result (Theorem 3.6) implies that $(\gamma(\tilde{x}, \tilde{\xi}), \gamma(\tilde{y}, -\tilde{\eta})) \in WF^{-\frac{1}{2}-\epsilon+\tau}(K_A)$, where $\gamma(\tilde{x}, \tilde{\xi})$ is the null bicharacteristic with initial data $(\tilde{x}, \tilde{\xi})$ and $\gamma(\tilde{y}, -\tilde{\eta})$ is the null bicharacteristic with initial data $(\tilde{y}, -\tilde{\eta})$. As a consequence of Lemma 6.13, Lemma 6.4, the fact that $(\partial_t + \partial_s)K_G = 0$ and the inclusion $WF^s(K_A) \subset WF^{s-1}(K_G) \cup \mathrm{Char}(\partial_t)$ for all $s \in \mathbb{R}$, we have $(\gamma(\tilde{x}, \tilde{\xi}), \gamma(\tilde{y}, -\tilde{\eta})) \in WF^{-\frac{3}{2}-\epsilon+\tau}$ (K_G). Then, we can apply Theorem 3.7 combined with Lemma 6.14 to obtain the result. The proof is along the lines [54, Theorem 4.17]. □

For the analysis of adiabatic states, it is enough to work with the inclusion shown above. However, in the smooth case we have an equality of sets. In Theorem 6.17, we show that this equality holds under stronger regularity assumptions on the metric.

First we show the following lemma

Lemma 6.16. *Let $(\tilde{x}, \tilde{\xi}) \in \mathrm{Char}(P)$ with P as in Eq. (6.4). Then $(\tilde{x}, \tilde{x}, \tilde{\xi}, -\tilde{\xi}) \in WF^{\frac{3}{2}+\epsilon}(K_G)$ for all $\epsilon > 0$.*

Proof. Since $WF^{s_1} \subset WF^{s_2}$ for $s_1 \leq s_2$, it is enough to show the result for small ϵ. Let $Q := \mathbb{R} \times \Sigma^2$. We define the embedding $f : Q \to M \times M$ by $f(s, x, y) = (s, x, s, y)$. The set of normals of the map f is

$$\begin{aligned} N_f &= \{(f(s, x, y), \tilde{\xi}, \tilde{\eta}) \in T^*(M \times M); {}^t f'(s, x, y)(\tilde{\xi}, \tilde{\eta}) = 0\} \\ &= \{(s, x, s, y, \xi^0, 0, -\xi^0, 0) \in T^*(M \times M)\}, \end{aligned}$$

where ${}^t f'$ is the transpose of the differential of f. In particular, $N_f \cap (\mathrm{Char}\, P \times \mathrm{Char}\, P) = \emptyset$. By Lemma 6.13,

$$WF^{\frac{3}{2}+\epsilon}(K_G) \cap N_f = \emptyset$$

and therefore

$$WF^{\frac{1}{2}+\epsilon}(\partial_t K_G) \cap N_f \subset WF^{\frac{3}{2}+\epsilon}(K_G) \cap N_f = \emptyset$$

for suitably small $\epsilon > 0$. Therefore, Proposition B.7 from [38] implies that the restriction of $\partial_t K_G$ to Q is defined and satisfies

$$\begin{aligned} WF^{\epsilon}(\partial_t K_G|_Q) &\subset f^*(WF^{\frac{1}{2}+\epsilon}(\partial_t K_G)) \\ &= \{(s,x,y,{}^t f'(\tilde{\xi},\tilde{\eta})) \in T^*Q; (f(s,x,y),\tilde{\xi},\tilde{\eta}) \in WF^{\frac{1}{2}+\epsilon}(\partial_t K_G)\}. \end{aligned} \tag{6.9}$$

As a distribution, $\partial_t K_G|_Q$ is given by

$$\partial_t K_G|_Q(s,x,y) = -\sum_j \phi_j(x)\phi_j(y),$$

i.e., it acts on the non-smooth density $\psi_1(s)\psi_2(x)\psi_3(y)\sqrt{h}(x)\sqrt{h}(y)dxdy$, by

$$\langle \partial_t K_G|_Q, \psi_1\psi_2\psi_3\rangle = -\int_{-\infty}^{\infty} \psi_1(p)dp \int_{\Sigma} \psi_2(w)\psi_3(w)\sqrt{h(w)}dw. \tag{6.10}$$

Therefore, its Fourier transform is given by

$$(\mathcal{F}(\partial_t K_G|_Q))(\chi,\xi,\eta) = \delta_0(\chi) \otimes \int_{\Sigma} e^{-iw(\xi+\eta)}\sqrt{h(w)}dw. \tag{6.11}$$

Moreover, we have $(\partial_t K_G|_Q - 1\otimes\delta(x-y))(\psi) = 0$ for all smooth densities on $\mathbb{R}\times\Sigma\times\Sigma$. Therefore, $\partial_t K_G|_Q = 1\otimes\delta(x-y)$ as elements of $\mathcal{D}'(\mathbb{R}\times\Sigma\times\Sigma)$. This implies

$$WF^s(\partial_t K_G|_Q) = \begin{cases} \emptyset, s < -\frac{3}{2} \\ (s,x,x,0,\xi,-\xi) \text{ for all } \xi \in T^*_x\Sigma, s \geq -\frac{3}{2}. \end{cases}$$

Using Eq. (6.9), we find that there exists ξ_0 such that $(s,x,s,x,\xi^0,\xi,-\xi^0,-\xi) \in WF^{\frac{1}{2}+\epsilon}(\partial_t K_G)$ for each $\xi \in T^*\Sigma$.

According to Proposition B.3 from [38],

$$WF^{\frac{1}{2}+\epsilon}(\partial_t K_G) \subset WF^{\frac{3}{2}+\epsilon}(K_G). \tag{6.12}$$

Since the wavefront set is contained in $\mathrm{Char}(P)\times\mathrm{Char}(P)$, we obtain from Lemma 6.13 $(s,x,s,x,\xi^0,\xi,-\xi^0,-\xi) \in \mathrm{Char}(P)\times\mathrm{Char}(P)$ with $\xi_0^2 = h^{ij}\xi_i\xi_j$. Without loss of generality, we choose a sign for ξ_0, i.e. $\xi_0 := \sqrt{h^{ij}\xi_i\xi_j}$.

Now we show that if $(s,x,s,x,\xi^0,\xi,-\xi^0,-\xi) \in WF^{\frac{3}{2}+\epsilon}(K_G)$, then $(s,x,s,x,-\xi^0,-\xi,\xi^0,\xi) \in WF^{\frac{3}{2}+\epsilon}(K_G)$. The diffeomorphism $f_1(t,x,s,y) = (s,y,t,x)$ has the set of normals $N_{f_1} = \{(s,y,t,x,0,0,0,0) \in T^*(M\times M)\}$ which has empty intersection with $WF(K_G)$. Then, [35, Theorem 8.2.3] and the invariance of the Sobolev wavefront set implies that

$$WF^{\frac{3}{2}+\epsilon}(f_1^*K_G) = f_1^*WF^{\frac{3}{2}+\epsilon}(K_G). \tag{6.13}$$

Moreover, $f_1^*K_G = -K_G$ which gives

$$WF^{\frac{3}{2}+\epsilon}(K_G) = f_1^*WF^{\frac{3}{2}+\epsilon}(K_G). \tag{6.14}$$

Now since $(s,x,s,x,\xi^0,\xi,-\xi^0,-\xi) \in WF^{\frac{3}{2}+\epsilon}(K_G)$, then we have $(s,x,s,x,-\xi^0,-\xi,\xi^0,\xi) \in WF^{\frac{3}{2}+\epsilon}(K_G)$ by Eq. (6.14).

Notice that we also have to show that $(s,x,s,x,-\xi^0,\xi,\xi^0,-\xi)$ and $(s,x,s,x,\xi^0,-\xi,-\xi^0,\xi)$ are in $WF^{\frac{3}{2}+\epsilon}(K_G)$.

In this case, we use the diffeomorphism $f_2(t,x,s,y)=(s,x,t,y)$ that has the set of normals $N_{f_2}=\{(s,x,t,y,0,0,0,0)\in T^*(M\times M)\}$ which has empty intersection with $WF(K_G)$. Then, [35, Theorem 8.2.3] and the invariance of the Sobolev wavefront set implies that

$$WF^{\frac{3}{2}+\epsilon}(f_2^*K_G)=f_2^*WF^{\frac{3}{2}+\epsilon}(K_G). \tag{6.15}$$

Moreover, $f_2^*K_G=-K_G$ which gives

$$WF^{\frac{3}{2}+\epsilon}(K_G)=f_2^*WF^{\frac{3}{2}+\epsilon}(K_G). \tag{6.16}$$

Now since $(s,x,s,x,\xi^0,\xi,-\xi^0,-\xi)\in WF^{\frac{3}{2}+\epsilon}(K_G)$ then we have $(s,x,s,x,-\xi^0,\xi,\xi^0,-\xi)\in WF^{\frac{3}{2}+\epsilon}(K_G)$ by Eq. (6.16). Using f_1, we obtain $(s,x,s,x,\xi^0,-\xi,-\xi^0,\xi)\in WF^{\frac{3}{2}+\epsilon}(K_G)$. This gives the desired result. □

Now we show the equality of sets as in the smooth case.

Theorem 6.17. *Let (M,g) be a C^τ ultrastatic spacetime with $\tau>3$ and K_G the causal propagator. Then, $C\subset WF'^{-\frac{3}{2}+\tau-\tilde\epsilon}(K_G)$ for all $\tilde\epsilon<\tau-3$ and C as in Eq. (1.1). In particular, we have $C\subset WF'^{s}(K_G)$ for all $s>\frac{3}{2}$.*

Proof. Under the additional regularity assumption and arguing locally as in Theorem 3.3, we have $P^b_{(t,x)}K_A, P^b_{(s,y)}K_A\in H^{\frac{3}{2}+\tilde\epsilon}(M\times M)$ and therefore for $(\tilde x,\tilde\xi,\tilde y,\tilde\eta)\in WF^{\frac{3}{2}+\tilde\epsilon}(K_G)\subset WF^{\frac{5}{2}+\tilde\epsilon}(K_A)$ we can choose $s=\frac{3}{2}+\tilde\epsilon$ in Theorem 3.6.

Now if $(\mathbf{x},\boldsymbol{\xi})=(\tilde x,\tilde\xi,\tilde y,-\tilde\eta)\in C'$ then there is a null geodesic γ such that $\gamma(t_1)=\tilde x,\gamma(t_2)=\tilde y$ and $g(\cdot,\dot\gamma)|_{T_{\tilde x}M}=\tilde\xi, g(\cdot,\dot\gamma)|_{T_{\tilde y}M}=\tilde\eta$. Now, $(\tilde x,\tilde\xi,\tilde x,-\tilde\xi)\in C'$ and by Lemma 6.16 $(\tilde x,\tilde\xi,\tilde x,-\tilde\xi)\in WF^{\frac{3}{2}+\epsilon}(K_G)$ for $\epsilon>0$ which implies for $\tilde\epsilon<\tau-3$ that $(\tilde x,\tilde\xi,\tilde x,-\tilde\xi)\in WF^{-\frac{3}{2}+\tau-\tilde\epsilon}(K_G)\subset WF^{-\frac{1}{2}+\tau-\tilde\epsilon}(K_A)$. Applying Theorem 3.6 to $P_{(t,x)}K_A,P_{(s,y)}K_A$ with the s described above we have $(\gamma(\tilde x,\tilde\xi),\gamma(\tilde x,-\tilde\xi))\in WF^{-\frac{1}{2}+\tau-\tilde\epsilon}(K_A)$. Using the same argument as in Theorem 6.15, this implies $(\tilde x,\tilde\xi,\tilde y,-\tilde\eta)=(\mathbf{x},\boldsymbol{\xi})\in WF^{-\frac{3}{2}+\tau-\tilde\epsilon}(K_G)$. □

Remark 6.18. The combination of Theorem 6.15 with Theorem 6.17 gives

$$WF'^{-\frac{3}{2}+\tau-\tilde\epsilon}(K_G)=C$$

for $\tau>3$ and $\tilde\epsilon<\tau-3$.

6.4.3. The $C^{1,1}$ Case. The following theorem states the result for the case of $C^{1,1}$ regularity.

Theorem 6.19. *Let (M,g) be a $C^{1,1}$ ultrastatic spacetime and K_G the causal propagator. Then, $WF'^{\frac{1}{2}-\tilde\epsilon}(K_G)\subset C$ for all $\tilde\epsilon>0$.*

Proof of Theorem 6.19. In order to show the theorem, we will state how different results of the paper change under this regularity.

From the comment above Theorem 3.6, we know that Theorem 3.6 still holds. Notice that $C^{1,1}\subset C^2_*$ [62, Chapter 1, Eq.(1.21)].

Also, notice that a $C^{1,1}$ metric guarantees the existence and uniqueness of the Hamiltonian flow which is critical for the proof. Theorem 3.3 holds even for $\tau>1$.

Lemma 4.1 requires no modification, since the results on global hyperbolicity still hold for this regularity [53, Corollary 3.4]. The hypothesis in [66, Theorem 1.1] is the requirement that the coefficients of the principal part have one derivative that is Lipschitz which is clearly satisfied in the $C^{1,1}$ case. Hence, Lemma 6.11 holds.

For Lemma 6.14 and Theorem 6.15, the only thing to notice is that in this case $P^b_{(t,x)}K_A, P^b_{(s,y)}K_A \in H^{\frac{1}{2}-\tilde{\epsilon}}(M \times M)$ (arguing locally as in Theorem 3.3) and therefore we can apply Theorem 3.6 for $s = \frac{1}{2} - \tilde{\epsilon}$. In this section, we have applied the version of Theorem 3.6 after [62, Proposition 11.4]. □

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