Invertible Functorial Field Theory for Symmetry Breaking and Interactions in Quantum Field Theory

by

Cameron Krulewski

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ABSTRACT

We apply invertible field theories to study two questions in quantum field theory. Specifically, we study reflection-positive fully-extended invertible field theories on manifolds with twisted spin structures, which are computed as Anderson-dual bordism groups [1, 2].

In high energy physics, invertible field theories represent anomalies of quantum field theories. Our first application is toward 't Hooft anomaly matching—a method first developed in the 1980s in which one treats anomalies as invariants of theories of interest and uses them to compute how quantum field theories change under physical processes. Specifically, we model three related processes around a form of spontaneous symmetry breaking via a charged order parameter using a twisted Gysin sequence of Anderson-dual bordism groups. We study the *Smith maps* of Madsen-Tillmann spectra that underlie the sequence, collecting examples and cataloging periodicities. Finally, we compute an extensive set of examples of physical interest and draw physical predictions from the results.

In condensed matter physics, invertible field theories model the low energy field theories of symmetry-protected topological phases (SPTs). In this second application, we develop and compute homotopical *free-to-interacting maps* to compare two classifications of fermionic SPTs: those for free (i.e. non-interacting) models, and more general interacting classifications. These maps contribute to what has been a prolific line of research in the physics literature for the past fifteen years. Generalizing [1], we construct maps from K-theory to twisted spin IFTs using T-duality and twisted versions of the spin orientation of K-theory [3]. We focus on two situations: weak phases [4, 5], which are SPTs protected by discrete translation symmetry, and primed phases [6], which are closely related to the famous tenfold way [7, 8], but which have a very different interacting classification. In the latter case, we demonstrate the dependence of the interacting classification on more than the Morita class of the symmetry algebra.

Thesis supervisor: Michael J. Hopkins
Title: Professor of Mathematics

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Contents

Chapter 1

Introduction

Invertible field theories lend insight to a wide range of physical problems, from symmetries of quantum field theories, to consistency of supergravity theories, to classification and prediction of quantum phases of matter called *symmetry-protected topological phases* (SPTs). In this thesis, we discuss two applications: in Chapter 3, we use invertible field theories to model a certain form of *symmetry breaking* for quantum field theories, and in Chapter 4 we construct maps from K-theory to invertible field theories model *free-to-interacting maps* for SPTs. Each of those chapters has its own longer introduction.

This thesis is compiled from several collaborative papers, the last of which is in progress:

- Arun Debray, Sanath Devalapurkar, CK, Yu Leon Liu, Natalia Pacheco-Tallaj, and Ryan Thorngren. *The Smith Fiber Sequence of Invertible Field Theories.* May 2024. arXiv: `2405.04649 [math.AT,math-ph]`. Appears here mostly in Chapter 3.

- Arun Debray, Sanath Devalapurkar, CK, Yu Leon Liu, Natalia Pacheco-Tallaj, and Ryan Thorngren. *A Long Exact Sequence in Symmetry Breaking: order parameter constraints, defect anomaly-matching, and higher Berry phases.* Sept. 2023. arXiv: `2309.16749 [hep-th,cond-mat.str-el,math-ph]`. To appear in the *Journal of High Energy Physics.* Appears with edits here in Section 3.7.

- Omar Antolín Camarena, Arun Debray, CK, Natalia Pacheco-Tallaj, Daniel Sheinbaum, and Luuk Stehouwer. *Weak Topological Phases in the Presence of Interactions.* Oct. 2024. arXiv: `2410.10031 [math-ph,cond-mat.str-el,hep-th]`. Appears here in Section 4.1.

- Arun Debray, CK, Natalia Pacheco-Tallaj, and Luuk Stehouwer. *Unraveling the Bott Spiral.* In preparation. Partially appears here in Section 4.2.

The first two papers address the symmetry breaking application, while the last two address the application to SPTs.

Chapter 2 provides background information on invertible field theories and includes new material. The classification of invertible field theories is discussed in Section 2.4.2 and physical interpretations are introduced in Section 2.5.

Chapter 3 discusses the twisted Gysin sequence of invertible field theories applied to symmetry breaking and incorporates the paper [9] mostly unchanged and the paper [10] with some modifications for consistency. The main results of [9] are the long exact sequence of field theories (Corollary 3.7.6), equivalence of different definitions of Smith homomorphisms (Corollary 3.3.39), and a systematic understanding of periodicities in Smith homomorphisms (Section 3.5). The main results of [10] are physical interpretations of the maps in the long exact sequence for obstructions to gapping theories (Section 3.8.1), defect anomaly matching (Section 3.8.2), and generalized Berry phases (Section 3.8.3), as well as a wide range of physical examples including long exact sequences of anomalies in Section 3.8.5.

Chapter 4 discusses free-to-interacting maps for two situations: weak phases (Section 4.1), from [11], and the Bott spiral (Section 4.2), from forthcoming work [12]. The main results of [11] are an ansatz for the free-to-interacting map for weak topological phases (Ansatz 4.1.73) and a collection of computations (Section 4.1.8). The main results of [12] are a generalization of Atiyah–Bott–Shapiro to spin bordism twisted by representations of $(\mathbb{Z}/2)^n$ (Definition 4.2.153), models for the free-to-interacting map for primed Altland–Zirnbauer classes (Definition 4.2.171) and for a specific form of dimensional reduction on invertible field theories (Definition 4.2.199), as well as computations of these maps in Theorem 4.2.266 and Theorem 4.2.306, resp.

Appendix A, from [9], works out an example long exact sequence in bordism for use in Chapter 3, while Appendix B, from [11], works out a twisted Atiyah–Bott–Shapiro map for use in Section 4.1.

Chapter 2

Invertible Field Theories

A functorial field theory[1] is a representation of a bordism category, in the sense that it is a functor from a bordism category to a target category. There are several types of functorial field theories, and our aim in this chapter is to narrow down to the case of *reflection-positive*, *invertible*, *fully-extended* field theories on manifolds with *twisted spin tangential structures*, which are our focus in this thesis.

2.1 Functorial Field Theory Basics

The most basic version of a functorial field theory assumes that the source and target are symmetric monoidal 1-categories. This is the *non-extended* case.

Definition 2.1.1. Let $\mathsf{Bord}_{\langle n-1,n\rangle}$ be the symmetric monoidal category whose objects are closed $(n-1)$-manifolds and morphisms are n-dimensional (diffeomorphism classes of) bordisms, with monoidal product given by disjoint union of manifolds. The monoidal unit is the empty set $\emptyset^{n-1}$, which is considered to be a closed $(n-1)$-manifold.

Example 2.1.2. The objects of the category $\mathsf{Bord}_{\langle 1,2\rangle}$ are all of the form $\bigsqcup_i S^1_i$; i.e. (possibly empty) disjoint unions of circles S^1. Bordisms include cylinders, cups, caps, and the famous pair of pants, as well as closed 2-manifolds. ◇

Definition 2.1.3. Let $\mathsf{Vect}_{\mathbb{C}}$ be the symmetric monoidal category whose objects are complex vector spaces and morphisms are linear maps, with monoidal product given by tensor product.

Definition 2.1.4. Let $\mathsf{sVect}_{\mathbb{C}}$ be the symmetric monoidal category whose objects are complex super vector spaces (i.e. $\mathbb{Z}/2$-graded complex vector spaces with the braiding $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$) and whose morphisms are grading-preserving linear maps.

Definition 2.1.5. Let $\mathsf{Line}_{\mathbb{C}}$ (resp. $\mathsf{sLine}_{\mathbb{C}}$) be the symmetric monoidal subcategory of $\mathsf{Vect}_{\mathbb{C}}$ (resp. $\mathsf{sVect}_{\mathbb{C}}$) of one-dimensional (resp. super) complex vector spaces.

---

[1] We use the term "functorial field theory" instead of "topological field theory" here because we will eventually compute with non-topological theories as well. See the end of Section 2.5.1, or e.g. [13, Lecture 9] for more explanation.

Note that $\mathsf{Line}_{\mathbb{C}}$ is the Picard groupoid of $\mathsf{Vect}_{\mathbb{C}}$, and analogously in the super case. The objects of $\mathsf{sLine}_{\mathbb{C}}$ are the even complex line $\mathbb{C}$ and its parity-reversed partner $\Pi\mathbb{C}$.

Definition 2.1.6. Let C be a symmetric monoidal category. A (non-extended) n-dimensional *functorial field theory* with target C is a symmetric monoidal functor

$$Z\colon \mathsf{Bord}_{\langle n-1,n\rangle} \longrightarrow \mathsf{C}. \tag{2.1.7}$$

For the rest this section, we will take $\mathsf{C} = \mathsf{Vect}_{\mathbb{C}}$. Then this definition is matches the original formulation of topological quantum field theories by Atiyah and Segal [14].

See e.g. [13] for more details. Observe that a closed $n-1$ manifold X is sent to a vector space $Z(X)$, which is called the *state space* of the theory on X, while a bordism $W\colon X \to Y$ between two closed $(n-1)$-manifolds is sent to a linear map $Z(W)\colon Z(X) \to Z(Y)$ between their state spaces. In the case that $X = Y = \emptyset^{n-1}$, the bordism W is itself a closed n-manifold, and the function $Z(W)$ is given by multiplication by a number. The assignment $W \mapsto Z(W) \in \mathbb{C}$ defines the *partition function* of the theory.

Example 2.1.8 (Finite Gauge Theory)**.** Let G be a finite group, and let n be a dimension. For a topological space X, let $\mathsf{Bun}_G(X)$ denote the groupoid of principal G-bundles on X. Its objects are principal G-bundles, and its morphisms are *equivalences* of principal G-bundles over X.

For a bordism $X\colon Y_0 \to Y_1$, let $s\colon Y_0 \hookrightarrow X$ and $t\colon Y_1 \hookrightarrow X$ be the inclusions of the boundaries. Restrictions of principal bundles along these maps is always well-defined. Over M a compact manifold, the groupoid $\mathsf{Bun}_G(M)$ has finitely many isomorphism classes of objects, each with finitely many automorphisms. In this case we have a nice formula for the pushforward, which is a finite version of the Feynman path integral: for $t\colon Y_1 \to M$ the pushforward is given by

$$t_*\colon \mathsf{Bun}_G(Y_1) \longrightarrow \mathsf{Bun}_G(M) \tag{2.1.9}$$

$$P \longmapsto \sum_{[P]\in\pi_0\mathsf{Bun}_G(Y_1)} \frac{t(P)}{\#\mathrm{Aut}(P)}. \tag{2.1.10}$$

Finite gauge theory is the functor

$$\begin{array}{rcl} Z\colon \quad \mathsf{Bord}_{\langle n-1,n\rangle} & \xrightarrow{\qquad\qquad\qquad} & \mathsf{Vect}_{\mathbb{C}} \\[1em] \text{closed } (n-1)\text{-manifold } M & \xmapsto{\qquad\qquad\qquad} & \mathbb{C}[\mathsf{Bun}_G(M)] \\[1em] \text{bordism } W\colon M_0 \to M_1 & \xmapsto{\qquad} & (t_* \circ s^*)\colon \mathbb{C}[\mathsf{Bun}_G(M_0)] \to \mathbb{C}[\mathsf{Bun}_G(M_1)]. \end{array} \tag{2.1.11}$$

See e.g. [13, Example 1.23], [15]. This theory is also called (untwisted) Dijkgraaf–Witten theory. ◇

Example 2.1.12 (Finite Homotopy Theory)**.** A generalization of the previous example takes in a π-finite space X. Recall that X is π-finite if and only if $\pi_0 X$ is finite, all homotopy groups $\pi_i X$ are finite, and only finitely many $\pi_i X$ are nonzero. A finite homotopy theory for such an X assigns objects as follows:

$$\begin{tikzcd}
Z_X\colon \quad \mathsf{Bord}_{\langle n-1,n\rangle} \arrow[r] & \mathsf{Vect}_{\mathbb{C}} \\
\text{closed } (n-1)\text{-manifold } M \arrow[r, mapsto] & \mathbb{C}[\mathrm{Maps}(M,X)].
\end{tikzcd} \tag{2.1.13}$$

With enough care, one can generalize the finite path integral of the previous example to give a similar formula for Z_X evaluated on a bordism W. See [16–19].

Note that the choice $X = BG$ recovers Example 2.1.8. Other examples of finite homotopy theories include topological sigma models with target X [20]. $\diamond$

Example 2.1.14 (Euler Theory)**.** Fix a nonzero complex number $\lambda \in \mathbb{C}^\times$. For M a compact manifold (possibly with boundary) let $\chi(M) \in \mathbb{Z}$ denote its Euler characteristic. The Euler theory Z_λ is such that

$$\begin{tikzcd}
Z_\lambda\colon \quad \mathsf{Bord}_{\langle n-1,n\rangle} \arrow[r] & \mathsf{Vect}_{\mathbb{C}} \\
\text{closed } (n-1)\text{-manifold } M \arrow[r, mapsto] & \mathbb{C} \\
\text{bordism } W\colon M_0 \to M_1 \arrow[r, mapsto] & (- \cdot \lambda^{\chi(W)-\chi(M_0)})\colon \mathbb{C} \to \mathbb{C}.
\end{tikzcd} \tag{2.1.15}$$

See [21]. $\diamond$

2.2 Tangential Structures, Bordism, and Thom Spectra

Functorial field theories become much more diverse after we allow some geometry into the picture. We shall consider very general stable tangential structures, but in this thesis there is a preferred set of ten structures—see the later Tables 4.1 and 4.3.

Definition 2.2.1. Let Top denote the ∞-category of spaces. A *stable tangential structure* is an element of the slice category $\mathsf{Top}_{/BO}$ of spaces over BO.

Concretely, this means that a stable tangential structure is the data $\xi\colon X \to BO$ of a space X and a map ξ to BO. We will alternatively refer to this as a (X,ξ)-structure, a ξ-structure, or—for specific X—an X structure.

Definition 2.2.2. An (X,ξ)-structure on a manifold M is the data of a lift along ξ of the classifying map of the stable tangent bundle of M:

$$\begin{tikzcd}
 & X \arrow[d, "\xi"] \\
M \arrow[r, "TM"'] \arrow[ur, dashed] & BO
\end{tikzcd} \tag{2.2.3}$$

An (X,ξ)-manifold is a pair $(M,\, M \to X)$ of a manifold M and this map to X lifting ξ.

Example 2.2.4. An *orientation* in the traditional sense corresponds to an SO-structure, where SO is the stable special orthogonal group. ◇

Example 2.2.5. We give a very general definition of *Clifford algebra* in Definition 4.2.24.

The *pin groups* are the subgroups of the group of units consisting of norm-one elements, and the *spin group* is the even or oriented part of the pin group. Specifically, let τ be the involution acting on a homogeneous element as

$$\tau\colon s_1 \otimes s_2 \otimes \cdots \otimes s_n \longmapsto s_n \otimes s_{n-1} \otimes \cdots \otimes s_1. \tag{2.2.6}$$

The *Clifford norm* of $x \in C\ell(F, S, \sigma)$ is $N(x) = \tau(x) \cdot x$.

Let $V = \mathbb{R}^n$ and consider the Clifford algebras $C\ell_{-n}$ and $C\ell_n$. Then

$$\mathrm{Pin}_n^+ \coloneqq \ker(N\colon C\ell_{+n} \to \mathbb{R}) \subset C\ell_{+n} \tag{2.2.7}$$
$$\mathrm{Pin}_n^- \coloneqq \ker(N\colon C\ell_{-n} \to \mathbb{R}) \subset C\ell_{-n} \tag{2.2.8}$$
$$\mathrm{Spin}_n \coloneqq \mathrm{Pin}_n^+ \cap C\ell_{+n}^{\mathrm{ev}} \cong \mathrm{Pin}_n^- \cap C\ell_{-n}^{\mathrm{ev}}. \tag{2.2.9}$$

The maps ξ for each structure type come from the fact that Pin^+ and Pin^- are $\mathbb{Z}/2$-extensions of O. We will study spin, pin^+, and pin^- structures going forward. ◇

Example 2.2.10. The complex version of spin can be defined using the complex Clifford algebra, or equivalently as the quotient

$$\mathrm{Spin}_n^c \coloneqq \mathrm{Spin}_n \times U(1)/((x,z) \sim (-x,-z)) \tag{2.2.11}$$

also written $\mathrm{Spin}_n \times_{\pm 1} \mathrm{U}(1)$. The unoriented version is defined analogously, and since the signature is invisible to the complex Clifford algebra, there is only one version of this pin^c group for $\mathbb{R}^n$. See e.g. [22]. ◇

Example 2.2.12. There is also a quaternionic version:

$$\mathrm{Spin}_n^h \coloneqq \mathrm{Spin}_n \times_{\pm 1} \mathrm{SU}(2) \tag{2.2.13}$$
$$\mathrm{Pin}_n^{h+} \coloneqq \mathrm{Pin}_n^+ \times_{\pm 1} \mathrm{Sp}(1) \tag{2.2.14}$$
$$\mathrm{Pin}_n^{h-} \coloneqq \mathrm{Pin}_n^- \times_{\pm 1} \mathrm{Sp}(1). \tag{2.2.15}$$

Note the exceptional isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$, and see also Remark 3.6.48. ◇

Remark 2.2.16. Many of the tangential structures we are interested in physically are twisted versions of the groups that appear in the Whitehead tower of the orthogonal group. Recall that the Whitehead tower $\cdots \to X_{(2)} \to X_{(1)} \to X$ of a space X is a tower of CW complexes that are increasingly connected approximations of X: they are such that $\pi_i X_{(n)} = 0$ for $i < n$, $\pi_i X_{(n)} \to \pi_i X$ is an isomorphism for $i > n$, and the sequences

$$K(\pi_n(X), n-1) \to X_{(n)} \to X_{(n-1)} \tag{2.2.17}$$

are fibrations [23].

The homotopy groups of $B\mathrm{O}$ to kill are as follows:

$$\begin{aligned}
\pi_0 B\mathrm{O} &= \mathbb{Z}\\
\pi_1 B\mathrm{O} &= \mathbb{Z}/2\\
\pi_2 B\mathrm{O} &= \mathbb{Z}/2\\
\pi_3 B\mathrm{O} &= 0\\
\pi_4 B\mathrm{O} &= \mathbb{Z}\\
\pi_5 B\mathrm{O} &= 0\\
\pi_6 B\mathrm{O} &= 0\\
\pi_7 B\mathrm{O} &= 0
\end{aligned}$$

and $\pi_{i+8} B\mathrm{O} \cong \pi_i B\mathrm{O}$ [24]. The groups in the first few steps in the tower and the obstruction classes are listed below.

$$\begin{tikzcd}
\vdots \arrow[d] & \\
B\mathrm{Fivebrane} \arrow[r, "\frac{1}{6}p_2"] \arrow[d] & B^8\mathbb{Z} \\
B\mathrm{String} \arrow[r, "\frac{1}{2}p_1"] \arrow[d] & B^4\mathbb{Z} \\
B\mathrm{Spin} \arrow[r, "w_2"] \arrow[d] & B^2\mathbb{Z}/2 \\
B\mathrm{SO} \arrow[r, "w_1"] \arrow[d] & B\mathbb{Z}/2 \\
B\mathrm{O} &
\end{tikzcd}$$

Each two-step ⌞-shaped subdiagram is a fiber sequence. We will see in later chapters how oriented theories correspond to theories of bosonic particles, while spin theories correspond to theories of fermions. String structures, naturally, are used to formulate string theories.

Definition 2.2.18. Let $\mathsf{Bord}^{(X,\xi)}_{\langle n-1,n\rangle}$ be the category whose objects are closed $(n-1)$-manifolds with (X,ξ) structure and whose morphisms are bordisms respecting this structure.[2]

We will often abbreviate the notation to just $\mathsf{Bord}^{X}_{\langle n-1,n\rangle}$ or $\mathsf{Bord}^{\xi}_{\langle n-1,n\rangle}$.

Definition 2.2.19. A (non-extended) n-dimensional (X,ξ)-*structured functorial field theory* with target C is a symmetric monoidal functor

$$Z\colon \mathsf{Bord}^{(X,\xi)}_{\langle n-1,n\rangle} \longrightarrow \mathsf{C}. \tag{2.2.20}$$

---

[2] That is, a bordism $W\colon M_0 \to M_1$ must carry an (X,ξ)-structure that reduces to each of the structures on M_0 and M_1 on each respective boundary.

Example 2.2.21 (Oriented Surfaces and Frobenius Algebras)**.** The classification of two-dimensional oriented functorial field theories is a classic story. See e.g. [25] or [26, Example 1.1.11]. An oriented two-dimensional functorial field theory

$$Z\colon \mathsf{Bord}^{\mathrm{SO}}_{\langle 1,2\rangle} \longrightarrow \mathsf{Vect}_{\mathbb{C}} \tag{2.2.22}$$

is determined by where it sends the oriented circle S^1. One can show that $Z(S^1)$, a priori just a vector space, additionally hosts the structure of a finite-dimensional commutative Frobenius algebra. The classification of such functorial field theories then reduces to the classification of commutative Frobenius algebras. ◇

Example 2.2.23 (Arf Theory)**.** Consider spin manifolds. There are only two connected spin 1-manifolds: the circle S^1_b with the bounding spin structure, which is nulbordant, and the circle S^1_{nb} with the non-bounding spin structure.[3] The *Arf invariant* of a closed spin surface S is a number $\mathrm{Arf}(S) \in \mathbb{Z}/2$ computed using the Arf invariant of the intersection form $H_1(S;\mathbb{Z}/2) \times H_1(S;\mathbb{Z}/2) \to H_2(S;\mathbb{Z}/2)$, which is in turn defined using the quadratic refinement of this form that corresponds to the given spin structure. One can also extend the definition of the Arf invariant to spin surfaces with boundary, so that it defines a multiplication-by-±1 map.

The (non-extended) *Arf theory* makes the following assignments:

$$\begin{array}{rcl}
\alpha\colon \mathsf{Bord}^{\mathrm{Spin}}_{\langle 1,2\rangle} & \longrightarrow & \mathsf{sLine}_{\mathbb{C}} \\
S^1_b & \longmapsto & \mathbb{C} \\
S^1_{nb} & \longmapsto & \Pi\mathbb{C} \\
\text{closed spin surface } S & \longmapsto & (-1)^{\mathrm{Arf}(S)} \\
\text{spin bordism } W\colon M_0 \to M_1 & \longmapsto & \left(- \cdot (-1)^{\mathrm{Arf}(W)}\right)\colon \alpha(M_0) \to \alpha(M_1).
\end{array} \tag{2.2.24}$$

See e.g. [27] and [13, Example 1.59].

◇

2.2.1 Construction of Thom spectra

This subsection is modified from [9, Section 2.2], which is joint work of the author with Arun Debray, Sanath Devalapurkar, Yu Leon Liu, Natalia Pacheco-Tallaj, and Ryan Thorngren.

Recall the classical construction of a *Thom space*: if $V \to X$ is a vector bundle, choose a Euclidean metric on V. Let $D(V)$ be the *disc bundle* of vectors in V of norm at most 1 and $S(V)$ be the *sphere bundle* of vectors of norm exactly 1; write $\mathrm{Th}(X;V) \coloneqq D(V)/S(V)$.

---

[3] S^1_b is also called the anti-periodic or Neveu-Schwarz spin circle, and S^1_{nb} is also called the periodic or Ramond spin circle.

Example 2.2.25. Let $\underline{\mathbb{R}}^n \to X$ be a trivial bundle and let X_+ be the space X with a disjoint basepoint. Then the Thom space is the n-fold suspension $\mathrm{Th}(X; \underline{\mathbb{R}}^n) \simeq \Sigma^n X_+$. ◇

Example 2.2.26. Let $X = \mathbb{RP}^n$ and let $V = \sigma$ be the tautological line bundle. Then the Thom space is $\mathrm{Th}(\mathbb{RP}^n; \sigma) \simeq \mathbb{RP}^{n+1}$. ◇

Proposition 2.2.27. *If X is compact, then $\mathrm{Th}(X; V)$ is the one point compactification of the disk bundle $D(V)$.*

Let $V \to X$ be a rank d real vector bundle (*not* merely a virtual vector bundle), and also write $V\colon X \to B\mathrm{O}(d)$ for the classifying map. Let Top denote the ∞-category of spaces and Top_* denote the ∞-category of pointed spaces. The action of $\mathrm{O}(d)$ on $\mathbb{R}^d$ induces an action on $S^d = S^{\mathbb{R}^d}$, and this induces a functor from the fundamental ∞-groupoid of $B\mathrm{O}(d)$ to Top_*.

Proposition 2.2.28. *The Thom space $\mathrm{Th}(X; V)$ is naturally homotopy equivalent to the colimit of the X-shaped diagram*[4]

$$X \xrightarrow{V} B\mathrm{O}(d) \longrightarrow \mathsf{Top}_*. \tag{2.2.29}$$

For virtual bundles on X, we must work in the category of spectra, which we denote Sp. We follow [28] in the rest of this section. By a *local system of spectra* over a space X we mean a functor $\mathcal{L}$ from the fundamental ∞-groupoid of X to spectra. We will usually denote this as $\mathcal{L}\colon X \to \mathsf{Sp}$. The fiber of a local system at a point $p \in X$ is obtained by composing $\mathcal{L}$ with the functor $\mathrm{pt} \to X$ given by inclusion at p; a functor out of pt is equivalent to a single spectrum, and we call this the fiber of $\mathcal{L}$ at p.

Definition 2.2.30 ([28, 29])**.** A *stable spherical fibration* is a local system of spectra valued in the full sub-∞-category of spectra with objects $\Sigma^n\mathbb{S}$, $n \in \mathbb{Z}$.

Here $\mathbb{S}$ denotes the *sphere spectrum.*

Definition 2.2.31. Let X be a space and $V \to X$ be a vector bundle of rank r. Let $\mathbb{S}^V \to X$ denote the associated stable spherical fibration, whose fiber at a point $x \in X$ is the suspension spectrum of the one-point compactification of V_x.

Now fix a base space X and a virtual vector bundle $V \to X$, which is equivalently a map $V\colon X \to BO \times \mathbb{Z}$. There is a canonical spectrum called the *Thom spectrum* X^V associated to X, V constructed as follows. There is a functor $J\colon BO \times \mathbb{Z} \to \mathsf{Sp}$, generalizing the map $BO(d) \to \mathsf{Top}_*$ above. It maps into spectra now, instead of spaces, because for a virtual bundle $V_1 - V_2$, we want to assign the sphere $S^{V_1} \wedge S^{-V_2}$, but S^{-V_2} doesn't make sense as a space. However, the sphere spectrum $\mathbb{S}$ can be desuspended, and the Thom spectrum associated to a virtual bundle is defined as follows.

Definition 2.2.32. Given a virtual bundle $V\colon X \to BO \times \mathbb{Z}$, the Thom spectrum X^V is the colimit (in spectra) of the composite $X \xrightarrow{V} BO \times \mathbb{Z} \xrightarrow{J} \mathsf{Sp}$.

Here's the compatibility between the Thom space and Thom spectrum construction:

---

[4] Note that "X-shaped diagram" means a functor out of the fundamental ∞-groupoid of X.

Lemma 2.2.33. *Let* $V\colon X \to BO(d)$ *be a vector bundle and let* $\xi\colon X \to BO(d) \to BO \times \mathbb{Z}$ *be the corresponding virtual bundle. Then the Thom spectrum of* ξ *is the suspension spectrum of the Thom space of* V*; i.e.* $X^\xi \simeq \Sigma^\infty_+ X^V$*.*

Here by Σ^∞_+ we mean first taking the disjoint union with a single point, which we take as the basepoint, then taking the suspension spectrum.

Proof. This follows from the fact that $\Sigma^\infty_+\colon \mathsf{Top} \to \mathsf{Sp}$ preserves colimits. ☐

Using Lemma 2.2.33, one can directly check that the Thom spectrum of the trivial bundle $\underline{\mathbb{R}}^n \to X$ is homotopy equivalent to a suspension of the suspension spectrum $\Sigma^n\Sigma^\infty_+ X$.

Lemma 2.2.34 ([30, Lemma 2.3])**.** *Let* $V \to X$ *and* $W \to Y$ *be virtual vector bundles. Then the Thom spectrum of* $V \boxplus W \to X \times Y$ *is homotopy equivalent to* $X^V \wedge Y^W$*.*

Here $\boxplus$ is the external direct sum, i.e. the direct sum of the pullbacks of V and W across the projection maps $X \times Y \to X$, resp. $X \times Y \to Y$.

One can often combine Lemma 2.2.34 with the observation that Thom spectra of trivial bundles are suspensions to simplify Thom spectra appearing in examples. For example, $X^{V \oplus \underline{\mathbb{R}}^n}$, often denoted X^{V+n}, is homotopy equivalent to $\Sigma^n X^V$. Since we are working with virtual vector bundles, n may be any integer.

Let us discuss a variant for tangential structures.

Definition 2.2.35. Let $\xi\colon B \to B\mathrm{O}$ be a tangential structure. Then its inverse (as a virtual vector bundle) $-\xi$ is often denoted $\xi^\perp$. Equivalently, $\xi^\perp$ is the composition of ξ with the map $-1\colon B\mathrm{O} \to B\mathrm{O}$, which is the inverse map in the E_∞-structure on $B\mathrm{O}$ induced by direct sum. Therefore $\xi^\perp$ is also a tangential structure; its Thom spectrum $B^{-\xi}$ is called a *Madsen-Tillmann spectrum* [31, 32] and is often denoted $MT\xi$. If $B \to B\mathrm{O}$ is obtained from a family of Lie group homomorphisms $H(n) \to \mathrm{O}(n)$ in the (co)limit $n \to \infty$, $MT\xi$ is often written MTH.

Likewise, the Thom spectrum of the pullback of $-V_n \to B\mathrm{O}(n)$ across a map $\xi_n\colon B_n \to B\mathrm{O}(n)$ is denoted $MT\xi_n$; if $B = BH(n)$ for a Lie group $H(n)$, this is often written $MTH(n)$.

$MT\xi$ has two key properties:

1. (Pontrjagin-Thom theorem) There is a natural isomorphism $\pi_n(MT\xi) \cong \Omega_n^\xi$, where the RHS is a *bordism group*, discussed in Section 2.2.2.[5]
2. (Thom isomorphism theorem) Let A be a commutative ring. Then there is a natural isomorphism $H^*(B; A_{w_1}) \xrightarrow{\cong} H^*(MT\xi; A)$, where A_{w_1} denotes the pullback by ξ of the orientation local system on $B\mathrm{O}$.

In the Thom isomorphism theorem, the use of twisted cohomology can be avoided by assuming $A = \mathbb{Z}/2$ or by choosing an orientation of the virtual vector bundle classified by the map ξ. When ξ is the result of applying the classifying space functor to a group homomorphism $G \to \mathrm{O}$, we often write MTG for $MT\xi$.

---

[5] It is most common to define Thom spectra and bordism in terms of the stable normal bundle, rather than the tangent bundle; the resulting spectra are written $M\xi$. The spectra $MT\xi$ and $M\xi$ coincide for the tangential structures O, SO, Spin^c and Spin, but not in general: $MT\mathrm{Pin}^\pm \simeq M\mathrm{Pin}^\mp$. By composing with the map $-1\colon B\mathrm{O} \to B\mathrm{O}$, one can pass between normal bordism and tangential bordism and therefore pass between our definition and the standard one.

2.2.2 Bordism Groups

In the Section 2.4.2, we will reduce our computational question about invertible field theories, which involve the bordism category, to a question about bordism *groups* Ω_*^ξ.

Definition 2.2.36. Let (X, ξ) be a stable tangential structure. Then n-dimensional ξ-*bordism group* is

$$\Omega_n^\xi \coloneqq (\{\text{closed } n\text{-dimensional } \xi\text{-manifolds}\}/\sim, \sqcup) \tag{2.2.37}$$

the set of ξ-manifolds modulo bordism and equipped with the operation of disjoint union. Two closed n-dimensional ξ-manifolds M and N are bordant if there exists a ξ-$(n+1)$-manifold with boundary W such that $\partial W = M \sqcup \bar{N}$, where $\bar{N}$ denotes the opposite ξ structure.

We write Ω_*^ξ for the sum $\bigoplus_{n=0}^\infty \Omega_n^\xi$ equipped with Cartesian product. When $MT\xi$ is multiplicative, Ω_*^ξ is a ring.

Remark 2.2.38. Note the difference between bordism *groups* Ω_n^ξ and the bordism *category* $\mathsf{Bord}^\xi_{\langle n-1,n\rangle}$. In the first, elements are manifolds modulo the equivalence relation of bordism, while in the second, objects are manifolds and bordisms are morphisms—the second is a categorified version of the first. We say more about their relationship in Section 2.4.2.

Often, just low-dimensional groups suffice to answer our questions of interest, so we collect a few common groups below. We compiled many of these results using the Manifold Atlas [33].

Example 2.2.39 (Unoriented Bordism)**.** Write $\mathcal{N}_* \coloneqq \Omega_*^O \cong \pi_* MT\mathrm{O}$ for unoriented bordism. This has been fully computed [34], and it's a polynomial ring:

$$\mathcal{N}_* \cong \mathbb{Z}/2[x_2, x_4, x_5, x_6, x_8, ...] \tag{2.2.40}$$

on generators x_i in degrees $i \neq 2^k - 1$. Manifold representatives are also known [35]: for even i, take $x_i = [\mathbb{RP}^i]$, and for odd $i = 2^r(2s+1)$ take $x_i = [P(2^r - 1, s2^r)]$ where $P(m,n)$ is the *Dold manifold*

$$P(m,n) = (S^m \times \mathbb{CP}^n)/((x, [(y_0, \dots, y_n)]) \sim (-x, [(\bar{y}_0, \dots, \bar{y}_n)]). \tag{2.2.41}$$

Two closed unoriented manifolds are bordant if and only if all Stiefel-Whitney numbers agree [34]. ◇

Example 2.2.42 (Oriented Bordism)**.** By Pontrjagin-Thom, $\Omega_*^{\mathrm{SO}} \cong \pi_*(MT\mathrm{SO})$. All torsion in Ω_*^{SO} is of exponent 2 [36], and rationally the bordism ring is [34]

$$\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \cong \pi_*(MT\mathrm{SO}) \otimes \mathbb{Q} \cong \pi_*(B\mathrm{SO}) \otimes \mathbb{Q} \cong H^*(B\mathrm{SO}; \mathbb{Q}) \cong \mathbb{Q}[p_i] \tag{2.2.43}$$

where the p_i in degrees $4i$ are the Pontryagin classes. In terms of manifold generators,

$$\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[\,[\mathbb{CP}^{2i}]\,]. \tag{2.2.44}$$

The entire ring has been determined [36], and in low degrees the groups are

$$\begin{aligned}
\Omega_0^{\mathrm{SO}} &= \mathbb{Z} \ \text{ generated by } \mathrm{pt}_+ \\
\Omega_1^{\mathrm{SO}} &= 0 \\
\Omega_2^{\mathrm{SO}} &= 0 \\
\Omega_3^{\mathrm{SO}} &= 0 \\
\Omega_4^{\mathrm{SO}} &= \mathbb{Z} \ \text{ generated by } [\mathbb{CP}^2] \\
\Omega_5^{\mathrm{SO}} &= \mathbb{Z}/2 \ \text{ generated by the } \textit{Wu manifold } \ [\mathrm{SU}_3/\mathrm{SO}_3] \\
\Omega_6^{\mathrm{SO}} &= 0 \\
\Omega_7^{\mathrm{SO}} &= 0 \\
\Omega_8^{\mathrm{SO}} &= \mathbb{Z}^2 \ \text{ generated by } [\mathbb{CP}^4] \text{ and } [\mathbb{CP}^2 \times \mathbb{CP}^2].
\end{aligned}$$

Two closed oriented manifolds are bordant if and only if they have the same Stiefel-Whitney and Pontrjagin numbers. ◇

Example 2.2.45 (Spin Bordism)**.** $\Omega_*^{\mathrm{Spin}} \cong \pi_*(MT\mathrm{Spin})$. That the Atiyah–Bott–Shapiro orientation (see Section 4.2.4.1) $MT\mathrm{Spin} \to ko$ is 7-connected ensures that the spin bordism groups agree with KO-theory up to degree 7, and this constitutes the first piece of the Anderson–Brown–Peterson splitting of $MT\mathrm{Spin}$ into shifts of ko and of $H\mathbb{Z}/2$ [37]. Let $\mathcal{P}(n)$ denote the set of partitions of n and $\mathcal{P}_1(n)$ denote the set of partitions where 1 is not a summand. Then there is a collection of cohomology classes $Z \subset H^*(MT\mathrm{Spin}; \mathbb{Z}/2)$ and a 2-local equivalence

$$MT\mathrm{Spin} \xrightarrow{\simeq} \bigvee_{k=0}^{\infty} \left(\bigvee_{\mathcal{P}_1(2k)} ko\langle 8k\rangle \quad \vee \bigvee_{\mathcal{P}_1(2k+1)} ko\langle 8k+2\rangle \right) \vee \bigvee_{z \in Z} \Sigma^{|z|} H\mathbb{Z}/2. \tag{2.2.46}$$

Here, for a spectrum X, $X\langle n\rangle$ indicates its *n-connective cover*: the spectrum with $\pi_k X\langle n\rangle = 0$ for $k < n$ and a map $X\langle n\rangle \to X$ inducing isomorphisms $\pi_k X\langle n\rangle \cong \pi_k X$ for $k > n$.

In low dimensions,

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}} &= \mathbb{Z} \ \text{ generated by } \mathrm{pt}_+ \\
\Omega_1^{\mathrm{Spin}} &= \mathbb{Z}/2 \ \text{ generated by } [S^1_{nb}] \\
\Omega_2^{\mathrm{Spin}} &= \mathbb{Z}/2 \ \text{ generated by } [S^1_{nb} \times S^1_{nb}] \\
\Omega_3^{\mathrm{Spin}} &= 0 \\
\Omega_4^{\mathrm{Spin}} &= \mathbb{Z} \ \text{ generated by the } \textit{K3 surface } [K3] \\
\Omega_5^{\mathrm{Spin}} &= 0 \\
\Omega_6^{\mathrm{Spin}} &= 0 \\
\Omega_7^{\mathrm{Spin}} &= 0 \\
\Omega_8^{\mathrm{Spin}} &= \mathbb{Z}^2 \ \text{ generated by } [\mathbb{HP}^2] \text{ and } 1/4[K3 \times K3]
\end{aligned}$$

Rationally, $\Omega_*^{\mathrm{Spin}} \otimes \mathbb{Q} \cong \Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$, and

$$H^*(B\mathrm{Spin}; \mathbb{Q}) \cong H^*(B\mathrm{SO}; \mathbb{Q}). \tag{2.2.47}$$

Generators in all dimensions are known, and two closed spin manifolds are bordant if and only if their Stiefel-Whitney and KO-Pontrjagin numbers agree [37]. The ring structure is not fully known. ◇

Example 2.2.48 (Spin^c Bordism)**.** Anderson–Brown–Peterson derived a similar splitting of the spectrum $MT\mathrm{Spin}^c$: there is a set of cohomology classes $Z \subset H^*(MT\mathrm{Spin}^c; \mathbb{Z}/2)$ and a 2-local homotopy-equivalence [37]

$$MT\mathrm{Spin}^c \xrightarrow{\simeq} \bigvee_{I \in \mathcal{P}} ku\langle 4|I|\rangle \vee \bigvee_{z \in Z} \Sigma^{|z|} H\mathbb{Z}/2. \tag{2.2.49}$$

In low dimensions,

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}^c} &= \mathbb{Z} \quad \text{generated by } \mathrm{pt}_+ \\
\Omega_1^{\mathrm{Spin}^c} &= 0 \\
\Omega_2^{\mathrm{Spin}^c} &= \mathbb{Z} \quad \text{generated by } [\mathbb{CP}^1] \\
\Omega_3^{\mathrm{Spin}^c} &= 0 \\
\Omega_4^{\mathrm{Spin}^c} &= \mathbb{Z}^2 \quad \text{generated by } [\mathbb{CP}^1 \times \mathbb{CP}^1] \text{ and } [\mathbb{CP}^2] \\
\Omega_5^{\mathrm{Spin}^c} &= 0 \\
\Omega_6^{\mathrm{Spin}^c} &= \mathbb{Z}^2 \\
\Omega_7^{\mathrm{Spin}^c} &= 0 \\
\Omega_8^{\mathrm{Spin}^c} &= \mathbb{Z}^4
\end{aligned}$$

◇

Example 2.2.50 (Spin^h Bordism)**.** For spin^h bordism, as originally computed in low dimensions by Freed–Hopkins [1] and Hu [38], Buchanan–McKean and Mills showed a similar splitting [39, Theorem 1.1] [40, Theorem 1.1]: there are cohomology classes $Z \subset H^*(M\mathrm{Spin}^h; \mathbb{Z}/2)$ such that there is a 2-local homotopy equivalence

$$MT\mathrm{Spin}^h \longrightarrow \bigvee_{I \in \mathcal{P}_{\mathrm{even}}} ksp\langle 4|I|\rangle \quad \vee \bigvee_{I \in \mathcal{P}_{\mathrm{odd}}} \Sigma^{4|I|} F \quad \vee \bigvee_{z \in Z} \Sigma^{|z|} H\mathbb{Z}/2, \tag{2.2.51}$$

where F is the fiber of the map $ko \to H\mathbb{Z}/2$ that classifies the nonzero element of $H^0(ko;\mathbb{Z}/2)$.

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}^h} &= \mathbb{Z} \ \text{ generated by } \mathrm{pt}_+ \\
\Omega_1^{\mathrm{Spin}^h} &= 0 \\
\Omega_2^{\mathrm{Spin}^h} &= 0 \\
\Omega_3^{\mathrm{Spin}^h} &= 0 \\
\Omega_4^{\mathrm{Spin}^h} &= \mathbb{Z}^2 \ \text{ generated by } [\mathbb{HP}^1] \text{ and } [\mathbb{CP}^2] \ [41] \\
\Omega_5^{\mathrm{Spin}^h} &= (\mathbb{Z}/2)^2 \ \text{ generated by the Wu manifold } [\mathrm{SU}(3)/\mathrm{SO}(3)] \text{ and } [S^1 \times S^4] \\
\Omega_6^{\mathrm{Spin}^h} &= (\mathbb{Z}/2)^2 \\
\Omega_7^{\mathrm{Spin}^h} &= 0 \\
\Omega_8^{\mathrm{Spin}^h} &= \mathbb{Z}^4
\end{aligned}$$

Most spin^h bordism generators are unknown, but we showed with Debray [42] that rank-$4k$ rational spin^h generators are induced from rational rank-$4k$ spin^c generators by taking the same underlying manifold and inducing the spin^h structure from the spin^c structure along the inclusion $\mathrm{U}(1) \simeq \mathrm{SO}(2) \hookrightarrow \mathrm{SO}(3)$. Two spin^h manifolds are bordant if and only if their KSp- and $H\mathbb{Z}/2$-characteristic numbers match [39, Theorem 1.2].

◇

Example 2.2.52 (String Bordism)**.** Since BString is 7-connected, its first 6 bordism groups and generators agree with the framed bordism groups in the next example. So, we focus on the higher-degree groups. Generators of the torsion subgroups below are all exotic spheres.

$$\begin{aligned}
\Omega_7^{\mathrm{String}} &= 0 \\
\Omega_8^{\mathrm{String}} &= \mathbb{Z} \oplus \mathbb{Z}/2 \ \text{ generated by a \textit{Bott manifold} } [43] \\
\Omega_9^{\mathrm{String}} &= (\mathbb{Z}/2)^2 \\
\Omega_{10}^{\mathrm{String}} &= (\mathbb{Z}/6) \\
\Omega_{11}^{\mathrm{String}} &= 0 \\
\Omega_{12}^{\mathrm{String}} &= \mathbb{Z} \ \text{ generated by a manifold of known signature} \\
\Omega_{13}^{\mathrm{String}} &= \mathbb{Z}/3 \\
\Omega_{14}^{\mathrm{String}} &= \mathbb{Z}/2.
\end{aligned}$$

For most applications in string theory, groups in degrees up to 14 suffice. See e.g. [44] for these groups.

◇

Example 2.2.53 (Framed Bordism)**.** By Pontryagin-Thom, framed bordism[6] is isomorphic

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[6] Here meaning manifolds with a framing of the stable tangent bundle, which some might call *stably framed*.

to the stable homotopy groups of spheres: $\Omega_*^{\mathrm{fr}} \cong \pi_*^s = \pi_*\mathbb{S}$. In low degrees, these are

$$\begin{aligned}
\Omega_0^{\mathrm{fr}} &= \mathbb{Z} \ \text{ generated by } [\mathrm{pt}] \\
\Omega_1^{\mathrm{fr}} &= \mathbb{Z}/2 \ \text{ generated by } [S^1_{\mathrm{Lie}}] = [\mathrm{U}(1)] \\
\Omega_2^{\mathrm{fr}} &= \mathbb{Z}/2 \ \text{ generated by } [S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}] \\
\Omega_3^{\mathrm{fr}} &= \mathbb{Z}/24 \ \text{ generated by } [S^3_{\mathrm{Lie}}] = [\mathrm{SU}_2] \\
\Omega_4^{\mathrm{fr}} &= 0 \\
\Omega_5^{\mathrm{fr}} &= 0 \\
\Omega_6^{\mathrm{fr}} &= \mathbb{Z}/2 \ \text{ generated by } [S^3_{\mathrm{Lie}} \times S^3_{\mathrm{Lie}}] \\
\Omega_7^{\mathrm{fr}} &= \mathbb{Z}/240 \ \text{ generated by } S^7 \text{ with framing from the generator of } \pi_7\mathrm{O} \\
\Omega_8^{\mathrm{fr}} &= (\mathbb{Z}/2)^2 \\
\Omega_9^{\mathrm{fr}} &= (\mathbb{Z}/2)^3 \\
\Omega_{10}^{\mathrm{fr}} &= \mathbb{Z}/6 \\
\Omega_{11}^{\mathrm{fr}} &= \mathbb{Z}/504 \\
\Omega_{12}^{\mathrm{fr}} &= 0
\end{aligned}$$

where the subscript Lie indicates to take the Lie group framing. The stable homotopy groups of spheres are known to all be finite [45] and at the moment have been computed up to degree 83 [46] with partial results up to 90 [47]. ◇

2.3 Fully-Extended Field Theories

Our next step will be to encode *locality*, or more specifically, *cluster decomposition*: the physical expectation that quantities evaluated from the data of a quantum field theory—like the partition function, correlation functions, and so on—should be able to be evaluated locally and then glued together. That is, these quantities should be able to be evaluated using various decompositions of the spacetime manifold. In the factorization algebras approach to formalizing quantum field theory, for example, this desired property is encoded through the definition of factorization algebra. In functorial field theory, locality can be encoded using higher categorical structure in the bordism and target categories. Extended field theory is also important for modeling extended operators and defects of various codimensions, which are of particular interest in the current research program exploring categorical symmetries in quantum field theory; see e.g. [48].

Our focus will be fully-extended field theories, so we shall skip to the (∞, n)-category of bordisms without including truncated versions for comparison.

Definition Sketch 2.3.1 ([26, Definition Sketch 1.4.6])**.** The (∞, n)-*category of bordisms* is the symmetric monoidal (∞, n)-category, written $\mathsf{Bord}_{(\infty,n)}$, whose objects are 0-manifolds and whose k-morphisms are manifolds with corners, which represent bordisms of bordisms.

For a complete definition, see Calaque–Scheimbauer [49, Section 5], who use the Segal space model of higher categories, or Ayala–Francis [50], who use flaggings and stratified spaces. See also [51, Definition A.6].

Definition 2.3.2. Let C be a symmetric monoidal (∞, n)-category. A *fully-extended n-dimensional functorial field theory* with target C is a symmetric monoidal functor

$$F\colon \mathsf{Bord}_{(\infty,n)} \longrightarrow \mathsf{C}. \tag{2.3.3}$$

Definition 2.3.4. Let C be as above and let (X, ξ) be a stable tangential structure. A *fully-extended (X, ξ)-structured functorial field theory* with target C is a symmetric monoidal functor

$$F\colon \mathsf{Bord}_{(\infty,n)}^{(X,\xi)} \longrightarrow \mathsf{C}. \tag{2.3.5}$$

Such theories are determined by their value on a point, according to the cobordism hypothesis. See [52, 53] and e.g. [54].

Theorem 2.3.6 ([26, Theorem 2.4.6])**.** *Let C be a symmetric monoidal (∞, n)-category with duals, and write Core(C) for its maximal ∞-groupoid. Let $\mathsf{Bord}_n^{\mathrm{fr}}$ be the symmetric monoidal (∞, n)-category of bordisms with n-framing. Then there is an equivalence of (∞, n)-categories*

$$\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{fr}}, \mathsf{C}) \longrightarrow \mathrm{Core}(\mathsf{C}) \tag{2.3.7}$$

$$Z \longmapsto Z(\mathrm{pt}) \tag{2.3.8}$$

There is also a version for other tangential structures. We will not discuss the cobordism hypothesis further in this thesis, but mention it because it is an important result in the context of extended theories.

Example 2.3.9. We will give an incomplete sketch of the extended Arf theory. This is a symmetric monoidal functor of 2-categories from the source $\mathsf{Bord}_{\langle 0,1,2\rangle}^{\mathrm{Spin}}$, the 2-category with objects spin 0-manifolds (points), 1-morphisms spin cobordisms between points (i.e. spin 1-manifolds with boundary), and 2-morphisms cobordisms of 1-morphisms that are trivial on the boundary (i.e. spin 2-manifolds with corners), to the target $\mathsf{sAlg}_{\mathbb{C}}$, the 2-category with objects complex superalgebras, 1-morphisms graded bimodules, and 2-morphisms intertwiners. Let s denote the bordism implementing the spin flip. The extended Arf theory makes the following assignments:

$$\begin{aligned} Z\colon \quad \mathsf{Bord}_{\langle 0,1,2\rangle}^{\mathrm{Spin}} &\longrightarrow \mathsf{sAlg}_{\mathbb{C}} \\ \mathrm{pt}_+ &\longmapsto C\ell_{+1} \\ \mathrm{pt}_- &\longmapsto C\ell_{-1} \\ \mathrm{id}_{\mathrm{pt}_\pm} &\longmapsto \mathbb{C} \\ s_{\mathrm{pt}_\pm} &\longmapsto \Pi\mathbb{C} \end{aligned} \tag{2.3.10}$$

and so on. See [27]. See also [55] for the related extended Arf–Brown TQFT. ◇

2.4 Unitarity and Invertibility

Unitarity—the property that time evolution is implemented by a unitary operator—is fundamental to quantum physics. In functorial field theory, it also enforces surprising mathematical constraints.

2.4.1 Dagger Categories for Unitarity

This section borrows from [56], which is joint work of the author with Gio Ferrer, Brett Hungar, Theo Johnson-Freyd, Lukas Müller, Nivedita, David Penneys, David Reutter, Claudia Scheimbauer, Luuk Stehouwer, and Chetan Vuppulury.

Intuitively, when time is reversed, the time evolution operator should be inverted, yielding—if it is a unitary operator—its adjoint. *Dagger categories* generalize this notion of having an adjoint. The easiest definition of dagger category to remember is the following.

Definition 2.4.1. A *dagger 1-category* is a category C equipped with an anti-involution $(-)^\dagger \colon \mathsf{C} \to \mathsf{C}^{op}$ that is the identity on objects.

This means there is a bijection $(-)^\dagger \colon \mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{C}}(y, x)$ for each pair of objects x, y in C, such that for each morphism f, $(f^\dagger)^\dagger = f$, and for each pair f, g of composable morphisms, $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$.

Example 2.4.2. The category of finite-dimensional complex Hilbert spaces $\mathsf{Hilb}_{\mathbb{C}}$ is made by equipping the category $\mathsf{Vect}_{\mathbb{C}}$ with inner products and with the involution $(-)^\dagger$ corresponding to taking adjoints of linear maps. ◇

However, since Definition 2.4.1 explicitly references objects, it is *evil*, meaning that it is not transported across equivalences of categories. One coherent definition of dagger 1-category uses anti-involutive categories, which are homotopy fixed points of the category of $(2, 1)$-categories under the $\mathbb{Z}/2$ action of taking opposites. Let $\iota_0\mathsf{C}$ denote the groupoid of objects in a category C.

Definition 2.4.3 ([57])**.** A *coherent dagger 1-category* is an anti-involutive category $(\mathsf{C}, (-)^\dagger)$ with a fully-faithful subgroupoid $\mathsf{C}_0 \hookrightarrow (\iota_0\mathsf{C})^{\mathbb{Z}/2}$ such that the composition $\mathsf{C}_0 \hookrightarrow (\iota_0\mathsf{C})^{\mathbb{Z}/2} \to \mathsf{C}$ is essentially surjective.

See e.g. [57] or [56] for more details. Note that C_0 picks out a space of preferred objects, which are ones that we consider to be "positive."

Example 2.4.4. If $\mathsf{C} = \mathsf{Vect}_{\mathbb{C}}$ and $(-)^\dagger \colon V \mapsto \bar{V}^\vee$ is the functor mapping a vector space to its complex conjugate dual, then $(\iota_0\mathsf{C})^{\mathbb{Z}/2}$ is the category $\mathsf{Herm}_{\mathbb{C}}$ of finite-dimensional *Hermitian vector spaces*, which are vector spaces equipped with a nondegenerate sesquilinear form. To get the category $\mathsf{Hilb}_{\mathbb{C}}$, we choose C_0 to pick out the Hermitian vector spaces with *positive-definite* forms. ◇

The category $\mathsf{Hilb}_{\mathbb{C}}$ and its super analog $\mathsf{sHilb}_{\mathbb{C}}$ provide targets for non-extended unitary field theories. We also need a dagger categorical source. To that end, consider the *reflection* *action* on (X, ξ)-manifolds that preserves the underlying manifold but reverses ξ-structures; see [1, Section 4.1] for a restricted set of ξ-structures and [56, Section 6] for general stable tangential structures. This action gives rise to involutions on $\mathsf{Bord}^{(X,\xi)}_{\langle n-1,n\rangle}$ and $\mathsf{Bord}^{(X,\xi)}_{(\infty,n)}$, as well as on the Madsen-Tillmann spectra $\Sigma^n MT\xi_n$ and $MT\xi$.

Definition 2.4.5. A non-extended theory with *reflection structure* is a $\mathbb{Z}/2$-equivariant functor $Z\colon \mathsf{Bord}^X_{\langle n-1,n\rangle} \longrightarrow \mathsf{Vect}_{\mathbb{C}}$.[7]

See [1, Definition 4.14] and [56, Section 6].

Definition 2.4.6. A non-extended theory $Z\colon \mathsf{Bord}^X_{\langle n-1,n\rangle} \longrightarrow \mathsf{Vect}_{\mathbb{C}}$ with reflection structure is *reflection-positive* if it induces a dagger functor $Z\colon \mathsf{Bord}^X_{\langle n-1,n\rangle} \longrightarrow \mathsf{Hilb}_{\mathbb{C}}$.

Consequently, for any object Y^{n-1}, the Hermitian vector space $Z(Y^{n-1})$ must be positive definite. See e.g. [1, Definition 4.18].

Remark 2.4.7. Reflection positivity is the Wick-rotated analog of unitarity. We will not discuss the interesting distinction between these concepts and instead use the terms interchangeably.

Example 2.4.8. Assume that n is even. The n-dimensional Euler theory Z_λ of Example 2.1.14 equipped with a reflection structure is reflection positive only if $\lambda \in \mathbb{R}$. By definition, $Z_\lambda(S^n) = \lambda^{\chi(S^n)} = \lambda^2$. However, viewing S^n as the union of the cup and the cap bordisms made out of disks D^n, we can interpret $Z_\lambda(S^n)$ as giving the inner product of $Z_\lambda(D^n)$ with itself with respect to the Hermitian form on S^{n-1}. For $Z_\lambda(S^{n-1})$ to be positive-definite, $Z_\lambda(S^n)$ must be a (positive) real number. ◇

For invertible extended theories, which will be defined in the next subsection, reflection positivity is encoded using $\mathbb{Z}/2$ actions on the source and target spectra MTH and $I_{\mathbb{Z}}$. We will not give the details and instead cite [1]. For non-invertible extended theories, we proposed a definition in [56] as part of the Higher Dagger Categories workshop. However, for this thesis we only need the invertible case.

Remark 2.4.9. In [58], we extended the reflection action on the source and target of fully-extended invertible theories to an action of the orthogonal group O to give an extended version of the *spin statistics theorem*.

2.4.2 Invertible Theories and Classification

This section is modified from [9, Section 8.2.2], which is joint work of the author with Arun Debray, Sanath Devalapurkar, Yu Leon Liu, Natalia Pacheco-Tallaj, and Ryan Thorngren.

In the case of invertible field theories, the physical constraint of unitarity has a profound mathematical consequence: stability. More precisely, the classification of reflection-positive, invertible field theories reduces to bordism invariants (Theorem 2.4.21). See Freed [59, Lectures 6–9] and Galatius [60] for more detailed reviews of this story. For readability in this section, we abbreviate $\mathsf{Bord}^{(X,\xi)}_{(\infty,n)}$ to Bord^{ξ}_n.

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[7] Here, the $\mathbb{Z}/2$-actions on the source and target are the involutions described above, not just any actions.

Example 2.4.10. The *trivial theory* $\mathbf{1}\colon \mathsf{Bord}_n^\xi \to \mathsf{C}$ is given by the constant functor valued at the monoidal unit of C, the identity 1-morphism, identify 2-morphism, and so on. ◇

Definition 2.4.11 ([61, Definition 5.7])**.** A field theory $Z\colon \mathsf{Bord}_n^\xi \to \mathsf{C}$ is *invertible* if there is some other theory Z^{-1} such that $Z \otimes Z^{-1}$ is the trivial theory.

This tensor product is evaluated "pointwise," meaning that $(Z \otimes Z^{-1})(M) := Z(M) \otimes Z^{-1}(M)$, where M is an object, 1-morphism, etc. in the bordism category. Therefore, invertibility implies that the functor Z factors through the Picard sub-k-groupoid of units $\mathsf{C}^\times$ inside C, meaning that if X is any object, morphism, or higher morphism in Bord_n^ξ, $Z(X)$ is invertible: $\otimes$-invertible if X is an object, and composition-invertible if X is a (higher) morphism. If X is invertible, then we must have data of an isomorphism $Z(X^{-1}) \xrightarrow{\cong} Z(X)^{-1}$ because Z is symmetric monoidal; thus, even if X is not invertible, we can heuristically *define* $Z(X^{-1}) := Z(X)^{-1}$ as if X^{-1} existed. These definitions are compatible as X varies, in the sense that Z extends to the *Picard k-groupoid completion* $\overline{\mathsf{Bord}_n^\xi}$ of Bord_n^ξ: the Picard k-groupoid defined by formally adding inverses to all objects, morphisms, higher morphisms, etc. of Bord_n^ξ. Thus, an invertible field theory $Z\colon \mathsf{Bord}_n^\xi \to \mathsf{C}$ is equivalent data to a morphism of Picard k-groupoids

$$Z\colon \overline{\mathsf{Bord}_n^\xi} \longrightarrow \mathsf{C}^\times. \tag{2.4.12}$$

An invertible theory valued in $\mathsf{Vect}_\mathbb{C}$ must land only in the Picard groupoid $\mathsf{Line}_\mathbb{C}$ of one-dimensional vector spaces. So, it must assign 1-dimensional state spaces to all input manifolds. This criterion immediately rules out models with ground state degeneracy.

Example 2.4.13. Finite gauge theory (Example 2.1.8) is not invertible in general, since the state spaces $\mathbb{C}[\mathsf{Bun}_G(M)]$ assigned to manifolds M will be multi-dimensional whenever M hosts nontrivial G-bundles. ◇

Example 2.4.14. The Arf theory of Example 2.2.23 is invertible, as we may guess by observing that it lands in $\mathsf{sLine}_\mathbb{C}$. It is actually self-inverse. ◇

Example 2.4.15. The Euler theory of Example 2.1.14 Z_λ is invertible, with inverse $Z_{\lambda^{-1}}$. ◇

To compute deformation classes of invertible field theories, we need to compute the groups of symmetric monoidal functors between these Picard k-groupoids, modulo natural isomorphisms. The homotopy theory of Picard groupoids embeds in the usual stable homotopy category: if D is a Picard groupoid, the geometric realization $|N\mathsf{D}|$ of the nerve of D has an E_∞-structure arising from the monoidal product on D, and the Picard condition implies $|N\mathsf{D}|$ is grouplike. Therefore it is equivalent data to a connective spectrum $|\mathsf{D}|$, which we call the *classifying spectrum* of D. This turns out to be a complete invariant of Picard k-groupoids.

Theorem 2.4.16 (Stable homotopy hypothesis (Moser–Ozornova–Paoli–Sarazola–Verdugo [62]))**.** *There is an equivalence of ∞-categories between the ∞-category of Picard k-groupoids and the ∞-category of spectra whose homotopy groups vanish outside of* $[0, k]$.

Remark 2.4.17. For $k = 1$, the stable homotopy hypothesis was originally a folklore theorem: proofs or sketches appear in [63–68]. For $k = 2$, the stable homotopy hypothesis was proven by Gurski–Johnson–Osorno [69].

Therefore we need to compute the group of homotopy classes of maps of spectra $|\overline{\mathsf{Bord}}_n^\xi| \to |\mathsf{C}^\times|$. A reasonable first step would be to identify these two classifying spectra. For the domain, the Picard k-groupoid completion of the bordism category, this is due to Galatius-Madsen-Tillmann-Weiss [70] and Nguyen [71] for the bordism $(\infty, 1)$-category and to Schommer-Pries [72] for more general (∞, k)-categories.

Theorem 2.4.18 (Galatius–Madsen–Tillmann–Weiss [70], Nguyen [71], Schommer-Pries [72])**.** *If* Bord_n^ξ *denotes the* (∞, k)*-category of bordisms of* ξ_n*-structured manifolds in dimensions* $n-k, \dots, n$*, then there is a natural equivalence* $|\overline{\mathsf{Bord}}_n^\xi| \simeq \Sigma^k MT\xi_n$*.*

Here $MT\xi_n$ is a Madsen-Tillmann spectrum as in Definition 2.2.35.

Freed–Hopkins–Teleman [73] then applied this result to classify invertible field theories in terms of $MT\xi_n$. The form of $|\mathsf{C}^\times|$, depends on the choice of C—Freed–Hopkins [1, §5.3] argue that the (shifted) *character dual of the sphere spectrum* $\Sigma^n I_{\mathbb{C}^\times}$ is a universal choice, and that a related object called the (shifted) *Anderson dual of the sphere spectrum* $\Sigma^{n+1} I_{\mathbb{Z}}$ should appear when one wants to classify deformation classes of invertible field theories. For applications to anomalies, we are interested in deformation classes, so use $\Sigma^{n+1} I_{\mathbb{Z}}$.

The Anderson dual $I_{\mathbb{Z}}$ is characterized by its universal property that for any spectrum $\mathcal{X}$, there is a short exact sequence [74, 75]

$$0 \longrightarrow \mathrm{Tors}(\mathrm{Hom}(\pi_{n+1}\mathcal{X}, \mathbb{C}^\times)) \xrightarrow{\varphi} [\mathcal{X}, \Sigma^{n+2} I_{\mathbb{Z}}] \xrightarrow{\psi} \mathrm{Hom}(\pi_{n+2}\mathcal{X}, \mathbb{Z}) \longrightarrow 0 \qquad (2.4.19)$$

Using this universal property and equivariant homotopy theory, Freed–Hopkins computed the classification of the topological theories of our interest.

Theorem 2.4.20 (Freed–Hopkins [1, Theorem 2.19])**.** *If* $n \geq 3$ *and* $\xi_n \colon BH(n) \to B\mathrm{O}(n)$ *is a tangential structure arising from a representation* $\rho \colon H(n) \to \mathrm{O}(n)$ *with* $H(n)$ *a compact Lie group and* $\mathrm{SO}(n) \subset \mathrm{Im}(\rho)$*, then there is a stable tangential structure* $\xi \colon BH \to BO$ *such that* ξ_n *is the pullback of* ξ *along* $B\mathrm{O}(n) \to B\mathrm{O}$*.*

Theorem 2.4.21 (Freed–Hopkins [1, Theorem 5.23], Grady [76])**.** *Suppose* $\xi \colon BH(n) \to B\mathrm{O}(n)$ *satisfies the hypotheses of Theorem 2.4.20. The abelian group of deformation classes of* n*-dimensional, reflection positive invertible field theories on manifolds with* ξ*-structure is naturally isomorphic to* $[MT\xi, \Sigma^{n+1} I_{\mathbb{Z}}]$*.*

So after we require reflection positivity, the classification changes from Madsen-Tillmann bordism to bordism in the usual sense, which is easier to calculate.

Notation 2.4.22*.* We will often denote $[MT\xi, \Sigma^{n+1} I_{\mathbb{Z}}]$ by $\mho_\xi^n$. Correspondingly, we denote the group of deformation classes of families of theories over Y, formally $[MT\xi \wedge Y_+, \Sigma^n I_{\mathbb{Z}}]$, by $\mho_\xi^n(Y)$. The reason for this choice in notation is that the degree in $\mho_\xi^n$ matches the dimension of the torsion subgroup, so that in the case the group of field theories is torsion, the group matches the corresponding bordism group.

Note that in our usage, $\mho^*$ indicates Anderson-dual bordism, not *co*bordism, which is what we choose to call the theory that is Poincaré dual to bordism.

Remark 2.4.23*.* There are some other approaches to the classification of invertible topological field theories, due to Yonekura [77], Rovi–Schoenbauer [78], Kreck–Stolz–Teichner (unpublished), and Hoekzema–Stehouwer–Veselá [79].

2.5 Physical Context

In this thesis we use the same mathematical objects—invertible field theories as defined above—to study physics questions in two different contexts: high energy theory and condensed matter.

2.5.1 Anomaly Theories

Chapter 3 relies on the interpretation of invertible field theories as *'t Hooft anomalies* of symmetries, which we use as invariants of their boundary quantum field theories. The following is adapted from [9, Section 8.1] and [10, Section II.A], which are joint works of the author with Arun Debray, Sanath Devalapurkar, Yu Leon Liu, Natalia Pacheco-Tallaj, and Ryan Thorngren.

We first discuss 't Hooft anomalies of the symmetry of a single theory. Physically, we begin with a k-dimensional quantum field theory with partition function Z and a symmetry group G, which is usually a compact Lie group. We can couple the theory to a background G-gauge field A, meaning that we parameterize the theory over BG. Then, the G-symmetry is *anomalous* if the partition function evaluated on a pair of a closed k-manifold M and a background gauge field with connection A is *not* gauge invariant but rather transforms with a phase. That is, under a gauge transformation $A \mapsto A^g$, the partition function transforms as

$$Z(M, A^g) = e^{i\alpha(M,A,g)} Z(M, A), \tag{2.5.1}$$

where $e^{i\alpha(M,A,g)}$ is a phase factor that cannot be canceled by local counter-terms. Mathematically, this means that when we evaluate the theory on a k-dimensional manifold M with a G-principal bundle with connection A, the partition function is a section of a non-trivial line bundle. The non-triviality of the line bundle is the anomaly.

To relate anomalies to invertible field theories, we use the notion of *anomaly inflow* [80, 81]: under mild hypotheses, and in all known cases, there is a local counterterm $e^{i\omega(K,A)}$ defined in one dimension higher, so that if K is a $(k+1)$-dimensional manifold with boundary and $\partial K = M$ (and A extends into K), then

$$e^{i\omega(K,A^g) - i\omega(K,A)} = e^{i\alpha(M,A,g)}. \tag{2.5.2}$$

We may interpret $e^{i\omega(K,A)}$ as the partition function of a $k+1$-dimensional invertible field theory with G symmetry. It is invertible, as stacking with the $e^{-i\omega(K,A)}$ theory gives the trivial theory. Furthermore, Z naturally lives at the boundary of this invertible theory, and together they are gauge invariant by Equation (2.5.2). Therefore we can interpret the existence of the G-anomaly, which is the failure of $Z(M, A)$ to be gauge invariant, as the statement that Z is a boundary theory of a non-trivial $(k+1)$-dimensional invertible field theory, which we call the bulk theory. This perspective, formulated mathematically by Freed–Teleman [82], is the link between anomalies and invertible field theories; see also Freed [83].

Example 2.5.3. A famous example is the chiral anomaly in 1+1d. Consider a theory of a free Dirac fermion with independently conserved left-movers and right-movers, corresponding to a symmetry group $G = U(1)_L \times U(1)_R$ with generators L and R. If we turn on background gauge fields A_L and A_R each with 2π magnetic flux through spacetime X, then there will be fermion zero modes that must be subtracted from the path integral measure, leading to an imbalance of "axial" $L - R$ charge and a nontrivial gauge variation of $Z_{\mathrm{Dirac}}(X, A_L, A_R)$. This variation is equivalent to the boundary variation of the 2+1d Chern-Simons term [84, 85]

$$\omega(Y^3, A) = \frac{1}{4\pi} \int_{Y^3} A_L dA_L - A_R dA_R. \tag{2.5.4}$$

This Chern Simons theory is one of the generators of the group $\mho^3_{\mathrm{Spin}^c} \cong \mathbb{Z}^2$.

We can think of the Dirac fermion as living at the *boundary* of a theory with this partition function. If we make symmetric deformations of the Dirac fermion, such as adding Luttinger interactions, the bulk cannot be affected even at strong coupling, and hence the anomaly does not change, since it is determined by the bulk. This property, known as anomaly matching, makes anomalies very useful for studying phase diagrams of theories and renormalization group flows. ◇

We also study anomalies of families of field theories parameterized by a topological space X. An anomaly of an X-family of field theories indicates a failure of the partition function to be consistently defined over the space of background X-fields.[8] In this case, the bulk theory is an one-dimension-higher X-family of invertible field theories.

Let us return to the classification from the previous section. Theorem 2.4.21 has a nice interpretation from the point of view of anomalies. Using the defining property of $I_{\mathbb{Z}}$, there is a short exact sequence

$$0 \longrightarrow \mathrm{Tors}(\mathrm{Hom}(\Omega^\xi_{n+1}, \mathbb{C}^\times)) \xrightarrow{\varphi} [MT\xi, \Sigma^{n+2} I_{\mathbb{Z}}] \xrightarrow{\psi} \mathrm{Hom}(\Omega^\xi_{n+2}, \mathbb{Z}) \longrightarrow 0, \tag{2.5.5}$$

where Tors(–) denotes the torsion subgroup. The first and third terms in this short exact sequence have anomaly-theoretic interpretations:

- The quotient $\mathrm{Hom}(\Omega^\xi_{n+2}, \mathbb{Z})$ is a free abelian group consisting of characteristic classes of $(n + 2)$-dimensional ξ-manifolds; under this identification, the map ψ sends an anomaly field theory to the corresponding anomaly polynomial, which is one degree higher, such as Chern-Simons and Chern-Weil forms. This data is visible to perturbative techniques, and is sometimes called the *local anomaly.*

- The subgroup $\mathrm{Tors}(\mathrm{Hom}(\Omega^\xi_{n+1}, \mathbb{C}^\times))$ is identified with the torsion subgroup of $[MT\xi, \Sigma^{n+2} I_{\mathbb{Z}}]$; these are the reflection positive invertible field theories that are *topological.* Such field theories' partition functions are bordism invariants, and the identification of these reflection positive invertible TFTs with $\mathrm{Tors}(\mathrm{Hom}(\Omega^\xi_{n+1}, \mathbb{C}^\times))$ assigns to a reflection positive invertible TFT its partition function. Typically this data is invisible to perturbative methods and is called the *global anomaly.*

Yamashita-Yonekura [86] and Yamashita [87] relate the short exact sequence (2.5.5) to a differential refinement of $\mathrm{Map}(MT\xi, \Sigma^{n+2} I_{\mathbb{Z}})$.

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[8] Typically in physics, X carries more structures, such as a smooth structure or Riemannian metric. The anomalies we consider here will not depend on those structures.

2.5.2 Symmetry-Protected Topological Phases

In the context of condensed matter physics, *symmetry-protected topological phases* are short-range entangled quantum phases of matter at zero temperature. They feature a bulk gap—meaning that the Hamiltonian describing the material has a gap in its spectrum between the ground state and the first excited state when restricted to the interior of the material—and nontrivial boundary states that are protected by symmetry. To be protected by symmetry means that symmetry-breaking deformations to the model can destroy the boundary state. See e.g. [88] for a physics review, and see e.g. [89, Section 1] for a mathematical discussion and more references.

SPT phases are of interest for their applications to quantum devices and quantum information. Many have been realized in experiment, like (class A) Chern insulators [90], quantum spin Hall effect insulators (class AII) [91], topological solitons in class AIII [92]—and some, like the integer quantum Hall effect, which were discovered experimentally [93] and only subsequently theoretically described theoretically [94]. Experimental work is ongoing to detect signatures of Majorana fermions, which feature in many famous SPT models, including the Majorana chain [95].

Mathematically, we model SPTs as reflection-positive invertible field theories, and the choice of tangential structure to impose on manifolds in the theory depends on the symmetry of the SPT. This mathematical model is really a special case of the interpretation in Section 2.5.1, since the invertible theory we study is the anomaly theory of the gapless quantum field theory describing the boundary modes of the SPT. This perspective is fairly recent in the context of condensed matter, but is supported by a growing body of computational evidence [1, 89, 96–103] and provides new insights.

However, since condensed matter models of SPTs often come in the form of lattice models, we would also like to discuss the role of invertible field theories as *low energy field theories.* Often, SPT phases are studied in condensed matter physics using lattice models. We will not attempt a mathematical definition of such models here, but just remark that they are often specified by Hamiltonian operators that may have excited states as well as the ground state. The invertible field theory is meant to capture just the information of the ground state—the low energy behavior. We modify the following ansatz from [55].

Ansatz 2.5.6. If a $d+1$-dimensional invertible field theory Z is the low energy field theory of a certain lattice model of a specified symmetry type, we expect that

1. On all objects Y^d, $Z(Y^d)$ gives the ground state of the lattice model, and
2. On a mapping torus M_φ for a diffeomorphism $\varphi\colon Y^d \to Y^d$, $Z(M_\varphi)\colon Z(Y^d) \to Z(Y^d)$ gives the action of the Hamiltonian on the ground state.

Example 2.5.7. The Majorana chain [95] defined by Kitaev is a fermionic SPT in $d = 1$ with “no symmetry,” meaning no symmetry in addition to the spin structure, which is required to define fermions. It has two phases, which we refer to as trivial and nontrivial, so its phase classification is given by the group $\mathbb{Z}/2$. Its nontrivial phase on a finite-length wire features Majorana zero modes on either end. Its low energy field theory is the Arf theory of Example 2.2.23, which is the generator of $\mho^3_{\mathrm{Spin}} \cong \mathbb{Z}/2$. The interesting ground state of the nontrivial phase corresponds to the state space of the Arf theory on S^1_{nb}. This result follows from [55]. ◇

The SPT phase classification of a model depends on the choice of symmetries to impose, just like the deformation class of an invertible field theory depends on the tangential structures required.

Example 2.5.8. The time-reversal symmetric Majorana chain, with time reversal squaring to the identity, has a different phase classification. It has the same underlying lattice model but now with time-reversing terms prohibited. Fidkowski–Kitaev showed that this model has a $\mathbb{Z}/8$ (interacting) classification [104]. The low-energy field theory for this model is the Arf–Brown–Kervaire theory [55] (see **??** and Example 4.2.116), which generates the group $\mho^3_{\mathrm{Pin}^-} \cong \mathbb{Z}/8$. The nontrivial phase of the lattice model should be detected on a generator $\mathbb{RP}^2$ of the bordism group $\Omega_2^{\mathrm{Pin}^-}$, though this is difficult to compute. ◇

Example 2.5.9. The Su-Schrieffer-Heeger model (SSH) model [105] is a $d = 1$ model for an insulator with particle-hole symmetry, modeling phases in polyacetylene. We study the low energy field theory of this model in forthcoming work with Stehouwer. ◇

Low energy field theories are not restricted to invertible theories, though we phrased Ansatz 2.5.6 only for these. In contrast to SPT phases, *topological orders* derive their interesting properties precisely from their lack of invertibility—they feature ground state degeneracy, and quantum information can be stored using the disparate ground states, or superselection sectors. Both SPTs and topological orders are kinds of topological phases, but the former see interesting topology as imposed by the symmetry group, while the latter see interesting topology in the manifold the model is defined on.

Example 2.5.10 (Toric Code)**.** The toric code [106] is a model defined by Kitaev in the context of quantum information theory. It has a Hamiltonian model defined on a square lattice with local interaction terms and its ground states are labeled by the cohomology group $H^0(\mathbb{T}^2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^2$. It features (abelian) anyons of four types, corresponding to the four possible ground states. The low energy field theory of the toric code is $\mathbb{Z}/2$-gauge theory [107] (Example 2.1.8), which indeed is not an invertible theory. ◇

We will not discuss topological orders further, but just included this example for comparison.

Our focus in Chapter 4 will be on SPTs known as topological insulators—materials that are insulating in the bulk but that conduct electricity on the surface—and topological superconductors—materials with a bulk gap but with gapless superconducting states on their surfaces. We study these phases by studying their low energy field theories, and more specifically use K-theory and invertible field theories to compare their free and interacting phase classifications.

Chapter 3

Smith Maps and Symmetry Breaking

The material in this chapter represents joint work of the author with Arun Debray, Sanath Devalapurkar, Yu Leon Liu, Natalia Pacheco-Tallaj, and Ryan Thorngren, which may be found in the papers [9, 108].

3.1 Introduction

Let M be a closed, smooth n-manifold equipped with a real line bundle $\pi\colon L \to M$. For any section $s\colon M \to L$ of π transverse to the zero section, standard theorems in differential topology imply $N_s \coloneqq s^{-1}(0)$ is a smooth, $(n-1)$-dimensional submanifold of M. We would like to make N_s into an invariant of M and L, but its diffeomorphism type depends on s: consider the trivial line bundle over S^1 with the constant section valued in 1 versus any section intersecting the zero section. However, all choices of N_s are *bordant*: given two sections $s_1, s_2\colon M \to L$, there is a compact n-manifold X whose boundary is diffeomorphic to $N_{s_1} \amalg N_{s_2}$. Thus the image of N_s in the *bordism group* $\Omega_{n-1}^{\mathrm{O}}$, the set of bordism equivalence classes with group structure given by disjoint union, is a well-defined invariant of M and L.

More is true: one can refine the bordism equivalence relation to extend the line bundle $L|_{N_{s_1}} \amalg L|_{N_{s_2}}$ across X, obtaining an invariant valued in the larger group $\Omega_{n-1}^{\mathrm{O}}(B\mathrm{O}(1))$ of bordism classes of the data of a closed $(n-1)$-manifold and a real line bundle. The value of this invariant also only depends on the bordism class of M and L and is additive in disjoint unions, re-expressing our invariant as a homomorphism of abelian groups

$$\mathrm{sm}_\sigma\colon \Omega_n^{\mathrm{O}}(B\mathrm{O}(1)) \longrightarrow \Omega_{n-1}^{\mathrm{O}}(B\mathrm{O}(1)). \tag{3.1.1}$$

This map was first studied by Conner-Floyd [109, Theorem 26.1], who called it the *Smith homomorphism* after P. A. Smith. Subsequently, many authors studied similarly-defined maps between other bordism groups, focusing on two methods of generalization:[1]

1. Generalize from real line bundles to real or complex vector bundles of other ranks.
2. Keep track of other topological data on M, and how it is affected by passing to M.

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[1] Kirby-Taylor [110, Theorem 6.11, Remark 6.15] generalize the Smith homomorphism in a different direction in the setting of *characteristic bordism*; that generalization is out of scope of this paper.

For example, suppose we give M an orientation structure. The submanifold N_s does not inherit an induced orientation, and can be unorientable. However, since $TM|_{N_s} = TN_s \oplus L|_{N_s}$ has an orientation, it follows that $L|_{N_s}$ is the orientation line bundle of N. It follows that $L|_{N_s}$ gives no additional structure at all, and this variant of the Smith homomorphism factors through $\Omega_{n-1}^{\mathrm{O}} \subset \Omega_{n-1}^{\mathrm{O}}(B\mathrm{O}(1))$. That is, we have a map

$$\mathrm{sm}_\sigma \colon \Omega_n^{\mathrm{SO}}(B\mathrm{O}(1)) \longrightarrow \Omega_{n-1}^{\mathrm{O}}. \tag{3.1.2}$$

To the best of our knowledge, this was first written down by Komiya [111, §5]. Other examples in the literature show the same phenomenon: if one places a tangential structure on M in the sense of Lashof [112], the Smith homomorphism lands in a bordism group whose degree and tangential structure are in general different from those of the domain. We spend considerable time in this chapter developing a general theory of Smith homomorphisms: providing several equivalent definitions, studying periodicities of twists, and computing associated long exact sequences. There are long exact sequences both in bordism and in Anderson-dual invertible field theories associated to Smith maps, and the latter are remarkably useful for anomaly matching in high energy physics.

Continuing a program begun by Hason–Komargodski–Thorngren [113] and Córdova-Ohmori-Shao-Yan [114], we use Smith homomorphisms to model a physical process of defect anomaly matching across a symmetry-breaking process. As invertible field theories can be understood as anomalies of quantum field theories (Section 2.5.1), the Anderson-dualized Smith homomorphism provides an anomaly-matching formula expressing the anomalies of certain QFTs in terms of anomalies of lower-dimensional *defect theories* [113, 114]. In [113], they noted in section 4.4 that they were missing a mathematical way to compute their physical homomorphisms of interest. The fiber sequence of spectra we studied to answer this question led to more interesting physics: in forming a fiber sequence, hence a long exact sequence, we found a tool to easily compute Smith homomorphisms. Furthermore, the two new maps in the long exact sequence also have interesting physical interpretations in the context of spontaneous symmetry breaking.

Specifically, let $\mho_\xi^*(-)$ denote the generalized cohomology theory that is Anderson dual to ξ-bordism. The group $\mho_\xi^*(X)$ classifies anomalies of X-families of field theories with ξ tangential structure. In particular, when $X = BG$, this classifies field theories with G-symmetry. Using the fiber sequence, we show the following.

Corollary 3.7.6. *Let X, V, W, and ξ be as above. There is a long exact sequence of invertible field theories:*

$$\cdots \longrightarrow \mho_\xi^{n-1}(S(W)^{p^*V-r_V}) \xrightarrow{\mathrm{Ind}_W} \mho_\xi^{n-r_W}(X^{V\oplus W-r_V-r_W}) \xrightarrow{\mathrm{Def}_W} \mho_\xi^{n}(X^{V-r_V}) \xrightarrow{\mathrm{Res}_W} \mho_\xi^{n}(S(W)^{p^*V-r_V}) \longrightarrow \cdots \tag{3.1.3}$$

In a companion physics paper [10], we called this sequence the *symmetry-breaking long exact sequence* (SBLES). The map dual to the Smith map sm_W is the defect anomaly matching map Def_W studied in [113] and [114]. Our framework allows for the study of more physical examples, as well as provides a way of explicitly computing each map and thus extracting more physical information. Moreover, it expands our physical understanding of this form of symmetry breaking to include two other processes: residual anomaly obstructions, which prevent certain symmetry breaking patterns, and index anomaly matching. We study the entire *symmetry breaking long exact sequence* in Section 3.8

3.2 Maps of spectra inducing Smith homomorphisms

Here we provide a general definition of the Smith homomorphism, then lift it to a map sm of bordism spectra. The map of spectra has been studied, though its identification with the Smith homomorphism is new. We use it to write down the cofiber of sm (Theorem 3.4.1) and therefore obtain Smith long exact sequences of bordism groups and Anderson-dualized bordism groups (Corollaries 3.4.8 and 3.7.6).

3.2.1 (X, V)-twisted tangential structures

Twisted tangential structures are an important ingredient in the Smith homomorphism—they determine its domain and codomain. Throughout this subsection, we fix a topological space X, a vector bundle $V \to X$ of rank r, and a tangential structure $\xi\colon B \to B\mathrm{O}$.

Definition 3.2.1. Let $W \to Y$ be a vector bundle. An (X, V)-*twisted* ξ-*structure* on W is the data of a map $f\colon Y \to X$ and a ξ-structure on $W \oplus f^*(V)$.

There is a space of (X, V)-twisted ξ-structures on W, and just like for tangential structures, we will think of two such structures as the same if they lie in the same connected component.

Twisted ξ-structures provide a convenient way to describe a more complicated tangential structure in terms of a simpler one.

Example 3.2.2. Recall that a spin^c structure on an oriented vector bundle $W \to Y$ is the data of a complex line bundle $L \to Y$ and an identification $w_2(L) = w_2(W)$. The data of L is equivalent to a map $Y \to B\mathrm{U}(1)$ such that L is the pullback of the tautological complex line bundle $S \to B\mathrm{U}(1)$. The identification $w_2(L) = w_2(W)$ is equivalent by the Whitney sum formula to $w_2(W \oplus L) = 0$.

Choosing a spin structure on $W \oplus L$ first provides an orientation of $W \oplus L$, which since L is canonically oriented by its complex structure is equivalent to an orientation of W; then it additionally provides an identification $w_2(W \oplus L) = 0$. Therefore the data of a spin^c structure on W is equivalent to the data of L and a spin structure on $W \oplus L$, meaning that a spin^c structure is equivalent to a $(B\mathrm{U}(1), S)$-twisted spin structure. ◇

Example 3.2.3. In a similar way, one can show that if $\sigma \to B\mathbb{Z}/2$ is the tautological real line bundle, pin$^-$ structures are equivalent to $(B\mathbb{Z}/2, \sigma)$-twisted spin structures, pin$^+$ structures are equivalent to $(B\mathbb{Z}/2, 3\sigma)$-twisted spin structures, and pin^c structures are equivalent to $(B\mathbb{Z}/2, \sigma)$-twisted spin^c structures. ◇

It turns out that all of these twisted tangential structures can also be "untwisted" into ordinary tangential structures.

Lemma 3.2.4 (Shearing). *Let $T \to B\mathrm{O}$ denote the tautological rank-zero virtual vector bundle and $\zeta\colon B \times X \to B\mathrm{O}$ be classified by the rank-zero virtual vector bundle $\xi^*(T) \boxplus (V - r)$. Then (X, V)-twisted ξ-structures are equivalent to ζ-structures.*

The proof is given in [115, Lemma 10.18] for $\xi = \mathrm{Spin}$; the general case is completely analogous. Invoking the Pontrjagin-Thom theorem, we then learn:

Corollary 3.2.5. *There is a bordism theory of manifolds with (X, V)-twisted ξ-structures, corresponding to the Thom spectrum $MT\xi \wedge X^{V-r_V}$. The bordism groups of these manifolds are thus in isomorphism with $\Omega_*^\xi(X^{V-r_V})$.*

Here we use the fact that the Thom spectrum functor sends external direct sums to smash products, which is Lemma 2.2.34.

Lemma 3.2.6. *Suppose X is a closed smooth manifold with a ξ-structure and $M \subset X$ is an embedded submanifold such that the image of the mod 2 fundamental class of M in $H_*(X; \mathbb{Z}/2)$ is Poincaré dual to $e(V) \in H^r(X; \mathbb{Z}/2)$. Then M has a canonical (X, V)-twisted ξ-structure.*

Proof. Because the homology class of M is Poincaré dual to the mod 2 Euler class of V, the normal bundle to $M \hookrightarrow X$ is isomorphic to $V|_M$. Choose a Riemannian metric on X; this is a contractible choice, so will not change the connected component of the data we obtain, so as discussed above different choices of metric lead to the same (X, V)-twisted ξ-structure in the end.

Using the Levi-Civita connection induced by the metric, we may split the short exact sequence of vector bundles over M,

$$0 \longrightarrow TM \longrightarrow TX|_M \longrightarrow \nu \longrightarrow 0, \tag{3.2.7}$$

thereby obtaining an isomorphism $TM \oplus V|_M \cong TX|_M$. Since TX has a ξ-structure, this implies $TM \oplus V|_M$ has a chosen ξ-structure, i.e. that we have put a (X, V)-twisted ξ-structure on M. ☐

3.2.2 Smith homomorphisms induced by maps of Thom spectra

Fix a tangential structure $\xi\colon B \to B\mathrm{O}$ such that its bordism spectrum $MT\xi$ is a ring spectrum (e.g. O, SO, Spin^c, Spin). Fix also a virtual vector bundle $V \to X$ of rank r_V and $W \to X$ a vector bundle of rank r_W.

Definition 3.2.8. The *Smith homomorphism* associated to ξ, V, and W is the homomorphism

$$\mathrm{sm}_W\colon \Omega_n^\xi(X^{V-r_V}) \longrightarrow \Omega_{n-r_W}^\xi(X^{V\oplus W-r_V-r_W}) \tag{3.2.9}$$

that sends a closed n-manifold $[M]$ to the bordism class $[N]$, where $N \subset M$ is the submanifold defined as follows: pull back W from X to M and choose a section $s\colon M \to f^*W$ transverse to the zero section. Then, $N \coloneqq s^{-1}(0)$ is an $(n - r_W)$-dimensional manifold whose mod 2 homology class is Poincaré dual to $e(W)$, hence by Lemma 3.2.6 has a $(X, V \oplus W)$-twisted ξ-structure, and we define $\mathrm{sm}_W([M]) \coloneqq [N]$.

Proposition 3.2.10 ([113] §4.2)**.** *The bordism class $[N] \in \Omega_{n-r_W}^\xi(X^{V\oplus W-r_V-r_W})$ is independent of the choice of section.*

Example 3.2.11. Let $\xi\colon B\mathrm{Spin} \to B\mathrm{O}$, $X = B\mathbb{Z}/2$, $V = 0$, and $W = \sigma \to B\mathbb{Z}/2$, where σ is the tautological line bundle. The corresponding Smith homomorphism is

$$\Omega_n^{\mathrm{Spin}}(B\mathbb{Z}/2) \xrightarrow{\mathrm{sm}_\sigma} \Omega_{n-1}^{\mathrm{Spin}}((B\mathbb{Z}/2)^{\sigma-1}). \tag{3.2.12}$$

After shearing (Lemma 3.2.4), we recognize this as

$$\Omega_n^{\mathrm{Spin}\times\mathbb{Z}/2} \xrightarrow{\mathrm{sm}_\sigma} \Omega_{n-1}^{\mathrm{Pin}^-}. \tag{3.2.13}$$

Letting $V = 0$, σ, 2σ, and 3σ produces the maps in the four-periodic family discussed in Example 3.6.8. ◇

Now, to the maps of spectra inducing Smith homomorphisms. Let X be a topological space and V be a rank r real vector bundle on X. We abuse notation and also denote the associated classifying map by $V\colon X \to BO(r)$. The inclusion $0 \hookrightarrow W$ induces a zero section map $X \to X^W$. More generally, we have the following.

Definition 3.2.14. Let V and W be vector bundles on X. Let $S^V \to S^{V\oplus W}$ be the map of finite-dimensional spheres over X induced by the zero section map on W. The *Smith map* associated to X, V, and W is the map of Thom spaces

$$\mathrm{sm}_W\colon \mathrm{Th}(X;V) \to \mathrm{Th}(X;V\oplus W) \tag{3.2.15}$$

formed as the colimit of the map of spheres.

Definition 3.2.16. In the case that of a virtual bundle V, the zero section map induces a map of stable spherical fibrations $\mathbb{S}^V \to \mathbb{S}^{V\oplus W} \simeq \mathbb{S}^V \wedge \mathbb{S}^W$ over X. Taking the colimit, we get a map of Thom spectra

$$\mathrm{sm}_W\colon X^V \to X^{V\oplus W} \tag{3.2.17}$$

which we also call a *Smith map.*

Proposition 3.2.18. *The map on ξ-bordism groups induced by the map (3.2.17) of spectra is equal to the Smith homomorphism as defined in Definition 3.2.8.*

This follows by unpacking the Pontrjagin-Thom isomorphism.

3.3 Euler classes and Smith homomorphisms

Since the Smith maps of spectra in the previous section are defined as zero-section maps, it is natural to discuss them in terms of bordism Euler classes. However, one of the main elements that make Smith homomorphisms interesting are the *twistings* of tangential structures, and in the general twisted case we do not have a Thom class to work with.

3.3.1 Euler classes in generalized cohomology

Fix $\xi\colon X \to BO$ a tangential structure and $W\colon X \to BO(r_W)$ a vector bundle on X. We might like to describe the Smith homomorphism on ξ bordism groups as taking a manifold $(M, p\colon M \to X)$ with ξ-structure to a smooth representative of the Poincaré dual of $e(p^*W)$, where $e(p^*W) \in H^{r_W}(M;\mathbb{Z})$ is the cohomology Euler class of W. This, however, is *not* true in general—see [9, Appendix B] for a detailed counterexample.

Instead, we need an Euler class living in twisted cobordism. More generally, for $\mathcal{R}$ an $\mathbb{E}_1$ ring spectrum, we define a $\mathcal{R}$-valued Euler class in the $\mathcal{R}$-cohomology of X^{-W}. In the case where there is a $\mathcal{R}$-orientation on W, in Lemma 3.3.24 we show that the *untwisted* Euler class is the pullback of the Thom class $U^{\mathcal{R}}(W) \in \mathcal{R}^{r_W}(\mathrm{Th}(X;W))$ along the zero section $X \to \mathrm{Th}(X;W)$ (e.g. in [116, §13]), so that our definition deserves to be called an Euler class. We also generalize to the twisted setting where there is no Thom class.

Recall the setup of Definition 3.2.16. Let 0 be the vector bundle over X of rank zero. The zero section gives a map $0 \to W$ of vector bundles over X. Therefore we get a map of stable spherical fibrations

$$z\colon \mathbb{S}^0 \longrightarrow \mathbb{S}^W, \tag{3.3.1a}$$

i.e. a fiberwise map of spectra. Because 0 is the trivial rank-zero vector bundle, $\mathbb{S}^0$ is the constant stable spherical fibration $\underline{\mathbb{S}}$ with fiber $\mathbb{S}$.

Apply the duality $\mathrm{Map}(-, \mathbb{S})$ fiberwise to obtain another map

$$z^\vee\colon \mathbb{S}^{-W} \longrightarrow \mathbb{S}^0. \tag{3.3.1b}$$

Because the codomain of $z^\vee$ is constant as a functor $X \to \mathsf{Sp}$, there is an induced map of spectra:

$$e^{\mathbb{S}}(W)\colon X^{-W} = \operatorname{colim}_X \mathbb{S}^{-W} \to \mathbb{S}. \tag{3.3.1c}$$

Definition 3.3.2. The class $e^{\mathbb{S}}(W)$ is called the *stable cohomotopy Euler class* of W. Usually, we will interpret generalized cohomology of $X^{r_W - W}$ as the $(-W)$-twisted cohomology of X, meaning $e^{\mathbb{S}}(W)$ is an element of the degree-r_W $(-W)$-twisted stable cohomotopy of X.

Remark 3.3.3. This cohomology class of $e^{\mathbb{S}}(W)$ lives in $(\mathbb{S})^0(X^{-W})$. By the Pontrjagin-Thom isomorphism, this is equivalent to the twisted cobordism group $\Omega^0_{\mathrm{fr}}(X, -W)$.

Definition 3.3.4. Let $\mathcal{R}$ be an $(\mathbb{E}_1)$-ring spectrum, so that there is a unique ring map $1_{\mathcal{R}}\colon \mathbb{S} \to \mathcal{R}$. The *$\mathcal{R}$-cohomology Euler class of W*, denoted $e^{\mathcal{R}}(W)$, is the composition $1_{\mathcal{R}} \circ e^{\mathbb{S}}(W)$. As in the previous definition, we interpret this as an element of the degree-r_W $(-W)$-twisted R-cohomology of X.

Now we see how the Euler class and Smith homomorphism are related:

Proposition 3.3.5.

1. *Let 0 be the trivial rank 0 vector bundle on X; then $e^{\mathbb{S}}(0)\colon \Sigma^\infty_+ X \to \mathbb{S}$ is the infinite suspension of the crush map $X \to *$.*

2. *Let W be a vector bundle on X and $\mathrm{sm}_W\colon X^{-W} \to X$ be the Smith map. Then $e^{\mathbb{S}}(W) = (\mathrm{sm}_W)^*(e^{\mathbb{S}}(0))$.*

Proof. For part 1: 0 defines the trivial stable spherical fibration on X, which factors through a point. Therefore the Euler class of 0 is the pullback of the Euler class of the trivial bundle over a point.

For part 2: this follows from the fact that $e^{\mathbb{S}}(W)\colon X^{-W} \to \mathbb{S}$ factors through

$$X^{-W} \xrightarrow{\mathrm{sm}_W} X \xrightarrow{e^{\mathbb{S}}(0)} 0. \qquad \square$$

We immediately learn that Smith maps pull back Euler classes.

Corollary 3.3.6. *Given a virtual vector bundle V and a vector bundle W, let sm_W denote the Smith map $\mathrm{sm}_W\colon X^{-V\oplus -W} \to X^{-V}$. Then*

$$\mathrm{sm}_W^*(e^{\mathbb{S}}(V)) = e^{\mathbb{S}}(V \oplus W). \tag{3.3.7}$$

We can thus recover the Smith homomorphism from capping with the twisted Euler class.

Proposition 3.3.8. *For any virtual bundle V on X, the Smith map $X^V \to X^{V\oplus W}$ can be defined as the following composition:*

$$X^V \simeq X^{(V\oplus W)\oplus -W} \xrightarrow{\Delta} (X\times X)^{(V\oplus W)\boxplus -W} \simeq X^{V\oplus W} \wedge X^{-W} \xrightarrow{e^{\mathbb{S}}(W)} X^{V\oplus W}. \tag{3.3.9}$$

The map $X^{(V\oplus W)\oplus -W} \xrightarrow{\Delta} (X\times X)^{(V\oplus W)\boxplus -W}$ is induced by the diagonal map $\Delta\colon X \to X\times X$.

Proof. The Euler map for the trivial rank 0 vector bundle

$$X^0 \simeq \Sigma^\infty_+ X \xrightarrow{e^{\mathbb{S}}(0)} \mathbb{S}. \tag{3.3.10}$$

is the counit for the $\mathbb{E}_\infty$-coalgebra structure on $\Sigma^\infty_+ X$. By Proposition 3.3.5, the Euler class $e^{\mathbb{S}}(W)$ factors through (3.3.10) as

$$X^{-W} \longrightarrow X^{-W\oplus W} \simeq \Sigma^\infty_+ X \xrightarrow{e^{\mathbb{S}}(0)} \mathbb{S}. \tag{3.3.11}$$

This implies that (3.3.9) can be written as

$$\begin{tikzcd}
X^V \arrow[r, phantom, "\simeq"] & X^{(V\oplus W)\oplus -W} \arrow[r, "\Delta"] \arrow[rr, bend right, "\phi"', start anchor=south, end anchor=west, to=2-4] & (X\times X)^{(V\oplus W)\boxplus -W} \arrow[r] & (X\times X)^{(V\oplus W)\boxplus 0} \arrow[r, "\simeq"] & X^{V\oplus W}\wedge \Sigma^\infty_+ X \arrow[r, "e^{\mathbb{S}}(W)"] & X^{V\oplus W} \\
 & & & X^{V\oplus W} \arrow[u, "\Delta"] & &
\end{tikzcd} \tag{3.3.12}$$

Since the map $X^{V\oplus W} \to (X\times X)^{(V\oplus W)\boxplus 0} \simeq X^{V\oplus W}\wedge \Sigma^\infty_+ X$ comes from the comodule structure of $X^{V\oplus W}$ over $\Sigma^\infty_+ X$, the composite $X^{V\oplus W} \to (X\times X)^{(V\oplus W)\boxplus 0} \to X^{V\oplus W}$ is the identity map. Therefore it is sufficient to show that the map ϕ in (3.3.12) is homotopy equivalent to the spectral Smith map sm_W, and this follows by restricting to the diagonal in the map $(X\times X)^{(V\oplus W)\boxplus -W} \to (X\times X)^{(V\oplus W)\boxplus 0}$ along the top of (3.3.12), which is induced from $\mathrm{id}\boxplus \mathrm{sm}_W$. ☐

The dual version of Proposition 3.3.8 also holds.

Proposition 3.3.13. *Let $\mathcal{R}$ be a ring spectrum. Then the pullback map on $\mathcal{R}$-cohomology $\mathrm{sm}_W^*\colon \mathcal{R}^*(X^{V\oplus W}) \to \mathcal{R}^*(X^V)$ is equal to the cup product with $e^{\mathcal{R}}(W)$.*

Remark 3.3.14. The long exact sequence of field theories we shall discuss in Section 3.8 is cohomological in nature: it is given by applying $I_{\mathbb{Z}}MT\xi$-cohomology to sm_W. However, Proposition 3.3.13 does not apply: the Smith homomorphism there cannot be described as taking the product with an $I_{\mathbb{Z}}\mathcal{R}$-Euler class. This is because if $\mathcal{R}$ is a ring spectrum, $I_{\mathbb{Z}}\mathcal{R}$ usually admits no ring spectrum structure. However, $I_{\mathbb{Z}}\mathcal{R}$ is an $\mathcal{R}$-module, so we do learn from Proposition 3.3.13 that this Smith homomorphism is the cup product with $e^{\mathcal{R}}(W)$ using the $\mathcal{R}$-module structure. For example, when we study fermionic invertible phases, we will typically choose $\mathcal{R} = MT\mathrm{Spin}$.

Next we discuss orientability.

Definition 3.3.15. Let W be a vector bundle of rank n on X. Fix $\mathcal{R}$ an $\mathbb{E}_1$-algebra in spectra and let Mod_R be the ∞-category of $\mathcal{R}$-module spectra. An $\mathcal{R}$-*orientation* of W is a natural isomorphism ϕ of functors between

$$\mathcal{R}^W : X \xrightarrow{W} BO(n) \to \mathsf{Sp} \xrightarrow{-\wedge\mathcal{R}} \mathsf{Mod}_{\mathcal{R}} \tag{3.3.16}$$

and the constant functor valued in $\Sigma^n\mathcal{R}$. An $\mathcal{R}$-orientation of a manifold M means an $\mathcal{R}$-orientation of TM.

Remark 3.3.17. The map $z^\vee$ from (3.3.1b) is similar to an orientation on $-W$, in the sense of Ando-Blumberg-Gepner-Hopkins-Rezk, except that $z^\vee$ is in general non-invertible and between different suspensions of the sphere spectrum.

An $\mathcal{R}$-orientation ϕ on W induces an equivalence

$$\mathrm{colim}_X\mathcal{R}^W \simeq \Sigma^\infty_+\mathrm{Th}(X;W)\wedge\mathcal{R} \simeq X\wedge\Sigma^n\mathcal{R} \simeq \Sigma^n\Sigma^\infty_+X\wedge\mathcal{R}. \tag{3.3.18}$$

Definition 3.3.19. The composite

$$U\colon \Sigma^\infty_+\mathrm{Th}(X;W) = X^W \to \Sigma^\infty_+\mathrm{Th}(X;W)\wedge\mathcal{R} \simeq \Sigma^n\Sigma^\infty_+X\wedge\mathcal{R} \to \Sigma^n\mathcal{R} \tag{3.3.20}$$

is the *Thom class*. Often we think of U through its homotopy class, which lives in $\mathcal{R}^n(\mathrm{Th}(X;W))$.

Given a $\mathcal{R}$-orientation on W, we can also define the (untwisted) Euler class of W. This is a standard definition (e.g. [116, §13]).

Definition 3.3.21. Given an $\mathcal{R}$-orientation, we have a natural isomorphism of functors $X \to \mathsf{Mod}_R$

$$R^{-W} \simeq \Sigma^{-n}\underline{\mathcal{R}}, \tag{3.3.22}$$

where $\Sigma^{-n}\underline{\mathcal{R}}$ is the constant functor valued in $\Sigma^{-n}\mathcal{R}$. The composite

$$\Sigma^{-n}X \longrightarrow \Sigma^{-n}X\wedge\mathcal{R} \simeq X^{-W}\wedge\mathcal{R} \xrightarrow{\mathrm{sm}_W} X\wedge\mathcal{R} \to \mathcal{R} \tag{3.3.23}$$

is called the (untwisted) *Euler class* of W.

Unlike the twisted Euler class, this untwisted Euler class depends on the $\mathcal{R}$-orientation.

Finally, we can prove that our definition of the Euler class, Definition 3.3.4, coincides with the more standard Definition 3.3.21 when they overlap (i.e. when there is an $\mathcal{R}$-orientation chosen on V).

Lemma 3.3.24. *Suppose W is $\mathcal{R}$-oriented, and let $U \in \mathcal{R}^r(\mathrm{Th}(X;W))$ denote the Thom class. Then $\mathrm{e}^{\mathcal{R}}(W) = z_W^*U$, where $z_W\colon X \to \mathrm{Th}(X;W)$ is the inclusion as the zero section.*

Proof. After suspending, the zero section map becomes the Smith map. Therefore it suffices to show that the following diagram commutes.

$$\begin{tikzcd}
\Sigma^\infty_+ X \arrow[r, "- \wedge \mathcal{R}"] \arrow[d, "\mathrm{sm}_W"] & \Sigma^\infty_+ X \wedge \mathcal{R} \arrow[r, "\simeq"] \arrow[d, "\mathrm{sm}_W \wedge \mathrm{id}_\mathcal{R}"] & \Sigma^n X^{-W} \wedge \mathcal{R} \arrow[d, "\mathrm{sm}_W"] \\
\Sigma^\infty_+ \mathrm{Th}(X;W) \simeq X^W \arrow[r, "- \wedge \mathcal{R}"] & X^W \wedge \mathcal{R} \arrow[r, "\simeq"] & \Sigma^n X \wedge \mathcal{R}.
\end{tikzcd} \tag{3.3.25}$$

Here the equivalences in the right square are the ones induced by the orientation ϕ.

The left-hand square commutes because smashing with $\mathcal{R}$ is a functor. The right-hand square commutes because the following diagram commutes in $\mathsf{Fun}(X, \mathsf{Mod}_R)$:

$$\begin{tikzcd}
\underline{\mathcal{R}} \arrow[r, "(\phi \wedge \mathcal{R}^{-W})"] \arrow[d, "z^\vee \wedge \mathcal{R}"] & \Sigma^n \mathcal{R}^{-W} \arrow[d, "z^\vee \wedge \mathcal{R} \wedge \mathcal{R}^W"] \\
\mathcal{R}^W \arrow[r, "\phi"] & \Sigma^n \underline{\mathcal{R}},
\end{tikzcd} \tag{3.3.26}$$

which follows from naturality. Recall that $z^\vee : \underline{\mathcal{R}} \to \mathcal{R}^W$ is the map of spherical fibrations over X that induces the Smith map. ☐

3.3.2 Smith homomorphisms defined via Atiyah-Poincaré dual of the generalized Euler classes

Now equipped with the theory of Euler classes, we can give another alternate definition of the Smith homomorphism. Fix $\xi \colon B \to BO$, $V \to X$ of rank r_V, and $W \to X$ of rank r_W as in Definition 3.2.8. Recall that by Corollary 3.2.5, a class $c \in \Omega^\xi_n(X^{V-r_V})$ can be represented by a closed n-manifold M with an (X, V)-twisted ξ-structure, which includes the data of a map $f \colon M \to X$.

In this subsubsection, we assume that $MT\xi$ is a ring spectrum.

Definition 3.3.27. The *Smith homomorphism* associated to ξ, V, and W is the homomorphism

$$\mathrm{sm}_W \colon \Omega^\xi_n(X^{V-r_V}) \longrightarrow \Omega^\xi_{n-r_W}(X^{V\oplus W-r_V-r_W}) \tag{3.3.28}$$

sending the class $[M]$ to the Poincaré dual of the cobordism Euler class $e^{MT\xi}(f^*W)$.

To show this, we first recall Atiyah duality. There is the standard notion of duals in any symmetric monoidal category C [117–119]. Here for C we take the homotopy category of spectra, which is monoidal with respect to the smash product $\wedge$. If A, B have duals $A^\vee, B^\vee$, then a morphism $f \colon A \to B$ induces a dual morphism, which we write as $f^\vee \colon B^\vee \to A^\vee$.

Theorem 3.3.29 (Atiyah duality [120, Proposition 3.2 and Theorem 3.3])**.** *Let M be a compact manifold; then $(M/\partial M)^\vee \simeq M^{-TM}$. If M is closed and $V \to M$ is a virtual vector bundle, then $(M^V)^\vee \simeq M^{-TM-V}$.*

Furthermore, dual spectra provide isomorphisms between homology and cohomology groups: let X be a spectrum with a dual $X^\vee$; then, for any spectrum $\mathcal{R}$, we have a canonical isomorphism

$$\mathcal{R}_*(X) \xrightarrow{\cong} \mathcal{R}^{-*}(X^\vee). \tag{3.3.30}$$

We call two classes $\alpha \in \mathcal{R}_*(X)$ and $\beta \in \mathcal{R}^{-*}(X^\vee)$ *Atiyah-Poincaré dual* if $\alpha \mapsto \beta$ under the isomorphism (3.3.30). This is functorial: given a map $f\colon X \to Y$ of dualizable spectra, let $f^\vee\colon Y^\vee \to X^\vee$ be the dual map. We have a commutative square:

$$\begin{tikzcd}
\mathcal{R}_*(X) \arrow[r, "f_*"] \arrow[d, "\simeq"] & \mathcal{R}_*(Y). \arrow[d, "\simeq"] \\
\mathcal{R}^{-*}(X^\vee) \arrow[r, "(f^\vee)^*"] & \mathcal{R}^{-*}(Y^\vee).
\end{tikzcd} \tag{3.3.31}$$

Let $\Omega_*^{\mathrm{fr}}(X)$ denote the stably framed bordism of X, as in Example 2.2.53, which Pontrjagin-Thom identifies with the stable homotopy groups of X.

Lemma 3.3.32. *Let M be a closed compact d-dimensional manifold. Then M defines a canonical class in $\Omega_d^{\mathrm{fr}}(M, -TM) = \mathbb{S}_0(M^{-TM})$. This is the Atiyah-Poincaré dual to the Euler class for the trivial bundle $e^{\mathbb{S}}(0) \in \mathbb{S}^0(M)$.*

Proof. The Euler class is represented by a map $e\colon M_+ \to S^0$ taking $+$ to the basepoint of S^0 and the entirety of M to the other point. On the other hand, consider an embedding $\iota\colon M \to \mathbb{R}^N$ and let ν be the normal bundle. Then $\Sigma_+^\infty \mathrm{Th}(M;\nu) \simeq \Sigma^{-N} M^{-TM}$. By the Pontrjagin-Thom construction, the tautological class $[M] \in \Omega_d^{\mathrm{fr}}(M, -TM)$ comes from the collapse map $S^N = (\mathbb{R}^N)^+ \to \mathrm{Th}(M;\nu)$, where $(-)^+$ denotes the one point compactification.

The result follows from the finite-dimensional description of the evaluation and co-evaluation map of M and M^{-TM} [30]: we have an evaluation map $S^N \to M_+ \wedge \mathrm{Th}(M;\nu)$, representing $\mathbb{S} \to M \wedge M^{-TM}$. The composite $S^N \to M_+ \wedge \mathrm{Th}(M;\nu) \xrightarrow{e} S^0 \wedge \mathrm{Th}(M;\nu) = \mathrm{Th}(M;\nu)$ is precisely the Pontrjagin-Thom collapse map. ☐

Now we see how Atiyah duality interacts with Smith homomorphisms on compact manifolds:

Lemma 3.3.33. *Fix a closed compact manifold M. Given a virtual bundle $V \to M$ and a vector bundle $W \to M$, the Atiyah dual $(\mathrm{sm}_W)^\vee$ of the Smith map*

$$\mathrm{sm}_W\colon M^V \longrightarrow M^{V\oplus W} \tag{3.3.34}$$

is the Smith map associated to $-TM - V - W$:

$$\mathrm{sm}_W\colon M^{-TM-V-W} \longrightarrow M^{-TM-V}. \tag{3.3.35}$$

Proof. Let us do the case $V = 0$; the general case follows in the same way. First we give a space-level description of the Atiyah dual map. Consider the manifold with boundary $D_M(W)$, the disc bundle of W. Its tangent bundle is $T(D_M(W)) \cong TM \oplus W$, where we are implicitly pulling back W to $D_M(W)$. Now consider an embedding $\mu_D\colon D_M(W) \to \mathbb{R}^N$. Then M, sitting as the zero section, also gets an embedding $\mu_M\colon M \to D_M(W) \to \mathbb{R}^N$.

Let ν_D, resp. ν_M be the normal bundle of μ_D, resp. μ_M. As virtual bundles,

$$\nu_D \cong \mathbb{R}^N - TM - W \tag{3.3.36a}$$

$$\nu_M \cong \mathbb{R}^N - TM. \tag{3.3.36b}$$

Note that $\nu_M = \nu_D \oplus W$. Now let $N_D(\mu)$ be a tubular neighborhood of $D_M(W)$ and $N_M(\mu)$ the same for M. Observe that $N_D(\mu)$ and $N_M(\mu)$ are diffeomorphic to μ_D, resp. μ_M.

Using the standard Pontrjagin-Thom collapse argument, the open embedding $i\colon N_M(\mu) \to N_D(\mu)$ induces a map of one-point compactifications $i^+\colon N_D(\mu)^+ \to N_M(\mu)^+$. By Proposition 2.2.27, we can write this as $\mathrm{Th}(D_M(W); \nu_D) \simeq \mathrm{Th}(M; \nu_D) \to \mathrm{Th}(M; \nu_D)$. Recall that $D_M(W)$ is homotopy equivalent to W.

After passing to spectra, Equation (3.3.36) gives a map

$$\Sigma^n M^{-TM-W} \longrightarrow \Sigma^n M^{-TM}. \tag{3.3.37}$$

This is the Atiyah dual map of the Smith map.

To see this is the Smith map for $-TM - V - W$ as claimed, notice that the composite $\mathrm{Th}(M; \nu_D) \to \mathrm{Th}(D_M(W); \nu_D) \to \mathrm{Th}(M; \nu_D \oplus W)$ is induced by the inclusion of disk bundles, a.k.a. the Smith homomorphism on Thom spaces, which suspends to the Smith map on Thom spectra. ☐

Lemma 3.3.38. *Let W be a rank r_W vector bundle on a closed compact d-manifold M, and let $[M] \in \Omega_d^{\mathrm{fr}}(M, -TM)$ be the tautological class. Then $(\mathrm{sm}_W)_*([M]) \in \Omega_d^{\mathrm{fr}}(M, -TM + W) = \mathbb{S}_0(M^{-TM+W})$ is the Atiyah-Poincaré dual of the Euler class $e^{\mathbb{S}}(W) \in \Omega^{r_W - d}_{\mathrm{fr}}(M, -W)$.*

Proof. By Equation (3.3.31), $\mathrm{sm}_{W*}([M])$ is Atiyah-Poincaré dual to $((\mathrm{sm}_W)^\vee)^*(e^{\mathbb{S}}(0))$, where 0 denotes the zero vector bundle. By Lemma 3.3.33, $(\mathrm{sm}_W)^\vee$ is still sm_W. By Proposition 3.3.5, $(\mathrm{sm}_W)^*(e^{\mathbb{S}}(0))$ is $e^{\mathbb{S}}(W)$. ☐

Now we can show that Definitions 3.2.16 and 3.3.27 are equivalent definitions of the Smith homomorphism. Below, $\Omega_d^{\mathrm{fr}}(X, V)$ indicates (X, V)-twisted framed bordism.

Corollary 3.3.39. *Let $V \to X$ be a virtual vector bundle and $W \to X$ be a rank r_W vector bundle. Choose a bordism class in $\Omega_d^{\mathrm{fr}}(X, V)$ and let M be a closed manifold representative of that class. Let $[N] \in \Omega_d^{\mathrm{fr}}(M, -TM \oplus W)$ be the Atiyah-Poincaré dual of the Euler class $e^{\mathbb{S}}(W|_M)$. Then the image of $[N]$ in $\Omega_d^{\mathrm{fr}}(X, V \oplus W)$ is $\mathrm{sm}_W([M])$.*

Proof. Since M has a (X, V)-twisted framing, the map $M \to X$ Thomifies to a map $f\colon M^{-TM} \to X^V$. The Smith map is functorial, so we get a commutative square:

$$\begin{tikzcd}
M^{-TM} \arrow[r, "f"] \arrow[d, "\mathrm{sm}_{W|_M}"] & X^V \arrow[d, "\mathrm{sm}_W"] \\
M^{-TM\oplus W} \arrow[r, "f"] & X^{V\oplus W}.
\end{tikzcd} \tag{3.3.40}$$

Furthermore, $[M] \in \Omega_d^{\mathrm{fr}}(X, V)$ is the f-pushforward of the tautological class in $\Omega_d^{\mathrm{fr}}(M, -TM)$. The result now follows from Lemma 3.3.38. ☐

Remark 3.3.41. This tells us that given a bordism class represented by M, $\mathrm{sm}_W([M])$ is represented by a manifold N that is Atiyah-Poincaré dual (in the bordism homology theory) to the twisted cobordism Euler class of M.

3.4 The Smith fiber sequence

In this section we extend the Smith map into a fiber sequence, which allows us to derive a long exact sequence of bordism groups and, dually, a long exact sequence of field theories.

For any vector bundle $E \to X$ of rank r, let E also denote the classifying map $X \to B\mathrm{O}(r)$. In this section, we will write $S_X(E)$ and $D_X(E)$ for the sphere, resp. disc bundles of E, because there will be places where it will help to remember which base space we work over. The following result appears, e.g., in [121, Remark 3.14], where it is attributed to James.

Theorem 3.4.1. *Let V, W be real vector bundles over X. Then there is a cofiber sequence in pointed spaces:*

$$S_X(W)^V \to X^V \to X^{W\oplus V}. \tag{3.4.2}$$

Similarly, if V is a virtual bundle, we have a (co)fiber sequence in spectra:

$$S_X(W)^V \to X^V \to X^{V\oplus W}. \tag{3.4.3}$$

Proof. We will do the case where V is an actual vector bundle; the virtual bundle case is analogous. Given an r-dimensional vector space W, we have a cofiber sequence in pointed spaces:

$$S(W)_+ \to D(W)_+ \simeq S^0 \to S^W. \tag{3.4.4}$$

Now since $\mathrm{Aut}(W) \cong \mathrm{O}(r)$ acts on W, we can upgrade (3.4.4) to a cofiber sequence of *spaces with* $\mathrm{O}(r)$*-actions*; equivalently, (3.4.4) is a cofiber sequence of functors $B\mathrm{O}(r) \to \mathsf{Top}_*$. Pull back to X via the map $X \to B\mathrm{O}(r)$ classifying W to get a cofiber sequence of functors $X \to \mathsf{Top}_*$. Then smash with S^V to get a cofiber sequence of the form

$$S(W)_+ \wedge S^V \to D(W)_+ \wedge S^V \to S^W \wedge S^V \simeq S^{V\oplus W}. \tag{3.4.5}$$

This cofiber sequence is in the category of functors $X \to \mathsf{Top}_*$.

Since taking the colimit over X preserves cofiber sequences, it is sufficient to show that the colimit of (3.4.5) over X is

$$S_X(W)^V \longrightarrow X^V \longrightarrow X^{V\oplus W}. \tag{3.4.6}$$

For the last term $S^{V\oplus W}$ in (3.4.5), this follows directly from the definition of the Thom spectrum (Definition 2.2.32).

For $\mathrm{colim}_X(D(W)_+ \wedge S^V)$, note that $D(W)_+ \simeq S^0$, so $D(W)_+ \wedge S^V \simeq S^V$, so Definition 2.2.32 once again tells us the colimit is X^V. It also follows that the map $X^V \to X^{V\oplus W}$ on colimits is the Smith map.

Lastly, the colimit of $S(W)_+$ over X is the associated sphere bundle $S_X(W)$. It follows that the colimit of $S(W)_+ \wedge S^V$ over X is equivalent to the colimit of (the pullback of) S^V over $S_X(W)$, which is $S_X(W)^V$. ☐

Remark 3.4.7. Everything here is functorial, so given a map $Y \to X$, we get maps between cofiber sequences and therefore a map of long exact sequences of homotopy groups.

Corollary 3.4.8. *Applying π_* to the fiber sequence gives a long exact sequence of bordism groups:*

$$\cdots \longrightarrow \Omega_k^\xi(S_X(W)^V) \longrightarrow \Omega_k^\xi(X^V) \longrightarrow \Omega_{k-r}^\xi(X^{V+W-r}) \longrightarrow \Omega_{k-1}^\xi(S_X(W)^V) \longrightarrow \cdots \tag{3.4.9}$$

Though written as bordism groups of Thom spectra, these are also twisted ξ-bordism groups thanks to Corollary 3.2.5. We work through an explicit example long exact sequence on the level of manifold generators in Appendix A.

Remark 3.4.10. Suppose $X = BG$ for a compact Lie group G and that $W \to X$ is the vector bundle associated to an orthogonal representation of G such that G acts transitively on the unit sphere in W. Then the sphere bundle has a particularly simple form: if G_v is the stabilizer subgroup for a point $v \in S(W)$, then the bundle map $S_X(W) \to X$ is homotopy equivalent to the map $BG_v \to BG$ induced by the inclusion $G_v \hookrightarrow G$. Thus, as we discuss more later, the obstruction for an invertible field theory to be in the image of the Anderson-dualized Smith homomorphism is its restriction from manifolds with G-bundles (and some sort of tangential structure) to manifolds with G_v-bundles (and the corresponding tangential structure).

Remark 3.4.11. The Smith long exact sequence above is a generalization of the Gysin long exact sequence to arbitrary vector bundle twists of generalized cohomology theories. For a comparison with the long exact sequence of Conner-Floyd [122], see [9, Section 5.1].

3.5 Periodicity of twists and shearing

Here, we study tangential structures twisted over a fixed base space, and explain periodic phenomena in these twists. These periodic phenomena lead to repeated symmetry types in iterated symmetry-breaking processes in physics.

3.5.1 Families of Smith homomorphisms

Definition 3.5.1. Fix a space X, a virtual vector bundle $V \to X$ of rank r_V, a vector bundle $W \to X$ of rank r_W, and a tangential structure ξ. The *family of Smith homomorphisms* associated to this data is the set of Smith homomorphisms

$$\mathrm{sm}_W \colon \Omega_n^\xi(X^{V-r_V+k(W-r_W)}) \longrightarrow \Omega_{n-r_W}^\xi(X^{V-r_V+(k+1)(W-r_W)}) \tag{3.5.2}$$

for $k \in \mathbb{Z}$, i.e. the Smith homomorphisms from $(X, V \oplus kW)$-twisted ξ-bordism to $(X, V \oplus (k+1)W)$-twisted ξ-bordism.

If there is some $\ell \in \mathbb{Z}$ and an identification of $(X, V \oplus kW)$-twisted ξ-structures with $(X, V \oplus (k+\ell)W)$-twisted tangential structures for all k that commutes with the Smith homomorphisms (3.5.2), we say this Smith family is *periodic* with period the smallest positive such ℓ.

Many of the families we apply toward physics applications end up being periodic.

The main new result in this section is Proposition 3.5.10, providing a way to calculate the periodicity of a family of Smith homomorphisms. We also review the theory of shearing in and around Lemma 3.5.18, which is a convenient way to split the Thom spectra for a wide class of twisted bordism theories, and is an essential step in identifying the terms in Smith long exact sequences. Our perspective on shearing follows [123, §1], so see there for some more details; see also [115, 124–127] for additional approaches.

Definition 3.5.3. Let $\xi\colon B \to BO$ be a tangential structure. *Two-out-of-three data* for ξ is the data of:

- for each pair of ξ-structured virtual vector bundles $V, W \to X$, a natural ξ-structure on $V \oplus W$; and
- for each ξ-structured virtual vector bundle $V \to X$, a natural ξ-structure on $-V \to X$.

The reason for this name is that, given this data, a ξ-structure on any two of V, W, and $V \oplus W$ induces a ξ-structure on the third. Unfortunately, this is sometimes called a "two-out-of-three property."

Example 3.5.4. The tangential structures O, SO, Spin^c, Spin, String, U, SU, and Sp all have two-out-of-three data. Pin$^\pm$ and Pin^c do not. ◇

If M and N are manifolds, $T(M \times N) \cong p_1^*(TM) \oplus p_2^*(TN)$, where p_1 and p_2 are the projections of $M \times N$ onto M, resp. N, so two-out-of-three data induces a ring structure on Ω_*^ξ given by the direct product of manifolds. More abstractly, this data makes B into a grouplike E_∞-space and ξ into an E_∞-map, where BO has the direct sum E_∞-structure. This implies by work of Lewis [128, Theorem IX.7.1] (see also [129, 130]) that $MT\xi$ is an E_∞-ring spectrum.

For R an E_∞-ring spectrum, May [129, §III.2] defines a grouplike E_∞-space $\mathrm{GL}_1(R)$, and Ando-Blumberg-Gepner-Hopkins-Rezk [28, 131] associate to a map $f\colon X \to B\mathrm{GL}_1(R)$, which we call a *twist* of R, a Thom spectrum $Mf \in \mathsf{Mod}_R$. The f-twisted R-homology groups of X are by definition the homotopy groups of Mf [28, Definition 2.27]. Homotopy-equivalent twists induce equivalent Thom spectra. All of this generalizes our discussion around Definition 2.2.32, for which $R = \mathbb{S}$.

Example 3.5.5 (Vector bundle twists)**.** We have been using (rank-zero virtual) vector bundles to define twists of bordism theories, and these two notions of twist are compatible: rank-zero virtual vector bundles $V \to X$ are classified by maps $f_V\colon X \to BO$, and the J-homomorphism is a map $BO \to B\mathrm{GL}_1(\mathbb{S})$; then, if ξ is a tangential structure with two-out-of-three data, the unit map $e\colon \mathbb{S} \to MT\xi$ induces a map $e\colon B\mathrm{GL}_1(\mathbb{S}) \to B\mathrm{GL}_1(MT\xi)$. The Thom spectrum for (X, V)-twisted ξ-bordism, as we characterized it in Corollary 3.2.5, is naturally equivalent to the Thom spectrum $M(e \circ J \circ f_V)$ of the map

$$X \xrightarrow{f_V} BO \xrightarrow{J} B\mathrm{GL}_1(\mathbb{S}) \xrightarrow{e} B\mathrm{GL}_1(MT\xi). \tag{3.5.6}$$

This is a combination of theorems of Lewis [128, Chapter IX] and Ando-Blumberg-Gepner-Hopkins-Rezk (see [28, Corollary 3.24] and [131, §1.2]). ◇

Theorem 3.5.7 (Beardsley [125, Theorem 1])**.** *There is a canonical null-homotopy of the map*

$$e \circ J \circ \xi \colon B \to B\mathrm{GL}_1(MT\xi), \tag{3.5.8}$$

and therefore $e \circ J$ factors through the cofiber $B\mathrm{O}/B$,[2] *and in fact through $B\mathrm{GL}_1(\mathbb{S})/B$.*

In other words, the homotopy type of the Thom spectrum for (X, V)-twisted ξ-bordism depends only on the image of V in $B\mathrm{O}/B$. And the key slogan is that the orders of elements in $[X, B\mathrm{O}/B]$ control the periodicity of families of Smith homomorphisms for twisted ξ-bordism; the group structure on $[X, B\mathrm{O}/B]$ uses the fact that $B\mathrm{O}/B$ is the cofiber of a map of grouplike E_∞-spaces, hence is also a grouplike E_∞-space, so homotopy classes of maps into $B\mathrm{O}/B$ naturally form abelian groups.

Definition 3.5.9 (Bhattacharya-Chatham [132, Definition 2.9])**.** The *$MT\xi$-orientation order* of a virtual vector bundle $V \to X$, written $\Theta(V, MT\xi)$, is the smallest positive integer k such that $V^{\oplus k}$ is $MT\xi$-oriented, or infinity if no such k exists.

Equivalently, $\Theta(V, MT\xi)$ is the order of the classifying map of $e \circ J \circ f_V \colon X \to B\mathrm{GL}_1(MT\xi)$, where $f_V \colon X \to B\mathrm{O}$ is the classifying map of V. By Theorem 3.5.7, $e \circ J \circ f_V$ factors through $[X, B\mathrm{O}/B]$, so $\Theta(V, MT\xi)$ divides the exponent of $[X, B\mathrm{O}/B]$. We will use this fact below to make quick estimates of orientation orders.

Proposition 3.5.10. *Let $V \to X$ be a vector bundle. If $\epsilon \coloneqq \Theta(V, MT\xi)$ is finite, the Smith homomorphism family of (X, kV)-twisted ξ-bordism is ϵ-periodic.*

This bound is not sharp, as we discuss in §3.5.4.

Proof. The image $\overline{f_V}$ of the classifying map $f_V \colon X \to B\mathrm{O}$ in $[X, B\mathrm{O}/B]$ satisfies $(k+\epsilon)\overline{f_V} = k\overline{f_V}$. Since the homotopy type of the Thom spectrum for (X, W)-twisted ξ-bordism only depends on $\overline{f_W} \in [X, B\mathrm{O}/B]$, this implies that the notions of (X, kV)-twisted ξ-bordism and $(X, (k+\epsilon)V)$-twisted spin bordism coincide, so the Smith family $\{(X, kV) : k \in \mathbb{Z}\}$ is ϵ-periodic. ☐

3.5.2 Examples of periodic Smith families

Though Proposition 3.5.10 seems abstract, it lends itself readily to examples.

Example 3.5.11 (Unoriented bordism families are 1-periodic)**.** Proposition 3.5.10 implies that when $\xi = \mathrm{id} \colon B\mathrm{O} \to B\mathrm{O}$, the periodicity of a Smith family of (X, kV)-twisted unoriented bordism divides the exponent of $[X, B\mathrm{O}/B\mathrm{O}] = 0$. In other words, all Smith families of twisted unoriented bordism are 1-periodic.

We will see some examples of Smith families for unoriented bordism in Examples 3.6.4, 3.6.22, and 3.6.39. ◇

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[2] Beardsley's proof is more abstract, more general, and more powerful than this statement: see [123, Lemma 1.13] for a simpler proof of just this part of Beardsley's theorem.

Example 3.5.12 (Oriented bordism families are 2-periodic)**.** Because $B\mathrm{SO}$ is the fiber of $w_1\colon B\mathrm{O} \to K(\mathbb{Z}/2, 1)$, and the Whitney sum formula implies w_1 is a map of E_∞-spaces, the cofiber $B\mathrm{O}/B\mathrm{SO}$ is equivalent to $K(\mathbb{Z}/2, 1)$ as grouplike E_∞-spaces. Thus for all spaces X, $[X, B\mathrm{O}/B\mathrm{SO}]$ is annihilated by 2, so all Smith families for twisted oriented bordism are 2-periodic (or 1-periodic).

We will see some examples of Smith families for oriented bordism in Examples 3.6.6, 3.6.13, 3.6.22, 3.6.35, and 3.6.39. ◇

Example 3.5.13 (Complex and spin^c bordism families are 2-periodic)**.** If V is a real vector bundle, then $V \oplus V$ has a canonical complex structure (think of this bundle as $V \oplus iV$), and therefore also a canonical spin^c structure. Therefore for any map $f\colon X \to B\mathrm{O}$, $2f$ lifts to $B\mathrm{U}$ and to $B\mathrm{Spin}^c$. Therefore the image of the map $[X, B\mathrm{O}] \to [X, B\mathrm{O}/B\mathrm{U}]$ has exponent 2 (and likewise for Spin^c), so by Proposition 3.5.10 all Smith families of complex and spin^c bordism are at most 2-periodic.

For examples of Smith families for spin^c bordism, see Examples 3.6.11, 3.6.22, and 3.6.39 and Footnote 13. ◇

Example 3.5.14 (Spin bordism families are 4-periodic)**.** $B\mathrm{O}/B\mathrm{Spin}$ is not equivalent to a product of Eilenberg-Mac Lane spaces even as an E_1-space [123, Lemma 1.37], so we cannot reuse the strategy of (3.5.12). However, there is a cofiber sequence of grouplike E_∞-spaces (heuristically, an extension of abelian ∞-groups) [123, §1.2.3]

$$K(\mathbb{Z}/2, 2) \longrightarrow B\mathrm{Spin}/B\mathrm{O} \longrightarrow K(\mathbb{Z}/2, 1), \tag{3.5.15}$$

inducing a long exact sequence on $[X, –]$. Since $[X, K(\mathbb{Z}/2, 2)]$ and $[X, K(\mathbb{Z}/2, 1)]$ both have exponent at most 2 for any X, exactness implies $[X, B\mathrm{O}/B\mathrm{Spin}]$ has exponent at most 4. Thus using Proposition 3.5.10 we conclude that all twisted spin bordism Smith families are at most 4-periodic; in fact, Example 3.6.8 has period exactly 4, which implies (3.5.15) does not split. One could also argue 4-periodicity similarly to Example 3.5.13.

If we restrict to oriented vector bundles, we can do better, as periodicity is controlled by maps into $B\mathrm{SO}/B\mathrm{Spin} \simeq K(\mathbb{Z}/2, 2)$ (the argument is similar to $B\mathrm{O}/B\mathrm{SO} \simeq K(\mathbb{Z}/2, 1)$ from Example 3.5.12). Therefore we conclude that twisted spin Smith families using an oriented vector bundle are 2-periodic.

We will discuss several examples of 1-, 2-, and 4-periodic Smith families for spin bordism in Examples 3.6.8, 3.6.26, 3.6.33, 3.6.35, 3.6.39, and 3.6.45. ◇

Remark 3.5.16. Periodicity for spin bordism also implies periodicity for *ko* and *KO*. Bhattacharya-Chatham [132, Main Theorem 1.5] generalize this periodicity of *KO*-orientability to higher real *K*-theories EO_Γ.

Example 3.5.17 (Families of twisted string structures)**.** The space $B\mathrm{Spin}$ is 3-connected, with $\pi_4(B\mathrm{Spin}) \cong \mathbb{Z}$; if one kills this homotopy group by taking the 4-connected cover, one gets a space $B\mathrm{String}$, and the corresponding tangential structure is called a *string structure* [133, Definition 5.0.3] (see also [134, §1]). The generator of $H^4(B\mathrm{Spin}; \mathbb{Z})$ is not the first Pontrjagin class p_1, but rather is a class λ with $2\lambda = p_1$ [135, Theorem 1.2]. Thus a string structure on a spin vector bundle is equivalent data to a trivialization of λ.

As grouplike E_∞-spaces, $B\mathrm{O}/B\mathrm{String}$ is an extension of $B\mathrm{O}/B\mathrm{Spin}$ by $B\mathrm{Spin}/B\mathrm{String} \simeq K(\mathbb{Z},4)$ (see [123, §1.2.4]); since $B\mathrm{O}/B\mathrm{Spin}$ is itself an extension of $K(\mathbb{Z}/2,1)$ by $K(\mathbb{Z}/2,2)$, if X is a 3-connected space, $[X, B\mathrm{O}/B\mathrm{String}] \cong H^4(X;\mathbb{Z})$. Thus for a general space X, Proposition 3.5.10 provides no information on Smith families for twisted string bordism: it reports that the period is at most infinity. We will nevertheless prove in Corollary 3.5.33 that all twisted string Smith families have finite period, though our proof does not provide an effective computation of the period.

In special cases, though, Proposition 3.5.10 allows us to provide sharper bounds: for example, because $H^*(B\mathbb{Z}/2;\mathbb{Z})$ is 2-torsion in positive degrees and $[B\mathbb{Z}/2, B\mathrm{O}/B\mathrm{Spin}]$ has exponent 4, the long exact sequence associated to the cofiber sequence $K(\mathbb{Z},4) \to B\mathrm{O}/B\mathrm{String} \to B\mathrm{O}/B\mathrm{Spin}$ implies $[B\mathbb{Z}/2, B\mathrm{O}/B\mathrm{String}]$ has exponent at most 8, implying that all Smith families of $(B\mathbb{Z}/2, V)$-twisted string bordism are at most 8-periodic; an 8-periodic example appears in Example 3.6.10. ◇

3.5.3 Examples of twisted bordism

In this subsection, we discuss how to use the perspective we have been developing to concretely identify examples of twists of ξ-bordism for the tangential structures SO, Spin^c, and Spin.

Lemma 3.5.18 (Shearing [131, §1.2])**.** *If a twist $f\colon X \to B\mathrm{GL}_1(M\xi)$ factors through a map $g_V\colon X \to B\mathrm{O}$ classifying a rank-zero virtual vector bundle $V \to X$ as in* (3.5.6)*, then $Mf \simeq MT\xi \wedge X^V$.*

We will use this lemma as follows: first, for the four tangential structures $\xi\colon BG \to B\mathrm{O}$ mentioned above, we compute the homotopy type of $B\mathrm{O}/BG$ and understand the map $B\mathrm{O} \to B\mathrm{O}/BG$, to recognize when a map $X \to B\mathrm{O}/BG$ comes from a (virtual rank-zero) vector bundle $V \to X$. In that situation, Lemma 3.5.18 describes the corresponding twisted ξ-bordism groups as $\Omega_*^\xi(X^V)$, so we can use the Smith homomorphism tools we developed in this paper.

Example 3.5.19 (Twists of oriented bordism)**.** Recall from Example 3.5.12 that $B\mathrm{O}/B\mathrm{SO} \simeq K(\mathbb{Z}/2,1)$; the argument there implies the map $B\mathrm{O} \to B\mathrm{O}/B\mathrm{SO} \xrightarrow{\simeq} K(\mathbb{Z}/2,1)$ is the first Stiefel-Whitney class. Given a map $a\colon X \to B\mathrm{O}/B\mathrm{SO}$, the Thom spectrum of the corresponding twist $f_a\colon X \to B\mathrm{GL}_1(MT\mathrm{SO})$ of $MT\mathrm{SO}$ is the bordism spectrum whose homotopy groups are the bordism groups of manifolds M with a map $\phi\colon M \to X$ and a trivialization of $w_1(M) - \phi^*(a)$.[3]

Every class $a \in H^1(X;\mathbb{Z}/2)$ is the first Stiefel-Whitney class of some line bundle $L_a \to X$, so for any twist $f\colon X \to B\mathrm{GL}_1(MT\mathrm{SO})$ described by a map $f_a\colon X \xrightarrow{a} K(\mathbb{Z}/2,1) \simeq B\mathrm{O}/B\mathrm{SO} \to B\mathrm{GL}_1(MT\mathrm{SO})$, there is a homotopy equivalence

$$Mf \xrightarrow{\simeq} MT\mathrm{SO} \wedge X^{L_a - 1}. \tag{3.5.20}$$

For example, unoriented bordism is an example of such a twist: every manifold M has a canonical map to $K(\mathbb{Z}/2,1)$, given by $w_1(M)$, and $w_1(M) - w_1(M)$ has a canonical trivialization.

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[3] Strictly speaking, what one trivializes is $w_1(\nu) - \phi^*(a)$, where $\nu \to M$ is the stable normal bundle, but there is a canonical identification of $w_1(M)$ and $w_1(\nu)$. This nuance will matter for spin structures.

Therefore unoriented bordism is twisted oriented bordism for the twist $K(\mathbb{Z}/2,1) \xrightarrow{\simeq} B\mathrm{O}/B\mathrm{SO}$, and Lemma 3.5.18 implies $MT\mathrm{O} \simeq MT\mathrm{SO} \wedge (K(\mathbb{Z}/2,1))^{\sigma-1}$, where $\sigma \to B\mathbb{Z}/2 \simeq K(\mathbb{Z}/2,1)$ is the tautological line bundle; this is a theorem of Atiyah [30, Proposition 4.1].

For another example of how to use Lemma 3.5.18, let $\mathcal{W}$ denote the Thom spectrum for the notion of bordism of manifolds M equipped with a lift of $w_1(M)$ to a class $\alpha \in H^1(M;\mathbb{Z})$. The class α is equivalent to a map $\phi\colon M \to B\mathbb{Z} = S^1$, and $\alpha = \phi^*x$, where $x \in H^1(S^1;\mathbb{Z})$ is the generator; rephrased in this way, the condition that $\alpha \bmod 2 = w_1(M)$ is equivalent to a trivialization of $w_1(M) - \phi^*(x \bmod 2)$. Therefore $\mathcal{W}$-bordism is twisted oriented bordism for $(S^1, x \bmod 2)$, and as $x \bmod 2$ is w_1 of the Möbius bundle $\sigma \to S^1$, we learn from Lemma 3.5.18 that $\mathcal{W} \simeq MT\mathrm{SO} \wedge (S^1)^{\sigma-1}$. This is also due to Atiyah [30, §4]. ◇

Example 3.5.21 (Twists of spin^c bordism)**.** There is an equivalence of spaces, but not E_1-spaces, $B\mathrm{O}/B\mathrm{Spin}^c \simeq K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3)$ [123, Proposition 1.20, Lemma 1.30], and the map $B\mathrm{O} \to B\mathrm{O}/B\mathrm{Spin}^c$ is picked out by $(w_1, \beta(w_2))$, where β is the integral Bockstein. The fact that $\beta(w_2)$ is not linear in the direct sum of vector bundles is why this decomposition of $B\mathrm{O}/B\mathrm{Spin}^c$ does not respect the E_1-structure.

Given data $a \in H^1(X;\mathbb{Z}/2)$ and $c \in H^3(X;\mathbb{Z})$, if $Mf_{a,c}$ is the Thom spectrum for the corresponding twist

$$f_{a,c}\colon X \xrightarrow{(a,c)} K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3) \simeq B\mathrm{O}/B\mathrm{Spin}^c \longrightarrow B\mathrm{GL}_1(MT\mathrm{Spin}^c), \tag{3.5.22}$$

then the homotopy groups of $Mf_{a,c}$ are the bordism groups of manifolds M with maps $\phi\colon M \to X$ and trivializations of $w_1(M) - \phi^*(a)$ and $\beta(w_2(M)) - \phi^*(c)$; the proof is essentially the same as Hebestreit-Joachim's [136, Corollary 3.3.8] (Footnote 3 still applies: what appears is the stable normal bundle, but the characteristic classes are the same). If there is a (rank-zero, virtual) vector bundle $V \to X$ with $w_1(V) = a$ and $\beta(w_2(V)) = c$, then Lemma 3.5.18 implies $Mf_{a,c} \simeq MT\mathrm{Spin}^c \wedge X^V$ and we can invoke the Smith homomorphism on V.

For example, a pin^c structure on a manifold M is a trivialization of $\beta(w_2(M))$ (i.e. the spin^c condition without the trivialization of w_1). Thus a pin^c structure is equivalent to a twisted spin^c structure where $X = B\mathbb{Z}/2$, a is the generator of $H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$, and $c = 0$: as in Example 3.5.19, $w_1(M)$ gives us a canonical map to $B\mathbb{Z}/2$, there is a canonical trivialization of $w_1(M) - w_1(M)$, and $c = 0$ means this twisted spin^c condition does not modify $\beta(w_2)$. So this twisted spin^c condition is that $\beta(w_2) = 0$ and w_1 is arbitrary, i.e. a pin^c structure. And if $\sigma \to B\mathbb{Z}/2$ is the tautological line bundle, $w_1(\sigma) = a$ and $\beta(w_2(\sigma)) = 0 = c$, so Lemma 3.5.18 implies $MT\mathrm{Pin}^c \simeq MT\mathrm{Spin}^c \wedge (B\mathbb{Z}/2)^{\sigma-1}$, reproving a theorem of Bahri-Gilkey [137, 138].

Other examples of twists of spin^c bordism which can be realized by vector bundles include the spin-U(2) bordism of Davighi-Lohitsiri [139, 140] and the tangential structure corresponding to Stehouwer's alternate class AI fermionic groups [126, §2.2].

Not every choice of (a, c) can be realized by a vector bundle; for example, $\beta(w_2)$ is always 2-torsion, but c need not be. There are also examples with 2-torsion c, as a consequence of work of Gunawardena-Kahn-Thomas [141, §2]. ◇

Example 3.5.23 (Twists of spin bordism)**.** The most commonly studied examples of twisted ξ-bordism in mathematical physics are twists of spin bordism. The story is closely analogous to Example 3.5.21, with $K(\mathbb{Z},3)$ replaced with $K(\mathbb{Z}/2,2)$, and the map $B\mathrm{O} \to B\mathrm{O}/B\mathrm{Spin} \simeq K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)$ is (w_1, w_2). Given classes $a \in H^1(X;\mathbb{Z}/2)$ and $b \in H^2(X;\mathbb{Z}/2)$, the homotopy groups of the Thom spectrum of the corresponding twist $f_{a,b}\colon X \to B\mathrm{GL}_1(MT\mathrm{Spin})$ are the bordism groups of manifolds M with maps $\phi\colon M \to X$ and trivializations of $w_1(\nu) - \phi^*(a)$ and $w_2(\nu) - \phi^*(b)$ [136, Corollary 3.3.8], where $\nu \to M$ is the stable normal bundle. Now, unlike in Footnote 3, the distinction between TM and ν matters: $w_1(TM) = w_1(\nu)$, but $w_2(TM) + w_1(TM)^2 = w_2(\nu)$, providing a formula for the nontrivial transition from tangential to normal data. If $a = w_1(V)$ and $b = w_2(V)$ for a rank-zero virtual vector bundle $V \to X$, Lemma 3.5.18 implies $Mf_{a,b} \simeq MT\mathrm{Spin} \wedge X^V$. See [123, §1.2.3] for more information.

Many commonly studied tangential structures arise as vector bundle twists of spin structures.

1. A pin$^-$ structure is a trivialization of $w_2(M) + w_1(M)^2$, with no condition on w_1. Thus this is equivalent to a trivialization of $w_2(\nu)$. Like in Examples 3.5.19 and 3.5.21, we can ask for a map $\phi\colon M \to B\mathbb{Z}/2$ and a trivialization of $w_1(\nu) - \phi^*(a)$, where $a \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ is the generator, and this is no data at all; then we also want to impose $w_2(\nu) = 0$. So pin$^-$ bordism is the Thom spectrum of the twist $f_{a,0}\colon B\mathbb{Z}/2 \to B\mathrm{GL}_1(MT\mathrm{Spin})$. The classes a and 0 are w_1 and w_2 of $\sigma \to B\mathbb{Z}/2$, so we learn that $MT\mathrm{Pin}^- \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{\sigma-1}$, a splitting first written down by Peterson [142, §7].

2. A pin$^+$ structure is a trivialization of $w_2(M)$, with no condition on w_1. Switching to the stable normal bundle, we want a trivialization of $w_2(\nu) + w_1(\nu)^2$. Just as for pin$^-$ structures, pick a map $\phi\colon M \to B\mathbb{Z}/2$ and ask for a trivialization of $w_1(\nu) - \phi^*(a)$, which is no data; then we want to trivialize $w_2(\nu) + \phi^*(a^2)$. Thus pin$^+$ bordism is the Thom spectrum of the twist $f_{a,a^2}\colon B\mathbb{Z}/2 \to B\mathrm{GL}_1(MT\mathrm{Spin})$. The classes a and a^2 are w_1, resp. w_2 of the virtual vector bundle $-\sigma$, so Lemma 3.5.18 tells us $MT\mathrm{Pin}^+ \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{1-\sigma}$, a result of Stolz [143, §8].[4]

3. A spin^c structure is data of a trivialization of $w_1(TM)$ and a class $c_1 \in H^2(M;\mathbb{Z})$ such that $c_1 \bmod 2 = w_2(TM)$; in this case there is no difference between $w_2(TM)$ and $w_2(\nu)$. This is a twisted spin structure where $X = B\mathrm{U}(1) = K(\mathbb{Z},2)$, $a = 0$, and b is the generator of $H^2(K(\mathbb{Z},2);\mathbb{Z}/2) \cong \mathbb{Z}/2$. As 0, resp. b are the first and second Stiefel-Whitney classes of the tautological complex line bundle $L \to B\mathrm{U}(1)$, Lemma 3.5.18 implies $MT\mathrm{Spin}^c \simeq MT\mathrm{Spin} \wedge (B\mathrm{U}(1))^{L-2}$, which is known due to Bahri-Gilkey [137, 138].

4. A spin-$\mathbb{Z}/2k$ structure on a manifold M is data of a principal $\mathbb{Z}/k$-bundle $P \to M$ together with trivializations of $w_1(M)$ and $w_2(M) - w_2(V_P)$, where V is the standard one-dimensional complex representation of $\mathbb{Z}/k$ as rotations and $V_P \to M$ is the associated complex line bundle to P. Thus, analogous to the spin^c argument above, this structure is a twisted spin structure for $X = B\mathbb{Z}/k$, $a = 0$, and $b = w_2(V)$, and Lemma 3.5.18 implies $MT(\mathrm{Spin}\text{-}\mathbb{Z}/2k) \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/k)^{V-2}$, reproving a theorem of Campbell [96, §7.9].

5. A spin^h structure is data of a trivialization of $w_1(M)$ and a rank-3 oriented vector bundle $E \to M$ and a trivialization of $w_2(M) - w_2(E)$. Again tangential vs. normal

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[4] As $[B\mathbb{Z}/2, BO/B\mathrm{Spin}]$ has exponent 4 by Example 3.5.14, $[1-\sigma] = [3\sigma - 3]$, so the reader who prefers to avoid virtual vector bundles can write $MT\mathrm{Pin}^+ \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{3\sigma-3}$.

does not matter here, and one can use the same line of reasoning to show that spin^h structures are twisted spin structures for $X = B\mathrm{SO}_3$, $a = 0$, and $b = w_2$. As these are w_1, resp. w_2 of the tautological vector bundle $V \to B\mathrm{SO}_3$, Lemma 3.5.18 tells us $MT\mathrm{Spin}^h \simeq MT\mathrm{Spin} \wedge (B\mathrm{SO}_3)^{V-3}$, which is due to Freed–Hopkins [124, §10].

There are many more examples of vector bundle twists of spin bordism, including the examples in, e.g., [115, 124, 126, 139, 140, 144–146]. But one can find twists of spin bordism not described by vector bundle twists, even in physically-motivated examples: see [147, Theorem 4.2] for an example where $X = B\mathrm{SU}_8/\{\pm 1\}$, with a few more examples given in [123, §3.1]. The Smith-theoretic techniques in our paper do not apply in those situations. ◇

Example 3.5.24 (James periodicity as Smith periodicity)**.** James periodicity [148] is a classical result in homotopy theory that the homotopy types of the *stunted projective spaces* $\mathbb{RP}^n_k := \mathbb{RP}^n/\mathbb{RP}^k$ (here $k < n$) are periodic, with periodicity dependent on n and k. There are also results for the analogously defined stunted complex and quaternionic projective spaces $\mathbb{CP}^n_k := \mathbb{CP}^n/\mathbb{CP}^k$ and $\mathbb{HP}^n_k := \mathbb{HP}^n/\mathbb{HP}^k$. These periodicities can be thought of in terms of periodic Smith families for framed bordism—or conversely, the periodicities in the previous several examples can be thought of as generalizations of James periodicity over other ring spectra than $\mathbb{S}$.

Proposition 3.5.10 is the engine behind our periodicity results; its key idea is that vector bundles inducing equivalent maps to $B\mathrm{GL}_1(R)$ have equivalent R-module Thom spectra. For framed bordism, where $R = \mathbb{S}$, we therefore should look at the image of the homomorphism $[X, BO] \to [X, B\mathrm{GL}_1(\mathbb{S})]$; following Atiyah [120, §1], this image is typically denoted $J(X)$. Atiyah (*ibid.*, Lemma 2.5) proves that if $V, W \to X$ have equal images in $J(X)$, then $X^V \simeq X^W$.[5] Therefore we can obtain framed bordism Smith periodicities, or equivalences of Thom spectra, by calculating the groups $J(X)$. Atiyah (*ibid.*, Proposition 1.5) shows that when X is a finite CW complex, $J(X)$ is a finite group, implying the existence of many framed Smith families.

For James periodicity specifically, choose $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Stunted projective spaces are Thom spectra: if $L \to F\mathbb{P}^k$ denotes the tautological (real, complex, or quaternionic) line bundle, there is an equivalence $\Sigma^\infty F\mathbb{P}^n_k \simeq (F\mathbb{P}^k)^{(n-k)L}$ [120, Proposition 4.3], reducing the proof of James periodicity to the computation of the order of L in $J(F\mathbb{P}^k)$. For example, for $F = \mathbb{R}$ Adams calculates the order of L in $J(\mathbb{RP}^k)$ in [150, Theorem 7.4] and [151, Example 6.3] to be $2^{\phi(k)}$, where $\phi(k)$ is the number of integers s with $0 < s \leq k$ and $s \equiv 0$, 1, 2, or 4 mod 8. Therefore for all k and n, there is a homotopy equivalence

$$\Sigma^\infty \mathbb{RP}^{n+2^{\phi(k)}}_k \xrightarrow{\simeq} \Sigma^\infty \Sigma^{2^{\phi(k)}} \mathbb{RP}^n_k. \tag{3.5.25}$$

(and in fact this is true even before applying Σ^∞ [152]). Additional computations in J-groups of $F\mathbb{P}^k$ are done by Adams-Walker [153], Lam [154], Federer-Gitler [155, 156], Sigrist [157], Walker [158], Crabb-Knapp [159], Dibağ [160, 161], Obiedat [162], and Randal-Williams [163, §5.3]. ◇

Remark 3.5.26. There are many other tangential structures ξ that one can study twists of. See [123, 132, 164–174] for more examples.

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[5] See Held-Sjerve [149, Theorem 1.2] for a partial converse to this result.

3.5.4 Lower-than-expected periodicities

The bound in Proposition 3.5.10 is not sharp. In this subsection, we discuss ways in which Proposition 3.5.10 loses information, yielding Smith families with lower-than-expected periodicity. This in particular occurs for twisted string structures.

Proposition 3.5.10 estimates the periodicity of a Smith family for a tangential structure ξ and vector bundle $V \to X$ in terms of the minimal positive integer k such that $V^{\oplus k}$ has a ξ-structure. As we have seen above in Examples 3.5.11, 3.5.12, 3.5.13, and 3.5.14, k is finite in many examples of interest, including O-, SO-, Spin^c-, Spin-, and U-structures. However, k is not always finite.

Lemma 3.5.27.

1. *Let $V \to X$ be a real vector bundle whose rational first Pontrjagin class $p_1(V) \in H^4(X;\mathbb{Q})$ is nonzero. Then for $k \neq 0$, $V^{\oplus k}$ does not admit a string structure.*
2. *Let $V \to X$ be a complex vector bundle whose rational first Chern class $c_1(V) \in H^2(X;\mathbb{Q})$ is nonzero. Then for $k \neq 0$, $V^{\oplus k}$ does not admit an* SU*-structure.*

Proof. For part (1), if $E \to M$ is a string vector bundle, then $\lambda(E) = 0$ implies $p_1(E) = 0 \in H^4(M;\mathbb{Z})$ (since $2\lambda = p_1$), which implies the image of $p_1(E)$ in $H^4(M;\mathbb{Q})$ is also 0, so it suffices to show $p_1(V^{\oplus k})$ has nonzero image in $H^q(X;\mathbb{Q})$ for $k \neq 0$. Since $p_1(V)$ is nonzero in this group, it is in particular nontorsion, and the Whitney sum formula implies that in $\mathbb{Q}$-cohomology $p_1(V^{\oplus k}) = kp_1(V)$, so it is also nonzero. Part (2) is analogous, using that the complete obstruction for lifting from a U-structure to an SU-structure is $c_1 \in H^2(B\mathrm{U};\mathbb{Z})$. ☐

There are many bundles satisfying the hypotheses of Lemma 3.5.27: for example, both are true for the tautological complex line bundle over $B\mathrm{U}(1)$.

Definition 3.5.28 (Lashof [112, §3])**.** Let $B\mathrm{U}\langle 6\rangle$ denote the 5-connected cover of $B\mathrm{U}$, and let $\xi\langle 6\rangle\colon B\mathrm{U}\langle 6\rangle \to B\mathrm{O}$ be the composition of the covering map $B\mathrm{U}\langle 6\rangle \to B\mathrm{U}$ and the map $B\mathrm{U} \to B\mathrm{O}$ forgetting the complex structure. We will refer to $\xi\langle 6\rangle$-structures as *$U\langle 6\rangle$-structures.*

A $U\langle 6\rangle$-structure induces both an SU-structure and a string structure; the former because the 5-connected covering map always factors through the 3-connected cover, which for $B\mathrm{U}$ is $B\mathrm{SU} \to B\mathrm{U}$, and the latter because the map $B\mathrm{U}\langle 6\rangle \to B\mathrm{O}$ must factor through the 5-connected cover of $B\mathrm{O}$, which is $B\mathrm{String}$.

One can construct two-out-of-three data for $U\langle 6\rangle$-structures using the 5-connected covers of the maps in the two-out-of-three data for $B\mathrm{U}$. Since this data is constructed in this universal way, it is compatible with the two-out-of-three data we have already used for U, SU, String, etc.

Proposition 3.5.29 (Bauer [175, Lemma 2.1])**.** *Let ξ be a tangential structure with two-out-of-three data admitting a map $\xi\langle 6\rangle \to \xi$ compatible with two-out-of-three data. If $L \to B\mathrm{U}(1)$ denotes the tautological complex line bundle, then there is a homotopy equivalence*

$$MT\xi \wedge (B\mathrm{U}(1))_+ \xrightarrow{\simeq} MT\xi \wedge (BU(1))^{24L^*-48}. \tag{3.5.30}$$

Bauer's statement in [175, Lemma 2.1] is only a corollary of this, obtained by base-changing from $M\mathrm{U}\langle 6\rangle$ to *tmf*, but in his proof he proves the version we provide here. One can also pass from L^* to L by pulling back along the complex conjugation map $B\mathrm{U}(1) \to B\mathrm{U}(1)$.

Corollary 3.5.31. *With ξ as in Proposition 3.5.29, in particular including string and* SU*-structures, every Smith family for ξ-structures and a complex line bundle is at most* 24*-periodic. In particular, Proposition 3.5.10 is not sharp.*

What went wrong? In this specific case, we didn't use all of the available information in Proposition 3.5.10. Looking into Bauer's proof, one learns that the map $[B\mathrm{U}(1), B\mathrm{O}/B\mathrm{String}] \to [B\mathrm{U}(1), B\mathrm{GL}_1(MT\mathrm{String})]$ from Theorem 3.5.7 sends the class of L^*, which is *infinite-order* in $[B\mathrm{U}(1), B\mathrm{O}/B\mathrm{String}]$, to a *finite-order* class in $[B\mathrm{U}(1), B\mathrm{GL}_1(MT\mathrm{String})]$. Specifically, because this twist came from a vector bundle, it also factors through $[B\mathrm{U}(1), B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}]$ (see Theorem 3.5.7), and the image of L^* in this group has finite order.

Proposition 3.5.32. *For any space X homotopy equivalent to a CW complex with finitely many cells in each degree, let $J_*\colon [X, B\mathrm{O}] \to [X, B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}]$ denote the map induced by the J-homomorphism $B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S})$ followed by taking the cofiber of $B\mathrm{String} \to B\mathrm{O} \xrightarrow{J} B\mathrm{GL}_1(\mathbb{S})$. Then all classes in $\mathrm{Im}(J_*)$ have finite order.*

Corollary 3.5.33. *Every twisted string structure Smith family over a base space X as in Proposition 3.5.32 has finite periodicity.*

Proof of Proposition 3.5.32. Because $B\mathrm{O}$ and $B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}$ are grouplike E_∞-spaces,[6] one can prove the proposition by lifting the map $B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}$ to the equivalent data of a map of spectra $j_{/bstring}\colon bo_0 \to bg\ell_1(\mathbb{S})/bstring$.[7]

The map $j_{/bstring}$ was induced from a map of grouplike E_∞-spaces that factored through $B\mathrm{GL}_1(\mathbb{S})$, and therefore $j_{/bstring}$ factors through $bg\ell_1(\mathbb{S})$. The homotopy groups of this spectrum are torsion, which follows from May's definition [129, §III.2] of $\mathrm{GL}_1(\mathbb{S})$ and the triviality of the positive-degree rational stable homotopy groups of the sphere [176]. Thus the rationalization $bg\ell_1(\mathbb{S}) \wedge H\mathbb{Q} \simeq 0$. For any class $x \in (bo_0)^0(X)$, if $J_*(x)$ has infinite order, its image in the rationalized $bg\ell_1(\mathbb{S})/bstring$-cohomology of X must be nonzero: because X has finitely many cells in each dimension, its generalized cohomology groups for any finite-type spectrum (including all spectra appearing in this proof) are finitely generated, so infinite-order elements persist through rationalization. Rationally, though, J_* passes through the zero spectrum. ☐

Remark 3.5.34. One way to interpret this phenomenon is that, even though the first Pontrjagin class $p_1\colon B\mathrm{O} \to K(\mathbb{Z}, 4)$ descends to a map $p_1\colon B\mathrm{O}/B\mathrm{String} \to K(\mathbb{Z}, 4)$, this map does not extend to a rationally nontrivial map out of $B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}$. In the language of [123], "a fake vector bundle with respect to twisted string structures has a first Pontrjagin class, but a fake spherical fibration does not."

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[6] It is not immediately obvious that $B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}$ is a grouplike E_∞-space, but one can prove it by modifying the argument in [123, Proof of Proposition 1.20].

[7] The name bo_0 instead of bo is because it is traditional to use bo to refer to the spectrum corresponding to the grouplike E_∞-space $\mathbb{Z} \times B\mathrm{O}$, i.e. the spectrum *ko*.

Remark 3.5.35 ($\mathrm{O}\langle n\rangle$-families' periodicity and Bernoulli numbers?). Proposition 3.5.32 and Corollary 3.5.33 generalize *mutatis mutandis* to tangential structures further up the Whitehead tower of $B\mathrm{O}$. To wit, given a natural number n, let $\xi\colon B\mathrm{O}\langle n\rangle \to B\mathrm{O}$ be the $(n-1)$-connected covering map. This defines a tangential structure commonly called a $\mathrm{O}\langle n\rangle$*-structure*, and the two-out-of-three data for $B\mathrm{O}$ pull back by the universal property of the $(n-1)$-connected cover to define two-out-of-three data for $\mathrm{O}\langle n\rangle$-structures. (Compare Definition 3.5.28.) Thus Definition 3.5.1, Example 3.5.5, , and Theorem 3.5.7 define vector bundle twists, non-vector-bundle twists, and Smith families for twisted $\mathrm{O}\langle n\rangle$-structures just like for string structures.

For $n > 4$, the homotopy groups of $B\mathrm{O}/B\mathrm{O}\langle n\rangle$ are not all torsion, which is downstream from the isomorphism $\pi_4(B\mathrm{O}) \cong \mathbb{Z}$ [177, §24, §25]. Therefore, like for $B\mathrm{O}/B\mathrm{String}$, the order of a twist in $[X, B\mathrm{O}/B\mathrm{O}\langle n\rangle]$ is not a particularly good estimate for the value of the corresponding Smith family's period. Indeed, Proposition 3.5.32 and Corollary 3.5.33 and their proofs generalize directly from $B\mathrm{String}$ to $B\mathrm{O}\langle n\rangle$, showing all Smith families of twisted $\mathrm{O}\langle n\rangle$-structures have finite order, provided they are over spaces homotopy equivalent to CW complexes with finitely many cells in each dimension. One can also generalize this whole story to the limiting case as $n \to \infty$, which is the tangential structure $E\mathrm{O} \to B\mathrm{O}$, i.e. a stable framing.

Our proof did not give any estimates on the orders of these Smith families, just finiteness. It would be interesting to bound or compute these orders; for example, one could investigate the map of Atiyah-Hirzebruch spectral sequences induced by $j_{/bo\langle n\rangle}$, the generalization of $j_{/bstring}$ in the proof of Proposition 3.5.32, or generalize Bauer's proof in [175, Lemma 2.1]. We suspect that sharp estimates for periodicity for Smith families of twisted $\mathrm{O}\langle n\rangle$-structures will have formulas involving Bernoulli numbers, because of their appearance in Adams' seminal work computing the image of the J-homomorphism in $\pi_*(\mathbb{S})$ [151, 178–180]. We would be interested in learning whether this is the case.

Recall the Whitehead tower from Remark 2.2.16. For $n \leq 16$, twisted $\mathrm{O}\langle n\rangle$-structures as defined here recover familiar twists of familiar tangential structures.

0. $\mathrm{O}\langle 0\rangle$- and $\mathrm{O}\langle 1\rangle$-structures are canonically equivalent to O-structures, i.e. no data, and so this story is vacuous.

2. $\mathrm{O}\langle 2\rangle$-structures are equivalent to SO-structures, and this story recovers the twists in Example 3.5.19.

3. $\pi_3(B\mathrm{O}) = 0$ [181, §IV], and $B\mathrm{O}\langle 3\rangle$ and $B\mathrm{O}\langle 4\rangle$ are equivalent to $B\mathrm{Spin}$. This story recovers the notion of twisted spin structure we discussed in Example 3.5.23.

5. $\pi_k(B\mathrm{O})$ vanishes for $k = 5$ [177, Remarks 24.11], $k = 6$ [182, p. 3.72], and $k = 7$ [183, Proposition 19.5], so $B\mathrm{O}\langle 5\rangle$, $B\mathrm{O}\langle 6\rangle$, $B\mathrm{O}\langle 7\rangle$, and $B\mathrm{O}\langle 8\rangle$ all coincide, and are $B\mathrm{String}$. This story recovers the standard story of twists of string bordism that we mentioned in Example 3.5.17.

9. Sati-Schreiber-Stasheff [164, Definition 1] call an $\mathrm{O}\langle 9\rangle$-structure a *fivebrane structure*, and in a sequel paper [169, §2.3], they introduce twisted fivebrane structures over a space X given by data of a map $X \to K(\mathbb{Z}, 8)$. It is possible to show that

their definition is a special case of ours. Specifically, similarly to the identification $K(\mathbb{Z},4) \simeq B\mathrm{Spin}/B\mathrm{String}$ producing a map from the $K(\mathbb{Z},4)$-twists of string bordism to the more general group of $B\mathrm{O}/B\mathrm{String}$ twists of string bordism [123, (1.45)], the characteristic class $\frac{1}{6}p_2\colon B\mathrm{String} \to K(\mathbb{Z},8)$ induces an equivalence of grouplike E_∞-spaces $B\mathrm{String}/B\mathrm{O}\langle 9\rangle \to K(\mathbb{Z},8)$, and this equivalence leads to a map $K(\mathbb{Z},8) \simeq B\mathrm{String}/B\mathrm{O}\langle 9\rangle \to B\mathrm{O}/B\mathrm{O}\langle 9\rangle$ carrying Sati-Schreiber-Stasheff's twisted fivebrane structures to a subgroup of the fake vector bundle twists of fivebrane structure.

10. Sati [170, Definition 2.4] calls $\mathrm{O}\langle 10\rangle$-structures "2-orientations." and studies their twists in (*ibid.*, Definition 5.1). Like for twisted fivebrane structures, Sati's twists are classified by maps to $K(\mathbb{Z}/2,9) \simeq B\mathrm{O}\langle 10\rangle/B\mathrm{O}\langle 9\rangle$, and map to the twists we considered via the map $B\mathrm{O}\langle 10\rangle/B\mathrm{O}\langle 9\rangle \to B\mathrm{O}/B\mathrm{O}\langle 9\rangle$.

13. As $\pi_{11}(B\mathrm{O}) = 0$ [184], $\mathrm{O}\langle 11\rangle$- and $\mathrm{O}\langle 12\rangle$-structures coincide. Sati refers to this as a "2-spin structure" [170, Definition 2.5], and in (*ibid.*, Definition 5.2) introduces twisted 2-spin structures corresponding to $K(\mathbb{Z}/2,10) \simeq B\mathrm{O}\langle 11\rangle/B\mathrm{O}\langle 10\rangle \to B\mathrm{O}/B\mathrm{O}\langle 10\rangle$.

14. As $\pi_k(B\mathrm{O}) = 0$ for $k = 13$, 14, and 15 [184], $B\mathrm{O}\langle 13\rangle = \cdots = B\mathrm{O}\langle 16\rangle$. Sati [170, Definition 3.1] names this tangential structure a *ninebrane structure*, and produces twisted ninebrane structures classified by the fractional Pontrjagin class $(1/240)p_3\colon B\mathrm{O}\langle 12\rangle \to K(\mathbb{Z},12)$ (*ibid.*, Definition 5.3). As in the previous cases, the map $K(\mathbb{Z},12) \simeq B\mathrm{O}\langle 13\rangle/B\mathrm{O}\langle 11\rangle \to B\mathrm{O}/B\mathrm{O}\langle 11\rangle$ sends Sati's twists to ours.

Passing to the infinite limit, we obtain an interpretation of maps to $B\mathrm{O}/E\mathrm{O}$, i.e. to $B\mathrm{O}$, as classifying vector bundle twists of framed bordism. These twists of framed bordism are studied in [185, 186].

3.6 Examples of Smith fiber sequences

3.6.1 Twisting by real line bundles

Our first family of examples use the tautological line bundle $\sigma \to B\mathbb{Z}/2$; its sphere bundle is the tautological $\mathbb{Z}/2$-bundle $E\mathbb{Z}/2 \to B\mathbb{Z}/2$, whose total space is contractible. Therefore by Theorem 3.4.1, for any $k \in \mathbb{Z}$, we have a cofiber sequence

$$\mathbb{S} \longrightarrow (B\mathbb{Z}/2)^{k(\sigma-1)} \xrightarrow{\mathrm{sm}_\sigma} \Sigma(B\mathbb{Z}/2)^{(k+1)(\sigma-1)}, \tag{3.6.1}$$

where sm_σ is the Smith homomorphism associated to σ. When $k = 0$, the middle spectrum is $\Sigma^\infty_+ B\mathbb{Z}/2 \simeq \mathbb{S} \vee \Sigma B\mathbb{Z}/2$ and the map $\mathbb{S} \to \mathbb{S} \vee \Sigma B\mathbb{Z}/2$ is the inclusion of the first factor of the wedge sum, leading to a Smith *isomorphism* $\mathrm{sm}_\sigma\colon \Sigma^\infty B\mathbb{Z}/2 \xrightarrow{\simeq} (B\mathbb{Z}/2)^\sigma$.[8]

Remark 3.6.2. The Thom spectrum $(B\mathbb{Z}/2)^{k\sigma}$ is often denoted in the homotopy theory literature by $\mathbb{RP}^\infty_k$, so that $(B\mathbb{Z}/2)^{k(\sigma-1)}$ can be identified with its desuspension $\Sigma^{-k}\mathbb{RP}^\infty_k$. One justification for this notation stems from (3.6.1): suspending it k times gives a cofiber sequence

$$\mathbb{S}^k \longrightarrow \mathbb{RP}^\infty_k \longrightarrow \mathbb{RP}^\infty_{k+1}, \tag{3.6.3}$$

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[8] This equivalence is well-known; see Kochman [187, Lemma 2.6.5] for a proof.

which exhibits $\mathbb{RP}^\infty_{k+1}$ as the spectrum obtained by crushing the bottom cell of $\mathbb{RP}^\infty_k$.

Example 3.6.4. Smash (3.6.1) with $MT\mathrm{O}$. As every virtual bundle has a unique $MT\mathrm{O}$-orientation, this cofiber sequence simplifies to

$$MT\mathrm{O} \longrightarrow MT\mathrm{O} \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} MT\mathrm{O} \wedge \Sigma(B\mathbb{Z}/2)_+. \tag{3.6.5}$$

This was the first Smith homomorphism studied; it was defined and named the Smith homomorphism by Conner-Floyd [109, Theorem 26.1]. Thom's celebrated calculation of Ω_*^{O} implies that $MT\mathrm{O}$ is a sum of shifts of $H\mathbb{Z}/2$; on each of these copies, the Smith map (3.6.5) is the cap product with the nonzero element of $H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$.

Stong [188, Proposition 5] and Uchida [189] study related examples, where one smashes (3.6.5) with spaces X; they identify the fiber $MT\mathrm{O} \wedge X$ and show that the long exact sequence of homotopy groups splits. Their papers are among the earliest examples identifying the Smith long exact sequence.[9] ◇

Example 3.6.6. Smash (3.6.1) with $MT\mathrm{SO}$. Since σ is not orientable, but 2σ is oriented (see Example 3.5.12), we obtain a 2-periodic series of codimension-1 Smith homomorphisms between the oriented bordism of $B\mathbb{Z}/2$ and $(B\mathbb{Z}/2,\sigma)$-twisted oriented bordism. The latter can be identified with unoriented bordism: a $(B\mathbb{Z}/2,\sigma)$-twisted orientation on V is data of a line bundle on L and an orientation of $V \oplus L$, which is no data at all: this identifies $L \cong \mathrm{Det}(V)^* \cong \mathrm{Det}(V)$ up to a contractible space of choices, and $V \oplus \mathrm{Det}(V)$ is canonically oriented. So every vector bundle has a canonical $(B\mathbb{Z}/2,\sigma)$-twisted orientation.

Therefore by Theorem 3.4.1 we obtain a 2-periodic sequence of codimension-1 Smith homomorphisms:

$$MT\mathrm{SO} \longrightarrow MT\mathrm{SO} \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{O} \tag{3.6.7a}$$

$$MT\mathrm{SO} \longrightarrow MT\mathrm{O} \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{SO} \wedge (B\mathbb{Z}/2)_+. \tag{3.6.7b}$$

These maps are obtained by taking smooth representatives of Poincaré duals of w_1 either of the manifold (when the domain is Ω_*^{O}) or of the principal $\mathbb{Z}/2$-bundle (when the domain is $\Omega_*^{\mathrm{SO}}(B\mathbb{Z}/2)$). See Section 3.8.5.2 for the physical interpretation of the corresponding long exact sequence of Anderson dual groups.

These Smith homomorphisms were first introduced by Komiya [111, §5]; see also Shibata [190, Proposition 2.1]. See Córdova-Ohmori-Shao-Yan [114, Appendix A], Hason–Komargodski–Thorngren [113, §4.4], and Fidkowski-Haah-Hastings [191] for applications of these Smith homomorphisms to physics. The splitting of the $k = 0$ case of (3.6.1) implies a homotopy equivalence $MT\mathrm{SO} \wedge B\mathbb{Z}/2 \xrightarrow{\cong} \Sigma MT\mathrm{O}$, a theorem of Atiyah [30, Proposition 4.1]. ◇

Example 3.6.8. Our primary examples of interest come from smashing (3.6.1) with $MT\mathrm{Spin}$. As we discussed in Example 3.5.14, the periodicity of this family is 1, 2, or 4; a Whitney

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[9] At the time, it was common to think of $\Omega_*^{\mathrm{O}}(B\mathbb{Z}/2)$ as the bordism groups of manifolds M equipped with a free involution τ, rather than manifolds with a principal $\mathbb{Z}/2$-bundle; Stong and Uchida's results are phrased in that language. To pass between these perspectives, rewrite (M,τ) as the principal $\mathbb{Z}/2$-bundle $M \to M/\tau$; in the other direction, take the deck transformation involution of the total space of a principal $\mathbb{Z}/2$-bundle.

sum formula calculation shows that $k\sigma$ is spin iff k is a multiple of 4, and therefore this Smith family is 4-periodic. The corresponding $(B\mathbb{Z}/2, k\sigma)$-twisted spin bordism groups can be identified with H-bordism for certain Lie groups H, as discussed in Example 3.5.23; specifically,

1. a $(B\mathbb{Z}/2, \sigma)$-twisted spin structure is equivalent to a pin$^-$ structure;
2. a $(B\mathbb{Z}/2, 2\sigma)$-twisted spin structure is equivalent to an H structure, where $H = \mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$; and
3. a $(B\mathbb{Z}/2, 3\sigma)$-twisted spin structure is equivalent to a pin$^+$ structure.

Using Theorem 3.4.1 once again, the 4-periodic sequence of codimension-1 Smith homomorphisms takes the form

$$MT\mathrm{Spin} \longrightarrow MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Pin}^- \tag{3.6.9a}$$

$$MT\mathrm{Spin} \longrightarrow MT\mathrm{Pin}^- \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4) \tag{3.6.9b}$$

$$MT\mathrm{Spin} \longrightarrow MT(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4) \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Pin}^+ \tag{3.6.9c}$$

$$MT\mathrm{Spin} \longrightarrow MT\mathrm{Pin}^+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)_+, \tag{3.6.9d}$$

with each sm_σ obtained by taking a smooth representative of a Poincaré dual of w_1 of the manifold or of a associated principal $\mathbb{Z}/2$-bundle, like in (3.6.7).

The splitting of the $k = 0$ Smith homomorphism in (3.6.1) gives us an equivalence $MT\mathrm{Spin} \wedge B\mathbb{Z}/2 \simeq MT\mathrm{Pin}^-$, a theorem of Peterson [142, §7].

This family of Smith homomorphisms has been discussed in the literature before. The piece involving $\mathrm{Spin} \times \mathbb{Z}/2$ and Pin^- was used by Peterson [142, §7] and Anderson-Brown-Peterson [192], who say that it was already “well-known.” The long exact sequence corresponding to (3.6.9c) appears in [193, Theorem 3.1], where it is attributed to Stong. The Smith homomorphism sm_σ in (3.6.9d) appears in Kreck [194, §4]. The long exact sequence induced by (3.6.9b) is used by Botvinnik–Rosenberg [195, §2], who also discuss (3.6.9c) and (3.6.9d). The composition of two maps in (3.6.9) in a row to go between pin$^+$ and pin$^-$ bordism appears in Kirby–Taylor [196, Lemma 7]. We work out the corresponding long exact sequences, as well as some physical consequences for all four Smith homomorphisms in Section 3.8.5.3.

The full family appears more recently in work of Hambleton–Su [197, §4.C], Kapustin–Thorngren–Turzillo–Wang [198, §8], Tachikawa–Yonekura [199, §3.1], Hason–Komargodski–Thorngren [113, §4.4], and Wan–Wang–Zheng [200, §6.7]. Ekholm [201] produces the 4-periodic sequence of tangential structures in a different setting but does not discuss the Smith homomorphism. ◇

Example 3.6.10. As we discussed in Example 3.5.17, for a general vector bundle $V \to X$, there is no guarantee that kV has a string structure. However, on $B\mathbb{Z}/2$, $k\sigma$ has a string structure iff $k \equiv 0 \bmod 8$, so there is an eight-periodic family of codimension-1 Smith homomorphisms between bordism groups of manifolds with $(B\mathbb{Z}/2, k\sigma)$-twisted string structures for various k.[10]

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[10] To prove the claimed fact about string structures on $k\sigma$, first use the Whitney sum formula to show that $w_1(k\sigma)$, $w_2(k\sigma)$, and $w_4(k\sigma)$ all vanish iff $k \equiv 0 \bmod 8$. The reduction mod 2 map $H^4(B\mathbb{Z}/2;\mathbb{Z}) \to H^4(B\mathbb{Z}/2;\mathbb{Z}/2)$ is an isomorphism, so the string obstruction $\lambda(k\sigma)$ vanishes iff its mod 2 reduction does, and $\lambda \bmod 2 = w_4$.

In Example 3.6.8, the four twisted spin structures turned out to be equivalent to G-structures for four Lie groups G. An analogous result is true here, but in the world of 2-groups, because the string group is a Lie 2-group [202]. One can show that for each $k \in \mathbb{Z}/8$, there is a Lie 2-group $\mathbb{G}[k]$ and a map $\xi\colon B\mathbb{G}[k] \to B\mathrm{O}$ such that $\mathbb{G}[k]$-structures on a smooth manifolds are naturally equivalent to $(B\mathbb{Z}/2, k\sigma)$-twisted string structures. These Lie 2-groups $\mathbb{G}[k]$ are extensions of $\mathrm{Spin} \times \mathbb{Z}/2$, $\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$, and $\mathrm{Pin}^\pm$ by $B\mathrm{U}(1)$; such extensions of a compact Lie group G by $B\mathrm{U}(1)$ are classified by $H^4(BG;\mathbb{Z})$ [202, 203], and the $\mathbb{G}[k]$ 2-groups' extension classes are λ of various spin vector bundles over $B\mathrm{Spin} \times B\mathbb{Z}/2$, $B(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4)$, and $B\mathrm{Pin}^\pm$. For example, $\mathbb{G}[4] = \mathrm{String} \times_{B\mathrm{U}(1)} \mathsf{sLine}$, where sLine is the abelian Lie 2-group of Hermitian super lines. ◇

Example 3.6.11. If one smashes (3.6.1) with $MT\mathrm{Spin}^c$, one obtains a very similar story to Example 3.6.6: twice any vector bundle is complex, hence spin^c, and $(B\mathbb{Z}/2, \sigma)$-twisted spin^c bordism is naturally identified with pin^c bordism, as we discussed in Example 3.5.21. So taking Poincaré duals of w_1 as in Example 3.6.6 defines a 2-periodic sequence of codimension-1 Smith homomorphisms

$$MT\mathrm{Spin}^c \longrightarrow MT\mathrm{Spin}^c \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Pin}^c \tag{3.6.12a}$$

$$MT\mathrm{Spin}^c \longrightarrow MT\mathrm{Pin}^c \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Spin}^c \wedge (B\mathbb{Z}/2)_+. \tag{3.6.12b}$$

To our knowledge, these long exact sequences first appear in Hambleton–Su [197, §4.C].

We also obtain an equivalence $MT\mathrm{Spin}^c \wedge B\mathbb{Z}/2 \xrightarrow{\simeq} \Sigma MT\mathrm{Pin}^c$, which was first observed by Bahri-Gilkey [137, §3]. See Shiozaki-Shapourian-Ryu [204, §E.1] and Kobayashi [205, §IV] for applications in condensed-matter physics and [206] for an application of a closely related Smith long exact sequence. ◇

Example 3.6.13. Pull back (3.6.1) along the map $B\mathbb{Z} \to B\mathbb{Z}/2$, i.e. $S^1 = \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^\infty$. The sphere bundle of $\sigma \to \mathbb{RP}^1$ is not contractible: it is the double cover $S^1 \to \mathbb{RP}^1$, and its Thom space is $\mathbb{RP}^2$. Therefore we obtain from Theorem 3.4.1 a cofiber sequence $\Sigma^\infty_+ S^1 \to \Sigma^\infty \mathbb{RP}^2 \to \Sigma^{1+\infty}_+ \mathbb{RP}^1$, which is a rotated version of the multiplication-by-2 cofiber sequence

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \longrightarrow \Sigma^{-1+\infty}\mathbb{RP}^2. \tag{3.6.14}$$

The same story applies to the complex, quaternionic, and octonionic Hopf fibrations: their cofibers are the respective projective planes $\Sigma^{-2+\infty}\mathbb{CP}^2$, $\Sigma^{-4+\infty}\mathbb{HP}^2$, and $\Sigma^{-8+\infty}\mathbb{OP}^2$, and in each case the map to the cofiber is a Smith homomorphism for the tautological line bundle over the respective projective line (which is a sphere). In the case of the complex Hopf fibration, after smashing with ko or KO, one obtains the Wood cofiber sequences [207] $\Sigma KO \xrightarrow{\eta} KO \to KU$ and $\Sigma ko \xrightarrow{\eta} ko \to ku$ as rotated versions of Smith cofiber sequences.

Smash (3.6.14) with $MT\mathrm{SO}$ and you obtain Wall's cofiber sequence [208, Theorem 3]

$$MT\mathrm{SO} \xrightarrow{2} MT\mathrm{SO} \longrightarrow \mathcal{W}, \tag{3.6.15}$$

where $\mathcal{W}$ is the Thom spectrum whose homotopy groups are the bordism groups of manifolds with an integral lift of w_1. This follows from Atiyah's identification of $\mathcal{W} \simeq \Sigma^{-1} MT\mathrm{SO} \wedge \mathbb{RP}^2$ [30, §4], but it is also easy to directly check that an integral lift of w_1 is equivalent data to a $(\mathbb{RP}^1, \sigma)$-twisted orientation, using that $\mathbb{RP}^1$ is a $B\mathbb{Z}$.

It is also interesting to smash (3.6.14) with $MT\mathrm{Spin}$; we work out the induced long exact sequence of bordism groups in low degrees in Figure A.1, and this long exact sequence also appears in [206]. ◇

Example 3.6.16. Let $\pi\colon E \to B$ be a principal $\mathbb{Z}/2$-bundle and $L \coloneqq E \times_{\mathbb{Z}/2} \mathbb{R} \to B$ be the associated line bundle. Then we have a Smith homomorphism $\mathrm{sm}_L\colon B^{-L} \to \Sigma^\infty_+ B$. The fiber is the Thom spectrum of the pullback of L to its sphere bundle; the sphere bundle is E and $\pi^*(L)$ is trivial, so Theorem 3.4.1 gives us a cofiber sequence

$$B^{-L} \xrightarrow{\mathrm{sm}_L} \Sigma^\infty_+ B \xrightarrow{\tau} \Sigma^\infty_+ E. \tag{3.6.17}$$

Lemma 3.6.18. *The map τ in (3.6.17) is the Becker-Gottlieb transfer [209–211] for π.*

Proof. It suffices to work universally with the Smith cofiber sequence $(B\mathbb{Z}/2)^{-\sigma} \to \Sigma^\infty_+ B\mathbb{Z}/2 \to \Sigma^\infty_+ E\mathbb{Z}/2$, i.e. $(\mathbb{RP}^\infty)^{-\sigma} \to \Sigma^\infty_+\mathbb{RP}^\infty \to \mathbb{S}$, and to show that the latter map is the transfer for $E\mathbb{Z}/2 \to B\mathbb{Z}/2$.

This transfer map admits the following description: consider the map of $\mathbb{Z}/2$-spectra[11] $f\colon \mathbb{S} \to \Sigma^{1-\sigma}(\mathbb{Z}/2)_+$, whose cofiber is $\mathbb{S}^{-\sigma}$. Upon taking homotopy orbits, we obtain a map $f_{h\mathbb{Z}/2}\colon \Sigma^\infty_+\mathbb{RP}^\infty \to \mathbb{S}$, and this is the transfer map.

If G is a finite group and $V \in RO(G)$, there is a natural equivalence of spectra $(\mathbb{S}^V)_{hG} \simeq (BG)^V$.[12] And taking homotopy orbits of G-spectra preserves cofiber sequences, so the fiber of the transfer $f_{h\mathbb{Z}/2}$ is the map $(\mathbb{RP}^\infty)^{-\sigma} \to \Sigma^\infty_+\mathbb{RP}^\infty$ given by the "inclusion" of virtual representations $-\sigma \hookrightarrow 0$, which is the Smith homomorphism we began with. ☐

In the case that B is a finite CW complex, one can prove Lemma 3.6.18 more classically by adapting Cusick's calculation [212, Corollary 2.11] identifying the cofibers of transfer maps for double covers. ◇

Remark 3.6.19. For another example along the lines of (3.6.17), Morisugi [213, Theorem 1.3] shows that the cofibers of certain Smith homomorphisms over compact Lie groups can be described as Becker-Schultz transfer maps [214, §4]. And Uchida [215], motivated by the study of immersions, works out the Smith long exact sequences of a few special cases of Example 3.6.16, where $E = B\mathrm{O}(k) \times B\mathrm{O}(k)$ and $B = B(\mathrm{O}(1) \ltimes (\mathrm{O}(k)^{\times 2}))$, where $\mathrm{O}(1)$ acts on $\mathrm{O}(k)^{\times 2}$ by swapping the two factors.

Remark 3.6.20. The ease of modifying the Smith long exact sequence by a vector bundle twist suggests that Example 3.6.16 could be generalized to some sort of twisted transfer map. The relevant twisted transfer maps have been constructed by Kashiwabara-Zare [121].

3.6.2 Twisting by complex line bundles

Now we consider the analogous family of examples arising from the tautological complex line bundle $L \to B\mathrm{U}(1)$. Its sphere bundle is $E\mathrm{U}(1) \to B\mathrm{U}(1)$, which is contractible, so just like

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[11] This fact, and our argument using it, works for both Borel and genuine $\mathbb{Z}/2$-spectra.

[12] One quick way to prove this uses the Ando-Blumberg-Gepner-Hopkins-Rezk approach to Thom spectra [28, 131]: both $(\mathbb{S}^V)_{hG}$ and $(BG)^V$ are both the colimit of the pt/G-shaped diagram whose value on pt is $\mathbb{S}^V$ and whose value on the morphism set G encodes the G-action on $\mathbb{S}^V$ [28, Theorem 1.17]. It is also possible to prove this more classically by working with Thom spaces.

in (3.6.1), we have for any $k \in \mathbb{Z}$ a cofiber sequence

$$\mathbb{S} \longrightarrow (B\mathrm{U}(1))^{k(L-2)} \xrightarrow{\mathrm{sm}_L} \Sigma^2 (B\mathrm{U}(1))^{(k+1)(L-2)}. \tag{3.6.21}$$

Again, when $k = 0$, this sequence splits, yielding another Smith isomorphism $\Sigma^\infty B\mathrm{U}(1) \xrightarrow{\cong} (B\mathrm{U}(1))^L$. This equivalence is well-known, e.g. [216, Example 2.1].

Example 3.6.22. Let G be one of O, SO, Spin^c, or U; then the tautological line bundle over $B\mathrm{U}(1)$ has a G-structure, and MTG is an E_∞-ring spectrum and we can make sense of G-orientations. The G-orientation on L untwists the Thom spectrum, so smashing (3.6.21) with MTG has a similar effect to Example 3.6.4: the result is a cofiber sequence

$$MTG \longrightarrow MTG \wedge (B\mathrm{U}(1))_+ \xrightarrow{\mathrm{sm}_L} \Sigma^2 MTG \wedge (B\mathrm{U}(1))_+. \tag{3.6.23}$$

For $G = \mathrm{U}$, this Smith homomorphism was first studied by Conner-Floyd [217, §5].

Lemma 3.6.24. *For $G = \mathrm{O}$, SO, Spin^c, or U,*

$$MTG \wedge (B\mathrm{U}(1))_+ \simeq \bigvee_{k\geq 0} \Sigma^{2k} MTG. \tag{3.6.25}$$

Proof. The zeroth step is splitting off the basepoint: $MTG \wedge (B\mathrm{U}(1))_+ \simeq MTG \vee MTG \wedge (B\mathrm{U}(1))$. As noted above, $\Sigma^\infty B\mathrm{U}(1) \simeq (B\mathrm{U}(1))^L$, and we have a Thom isomorphism $MTG \wedge (B\mathrm{U}(1))^L \simeq MTG \wedge \Sigma^2 (B\mathrm{U}(1))_+$. We are now in the same situation as at the beginning of the proof, but shifted up by 2, and we carry on in a similar way. ☐

◇

Example 3.6.26. Smash (3.6.21) with $MT\mathrm{Spin}$; the bundle $L \to B\mathrm{U}(1)$ is oriented but not spin, so $2L$ is spin, and therefore we obtain a 2-periodic, codimension-2 family of Smith homomorphisms between the spin bordism of $B\mathrm{U}(1)$ and $(B\mathrm{U}(1), L)$-twisted spin bordism. A $(B\mathrm{U}(1), L)$-twisted spin structure is equivalent data to a spin^c structure, as we discussed in Example 3.5.23, so this Smith family takes the form

$$MT\mathrm{Spin} \longrightarrow MT\mathrm{Spin}^c \xrightarrow{\mathrm{sm}_L} \Sigma^2 MT\mathrm{Spin} \wedge (B\mathrm{U}(1))_+ \tag{3.6.27a}$$

$$MT\mathrm{Spin} \longrightarrow MT\mathrm{Spin} \wedge (B\mathrm{U}(1))_+ \xrightarrow{\mathrm{sm}_L} \Sigma^2 MT\mathrm{Spin}^c. \tag{3.6.27b}$$

The long exact sequence arising from (3.6.27a) was identified by Kirby-Taylor [110, Corollary 6.12, Remark 6.14]. The splitting of (3.6.21) when $k = 0$ leads to an equivalence $MT\mathrm{Spin} \wedge B\mathrm{U}(1) \simeq \Sigma^2 MT\mathrm{Spin}^c$, a theorem due to Stong [218, Chapter XI]. We use the symmetry breaking long exact sequences corresponding to (3.6.27), i.e. the long exact sequences on cohomology for the Anderson dual cofiber sequences to (3.6.27), several times later, including Section 3.8.2.1, Section 3.8.3.1 and Section 3.8.3.3 where we apply it to symmetry breaking, and in Section 3.8.5.1, where we explicitly calculate the groups and maps in the long exact sequences in low dimensions. ◇

It would be interesting to study analogues of this example for pin^c or pin$^{\tilde{c}\pm}$ bordism and applications to invertible phases. Kirby-Taylor [110, Remark 6.15] consider two additional analogues of (3.6.27a), including a Smith long exact sequence for G-bordism where $G \coloneqq \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{O}(2)$. Guillou-Marin [219] and Stehouwer [126, §4] compute G-bordism groups in low dimensions, and G-bordism also appears in [115, 206, 220]. In addition, Hambleton-Kreck-Teichner [221, §2] study a pin$^-$ and pin^c analogue of Example 3.6.26.

Example 3.6.28. Pull back (3.6.21) along the inclusion $\mathbb{Z}/k \hookrightarrow \mathrm{U}(1)$, giving us Smith homomorphisms $(B\mathbb{Z}/k)^{k(L-2)} \to \Sigma^2(B\mathbb{Z}/k)^{(k+1)(L-2)}$, where L is the complex line bundle induced by the rotation representation of $\mathbb{Z}/k$ on $\mathbb{C}$. Recall from Theorem 3.4.1 the fiber sequence

$$S(V_2)^{V_1} \to X^{V_1} \to X^{V_1 \oplus V_2}. \tag{3.6.29}$$

For this example, we start with $X = B\mathbb{Z}/n$, $V_2 = i^*L - 2$ (for L as in the previous example and 2 the trivial complex line bundle), and $V_1 = k(i^*L - 2)$. We can compute the sphere bundle $S(i^*L)$ by fitting it into a pullback square:

$$\begin{tikzcd} S(i^*L) \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & S(L) \simeq * \arrow[d] \\ B\mathbb{Z}/n \arrow[r] & B\mathrm{U}(1). \end{tikzcd} \tag{3.6.30}$$

As noted above, $S(L)$ is contractible as it is the total space of the universal fibration. Therefore, the other three corners of the square form a fiber sequence. To compute the fiber of $B\mathbb{Z}/n \to B\mathrm{U}(1)$, we notice that applying the classifying space functor to the short exact sequence $\mathbb{Z}/n \hookrightarrow \mathrm{U}(1) \xrightarrow{\times n} \mathrm{U}(1)$ gives a fibration $B\mathbb{Z}/n \to B\mathrm{U}(1) \to B\mathrm{U}(1)$. Then, recognizing the map $B\mathrm{U}(1) \to B\mathrm{U}(1)$ as the classifying map for a principal $\mathrm{U}(1)$-bundle over $\mathrm{U}(1)$ with total space $B\mathbb{Z}/n$, we conclude that the fiber of the map $B\mathbb{Z}/n \to B\mathrm{U}(1)$ is exactly $\mathrm{U}(1)$. So, $S(i^*L) \simeq S^1$.

Next, we need to pull back V_1 along the projection $p\colon S(i^*L) \to B\mathbb{Z}/n$. We have that $p^*(k(i^*L - 1)) \cong \bigoplus_k p^*(i^*L)$. Since L is oriented as a real vector bundle, its pullbacks are as well, so p^*i^*L is oriented when considered as a real vector bundle over S^1, and thus it is the trivial 2-plane bundle.

Therefore, we recognize the Thom spectrum $S(i^*L)^{kp^*(i^*L)}$ as

$$\begin{aligned} S(i^*L)^{kp^*(i^*L)} &\simeq (S^1)^k \\ &\simeq \Sigma^{2k}\mathrm{Th}(S^1; 0) \\ &\simeq \Sigma^{2k}(\Sigma^\infty_+ S^1) \\ &\simeq \Sigma^{2k}(\Sigma^\infty S^1 \oplus \Sigma^\infty S^0) \\ &\simeq \Sigma^{2k+1}\mathbb{S} \vee \Sigma^{2k}\mathbb{S}. \end{aligned}$$

Thus for each $k \geq 0$ we have a Smith cofiber sequence

$$\Sigma^{2k+1}\mathbb{S} \vee \Sigma^{2k}\mathbb{S} \longrightarrow (B\mathbb{Z}/n)^{k \cdot i^*L} \longrightarrow B\mathbb{Z}/n^{(k+1)i^*L}. \tag{3.6.31}$$

Finally, we place V_1 in virtual dimension zero by taking $V_1 = k(i^*L - 2)$, to be consistent with the other examples in this section, and obtain the cofiber sequence

$$\Sigma\mathbb{S} \vee \mathbb{S} \longrightarrow (B\mathbb{Z}/n)^{k(i^*L-2)} \xrightarrow{\mathrm{sm}_L} \Sigma^2(B\mathbb{Z}/n)^{(k+1)(i^*L-2)}. \tag{3.6.32}$$

◇

Example 3.6.33. Smash (3.6.32) with $MT\mathrm{Spin}$. Like in Example 3.6.26, i^*L is oriented but not spin, and $2i^*L$ is spin, so we obtain a 2-periodic, codimension-2 family of Smith homomorphisms between the spin bordism of $B\mathbb{Z}/n$ and $(B\mathbb{Z}/n, i^*L)$-twisted spin bordism. Campbell [96, §7.9] identifies the latter as bordism for the tangential structure $\mathrm{Spin}\times_{\{\pm1\}}\mathbb{Z}/2n$, explicitly giving us Smith cofiber sequences

$$\Sigma MT\mathrm{Spin} \vee MT\mathrm{Spin} \longrightarrow MT(\mathrm{Spin}\times_{\{\pm1\}}\mathbb{Z}/2k) \xrightarrow{\mathrm{sm}_{i^*L}} \Sigma^2 MT\mathrm{Spin}\wedge(B\mathbb{Z}/k)_+ \tag{3.6.34a}$$

$$\Sigma MT\mathrm{Spin} \vee MT\mathrm{Spin} \longrightarrow MT\mathrm{Spin}\wedge(B\mathbb{Z}/k)_+ \xrightarrow{\mathrm{sm}_{i^*L}} MT(\mathrm{Spin}\times_{\{\pm1\}}\mathbb{Z}/2k). \tag{3.6.34b}$$

$\mathrm{Spin}\times_{\{\pm1\}}\mathbb{Z}/2k$ bordism appears in the mathematical physics literature in [96, 115, 140, 145, 206, 220, 222–230]; the case $k = 2$ also appears in [113, 199, 231–233]. The Smith homomorphisms in (3.6.34) for $n = 4$ appear in [115]. We work out the Anderson-dualized long exact sequences corresponding to (3.6.34) for the $k = 3$ case in Section 3.8.5.4, and for the $k = 4$ case in Section 3.8.5.5. ◇

Example 3.6.35. We elaborate on Example 3.6.33 when $n = 2$. The rotation representation is isomorphic to 2σ, where σ denotes the real sign representation; we will also let σ denote the associated bundle over $B\mathbb{Z}/2$.

Everything in Example 3.6.33 still works for $n = 2$, but now we have more options: we can start with an odd number of copies of σ. In this case, the fiber of the Smith map is the Thom spectrum of the Möbius bundle $(\sigma - 1) \to \mathrm{U}(1)$; one can directly check that the Thom space of σ is $\mathbb{RP}^2$, so the Thom spectrum of $\sigma - 1$ is $\Sigma^{-1+\infty}\mathbb{RP}^2$. Therefore we have a Smith cofiber sequence

$$\Sigma^{-1+\infty}\mathbb{RP}^2 \longrightarrow (B\mathbb{Z}/2)^{(2k-1)(\sigma-1)} \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2(B\mathbb{Z}/2)^{(2k+1)(\sigma-1)}. \tag{3.6.36}$$

Out of all the examples we have studied in this section, this is the first one where the pullback of V_2 to the sphere bundle is nontrivial.

As usual, we smash (3.6.36) with various bordism spectra. The map $\mathrm{sm}_{2\sigma}$ is the composition of two iterations of sm_σ from (3.6.1), so some of the resulting cofiber sequences look familiar from that perspective. We only discuss a few examples, but plenty more are out there.

- If we smash (3.6.36) with $MT\mathrm{SO}$, we obtain a cofiber sequence first discussed by Atiyah [30, (4.3)]:

  $$\mathcal{W} \longrightarrow MTO \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2 MTO, \tag{3.6.37}$$

  where $\mathcal{W}$ is Wall's bordism spectrum (see Example 3.6.13). Here we use the identifications $\Sigma MTO \simeq MT\mathrm{SO}\wedge B\mathbb{Z}/2$ and $\mathcal{W}\simeq MT\mathrm{SO}\wedge\Sigma^{-1}\mathbb{RP}^2$, both due to Atiyah [30, §4], that we discussed in Examples 3.6.6 and 3.6.13, respectively.

- If we instead smash (3.6.36) with $MT\mathrm{Spin}$, we obtain a cofiber sequence

$$MT\mathrm{Spin}\wedge\Sigma^{-1}\mathbb{RP}^2 \longrightarrow MT\mathrm{Pin}^{\pm} \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2 MT\mathrm{Pin}^{\mp}, \tag{3.6.38}$$

  which was first constructed by Kirby-Taylor [196, Lemma 7]. Here we have used the identifications of pin$^+$, resp. pin$^-$ bordism as $(B\mathbb{Z}/2, 3\sigma)$, resp. $(B\mathbb{Z}/2,\sigma)$-twisted spin bordism that we discussed in Example 3.6.8. In Figure A.2, we calculate the long exact sequence on bordism groups corresponding to (3.6.38) (specifically, the pin$^-$ to pin$^+$ case) in low degrees. See [115] for an application of a related but different Smith homomorphism in physics.

  The Smith homomorphism in (3.6.38) is the composition of two of the Smith homomorphisms in the 4-periodic collection of Example 3.6.8, where we go from pin$^+$ to $\mathrm{Spin}\times\mathbb{Z}/2$ to pin$^-$, or from pin$^-$ to $\mathrm{Spin}\times_{\{\pm1\}}\mathbb{Z}/4$ to pin$^+$. The other two compositions, which exchange the spin bordism of $B\mathbb{Z}/2$ with $\mathrm{Spin}\times_{\{\pm1\}}\mathbb{Z}/4$ bordism, are (3.6.34) for $n=2$.

◇

3.6.3 A few more examples

In this section, we record some examples of Smith cofiber sequences that do not arise from real or complex line bundles.

Example 3.6.39. Like our previous examples over $B\mathbb{Z}/2$ and $B\mathrm{U}(1)$, we can study Smith homomorphisms for the tautological quaternionic line bundle $V_4\to B\mathrm{SU}(2)$. Once again, the sphere bundle of V is contractible, as it is $E\mathrm{SU}(2)\to B\mathrm{SU}(2)$, so we obtain Smith cofiber sequences like in (3.6.1) and (3.6.21):

$$\mathbb{S}\longrightarrow (B\mathrm{SU}(2))^{k(V_4-4)} \xrightarrow{\mathrm{sm}_{V_4}} \Sigma^4(B\mathrm{SU}(2))^{(k+1)(V_4-4)}. \tag{3.6.40}$$

For $k=0$, this sequence splits, yielding a third Smith isomorphism $\Sigma^\infty B\mathrm{SU}(2)\xrightarrow{\cong}(B\mathrm{SU}(2))^{V_4}$. This equivalence is well-known, e.g. [234, §2].

This bundle has a G-structure for G including O, SO, Spin, Spin^c, U, SU, and Sp, and in all of these cases, smashing with MTG produces Smith homomorphisms similar to those in Examples 3.6.4 and 3.6.22. The proof of Lemma 3.6.24 still works in this setting, and for these G we obtain splittings

$$MTG\wedge(B\mathrm{SU}(2))_+\simeq\bigvee_{k\geq0}\Sigma^{4k}MTG. \tag{3.6.41}$$

The Smith map (3.6.40), after smashing with $MT\mathrm{Sp}$, was studied by Landweber [235, §5]. ◇

Example 3.6.42. As we mentioned in Example 3.6.10, Lie 2-group extensions of a compact Lie group G by $B\mathrm{U}(1)$ are classified by $H^4(BG;\mathbb{Z})$ [202, 203]. Let String-SU(2) be the Lie 2-group belonging to the extension

$$B\mathrm{U}(1)\longrightarrow \text{String-SU}(2)\longrightarrow \mathrm{Spin}\times\mathrm{SU}(2) \tag{3.6.43}$$

classified by $\lambda + p_1^{\mathbb{H}} \in H^4(B\mathrm{Spin}\times B\mathrm{SU}(2);\mathbb{Z})$; here $\lambda \in H^4(B\mathrm{Spin};\mathbb{Z})$ and $p_1^{\mathbb{H}} \in H^4(B\mathrm{SU}(2);\mathbb{Z})$ are the canonical generators of H^4 of a connected, simply connected, simple Lie group. (For $B\mathrm{Spin}$, we may use $B\mathrm{Spin}(n)$ with $n \gg 0$.) Using the usual map $\mathrm{Spin} \to \mathrm{O}$, we obtain a tangential structure $B(\text{String-SU}(2)) \to B\mathrm{O}$; by an argument similar to the one in [115, §10.4] (see [236, §3.3.1]), $MT(\text{String-SU}(2)) \simeq MT\mathrm{String} \wedge (B\mathrm{SU}(2))^{L-4}$, where $L \to B\mathrm{SU}(2)$ is the tautological quaternionic line bundle. Thus (3.6.40) with $k = 0$, smashed with $MT\mathrm{String}$, produces a Smith isomorphism

$$\mathrm{sm}_L\colon \widetilde{\Omega}_*^{\mathrm{String}}(B\mathrm{SU}(2)) \xrightarrow{\cong} \Omega_{*-4}^{\text{String-SU}(2)}. \tag{3.6.44}$$

As $L \to B\mathrm{SU}(2)$ is not string, the argument for (3.6.41) does not apply, and indeed one can show $MT(\text{String-SU}(2))$ does not split in that way. Thus this Smith isomorphism is expressing something nontrivial about string-SU(2) bordism. To our knowledge, (3.6.44) is the first such nontrivial quaternionic Smith isomorphism known.

String-SU(2) bordism appears in [236, §3] as an intermediary to other twisted string bordism computations, and Bruner-Rognes [237, §1.4, Chapter 8, §12.3, Appendix D] study a closely related object called *tmf*/ν. ◇

Example 3.6.45. Consider the Smith homomorphisms coming from the tautological rank-3 vector bundle $V_3 \to B\mathrm{SO}(3)$. Then, like in Example 3.6.49, the one-point compactification of $\mathfrak{so}(3)/\mathfrak{u}_1$ is isomorphic to $\mathrm{SO}(3)/\mathrm{U}(1) \cong S^2$. Since $\mathfrak{so}(3)/\mathfrak{u}_1 \oplus \mathbb{R}$ is isomorphic to the defining representation V of SO(3), we obtain a cofiber sequence of spectra

$$(B\mathrm{U}(1))^{k(L-2)} \longrightarrow (B\mathrm{SO}(3))^{k(V_3-3)} \longrightarrow \Sigma^3(B\mathrm{SO}(3))^{(k+1)(V_3-3)}. \tag{3.6.46}$$

We are most interested in smashing this sequence with $MT\mathrm{Spin}$.[13] Note that V_3 is not spin, but because V_3 is oriented, $2V_3$ is spin; therefore we obtain a 2-periodic family of codimension-3 Smith homomorphisms exchanging the spin bordism of $B\mathrm{SO}(3)$ and $(B\mathrm{SO}(3), V_3)$-twisted spin bordism. Freed–Hopkins [124, (10.20)] identify $(B\mathrm{SO}(3), V_3)$-twisted spin bordism with bordism for the group $G^0 := \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}(2)$, which is in various sources called spin^h bordism, spin^q bordism, spin-SU(2) bordism, or G^0 bordism.[14] The fiber we've seen before in Example 3.6.26: spin^c bordism when k is odd in (3.6.46), and the spin bordism of $B\mathrm{U}(1)$ when k is even.

In summary, we have two Smith cofiber sequences

$$MT\mathrm{Spin}^c \longrightarrow MT\mathrm{Spin}^h \xrightarrow{\mathrm{sm}_{V_3}} \Sigma^3 MT\mathrm{Spin} \wedge (B\mathrm{SO}(3))_+ \tag{3.6.47a}$$

$$MT\mathrm{Spin} \wedge (B\mathrm{U}(1))_+ \longrightarrow MT\mathrm{Spin} \wedge (B\mathrm{SO}(3))_+ \xrightarrow{\mathrm{sm}_{V_3}} \Sigma^3 MT\mathrm{Spin}^h. \tag{3.6.47b}$$

---

[13] It is also interesting to smash (3.6.46) with $MT\mathrm{Spin}^c$: in this case one obtains a codimension-3, 2-periodic family of Smith homomorphisms exchanging the spin^c bordism of $B\mathrm{SO}(3)$ with "spin-U(2) bordism," i.e. bordism of the group $\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{U}(2) \cong \mathrm{Spin}^c \times_{\{\pm 1\}} \mathrm{SU}(2)$. Davighi-Lohitsiri [139, 140] introduced Spin-U(2) bordism and calculated it in low dimensions; spin-U(2) structures also appear in Seiberg-Witten theory (e.g. [238, 239]) under the name *spin^u structures.*

[14] To the best of our knowledge, spin^h structures were first studied in [240] in the context of quantum gravity; they have also been applied to Seiberg-Witten theory [241], index theory, e.g. in [124, 242–245], almost quaternionic geometry, e.g. in [243, 244, 246], immersion problems [244, 246], and the study of invertible field theories [98, 124, 147, 247]. See [248] for a review and [249–252] for additional related work.

The long exact sequence of bordism groups associated to (3.6.47a) appears in [9, Appendix B] as an example where one must use the cobordism Euler class to calculate the Smith homomorphism: ordinary cohomology Euler classes give the wrong answer. Other works studying anomalies of spin^h QFTs include [124, 139, 147, 200, 206, 247, 253–256]. ◇

Remark 3.6.48. Freed–Hopkins [124] also study two unoriented analogues of spin^h structures, called pin$^{h\pm}$ or $G^\pm$ structures, corresponding to the groups $\mathrm{Pin}^\pm \times_{\{\pm 1\}} \mathrm{SU}(2)$. It would be interesting to work out analogues of the Smith homomorphisms such as the ones in Examples 3.6.8 and 3.6.45 for pin$^{h\pm}$ structures and apply them to symmetry breaking; see our paper with Debray [42] for some work in that direction. Pin$^{h\pm}$ manifolds are also studied in [98, 206, 246, 257, 258].

Example 3.6.49. If we pull Example 3.6.45 back to $B\mathrm{SU}(2)$, we obtain a Smith long exact sequence which makes an appearance both in §3.8.5.6 and in [9, Appendix B].

The tautological quaternionic line bundle over $B\mathrm{SU}(2)$ is *not* isomorphic to the bundle associated to $\mathfrak{su}_2 \oplus \mathbb{R}$, where $\mathfrak{su}_2$ is the adjoint representation of $\mathrm{SU}(2)$. Rather, since $\mathfrak{su}_2 \cong \mathbb{R} \oplus \mathfrak{su}_2/\mathfrak{u}_1$, the map $B\mathrm{U}(1) \to B\mathrm{SU}(2)$ exhibits $B\mathrm{U}(1)$ as the unit sphere bundle in the adjoint representation of $\mathrm{SU}(2)$. It follows that there is a cofiber sequence

$$B\mathrm{U}(1) \longrightarrow B\mathrm{SU}(2) \xrightarrow{\mathrm{sm}_V} \Sigma^3 (B\mathrm{SU}(2))^{V-3}, \tag{3.6.50}$$

where $V \to B\mathrm{SU}(2)$ is the vector bundle associated to $\mathfrak{su}(2)$. We claim the first map is induced by the inclusion of a maximal torus into $\mathrm{SU}(2)$. To see that the sphere bundle is $B\mathrm{U}(1)$ as claimed, identify $\mathrm{SU}(2) \to \mathrm{SO}(3)$ with $\mathrm{Spin}(3) \to \mathrm{SO}(3)$ and $\mathrm{U}(1) \to \mathrm{SU}(2)$ with $\mathrm{Spin}(2) \to \mathrm{Spin}(3)$; by the third isomorphism theorem, $\mathrm{Spin}(3)/\mathrm{Spin}(2) \cong \mathrm{SO}(3)/\mathrm{SO}(2)$, and in Example 3.6.45 we identified that quotient with the unit sphere inside $\mathbb{R}^3$. Therefore taking associated bundles, we end up with $B\mathrm{Spin}(2)$ as the fiber in (3.6.50).

Since $\mathrm{SU}(2)$ is simply connected, $B\mathrm{SU}(2)$ is 2-connected and therefore all of its vector bundles admit spin structures. Thus, when we smash (3.6.50) with $MT\mathrm{Spin}$, we obtain a cofiber sequence

$$MT\mathrm{Spin} \wedge (B\mathrm{U}(1))_+ \longrightarrow MT\mathrm{Spin} \wedge (B\mathrm{SU}(2))_+ \xrightarrow{\mathrm{sm}_V} \Sigma^3 MT\mathrm{Spin} \wedge (B\mathrm{SU}(2))_+. \tag{3.6.51}$$

The Thom spectrum $(B\mathrm{SU}(2))^{\mathfrak{su}(2)}$ is known as James' "quasiprojective space" (see [259]).

The same Thom isomorphism applies for any $MT\mathrm{Spin}$-oriented ring spectrum, such as $MT\mathrm{SO}$ or ko. ◇

Example 3.6.52. In Section 3.8.2.2, we study the SBLES in twisted spin bordism corresponding to the vector bundle $2L \to B\mathrm{U}(1)$, where L denotes the tautological bundle. Since $2L$ is spin, we obtain a one-periodic family of Smith homomorphisms of the form

$$S(2L) \longrightarrow B\mathrm{U}(1) \xrightarrow{\mathrm{sm}_{2L}} \Sigma^4 (B\mathrm{U}(1))^{2L-4}. \tag{3.6.53a}$$

The new wrinkle is showing that $S(2L) \to B\mathrm{U}(1)$ is homotopy equivalent to the map $S^2 \to B\mathrm{U}(1)$ given by the inclusion of the 2-skeleton. But this is not so hard: using the long exact sequence in cohomology associated to the cofiber sequence, one learns that if C is the cofiber of sm_{2L}, $\widetilde{H}^*(C;\mathbb{Z})$ vanishes except in degree 3, where it is $\mathbb{Z}$; this characterizes S^3, so the fiber, which is the total space of the sphere bundle, is S^2. Stably this splits as $\mathbb{S}\vee\Sigma^2\mathbb{S}$, so our cofiber sequence is

$$\mathbb{S}\vee\Sigma^2\mathbb{S}\longrightarrow\Sigma_+^\infty B\mathrm{U}(1)\xrightarrow{\mathrm{sm}_{2L}}(B\mathrm{U}(1))^{2L-4}. \tag{3.6.53b}$$

This cofiber sequence is a complexified version of (3.6.32). One therefore wonders what happens if we consider it within its family

$$\mathrm{sm}_{2L}\colon(B\mathrm{U}(1))^{kL-2k}\longrightarrow\Sigma^4(B\mathrm{U}(1))^{(k+1)L-2k-2}. \tag{3.6.54}$$

If we smash with $MT\mathrm{Spin}$, this is a 2-periodic family: it only matters whether k is odd or even. For k even we reduce to (3.6.53b) above; for k odd, we have a very similar cofiber sequence, but the sphere bundle does not split: we obtain for the fiber $(\mathbb{CP}^1)^{\mathcal{O}(-1)-2}\simeq\mathbb{CP}^2$:

$$MT\mathrm{Spin}\wedge\mathbb{CP}^2\longrightarrow MT\mathrm{Spin}^c\longrightarrow\Sigma^4MT\mathrm{Spin}^c, \tag{3.6.55}$$

using the identification $MT\mathrm{Spin}\wedge(B\mathrm{U}(1))^{L-2}\simeq MT\mathrm{Spin}^c$ from Example 3.5.23. This is the complex analogue of (3.6.38). ◇

Remark 3.6.56. There is a related example where one uses $L\oplus L^*\to B\mathrm{U}(1)$ instead of $2L$; the corresponding long exact sequence in twisted SU-bordism was studied by Conner-Floyd [217, §§6, 14, 17]. When L is odd, the third term in the long exact sequence, corresponding to the sphere bundle, is the bordism of manifolds with c_1-*aspherical structures* or *complex Wall structures*, first introduced by Conner-Floyd [217], and also discussed by Stong [218, Chapter VIII]. Complex Wall bordism plays an important role in the calculation of Ω_*^{SU} via the Adams-Novikov spectral sequence [260, §7], and has also been studied in the context of complex orientations [261–263].

Example 3.6.57. The unit sphere bundle to the tautological bundle $V_{n+1}\to B\mathrm{O}(n+1)$ is homotopy equivalent to the map $B\mathrm{O}(n)\to B\mathrm{O}(n+1)$. This is because $S^n\cong\mathrm{O}(n+1)/\mathrm{O}(n)$, so the unit sphere bundle can be described by the mixing construction

$$S^n\times_{\mathrm{O}(n+1)}E\mathrm{O}(n+1)\cong(\mathrm{O}(n+1)/\mathrm{O}(n))\times_{\mathrm{O}(n+1)}E\mathrm{O}(n+1)\cong E\mathrm{O}(n+1)/\mathrm{O}(n)\cong B\mathrm{O}(n). \tag{3.6.58}$$

More generally, if $\xi_{n+1}\colon B_{n+1}\to B\mathrm{O}(n+1)$ is an unstable tangential structure and $\xi_n\colon B_n\to B\mathrm{O}(n)$ is the pullback of ξ_{n+1} by $B\mathrm{O}(n)\to B\mathrm{O}(n+1)$, the sphere bundle of $\xi_{n+1}^*V_{n+1}$ is the pullback of $S(V_{n+1})=B\mathrm{O}(n)$ by ξ_{n+1}, which is ξ_n. If you then pull $\xi_{n+1}^*V_{n+1}$ back across $B_n\to B_{n+1}$, it splits as $V_n\oplus\underline{\mathbb{R}}$, so there is a Smith cofiber sequence

$$\Sigma^{-1}B_n^{n-V_n}\longrightarrow B_{n+1}^{n+1-V_{n+1}}\longrightarrow\Sigma_+^\infty B_{n+1}. \tag{3.6.59}$$

This cofiber sequence is due to Galatius-Madsen-Tillmann-Weiss [264, (3.3), §5]. The spectrum $\Sigma^nB_n^{n-\xi_n^*V_n}$ is often denoted $MT\xi_n$. ◇

3.7 Long exact sequence of invertible field theories

Just as the map of spectra of Section 3.2.2 induces a Smith homomorphism on bordism, it dually induces a map of invertible field theories. Explicitly, we obtain this by mapping into the Anderson dual and taking homotopy groups. As in Section 3.2.1, consider a topological space X, a tangential structure $\xi\colon B \to B\mathrm{O}$, a virtual bundle $V \to X$, and a vector bundle $W \to X$ of rank r. Recall that the tangential structure encodes both symmetries of the theory as well as the parameter space.

Definition 3.7.1. The *defect anomaly map*, generalized from [113] section 4.2, is the map

$$\mho_\xi^{k-r}(X^{V+W-r}) \xrightarrow{\mathrm{Def}_W} \mho_\xi^k(X^V) \tag{3.7.2}$$

of invertible field theories induced by the map $\mathrm{sm}_W\colon X^V \to X^{V+W}$.

Physically, we interpret $\mho_\xi^k(X^V)$ to be classifying anomalies of QFTs in dimension $k-1$ with symmetry according to a (X,V)-twisted ξ-structure. The group mapping in, $\mho_\xi^{k-r}(X^{V+W-r})$, classifies the anomalies of defect theories in dimension $k-r-1$. These defect theories are created from the bulk theory, physically, by setting a non-trivial boundary condition on a symmetry-breaking order parameter, which corresponds to a section of the vector bundle W. See Section 3.8 for further explanation.

For clarity, we begin a running example.

Example 3.7.3. Recall the $\mathbb{Z}/2$ family of examples from Example 3.6.8 and specifically Equation (3.6.9d):

$$MT\mathrm{Pin}^+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)_+. \tag{3.7.4}$$

Here, we take ξ according to spin bordism and twist with the tautological line bundle $W = \sigma$ over $X = B\mathbb{Z}/2$. We take $V = 3\sigma$ and apply the results of Section 3.5 to simplify (i.e., we use the that 4σ is spin). The corresponding map on invertible field theories is

$$\mho^{k-1}_{\mathrm{Spin}\times\mathbb{Z}/2} \xrightarrow{\mathrm{Def}_\sigma} \mho^k_{\mathrm{Pin}^+}. \tag{3.7.5}$$

Physically, we begin with a field theory with a pin$^+$ symmetry; that is, a fermionic theory with an additional time reversal symmetry T that squares to fermion parity: $T^2 = (-1)^F$. We require the physical assumption that there is a $\mathbb{Z}/2$-odd bosonic operator ϕ such that the theory is gapped when the theory is deformed by ϕ. One example is the Majorana mass term for $2+1$D Majorana fermions. We can define the theory on any manifold M with a pin$^+$ structure $P\colon M \to B\mathrm{Pin}^+$ on its stable tangent bundle. If we choose generic configuration for the ϕ field, i.e., choose a section of the tautological line bundle $P^*\sigma$, we arrive at an effective theory whose excitations are localized at the zero set of ϕ. We view this as a theory in one dimension lower, which is called a *domain wall theory.* Note that the bordism class of the zero set of ϕ is precisely the image of $[M]$ under sm_σ in (3.7.4). The domain wall theory no longer has the symmetry of the bulk theory: instead, it is a fermionic theory with a unitary internal symmetry U squaring to 1; i.e. it has $\mathrm{Spin}\times\mathbb{Z}/2$ tangential structure. See Section 3.8.2 and [113, §3.1] for more details.

◇

In this context, anomaly matching refers to the process of identifying pairs of preimages and images of anomaly classes under the defect anomaly map Def_W. To perform anomaly matching, we must understand not only the two groups classifying the possible bulk and

Smith fiber sequences	SBLES
Example 3.6.26	[10, §IV.A]
Example 3.6.6	[10, §IV.B]
Example 3.6.8	[10, §IV.C]
Example 3.6.33	[10, §IV.D & E]
Examples 3.6.39 and 3.6.45	[10, §IV.F]

Table 3.1: Here we present a cross-list of Smith fiber sequences from [9] with the corresponding symmetry breaking long exact sequences (SBLES) in [10, §IV]. We have added internal cross references within this thesis as well.

defect theories, but also the kernel and cokernel of the map Def_W. In some cases, one may deduce that information from an understanding of explicit bordism generators, but in general this approach is difficult. To address this question, we derived the map of spectra and identified its fiber, forming the Smith fiber sequence of Section 3.4. Now, just as for bordism, we may form a long exact sequence.

Corollary 3.7.6. *Applying $I_{\mathbb{Z}}$ to the cofiber sequence (3.4.3), we obtain the following long exact sequence of Anderson-dualized bordism groups, or in light of Theorem 2.4.21, groups of invertible field theories:*

$$\cdots \longrightarrow \mho_\xi^{k-r}(X^{V+W-r}) \xrightarrow{\mathrm{Def}_W} \mho_\xi^{k}(X^{V}) \xrightarrow{\mathrm{Res}_W} \mho_\xi^{k}(S_X(W)^{V}) \xrightarrow{\mathrm{Ind}_W} \mho_\xi^{k-r+1}(X^{V+W-r}) \longrightarrow \cdots \tag{3.7.7}$$

Recall the notation from Notation 2.4.22.

This long exact sequence is our mathematical model for the symmetry-breaking long exact sequence (SBLES) of Section 3.8. In addition to the defect anomaly map Def_W defined in Definition 3.7.1, we call Res_W the *residual anomaly map* and Ind_W the *index anomaly map*.

$$\begin{tikzcd}
 & \mho^k_{\mathrm{Spin}\times\mathbb{Z}/2} \arrow[r, "\mathrm{Def}_\sigma"] & \mho^{k+1}_{\mathrm{Pin}^+} \arrow[r, "\mathrm{Res}_\sigma"] & \mho^{k+1}_{\mathrm{Spin}} \\
-1 & 0 & 0 & \mathbb{Z} \arrow[dll, rounded corners] \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 & 0 \\
1 & 0 & 0 & \mathbb{Z}/2 \arrow[dll, rounded corners] \\
2 & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 & \mathbb{Z}/2 \arrow[dll, rounded corners] \\
3 & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 & \mathbb{Z} \arrow[dll, rounded corners] \\
4 & \mathbb{Z}\oplus\mathbb{Z}/8 \arrow[r] & \mathbb{Z}/16 & 0
\end{tikzcd}$$

Figure 3.1: Long exact sequence of field theories associated to Equation (3.6.9d). Observe that all maps in low degrees are determined by exactness. This long exact sequence also appears in [10, §IV.C]

Example 3.7.8. In degree $k = 4$, there is a map of invertible field theories

$$\mho^3_{\mathrm{Spin}\times\mathbb{Z}/2} \cong \mathbb{Z}\oplus\mathbb{Z}/8 \xrightarrow{\mathrm{Def}_\sigma} \mathbb{Z}/16 \cong \mho^4_{\mathrm{Pin}^+}. \tag{3.7.9}$$

The $\mathbb{Z}/16$ classifies anomalies of Majorana fermions in $2+1$ dimensions. An associated domain wall theory often has $1+1$d chiral fermion modes. To answer the question of what particular chiral fermions can live on the domain wall, we need to identify the map Def_σ in (3.7.9). To do so, we turn to the long exact sequence of invertible field theories, which we draw out in Figure 3.1.

By exactness, we deduce that the defect matching map in degree $k = 4$ sends $(a, b) \in \mathbb{Z}\oplus\mathbb{Z}/8$ to $-a + 2b \in \mathbb{Z}/16$. This matches with the physical computation in [113, §3.1], and its physical significance is discussed further there and in Section 3.8.3.2.

◇

As a computational tool, the long exact sequence allows us to determine the defect matching maps with ease. Moreover, the other two maps in the SBLES (3.7.7) also have physical interpretations. The residual anomaly map classifies the obstruction to gapping a QFT after symmetry breaking, as we explain in Section 3.8.1, while the index anomaly map generalizes the relationship between Berry phases and the ground-state degeneracy in $0+1$D systems; see Section 3.8.3.

3.8 Physics of the Symmetry Breaking Long Exact Sequence

This section is adapted from [10, Section III], which is joint work of the author with Arun Debray, Sanath Devalapurkar, Yu Leon Liu, Natalia Pacheco-Tallaj, and Luuk Stehouwer.

We now provide more detailed physical interpretations of the maps of field theories in the sequence (3.7.7). We thus specialize the long exact sequence and make a change of notation to align more with the physics literature, and specifically [265]. We narrow our focus to the following situations:

- $MT\xi$ is $MT\mathrm{Spin}$ or $MT\mathrm{SO}$,
- $X = BG$ for G a finite or Lie group,
- V and W arise from G-representations. Often, V and W will both be multiples of a fixed representation.

The physical process of symmetry breaking that we describe depends on the data of a *symmetry breaking pattern*, which is a pair of a group G and an orthogonal representation ρ of G. In our examples, G will be a finite or Lie group. We will write $W \to BG$ for the associated vector bundle to ρ. Many of the physical theories we discuss are specified by their Hamiltonian operators, and the process of symmetry breaking—modeled by the defect anomaly map (Definition 3.7.1)—involves deforming the original Hamiltonian by adding *symmetry-breaking operators*, which, instead of being preserved under an action of G, transform under ρ.

Here, in the degree, we replace k with $D+2$, where D is interpreted to be the *spacetime* dimension of the anomalous quantum field theory of study, so that $D+1$ is the dimension of the invertible field theory in one dimensional higher that mathematically models its anomaly.

Definition 3.8.1 (Symmetry Breaking Long Exact Sequence (SBLES))**.** When $X = BG$, W is of rank r, and $MT\xi = MT\mathrm{Spin}$, the long exact sequence of invertible field theories specializes to

$$\cdots \longrightarrow \mho^{D+1-r}_{\mathrm{Spin}}(BG^{V+W-r}) \xrightarrow{\mathrm{Def}_W} \mho^{D+1}_{\mathrm{Spin}}(BG^{V}) \xrightarrow{\mathrm{Res}_W} \mho^{D+1}_{\mathrm{Spin}}(S_{BG}(W)^{V}) \xrightarrow{\mathrm{Ind}_W} \mho^{D+1-r+1}_{\mathrm{Spin}}(BG^{V+W-r}) \longrightarrow \cdots \tag{3.8.2}$$

This sequence models symmetry breaking for *fermionic* theories. For *bosonic* theories, we instead take $MT\xi = MT\mathrm{SO}$, and get the sequence

$$\cdots \longrightarrow \mho^{D+1-r}_{\mathrm{SO}}(BG^{V+W-r}) \xrightarrow{\mathrm{Def}_W} \mho^{D+1}_{\mathrm{SO}}(BG^{V}) \xrightarrow{\mathrm{Res}_W} \mho^{D+1}_{\mathrm{SO}}(S_{BG}(W)^{V}) \xrightarrow{\mathrm{Ind}_W} \mho^{D+1-r+1}_{\mathrm{SO}}(BG^{V+W-r}) \longrightarrow \cdots \tag{3.8.3}$$

By anomaly inflow (Section 2.5.1), we can look at this long exact sequence either from the $D+1$-dimensional point of view of the invertible field theories, or from the D-dimensional point of view of the anomalous theories. From the latter perspective, these groups are

- $\mho^{D+1-r}_{\xi}(BG^{V+W-r})$: anomalies of fermionic ($\xi = \mathrm{Spin}$) or bosonic ($\xi = \mathrm{SO}$) G-symmetric theories in $D-r$ spacetime dimensions, twisted by $V+W$,

- $\mho_\xi^{D+1}(BG^V)$: anomalies of fermionic ($\xi = \mathrm{Spin}$) or bosonic ($\xi = \mathrm{SO}$) G-symmetric theories in D spacetime dimensions, twisted by V, and
- $\mho_\xi^{D+1}(S_{BG}(W)^V)$: anomalies of G-equivariant fermionic ($\xi = \mathrm{Spin}$) or bosonic ($\xi = \mathrm{SO}$) families of theories parameterized by the unit sphere $S(W) \simeq S^{r-1}$ twisted by V.

We physically interpret the three maps in the sequence as

- Res_W: Measures the *residual family anomaly* of the D-dimensional theory after breaking the symmetry by an operator transforming in the representation ρ whose associated bundle is W. (Section 3.8.1)
- Def_W: Describes the reconstruction of the bulk anomaly from the anomaly on a certain *defect* associated with this symmetry breaking, such as a domain wall. (Section 3.8.2)
- Ind_W: Encodes a generalized index theorem which associates an anomalous defect to a certain winding configuration in the space of symmetry-broken states. (Section 3.8.3)

Exactness of the sequence has several physical consequences. For example, the anomalies that have no residual family anomaly, and so live in the kernel of Res_W, are precisely those which can be associated with a special defect (which we call a ρ-defect or W-defect) whose anomaly recovers the original anomaly by the map Def_W. We examine some long subsequences of the whole structure in Section 3.8.4.

3.8.1 Residual family anomalies

This subsection is adapted from [10, IIIA].

Consider the second map in the long exact sequence, Res_W, which is induced by the pullback of the bundle V to the sphere bundle $S_{BG}(W)$. We call this map the *residual anomaly map*, since we view the image of this map as the anomaly left over after a theory is perturbed by symmetry-breaking operators. The residual anomaly map

$$\mathrm{Res}_W\colon \mho_\xi^{D+2}(BG^V) \longrightarrow \mho_\xi^{D+2}(S_{BG}(W)^V) \tag{3.8.4}$$

sends a class ω, which is the anomaly of a D-dimensional QFT, to a class $\mathrm{Res}_W(\alpha)$, which is the anomaly of a D-dimensional *family* of QFTs formed from the original theory by extending over the sphere $S_{BG}(W)$.

Note that the residual family anomaly generalizes the anomaly of the original theory with unbroken symmetry. The element of the family at some fixed vector $v \in S(W)^V$ is a theory which may have an anomaly for some unbroken subgroup of G, which prevents it from being gapped. However, even if each individual element of the family has zero anomaly, the family itself can still have a nontrivial anomaly even if the symmetry is completely broken. See Section 3.8.1.1.

One situation in which the residual anomaly for a fixed $v \in S(W)^V$ *does* determine the residual family anomaly is when the group G acts transitively on the representation sphere $S(\rho)$. In this case, there is an isomorphism[15]

$$\mho_\xi^{D+1}(S_{BG}(W)^V) \cong \mho_\xi^{D+1} \tag{3.8.5}$$

---

[15] Recall our stipulation that V is virtual dimension zero.

In our worked examples, this will occur for the following cases:

1. $G = \mathbb{Z}/2$ and $\rho = \sigma$ is the 1-dimensional sign representation.
2. $G = \mathrm{U}(1)$ and ρ is the (real) 2-dimensional charge 1 representation.
3. $G = \mathrm{SU}(2)$ and $\rho = V_4$ is the (real) 4-dimensional fundamental representation.

In each of these cases, BG has the form of an infinite projective space, and the associated bundle $W \to BG$ is the tautological bundle. The untwisting of the group over the twisted sphere is related to the triplet of Smith isomorphisms in Equations (3.6.1), (3.6.21), and (3.6.40) For these cases, the family anomaly reduces to pure gravitational anomaly. We will repeatedly use this in Section 3.8.5.

Next, we explain how this particular form of anomaly tracking fits into the general program of anomalies and gapping in high energy physics. Recall that a nontrivial 't Hooft anomaly for a theory with G symmetry obstructs the deformation of that theory into a nondegenerate gapped phase. In the absence of gravitational anomalies, a theory can always be nondegenerately gapped if the symmetry is allowed to be broken down to a (possibly trivial) anomaly-free subgroup $H \subset G$. The residual anomalies defined above target a more refined question: instead of allowing the symmetry G to be broken down to an arbitrary subgroup, we can ask whether a theory becomes gappable after being deformed by a certain family of symmetry-breaking operators transforming according to a representation ρ of G.

Physics Definition 3.8.6. A theory is *(nondegenerately) ρ-gappable* if there exists an operator transforming in the representation ρ such that for all large enough perturbations by this operator, the theory has a (nondegenerate) gapped ground state.

Equivalently, the ground state for all large enough symmetry breaking fields is *uniformly (nondegenerately) gapped*, meaning there is a uniform lower bound on the energy gap above the ground states (and further the ground state is unique). For this paper, "nondegenerately" will always be implied. This ρ-gappability condition is equivalent to the existence of a local "ρ-defect", defined in Section 3.8.2 below, and whose creation we mathematically model using the map Def_W. More specifically, we posit that a theory admits a ρ-defect if and only if it is in the image of the map Def_W.

This ρ-gappability condition is nontrivial: theories are not always ρ-gappable for every choice of (G, ρ). The most obvious obstruction to ρ-gappability occurs when the modified symmetry group is still anomalous. However, there are additional obstructions that can exist even when all remaining symmetries are anomaly-free, which arise as nontrivial configurations of a family of theories. We refer to these as parameter space anomalies and higher Berry phases [266–270]. Using the SBLES (Definition 3.8.1), we can immediately pose a very general obstruction to ρ-gappability: a D-dimensional theory ω with anomaly $\omega \in \mho_\xi^{D+1}(BG^V)$ is ρ-gappable only if $\mathrm{Res}_W(\omega) = 0$: by exactness, this is the condition that ω lies in the image of the map Def_W.

3.8.1.1 Example: $2 + 1$D Majoranas

Here is a simple example of a theory with a residual family anomaly that is nontrivial even though the symmetry is completely broken. Consider a single Majorana fermion[16] ψ in 2+1D transforming under time reversal with $T^2 = (-1)^F$. This theory is known to be anomalous, and is associated with the generator the group $\mho^4_{\mathrm{Pin}^+} = \mathbb{Z}/16$ of 3+1D SPTs [271]. This and related symmetry breaking patterns are discussed later in Section 3.8.5.3.

The mass term $m\bar{\psi}\psi$ is T-odd and completely gaps the theory, so for σ the sign representation of $\mathbb{Z}/2$, the theory is σ-gappable. Indeed, the map

$$\begin{gathered} \mho^4_{\mathrm{Spin}}((B\mathbb{Z}/2)^{3\sigma-3}) \overset{(3.2.3)}{\cong} \mho^5_{\mathrm{Pin}^+} \xrightarrow{\mathrm{Res}_\sigma} \mho^4_{\mathrm{Spin}}(S(\sigma)^{3\sigma-3}) \overset{(3.8.5)}{\cong} \mho^4_{\mathrm{Spin}} \\ \mathbb{Z}/16 \xrightarrow{\qquad\qquad\mathrm{Res}_\sigma\qquad\qquad} 0, \end{gathered} \tag{3.8.7}$$

which corresponds to the sequence induced by (3.6.9d), is zero. However, if we take 2σ, or equivalently the π-rotation representation of $\mathbb{Z}/2$, this theory turns out not to be 2σ-gappable, since the map

$$\begin{gathered} \mho^4_{\mathrm{Pin}^+} \xrightarrow{\mathrm{Res}_{2\sigma}} \mho^4_{\mathrm{Spin}}(S(2\sigma)^{3\sigma-3}) \cong \tilde{\mho}^5_{\mathrm{Spin}}(\mathbb{RP}^2) \\ \mathbb{Z}/16 \xrightarrow{\qquad\mathrm{mod}\ 2\qquad} \mathbb{Z}/2 \end{gathered} \tag{3.8.8}$$

of row 4 of the last sequence in Section 3.8.5.3 is nonzero on the generator. This means that for *any* pair of T-odd operators $\mathcal{O}_1$, $\mathcal{O}_2$, and for any radius r, there exists a parameter θ such that with the symmetry breaking field

$$r\cos\theta\mathcal{O}_1 + r\sin\theta\mathcal{O}_2, \tag{3.8.9}$$

the theory is *not* nondegenerately gapped. As a somewhat trivial example, if we take $\mathcal{O}_1$ and $\mathcal{O}_2$ to both be the (same) T-odd mass term, then we can always balance the coefficients so they cancel and we have the massless Majorana. This gives a phase diagram as in Fig. 3.2.

We interpret the phase diagram as follows. Going around the circle by an angle of π is equivalent to changing the sign of the mass. Majoranas with opposite masses differ by an invertible phase known as the $p + ip$ *superconductor*, so we can say that the invertible family pumps a $p + ip$ superconductor or its inverse, a $p - ip$ superconductor, to the boundary as it crosses the $m = 0$ values of the angle—see Fig. 3.2. Observe that nothing is pumped going around the *entire* circle[17], since the $p + ip$ and $p - ip$ are inverse phases and thus cancel. However, this *family* is still nontrivial, which can be seen as follows. In such a family we can go adiabatically half way around the circle then return to where we started by applying time reversal, which acts as a π -rotation. The invertible phase pumped to the boundary over such a cycle is a sort of equivariant generalization of the Thouless charge pump.

---

[16] That is, a real fermion, which comes from a real Clifford algebra representation that restricts to the spin group.

[17] We will see this is a general feature of families occurring in the image of the gapping obstruction in Section 3.8.4.

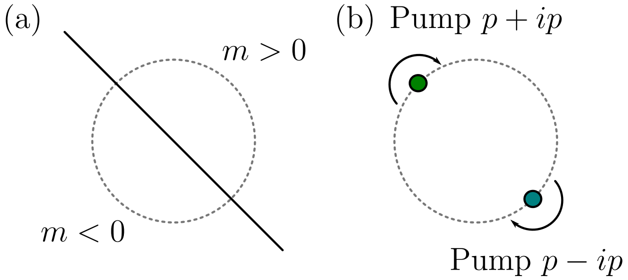

Figure 3.2: (a) The phase diagram of a single 2+1D Majorana with redundant mass term. Time reversal acts on this phase diagram by a π rotation. The solid black line marks where the Majorana is massless, and the dotted circle represents $S(2\sigma)$. (b) A representation of the 3+1D invertible family, where upon crossing either the green or blue dot, a $p+ip$ or $p-ip$ superconductor is pumped to the boundary. Observe that there is no total pump in going around the entire circle. However, with time reversal, this family is non-trivial, as can be measured by going half-way around the circle and then applying time reversal to return to the starting point. The number of $p+ip$'s pumped mod 2 this way is an invariant of the equivariant family.

The fact that this family is nontrivial implies that the Majorana is not 2σ-gappable for *any* pair of T-odd operators, not just the redundant mass terms. For example, we may take $\mathcal{O}_1$ to be the mass term and $\mathcal{O}_2$ to be any other T-odd operator, such as $(\bar{\psi}\psi)^3$.

3.8.2 Defect anomaly matching

This subsection is adapted from [10, IIIB].

The next map in the long exact sequence is the defect anomaly map (Definition 3.7.1)

$$\mho_\xi^{D+2-r}(BG^{V+W-r}) \xrightarrow{\mathrm{Def}_W} \mho_\xi^{D+2}(BG^V), \tag{3.8.10}$$

which is the map induced by the Smith map sm_W. We already discussed this map in Section 3.7, but here we give some additional physical interpretations.

Assume now that there is no residual family anomaly; i.e. that $\mathrm{Res}_W(\alpha) = 0$, so we assume the system is nondegenerately gapped for all values of the symmetry breaking parameter $v \in S(\rho)$. In this case, we argued in the previous section that we should be able to construct a localized *ρ-defect*. Explicitly, this defect is created by adding a term to the Hamiltonian that depends on a symmetry breaking order parameter ϕ, which after we set boundary conditions in the theory, gives a choice of section $\phi\colon BG \to W$. Condensing the order parameter to zero, then, has the effect of intersecting this section ϕ with the zero section of W, thus concentrating the defect theory on the submanifold in the image of the Smith homomorphism sm_W.

We view (3.8.10) as modeling the inverse process of reconstructing the bulk anomaly α from the defect anomaly. Following [113], this can be done on the level of Hamiltonian operators by comparing the symmetries of each theory. By combining the G action with Lorentz symmetries and using the CPT theorem, in many examples a new "G_ρ" symmetry can be defined that holds in the effective $D-r$-dimensional defect theory. The anomaly of the defect theory can be physically observed by computing the transformations of the localized modes of the theory under the symmetry G_ρ.

Remark 3.8.11. We would like to compare and contrast our situation with a more standard story. In spontaneous symmetry breaking, the topologically protected defects of codimension n are classified by $\pi_{n-1}(G/H)$ [272], where H is the unbroken symmetry group. This coincides with our ρ-defect (with $k = n$) precisely in the case where G acts transitively on the sphere $S(\rho)$, in which case $S(\rho) \cong G/H$ and the ρ defect is a G-symmetric generator of $\pi_{k-1}(S(\rho)) = \mathbb{Z}$. In cases where $S(\rho)$ consists of many G orbits, the ρ-defect is not among these topologically protected defects, but is still of interest for anomaly-matching, as we will see.

Note that the defect anomaly α determines the bulk anomaly ω, but not vice versa, and in particular even anomaly-free symmetries can have anomalous ρ-defects, as we see in in Section 3.8.3. Note also that exactness requires that $\mathrm{Res}_W \circ \mathrm{Def}_W = 0$, meaning that the anomalies in the kernel of Res_W—the anomalies that we say do not have a residual family anomaly–are precisely the anomalies in the image of Def_W—the anomalies that we expect can be reconstructed from the ρ-defect. For this reason, we expect that Res_W is the only obstruction to ρ-gappability as defined earlier. This condition generalized Theorem 4.2 in [113] from finite cyclic groups to arbitrary Lie groups, answering the question posed there about the cokernel of Def_W.

3.8.2.1 3+1D Dirac fermion

Let us analyze the sequence

$$\begin{tikzcd}
\mho^4_{\mathrm{Spin}}(B\mathrm{U}(1)^{L-2}) \cong \mho^4_{\mathrm{Spin}^c} \arrow[r, "\mathrm{Def}_L"] & \mho^6_{\mathrm{Spin}}(B\mathrm{U}(1)) \arrow[r, "\mathrm{Res}_L"] & \mho^6_{\mathrm{Spin}} \\
\mathbb{Z}^2 \arrow[r, "\cong"] & \mathbb{Z}^2 \arrow[r] & 0
\end{tikzcd} \tag{3.8.12}$$

induced by (3.6.27b). Recall that $L \to B\mathrm{U}(1)$ corresponds to the charge one representation of $\mathrm{U}(1)$.

Consider a 3+1D Dirac fermion ψ (with four complex components). This has an anomalous chiral symmetry $\mathrm{U}(1)_L$[18] which gives charge 1 to the two left-handed components of ψ and charge 0 to the two right handed ones. There are two Dirac masses $\bar{\psi}\psi$ and $i\bar{\psi}\gamma^5\psi$, which transform together under $\mathrm{U}(1)_L$ as a charge 1 doublet ρ. Any combination of the two mass terms completely gaps the fermion, so in this case there is no residual family anomaly and there should be a local ρ-defect, as we can see from the second map $\mathrm{Res}_L = 0$ in (3.8.12).

The ρ-defect in this theory is constructed by setting a spatially-varying mass profile of the form

$$x_1\bar{\psi}\psi + x_2 i\bar{\psi}\gamma^5\psi. \tag{3.8.13}$$

Solving the Dirac equation for localized modes with this mass profile yields a massless 1+1D Weyl fermion (with one complex component) propagating in the remaining coordinates [80].

---

[18] Here L stands for *left*, not the bundle over $B\mathrm{U}(1)$.

The residual symmetry $\mathrm{U}(1)_\rho$ acting on the 1+1D ρ-defect acts as a combination of a $\mathrm{U}(1)_L$ rotation and a compensating Spin(2) rotation, where Spin(2) is the rotation in the x_1, x_2 plane, such that the mass profile is invariant under their combination. In particular, a 2π $\mathrm{U}(1)_\rho$ rotation is equal to a 2π rotation of this plane, which equals the fermion parity $(-1)^F$. In other words, the 1+1D system has a $\mathrm{Spin}^c = (\mathrm{Spin} \times U(1)_\rho)/\mathbb{Z}/2$ structure. A general anomaly for such a theory is given by a Chern-Simons form associated with a 4D integer bordism invariant

$$\alpha = k_1\left(\frac{1}{8}(c_1^\rho)^2 - \frac{1}{24}p_1(TY)\right) + k_2(c_1^\rho)^2, \tag{3.8.14}$$

where $k_1, k_2 \in \mathbb{Z}$.

The map $\mathrm{Def}_\rho(\alpha)$ can be computed in terms of these 4D bordism invariants: since the map of invertible field theories we are considering in this case is torsion-free, we lose nothing by rationalizing (Section 2.2.2) and considering the corresponding map

$$H^4(B\mathrm{Spin} \times B\mathrm{U}(1); \mathbb{Q}) \to H^6(B\mathrm{Spin} \times B\mathrm{U}(1); \mathbb{Q}). \tag{3.8.15}$$

Explicitly, suppose X is a closed spin 6-manifold with a principal $\mathrm{U}(1)_L$-bundle P and that ϕ is a section of the complex line bundle $L = P \times_{\mathrm{U}(L)_L} \gamma$ associated to the charge 1 representation γ. Its image under the Smith homomorphism sm_L is the zero set of ϕ—i.e., the intersection of this section with the zero section—which is a 4-manifold Y whose homology class $[Y] \in H_4(X;\mathbb{Z})$ is Poincaré dual to the first Chern class $c_1^L \in H^2(X;\mathbb{Z})$.[19] Here, Y represents the spacetime submanifold on which the γ-defect theory is concentrated. For any class $\beta \in H^4(X,\mathbb{Z})$,

$$\int_X c_1^L \beta = \int_Y \beta. \tag{3.8.16}$$

To compute $\mathrm{Def}_\rho(\alpha)$, we must find β such that $\beta|_Y = \alpha$, so that by definition we then have $\mathrm{Def}_\rho(\alpha) = c_1^L\beta$. Working backwards, we first try to match the $(c_1^\rho)^2$ terms in (3.8.14). Since the $\mathrm{U}(1)_\rho$ bundle over Y is the restriction of the $\mathrm{U}(1)_L$ bundle over X, we have $c_1^L|_Y = c_1^\rho$. In terms of the defect anomaly map, this means to get $(c_1^\rho)^2$ we should take $\beta = (c_1^L)^2$, so

$$\mathrm{Def}_\rho((c_1^\rho)^2) = (c_1^L)^3. \tag{3.8.17}$$

Matching the *gravitational* term (which involves the spacetime curvature through $p_1(TY)$ and not the principal bundle) is more involved. Since Y was cut out of X as the zero set of ϕ, we have the normal sequence

$$TX|_Y = TY \oplus NY = TY \oplus E_\gamma|_Y, \tag{3.8.18}$$

where the normal bundle NY is identified with the restriction of the associated bundle $E_\gamma = L$. By the Whitney sum formula,

$$\begin{aligned} p_1(TX)|_Y &= p_1(TY) + p_1(E_\rho)|_Y \\ &= p_1(TY) + (c_1^L)^2|_Y \\ &= p_1(TY) + (c_1^\rho)^2. \end{aligned} \tag{3.8.19}$$

---

[19] As we observed in [10, Appendix B], this trick of considering the *cohomology* Euler class does not always suffice, and in general we need the Anderson-dual bordism Euler class. But here, it is fine.

So to get

$$\alpha = \frac{1}{8}(c_1^\rho)^2 - \frac{1}{24}p_1(TY), \tag{3.8.20}$$

we take

$$\beta = \frac{1}{6}(c_1^L)^2 - \frac{1}{24}p_1(TX), \tag{3.8.21}$$

hence

$$\mathrm{Def}_\rho\left(\frac{1}{8}(c_1^\rho)^2 - \frac{1}{24}p_1(TY)\right) = \frac{1}{6}(c_1^L)^3 - \frac{1}{24}c_1^L p_1(TX). \tag{3.8.22}$$

This turns out to precisely coincide with the $\mathrm{U}(1)_L$ anomaly of the 3+1D Dirac fermion. Thus defect anomaly matching requires $k_1 = 1$, $k_2 = 0$ in (3.8.14). This is consistent with a 1+1D Weyl fermion with $\mathrm{U}(1)_\rho$ charge 1.

The above calculation seems to rely on a choice of β. Actually, it does not, since if $\beta'|_Y = \alpha$, $(\beta - \beta')|_Y = 0$, and so, using Poincaré duality, $c_1^L\beta - c_1^L\beta' = c_1^L(\beta - \beta') = 0$. On the other hand, the existence of such a β is guaranteed by the vanishing of the residual family anomaly, since this guarantees that $\int_X \omega = \int_Y \beta$ for some β.

3.8.2.2 3+1D Weyl fermion

To see the importance of the choice of representation in the above computation, let us consider a closely related example, this time beginning with a left-handed Weyl fermion in $3+1$D. This has a $\mathrm{U}(1)_L$ symmetry with the same anomaly as the Dirac in Section 3.8.2.1 (since the right-handed Weyl does not contribute anything):

$$\omega = \frac{1}{6}(c_1^L)^3 - \frac{1}{24}c_1^L p_1(TX). \tag{3.8.23}$$

Above we studied the Dirac mass, which couples the two Weyl components. However, a single Weyl on its own has a Majorana mass that is *charge 2* under $\mathrm{U}(1)_L$. Solving the equations of motion for the associated ρ-defect we find a left-handed Majorana-Weyl fermion in 1+1D. This has one real component, so $\mathrm{U}(1)_\rho$ must act trivially on it.

Let us compute the defect anomaly map in this case and verify that this matches. Note that a 2π rotation in $\mathrm{U}(1)_\rho$ is a 4π rotation in V_ρ, which is 1 on the fermion, so there is no Spin^c business here. Anomalies of 1+1D fermions with $\mathrm{Spin} \times U(1)_\rho$ symmetry split between a pure gravity and a pure symmetry part, and take the form

$$\alpha = \frac{k_1}{48}p_1(TY) + k_2(c_1^\rho)^2. \tag{3.8.24}$$

The calculation proceeds as above, although now $[Y] \in H_4(X, \mathbb{Z})$ is Poincaré dual to $2c_1^L \in H^2(X, \mathbb{Z})$, since ρ is a charge 2 representation. Once we compute β such that $\beta|_Y = \alpha$, we will have $\mathrm{Def}_\rho(\alpha) = 2c_1^L\beta$.

Since 2γ is associated to $2L$, two copies of the tautological bundle over $B\mathrm{U}(1)$, this situation corresponds to the dual of the Smith map

$$\mathrm{sm}_{2L} \colon MT\mathrm{Spin} \wedge B\mathrm{U}(1) \longrightarrow MT\mathrm{Spin} \wedge B\mathrm{U}(1)^{2L} \simeq \Sigma^4 MT\mathrm{Spin} \wedge B\mathrm{U}(1). \tag{3.8.25}$$

We did not study this Smith map above because it is somewhat degenerate: since $2L$ is spin, this example does not feature a twisting of tangential structures. The map on invertible field theories is

$$\begin{tikzcd}
\mho^2_{\mathrm{Spin}}(B\mathrm{U}(1)) \arrow[r, "\mathrm{Def}_{2L}"] & \mho^6_{\mathrm{Spin}}(B\mathrm{U}(1)) \\
\mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r] & \mathbb{Z}.
\end{tikzcd} \tag{3.8.26}$$

Rationalizing, we study the map

$$\begin{tikzcd}
H^2(B\mathrm{Spin} \times B\mathrm{U}(1); \mathbb{Q}) \arrow[r] & H^2(B\mathrm{Spin} \times B\mathrm{U}(1); \mathbb{Q}) \\
\mathbb{Q} \arrow[r] & \mathbb{Q}^2.
\end{tikzcd} \tag{3.8.27}$$

Using $c_1^L|_Y = c_1^\rho$, we find

$$\mathrm{Def}_\rho((c_1^\rho)^2) = 2(c_1^L)^3. \tag{3.8.28}$$

We also have

$$\begin{aligned} p_1(TX)|_Y &= p_1(TY) + p_1(E_\rho)|_Y \\ &= p_1(TY) + 4(c_1^L)^2|_Y \\ &= p_1(TY) + 4(c_1^\rho)^2. \end{aligned} \tag{3.8.29}$$

Thus we find

$$\mathrm{Def}_\rho\left(\frac{1}{48}p_1(TY)\right) = \frac{1}{6}(c_1^L)^3 - \frac{1}{24}c_1^L p_1(TX), \tag{3.8.30}$$

so the defect anomaly matches correctly with $k_1 = 1$, $k_2 = 0$.

3.8.3 The index map and higher Berry phases

Above we described an anomaly matching condition in terms of the map Def_W, for which the image of the defect anomaly α is the bulk anomaly ω: $\mathrm{Def}_W(\alpha) = \omega$. Although the defect anomaly determines the bulk anomaly, the bulk anomaly does not determine the defect anomaly in the cases that Def_W is not injective. In particular, there can be anomalous defects ($\alpha \neq 0$) in anomaly-free bulk theories ($\omega = 0$).

We now turn to the third map in the sequence, the *index map*:

$$\mho_\xi^{D+1}(S_{BG}(W)^V) \xrightarrow{\mathrm{Ind}_W} \mho_\xi^{D+1-r+1}(BG^{V+W-r}). \tag{3.8.31}$$

Recall that as long as there is no residual family anomaly, we expect that we can perturb the theory so that for each large enough value of the symmetry-breaking field, we obtain a trivially gapped ground state. This defines a G-equivariant family of invertible field theories over the sphere $S(\rho)$. This family is not typically free of G-anomalies, but it is when $\omega = 0$. In this case, we can couple it to a G gauge field, and the invariant $\zeta \in \mho_\xi^{D+1}(S(W)^V)$ classifies its topological response. Given such a family with an anomaly ζ, we can construct the ρ-defect as before, and its anomaly is given by $\mathrm{Ind}_W(\zeta)$. Physically, we view this map as a compactification of ζ on the sphere $S(W) \simeq S^{r-1}$.

We can also consider elements of $\mho_\xi^{D+1}(S(W)^V)$ as D-dimensional counterterms that appear when relating different symmetry-breaking patterns of a given theory with the same representation ρ. In particular, we can compare two different G-equivariant $S(\rho)$-families of invertible field theories by stacking one with the orientation reversal of the other. The result is free of G-anomalies and defines an element of $\mho_{\xi}^{D+1}(S(W)^V)$. Thus, the image of Ind_W above describes both the ambiguity in the defect anomaly and the kernel of Def_W (answering the question about the kernel of the Smith homomorphism posed in [113]). This index map can also be thought of as an interacting generalization of the Callias index theorem [273, 274] which computes the fermion zero modes at the core of a mass defect. Our map gives the G-anomaly of those zero modes, which takes into account interactions.

Finally, the value of index map is the obstruction to extending an $S(W)$-family to a G-equivariant family over the disk bundle $D(W)$ In particular, the theory over the point $0 \in D(W)$ is a G-symmetric invertible field theory, which lives in the kernel of Ind_W and thus the image of Res_W. In terms of bulk-boundary correspondence, the index map is the obstruction to a G-equivariant family admitting a G-symmetry boundary condition which is *independent* of the parameters.

3.8.3.1 Thouless pump and vortices

We will consider the relationship between the index map and the famous *Thouless pump*, which is one of the simplest examples of topology in quantum physics [275]. In this kind of system, a periodic driving force or potential leads to a quantized amount of charge being *pumped* from one side of the system to the other.

Begin with a 1+1D Dirac fermion (with two complex components) with its anomaly-free U(1) symmetry

$$\psi \longmapsto e^{i\theta/2}\psi. \tag{3.8.32}$$

Then, add a U(1)-symmetric mass term

$$i((\cos\phi)\bar{\psi}\psi + i(\sin\phi)\bar{\psi}\gamma^c\psi), \tag{3.8.33}$$

where γ^c is the chirality operator $i\gamma^0\gamma^1$. The fact that this operator is symmetric, rather than symmetry-breaking, corresponds to the choice of ρ trivial, or $W = \underline{\mathbb{C}}$, the trivial line bundle. The mass term defines a U(1)-*symmetric* S^1-family of invertible field theories parametrized by ϕ. This family is nontrivial, and can be described by

$$\zeta(Z, A, \phi) = \frac{1}{2\pi}\int_Z d\phi\, A, \tag{3.8.34}$$

where Z is the 1+1D spacetime, A determines a spin^c structure, and $\phi\colon Z \to S^1$ is the order parameter. This term contributes an A current when adiabatically varying the S^1 parameter, leading to a quantized charge pump—the classic Thouless pump [275].

We expect the ρ-defect, which is the operator which creates a vortex in ϕ, to carry a unit A charge which matches the Thouless pump. In this case, the index map runs

$$\mathrm{Ind}_{\underline{\mathbb{C}}}\colon \mathbb{Z} \cong \mho^2_{\mathrm{Spin}^c}(S^1) \to \mho^1_{\mathrm{Spin}^c} \cong \mathbb{Z} \tag{3.8.35}$$

To compute the map, we will associate to ζ in (3.8.34) a partition function of 0+1D spacetimes Y (which are merely collections of oriented circles) equipped with a spin^c connection

A. We start by forming the associated $S(\rho)$ bundle over Y. Since ρ is trivial, this bundle is simply a product $W = S^1 \times Y$. The canonical section $\phi\colon W \to W \times S^1$ is the product of diagonal map $S^1 \to S^1 \times S^1$ and the identity map $Y \to Y$. In particular, $d\phi/2\pi$ is the volume form on the S^1 factor. It follows that

$$\begin{aligned}\mathrm{Ind}_\rho(\zeta)(Y, A, \phi) &= \zeta(W, \pi^* A, \phi) \\ &= \frac{1}{2\pi}\int_W d\phi\, \pi^* A \\ &= \int_{S^1} \frac{d\phi}{2\pi} \int_Y A \quad = \int_Y A,\end{aligned} \tag{3.8.36}$$

which is the generator of $\mho^1_{\mathrm{Spin}^c}$, as expected.

3.8.3.2 Time reversal domain wall for 2+1D Majorana fermions

Let us analyze an example from [113] of a situation with ambiguous defect anomaly. We study N_f 2+1D Majorana fermions ψ_j with time reversal

$$T\psi_j = \gamma^0 \psi_j, \tag{3.8.37}$$

which satisfies $T^2 = (-1)^F$. This has an anomaly $\omega = N_f \omega_4 \in \mho^4_{\mathrm{Spin}}(B\mathbb{Z}/2, 3\sigma) = \mho^4_{\mathrm{Pin}^+} \cong \mathbb{Z}/16$, where ω_4 is the generator corresponding to $N_f = 1$—it can be expressed as an eta invariant of the Dirac operator [276]. This example is also a member of the 4-periodic family discussed later in Section 3.8.5.5. It fits into the sequence induced by the Smith map Equation (3.6.9d).

Let us consider $N_f = 2$. Time reversal can be broken by mass terms like

$$\bar\psi_1 \psi_1 \pm \bar\psi_2 \psi_2. \tag{3.8.38}$$

(Note that each T-odd mass term transforms in the sign representation, which is ρ here.) On the time reversal domain wall there is a unitary $\mathbb{Z}/2$ symmetry U, whose anomaly group is given by $\mho^3_{\mathrm{Spin}\times\mathbb{Z}/2} \cong \mathbb{Z} \oplus \mathbb{Z}/8$, where the first part α_3 is purely gravitational and the second part $\alpha_3^{\mathbb{Z}/2}$ involves the internal symmetry U. It turns out that depending on the relative sign, the domain wall has different anomalous modes. If the sign is the same, on the wall there two 1+1d Majorana modes of the same chirality. However, for opposing signs, there are two Majoranas with opposite chirality. These clearly have distinct gravitational anomalies, and it turns out they have distinct U anomalies as well, with U acting trivially in the first case and chirally in the second case.

Although they have different anomalies, both must satisfy the defect anomaly matching condition. Since the defect map is linear, we can use the two data points above to compute it, and find, in terms of generators $k_1 \in \mathbb{Z}$, $k_2 \in \mathbb{Z}/8$,

$$\mathrm{Def}_\sigma(k_1 \alpha_3 + k_2 \alpha_3^{\mathbb{Z}/2}) = (k_1 - 2k_2)\omega_4, \tag{3.8.39}$$

where (k_1, k_2) is $(2, 0)$ or $(0, 1)$ in the two domain walls above, and both match the anomaly $2\omega_4$ as expected. We see that the kernel of Def_σ is generated by $(2, 1)$, as was noted in [113].

We can view this kernel as an ambiguity arising from Ind_ρ as follows: start by considering 2+1D $\mathbb{Z}/2^T$-equivariant families of invertible field theories over $S(\sigma)$. In this case, $S(\sigma) = S^0$ is just two points that get exchanged by T. The generator $\zeta \in \mho^3_{\mathrm{Spin}}(S(\sigma)^{p^*(3\sigma-3)})$ is defined by taking the generator $\alpha_3 \in \mho^3_{\mathrm{Spin}} = \mathbb{Z}$ over one of the two points, and its time-reversed partner $-\alpha_3$ over the other point. To calculate Ind_σ, we study the interface between these two invertible theories. The result is two fermions of equal chirality (with gravitational anomaly $2\alpha_3$), which are swapped by the induced $\mathbb{Z}/2$ symmetry U. This swap has eigenvalues ±1, and its anomaly is $\alpha_3^{\mathbb{Z}/2}$. So if ζ is the class of the family above,

$$\mathrm{Ind}_\sigma(\zeta) = 2\alpha_3 + \alpha_3^{\mathbb{Z}/2}, \tag{3.8.40}$$

the image of which is indeed the kernel of Def_σ we computed above. This has a physical interpretation in terms of the two mass terms above: changing the sign of just the $\bar\psi_2\psi_2$ mass term corresponds to stacking with either α_3 or $-\alpha_3$, depending whether the sign change is from minus to plus or from plus to minus. This gives the invertible family ζ above.

3.8.3.3 Vortices in $p + ip$ superfluid

Now we will discuss the famous Majorana zero modes bound to the vortices of a $p + ip$ superfluid [277], which turn out to have an interesting description in terms of the index map.

We study a single Dirac fermion in 2+1D, carrying charge 1 under $G = U(1)$ symmetry, and undergoing symmetry breaking via a charge 1 complex order parameter coupling to the two Majorana masses. Such a spontaneous symmetry breaking scenario is typically referred to as a $p+ip$ superfluid.[20] The resulting $S(\rho) \cong S^1$ family has a unique gapped ground state for all nonzero values of the order parameter, and the U(1) symmetry is anomaly-free, and it represents a generator of

$$\mho^3_{\mathrm{Spin},U(1),\rho}(S(\rho)) \cong \mathbb{Z}. \tag{3.8.41}$$

We want to compute the index map

$$\mathrm{Ind}^{\mathrm{U}(1)}_\rho \colon \mho^3_{\mathrm{Spin},U(1),\rho}(S(\rho)) \to \mho^2_{\mathrm{Spin},U(1)} \cong \mathbb{Z}/2. \tag{3.8.42}$$

It is interesting to consider the map

$$f\colon \mho^3_{\mathrm{Spin},U(1),\rho}(S(\rho)) \to \mho^3_{\mathrm{Spin}}(S^1) \cong \mathbb{Z}\oplus\mathbb{Z}/2 \tag{3.8.43}$$

which forgets the U(1) action, since the index map of the latter, namely

$$\mathrm{Ind}_\rho\colon \mho^3_{\mathrm{Spin}}(S^1) \to \mho^2_{\mathrm{Spin}} \tag{3.8.44}$$

can be more easily understood. The generators of $\mho^3_{\mathrm{Spin}}(S^1)$ correspond to the generator of $\mho^3_{\mathrm{Spin}} \cong \mathbb{Z}$, with trivial parameter dependence, and $\mho^2_{\mathrm{Spin}} = \mathbb{Z}/2$, via a family which pumps this phase to the boundary as we go around S^1. The index map clearly sends the $\mathbb{Z}$ generator to zero and the $\mathbb{Z}/2$ generator to the generator of $\mho^2_{\mathrm{Spin}} = \mathbb{Z}/2$.

Because the SBLES is functorial in G, we have a commutative square

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[20] Note that there is a mixed U(1) and time reversal anomaly, and a choice of U(1) symmetric fermion regulator will break time reversal and select either a $p + ip$ or $p - ip$ superfluid.

$$\begin{tikzcd}
\mathbb{Z} \cong \mho^3_{\mathrm{Spin},U(1),\rho}(S(\rho)) \arrow[r, "\mathrm{Ind}_\rho^{\mathrm{U}(1)}"] \arrow[d, "f"] & \mho^2_{\mathrm{Spin},U(1)} \cong \mathbb{Z}/2 \arrow[d, "\sim"] \\
\mathbb{Z} \oplus \mathbb{Z}/2 \cong \mho^3_{\mathrm{Spin}}(S^1) \arrow[r, "\mathrm{Ind}_\rho"] & \mho^2_{\mathrm{Spin}} \cong \mathbb{Z}/2.
\end{tikzcd}$$

Combined with the information above, we learn $\mathrm{Ind}_\rho^{\mathrm{U}(1)}$ must be reduction mod 2. This is reasonable from the physical point of view, since it is known that a vortex in the $p + ip$ superfluid binds an odd number of Majorana zero modes, which carry the gravitational anomaly associated with the generator of $\mho^2_{\mathrm{Spin}}$. We also learn that the map f above sends the generator to the sum of the generators $(1, 1) \in \mathbb{Z} \oplus \mathbb{Z}/2$, which is a bit more surprising! We will verify both these facts directly from the definition of these maps.

First we study f. In terms of spacetime manifolds, we want to take a 3-manifold X with spin structure ξ and a map $\phi\colon X \to S^1$, and construct a Spin^c structure A on X under which ϕ has charge 2, so that A gets Higgs'd to a spin structure. In terms of equations we want

$$\begin{aligned} 2A &= d\phi \\ dA &= \pi w_2(TX) = \pi d\xi, \end{aligned} \tag{3.8.45}$$

which can be solved by

$$A = \pi\xi + \frac{1}{2}d\phi. \tag{3.8.46}$$

The two terms here is the essential reason why we get the sum of generators when we compute f. It means when ϕ has an odd winding number around a 1-cycle of X, we twist the spin structure ξ along that cycle, turning it from periodic to antiperiodic or vice versa.

We do the same thing when we compute $\mathrm{Ind}_\rho^{\mathrm{U}(1)}$ according to the recipe given at the beginning of this subsection. There, from a spin surface Y we form the manifold $X = Y \times S^1$ with ϕ winding once around the S^1 factor. The spin structure along this S^1 factor becomes twisted. When we evaluate the $\mathbb{Z}$ generator of $\mho^3_{\mathrm{Spin}}$ on this spin 3-manifold, we get the Arf invariant of Y and its spin structure, which is the nontrivial element of $\mho^2_{\mathrm{Spin}}$.

3.8.4 Completing the circle

Consider the last piece of the sequence:

$$\mho_\xi^{D+1}(BG^V) \xrightarrow{\mathrm{Res}_W} \mho_\xi^{D+1}(S_{BG}(W)^V) \xrightarrow{\mathrm{Ind}_W} \mho_\xi^{D+1-r+1}(BG^{V+W-r}), \tag{3.8.47}$$

where exactness ensures that the kernel of Ind_W is the image of Res_W. Physically, suppose we start with a class $\omega \in \mho_\xi^{D+1}(BG^V)$, which we think of as a G-symmetric invertible theory. When we break the G-symmetry, we get an equivariant invertible family over the unit ball $D(\rho) \subset \rho$ of the representation. Restricting this family to the sphere $S(\rho)$ gives the class $\mathrm{Res}_W(\omega)$. Suppose we then compactify this family to form the invertible phase $\mathrm{Ind}_W(\mathrm{Res}_W(\omega))$ and study it on a manifold Y with boundary. If we place the family instead on the associated $D(\rho)$ bundle over ∂Y, then we get a trivial boundary condition of this invertible phase by gluing the two boundaries, which are both the associated $S(\rho)$ bundle over ∂Y. Since the ρ-defect in the invertible phase ω is trivial, it has trivial anomaly $\mathrm{Ind}_W\mathrm{Res}_W(\omega) = 0$. The converse follows from the Thom isomorphism.

3.8.5 Extended Examples

In this section we analyze a few longer segments of the SBLES, containing some of the examples of individual maps we have already seen.

3.8.5.1 U(1) symmetry breaking for fermions

Let us consider the symmetry breaking long exact sequence for a $\mathrm{U}(1)$ symmetry in a fermionic theory and an order parameter transforming in the charge 1 representation ρ, which is associated to the tautological bundle $L \to B\mathrm{U}(1)$. There are two cases to consider, depending on whether we have a spin-charge relation, meaning that fermionic operators have half-integer $\mathrm{U}(1)$ charge, or not. In either case the relevant groups of invertible field theories we will need are shown in Table 3.2. To calculate these groups, one applies the universal property of Anderson duality to the spin bordism groups, the $\mathrm{Spin} \times \mathrm{U}(1)$ bordism groups, and the Spin^c bordism groups, which are known: for spin bordism, see Milnor [278], for $\Omega_*^{\mathrm{Spin}}(B\mathrm{U}(1))$, see Wan-Wang [253, §3.1.5], and for spin^c bordism, see Bahri–Gilkey [137].

D	$\mho^D_{\mathrm{Spin}}$	$\mho^D_{\mathrm{Spin}}(B\mathrm{U}(1))$	$\mho^D_{\mathrm{Spin}^c}$
-1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0
1	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}$	$\mathbb{Z}$
2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
3	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
4	0	0	0
5	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$
6	0	0	0

Table 3.2: Classification of D-spacetime-dimensional fermionic invertible field theories with $\mathbb{Z}/2^F$, $\mathrm{U}(1) \times \mathbb{Z}/2^F$, and $\mathrm{U}(1)^F$ symmetry, respectively.

First we study the case with a spin-charge relation, where fermions carry half-charge under $\mathrm{U}(1)$ and bosons carry integer charge. We consider symmetry breaking by a charge 1 order parameter (charge $2e$ from the point of view of the fermions). We studied such an example in Section 3.8.3.3, the $p + ip$ superfluid, using the Smith map of (3.6.27b).

We organize the SBLES into rows associated with this symmetry breaking in each dimension D. The map from the first column to the second is the defect anomaly map Def_ρ, from the second to the third is the residual family anomaly Res_ρ, and the index maps Ind_ρ go from the third column of one row to the first column of the next. We omit arrows for maps that are zero. We use the isomorphism $\mho^D_{\mathrm{Spin}}(S(L)^{p^*L}) = \mho^D_{\mathrm{Spin}}$ (3.8.5) to substitute the latter group for the third map in the SBLES.

$$\begin{tikzcd}
D & \mho^{D-2}_{\mathrm{Spin}}(B\mathrm{U}(1)) \arrow[r, "\mathrm{Def}_L"] & \mho^{D}_{\mathrm{Spin}^c} \arrow[r, "\mathrm{Res}_L"] & \mho^{D}_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \\
2 & 0 & 0 & \mathbb{Z}/2 \arrow[dll, out=-30, in=150] \\
3 & \mathbb{Z}/2 \oplus \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \arrow[dll, out=-30, in=150] \\
4 & \mathbb{Z}/2 & 0 & 0 \\
5 & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 & 0
\end{tikzcd}$$

The long subsequence beginning in $D = 2$ is

$$\begin{tikzcd}
\mho^2_{\mathrm{Spin}} \cong \mathbb{Z}/2 \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & \mho^1_{\mathrm{U}(1),\mathrm{Spin}} \cong \mathbb{Z}/2 \oplus \mathbb{Z} \arrow[r, "\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)"] & \mho^3_{\mathrm{Spin}^c} \cong \mathbb{Z}^2 \arrow[dll, "(2\ 0)"] \\
\mho^3_{\mathrm{Spin}} \cong \mathbb{Z} \arrow[r, "1"] & \mho^2_{\mathrm{Spin}} \cong \mathbb{Z}/2 &
\end{tikzcd}$$

We discussed the last map in Section 3.8.3.3: it corresponds to the Majorana zero mode bound to the vortex of the $p + ip$ superfluid. Let us briefly discuss the computation of the other maps, although they are determined by exactness. The preceding map $\mho^3_{\mathrm{Spin}^c} \to \mho^3_{\mathrm{Spin}}$ measures the residual gravitational anomaly upon breaking the U(1) symmetry. The group $\mho^3_{\mathrm{Spin}^c}$ represents Chern-Simons terms associated with the four-dimensional invariants

$$k_1 \left(\frac{1}{8} c_1^2 - \frac{1}{24} p_1 \right) + k_2 c_1^2, \tag{3.8.48}$$

as in (3.8.14). Meanwhile, the generator of $\mho^3_{\mathrm{Spin}}$ is represented by $-\frac{1}{48} p_1$, so we see the map sends (k_1, k_2) to $2k_1$.

The defect anomaly map $\mho^1_{\mathrm{Spin}}(B\mathrm{U}(1)) \to \mho^3_{\mathrm{Spin}^c}$ tells us the fermion parity as well as the $\mathrm{U}(1)_\rho$ charge of the ρ-defect, i.e. the vortex of the order parameter. A physical model with anomaly $k_1 = 0$ and $k_2 = 1$ is the 1+1D compact boson with U(1) acting only on the left movers. The vortex clearly is parity-even since there are no fermions in the model. However, it carries unit U(1) charge, as is well-known from the chiral anomaly.

Finally, the index map $\mho^2_{\mathrm{Spin}} \to \mho^1_{\mathrm{Spin}}(B\mathrm{U}(1))$ can be understood in terms of the "topological superfluid" in 1+1D. This can be thought of as a U(1)-charged Dirac fermion with the U(1) symmetry broken by the two Majorana masses, which form a doublet. This is in the same phase as the Majorana chain. A vortex operator in this phase, which changes the winding number of the order parameter, also changes the boundary conditions for the fermions, and therefore toggles the fermion parity of the ground state. This is captured by the nonzero index map, landing on the generator of $\mho^1_{\mathrm{Spin}} \cong \mathbb{Z}/2$ inside $\mho^1_{\mathrm{Spin}}(B\mathrm{U}(1))$, which gives the "anomaly" of the vortex operator, namely its fermion parity—compare with Section 3.8.3.1.

Next, we collect the SBLES for charge 1 breaking of a U(1) symmetry *without* a spin-charge relation, corresponding to the Smith map (3.6.27a):

$$\begin{tikzcd}
D & \mho^{D-2}_{\mathrm{Spin}^c} \arrow[r, "\mathrm{Def}_L"] & \mho^D_{\mathrm{Spin}}(B\mathrm{U}(1)) \arrow[r, "\mathrm{Res}_L"] & \mho^D_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
2 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
3 & \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \\
4 & 0 & 0 & 0 \\
5 & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 & 0
\end{tikzcd}$$

We studied the map in degree $D = 5$ in Section 3.8.2.1 when we considered breaking of chiral symmetry of a 4+1D Dirac fermion by its Dirac mass terms.

One general observation is that the index map always vanishes. The reason is that in the definition of the index map from Section 3.8.3, we produce an S^1 bundle Z with spin structure which extends to the disc bundle, since this S^1 always carries anti-periodic spin structure. Moreover, Def_L is an isomorphism from $\mho^{D-2}_{\mathrm{Spin}}(B\mathrm{U}(1)^{L-2})$ to the "reduced" part of $\mho^D_{\mathrm{Spin}}(B\mathrm{U}(1))$, namely those U(1) symmetric invertible phases with no pure gravitational response, in other words which become trivial upon breaking the U(1) symmetry. This the Smith isomorphism of [110], mentioned after (3.6.21).

3.8.5.2 $\mathbb{Z}/2$ symmetry breaking for bosons

Next we discuss what is perhaps the simplest example of the SBLES, which describes the breaking of a unitary $\mathbb{Z}/2$ symmetry of a bosonic system by a single real order parameter transforming in the sign representation σ. This situation is described using the Smith map Example 3.6.6. On the domain wall, this unitary symmetry is transmuted to an anti-unitary symmetry. For reference, the relevant classification groups are shown in Table 3.3, with $\mho^D_{\mathrm{SO}}$ denoting D-spacetime-dimensional bosonic invertible field theories, $\mho^D_{\mathrm{SO}}(B\mathbb{Z}/2)$ denoting those with a unitary $\mathbb{Z}/2$ symmetry, and $\mho^D_{\mathrm{SO}}((B\mathbb{Z}/2)^{\sigma-1})$ denoting those with an anti-unitary $\mathbb{Z}/2$ symmetry. As usual, these groups were obtained by applying Anderson duality to oriented bordism, unoriented bordism, and the oriented bordism of $B\mathbb{Z}/2$. See Thom [34, Théorèmes IV.9, IV.13] for oriented and unoriented bordism groups in low degrees. We do not know of an explicit reference for $\mho_*^{\mathrm{SO}}(B\mathbb{Z}/2)$, but it can be calculated using a result of Wall [208] that implies that the Atiyah-Hirzebruch spectral sequence for oriented bordism collapses for any space whose mod p cohomology vanishes for all odd p.

D	$\mho_{\mathrm{SO}}^D$	$\mho_{\mathrm{SO}}^D((B\mathbb{Z}/2)^{\sigma-1}) = \mho_O^D$	$\mho_{\mathrm{SO}}^D(B\mathbb{Z}/2)$
-1	$\mathbb{Z}$	0	$\mathbb{Z}$
0	0	$\mathbb{Z}/2$	0
1	0	0	$\mathbb{Z}/2$
2	0	$\mathbb{Z}/2$	0
3	$\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$
4	0	$(\mathbb{Z}/2)^2$	0
5	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$
6	0	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$
7	0	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$

Table 3.3: Classification of D-spacetime-dimensional bosonic invertible field theories with no symmetry, time reversal symmetry, and $\mathbb{Z}/2$ symmetry, respectively.

We collect the SBLES as follows. By (3.8.5) the third group in the SBLES simplifies: $\mho_{\mathrm{SO}}^D(S(\sigma)^{p^*\sigma}) = \mho_{\mathrm{SO}}^D$.

$$\begin{tikzcd}
 & \mho_O^{D-1} \arrow[r, "\mathrm{Def}_\sigma"] & \mho^D_{\mathrm{SO}}(B\mathbb{Z}/2) \arrow[r, "\mathrm{Res}_\sigma"] & \mho^D_{\mathrm{SO}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0 \\
2 & 0 & 0 & 0 \\
3 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \\
4 & 0 & 0 & 0 \\
5 & (\mathbb{Z}/2)^2 \arrow[r] & (\mathbb{Z}/2)^3 \arrow[r] & \mathbb{Z}/2 \\
6 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0 \\
7 & (\mathbb{Z}/2)^3 \arrow[r] & (\mathbb{Z}/2)^3 & 0
\end{tikzcd}$$

This has a similar structure to the $\mathrm{U}(1) \times \mathbb{Z}/2^F \to \mathbb{Z}/2^F$ breaking we studied above in Section 3.8.5.1, splitting into isomorphisms given by Def_σ (induced by a Smith isomorphism) and Res_σ, with Ind_σ vanishing. Meanwhile, the pure gravitational part is mapped isomorphically by Res_σ, since by definition we do not need the $\mathbb{Z}/2$ symmetry to detect it, and $\mathbb{Z}/2$ acts transitively on $S(\sigma)$, so the residual family anomaly is determined by the anomaly of the unbroken subgroup, which is just the gravitational part.

We can also compute the SBLES associated with breaking of a time reversal symmetry by a single real order parameter transforming in the sign representation. This turns out to be more interesting, since we no longer have a Smith isomorphism, and Ind_σ may be nonzero. There is a similar identification of the group over the sphere as $\mho^D_{\mathrm{SO}}(S(\sigma)^{p^*\sigma-1}) = \mho^D_{\mathrm{SO}}$.

$$\begin{tikzcd}
 & \mho^{D-1}_{\mathrm{SO}}(B\mathbb{Z}/2) \arrow[r, "\mathrm{Def}_\sigma"] & \mho^{D}_{\mathrm{O}} \arrow[r, "\mathrm{Res}_\sigma"] & \mho^{D}_{\mathrm{SO}} \\
-1 & 0 & 0 & \mathbb{Z} \arrow[dll, bend left] \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 & 0 \\
1 & 0 & 0 & 0 \\
2 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0 \\
3 & 0 & 0 & \mathbb{Z} \arrow[dll, bend left] \\
4 & \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & 0 \\
5 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
6 & (\mathbb{Z}/2)^3 \arrow[r] & (\mathbb{Z}/2)^3 & 0 \\
7 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0
\end{tikzcd}$$

Consider for example the 3rd to 4th rows. We have the sequence

$$\begin{tikzcd}
\mho^{3}_{\mathrm{SO}}(S(\sigma)^{p^*\sigma-1}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mho^{3}_{\mathrm{SO}}(B\mathbb{Z}/2) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mho^{4}_{\mathrm{O}} \arrow[d, phantom, sloped, "\cong"] \\
\mathbb{Z} \arrow[r, "\begin{pmatrix} 2 & 0 \end{pmatrix}"] & \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"] & \mathbb{Z}/2 \oplus \mathbb{Z}/2
\end{tikzcd}$$

The generator of the first nonzero group is the $S(\sigma) \simeq S^0$-family with an E_8 phase [279] at one point (the generator of $\mho^3_{\mathrm{SO}} \cong \mathbb{Z}$), and its inverse phase at the other point. To compute the index map, we study a domain wall between the E_8 and its inverse, which with the standard boundary conditions has chiral modes with $c_L = 16$, $c_R = 0$. The induced unitary $\mathbb{Z}/2$ symmetry is anomaly-free, since $k = 0 \bmod 8$ of the modes are charged. This theory represents the anomaly $(2, 0) \in \mathbb{Z} \oplus \mathbb{Z}/2 \cong \mho^3_{\mathrm{SO}}(B\mathbb{Z}/2)$.

The next map sends the E_8 state, representing $(1, 0)$ in that group, to the time-reversal symmetric phase described by a gravitational $\theta = \pi$ angle, or $\frac{1}{2}w_2^2$. This encodes the well-known fact that the time reversal domain wall at the boundary of that theory (known as $e_f m_f$ in [280]) hosts $c_L = 8 \bmod 16$ gapless chiral modes. Meanwhile, it sends the Levin–Gu SPT [281] associated to $\frac{1}{2}A^3$ and representing $(0, 1)$ in $\mho^3_{\mathrm{SO}}(B\mathbb{Z}/2)$, to the phase associated with $\frac{1}{2}w_1^4$.

3.8.5.3 $\mathbb{Z}/2$ symmetry breaking for fermions

Now we turn to the same $\mathbb{Z}/2$ symmetry breaking scenario for fermions. In the fermionic setting, there are four different types of $\mathbb{Z}/2$ symmetry: either unitary with $U^2 = 1$ or $U^2 = (-1)^F$, or time-reversing with $T^2 = 1$ or $T^2 = (-1)^F$. The relevant classifications are collected in Table 3.4, corresponding to low-degree bordism groups that are explicitly calculated in the following references: Milnor [278] (spin bordism), Giambalvo [193] (Pin^+ bordism), Kirby-Taylor [110] (Pin^- bordism), García–Etxebarria and Montero [225, (C.18)] ($\mathrm{Spin} \times \mathbb{Z}/2$ bordism),[21] and Giambalvo [231] ($\mathrm{Spin} \times_{\mathbb{Z}/2} \mathbb{Z}/4$ bordism). We freely use the shearing isomorphisms from Example 3.2.3 and Example 3.6.8.

D	$\mho^D_{\mathrm{Spin}}$	$\mho^D_{\mathrm{Spin}}(B\mathbb{Z}/2)$	$\mho^D_{\mathrm{Pin}^-}$	$\mho^D_{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/4}$	$\mho^D_{\mathrm{Pin}^+}$
-1	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
1	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	0
2	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/8$	0	$\mathbb{Z}/2$
3	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/8$	0	$\mathbb{Z}$	$\mathbb{Z}/2$
4	0	0	0	0	$\mathbb{Z}/16$
5	0	0	0	$\mathbb{Z}/16$	0
6	0	0	$\mathbb{Z}/16$	0	0

Table 3.4: Fermionic invertible field theories in D spacetime dimensions with symmetry $\mathbb{Z}/2^F$, $\mathbb{Z}/2^U \times \mathbb{Z}/2^F$, $\mathbb{Z}/2^T \times \mathbb{Z}/2^F$, $\mathbb{Z}/4^U$, or $\mathbb{Z}/4^T$, respectively.

There are four different SBLES concerning symmetry breaking by the order parameter σ, one for each of the four types of $\mathbb{Z}/2$ symmetry. We have computed an initial segment of each. Observe that by (3.8.5), we have $\mho^D_{\mathrm{Spin}}(S(\sigma)^{p^*(n\sigma-n)}) \cong \mho^D_{\mathrm{Spin}}$ for any n, so the third group in the long exact sequence classifies pure gravitation anomalies. First we study $\mathbb{Z}/2^U \times \mathbb{Z}/2^F$ breaking to $\mathbb{Z}/2^F$.

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[21] This calculation, or more precisely its equivalent analogue in ko-homology, was first done by Mahowald–Milgram [282], with $ko_*(B\mathbb{Z}/2)$ worked out explicitly by Bruner–Greenlees [283, Example 7.3.1], but the cited reference lists spin bordism groups specifically.

$$\begin{tikzcd}
 & \mho^{D-1}_{\mathrm{Pin}^-} \arrow[r, "\mathrm{Def}_\sigma"] & \mho^{D}_{\mathrm{Spin}}(B\mathbb{Z}/2) \arrow[r, "\mathrm{Res}_\sigma"] & \mho^{D}_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & \mathbb{Z}/2 \arrow[r] & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 \\
2 & \mathbb{Z}/2 \arrow[r] & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 \\
3 & \mathbb{Z}/8 \arrow[r] & \mathbb{Z}/8 \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \\
4 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 \\
7 & \mathbb{Z}/16 \arrow[r] & \mathbb{Z}/16 \oplus \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2
\end{tikzcd}$$

Next we study $\mathbb{Z}/2^T \times \mathbb{Z}/2^F$ breaking to $\mathbb{Z}/2^F$.

$$\begin{tikzcd}
D & \mho^{D}_{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/4} \arrow[r, "\mathrm{Def}_\sigma"] & \mho^{D}_{\mathrm{Pin}^-} \arrow[r, "\mathrm{Res}_\sigma"] & \mho^{D}_{\mathrm{Spin}} \\
-1 & 0 & 0 & \mathbb{Z} \arrow[dll, out=-20, in=160] \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 & 0 \\
1 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
2 & \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/8 \arrow[r] & \mathbb{Z}/2 \\
3 & 0 & 0 & \mathbb{Z} \arrow[dll, out=-20, in=160] \\
4 & \mathbb{Z} & 0 & 0 \\
5 & 0 & 0 & 0 \\
6 & \mathbb{Z}/16 \arrow[r] & \mathbb{Z}/16 & 0 \\
7 & 0 & 0 & \mathbb{Z}^2
\end{tikzcd}$$

One generator of $\mho^2_{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ is represented by a T-odd Majorana zero mode. Upon forgetting its T symmetry, this still has a gravitational anomaly, associated with $\mho^2_{\mathrm{Spin}} \cong \mathbb{Z}/2$. If we have two T-odd Majoranas γ_1 and γ_2, we can form the T-odd pairing term $i\gamma_1\gamma_2$ which leads to a unique ground state. Changing the sign of this term toggles the fermion parity of this ground state, so the associated operator has unit charge under the induced unitary symmetry U, since $U^2 = (-1)^F$. This "anomaly" represents a generator of $\mho^1_{\mathrm{SO}}((B\mathbb{Z}/2)^{2\sigma-2}) \cong \mathbb{Z}/4$.

Next, we have the breaking of a unitary symmetry U with $U^2 = (-1)^F$ down to $\mathbb{Z}/2^F$ as well as breaking of a time reversal symmetry T with $T^2 = (-1)^F$ down to $\mathbb{Z}/2^F$.

$$\begin{tikzcd}
D & \mho^{D-1}_{\mathrm{Pin}^+} \arrow[r, "\mathrm{Def}_\sigma"] & \mho^{D}_{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/4} \arrow[r, "\mathrm{Res}_\sigma"] & \mho^{D}_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/2 \arrow[dll] \\
2 & 0 & 0 & \mathbb{Z}/2 \arrow[dll] \\
3 & \mathbb{Z}/2 & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[dll] \\
4 & \mathbb{Z}/2 & 0 & 0 \\
5 & \mathbb{Z}/16 \arrow[r] & \mathbb{Z}/16 & 0
\end{tikzcd}$$

$$\begin{tikzcd}
D & \mho^{D-1}_{\mathrm{Spin}}(B\mathbb{Z}/2) \arrow[r, "\mathrm{Def}_\sigma"] & \mho^{D}_{\mathrm{Pin}^+} \arrow[r, "\mathrm{Res}_\sigma"] & \mho^{D}_{\mathrm{Spin}} \\
-1 & 0 & 0 & \mathbb{Z} \arrow[dll] \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 & 0 \\
1 & 0 & 0 & \mathbb{Z}/2 \arrow[dll] \\
2 & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 & \mathbb{Z}/2 \arrow[dll] \\
3 & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 & \mathbb{Z} \arrow[dll] \\
4 & \mathbb{Z} \oplus \mathbb{Z}/8 \arrow[r] & \mathbb{Z}/16 & 0
\end{tikzcd}$$

The short exact sequence from $D = 3$ to $D = 4$ was analyzed in Section 3.8.3.2 in the context of time reversal domain walls of $2+1$D Majorana fermions.

Finally, we may also consider composing two of the defect maps in this section, which produces a defect map and an SBLES corresponding to symmetry breaking by an order parameter transforming under 2σ. Note that an 2σ order parameter is a pair of $\mathbb{Z}/2$-odd operators. Unlike symmetry breaking by an σ order parameter, this process preserves the (anti)-unitarity of the symmetry operator, and exchanges the symmetry types $\mathbb{Z}/2^T \times \mathbb{Z}/2^F$ and $\mathbb{Z}/4^T$, and symmetry types $\mathbb{Z}/2^U \times \mathbb{Z}/2^F$ and $\mathbb{Z}/4^U$. We discuss the case of $\mathbb{Z}/4^T$ (Pin^+) breaking to $\mathbb{Z}/2^T \times \mathbb{Z}/2^F$ (Pin^-). Note that the residual family anomaly is more than just gravitational, as it falls outside of the cases considered in Equation (3.8.5). In particular, it is classified by $\mho^D_{\mathrm{Spin}}(S(2\sigma)^{p^*(3\sigma-3)}) \cong \mho^D_{\mathrm{Spin}}(S(2\sigma)^{p^*(\sigma-1)}) \cong \tilde{\mho}^{D+1}_{\mathrm{Spin}}(\mathbb{R}P^2)$, where we used that 2σ is trivial over $S(2\sigma) \simeq S^1$ and the discussion from Appendix A. This is related to Equation (3.6.38), but starting from the other pin group.

$$\begin{tikzcd}
 & \mho^{D-2}_{\mathrm{Pin}^-} \arrow[r, "\mathrm{Def}_{2\sigma}"] & \mho^{D}_{\mathrm{Pin}^+} \arrow[r, "\mathrm{Res}_{2\sigma}"] & \tilde{\mho}^{D+1}_{\mathrm{Spin}}(\mathbb{R}P^2) \\
-1 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
1 & 0 & 0 & \mathbb{Z}/2 \arrow[dll, rounded corners] \\
2 & \mathbb{Z}/2 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/4 \arrow[dll, rounded corners] \\
3 & \mathbb{Z}/2 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
4 & \mathbb{Z}/8 \arrow[r] & \mathbb{Z}/16 \arrow[r] & \mathbb{Z}/2 \\
5 & 0 & 0 & 0 \\
6 & 0 & 0 & 0
\end{tikzcd}$$

Consider the $D = 4$ short exact sequence from the table above. As noted above, the generator of $\mathbb{Z}/16$ can be represented by $2+1$D Majorana fermions. One Majorana fermion has a single T-odd Majorana mass term $\mathcal{O}_1$, and we have considered its symmetry breaking. However, as discussed in Section 3.8.1.1, we cannot find a second T-odd operator such that

$$r \cos\theta \mathcal{O}_1 + r \sin\theta \mathcal{O}_2 \tag{3.8.49}$$

is nondegenerately gapped for large r and all θ. This corresponds to the $\mathbb{Z}/2$ residual anomaly in $\mho^4_{\mathrm{Spin}}(S(2\sigma))$. However, if we have two Majorana fermions, then we can find two T-odd operators such that (3.8.49) gaps the system for all θ. The codimension-two vortex is precisely the Majorana zero modes, which have a $\mathbb{Z}/8$ classification.

3.8.5.4 $\mathbb{Z}/3$ symmetry breaking for fermions

D	$\mho^D_{\mathrm{Spin}}$	$\mho^D_{\mathrm{Spin}}(B\mathbb{Z}/3)$
-1	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0
1	$\mathbb{Z}/2$	$\mathbb{Z}/3 \oplus \mathbb{Z}/3$
2	$\mathbb{Z}/2$	$\mathbb{Z}/2$
3	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/3$
4	0	0
5	0	$\mathbb{Z}/9$
6	0	0
7	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}/9$
8	0	0
9	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/27$
10	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$
11	$\mathbb{Z}^3$	$\mathbb{Z}^3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/27$

Table 3.5: Classification of invertible field theories with $\mathbb{Z}/2^F$ and $\mathbb{Z}/3 \times \mathbb{Z}/2^F$ symmetry in D spacetime dimensions.

An interesting case that demonstrates some of the more typical complexity of the SBLES is $\mathbb{Z}/3$ symmetry breaking in fermionic systems via a charge 1 order parameter. Such a symmetry must be unitary and the symmetry group must have the product structure $\mathbb{Z}/3^U \times \mathbb{Z}/2^F$. This situation corresponds to the Smith map of Example 3.6.33, with $k = 3$, where we decided to denote the bundle by i^*L. The relevant classification is shown in Table 3.5; the new piece of information we need is $\mho^{\mathrm{Spin}}_*(B\mathbb{Z}/3)$, which is worked out in degrees 11 and below in [115, §12.2] using work of Bruner-Greenlees [283, Example 7.3.2]. Note that although in general the tangential structure *changes* from $\mathrm{Spin} \times \mathbb{Z}/k$ to $\mathrm{Spin} \times_{\mathbb{Z}/2} \mathbb{Z}/2k$ under this Smith map because the bundle i^*L is not in general spin, the case of $k = 3$ is simpler: since $H^1(B\mathbb{Z}/3; \mathbb{Z}/2)$ and $H^2(B\mathbb{Z}/3; \mathbb{Z}/2)$ vanish, i^*L over $B\mathbb{Z}/3$ has to be spin, and indeed since $\mathbb{Z}/6 = \mathbb{Z}/2 \times \mathbb{Z}/3$, we have an identification $\mathrm{Spin} \times \mathbb{Z}/3 \cong \mathrm{Spin} \times_{\mathbb{Z}/2} \mathbb{Z}/6$

The anomaly group over the sphere $S(i^*L)$ simplifies as $\mho^D_{\mathrm{Spin}}(S(i^*L)) = \mho^D_{\mathrm{Spin}} \oplus \mho^{D-1}_{\mathrm{Spin}}$, and the long exact sequence is the following:

$$\begin{tikzcd}
D & \mho^{D-2}_{\mathrm{Spin}}(B\mathbb{Z}/3) \arrow[r, "\mathrm{Def}_{i^*L}"] & \mho^{D}_{\mathrm{Spin}}(B\mathbb{Z}/3) \arrow[r, "\mathrm{Res}_{i^*L}"] & \mho^{D}_{\mathrm{Spin}} \oplus \mho^{D-1}_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r, "(1)"] & \mathbb{Z} \\
0 & 0 & 0 & \mathbb{Z} \arrow[dll, "(3)" description] \\
1 & \mathbb{Z} \arrow[r, "\binom{0}{1}"] & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \arrow[r, "{(1,0)}"] & \mathbb{Z}/2 \\
2 & 0 & \mathbb{Z}/2 \arrow[r] & (\mathbb{Z}/2)^2 \arrow[dll, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}" description] \\
3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}"] & \mathbb{Z} \oplus \mathbb{Z}/3 \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"'] & \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[dll, "{(0,1)}" description] \\
4 & \mathbb{Z}/2 & 0 & \mathbb{Z} \arrow[dll, "\binom{-3}{1}" description] \\
5 & \mathbb{Z} \oplus \mathbb{Z}/3 \arrow[r, "{(1,3)}"] & \mathbb{Z}/9 & 0 \\
6 & 0 & 0 & 0 \\
7 & \mathbb{Z}/9 \arrow[r, "{(0,1)}"] & \mathbb{Z}^2 \oplus \mathbb{Z}/9 \arrow[r, "\binom{1}{0}"] & \mathbb{Z}^2 \\
8 & 0 & 0 & \mathbb{Z}/2 \arrow[dll] \\
9 & \mathbb{Z}/2 \oplus \mathbb{Z}/9 \arrow[r] & (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/27 \arrow[r] & (\mathbb{Z}/2)^2 \\
10 & 0 & (\mathbb{Z}/2)^3 \arrow[r] & (\mathbb{Z}/2)^5 \arrow[dll] \\
11 & (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/27 \arrow[r] & \mathbb{Z}^3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/27 \arrow[r] & \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^3
\end{tikzcd}$$

Note that because there is no twist, $\mho^D_{\mathrm{Spin}}(B\mathbb{Z}/3) = \tilde{\mho}^D_{\mathrm{Spin}}(B\mathbb{Z}/3) \oplus \mho^D_{\mathrm{Spin}}$, where $\tilde{\mho}^D_{\mathrm{Spin}}(B\mathbb{Z}/3)$ denotes the subgroup of those phases which become trivial upon breaking $\mathbb{Z}/3$. This subgroup is finite and has no 2-torsion, so Res_{i^*L} is always zero on it, while it maps the $\mho^D_{\mathrm{Spin}}$ factor injectively. It follows that the long exact sequence splits into a series of short exact sequences of the form

$$0 \to \mho^{D-2}_{\mathrm{Spin}} \xrightarrow{\mathrm{Ind}_{i^*L}} \mho^{D-2}_{\mathrm{Spin}}(B\mathbb{Z}/3) \xrightarrow{\mathrm{Def}_{i^*L}} \tilde{\mho}^{D}_{\mathrm{Spin}}(B\mathbb{Z}/3) \to 0 \tag{3.8.50}$$

There are four interesting ones:

- $D = 1$: $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/3$
- $D = 2$: $\mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/3 \to \mathbb{Z}/3$.
- $D = 4$: $\mathbb{Z}/2 \to \mathbb{Z}/2$.
- $D = 5$: $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/3 \to \mathbb{Z}/9$.

Let us consider for example $D = 5$. The first $\mathbb{Z} \cong \mho^3_{\mathrm{Spin}} = \tilde{\mho}^4_{\mathrm{Spin}}(S(i^*L))$ is generated by a 3+1D family which pumps a generator of $\mho^3_{\mathrm{Spin}}$ to the boundary over each third of the $S(i^*L) \cong S^1$. When we compute the first map, the index map, we look at the vortex where the order parameter windings all the way around $S(i^*L)$. This has three 1+1D gapless Majorana modes of the same chirality, with $\mathbb{Z}/3$ acting as a permutation. This can be written as a neutral chiral Majorana and a charge 1 Weyl, so it has anomaly $(3, 1) \in \mathbb{Z} \oplus \mathbb{Z}/3$. The calculation of the next map, the defect anomaly map, follows Section 3.8.2.1.

$D = 1$ is also interesting. Since it involves phases in "negative dimension" we need to think in terms of families. The map $\mathrm{Def}_{i^*L} \colon \mathbb{Z} \to \mathbb{Z}/3$ says that if we have an S^2 family of quantum states, with $\mathbb{Z}/3$ acting as a $2\pi/3$ polar rotation, the difference in the $\mathbb{Z}/3$ charges of the states at the poles equals the Chern number mod 3.

3.8.5.5 $\mathbb{Z}/4$ symmetry breaking for fermions

Now we consider symmetry breaking of a unitary symmetry U with $U^4 = (-1)^F$ by a charge 1 order parameter, which we will just denote as ρ. This situation corresponds to Example 3.6.33 with $k = 4$. The relevant classifications are given in Table 3.6; the new bordism groups we need as input are $\mho^{\mathrm{Spin}}_*(B\mathbb{Z}/4)$ and $\mho^{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/8}_*$, which appear explicitly in [115, §12.1, §13.2] (the former building on a calculation of Bruner-Greenlees [283, Example 7.3.3]). Finally, there is an isomorphism $\mho^D_{\mathrm{Spin}}(S(\rho)^{p^*(\rho-2)}) \cong \mho^D_{\mathrm{Spin}} \oplus \mho^{D-1}_{\mathrm{Spin}}$.

The symmetry breaking long exact sequence is as follows:

D	$\mho^D_{\mathrm{Spin}}$	$\mho^D_{\mathrm{Spin}}(B\mathbb{Z}/4)$	$\mho^D_{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/8}$
-1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0
1	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/8$
2	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0
3	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8$	$\mathbb{Z} \oplus \mathbb{Z}/2$
4	0	0	0
5	0	$\mathbb{Z}/4$	$\mathbb{Z}/32 \oplus \mathbb{Z}/2$
6	0	0	0

Table 3.6: The classification of $\mathbb{Z}/4$ symmetric invertible field theories in D spacetime dimension. Here ρ is the charge one representation of $\mathbb{Z}/4$, giving a unitary symmetry class with $U^4 = (-1)^F$.

$$\begin{tikzcd}
D & \mho^{D-2}_{\mathrm{Spin}}(B\mathbb{Z}/4) \arrow[r, "\mathrm{Def}_\rho"] & \mho^D_{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/8} \arrow[r, "\mathrm{Res}_\rho"] & \mho^D_{\mathrm{Spin}} \oplus \mho^{D-1}_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r, "(1)"] & \mathbb{Z} \\
0 & 0 & 0 & \mathbb{Z} \arrow[dll, "(4)" description] \\
1 & \mathbb{Z} \arrow[r] & \mathbb{Z}/8 \arrow[r] & \mathbb{Z}/2 \\
2 & 0 & 0 & (\mathbb{Z}/2)^2 \arrow[dll, "\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}" description] \\
3 & \mathbb{Z}/2 \oplus \mathbb{Z}/4 \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}"] & \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r, "\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}"'] & \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[dll, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}" description] \\
4 & (\mathbb{Z}/2)^2 & 0 & \mathbb{Z} \arrow[dll] \\
5 & \mathbb{Z} \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/32 \oplus \mathbb{Z}/2 & 0 \\
6 & 0 & 0 & 0
\end{tikzcd}$$

Let us study the subsequence from $D = 2$ to $D = 4$. The first map is $\mathrm{Ind}_\rho \colon \mho^2_{\mathrm{Spin}}(S(\rho)^{p^*(\rho-2)}) \to \mho^1_{\mathrm{Spin}}(B\mathbb{Z}/4)$. The $\mho^2_{\mathrm{Spin}}$ generator is the 1+1D topological superfluid we discussed around

(3.8.5.1) test and gets mapped to the $\mho^1_{\mathrm{Spin}}$ generator as we discussed there. The other $\mathbb{Z}/2$ generator pumps four fermionic charges to the boundary when traversing $S(\rho) \simeq S^1$. Let $\mathcal{O}_i$ for $i = 1, 2, 3, 4$ be the four operators creating these charges, which anticommute. The $\mathbb{Z}/4$ symmetry acts on them by permuting $\mathcal{O}_i \mapsto \mathcal{O}_{i+1}$. The vortex operator of the whole family is the product $\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4$, which one can show is charge 2 under $\mathbb{Z}/4$. This corresponds to $2 \in \mathbb{Z}/4 \cong \tilde{\mho}^1_{\mathbb{Z}/4,\mathrm{Spin}}$.

The next group is $\mho^3_{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/8} \cong \mathbb{Z} \oplus \mathbb{Z}/2$. The $\mathbb{Z}$ generator represents the anomaly of a charge 1/2 (charge 1 under $\mathbb{Z}/8^F$) 1+1D Weyl fermion, while the $\mathbb{Z}/2$ generator represents that of a Dirac fermion with chiral charges $\pm 1/2$ for the left and right handed components. In the second case, if we break the symmetry by adding a Dirac mass (which transforms in the representation ρ) we get a Thouless pump with a unit $\mathbb{Z}/4$-charged vortex operator, matching the defect anomaly map $\mathbb{Z}/4 \to \mathbb{Z}/2$. Note that Res_ρ maps the $\mathbb{Z}$ generator to two times the $\mathbb{Z}$ generator of $\mho^3_{\mathrm{Spin}}$, since a Weyl fermion is two Majorana-Weyl fermions.

Another interesting subsequence goes from $D = 4$ to 5. Exactness requires the index map to be

$$\begin{tikzcd}
\mathrm{Ind}_\rho\colon \mho^4_{\mathrm{Spin}}(S(\rho)^{p^*(\rho-2)}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mho^3_{\mathrm{Spin}}(B\mathbb{Z}/4) \arrow[d, phantom, sloped, "\cong"] \\
\mathbb{Z} \arrow[r, "(4 \quad 1 \quad 0)^T"] & \mathbb{Z} \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2.
\end{tikzcd}$$

Let us provide a physical description. The generator of the source is a family which pumps the generator of $\mho^3_{\mathrm{Spin}} \cong \mathbb{Z}$ to the boundary over each quarter of the circle $S(\rho)$. When we form the ρ-defect, we have four co-propagating 1+1D chiral Majorana modes, with $\mathbb{Z}/4$ acting as a permutation. This corresponds to a charge 1 and a charge 2 left-handed Weyl. If this were a U(1) symmetry, its chiral anomaly would be $1^2 + 2^2 = 5$, which is indeed coprime to 8, so when U(1) is reduced to $\mathbb{Z}/4$, this is a generator of $\mathbb{Z}/8$.

3.8.5.6 SU(2) symmetry breaking for fermions

Now we discuss SU(2) and SO(3) symmetry breaking in fermionic systems. There are three cases of interest: $\mathrm{SU}(2)\times\mathbb{Z}/2^F$, $\mathrm{SO}(3)\times\mathbb{Z}/2^F$, and $\mathrm{SU}(2)^F$, where the latter has a spin-charge relation where fermions carry half integer spin and bosons carry integer spin. We will consider symmetry breaking by both spin-1/2 and spin-1 order parameters. The relevant classifications are shown in Table 3.7. As input, we need $\mho^{\mathrm{Spin}}_*$, as discussed above, and several more families of bordism groups.

- $\mho^{\mathrm{Spin}}_*(B\mathrm{SO}(3))$ was calculated in low degrees by Wan-Wang [253, §5.3.3].
- $\mho^{\mathrm{Spin}}_*(B\mathrm{SU}(2))$ was calculated in low degrees by Lee-Tachikawa [284, Appendix B.2].
- $\mho^{\mathrm{Spin}^h}_* \cong \mho^{\mathrm{Spin}}_*(B\mathrm{SO}(3)^{V_{\mathrm{taut}}-3})$ was calculated in low degrees by Freed–Hopkins [124, Theorem 9.97].

First we will consider $\mathrm{SU}(2) \times \mathbb{Z}/2^F$ symmetry breaking to $\mathbb{Z}/2^F$ by a complex spin-1/2 order parameter, which is the simplest case. This representation corresponds to the tautological bundle V_4 over $B\mathrm{SU}(2)$, which is considered in (3.6.40). We use that

D	$\mho^D_{\mathrm{Spin}}$	$\mho^D_{\mathrm{Spin}}(B\mathrm{SU}(2))$	$\mho^D_{\mathrm{Spin}}(B\mathrm{SO}(3))$	$\mho^D_{\mathrm{Spin}^h}$
-1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0	0
1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0
3	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
4	0	0	0	0
5	0	$\mathbb{Z}/2$	0	$(\mathbb{Z}/2)^2$
6	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$
7	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$

Table 3.7: Anomaly groups relevant to the SU(2) families of long exact sequences of field theories

$\mho^D_{\mathrm{Spin}}(S(V_4)^{p^*(V_4-4)}) = \mho^D_{\mathrm{Spin}}$ (see (3.8.5)). Note that this is another instance of the Smith *isomorphism*, where the long exact sequence splits and $\mho^D_{\mathrm{Spin}}(B\mathrm{SU}(2)^{V_4-4}) \cong \tilde{\mho}^D_{\mathrm{Spin}}(B\mathrm{SU}(2))$.

$$\begin{tikzcd}
D & \mho^{D-4}_{\mathrm{Spin}}(B\mathrm{SU}(2)) \arrow[r, "\mathrm{Def}_{V_4}"] & \mho^D_{\mathrm{Spin}}(B\mathrm{SU}(2)) \arrow[r, "\mathrm{Res}_{V_4}"] & \mho^D_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
2 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
3 & \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \\
4 & 0 & 0 & 0 \\
5 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0 \\
6 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0 \\
7 & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^2
\end{tikzcd}$$

The generator of $\mho^5_{\mathrm{Spin}}(B\mathrm{SU}(2)) \cong \mathbb{Z}/2$ corresponds to Witten's SU(2) anomaly [285]. For example, we can consider $N_f = 2$ QCD with chiral $\mathrm{SU}(2)_L \times SU(2)_R$ symmetry. In the usual chiral symmetry breaking scenario, the order parameters are mass terms and form a complex

SU(2) doublet. The defect anomaly map here captures the fact that skyrmions in this theory are fermions.

Next we study $\mathrm{SU}(2)\times\mathbb{Z}/2^F$ symmetry breaking to $\mathrm{U}(1)\times\mathbb{Z}/2^F$ by a real spin-1 order parameter, corresponding to the Smith map (3.6.51). Note that $\mho^D_{\mathrm{Spin}}(S_{B\mathrm{SU}(2)}(V)^{p^*V-3})\cong\mho^D_{\mathrm{Spin}}(B\mathrm{U}(1))$ since SU(2) acts surjectively on $S(V)$ with stabilizer group U(1) (see (3.8.5)).

$$\begin{tikzcd}
D & \mho^{D-3}_{\mathrm{Spin}}(B\mathrm{SU}(2)) \arrow[r, "\mathrm{Def}_V"] & \mho^{D}_{\mathrm{Spin}}(B\mathrm{SU}(2)) \arrow[r, "\mathrm{Res}_V"] & \mho^{D}_{\mathrm{Spin}}(B\mathrm{U}(1)) \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2\oplus\mathbb{Z} \arrow[dll, rounded corners] \\
2 & \mathbb{Z} & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
3 & 0 & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 \arrow[dll, rounded corners] \\
4 & \mathbb{Z}/2 & 0 & 0 \\
5 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & \mathbb{Z}^2 \arrow[dll, rounded corners] \\
6 & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}/2 & 0 \\
7 & 0 & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^4
\end{tikzcd}$$

The residual family anomaly in $D=3$ maps the gravitation Chern-Simons term associated with $\mho^3_{\mathrm{Spin}}\cong\mathbb{Z}$ to itself, while the level 1 SU(2) Chern-Simons term corresponding to the other generator of $\mho^3_{\mathrm{Spin}}(B\mathrm{SU}(2))$ maps to a level 2 Chern-Simons term for the unbroken U(1) subgroup. If we have a level 1 Chern-Simons term, the V-defect acts as a U(1) monopole (which is like an 't Hooft-Polyakov monopole), and is thus fermionic, which is captured by the index map.

Now we study $\mathrm{SO}(3)\times\mathbb{Z}/2^F$ symmetry breaking to $\mathrm{U}(1)\times\mathbb{Z}/2^F$ by a real spin 1 order parameter, corresponding to the fiber sequence (3.6.47b). Once again we use that $\mho^D_{\mathrm{Spin}}(S_{B\mathrm{SO}(3)}(\rho)^{p^*(V_3-3)})=\mho^D_{\mathrm{Spin}}(B\mathrm{U}(1))$.

$$\begin{tikzcd}
D & \mho^{D-3}_{\mathrm{Spin}^h} \arrow[r, "\mathrm{Def}_{V_3}"] & \mho^D_{\mathrm{Spin}}(B\mathrm{SO}(3)) \arrow[r, "\mathrm{Res}_{V_3}"] & \mho^D_{\mathrm{Spin}}(B\mathrm{U}(1)) \\
-1 & 0 & \mathbb{Z} \arrow[r, "(1)"] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & 0 & \mathbb{Z}/2 \arrow[r, "{(0,1)}"] & \mathbb{Z}\oplus\mathbb{Z}/2 \arrow[dll, "{(2,0)}" description] \\
2 & \mathbb{Z} \arrow[r, "{(1,0)}"'] & (\mathbb{Z}/2)^2 \arrow[r, "{(0,1)}"] & \mathbb{Z}/2 \\
3 & 0 & \mathbb{Z}^2 \arrow[r, "\cong"] & \mathbb{Z}^2 \\
4 & 0 & 0 & 0 \\
5 & 0 & 0 & \mathbb{Z}^2 \arrow[dll, "{(1,2)}" description] \\
6 & \mathbb{Z}^2 \arrow[r, "{(1,0)}"'] & \mathbb{Z}/2 & 0 \\
7 & 0 & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^4
\end{tikzcd}$$

Finally, we study $\mathrm{SU}(2)^F$ symmetry breaking to $\mathrm{U}(1)^F$ by a real spin 1 order parameter, corresponding to the Smith fiber sequence (3.6.47a). Note that $\mho^D_{\mathrm{Spin}}(S(V_3)^{p^*(V_3-3)}) \cong \mho^D_{\mathrm{Spin}^c}$.

$$\begin{tikzcd}
D & \mho^{D-3}_{\mathrm{Spin}}(B\mathrm{SO}(3)) \arrow[r, "\mathrm{Def}_{V_3}"] & \mho^D_{\mathrm{Spin}^h} \arrow[r, "\mathrm{Res}_{V_3}"] & \mho^D_{\mathrm{Spin}^c} \\
-1 & 0 & \mathbb{Z} \arrow[r, "(1)"] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & \mathbb{Z} \arrow[dll, "(1)" description] \\
2 & \mathbb{Z} & 0 & 0 \\
3 & 0 & \mathbb{Z}^2 \arrow[r, "{(2,1)}"] & \mathbb{Z}^2 \arrow[dll, "{(1,0)}" description] \\
4 & \mathbb{Z}/2 & 0 & 0 \\
5 & (\mathbb{Z}/2)^2 \arrow[r, "\cong"] & (\mathbb{Z}/2)^2 & \mathbb{Z}^2 \arrow[dll, "2" description] \\
6 & \mathbb{Z}^2 \arrow[r, "\mathrm{mod}\ 2"'] & (\mathbb{Z}/2)^2 & 0 \\
7 & 0 & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^4
\end{tikzcd}$$

Chapter 4

Free-to-Interacting Maps for Fermionic Symmetry-Protected Topological Phases

The material of Section 4.1 is taken from [11], which is joint work of the author with Omar Antolín Camarena, Arun Debray, Natalia Pacheco-Tallaj, Daniel Sheinbaum, and Luuk Stehouwer. Meanwhile, the material of Section 4.2 will appear in forthcoming work of the author with AD, NPT, and LS [12].

In the context of symmetry-protected topological phases (introduced in Section 2.5.2), *free-to-interacting maps* study how the behavior of a free (i.e., noninteracting) fermionic physical theory changes once higher-order interactions are allowed. Free models are convenient physically because they are often exactly-solvable, allowing for example for the ground states of the system to be computed. However, interacting models are more physically realistic. It surprised the physics community in 2009 when Fidkowski–Kitaev [104] provided the first example of the classification changing once interactions were allowed: they studied the a one-dimensional model called the time-reversal symmetric Majorana chain, finding that the $\mathbb{Z}$ classification of free phases breaks down to only a $\mathbb{Z}/8$ classification of interacting phases. Since then, many physics papers have followed to study this on a case-by-case basis for various SPTs; see e.g. [52, 286–292].

Free fermion SPTs are famously described by the *tenfold way*: a collection of ten combinations of symmetries to enforce on systems, including e.g. time-reversal, charge-conjugation, and chiral symmetry. Classifications of phases are periodic both in the spatial dimension of the system and in the number of (appropriately defined) symmetries. The tenfold way unified many known examples of SPTs, like the quantum spin Hall effect [91], as well as predicted the existence of new phases, spurring a large amount of experimental research.

A K-theoretic framework for classifying free-fermion phases in the tenfold way was introduced contemporaneously by Kitaev [7] and Ryu–Schnyder–Furusaki–Ludwig [293], and later expanded upon by Freed-Moore [294], Thiang [295], Alldridge-Max-Zirnbauer [296], and others. This framework encodes the ten Altland–Zirnbauer classes in the two shifts of complex K-theory (A and AIII) and the eight shifts of real KO-theory (D, BDI, AI, CI, C, CII, AII, DIII) according to Bott periodicity.

As discussed in Section 2.5.2, interacting SPTs may be modeled as invertible field theories, and then classified using results of Freed–Hopkins [1] and Grady [2] as Anderson-dual bordism groups. Remarkably, not just the free and interacting SPTs are modeled homotopy theoretically, but also the passage between them: in [1, Section 9.2], Freed–Hopkins constructed Anderson-dualized twisted Atiyah–Bott–Shapiro maps to model free-to-interacting maps for the tenfold way. They then computed the kernels and cokernels in low dimensions, finding agreement with a large collection of physics references. Our aim is to generalize Freed–Hopkins' approach to beyond the tenfold way, defining more general free-to-interacting maps as well as computing them. In this chapter, we present generalizations to two new physical situations, each of which require mathematical innovations. Our broader goal in future work is to treat SPTs with mixed spatial and internal symmetries.

4.1 Weak Phases

4.1.1 Introduction

Within a fixed Altland–Zirnbauer class, there are different kinds of phases, distinguished for example by the spatial symmetries of the phase. *Strong* phases, protected solely by internal symmetry, are robust even in the presence of symmetry-preserving strong disorder. On the other hand, *weak* phases, which are protected by a lattice translation symmetry, are a priori less robust in the presence of disorder. Weak topological phases can be formed from layers of lower-dimensional strong topological phases, and so invariants of weak phases are often built from invariants of lower-dimensional strong phases [5, 297, 298]. They are of interest for various exotic phenomena: their construction by stacking often results in anisotropic gapless edge modes [297–300], while translation symmetry defects called dislocations can trap topological bound states [301–305]. Weak SPTs also have the potential for producing non-abelian anyons [302, 303] and hosting helical edge states [306–310]. Experimental studies of weak phases include, e.g. in [311–313].

The question of whether weak phases persist in the presence of disorder or short-range interactions is an active area of research [297, 314–317]. In this section, we generalize the approach of Freed–Hopkins [1, Section 9.2] to construct maps between the *KR*-theory classification of free phases [7] and the Anderson-dual bordism classification of interacting phases [1, 100], modeling free-to-interacting maps for weak phases in each of the tenfold way classes. We also compute all the groups of free and interacting phases in low dimensions and explore some of the physical consequences.

4.1.2 The Ansatz for the Weak Free-to-Interacting Map

4.1.2.1 Fermionic symmetry groups

Fermionic groups In condensed matter, symmetry groups G_f of fermionic systems have the extra structure of what we will call a *fermionic group* [318, 319]. This means that G_f comes equipped with a central element $(-1)^F \in G_f$ of order two and a homomorphism $\phi : G_f \to \mathbb{Z}_2$ labeling time-reversing elements such that $(-1)^F$ is time-preserving.

Symmetries form a superalgebra A *superalgebra* is a $\mathbb{Z}_2$-graded algebra. Each Altland–Zirnbauer class specifies a set of symmetry operators, which generate a superalgebra over $\mathbb{R}$ or $\mathbb{C}$. The reader should be warned that the physical interpretation of the $\mathbb{Z}_2$-grading here is given by time-reversing versus time-preserving symmetries, as opposed to fermions versus bosons.

From superalgebra to fermionic group The superalgebras we obtain by the Altland–Zirnbauer classification are *super division algebras*, meaning all homogeneous elements are invertible. There are exactly ten such superalgebras [320]

$$D_i^{\mathbb{C}}, D_j^{\mathbb{R}} \quad i = 0, 1,\ j = 0, \ldots, 7, \tag{4.1.1}$$

which can be constructed explicitly as certain Clifford algebras. Given a super division algebra A, the set $S(A)$ of norm-1 elements of A acquires the structure of a compact Lie group from the multiplication on A. The grading operator defines a homomorphism $\phi\colon S(A) \to \mathbb{Z}_2$, and $(-1)^F$ generates a central $\mathbb{Z}_2$ subgroup of $S(A)$, making $S(A)$ into a fermionic group.

Example 4.1.2. The Altland–Zirnbauer class AII, corresponding to topological insulators with a time-reversal symmetry, has a time-reversal symmetry squaring to $(-1)^F$ and a charge Q generating a $\mathrm{U}(1)$ symmetry corresponding to conservation of particle number. These symmetries are subject to a *spin-charge relation*: the -1 in this $\mathrm{U}(1)$ is equal to $(-1)^F$, and time-reversal acts on $\mathrm{U}(1)$ by complex conjugation.

The algebra generated by T and Q over $\mathbb{R}$ is isomorphic to the Clifford algebra

$$C\ell_{-2} \coloneqq \mathbb{R}\langle e_1, e_2\rangle/(e_1^2 = e_2^2 = -1, e_1e_2 + e_2e_1 = 0). \tag{4.1.3}$$

This isomorphism can be explicitly realized by sending $T \mapsto e_1$ and $e^{i\pi\theta} \in \mathrm{U}(1)$ to $\cos(\theta) + e_1e_2\sin(\theta)$ [319, Example 4].

The second step is to find $S(C\ell_{-2})$, which by definition of the pin groups is equal to $\mathrm{Pin}^-(2)$. First consider the real superalgebra

$$A' \coloneqq \mathbb{C}[T]/(T^2 = -1, iT + Ti = 0), \tag{4.1.4}$$

where i is even and T is odd. There is an isomorphism $\phi\colon A' \to C\ell_{-2}$ of real superalgebras defined by setting $\phi(i) = e_1e_2$ and $\phi(T) = e_1$, then extending linearly to all of A'.

From the viewpoint of A', it is easier to find the homogeneous norm-one elements: the unit complex numbers, which generate a $\mathrm{U}(1)$ subgroup of $S(A')$, and the $\mathbb{Z}_4$ subgroup generated by T. The operator T acts on $\mathrm{U}(1)$ by complex conjugation, and $T^2 = -1$ is in $\mathrm{U}(1)$, so we see that

$$S(C\ell_{-2}) \cong S(A') \cong \frac{\mathrm{U}(1) \rtimes \mathbb{Z}_4^T}{\mathbb{Z}_2}. \tag{4.1.5}$$

The homomorphism ϕ is the unique one which is trivial when pulled back to $\mathrm{U}(1)$ and nontrivial when pulled back to $\mathbb{Z}_4^T$; $(-1)^F$ is the common central element. ◇

4.1.2.2 *K*-theory classifications of free fermion phases

The classification of SPT phases of complex free fermions can be connected to K-theory as follows [294]. For a symmetry group G, consider a one particle state space V, which furnishes a representation R of G. We want to understand the space of all gapped Hamiltonians H on V with symmetry G. After shifting the Fermi energy to zero, a gapped Hamiltonian is defined as a linear operator without kernel that intertwines R. This splits the representation $V = V_{valence} \oplus V_{conduction}$ into $\pm$-eigenspaces of H. Therefore H defines an element

$$V_{valence} - V_{conduction} \in K^0_G(\mathrm{pt}) \tag{4.1.6}$$

in the representation ring of G. If two Hamiltonians give different elements of $K^0_G(\mathrm{pt})$, a path between them must involve crossing the gap. Conversely, two different Hamiltonians with the same decomposition $V = V_{valence} \oplus V_{conduction}$ are in the same path component by spectral flattening. Therefore, the set of components π_0 essentially[1] equals $K^0_G(\mathrm{pt})$.

With enough care about the mathematical details, the above heuristic applies in various settings:

1. When G contains time-reversal symmetries, they act anti-unitarily on V. We have to accommodate for this in the definition of the representation ring.
2. Some symmetries have additional constraints relating to fermion parity, such as $T^2 = (-1)^F$ for a time-reversal symmetry. Since $(-1)^F$ acts by -1 on V, we have to enforce this relation in the representation ring.
3. In positive spatial dimension d, it is reasonable to require $G = \mathbb{Z}^d$ to be the symmetry group of a lattice of atoms. Stable homotopy theory for noncompact G is still in development. Since the group algebra of an infinite group will not suffice for these purposes, we define $K^0_G(\mathrm{pt})$ to be the K-theory of the complex group C^*-algebra $C^*(G)$ of G.

As argued by [295, Example 9.1-9.3], there is an isomorphism $K^0_{\mathbb{Z}^d}(\mathrm{pt}) \cong K^0(\mathbb{T}^d)$, closely related to Bloch's theorem. Here $K^0(\mathbb{T}^d)$ is the K-theory of the Brillouin zone torus. The isomorphism is given by a Fourier transform to momentum space, a special case of the Pontryagin duality isomorphisms of C^*-algebras

$$C^*(\mathbb{Z}^d) \cong C(\widehat{\mathbb{Z}^d}, \mathbb{C}) = C(\mathbb{T}^d, \mathbb{C}). \tag{4.1.7}$$

Here $C(X, \mathbb{C})$ denotes the ring of continuous functions on X and $\widehat{\mathbb{Z}^d} := \mathrm{Hom}(\mathbb{Z}^d, U(1)) = \mathbb{T}^d$ is the Pontryagin dual of $\mathbb{Z}^d$. Explicitly, a vector bundle E over $\mathbb{T}^d$ gives a $C^*(\mathbb{Z}^d)$-module $\Gamma(E)$ of continuous sections of E by mapping $\vec{n} \in \mathbb{Z}^d$ to the function $\mathbb{T}^d \to \mathbb{C}$ given by $e^{i\vec{n}\cdot\vec{k}}$. Here we used the common convention of identifying $\mathbb{T}^d$ with a quotient of the box $[-\pi, \pi]^d$ using the map $k \mapsto (\vec{n} \mapsto e^{i\vec{n}\cdot\vec{k}})$. We have therefore reproduced the fact [7] that class A topological insulators in spatial dimension d are classified by $K^0(\mathbb{T}^d)$.

---

[1] In the representation ring, we quotient out by additional relations such as $V - V = 0$ to ensure $K^0_G(\mathrm{pt})$ is a group. A priori, there is no physical justification for requiring this invertibility under stacking (which is given by direct sum since we are on a 1-particle space). However, here we restrict to invertible phases. Phases which are unstably nontrivial are called *fragile* phases [321, 322].

In order to address the question of which topological phases survive in the continuum limit, we redo the above argument for $G = \mathbb{R}^d$ the group of continuous translations. We again have the Fourier transformation isomorphism

$$C^*(\mathbb{R}^d) \cong C(\widehat{\mathbb{R}^d}, \mathbb{C}) = C(\mathbb{R}^d, \mathbb{C}), \tag{4.1.8}$$

so that

$$K^0_{\mathbb{R}^d}(\mathrm{pt}) = K_0(C^0(\mathbb{R}^d)) = \tilde{K}^0(S^d). \tag{4.1.9}$$

This agrees with the classification of strong class A topological insulators.

In the above discussion, we implicitly assumed our fermions are charged. In other words, we assumed the existence of a polarization giving the one particle spaces V and V^* of creation and annihilation operators, thus disallowing unpaired Majoranas. There is an analogous discussion for neutral fermions, resulting in KO- instead of K-theory. This approach can be formulated in the Bogoliubov-de-Gennes formalism [296]. Even though most of the condensed matter literature does not use Majorana fermions, we will focus on this perspective, following our main references [7] and [1].

The main difference in the new set-up will be that the complex one particle Hilbert space V is replaced by a real Hilbert space $\mathcal{M}$. The self-adjoint gapped Hamiltonian H is replaced with a skew-adjoint gapped operator Ξ on $\mathcal{M}$, which one should think of as $-iH$. Even though it is not possible to look at the positively imaginary and negatively imaginary eigenvalues of Ξ on $\mathcal{M}$, the operator does induce a complex structure $\Xi/|\Xi|$ on $\mathcal{M}$. Stably, the space of complex structures becomes a classifying space for KO^{-2}. Since by a spectral flattening procedure the space of such gapped skew-adjoint Ξ is homotopy equivalent to the space of complex structures, this hints towards a relationship between neutral phases and KO-theory. This discussion generalizes to arbitrary symmetry groups, taking into account that time-reversal symmetries should anti-commute with Ξ.

We can use the formalism of Karoubi triples [295, 323] to make this discussion mathematically precise: let A be the (real or complex) super C^*-algebra of symmetries, graded by time-reversal.[2] A *Karoubi triple* $(\mathcal{M}, \Xi_1, \Xi_2)$ consists of of a finitely generated (ungraded) A-module $\mathcal{M}$ and maps $\Xi_i : \mathcal{M} \to \mathcal{M}$ satisfying $\Xi^2 = -\mathrm{id}_{\mathcal{M}}$ and $\Xi_i a = (-1)^{|a|} a \Xi_i$ for all $a \in A$.[3] One can think of a Karoubi triple as a formal difference $[\Xi_1] - [\Xi_2]$ of Hamiltonians with A-symmetry. We now want to impose that $[\Xi_1] - [\Xi_2] = 0$ if Ξ_1 and Ξ_2 are in the same path component. So define a Karoubi triple to be *elementary* when Ξ_1 is in the same path component as Ξ_2 in the space of complex structures Ξ such that $\Xi a = (-1)^{|a|} a \Xi$ for all $a \in A$. Two Karoubi triples $(\mathcal{M}, \Xi_1, \Xi_2), (\mathcal{M}', \Xi_1', \Xi_2')$ are *isomorphic* if there exists an A-module isomorphism $\mathcal{M} \to \mathcal{M}'$ intertwining Ξ_i with Ξ_i' for $i = 1, 2$. Note that there is an obvious notion of direct sum $\oplus$ of Karoubi triples. We say two triples T_1, T_2 are *stably equivalent* if there exists an elementary triple T' such that $T_1 \oplus T'$ is isomorphic to $T_2 \oplus T'$. The set of Karoubi triples can be thought of as a stabilization of the space of A-symmetric Bogoliubov de-Gennes Hamiltonians Ξ.

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[2] In this work, we will restrict to the case where A is the tensor product of a tenfold way symmetry as explained in §4.1.2.1 with the group C^*-algebra of the Lie group of translation symmetries, either discrete $\mathbb{Z}^d$ or continuous $\mathbb{R}^d$.

[3] There is an infinite dimensional version of the Karoubi description given here, which can be shown to be equivalent [324, §4]. Using modules that are not finitely generated can be more suitable for physics, for example if we want to take the unbounded above valence band into account.

Definition 4.1.10. The group of *A-symmetric free SPT phases* is the set of Karoubi triples modulo stable equivalence under $\oplus$.

If $A = C^*(\mathbb{Z}^d; F) \otimes C\ell_{-s}$, where $F = \mathbb{R}$, resp. $F = \mathbb{C}$, we will refer to A-symmetric free SPT phases as *discrete translation-invariant free SPT phases of real, resp. complex Altland–Zirnbauer class s*. Similarly, $A = C^*(\mathbb{R}^d; F) \otimes C\ell_{-s}$ gives *continuous translation-invariant free SPT phases.*[4]

The following theorem and remark will be proven in upcoming work of Stehouwer [325]:

Theorem 4.1.11 ([325])**.** *The group of A-symmetric free SPT phases is isomorphic to $KO_2(A)$.*

Remark 4.1.12. Suppose A is a real super C^*-algebra containing a subalgebra $\mathbb{C}$, which is not necessarily in the center. We think of this subalgebra as generating charge. Suppose additionally that $A = A_+ \oplus A_-$ where $a_\pm \in A_\pm$ if and only if $a_\pm z = z^\pm a_\pm$ for all $a \in A_\pm$ and $z \in \mathbb{C}$. This defines a $\mathbb{Z}_2$-grading μ on A not necessarily equal to the $\mathbb{Z}_2$-grading ϕ given by time-reversal. Note that these this grading commutes with the other $\mathbb{Z}_2$-grading on A in the sense that the corresponding operators with eigenvalues ± 1 commute. Therefore, there is a product/diagonal $\mathbb{Z}_2$-grading c. Then $KO_0(A, c) \cong KO_2(A, \phi)$, where $K_i(A, \lambda)$ denotes the degree i K-theory of the algebra A with $\mathbb{Z}_2$-grading λ. This connects the description of Theorem 4.1.11 to the discussion of the beginning of this section and in particular to [294].

Example 4.1.13. Take $A = C^*(\mathbb{Z}^d; \mathbb{R}) \otimes C\ell_{-2}$ to be the tensor product of a d-dimensional discrete translation symmetry and the internal symmetry algebra of class AII (see Example 4.1.2). Using the fact that $K_i(A \otimes C\ell_{\pm 1}) \cong K_{i \mp 1}(A)$ [326], we see that the group of A-symmetric free SPT phases is given by $KO_2(A) \cong KO_4(C^*(\mathbb{Z}^d; \mathbb{R}))$. We can now apply arguments as above to relate $\mathbb{Z}^d$ to the torus, but there is one important subtlety. Namely, the Fourier transform crucially uses the complex numbers through the factor $e^{i\vec{n}\cdot\vec{k}}$ and

$$\overline{e^{i\vec{n}\cdot\vec{k}}} = e^{i\vec{n}\cdot(-\vec{k})}. \tag{4.1.14}$$

So under the isomorphism $C^*(\mathbb{Z}^d; \mathbb{C}) \cong C(\mathbb{T}^d; \mathbb{C})$ of complex C^*-algebras, complex conjugation on the left hand side gets mapped to the operation mixing complex conjugation with the involution $k \mapsto -k$ on the Brillouin zone. Therefore the K-theory of $C^*(\mathbb{Z}^d; \mathbb{R})$ is not the KO-theory of the torus, but its KR-theory for this involution. We obtain that the classification of class AII topological insulators is given by

$$KO_4(C^*(\mathbb{Z}^d; \mathbb{R})) \cong KR_4(C(\mathbb{T}^d; \mathbb{C})) \cong KR^{-4}(\mathbb{T}^d). \tag{4.1.15}$$

Replacing $\mathbb{Z}^d$ by a continuous translation symmetry $\mathbb{R}^d$, we obtain similarly that

$$KO_4(C^*(\mathbb{R}^d; \mathbb{R})) \cong \widetilde{KR}^{-4}(S^d). \tag{4.1.16}$$

For example, consider the $d = 3$ time-reversal invariant insulator in class AII studied first in [5, 327]. We classify its phases using (4.1.15) as

$$K_2(C^*(\mathbb{Z}^3) \otimes C\ell_{+2}) \cong KR^{-4}(\mathbb{T}^3) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus (\mathbb{Z}_2)^3, \tag{4.1.17}$$

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[4] Because $C\ell_{-s}$ is finite-dimensional, different notions of C^*-tensor product agree.

see [294, Theorem 11.14]. As observed in e.g. [7] and [328, Theorem 3.35], one $\mathbb{Z}_2$ invariant encodes the strong phase detected by the Fu-Kane-Mele invariant. The $\mathbb{Z}$ invariant counts the number of Kramers pairs of electrons, one $\mathbb{Z}_2$ invariant encodes the strong phase, and the $(\mathbb{Z}_2)^3$ vector invariant encodes the weak topological phases: phases protected by the discrete translation symmetry. These phases may be viewed as quantum spin Hall phases living on each two-dimensional cross section of the three-dimensional material. Indeed, for a continuous translation symmetry, we obtain $\widetilde{KR}^{-4}(S^3) \cong \mathbb{Z}_2$ and only the first $\mathbb{Z}_2$ survives. ◇

Example 4.1.13 generalizes to all tenfold classes to obtain the following corollary of Theorem 4.1.11:

Corollary 4.1.18. *Discrete (resp. continuous) translation-invariant free SPT phases of real Altland–Zirnbauer class s in spatial dimension d are classified by $KR^{s-2}(\mathbb{T}^d)$ (resp. $\widetilde{KR}^{s-2}(S^d)$). There is a similar statement for the two complex classes, replacing KR by complex K-theory.*

Remark 4.1.19. In our convention, class A weak SPT phases are classified by *unreduced K*-theory $K^0(\mathbb{T}^d)$. In the decomposition $K^0(\mathbb{T}^d) \cong K^0(\mathrm{pt}) \oplus \tilde{K}^0(\mathbb{T}^d)$, the first term corresponds to the 0-cell of the Brillouin zone. Physically, this $K^0(\mathrm{pt}) \cong \mathbb{Z}$-valued invariant is a comparison count of the number of bands below versus above the gap. An analogous argument applies to the other classes, where the invariant can also be $\mathbb{Z}/2$-valued or nonexistent depending on $KO^{s-2}(\mathrm{pt})$. This invariant is typically ignored in the condensed matter literature, but we would argue it should be included as a weak phase corresponding to a 0-dimensional strong phase.

4.1.2.3 Bordism classifications of interacting phases

As mentioned in the introduction, when we "turn on interactions" by regarding free fermion Hamiltonians in the context of all symmetry-protected gapped lattice Hamiltonians, (i.e. representatives of invertible topological phases) it is conjectured that deformation classes of invertible topological phases are classified by their low-energy behavior, captured by a reflection-positive invertible field theory (IFT). See [1, 329] for further discussion of this ansatz, which is supported by a strong body of computational evidence [1, 89, 96–103].

4.1.2.3.1 The classification of reflection-positive invertible field theories Reflection-positive IFTs are defined at a mathematical level of rigor by Freed–Hopkins [1, §8] in the topological case and Grady-Pavlov [330, §5] in the nontopological case.[5] They are classified using generalized cohomology; before we give the classification in Theorem 4.1.24, we review some key definitions.

Let O denote the infinite orthogonal group $\mathrm{colim}_n\, \mathrm{O}(n)$, and let $\rho\colon H \to \mathrm{O}$ be a homomorphism of topological groups; this is equivalent to a collection of topological groups H_n for

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[5] See [331–335] for more about reflection positivity in the noninvertible setting.

$n \geq 1$ and maps $i_n \colon H_n \to H_{n+1}$ and $\rho_n \colon H_{n+1} \to \mathrm{O}(n)$ such that the diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_n \arrow[r, "i_n"] \arrow[d, "\rho_n"'] & H_{n+1} \arrow[r] \arrow[d, "\rho_{n+1}"] & \cdots \\ \cdots \arrow[r] & \mathrm{O}(n) \arrow[r, "-\oplus 1"] & \mathrm{O}(n+1) \arrow[r] & \cdots \end{tikzcd} \tag{4.1.20}$$

commutes. For example, $\rho_n \colon H_n \to \mathrm{O}(n)$ could be the inclusion $\mathrm{SO}(n) \hookrightarrow \mathrm{O}(n)$, or the spin group with the vector representation $\mathrm{Spin}(n) \to \mathrm{O}(n)$.

Definition 4.1.21. Given $\rho \colon H \to \mathrm{O}$ as above, let $\Omega_*^H(-)$ denote the generalized homology theory called *H-bordism*: $\Omega_n^H(X)$ is the abelian group of closed n-manifolds with an H-structure [336] and a map to X under disjoint union, modulo bordisms of such data.

For example, Ω_*^{SO} is the bordism theory of oriented manifolds.

Definition 4.1.22. There is a duality on generalized homology and cohomology called *Anderson duality* [75, 337]. Given a generalized homology theory E_*, the *Anderson dual of E_** is the generalized cohomology theory $(I_{\mathbb{Z}}E)^*$ defined to satisfy the following universal property: for all spaces X, there is a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}(E_{n-1}(X), \mathbb{Z}) \longrightarrow (I_{\mathbb{Z}}E)^n(X) \longrightarrow \mathrm{Hom}(E_n(X), \mathbb{Z}) \longrightarrow 0. \tag{4.1.23}$$

One can check that (4.1.23) actually uniquely characterizes a generalized cohomology theory $(I_{\mathbb{Z}}E)^*$. Moreover, because Hom and Ext are contravariant functors in their first argument, Anderson duality defines a contravariant functor on cohomology theories: given a natural transformation $E_*(-) \Rightarrow F_*(-)$, there is a natural transformation $(I_{\mathbb{Z}}E)^*(-) \Leftarrow (I_{\mathbb{Z}}F)^*(-)$.

The short exact sequence (4.1.23) splits, but *not* naturally,[6] implying an isomorphism from $(I_{\mathbb{Z}}E)^n(X)$ to the direct sum of the torsion subgroup of $E_{n-1}(X)$ with the free part of $E_n(X)$.

For more on $I_{\mathbb{Z}}$ and its appearance in this context, see Freed–Hopkins [1, §5.3, 5.4].

Theorem 4.1.24 (Freed–Hopkins [1, Theorem 1.1], Grady [76, Theorem 1])**.** *Let $\mho_H^*$ denote the Anderson dual cohomology theory to Ω_*^H. Then there is a natural isomorphism from the abelian group of deformation classes of d-dimensional IFTs on manifolds with H-structure to $\mho_H^{d+2}$.*

As always in this chapter, d is the spatial dimension of the theory.

4.1.2.3.2 Spacetime symmetry groups for the tenfold way

Theorem 4.1.24 leads us to use the cohomology theory $\mho_H^*$ to model interacting phases, but we need to determine H and its map to O for the ten collections of symmetries we are interested in. The reference [319, §3.2 and 3.3] provides a unified way of doing this.

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[6] This is a generalization of the unnatural splitting of the short exact sequence in the universal coefficient theorem [337, §4].

There is a construction of a spacetime structure group $H(G)$ from an internal symmetry group G indicated in [1]; see [319] for a construction based on [338]. Given a fermionic group $(G, \phi, (-1)^F)$, one first takes the central product

$$\widetilde{H} \coloneqq \frac{G \times \mathrm{Pin}^-}{\langle((-1)^F, -1)\rangle}, \tag{4.1.25}$$

where $-1 \in \mathrm{Pin}^-$ is the nontrivial element in the kernel of the map to O. There is a homomorphism $\widetilde{\phi}\colon \widetilde{H} \to \mathbb{Z}_2$ defined by sending $(g, B) \in G \times \mathrm{Pin}^-$ to $\phi(g) + \det(B)$, where $\det\colon \mathrm{Pin}^- \to \mathbb{Z}_2$ corresponds to the homomorphism taking the $\{\pm 1\}$-valued determinant of B under the canonical isomorphism $\{\pm 1\} \cong \mathbb{Z}_2$.

Finally, the tangential structure $H(G)$ associated to G is the group $\widetilde{\phi}^{-1}(0)$, with the map to O induced by the map on Pin^-. It is easy to show that the corresponding family of topological groups $H_d(G)$ is obtained by replacing Pin^- by $\mathrm{Pin}^-(d)$ in the above discussion.

Proposition 4.1.26. *Let G be a fermionic group with $\phi = 0$ trivial and let $i : G \to G$ be an involution. Define the two fermionic groups*

$$G_\pm \coloneqq \frac{G \rtimes \mathrm{Pin}^\pm(1)}{\mathbb{Z}_2^F}, \tag{4.1.27}$$

where the semidirect product is defined using i and $\det\colon \mathrm{Pin}^\pm(1) \to \mathbb{Z}_2$. The ϕ is defined by projection onto the second factor. Then there is an isomorphism of fermionic structure groups

$$H_d(G_\pm) \cong \frac{G \rtimes \mathrm{Pin}^\mp(d)}{\mathbb{Z}_2^F}, \tag{4.1.28}$$

where the semidirect product is again defined using i and $\det : \mathrm{Pin}^\mp(d) \to \mathbb{Z}_2$

Proof. First some notation: denote the canonical odd element $T \in \mathrm{Pin}^\pm(1) \subset G_\pm$, so $T^2 = (\pm 1)^F$ and $gT = Ti(g)$ for $g \in G$ the elements with $\phi(g) = 0$. Given elements $x_1, x_2 \in \mathrm{Pin}^+(d)$, we define a new group structure (the 'graded opposite') by

$$x_1 * x_2 \coloneqq \begin{cases} (-1)^F x_1 x_2 & \text{both odd,} \\ x_1 x_2 & \text{otherwise.} \end{cases} \tag{4.1.29}$$

Then $(\mathrm{Pin}^+(d), *) \cong \mathrm{Pin}^-(d)$ as fermionic groups.

Define the map

$$\psi\colon \frac{G \rtimes \mathrm{Pin}^\mp(d)}{\mathbb{Z}_2^F} \to H(G_\pm) \subseteq \frac{G_\pm \times \mathrm{Pin}^-(d)}{\mathbb{Z}_2^F} \tag{4.1.30a}$$

by

$$\psi(g \rtimes x) = \begin{cases} (g, x) & \det(x) = 0, \\ (gT, x) & \det(x) = 1. \end{cases} \tag{4.1.30b}$$

This is well-defined because we quotient by all common $\mathbb{Z}_2^F$ on both sides, and ψ lands in $H(G_\pm)$ because $\det(x) + \phi(T) = 0$ if $\det(x) = 0$, and $\det(x) + \phi(gT) = 0$ if $\det(x) = 1$. To check that this is a homomorphism, we let $g_1 \rtimes x_1,\ g_2 \rtimes x_2 \in G \rtimes \mathrm{Pin}^{\mp}(d)$ and have to show $\psi((g_1 \rtimes x_1)(g_2 \rtimes x_2)) = \psi(g_1 \rtimes x_1)\psi(g_2 \rtimes x_2)$. There are four cases depending on $\det x_1$ and $\det x_2$. The most nontrivial case is the one for which both are 1:

$$(g_1 \rtimes x_1)(g_1 \rtimes x_2) = g_1 i(g_2) \rtimes (\mp 1)^F x_1 x_2, \tag{4.1.31}$$

where we have used the product $*$ in case we are working in $\mathrm{Pin}^-(d)$ and the normal product of $\mathrm{Pin}^+(d)$ otherwise. This element is indeed mapped to

$$(g_1 T, x_1)(g_2 T, x_2) = (g_1 T g_2 T, x_1 x_2) = ((\pm 1)^F 1 g_1 i(g_2), x_1 x_2). \tag{4.1.32}$$

The other three cases are easier. It is not hard to see that ψ is a bijection. □

Example 4.1.33. We illustrate how to use Proposition 4.1.26 to determine the tangential structures for symmetry classes BDI and DIII, which are the cases $s = 1$ and $s = -1$ respectively. There are isomorphisms of fermionic groups $S(C\ell_{\pm 1}) \cong \mathrm{Pin}^{\pm}(1)$; $\mathrm{Pin}^+(1) \cong \mathbb{Z}_2^F \times \mathbb{Z}_2^T$ and $\mathrm{Pin}^-(1) \cong \mathbb{Z}_4^T$, with $\mathbb{Z}_2^F \subset \mathbb{Z}_4^T$ the unique order-two subgroup. Now apply Proposition 4.1.26 with $G = \mathbb{Z}_2$ and the involution $i = \mathrm{id}$: the semidirect product $G \rtimes \mathrm{Pin}^{\pm}(1)$ simplifies to a direct product, and then $\mathbb{Z}_2$ cancels the $\mathbb{Z}_2$ in the denominator, so in (4.1.27), $G_{\pm} = \mathrm{Pin}^{\pm}(1)$. In exactly the same way, $H_d(G_{\pm})$ simplifies to $\mathrm{Pin}^{\mp}(d)$. Thus Proposition 4.1.26 reproduces a well-known fact in the physics literature: fermionic systems with a time-reversal symmetry T with $T^2 = 1$ correspond to putting pin$^-$ structures on spacetime, and with $T^2 = (-1)^F$ correspond to putting pin$^+$ structures on spacetime. ⋄

Example 4.1.34. We come back to class AII. In Example 4.1.2, we obtained the fermionic group $S(C\ell_{-2}) \cong (\mathbb{Z}_4^T \rtimes \mathrm{U}(1))/(\mathbb{Z}_2^F)$ from the symmetry algebra of this class. Using Proposition 4.1.26, we will compute the tangential structure group $H_d(S(C\ell_{-2}))$: there is an isomorphism $\mathbb{Z}_4^T \cong \mathrm{Pin}^-(1)$ of fermionic groups: both have underlying group isomorphic to $\mathbb{Z}_4$ with the map to O_1 nontrivial, and this characterizes $\mathbb{Z}_4^T$ up to isomorphism of fermionic groups. Therefore, we can apply Proposition 4.1.26 with $G = \mathrm{U}(1)$ and i equal to complex conjugation. Using that $G_- = S(C\ell_{-2})$, we conclude

$$H_d(S(C\ell_{-2})) \cong \frac{\mathrm{Pin}^+(d) \ltimes \mathrm{U}(1)}{\mathbb{Z}_2^F}. \tag{4.1.35}$$

Metlitski [339, §III.B] introduces this group in the context of invertible phases, and calls it $\mathrm{Pin}_{\tilde{c}}$. Its appearance in the tenfold way is due to Freed–Hopkins [1, (9.9)], who call this group $\mathrm{Pin}^{\tilde{c}+}(d)$. We will follow Freed–Hopkins' notation, as we will also need $\mathrm{Pin}^{\tilde{c}-}(d) := (\mathrm{Pin}^-(d) \ltimes \mathrm{U}(1))/\mathbb{Z}_2^F$. ⋄

The other seven classes in the tenfold way can be worked out in a similar manner. We summarize the results of each step in Table 4.1.[7]

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[7] These are not the only conventions for the superalgebras, fermionic groups, and spacetime tangential structures in the literature: see [319] and the references therein.

s	AZ class	A	$S(A)$	$H^c(s) = H(S(A))$
0	A	$\mathbb{C}$	$\mathrm{U}(1)$	$H^c(0) = \mathrm{Spin}^c$
1	AIII	$C\ell_1 \otimes \mathbb{C}$	$\mathrm{U}(1) \times \mathbb{Z}_2$	$H^c(1) = \mathrm{Pin}^c$

s	AZ class	A	$S(A)$	$H(s) = H(S(A))$
-3	CII	$C\ell_{-3}$	$\mathrm{Pin}^-(3)$	$H(-3) = \mathrm{Pin}^{h-} := \mathrm{Pin}^- \times_{\{\pm 1\}} \mathrm{SU}(2)$
-2	AII	$C\ell_{-2}$	$\mathrm{Pin}^-(2)$	$H(-2) = \mathrm{Pin}^{\tilde{c}+} := \mathrm{Pin}^+ \ltimes_{\{\pm 1\}} \mathrm{U}(1)$
-1	DIII	$C\ell_{-1}$	$\mathrm{Pin}^-(1)$	$H(-1) = \mathrm{Pin}^+$
0	D	$\mathbb{R}$	$\mathrm{Spin}(1)$	$H(0) = \mathrm{Spin}$
1	BDI	$C\ell_1$	$\mathrm{Pin}^+(1)$	$H(1) = \mathrm{Pin}^-$
2	AI	$C\ell_2$	$\mathrm{Pin}^+(2)$	$H(2) = \mathrm{Pin}^{\tilde{c}-} := \mathrm{Pin}^- \ltimes_{\{\pm 1\}} \mathrm{U}(1)$
3	CI	$C\ell_3$	$\mathrm{Pin}^+(3)$	$H(3) = \mathrm{Pin}^{h+} := \mathrm{Pin}^+ \times_{\{\pm 1\}} \mathrm{SU}(2)$
4	C	$C\ell_4$	$\mathrm{Spin}(3)$	$H(4) = \mathrm{Spin}^h := \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}(2)$

Table 4.1: Summary of the procedure outlined in §4.1.2.3.2 beginning with an Altland–Zirnbauer class (second column) and then building a super division algebra A (third column), a fermionic group $S(A)$ (fourth column), and a tangential structure $H(S(A))$ (fifth column). For the tangential structures, the maps to O are all trivial on U(1) and SU(2) and are the usual maps Spin → SO → O or $\mathrm{Pin}^\pm \to \mathrm{O}$ on the other factors; since $\{\pm 1\}$ is in the kernel of all of these maps, these maps descend across the quotient by $\{\pm 1\}$ to produce well-defined maps $H(s) \to \mathrm{O}$. First table: the two complex cases. Second table: the eight real cases. Tables adapted from [1, (9.24), (9.25)] and [319, Table 1].

4.1.2.3.3 What changes for weak phases? To the best of our knowledge, a mathematical model for the classification of weak phases with interactions has not been widely applied in the literature. In this subsubsection, we propose one in Ansatz 4.1.36, building from an ansatz of Freed–Hopkins [100, Ansatz 2.1] to obtain a homotopical model in Corollary 4.1.38 – families of invertible field theories over the spatial torus. In §4.1.7 we will further discuss and justify this ansatz from a physical point of view.

Ansatz 4.1.36. The data of a discrete translation-invariant topological phase is equivalent to a family of phases parametrized by the spatial torus $\mathbf{T}^d$.

See in particular [100, Example 2.3].

The appearance of the real (unit cell) torus $\mathbb{R}^d/\mathbb{Z}^d = \mathbf{T}^d$ is understood from the gauge theory point of view through the so-called crystalline equivalence principle [286], where if $\mathbb{Z}^d$ is a spatial symmetry group and a theory is defined on $\mathbb{R}^d$, there is a procedure for gauging the spatial symmetry and considering the emergent gauge theory on the quotient space $\mathbb{R}^d/\mathbb{Z}^d$. More generally, Freed and Hopkins propose an ansatz [100, Ansatz 3.3, Remark 2.6] that the invertible field theories with (spatial) dimension d on a compact d-dimensional manifold Y are classified by (a possibly twisted version of) $\mho_H^{d+2}(Y)$. They also consider stacks, and thus can obtain invertible field theories on any quotient $\mathbb{R}^d/G$ with G locally compact. In §4.1.7 we present a first-principles derivation in which the unit cell spatial torus

$\mathbf{T}^d$ must appear in many-body interacting systems that have discrete translation symmetry, without the particular need to appeal to field theory. Nevertheless, both points of view can be combined when we employ the Freed–Hopkins ansatz for the spectrum classifying SPT phases.

Ansatz 4.1.37 (Freed–Hopkins [100, Ansatz 2.1, Remark 2.6])**.** The classification of (interacting) invertible d-dimensional phases of symmetry type $\rho\colon H \to \mathrm{O}$ over a compact, stably framed manifold Y is naturally equivalent to the classification of d-dimensional reflection-positive IFTs of manifolds with an H-structure and a map to Y, i.e. the generalized cohomology group $\mho_H^{d+2}(Y)$.

Freed–Hopkins' ansatz is more general than ours; we include only the special case[8] we need.

Corollary 4.1.38. *Assuming Ansatzes 4.1.36 and 4.1.37, deformation classes of invertible discrete translation-invariant topological phases in (spatial) dimension d and Altland–Zirnbauer class s are classified by d-dimensional reflection-positive IFTs on $H(s)$-manifolds with a map to $\mathbf{T}^d$, i.e. by the generalized cohomology group $\mho_{H(s)}^{d+2}(\mathbf{T}^d)$.*

We will discuss this further in §4.1.7.

4.1.2.4 Freed–Hopkins' free-to-interacting map for strong phases

Freed and Hopkins connect the K-theoretic classification of free theories to the invertible-field-theoretic classification of interacting theories using a *free-to-interacting map* ([1] (9.71)). The kernel of this map comprises the theories that are nontrivial under two-body nearest-neighbor interactions, but which may be trivialized using higher-order interactions: a famous example of such a theory is eight copies of the time-reversal symmetric Majorana chain studied by Fidkowski-Kitaev [104]. The cokernel of this map consists of "interaction-enabled" phases: interacting phases that have no free analog. For example, there is a class CI superconductor in $d = 3$ with an intrinsically interacting phase generating a $\mathbb{Z}_2$ interaction-enabled classification [340, §V.B]. Thus the free-to-interacting map allows one to mathematically study the physical questions of whether a free phase is robust to interactions and whether new phases arise in the interacting setting.

The free-to-interacting map is built out of two main ingredients. The first one, the Atiyah–Bott–Shapiro (ABS) orientation, provides a way to get from a bordism class to a K-theory class. Then, bordism is Anderson-dual to the interacting IFT classification (recall Definition 4.1.22), so to land in IFTs instead of bordism we implement this duality and use the Anderson self-duality of KO-theory.

4.1.2.4.1 ABS Orientation We start with the ABS map in the real case. There is a classical ABS map from spin bordism Ω_*^{Spin} to the KO-theory of a point, first defined in [341, §11]. Here, we follow Freed–Hopkins [1, §9.6.3], who use a model for KO-theory developed in [342] and follow [343, §II.7]. An element in Ω_n^{Spin} is represented by an n-dimensional spin manifold, while an element in $KO_n(\mathrm{pt})$ is (the equivalence class of) a $C\ell_n$-module equipped

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[8] This case is studied for $s = 0$, $d = 2$ in their Ex. 2.3.

with a Clifford-linear Fredholm operator. Choose a spin manifold M with a Riemannian metric g, and let ∇ be the induced Levi-Civita connection on the *Dirac bundle*

$$\mathcal{S} := P_{\mathrm{Spin}} \times_{\mathrm{Spin}(n)} C\ell_n, \tag{4.1.39}$$

where P_{Spin} is the $\mathrm{Spin}(n)$-principal bundle associated to the spin structure on M. We obtain a Clifford-linear Dirac operator $\not{D}_M \colon C^\infty(\mathcal{S}) \to C^\infty(\mathcal{S})$ by acting by the covariant derivative followed by Clifford multiplication $c \colon TM \times \mathcal{S} \to \mathcal{S}$:

$$\mathcal{S} \xrightarrow{\nabla} T^*M \otimes \mathcal{S} \overset{g}{\simeq} TM \otimes \mathcal{S} \hookrightarrow C\ell(TM) \otimes \mathcal{S} \to \mathcal{S}. \tag{4.1.40}$$

This operator extends to an operator on the appropriate Sobolev completion $\overline{C^\infty(\mathcal{S})}$ of $C^\infty(\mathcal{S})$. In local coordinates, for $s \in C^\infty(\mathcal{S})$, $\not{D}_M$ has the formula

$$\not{D}_M(s) = \sum e_j \cdot \nabla_{e_j}(s). \tag{4.1.41}$$

The ABS map

$$\begin{aligned} \mathrm{ABS} \colon \Omega_n^{\mathrm{Spin}} &\to KO_n \\ M &\longmapsto (\overline{C^\infty(\mathcal{S})}, \not{D}_M) \end{aligned} \tag{4.1.42}$$

sends a spin manifold M to the Hilbert space $\overline{C^\infty(\mathcal{S})}$ equipped with the Dirac operator.

Freed and Hopkins [1, §9.2.2] develop its twisted generalizations

$$\mathrm{ABS}_s \colon \Omega_n^{H(s)} \to KO_{n+s} \tag{4.1.43}$$

by showing that an n-manifold M with $H(s)$-structure has a canonical twisted spinor bundle with a twisted $C\ell_{n+s}$-linear Dirac operator.[9]

Example 4.1.44 (Twisted ABS for class AII)**.** We go into the details of Freed–Hopkins' construction for the case $s = -2$: see [1, §9.2.2] for the proofs of these assertions.

In class AII, $H(s) = \mathrm{Pin}^{\tilde{c}+} := \mathrm{Pin}^+ \ltimes_{\{\pm 1\}} \mathrm{U}_1$ (Table 4.1).

An element of $\Omega_n^{\mathrm{Pin}^{\tilde{c}+}}$ is represented by an n-manifold with $\mathrm{Pin}_n^{\tilde{c}+}$-structure, which is the same as a lift of the classifying map of the tangent bundle $M \xrightarrow{TM} B\mathrm{O}(n)$ to a map $M \to B\mathrm{Pin}_n^{\tilde{c}+}$. This gives us a $\mathrm{Pin}_n^{\tilde{c}+}$ principal bundle $P_{\mathrm{Pin}_n^{\tilde{c}+}} \to M$. The group $\mathrm{Pin}_n^{\tilde{c}+}$ embeds into the superalgebra $C\ell_n \otimes C\ell_{-2}$, as follows from [1, Lemma 9.27]. We thus have a *twisted Dirac bundle*

$$\mathcal{S}' := P_{\mathrm{Pin}_n^{\tilde{c}+}} \times_{\mathrm{Pin}_n^{\tilde{c}+}} (C\ell_n \otimes C\ell_{-2}) \to M. \tag{4.1.45}$$

We can define a Clifford multiplication map

$$c \colon TM \otimes \mathcal{S}' \to \mathcal{S}' \tag{4.1.46}$$

by using the Clifford multiplication $TM \otimes C\ell(TM) \to C\ell(TM)$ and tensoring with $C\ell_{-2}$. Now choose a Riemannian metric on M, and choose a connection ∇ on the principal $\mathrm{Pin}_n^{\tilde{c}+}$-bundle of frames $P_{\mathrm{Pin}^{\tilde{c}+}} \to M$ whose induced connection on the principal $\mathrm{O}(n)$-bundle of

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[9] A construction of Stolz [338, §9.3] overlaps with Freed–Hopkins' definition for $s = \pm 1$: the index theory is the same, but Stolz does not turn it into a map of spectra. See also [143, 344, 345] for more on index theory on pin^+ and pin^- manifolds.

frames is the Levi-Civita connection. This induces a connection on the twisted Dirac bundle, which following tradition we also denote ∇. Now we can define a *twisted Clifford-linear Dirac operator* $\not{D}_M = e_i \cdot \nabla_{e_i}$ acting on sections of $\mathcal{S}'$ by taking the covariant derivative followed by Clifford multiplication. This acts $C\ell_n \otimes C\ell_{-2}$-linearly, so $(\overline{C^\infty(\mathcal{S}')}, \not{D}_M)$ gives an element of $KO_{n-2}(\mathrm{pt})$.

For example, on the pin$^{\tilde{c}+}$ manifold $\mathbb{CP}^1 \times \mathbb{CP}^1$, this twisted Dirac index evaluates to the generator of $KO_2(\mathrm{pt})$. We prove this in an indirect manner, using a Smith homomorphism, in Appendix B; it would be interesting to find an index-theoretic proof. $\diamond$

The twisted ABS map and twisted Dirac operators are discussed in full generality for all symmetry classes $H(s)$ in [1, §9.2.2, 9.2.3].

Just as for the real case, there is an ABS orientation landing in complex K-theory. The classical map

$$\mathrm{ABS}^c\colon \Omega_n^{\mathrm{Spin}^c} \to K_n \tag{4.1.47}$$

is from spin^c bordism to the K-homology of a point and sends a spin^c manifold M to the complex $\mathbb{C}\ell_n$-linear Dirac operator acting on smooth sections of the complex Dirac bundle $\mathcal{S} := P_{\mathrm{Spin}^c(n)} \times_{\mathrm{Spin}^c(n)} \mathbb{C}\ell_n$.

To incorporate the case $H^c(1) = \mathrm{Pin}^c$, Freed–Hopkins in [1] (9.44) develop the twisted generalization of this map:

$$\mathrm{ABS}_1^c\colon \Omega_n^{\mathrm{Pin}^c} \to K_{n+1}. \tag{4.1.48}$$

4.1.2.4.2 Anderson self-duality of K-Theory

The ABS orientation of the previous subsubsection defines a map from twisted spin bordism to KO-homology. However, the invertible field theories modeling interacting phases are classified by *Anderson-dual* twisted spin bordism (Theorem 4.1.24), while free theories are classified by *KO-cohomology* (Corollary 4.1.18). To reconcile these descriptions, we may apply Anderson duality (Definition 4.1.22) and exploit the Anderson self-duality of KO-theory and K-theory [337, Theorem 4.16].[10]

Corollary 4.1.49. *There is an isomorphism of cohomology theories*

$$(I_{\mathbb{Z}}KO)^* \cong KO^{*-4}. \tag{4.1.50}$$

For complex K-theory, there is actually an isomorphism $I_{\mathbb{Z}}K^* \cong K^*$ with no shift. However, by Bott periodicity, $K^* \cong K^{*+2}$, so we may choose to insert a fourfold shift.

Corollary 4.1.51. *There is an isomorphism of cohomology theories*

$$(I_{\mathbb{Z}}K)^* \cong K^{*-4}. \tag{4.1.52}$$

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[10] Anderson's proof appears in unpublished lecture notes, and it is also discussed in Yosimura [75, Theorem 4]. There are several proofs by a variety of different methods; for example, see Freed-Moore-Segal [346, Proposition B.11], Heard-Stojanoska [347, Theorem 8.1], Ricka [348, Corollary 5.8], and Hebestreit-Land-Nikolaus [349, Example 2.8].

4.1.2.4.3 Free-to-Interacting Maps We now have all of the ingredients we need to define the free-to-interacting maps.

Definition 4.1.53 (Freed–Hopkins [1, Conjecture 9.70])**.** Let $\mathrm{ABS}_s\colon \Omega_n^{H(s)} \to KO_{n+s}$ be the twisted ABS map (4.1.43). Applying Anderson duality (4.1.22) gives a map

$$I_{\mathbb{Z}}\mathrm{ABS}_s\colon I_{\mathbb{Z}}KO^{d+s-2}(-) \to I_{\mathbb{Z}}\Omega_{H(s)}^{d+2}(-) \tag{4.1.54a}$$

of cohomology theories. The right side is by definition $\mho_{H(s)}^{d+2}$. The left side, by Anderson self-duality of KO-theory (4.1.49), is identified with $KO^{d+s-2}(\mathrm{pt})$, so $I_{\mathbb{Z}}\mathrm{ABS}_s$ is a map of cohomology theories of the form

$$\mathrm{F2I}_s\colon KO^{d+s-2}(-) \xrightarrow{\cong} I_{\mathbb{Z}}KO^{d+s+2}(-) \xrightarrow{I_{\mathbb{Z}}\mathrm{ABS}_s} I_{\mathbb{Z}}\Omega_{H(s)}^{d+2}(-) = \mho_{H(s)}^{d+2}(-). \tag{4.1.54b}$$

The free to interacting map is the composition

$$\mathrm{F2I}_{s,strong} \coloneqq \mathrm{F2I}_s(\mathrm{pt})\colon KO^{d+s-2} \to \mho_{H(s)}^{d+2}. \tag{4.1.55}$$

The complex version of the free-to-interacting map is given by a similar composition, implicit in [1]. We define a natural transformation of cohomology theories $\mathrm{F2I}_s^c\colon K^{d+s+2}(-) \to \mho_{H^c(s)}^{d+2}(-)$ just as in Definition 4.1.53, then evaluate it on pt to define $\mathrm{F2I}_{s,strong}^c$.

Definition 4.1.56 (Freed–Hopkins [1])**.** Let s be a complex symmetry type. The free-to-interacting map for theories in spatial dimension d and of symmetry type s is the composition

$$\mathrm{F2I}_{s,strong}^c \coloneqq \mathrm{F2I}_s^c(\mathrm{pt})\colon K^{d+s-2} \xrightarrow{\cong} I_{\mathbb{Z}}K^{d+s+2} \xrightarrow{I_{\mathbb{Z}}\mathrm{ABS}_s^c} I_{\mathbb{Z}}\Omega_{H^c(s)}^{d+2} = \mho_{H^c(s)}^{d+2}, \tag{4.1.57}$$

where the first arrow is the Anderson self-duality of K-theory (4.1.51) and the second map is the Anderson dual of the twisted ABS map defined in (4.1.47), (4.1.48).

Ansatz 4.1.58 (Freed–Hopkins [1, §9.2.6])**.**

1. Under the identifications in Corollary 4.1.18 and Theorem 4.1.24 identifying the groups of strong free fermion phases, resp. reflection positive IFTs in dimension d and real Altland–Zirnbauer class s with KO^{d+s-2}, resp. $\mho_{H(s)}^{d+2}$, the homomorphism assigning to a free fermion Hamiltonian its low-energy invertible field theory is $\mathrm{F2I}_{s,strong}$.

2. The above is true mutatis mutandis for complex Altland–Zirnbauer class s with $H^c(s)$ in place of $H(s)$, K in place of KO, and $\mathrm{F2I}_{s,strong}^c$ in place of $\mathrm{F2I}_{s,strong}$.

Recall the motivation for free-to-interacting maps given in §4.1.2.4: knowing these maps allows us to determine both whether a free-fermion SPT phase is stable to interactions and whether there are interaction-enabled phases that one cannot represent using free-fermion models.

Example 4.1.59. Return to class AII in dimension $d = 3$. Let $x \in KO^{-1}(\mathrm{pt}) \cong \mathbb{Z}_2$ be a free theory with nontrivial Fu-Kane-Mele invariant, which is the generator of $\mathbb{Z}_2$. Such a theory models for example a conducting surface state of the 3d topological insulator BiSb [350]. Its image under the free-to-interacting map is the deformation class of the $\mathrm{pin}^{\tilde{c}+}$ topological field theory whose partition function is described by Witten in [351, §4.7]. In Theorem B.0.9, we show that, when evaluated on the generating manifolds $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$, and $\mathbb{RP}^4$ of $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}}$, this invariant is nontrivial on the first but trivial on the second two.

Since a free theory with the nontrivial Fu-Kane-Mele invariant is sent to a nontrivial interacting theory generating a $\mathbb{Z}_2$ subgroup of the interacting theories, we see that this strong phase is robust to interactions [1]. That the this invariant survives the addition of interactions was observed in [351, §4.7] and [1], and an interacting $\mathbb{Z}_2$ Fu-Kane-Mele index was recently developed in [352].

We have accounted for a $\mathbb{Z}_2$ subgroup of $\mho^5_{\mathrm{Pin}^{\tilde{c}+}} \cong (\mathbb{Z}_2)^3$; the remaining six elements are not in the image of the free-to-interacting map and thus are interaction-enabled phases. There is a generating set of $\mho^5_{\mathrm{Pin}^{\tilde{c}+}}$ given by the Fu-Kane-Mele theory described above, together with two theories whose partition functions are

$$X \longmapsto (-1)^{\int_X w_2(TX)^2}, \qquad X \longmapsto (-1)^{\int_X w_1(TX)^4}, \tag{4.1.60}$$

detected by $\mathbb{CP}^2$ and $\mathbb{RP}^4$ respectively. These theories are closely related to *classical Dijkgraaf-Witten theories* [353, §1],[11] in that they are given by a classical action which integrates a characteristic class. Unlike the classical actions of Dijkgraaf-Witten theories, the classes w_1^4 and w_2^2 do not depend on anything stronger that the homotopy type of X—in particular, they are independent of the choice of $\mathrm{pin}^{\tilde{c}+}$ structure. One could thus think of these theories as “bosonic;” frequently such theories are set aside by researchers investigating fermionic SPTs.

◇

4.1.2.4.4 The free-to-interacting map constrains the spectrum of SPT phases

The ansatz that interacting phases are classified in terms of invertible field theories – and therefore, thanks to Theorem 4.1.24, in terms of bordism – is not the only model for the classification of interacting phases. We point out that the existence of the free-to-interacting map, and one of its basic properties, strongly constrains the possible models for the classification of interacting phases.

As we discuss further in §4.1.7.2, Kitaev proposed the ansatz that invertible phases have the structure of a spectrum D: that is, D determines a generalized cohomology theory D^*, and the classification of G-symmetric SPT phases is the (possibly twisted) cohomology group $D^*(BG)$.[12] The model we have followed, which uses invertible field theories, chooses D^* to be the Anderson dual of spin bordism. But there are two more common choices for fermionic

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[11] The reason we write “closely related” instead of “are” is a few key differences between these theories and classical Dijkgraaf-Witten theories: in the former, we integrated a mod 2 characteristic class, and in the latter, one integrates an $\mathbb{R}/\mathbb{Z}$-valued cohomology class ω. Secondly, Dijkgraaf-Witten theory has a background principal bundle for a finite group G, and requires ω to be a characteristic class of G-bundles. Integrating $\mathbb{R}/\mathbb{Z}$-cohomology classes requires an orientation, but integrating mod 2 cohomology classes does not, so the theories in (4.1.60) are defined on any compact 4-manifold. See [354–357] for more information on unoriented generalizations of Dijkgraaf-Witten theory. Sometimes, theories given by integrating a characteristic class of the tangent bundle are called *gravitational theories*.

[12] It is predicted that there are two different versions of D, one for bosonic phases and the other for fermionic phases. In this paper we focus on the latter.

phases: *restricted supercohomology* SH_1 as introduced by Freed [358, §1] and Gu-Wen [359], and *extended supercohomology* SH_2 as defined by Kapustin-Thorngren [97] and Wang-Gu [99]. See also Gaiotto-Johnson-Freyd [89, §5.3, 5.4].[13] We will not need to know much about these generalized cohomology theories—only that $SH_1^*(\mathrm{pt})$ and $SH_2^*(\mathrm{pt})$ are concentrated in degrees 0 through 3.

Kitaev's argument producing the structure of a spectrum of invertible phases applies equally well for both the free and the interacting classifications, and the argument is compatible with the free-to-interacting map between them. Therefore we hypothesize that Kitaev's conjecture extends: that *the free-to-interacting map refines to a map of spectra*—indeed, this is how Freed–Hopkins [1, §9.2] construct their free-to-interacting maps. This does not constrain the spectrum of SPT phases very much, though: there are nontrivial maps from the K-theory spectrum to both restricted and extended supercohomology.

We can do better with one more piece of information: assume there is a procedure on phases of free fermion theories which is analogous to the field-theoretic process of compactification. After taking a continuum limit, one ought to be able to formulate a topological phase of (spatial) dimension d on a closed d-manifold M, together with some additional structure such as a lattice, a twisted spin structure for fermionic SPT, etc.[14] By choosing M to be a product $M = N_1 \times N_2$, we can compactify on N_1 to pass from a d-dimensional phase formulated on M to a $(d - \dim(N_1))$-dimensional phase formulated on N_2. With some care applied to the tangential structure on N_1,[15] this procedure is expected to define a homomorphism from d-dimensional SPT phases to $(d - \dim(N_1))$-dimensional phases, and it is routinely applied in the condensed-matter theory literature, e.g. [363–365]. As this procedure can be applied in the same ways to free and to interacting phases, we expect compactification to commute with the free-to-interacting map. Though this may not literally be compactification on free fermion phases, as it is not yet clear whether the process of putting the theory on a general manifold is possible before taking a continuum limit, we expect a homomorphism of this sort to exist for free fermion phases, and we will refer to this homomorphism as compactification.

Ansatz 4.1.61.

Physics version The free-to-interacting map commutes with the procedure of compactifying on closed spin manifolds.

Math version The free-to-interacting map is a map of $MT\mathrm{Spin}$-module spectra.

Here $MT\mathrm{Spin}$ is the spectrum whose associated generalized homology theory is spin bordism. The connection between the two versions of Ansatz 4.1.61 is discussed by Yamashita-Yonekura [362, §7.3]; see also Tachikawa-Yamashita [366, §2.2.6]. Freed–Hopkins' free-to-interacting maps satisfy Ansatz 4.1.61 [1, §10].

Proposition 4.1.62. *Assuming Ansatz 4.1.61, SH_1 and SH_2 cannot be the spectrum of fermionic phases.*

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[13] Wang-Gu [360] introduce another variant that we will not use here; it does not suffer the K3 problem we discuss here, but our argument can be adapted to their version of supercohomology.

[14] For example, we might to be able to glue a lattice model on M together from a Hamiltonian description on contractible patches.

[15] See Schommer-Pries [361, §9] for a careful general analysis of the tangential structures needed to compactify; we just need a special case addressed by Yamashita-Yonekura [362, §7.3].

Proof. Let K denote the K3 surface, which is a closed spin 4-manifold whose spin bordism class generates Ω_4^{Spin} [278]. The $MT\mathrm{Spin}$-module structure on KO is through the Atiyah–Bott–Shapiro map (4.1.42); since this is an isomorphism in degrees 0 through 7 [367], compactifying a free fermion theory represented by a class $x \in KO^m$ on the K3 surface is the same thing as multiplying x by a generator a of KO^{-4}: the minus sign is an artifact of the switch from homology to cohomology. In particular, this map is an isomorphism $KO^4 \to KO^0$.

As noted above, there is no n such that $SH_i^n(\mathrm{pt})$ and $SH_i^{n-4}(\mathrm{pt})$ are both nonzero, for $i = 1$ or $i = 2$. Therefore in both restricted and extended supercohomology, compactifying on K3 is the zero map.

Suppose SH_i with either $i = 1$ or $i = 2$ models the spectrum of interacting fermionic SPTs, and let τ denote the twist of supercohomology over $B\mathbb{Z}_2$ corresponding to Altland–Zirnbauer class DIII. Then compatibility of the free-to-interacting maps with compactification means that the following diagram must commute:

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, "\cong"'] & KO^4 \arrow[r, "\mathrm{F2I}"] \arrow[d, "a"] & SH_i^{5+\tau}(B\mathbb{Z}_2) \arrow[d, "0"] & \\
\mathbb{Z} \arrow[rrr, bend right, two heads, "\mathrm{mod}\ 2"'] & KO^0 \arrow[r, two heads, "\mathrm{F2I}"] & SH_i^{1+\tau}(B\mathbb{Z}_2) & \mathbb{Z}_2
\end{tikzcd} \tag{4.1.63}$$

For both SH_1 and SH_2, the classification of interacting class DIII phases in (spacetime) dimension 0 is $\mathbb{Z}_2$: see, e.g., Wang-Gu [360, §VII.E.2.d].[16] And the free-to-interacting map in dimension 0 in class DIII is well-known to be nonzero: see [1, §9.3.1] and the references therein. But this is not compatible with the compactification map $KO^4 \to KO^0$ being an isomorphism and the compactification map on supercohomology vanishing. ☐

We model interacting phases in class DIII with Anderson-dualized pin$^+$ bordism; therefore in (spacetime) dimension 4 we have $\mathbb{Z}_{16}$ [193, §2], in dimension 0 we have $\mathbb{Z}_2$ (*ibid.*), and the free-to-interacting maps in these dimensions are surjective [1, Corollary 9.83]. Therefore we learn that compactifying on K3 is the unique surjective map $\mathbb{Z}_{16} \to \mathbb{Z}_2$, which is dual to the fact that the K3 surface represents $8 \in \Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}_{16}$ [110, Lemma 5.3].

4.1.3 T-duality

Whereas the real torus $\mathbf{T}^d$ appears in our Ansatz 4.1.36 for interacting weak phases, non-interacting fermionic topological phases are traditionally formulated over the crystalline momentum space torus $\mathbb{T}^d$. These tori behave differently, particularly when symmetries are included. However, T-duality, a construction that originated in string theory [368], precisely relates these two tori in a manner that allows us to recast non-interacting results in terms of $\mathbf{T}^d$ and thus to define a free-to-interacting map. We note that T-duality has been employed many times to treat problems in non-interacting fermionic topological phases [369–373]. Associated to a d-dimensional lattice Π are the unit cell or spatial torus $\mathbf{T}^d := \mathbb{R}^n/\Pi$ and

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[16] As remarked above, this version of supercohomology is not the same as SH_1 or SH_2, but all three agree in dimension 0.

the momentum space torus or Brillouin zone $\mathbb{T}^d := \mathrm{Hom}(\Pi, \mathrm{U}(1))$. The Brillouin zone has a $\mathbb{Z}_2$-action given by complex conjugation on $\mathrm{U}(1)$. The Fourier transform between position and momentum space has a K-theoretic analog in the T-duality isomorphisms[17]

$$\mathrm{T}_{\mathbb{R}} \colon KO^\bullet(\mathbf{T}^d) \xrightarrow{\sim} KR^{\bullet-d}(\mathbb{T}^d) \qquad \text{TRS squares to 1} \qquad (4.1.64a)$$

$$\mathrm{T}_{\mathbb{H}} \colon KSp^\bullet(\mathbf{T}^d) \xrightarrow{\sim} KQ^{\bullet-d}(\mathbb{T}^d) \qquad \text{TRS squares to } -1 \qquad (4.1.64b)$$

$$\mathrm{T}_{\mathbb{C}} \colon K^\bullet(\mathbf{T}^d) \xrightarrow{\sim} K^{\bullet-d}(\mathbb{T}^d) \qquad \text{Chern insulators} \qquad (4.1.64c)$$

which can be defined in terms of a pull-convolve-push construction for topological bundles called the *Fourier-Mukai transform*. These isomorphisms have been well studied in the condensed matter literature; see for instance [7, 371, 372]. There is also a C^*-algebraic approach to this material: see [375]. Here we review the perspective of [371, 372].

The *Poincaré line bundle* $\mathcal{L}$ is the complex line bundle on $\mathbf{T}^d \times \mathbb{T}^d = \mathbb{R}^d/\Pi \times \mathrm{Hom}(\Pi, \mathrm{U}(1))$ obtained as the quotient of the trivial bundle $\mathbb{C} \times \mathbb{R}^n \times \mathbb{T}^d$ by the Π-action via characters

$$\pi \cdot (z, v, \chi) \sim (e^{2\pi i \chi(\pi)} z, v + \pi, \chi). \qquad (4.1.65)$$

Bloch waves come from sections of the restrictions of $\mathcal{L}$ to different momentum cross-sections $\mathbf{T}^d \times \{\chi\} \subset \mathbf{T}^d \times \mathbb{T}^d$. The T-duality map can then be expressed as a pull-push along the correspondence

$$\begin{tikzcd} & \mathbf{T}^d \times \mathbb{T}^d \arrow[dl, "p"'] \arrow[dr, "\widehat{p}"] & \\ \mathbf{T}^d & & \mathbb{T}^d \end{tikzcd} \qquad (4.1.66)$$

twisted by the Poincaré line bundle

$$E \longmapsto \widehat{p}_*(p^* E \otimes \mathcal{L}) \qquad (4.1.67)$$

where the pushforward $\widehat{p}_*$ is, intuitively, "integrating out the $\mathbf{T}^d$ direction" and thus reduces the dimension by d.[18] To recover (4.1.64a), note that in the presence of TRS squaring to 1, the involution $k \mapsto -k$ lifts to an antilinear action of T on $\mathcal{L}$, so $p^*E \otimes \mathcal{L}$ is a Real bundle in $KR(\mathbf{T}^d \times \mathbb{T}^d)$ and $\widehat{p}_*$ is the pushforward in KR-cohomology. Equations (4.1.64b) and (4.1.64c) follow from similar reasoning.[19]

4.1.4 Splitting the generalized cohomology of tori

The generalized cohomology of a torus $\mathbb{T}^d$ has a convenient description in terms of the generalized cohomology of spheres (interpreted as cells in a cellular decomposition of $\mathbb{T}^d$), using the fact that for spaces X, Y, there is a homotopy equivalence

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y). \qquad (4.1.68)$$

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[17] In (4.1.64b), KSp is the K-theory of quaternionic bundles and KQ is a Real-equivariant version of KSp introduced by Dupont [374].

[18] Rigorously, $\widehat{p}_*$ is fiber integration in the appropriate K-theory.

[19] Another way to see this is that, because $\mathbf{T}^d$ has trivial $\mathbb{Z}_2$ action, $KO(\mathbf{T}^d) \simeq KR(\mathbf{T}^d)$ and $KSp(\mathbf{T}^d) \simeq KQ(\mathbf{T}^d)$ so the T-duality maps (4.1.64a) and (4.1.64b) seem like they change the K-theory type but in fact they are a Fourier-Mukai transform internal to KR, KQ respectively, where one of the sides has trivial $\mathbb{Z}_2$-action and thus reduces to an ordinary nonequivariant K-theory.

For instance, $\Sigma T^2 \simeq \Sigma S^1 \vee \Sigma S^1 \vee \Sigma S^2$. If we iterate this equivalence over the d-fold product of circles $T^d = (S^1)^{\times d}$ and and use the suspension isomorphism for generalized cohomology, we get the following identity, where we've labelled the circle factors $\mathbb{T}^d \simeq S^1_1 \times \ldots \times S^1_d$.

Lemma 4.1.69 (James splitting of the torus)**.** *The generalized cohomology of the d-dimensional torus is*

$$\widetilde{E}_0(\mathbb{T}^d) \cong \bigoplus_{I\subset\{1,\ldots,n\}} \widetilde{E}_0(S^I) \cong \bigoplus_{n=1}^{d} \widetilde{E}_{-n}^{\oplus\binom{d}{n}} \tag{4.1.70}$$

we denote by $S^I := \wedge_{i\in I} S^1_i$ the $|I|$-dimensional sphere factor indexed by $I \subset \{1,\ldots,n\}$.

The James splitting interacts nicely with T-duality. Given a spatial torus $\mathbf{T}^d = S^1_1 \times \ldots \times S^1_d$, the Brillouin zone is $\mathbb{T}^d \simeq \widehat{S}^1_1 \times \ldots \times \widehat{S}^1_d$ where $\widehat{S}^1_i$ are the dual circles (1d "dual tori") to S^1_i. Then,

Lemma 4.1.71 (T-duality and James splitting)**.** *Under the James splitting isomorphisms $K(\mathbf{T}^d) \simeq \bigoplus_I K(S^I)$ and $K(\mathbb{T}^d) \simeq \bigoplus_J K(\widehat{S}^J)$, T-duality maps $K(S^I)$ to $K(\widehat{S}^{I^c})$, where $K = KO$, KR, KSp, KQ, KU as appropriate, and I^c is the complement of I.*

We refer the reader to [371, §6] for more details.

4.1.5 Comparing strong and weak phases

In the previous sections we have discussed how the generalized cohomology of the torus splits into several summands, some of which are strong, and some of which are weak. Here, we would like to emphasize that which cell—top or bottom—of the torus is associated to the strong phase depends on which torus we consider. On the real space torus $\mathbf{T}^d$, strong phases correspond to the summands in $KO^{d+s-2}(\mathrm{pt}) \subset KO^{d+s-2}(\mathbf{T}^d)$; i.e. the summands coming from the point, or 0-cell of the torus. On the dual torus, which forms the Brillouin zone, the strong phases instead come from the top cell, in the sense that if one crushes all cells except for the top one, the resulting space is $\mathbb{Z}_2$-equivariantly homeomorphic to $\bar{S}^d$, and the induced pullback map on phases is the inclusion of the strong phases in the group of weak phases. In summary, the inclusion of strong phases into the total classification including strong and weak phases interacts with T-duality in the way outlined in the following diagram.

$$\begin{tikzcd}
\widetilde{KR}^{s-2}(\bar{S}^d) \arrow[r, hook] & KR^{s-2}(\bar{\mathbb{T}}^d) \\
\widetilde{KO}^{d+s-2}(S^0) \arrow[r, hook] \arrow[u, "\mathrm{T}_{\mathbb{R}}", "\cong"'] & KO^{d+s-2}(\mathbf{T}^d) \arrow[u, "\mathrm{T}_{\mathbb{R}}", "\cong"']
\end{tikzcd} \tag{4.1.72}$$

Here the top horizontal arrow is induced by a $\mathbb{Z}_2$-equivariant collapse map $(\bar{\mathbb{T}}^d)_+ \to \bar{S}^d$ and the lower by the crush map $(\mathbf{T}^d)_+ \to S^0$. This, and the analogous complex K-theory diagram, commute by Lemma 4.1.71, as these maps pick out the top cell of $\mathbb{T}^d$ and the bottom cell of $\mathbf{T}^d$. These inclusions are split and so realize the strong phases as a direct summand of all phases.

4.1.6 The ansatz for the free-to-interacting map for weak phases

Now we have all the ingredients we need to state our main ansatz (Ansatz 4.1.73): a model for the free-to-interacting map in terms of generalized cohomology.

Fix a (spatial) dimension d and symmetry type s. In Corollary 4.1.18, we modeled the group of free weak phases as $KR^{s-2}(\overline{\mathbb{T}}^d)$ in the real case and $K^{s-2}(\mathbb{T}^d)$ in the complex case, and in Corollary 4.1.38 we modeled the group of interacting weak phases as $\mho^{d+2}_{H(s)}(\mathbf{T}^d)$ in the real case and $\mho^{d+2}_{H^c(s)}(\mathbf{T}^d)$ in the complex case.

For the moment restrict to real symmetry types. We need to get from the Brillouin torus to the spatial torus, so our first step is to use T-duality (4.1.64a) to get from $KR^{s-2}(\overline{\mathbb{T}}^d)$ to $KO^{d+s-2}(\mathbf{T}^d)$. After that, we simply apply the free-to-interacting map $\mathrm{F2I}_s\colon KO^{d-s-2}(\mathbf{T}^d) \to \mho^{d+2}_{H(s)}(\mathbf{T}^d)$ of Definition 4.1.53, evaluated on the spatial torus. For complex symmetry types, the story is completely analogous, using the T-duality isomorphism of (4.1.64c) and the free-to-interacting map from Definition 4.1.56.

Ansatz 4.1.73.

1. Let $x \in KR^{s-2}(\overline{\mathbb{T}}^d)$ be a discrete translation-invariant free fermion theory in d dimensions and of *real* symmetry type s. The long-range effective theory of x is given by the image of x under the composition

$$\mathrm{F2I}_{weak}\colon KR^{s-2}(\overline{\mathbb{T}}^d) \underset{(4.1.64a)}{\xrightarrow{\mathrm{T}_{\mathbb{R}}^{-1}}} KO^{d-s-2}(\mathbf{T}^d) \underset{(4.1.53)}{\xrightarrow{\mathrm{F2I}_s}} \mho^{d+2}_{H(s)}(\mathbf{T}^d). \tag{4.1.74a}$$

2. Let $x \in K^{s-2}(\mathbb{T}^d)$ be a discrete translation-invariant free fermion theory in d dimensions and of *complex* symmetry type s. The long-range effective theory of x is given by the image of x under the composition

$$\mathrm{F2I}^c_{weak}\colon K^{s-2}(\mathbb{T}^d) \underset{(4.1.64c)}{\xrightarrow{\mathrm{T}_{\mathbb{C}}^{-1}}} K^{d-s-2}(\mathbf{T}^d) \underset{(4.1.56)}{\xrightarrow{\mathrm{F2I}^c_s}} \mho^{d+2}_{H^c(s)}(\mathbf{T}^d). \tag{4.1.74b}$$

This ansatz has several consequences for free and interacting weak phases. The following consequence refines the observation that weak phases break up into a direct sum of strong phases, which is well-known in the physics literature (see e.g. [376, §7.3 and Proposition F.8] and [377, §9]), and applies it to our free-to-interacting map. The decomposition can be subtle; T-duality is essential for a clear understanding of this phenomenon.

Lemma 4.1.75 (Weak phases are built from strong phases)**.** *Write* $\mathrm{F2I}^d_{weak}$ *for the weak free-to-interacting map in dimension* d *from Ansatz 4.1.73, and* $\mathrm{F2I}^d_{strong}$ *for the strong free-to-interacting map in dimension* d *from Definition 4.1.53. We have that*

$$\mathrm{F2I}^d_{weak} = \bigoplus_{k=0}^{d} \binom{d}{k} \mathrm{F2I}^{d-k}_{strong}. \tag{4.1.76}$$

The analogous statement is true for the complex free-to-interacting maps.

Proof. The James splitting Lemma 4.1.69 of the Brillouin zone is equivariant with respect to the involutions on $\bar{\mathbb{T}}^d$ [294, Theorem 11.8]. Therefore there is a $\mathbb{Z}_2$-equivariant stable equivalence $\bar{\mathbb{T}}^d \simeq_{\text{stably}} \bigvee_{I\subseteq[d]} \bar{S}^I$, under which the element $x \in KR^{s-2}(\bar{\mathbb{T}}^d)$ splits into elements $x_I \in \widetilde{KR}^{s-2}(\bar{S}^I) = KO^{s-2+|I|}(\mathrm{pt})$. Under T-duality, by Lemma 4.1.71, we get elements $\mathrm{Dual}(x_I) = \bar{x}_{I^c} \in KO^{s-2+d-|I^c|} = KO^{s-2+|I|}$. Each I thus gives us a strong free-to-interacting map

$$\mathrm{F2I}^I_{\mathit{strong}}\colon \widetilde{KR}^{s-2}(\bar{S}^{|I|}) \to \mho^{|I|+2}_{H(s)}. \tag{4.1.77}$$

□

As a result, the kernel and cokernel of $\mathrm{F2I}^d_{\mathit{weak}}$ can be computed from those of $\mathrm{F2I}^k_{\mathit{strong}}$ as k varies from 0 to the dimension:

$$\ker \mathrm{F2I}^d_{\mathit{weak}} = \bigoplus_{k=0}^{d} \binom{d}{k} \ker \mathrm{F2I}^{d-k}_{\mathit{strong}}\,, \qquad \operatorname{coker} \mathrm{F2I}^d_{\mathit{weak}} = \bigoplus_{k=0}^{d} \binom{d}{k} \operatorname{coker} \mathrm{F2I}^{d-k}_{\mathit{strong}}. \tag{4.1.78}$$

This corollary makes the statement that weak phases are built from strong phases of lower dimension precise within our framework. There are two physical consequences from this result. First, if the kth strong phase is robust to interactions, then so is kth component of the weak phase, and vice versa. Similarly, all interaction-enabled weak phases arise from interaction-enabled strong phases in lower dimensions—in our model, there are no interaction-enabled phases that do not arise from lower-dimensional phenomena. We give more examples in §4.1.8.

Example 4.1.79. Freed–Hopkins [1, Corollary 9.93] calculated that in class AII, the strong free-to-interacting map is always injective in low degrees, including up to spatial dimension 3. From Lemma 4.1.75, we conclude that weak phases of translation-invariant class AII insulators in dimensions up to three are always robust to interactions.

In $d = 3$ in particular, the QSH phases associated to the three planar surfaces of the insulator are robust to interactions. Meanwhile, there are two interaction-enabled phases associated to the top-dimensional cell, coming from $\operatorname{coker} \mathrm{F2I}^3_{\mathit{strong}}$, as discussed in Example 4.1.59. There is also an interaction-enabled phase coming from the zero cell, encoded in $\operatorname{coker} \mathrm{F2I}^0_{\mathit{strong}}$; see [378] for a physics interpretation of this phase. ◇

Remark 4.1.80. Weak phases are protected by discrete translation symmetries. However, their free and interacting classifications can also be used to study situations in which a certain form of crystalline disorder called a dislocation is present. Dislocations are localized disruptions in the crystalline order that can host topologically protected modes—for example, three-dimensional TIs can host one-dimensional helical modes [379]. In [301], Ran developed a criterion for when these protected modes could occur: if $\vec{B}$ is the Burgers vector of the dislocation, and $\vec{M}$ is a vector of $(d-1)$-dimensional indices, then helical modes can exist if $\vec{B} \cdot \vec{M}$ takes the possible nonzero value.

Our framework can generalize this condition to the interacting setting and to other symmetry types. Consider the group $\widetilde{\mho}^{d+2}_H(S^1 \vee \cdots \vee S^1)$ classifying codimension-one weak indices for systems with symmetry type H. By the suspension isomorphism and wedge axiom, this group is isomorphic to $(\mho^{d+1}_H)^{\oplus d}$, so we may consider its elements to be d-vectors $\vec{M}$ of d spacetime-dimensional invertible field theories. We may still consider the Burgers vector

$\vec{B}$ to be a vector in the cubic lattice $\mathbb{Z}^d$. Then the generalized dislocation pairing $\vec{B}\cdot\vec{M}$ is valued in $\mho_H^{d+1}$. For example, the three-dimensional weak topological insulator and the helical modes condition of [301] concerns the pairing of $\vec{M} \in (\mho^4_{\mathrm{Pin}^{\tilde{c}+}})^{\oplus 3} \cong (\mathbb{Z}_2)^3 \subset \mho^5_{\mathrm{Pin}^{\tilde{c}+}}$ with a vector $\vec{B} \in \mathbb{Z}^3$, which takes a binary value in $\mho^4_{\mathrm{Pin}^{\tilde{c}+}} \cong \mathbb{Z}_2$.

4.1.7 Physical Justification

Why would the spatial unit cell torus appear in the classification of SPT phases? We know it appears under T-duality in the free fermion classification, yet so far we have not rigorously derived it on the interacting side. In the literature, this has been justified for the group cohomology classification through the crystalline equivalence principle, where it appears as the classifying space of spatial translations $B\mathbb{Z}^d$ [286], thus requiring a rather strong physical statement in order to employ the topology of the unit cell. Here we provide a general derivation from first principles and a functional analysis perspective as to why, independently of the choice of cohomology theory that classifies SPT phases, the unit cell spatial torus arises in the classification of discrete translation invariant topological phases.

4.1.7.1 Physical Interpretation

Let us consider the single particle Hilbert space in one dimension

$$\mathcal{H}_1 = L^2(\mathbb{R}). \tag{4.1.81}$$

There is a spatial decomposition of $\mathcal{H}_1$ (direct integral decomposition) using a unit cell $\mathbf{T} = B\mathbb{Z}$ [380]. Let

$$\mathcal{H}_1{}' = L^2([0, 2\pi]) \tag{4.1.82}$$

where $[0, 2\pi]$ is the unit cell (with boundary) and consider the direct integral over the periodic unit cell $\mathbf{T}$ [380]

$$\mathcal{V}_1 = \int_{\mathbf{T}}^{\oplus} \mathcal{H}_1{}' \frac{d\theta}{2\pi}. \tag{4.1.83}$$

There is a unitary equivalence $U\colon L^2(\mathbb{R}) \to \mathcal{V}_1$ given by

$$(U\psi)_\theta(\tilde{x}) = \sum_{n=-\infty}^{\infty} e^{-n\theta}\psi(\tilde{x} + 2\pi n). \tag{4.1.84}$$

This decomposition can be extended to arbitrary $\mathbb{R}^d$:

$$(U\psi)_{(\theta_1,\ldots,\theta_d)}(\tilde{x}) = \sum_{n\in\mathbb{Z}^d} e^{-\sum_{j=1}^d \theta_j n_j}\psi(\vec{\tilde{x}} + \sum_i^d n_i a_i), \tag{4.1.85}$$

where a_i are the chosen lattice generators that will be relevant according to the periodicity of the system's Hamiltonian $\mathcal{H}$.

Let us now consider N particles in d-dimensions:

$$\mathcal{H}_N = \bigotimes_{i=1}^{N} L^2(\mathbb{R}_i^d) \approx L^2(\mathbb{R}^{dN}). \tag{4.1.86}$$

Let us now consider the diagonal action of $\mathbb{Z}^d$ on $\mathbb{R}^{dN}$ given by

$$(\vec{x}_1, ..., \vec{x}_n) \longmapsto (\vec{x}_1 + \vec{a}, ..., \vec{x}_n + \vec{a}) \tag{4.1.87}$$

with $\vec{a} \in \mathbb{Z}^d$. Consider the quotient by the action $\mathbb{R}^{dN}/\mathbb{Z}^d$. This is homeomorphic to $\mathbf{T}^d \times \mathbb{R}^{d(N-1)}$; however, we have yet to choose a fundamental region for this action. We choose a convenient fundamental region $R(N,d)$ that is better described with the following coordinates:

$$\vec{y}_1 = \frac{1}{N}\sum_{i=1}^{N} \vec{x}_i, \tag{4.1.88}$$

$$\vec{y}_j = \vec{x}_j - \vec{x}_1 \ \forall j \neq 1. \tag{4.1.89}$$

Thus, we choose the fundamental region $R(N,d)$ in which our diagonal action simply becomes translation in the $\vec{y}_1$- direction. With this choice of fundamental region, we consider the Hilbert space $L^2(R(N,d)) = \mathcal{V}'_N$ and we can construct a direct integral Hilbert space $\mathcal{V}_N = \int^{\oplus}_{T^d} \mathcal{V}'_N d\theta$. Furthermore, there is a unitary equivalence $U\colon L^2(\mathbb{R}^{dN}) \to \mathcal{V}_N$ given by essentially the same formula

$$(U\psi)_{(\theta_1,...,\theta_d)}(\vec{y}_1, \vec{y}_2, ..., \vec{y}_N) = \sum_{m\in\mathbb{Z}^d} e^{-\sum_{j=1}^{d}\theta_j m_j}\psi\left(\vec{y}_1 + \sum_{i}^{d} m_i a_i, ..., \vec{y}_N\right). \tag{4.1.90}$$

Notice that exchanging the x_i's is equivalent to exchanging the $\vec{y}_i$'s whenever $i \neq 1$. However, exchanging $\vec{x}_1$ with $\vec{x}_j$ is equivalent to $\vec{y}_1 \mapsto \vec{y}_1$ and $\vec{y}_j \mapsto -\vec{y}_j$. Thus, we have the added bonus that the above decomposition works equally well for bosonic or fermionic wave functions. The above construction is well defined for every finite N and thus we can rewrite Fock space $\mathcal{F}(L^2(\mathbb{R}^d))$ as

$$\mathcal{V}_\infty = \bigoplus_{N\geq 0} \int^{\oplus}_{\mathbf{T}^d} \mathcal{V}'_N \frac{d\theta_1...d\theta_d}{(2\pi)^d}. \tag{4.1.91}$$

A system has discrete translation symmetry if for every finite N, its Hamiltonian $\mathcal{H}_N$ commutes with a representation of a lattice Λ in $\mathcal{F}(L^2(\mathbb{R}^d))$, i.e. that the Hamiltonian diagonalizes in the direct integral decomposition

$$U\mathcal{H}_N U^{-1} = \int^{\oplus}_{\mathbf{T}^d} \mathcal{H}_N(\theta_1, ..., \theta_d)\frac{d\theta_1...d\theta_d}{(2\pi)^d} \tag{4.1.92}$$

where the fiber Hamiltonian $\mathcal{H}(\theta_1, ..., \theta_d)$ must satisfy certain quasi-periodic boundary conditions on $[0, 2\pi]^d$. Thus our many-body Hamiltonian can be formally viewed as a map

$$\mathcal{H}\colon \mathbf{T}^d \to \mathcal{L}_d := \mathcal{L}\left(\bigoplus_{N\geq 0} L^2(R(N,d))\right) \tag{4.1.93}$$

to some subspace of the self-adjoint operators on $\bigoplus_{N\geq 0} L^2(R(N,d))$. (It is actually a section on the operators on a "Fock bundle" over $\mathbf{T}^d$ but for now this is not relevant.)

The standard definition of a topological phase is as an equivalence class of systems under adiabatic evolution, which generally is generally interpreted as homotopy classes of maps [7] between the Hamiltonians of different systems. For systems that have discrete translation symmetry, this is equivalent to

$$[\mathbf{T}^d, \mathcal{L}_d]. \tag{4.1.94}$$

A similar but slightly simpler analysis can be made for systems with full translation symmetry. Above, the torus arose as the quotient $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$; in the case of symmetry under the full translation group $\mathbb{R}^d$ we'd have instead just a point $\mathrm{pt} = \mathbb{R}^d/\mathbb{R}^d$. The end result would be that phases in dimension d protection by continuous translation symmetry are given by

$$[\mathrm{pt}, \mathcal{L}_d^{strong}], \tag{4.1.95}$$

where $\mathcal{L}_d^{strong}$ is a space of Hamiltonians with full translation symmetry.

For Hamiltonians with a unique ground state and short range interactions, there is a notion of symmetry protected topological phase (SPT), which must satisfy certain properties under stacking. This led Kitaev to conjecture that for SPT phases, the spaces of Hamiltonians $\{\mathcal{L}_d^{strong}\}$ form a spectrum; a modest extension of Kitaev's conjecture would be to conjecture that the $\{\mathcal{L}_d\}$ also form a spectrum. If one assumes these conjectures, then Equation (4.1.94) and Equation (4.1.95) become d-dimensional cohomology groups of the torus and of the point for the generalized cohomology theories given by the respective spectra. How are the two spectra, Kitaev's $\{\mathcal{L}_d^{strong}\}$, and $\{\mathcal{L}_d\}$ related? We can look for inspiration in the setting of free fermion systems, where, as we explain below in §4.1.7.2, the two spectra are the same: $\Sigma^{s-2}KO$ (where the parameter s specifies the symmetry class). It seems reasonable to further extend Kitaev's conjecture adding that the analogous phenomenon happens for SPT phases: the spectrum for weak and strong phases are equivalent—though this is not so easy to see from this analytical description in terms of Hamiltonians. Notice that if one assumes this extension of Kitaev's conjecture, then the James splitting (Lemma 4.1.69) implies that weak phases are built out of lower dimensional strong phases as in Lemma 4.1.75.

There are now quite a few proposals for what Kitaev's spectrum should be: see [89, 376, 381, 382] among others. Here we have been considering the ansatz proposed by Freed–Hopkins [1, 100] based on invertible TFTs as a low-energy limit. However, we have to interpret this ansatz carefully. On the one hand, the original conjecture in [1] is that *strong* SPT phases are classified by the group $\mho_H^{d+2}(\mathrm{pt})$. Our proposed extension of Kitaev's conjecture then says that weak SPT phases should then by classified by $\mho_H^{d+2}(\mathbf{T}^d)$, where the correct physical interpretation of $\mathbf{T}^d$ is the spatial torus.

4.1.7.2 Kitaev's conjecture for free fermions and T-duality

The different proposals for classifying SPT phases in $d+1$ dimensions generally satisfy Kitaev's conjecture [383, 384] that SPT phases form a spectrum in the sense of algebraic topology, i.e. there is always a map

$$\Omega SPT_{d+1}(G) \longrightarrow SPT_d(G), \tag{4.1.96}$$

which is often a homotopy equivalence. Any of the tenfold way classifications for type $H(s)$ in d dimensions can be essentially written in the form $KR^{s-2}(\mathbb{T}^d)$ (or $K^{s-2}(\mathbb{T}^d)$ in the complex cases) [7, 294] so that phases of free fermion systems in dimension d are classified by the group

$$FF_d(\mathbb{Z}^d, H(s)) \cong KR^{s-2}(\mathbb{T}^d). \tag{4.1.97}$$

As we can see, this a priori seems to contradict Kitaev's conjecture since the degree of the K-theory group, $s-2$, does not depend at all on the dimension of the system. However, if we use the T-duality isomorphism (4.1.64a) we have

$$KR^{s-2}(\mathbb{T}^d) \cong KO^{d+s-2}(\mathbf{T}^d), \tag{4.1.98}$$

which is the correct instantiation of Kitaev's conjecture when we have discrete translation invariance and weak phases. We can go on to see this satisfies Kitaev's original formulation of the conjecture for strong free fermion SPT phases by mapping to the bottom cell, i.e.

$$FF_d^{strong}(\mathbb{Z}^d, H(s)) \cong KO^{d+s-2}(\mathrm{pt}). \tag{4.1.99}$$

So the spectrum $\Sigma^{s-2}KO$ satisfies the conjecture. Hence T-duality plays an important role in Kitaev's conjecture for free fermions.

Remark 4.1.100. We can interpret Kitaev's proposal for strong phases on the interacting side field-theoretically as follows, compare [89, §3.2]. An element of $\mho^{d+1}_{H(s)}(X)$ is a d-space-dimensional invertible field theory of symmetry type s equipped with a background field valued in X. In particular, the suspension isomorphism $\mho^{d+1}_{H(s)}(S^k) \cong \mho^{d+1}_{H(s)} \oplus \mho^{d+1-k}_{H(s)}$ can be interpreted as follows. Given a d-dimensional invertible quantum field theory Z with background field valued in S^k, this gives a $(d-k)$-dimensional invertible quantum field theory by sending[20]

$$N^{d-k} \longmapsto Z(N \times S^k \xrightarrow{pr} S^k). \tag{4.1.101}$$

The factor $\mho^{d+1}_{H(s)}$ simply corresponds to elements of $\mho^{d+1}_{H(s)}(S^k)$ which do not depend on the S^k-valued background field.

We can in this way also reinterpret elements of $\mho^{d+2}_{H(s)}(\mathbf{T}^d)$ as $(d+1)$-dimensional field theories with target $\mathbf{T}^d$. By taking appropriate cells of $\mathbf{T}^d$, we can interpret the lower-dimensional terms in James splitting. Specifically, if $\mathbf{T}^k \subseteq \mathbf{T}^d$ is a subtorus corresponding to a subset of $\{1, \ldots, d\}$ of size k, we can define a map $\mho^{d+2}_{H(s)}(\mathbf{T}^d) \to \mho^{d+2-k}_{H(s)}$ as

$$N^{d-k} \longmapsto Z(N \times \mathbf{T}^k \to \mathbf{T}^d) \tag{4.1.102}$$

where the map to $\mathbf{T}^d$ is induced by the inclusion $\mathbf{T}^k \subseteq \mathbf{T}^d$.

4.1.8 Examples: the Tenfold Way

We now apply Lemma 4.1.75 to compute the groups of phases for weak topological insulators and superconductors in spatial dimensions 1, 2, and 3 with symmetry types according to the tenfold way.

For illustrative purposes, we will discuss Class AII in detail. Class AII includes some of the first weak phases studied in the literature, the weak topological insulators (WTIs) of [327]

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[20] If N is an $H(s)$-manifold, we use the stably framed structure on S^k arising from the standard isomorphism $TS^k \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{k+1}$ to make $N \times S^k$ into an $H(s)$-manifold.

and [5]. We focus on dimension $3+1$. As reviewed in Example 4.1.13, free phases of band insulators are given by the group $KR^{-4}(\overline{\mathbb{T}}^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}$. Under T-duality, this group is isomorphic to the real KO-theory group $KO^{-1}(\mathbf{T}^3)$ on the spatial torus. Using the James splitting (Lemma 4.1.69) we obtain an alternative computation of this group as

$$KO^{-1}(\mathbf{T}^3) \cong KO^{-1}(\mathrm{pt}) \oplus 3KO^{-2}(\mathrm{pt}) \oplus 3KO^{-3}(\mathrm{pt}) \oplus KO^{-4}(\mathrm{pt}) \tag{4.1.103}$$

$$= \mathbb{Z}_2 \oplus (\mathbb{Z}_2)^3 \oplus \mathbb{Z}. \tag{4.1.104}$$

Here blue summands are the strong phases, regarded as a subgroup of the group of weak phases (see §4.1.5). The red summands are captured by the invariant that counts the number of valence bands. This is generally not an interesting invariant and is often excluded in the physics literature; see Remark 4.1.19 for details. With respect to the James splitting, strong phases (blue summands) correspond to the bottom cell of the spatial torus, and the valence-band-counting invariant (red summands) corresponds to the top cell.[21]

In (4.1.103), the strong phase in the $\mathbb{Z}_2$ summand $KO^{-1}(\mathrm{pt})$ is detected by the Fu-Kane-Mele invariant [5], while the $\mathbb{Z}$ summand counts the number of valence bands. The remaining $(\mathbb{Z}_2)^3$ coming from $3KO^{-2}(\mathrm{pt})$ comprises the weak phases, which may be viewed as quantum spin Hall (QSH) phases localized to two-dimensional surfaces of the three-dimensional material.

We compute the interacting classification using the James splitting as well. We have

$$\begin{aligned}\mho^5_{\mathrm{Pin}^{\tilde{c}+}}(\mathbf{T}^3) &\cong \mho^5_{\mathrm{Pin}^{\tilde{c}+}}(\mathrm{pt}) \oplus 3\mho^4_{\mathrm{Pin}^{\tilde{c}+}}(\mathrm{pt}) \oplus 3\mho^3_{\mathrm{Pin}^{\tilde{c}+}}(\mathrm{pt}) \oplus \mho^2_{\mathrm{Pin}^{\tilde{c}+}}(\mathrm{pt}) \\ &\cong \mathbb{Z} \oplus (\mathbb{Z}_2)^3 \oplus (\mathbb{Z}_2)^3.\end{aligned} \tag{4.1.105}$$

Once again the colors illustrate which phases come from the top and bottom cells of $\mathbf{T}^d$. The first triplet of $\mathbb{Z}_2$'s comes from the interacting weak phases, while the second triplet of $\mathbb{Z}_2$'s comes from the interacting strong phases in 2d. That the weak $(\mathbb{Z}_2)^3$ injects into $\mho^5_{\mathrm{Pin}^{\tilde{c}+}}(\mathbf{T}^3)$ corroborates the expectation that these weak phases are stable under interactions [385, §III.A], [386].

In $d=3$, there is also a $(\mathbb{Z}_2)^2$ classification of interaction-enabled phases. These phases are all strong; i.e. they arise from $\mho^5_{\mathrm{Pin}^{\tilde{c}+}}$ applied to the bottom cell of the spatial torus. These interaction-enabled phases were originally found in the physics literature in [378] and connected to bordism theory in [339].

Corollary 4.1.106 includes the classification for all three relevant dimensions.

Corollary 4.1.106 (Symmetry class AII, $s=-2$)**.** *The free-to-interacting map for the groups of weak phases in Altland–Zirnbauer type AII is:*

d	$\ker(\mathrm{F2I}) \longrightarrow$	$KO^{d-4}(\mathbf{T}^d) \xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Pin}^{\tilde{c}+}}(\mathbf{T}^d) \longrightarrow$	$\mathrm{coker}\,(\mathrm{F2I})$
1	0	$\mathbb{Z}$	$\mathbb{Z}$	0
2	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0
3	0	$\mathbb{Z} \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}_2^2$

(4.1.107)

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[21] As T-duality incorporates Poincaré duality, the description is opposite for the Brillouin torus: the strong phases correspond to the top cell of $\overline{\mathbb{T}}^d$, and the valence-band-counting invariant corresponds to the bottom cell, see §4.1.5.

Literature Note 4.1.108. The classification of these free weak phases has been studied from many perspectives in the literature: see, for example, De Nittis-Gomi [387–391] , Fiorenza-Monaco-Panati [392], and Kaufmann-Li-Wehefritz-Kaufmann [393–396]. $\mathrm{Pin}^{\tilde{c}+}$ bordism groups in these dimensions were first computed by Freed–Hopkins [1, Theorem 9.87].

We continue with the seven other real symmetry types and the two complex symmetry types.

Corollary 4.1.109 (Symmetry class D, $s = 0$)**.** *The free-to-interacting map for the groups of weak phases in Altland–Zirnbauer type D is:*

d	$\ker(\mathrm{F2I}) \longrightarrow$	$KO^{d-2}(\mathbf{T}^d) \xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Spin}}(\mathbf{T}^d) \longrightarrow$	$\operatorname{coker}(\mathrm{F2I})$
1	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0
2	0	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^2$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^2$	0
3	0	$\mathbb{Z}^3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^3$	0

(4.1.110)

Literature Note 4.1.111. Hughes [317, 6m29s] gives a classification of the weak free phases in these dimensions, modulo the band-counting $\mathbb{Z}_2$ subgroup. Freed–Hopkins [100, Example 2.3] study class D phases on a torus in dimension 2, and observe that the free-to-interacting map is an isomorphism in that dimension; this example is also studied by Ran [301]. The spin bordism groups used in Corollary 4.1.109 were first computed by Milnor [278].

Corollary 4.1.112 (Symmetry class BDI, $s = 1$)**.** *The free-to-interacting map for the groups of weak phases in Altland–Zirnbauer type BDI is:*

d	$\ker(\mathrm{F2I}) \to$	$KO^{d-1}(\mathbf{T}^d) \overset{\mathrm{F2I}}{\to}$	$\mho^{d+2}_{\mathrm{Pin}^-}(\mathbf{T}^d) \to$	$\operatorname{coker}(\mathrm{F2I})$
1	$8\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	0
2	$(8\mathbb{Z})^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8^2 \oplus \mathbb{Z}_2$	0
3	$(8\mathbb{Z})^3$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8^3 \oplus \mathbb{Z}_2$	0

(4.1.113)

Literature Note 4.1.114. Corollary 4.1.112 is in agreement with work of Xiao-Kawabata-Luo-Ohtsuki-Shindou [397], who study 3d class BDI weak phases and conclude that the three $\mathbb{Z}$-valued invariants of weak topological phases remain nontrivial in the presence of interactions.

The pin^- bordism groups used in this computation were first computed by Anderson–Brown–Peterson [192]. The Majorana chain with its time-reversal symmetry is a 1-dimensional strong phase in class BDI, generating the $\mathbb{Z}$ summand of free phases and the $\mathbb{Z}_8$ summand of interacting phases in $d = 1$ [104, 398–400]; this phase thus defines higher-dimensional weak phases and so contributes to the kernel of the free-to-interacting map in all higher degrees. There are no interaction-enabled phases in dimensions 6 and below.

Corollary 4.1.115 (Symmetry class AI, $s = 2$)**.** *The free-to-interacting map for the groups of weak phases in Altland–Zirnbauer type AI is:*

d	$\ker(\mathrm{F2I})$	$\to KO^d(\mathbf{T}^d)$	$\xrightarrow{\mathrm{F2I}} \mho^{d+2}_{\mathrm{Pin}^{\tilde{c}-}}(\mathbf{T}^d)$	$\to \operatorname{coker}(\mathrm{F2I})$
1	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
3	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2^4$	$\mathbb{Z}_2^4$

(4.1.116)

Literature Note 4.1.117. The pin$^{\tilde{c}-}$ bordism groups used in this computation were computed by Freed–Hopkins [1, Theorem 9.87]. Like in Corollary 4.1.112, the interaction-enabled weak phases in dimensions 2 and 3 are a consequence of the interaction-enabled *strong* phase in dimension 1 in this class; this strong phase appears in Freed–Hopkins [1, Corollary 9.95] (they use spacetime dimension, so call that phase 2-dimensional). De Nittis-Gomi [401, 402] classify the free weak phases in class AI using Real-equivariant vector bundles.

Corollary 4.1.118 (Symmetry class CI, $s = 3$)**.** *The free-to-interacting map for the groups of weak phases in Altland–Zirnbauer type CI is:*

d	$\ker(\mathrm{F2I})$	$\to KO^{d+1}(\mathbf{T}^d)$	$\xrightarrow{\mathrm{F2I}} \mho^{d+2}_{\mathrm{Pin}^{h+}}(\mathbf{T}^d)$	$\longrightarrow \operatorname{coker}(\mathrm{F2I})$
1	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
3	$4\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2^3$

(4.1.119)

Literature Note 4.1.120. The pin^{h+} bordism groups appearing in this computation were computed by Freed–Hopkins [1, Theorem 9.97]; they use the notation G^+ for Pin^{h+}. Just as in Corollary 4.1.115, the interaction-enabled strong phase in dimension 1, first studied by Freed–Hopkins [1, Corollary 9.101], gives rise to interaction-enabled weak phases in higher dimensions.

Corollary 4.1.121 (Symmetry class C, $s = 4$)**.** *The free-to-interacting map for the groups of weak phases in Altland–Zirnbauer type C is:*

d	$\ker(\mathrm{F2I})$	$\to KO^{d+2}(\mathbf{T}^d)$	$\xrightarrow{\mathrm{F2I}} \mho^{d+2}_{\mathrm{Spin}^h}(\mathbf{T}^d)$	$\to \operatorname{coker}(\mathrm{F2I})$
1	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
3	0	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^3$

(4.1.122)

Literature Note 4.1.123. Both a free weak phase and an interaction-enabled weak phase contribute to the classification in $d = 3$. The spin^h bordism groups used in the computation in Corollary 4.1.121 were first computed by Freed–Hopkins [1, Theorem 9.97], though they use the notation G^0 for Spin^h.

Corollary 4.1.124 (Symmetry class CII, $s = -3$)**.** *The free-to-interacting map for the groups of weak phases in Altland–Zirnbauer type CII is:*

d	$\ker(\mathrm{F2I})$	$\to KO^{d+3}(\mathbf{T}^d)$	$\overset{\mathrm{F2I}}{\to} \mho^{d+2}_{\mathrm{Pin}^{h-}}(\mathbf{T}^d)$	$\to \mathrm{coker}\,(\mathrm{F2I})$
1	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
2	$(2\mathbb{Z})^2$	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	0
3	$(2\mathbb{Z})^3$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^2$

(4.1.125)

Literature Note 4.1.126. Just as in Corollary 4.1.112, the generator of the group of one-dimensional strong phases becoming torsion in the interacting classification ([1, Corollary 9.103]) gives rise to weak phases in the kernel of the free-to-interacting map in higher dimensions. In $d = 3$ there are also two interaction-enabled phases. Xiao-Kawabata-Luo-Ohtsuki-Shindou [397] briefly discuss 3d weak interacting phases in class CII: they claim that the three $\mathbb{Z}$-valued topological indices of free phases remain nontrivial in the presence of interactions, which our computations verify.

The pin^{h-} bordism groups that we used in this computation are computed by Freed–Hopkins [1, Theorem 9.97]; they write G^- for Pin^{h-}.

Corollary 4.1.127 (Symmetry class DIII, $s = -1$)**.** *The free-to-interacting map for the groups of weak phases in Altland–Zirnbauer type DIII is:*

d	$\ker(\mathrm{F2I})$	$\to KO^{d-3}(\mathbf{T}^d)$	$\overset{\mathrm{F2I}}{\to} \mho^{d+2}_{\mathrm{Pin}^{+}}(\mathbf{T}^d)$	$\to \mathrm{coker}\,(\mathrm{F2I})$
1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
2	0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_2$	0
3	$16\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2^6$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_2^6$	0

(4.1.128)

Literature Note 4.1.129. The pin$^+$ bordism groups used in this computation were computed by Giambalvo [193, §2]. De Nittis-Gomi [403] classify the free weak phases in this class using equivariant cohomology. The weak phases arising from $d = 1$ are stable to interactions. The strong phase in $d = 3$ breaks from generating a $\mathbb{Z}$ of free phases to a $\mathbb{Z}_{16}$ of interacting phases; this has been argued in many different ways: see for example [1, 340, 351, 360, 382, 384, 404–409].

Corollary 4.1.130 (Symmetry class A, $s = 0$)**.** *The free-to-interacting map for the groups of weak phases in Altland–Zirnbauer type A is:*

d	$\ker(\mathrm{F2I})$	$\to K^{d}(\mathbf{T}^d)$	$\overset{\mathrm{F2I}}{\to} \mho^{d+2}_{\mathrm{Spin}^{c}}(\mathbf{T}^d)$	$\to \mathrm{coker}\,(\mathrm{F2I})$
1	0	$\mathbb{Z}$	$\mathbb{Z}$	0
2	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}^2$	$\mathbb{Z}$
3	0	$\mathbb{Z} \oplus \mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}^6$	$\mathbb{Z}^3$

(4.1.131)

Literature Note 4.1.132. Varjas-de Juan-Lu [410, §II] observe that the Hall conductivity, a $\mathbb{Z}$-valued invariant of weak free class D phases in 3d, remains a well-defined, $\mathbb{Z}$-valued invariant of interacting systems, which is consistent with our computations. The $\mathbb{Z}$ summand in $d = 2$ corresponding to the integer quantum Hall effect (a strong phase) is stable under interactions and contributes to weak phases in $d = 3$. There are also interaction-enabled phases in $d = 2$, which contribute to weak interaction-enabled phases in the $d = 3$ Chern insulator. The calculation of spin^c bordism groups is attributed to Anderson–Brown–Peterson [367]; see Bahri-Gilkey [137, §1] for an explicit description.

Corollary 4.1.133 (Symmetry class AIII, $s = 1$)**.** *The free-to-interacting map for the groups of weak phases in Altland–Zirnbauer type A is:*

d	$\ker(\mathrm{F2I}) \longrightarrow$	$K^{d-1}(\mathbf{T}^d) \xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Pin}^c}(\mathbf{T}^d) \longrightarrow$	$\operatorname{coker}(\mathrm{F2I})$
1	$4\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_4$	0
2	$(4\mathbb{Z})^2$	$\mathbb{Z}^2$	$\mathbb{Z}_4^2$	0
3	$8\mathbb{Z} \oplus (4\mathbb{Z})^3$	$\mathbb{Z} \oplus \mathbb{Z}^3$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4^3$	$\mathbb{Z}_2$

(4.1.134)

Literature Note 4.1.135. Corollary 4.1.133 is in agreement with work of Xiao-Kawabata-Luo-Ohtsuki-Shindou [397], who discuss how the weak $\mathbb{Z}$-valued indices in 3d remain nontrivial in the presence of interactions. See also Claes-Hughes [297], who study the behavior of these indices under disorder. De Nittis-Gomi [411] study free weak phases in this class in terms of objects called *chiral vector bundles*.

Like in Corollaries 4.1.112 and 4.1.124, the nontrivial kernel of the one-dimensional strong free-to-interacting map ([1, Corollary 9.91]) produces phases in the kernel of the weak free-to-interacting map in higher dimensions. The generator of the group of one-dimensional strong phases is the class of the Su-Schrieffer-Heeger model [412]. In three dimensions there is an additional free phase that breaks down, as well as an interaction-enabled phase. Pin^c bordism groups were first computed by Bahri-Gilkey [137, 138].

4.2 The Bott Spiral

4.2.1 Introduction

In this section, we develop free-to-interacting maps for a generalization of the tenfold way due to Queiroz–Khalaf–Stern [6]. Working with symmetry classes that slightly generalize the usual Altland–Zirnbauer classes for the tenfold way, they studied the interacting classifications of free-fermion models. They computed via a *dimensional reduction* approach that the order of the interacting class of their models was a power of two dependent on both the symmetry class and the dimension. Critically, in contrast with the eight-periodic "Bott clock" of the free phase classification, the interacting order grows with the degree, forming an increasing "Bott spiral" of real phases, and similarly for the complex case.

Our homotopical model for their work not only reproduces their results, but also generalizes their methods—we develop an invertible field theory model for their dimensional reduction map, showing that it commutes with the free-to-interacting maps we define. We interpret the "spiraling" nature of the interacting phase classification as a failure of Morita invariance. Finally, we compute the relevant subgroups of invertible field theories modeling primed phases in all dimensions.

4.2.2 Fermionic groups, *K*-theory, and bordism

In this work, we study the relationship between free fermionic phases, which are classified by K-theory groups [7, 294], and interacting fermionic phases, which under the SPT-bordism conjecture [1, 413, 414] are classified by the Anderson dual of bordism. We present the first unified account to symmetries in both classifications using the newly developed language of fermionic groups.

4.2.2.1 The definition of a fermionic group

Definition 4.2.1 (Benson [318, §7])**.** A *fermionic group* consists of

- a compact Lie group G_f,
- a central element $(-1)^F$ of order two called *fermion parity*,
- and a homomorphism $\theta\colon G_f \to \mathbb{Z}_2$

such that $\theta((-1)^F) = 0$.

A morphism of fermionic groups $(G, ((-1)^F)_G, \theta_G) \to (H, ((-1)^F)_H, \theta_H)$ is a group homomorphism $\varphi\colon G \to H$ such that $\varphi(((-1)^F)_G) = ((-1)^F)_H$ and $\theta_G = \theta_H \circ \varphi$. With these morphisms, fermionic groups form a category $\mathsf{FermGrp}$.

Let $\mathbb{Z}_2^F \subseteq G_f$ denote the subgroup generated by $(-1)^F$. Physically, G_f represents the group of symmetries of a given physical systems with fermions. This group potentially contains time-reversing (or antiunitary) symmetries, which are sent to -1 by θ. We denote by $G_b \coloneqq G_f/\mathbb{Z}_2^F$ the associated *bosonic symmetry group*.

Definition 4.2.2. Let $(-1)^F \in G_f \xrightarrow{\theta} \mathbb{Z}_2$ be a fermionic group. The *associated twist* τ is the element

$$(\theta, \omega) \in H^1(BG_b; \mathbb{Z}_2) \times H^2(BG_b; \mathbb{Z}_2), \tag{4.2.3}$$

where the first element is the induced map $\theta\colon BG_b \to B\mathbb{Z}_2$ and the second element, ω, classifies the extension

$$1 \longrightarrow \mathbb{Z}_2^F \longrightarrow G_f \longrightarrow G_b \longrightarrow 1. \tag{4.2.4}$$

The terminology 'twist' comes from twisted cohomology, as τ also specifies a twist of G_b-equivariant KO-theory (see [415], [324, §3.4.2]) and spin bordism of BG_b. We will see in later sections how such G_b-equivariant cohomology theories classify G_b-protected SPT phases. The twists will properly implement the desired physical properties such as how time-reversal symmetry T reverses the orientation of spacetime and how the algebra of symmetries is related to $(-1)^F$ (e.g. for class DIII, $T^2 = (-1)^F$).

Example 4.2.5. If $\rho\colon G_b \to \mathrm{O}_n$ is a representation, there are two associated fermionic groups defined by pulling back the $\mathrm{Pin}^\pm$ extensions of O_n along ρ:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_2^F \arrow[r] \arrow[d, equal] & G_f \arrow[r] \arrow[d] & G_b \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & \mathbb{Z}_2^F \arrow[r] & \mathrm{Pin}_n^\pm \arrow[r] & \mathrm{O}_n \arrow[r] & 1.
\end{tikzcd} \tag{4.2.6}$$

For the pin^+ twist, $\theta = w_1(\rho)$ and $\omega = w_2(\rho)$; for the pin^- twist, $\theta = w_1(\rho)$ and $\omega = w_2(\rho) + w_1(\rho)^2$. The universal case $G_b = \mathrm{O}_n$ and $\rho = \mathrm{id}$ defines fermionic group structures on $\mathrm{Pin}_n^\pm$ with $(\mathrm{Pin}_n^\pm)_b \cong \mathrm{O}_n$. ◇

The fermionic tensor product of G and H is a certain fermionic group whose underlying bosonic group is $G_b \times H_b$:

Definition 4.2.7 (Benson [318, §7])**.** The *fermionic tensor product* $G \hat{\times} H$ of (G, θ_G, ω_G), (H, θ_H, ω_H) is the set

$$\frac{G \times H}{\mathbb{Z}_2} \tag{4.2.8}$$

equipped with the operation

$$(g_1 \hat{\otimes} h_1)(g_2 \hat{\otimes} h_2) = \begin{cases} (-1)^F g_1 g_2 \hat{\otimes} h_1 h_2, & \text{if } \theta(h_1) = 1 \text{ and } \theta(g_2) = 1 \\ g_1 g_2 \hat{\otimes} h_1 h_2, & \text{if } \theta(h_1) = 0 \text{ or } \theta(g_2) = 0. \end{cases} \tag{4.2.9}$$

$G \hat{\times} H$ has the structure of a fermionic group, with the following data.

1. $\theta(g, h) = \theta_G(g) + \theta_H(h)$ for $g \in G$ and $h \in H$; since θ_G and θ_H vanish on the $\mathbb{Z}/2$ subgroup we quotiented by, this is indeed well-defined.
2. The central subgroup $\mathbb{Z}_2^F \times \mathbb{Z}_2^F \subset G \times H$ maps to a central subgroup of $G \hat{\times} H$ isomorphic to $\mathbb{Z}/2$ under the quotient map; we define $\mathbb{Z}_2^F$ of $G \hat{\times} H$ to be this $\mathbb{Z}/2$ subgroup.

Proposition 4.2.10. $\mathsf{FermGrp}$ *is symmetric monoidal with respect to the fermionic tensor product; the unit is* $\mathbb{Z}_2^F$*.*

Lemma 4.2.11. *Let G_f and H_f be fermionic groups.*

1. *There is a natural isomorphism $(G_f \hat{\times} H_f)_b \xrightarrow{\cong} G_b \times H_b$ of groups.*
2. *If (θ_G, ω_G), resp. (θ_H, ω_H) denote the associated twists to G_f, resp. H_f, then inside $H^*(B(G_b \times H_b); \mathbb{Z}/2)$, the associated twist to $G_f \hat{\times} H_f$ is*

$$(\theta_G + \theta_H, \omega_G + \theta_G\theta_H + \omega_H). \tag{4.2.12}$$

Proof. Part (1) follows directly from Definition 4.2.7. The formula for θ of $G_f \hat{\times} H_f$ follows from the natural identification $H^1(BK; \mathbb{Z}/2) \cong \mathrm{Hom}(K, \mathbb{Z}/2)$ of abelian groups, where K is a discrete group, and the fact that we defined θ of $G_f \hat{\times} H_f$ by adding θ_G and θ_H.

The formula for ω is less trivial. It must be a natural formula in the data of two fermionic groups; the only natural cohomology classes associated to this data are θ_G, θ_H, ω_G, and ω_H and classes built out of them (e.g. products or Steenrod squares). Moreover, the formula must be symmetric in G_f and H_f, because the fermionic tensor product is symmetric monoidal. Therefore there are constants $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}/2$ such that

$$\omega_{G_f \hat{\times} H_f} = \lambda_1(\omega_G + \omega_H) + \lambda_2(\theta_G^2 + \theta_H^2) + \lambda_3(\theta_G\theta_H). \tag{4.2.13}$$

We will show $\lambda_1 = 1$, $\lambda_2 = 0$, and $\lambda_3 = 1$.

1. To show $\lambda_1 = 1$, let G_f be any fermionic group with $\theta_G = 0$ but $\omega_G \neq 0$ and let $H_f = \mathbb{Z}_2^F$. For example, we could take $G_f = \mathrm{SU}_2$ with the fermionic group structure discussed in [319, Example 9]. Then $G_f \hat{\times} H_f \cong G_f$ as fermionic groups, so $\omega_{G_f \hat{\times} H_f} = \omega_G$.

2. To show $\lambda_2 = 0$, let G_f be any fermionic group with $\theta_G \neq 0$, $\theta_G^2 \neq 0$, and $\omega_G = 0$, and let $H_f = \mathbb{Z}_2^F$ again. For example, we could take $G = \mathrm{Pin}_1^+$. Then $G_f \hat{\times} H_f \cong G_f$ again, and now $\omega_{G_f \hat{\times} H_f} = 0$ and $\theta_G^2 \neq 0$, forcing $\lambda_2 = 0$.

3. To show $\lambda_3 = 1$, let $G_f = H_f = \mathbb{Z}/4 \times \mathbb{Z}_2^F$, so that $G_b = H_b = \mathbb{Z}/4$, with $\theta_G, \theta_H \colon \mathbb{Z}/4 \to \mathbb{Z}/2$ both equal to the unique nontrivial homomorphism. We also have $\omega_G = \omega_H = 0$. There is an isomorphism $H^*(B\mathbb{Z}/4; \mathbb{Z}/2) \cong \mathbb{Z}/2[x, y]/(x^2)$ with $|x| = 1$ and $|y| = 2$; since θ_G and θ_H are nontrivial, they both equal x, and we deduce $\theta_G^2 = \theta_H^2 = 0$.

   Thus for this choice of G_f and H_f, (4.2.13) simplifies to

   $$\omega_{G_f \hat{\times} H_f} = \lambda_3\theta_G\theta_H, \tag{4.2.14}$$

   and all we have to do is show that the extension

   $$1 \longrightarrow \mathbb{Z}_2^F \longrightarrow G_f \hat{\times} H_f \longrightarrow \mathbb{Z}/4 \times \mathbb{Z}/4 \longrightarrow 1, \tag{4.2.15}$$

   which is classified by $\omega_{G_f \hat{\times} H_f}$, is not split. The split extension $\mathbb{Z}_2^F \times \mathbb{Z}/4 \times \mathbb{Z}/4$ is abelian, so it suffices to show $G_f \hat{\times} H_f$ is not abelian, which can be checked directly from Definition 4.2.7. ☐

Remark 4.2.16. In other words, the symmetric monoidal category $\mathsf{FermGrp}$ is equivalent to the category of data (G_b, θ, ω) with the monoidal structure on θ and ω satisfying (4.2.12).

Remark 4.2.17. It is no coincidence that (4.2.12) looks like the Whitney sum formula. Indeed, recall that BO is a grouplike E_∞-space under $\oplus$. The quotient $BO/B\mathrm{Spin}$ with respect to the subgroup $B\mathrm{Spin}$ is equivalent as a space to the 2-type $B\mathbb{Z}_2 \times B^2\mathbb{Z}_2$. However, $\oplus$ introduces a nontrivial group structure on this space via the Whitney sum formula.

This group structures also arises naturally from the following context. Let $\mathsf{sLine}_\mathbb{R}$ be the category of real superlines, i.e. one-dimensional real supervector spaces with invertible linear maps. It obtains an E_∞-structure from the fact that it is the maximal Picard groupoid inside the symmetric monoidal category $\mathsf{sVect}_\mathbb{R}$ with its interesting braiding 4.2.21. This makes the inclusion $B\mathrm{Spin}/BO = B\mathbb{Z}_2 \times B^2\mathbb{Z}_2 \hookrightarrow B\mathbb{Z}_2 \times B^2\mathbb{R}^\times \cong B\mathsf{sLine}_\mathbb{R}$ into an E_∞-map.

Now suppose G_f and H_f are fermionic groups with associated twists $BG_b \to B\mathbb{Z}_2 \times B^2\mathbb{Z}_2$ and $BH_b \to B\mathbb{Z}_2 \times B^2\mathbb{Z}_2$ We get a new twist on $BG_b \times BH_b$ as

$$BG_b \times BH_b \to (B\mathbb{Z}_2 \times B^2\mathbb{Z}_2) \times (B\mathbb{Z}_2 \times B^2\mathbb{Z}_2) \xrightarrow{\oplus} B\mathbb{Z}_2 \times B^2\mathbb{Z}_2. \tag{4.2.18}$$

It follows by Lemma 4.2.11 that this is the twist associated to $G_f \mathbin{\hat{\times}} H_f$. The following fermionic group will play a prominent role in the rest of this chapter:

Definition 4.2.19. We define the fermionic group $E_{\ell,k}$ to be

$$E_{\ell,k} \coloneqq \underbrace{\mathrm{Pin}_1^+ \mathbin{\hat{\times}} \cdots \mathbin{\hat{\times}} \mathrm{Pin}_1^+}_{\ell \text{ copies}} \mathbin{\hat{\times}} \underbrace{\mathrm{Pin}_1^- \mathbin{\hat{\times}} \cdots \mathbin{\hat{\times}} \mathrm{Pin}_1^-}_{k \text{ copies}}. \tag{4.2.20}$$

Thus by Lemma 4.2.11 $(E_{\ell,k})_b$ is the elementary abelian 2-group[22] $(\mathbb{Z}/2)^{\ell+k}$.

4.2.2.2 Superalgebras and Clifford algebras

In this paper we frequently work with $\mathbb{Z}/2$-graded objects and require the tensor product to obey the Koszul sign rule with respect to the grading. We summarize the definitions we use; the reader who is familiar with superalgebras and Clifford algebras is welcome to jump to Definition 4.2.28.

Fix a field F of characteristic not equal to 2; in this paper we will only need $F = \mathbb{R}$ and $F = \mathbb{C}$. A *super vector space* V over F is a $\mathbb{Z}/2$-graded vector space over F. The tensor product $V \mathbin{\hat{\otimes}} W$ of two super vector spaces V and W is given the following grading: if $v \in V$ and $w \in W$ are homogeneous elements, then $v \mathbin{\hat{\otimes}} w$ has degree $|v| + |w| \in \mathbb{Z}/2$. We use the Koszul sign rule isomorphism $V \mathbin{\hat{\otimes}} W \xrightarrow{\cong} W \mathbin{\hat{\otimes}} V$, defined on pure tensors of homogeneous elements by

$$v \mathbin{\hat{\otimes}} w \longmapsto (-1)^{|v||w|} w \mathbin{\hat{\otimes}} v. \tag{4.2.21}$$

This formula extends uniquely to define an isomorphism on the entire tensor product, and is the symmetry data in a symmetric monoidal category sVect_F of super vector spaces and grading-preserving F-linear maps.

A *superalgebra* A over F is a super vector space that is also an algebra, such that the unit has even grading and the multiplication $A \mathbin{\hat{\otimes}} A \to A$ preserves the grading. The tensor product of superalgebras is canonically a superalgebra with the product map defined to extend linearly from the following formula on pure tensors of homogeneous elements:

$$(v_1 \mathbin{\hat{\otimes}} w_1) \cdot (v_2 \mathbin{\hat{\otimes}} w_2) \coloneqq (-1)^{|v_2||w_1|} v_1 v_2 \mathbin{\hat{\otimes}} w_1 w_2. \tag{4.2.22}$$

The *opposite* A^{op} is defined to be equal to A as a graded vector space but with multiplication

$$a^{\mathsf{op}} \cdot b^{\mathsf{op}} = (-1)^{|a||b|} (ba)^{\mathsf{op}}. \tag{4.2.23}$$

A straightforward computation shows $(A \mathbin{\hat{\otimes}} B)^{\mathsf{op}} \cong A^{\mathsf{op}} \mathbin{\hat{\otimes}} B^{\mathsf{op}}$. A *superhomomorphism* $A \to B$ between two superalgebras is a grading-preserving algebra homomorphism. A *supermodule*

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[22] Here and throughout this chapter, "2-groups" are in the sense of p-groups, i.e. finite groups of order 2^s for some s. We do not use "n-groups" in the sense of, e.g., [416].

over a superalgebra A is a super vector space M which is also a A-module, such that the action map preserves the grading.[23] One can thus define super bimodules and bimodule homomorphisms to be grading-preserving linear maps which commute with the action maps.

Definition 4.2.24. Let S be a finite set, $\sigma\colon S \to \{\pm 1\}$ be a function, and $F[S]$ denote the k-vector space of functions $S \to F$. The *Clifford algebra* $C\ell(F, S, \sigma)$ is the F-algebra

$$C\ell(F, S, \sigma) := T(F[S])/(s^2 = \sigma(s), st = -ts \mid s, t \in S), \tag{4.2.25}$$

where $T(V)$ denotes the tensor algebra on a vector space V. We make the following notational shortcuts:

- If $S = \{1, \dots, \ell\} \cup \{-1, \dots, -k\}$ and $\sigma(x) = \mathrm{sign}(x)$, we will write $C\ell_{\ell,k}(F)$ for $C\ell(F, S, \sigma)$.
- If $k = 0$ in the previous example, we will just write $C\ell_\ell(F)$; if instead $\ell = 0$ we will write $C\ell_{-k}(F)$.
- When $F = \mathbb{R}$ we will just write $C\ell_{\ell,k}$, $C\ell_\ell$, and $C\ell_{-k}$ for $C\ell_{\ell,k}(\mathbb{R})$, $C\ell_\ell(\mathbb{R})$, and $C\ell_{-k}(\mathbb{R})$, respectively.
- When $F = \mathbb{C}$ we will write $\mathbb{C}\ell_s$ for $C\ell_s(\mathbb{C})$.

We give $C\ell(F, S, \sigma)$ the structure of a super vector space by specifying that a product of n elements of S has grading n mod 2; since the quotient in (4.2.25) is by an ideal spanned by products of even numbers of elements of S, this grading, a priori defined on $T(F[S])$, descends to $C\ell(F, S, \sigma)$ as claimed. Moreover, multiplication preserves the grading and therefore $C\ell(F, S, \sigma)$ is a superalgebra. Note that $C\ell_{\ell,k}(\mathbb{C}) \cong C\ell_{\ell+k}(\mathbb{C})$ as superalgebras, since multiplying generators by i changes the sign of their square.

Lemma 4.2.26 (Atiyah–Bott–Shapiro [3, Proposition 1.6])**.** *Let S and σ be as in Definition 4.2.24, and suppose $S = S_1 \amalg S_2$. Then the inclusions $S_1 \to S$ and $S_2 \to S$ extend to a natural isomorphism of superalgebras*

$$C\ell(F, S_1, \sigma|_{S_1}) \mathbin{\hat{\otimes}_F} C\ell(F, S_2, \sigma|_{S_2}) \xrightarrow{\cong} C\ell(F, S, \sigma). \tag{4.2.27}$$

In particular, for $\ell_1, m_2, k_1, k_2 \geq 0$, $C\ell_{\ell_1,k_1} \mathbin{\hat{\otimes}} C\ell_{\ell_2,k_2} \cong C\ell_{\ell_1+\ell_2,k_1+k_2}$. Note that ${C\ell_{\ell,k}}^{\mathsf{op}} \cong C\ell_{k,\ell}$.

4.2.2.3 Twisted group superalgebras

Every fermionic group has a canonical group superalgebra associated to it:

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[23] Note that this does *not* mean that a superalgebra A acts on a supermodule M by grading-preserving endomorphisms! Instead, it means that if $a \in A$ and $m \in M$ are odd, then $a \cdot m$ has the same grading as $a \mathbin{\hat{\otimes}} m$ in $A \mathbin{\hat{\otimes}} M$: even.

Definition 4.2.28 (Benson [318, §7])**.** Let G_f be a finite fermionic group. The *fermionic group algebra of G_f* is the real superalgebra

$$\mathbb{R}^f[G_f] \coloneqq \frac{\mathbb{R}[G_f]}{((-1)^F + 1)} \tag{4.2.29}$$

graded by θ.

This construction is covariantly functorial in G_f.

Remark 4.2.30. An alternative but equivalent way to define $\mathbb{R}^f[G_f]$ is to start with the slightly smaller graded group algebra of G_b and twist the multiplication. Specifically, choose a group 2-cocycle[24] $\widetilde{\omega}\colon G_b \times G_b \to \mathbb{Z}/2$ representing the cohomology class $\omega \in H^2(BG_b; \mathbb{Z}/2)$; then $\mathbb{R}^f[G_f, \widetilde{\omega}]$ is defined to be the vector space $\mathbb{R}[G_b]$ with the multiplication modified to satisfy the formula

$$g_1 \cdot_{\mathbb{R}^f[G_f,\widetilde{\omega}]} g_2 \coloneqq (-1)^{\widetilde{\omega}(g_1,g_2)} g_1 g_2. \tag{4.2.31}$$

As G_b is a natural basis for $\mathbb{R}^f[G_f, \widetilde{\omega}]$, this formula extends uniquely to define a product on $\mathbb{R}^f[G_f, \widetilde{\omega}]$; one can check this product is associative and that $e \in G_b$ is the unit for it. One can use θ to define a $\mathbb{Z}/2$-grading, and the resulting superalgebra is isomorphic to $\mathbb{R}^f[G_f]$ as defined in Definition 4.2.28. Moreover, the data of a coboundary $\partial\alpha = \widetilde{\omega} - \widetilde{\omega}'$ in $C^2(G_b; \mathbb{Z}/2)$ can be used to construct a natural isomorphism $\psi_\alpha\colon \mathbb{R}^f[G_f, \widetilde{\omega}] \to \mathbb{R}^f[G_f, \widetilde{\omega}']$, so this construction does not depend on the choice of cocycle representative $\widetilde{\omega}$.

Remark 4.2.32. The major reason for restricting to finite groups is that the group algebra of infinite groups is not suitable for our application. Many results will generalize at least to compact Lie groups if we replace the fermionic group algebra by a twisted group C^*-algebra, with the notable exception of the wrong-way map for comparing symmetries of Corollary 4.2.57.

Example 4.2.33. We calculate $\mathbb{R}^f[\mathrm{Pin}_1^\pm] \cong C\ell_{\pm 1}$ using Remark 4.2.30. This computation is also done a different way by Stolz [419, Proposition 8.3] and Albuquerque–Majid [420].

Since $(\mathrm{Pin}_1^\pm)_b \cong \mathrm{O}_1 \cong \mathbb{Z}/2$, we need to compute $\mathbb{Z}/2$-valued cocycles on $\mathbb{Z}/2$. For any $n \geq 0$, $H^n(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2$. In general, if M carries the trivial G-action, the group of cocycles $Z^1(G; M)$ is the group of homomorphisms $G \to M$; thus we may represent the nontrivial element $x \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ by $\mathrm{id}\colon \mathbb{Z}/2 \to \mathbb{Z}/2$. Thus $x^2 \in H^2(B\mathbb{Z}/2; \mathbb{Z}/2)$ is represented by the 2-cocycle $\mathrm{id} \smile \mathrm{id}\colon \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$ with the formula

$$(\mathrm{id} \smile \mathrm{id})(a, b) \coloneqq a \cdot b. \tag{4.2.34}$$

The bundle $\sigma \to B\mathbb{Z}/2$ is a nontrivial line bundle, so $w_1(\sigma) \neq 0$, which forces $w_1(\sigma) = x$, and $w_2(\sigma) = 0$. Thus Pin_1^+ has $\theta = x$ and $\omega = 0$. That is, we do not modify the multiplication in $\mathbb{R}[\mathrm{O}_1]$:

$$\mathbb{R}^f[\mathrm{Pin}_1^+] \cong \mathbb{R}[\mathrm{O}_1] \cong \mathbb{R}[a]/(a^2 = 1), \tag{4.2.35}$$

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[24] The cocycle condition is necessary in order for the multiplication in (4.2.31) to be associative. An alternative approach is to allow all cochains and construct nonassociative algebras; for example, Albuquerque–Majid [417, 418] construct the octonion algebra as a twisted group $\mathbb{R}$-algebra for $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ with a cochain that is not a cocycle.

with $\mathbb{Z}/2$-grading odd on a and even on scalars. This is $C\ell_1$ by definition.

For Pin_1^-, we modify the multiplication by the formula in (4.2.34). This cocycle is only nonzero when evaluated on $(1,1)$, so only has the effect of multiplying a^2 by $(-1)^{(\mathrm{id}\smile\mathrm{id})(1,1)} = -1$. Since θ is the same as it was for Pin_1^+, we again have a odd and scalars even; thus

$$\mathbb{R}^f[\mathrm{Pin}_1^-] \cong \mathbb{R}[a]/(a^2=-1) \cong C\ell_{-1}. \tag{4.2.36}$$

◇

Example 4.2.37 (Stolz [419, Proposition 8.3])**.** Consider the fermionic group $C := \{\pm 1, \pm i\} \cong \mathbb{Z}/4$ with θ trivial and $(-1)^F = -1 \in C$. This fermionic group has $G_b \cong \mathbb{Z}/2$ and ω nontrivial, so following a similar line of reasoning as in Example 4.2.33 one learns that as superalgebras, $\mathbb{R}^f[C] \cong \mathbb{C}$ in purely even grading. $\mathbb{C}$ and $C\ell_{-1}$ are isomorphic as algebras, but not as superalgebras, as the gradings do not match. ◇

Maschke's theorem generalizes readily to the setting of twisted group superalgebras.

Lemma 4.2.38 (Ganter-Kapranov [68, Example 3.5.1])**.** *For all finite fermionic groups G_f, $\mathbb{R}^f[G_f]$ and $\mathbb{R}^f[G_f]\otimes\mathbb{C}$ are semisimple.*

Proposition 4.2.39. *The functor $G_f \mapsto \mathbb{R}^f[G_f]$ from $\mathsf{FermGrp}$ to the category sAlg_F^1 of superalgebras and super algebra homomorphisms is symmetric monoidal.*

Proof. The proof follows as in the ungraded setting, with the exception that we must construct a natural isomorphism

$$\mathbb{R}^f[G_f \hat{\times} H_f] \xrightarrow{\cong} \mathbb{R}^f[G_f] \hat{\otimes} \mathbb{R}^f[H_f] \tag{4.2.40}$$

using the graded tensor product. We will use the point of view of Remark 4.2.30, meaning we will choose group 2-cocycle representatives $\widetilde{\omega}_G \colon G_b \times G_b \to \mathbb{Z}/2$ and $\widetilde{\omega}_H \colon H_b \times H_b \to \mathbb{Z}/2$ for ω_G, resp. ω_H. Thanks to Lemma 4.2.11, this induces a cocycle representative of $\omega_{G_f \hat{\times} H_f}$, namely

$$\begin{aligned} \widetilde{\omega}_{G_f \hat{\times} H_f}((g_1 \hat{\otimes} h_1),(g_2 \hat{\otimes} h_2)) &:= \widetilde{\omega}_G(g_1,g_2) + (\theta_G \times \theta_H)(((g_1 \hat{\otimes} h_1),(g_2 \hat{\otimes} h_2))) + \widetilde{\omega}_H(h_1,h_2) \\ &= \widetilde{\omega}_G(g_1,g_2) + \theta_H(h_1)\theta_G(g_2) + \widetilde{\omega}_H(h_1,h_2). \end{aligned} \tag{4.2.41}$$

There is a super vector space isomorphism $\Psi \colon \mathbb{R}^f[G_f \hat{\times} H_f, \widetilde{\omega}_{G_f \hat{\times} H_f}] \to \mathbb{R}^f[G_f,\widetilde{\omega}_G] \hat{\otimes} \mathbb{R}^f[H_f,\widetilde{\omega}_H]$ defined by observing that each has $G_b \times H_b$ as a canonical basis.

Given $g_1, g_2 \in G_b$, $h_1, h_2 \in H_b$, the product in $\mathbb{R}^f[G_f \hat{\times} H_f, \widetilde{\omega}_{G_f \hat{\times} H_f}]$ has the formula

$$\begin{aligned} (g_1 \hat{\otimes} h_1)(g_2 \hat{\otimes} h_2) &\overset{(4.2.31)}{=} (-1)^{\widetilde{\omega}_{G_f \hat{\times} H_f}((g_1 \hat{\otimes} h_1),(g_2 \hat{\otimes} h_2))} g_1 g_2 \hat{\otimes} h_1 h_2 \\ &\overset{(4.2.11)}{=} (-1)^{\widetilde{\omega}_G(g_1,g_2)+\widetilde{\omega}_H(h_1,h_2)+\theta_H(h_1)\theta_G(g_2)} g_1 g_2 \hat{\otimes} h_1 h_2, \end{aligned} \tag{4.2.42}$$

which is $(-1)^{\theta(h_1)\theta(g_2)}$ times the product in $\mathbb{R}^f[G_f,\widetilde{\omega}_G]$ of g_1 and g_2 tensor the product in $\mathbb{R}^f[H_f,\widetilde{\omega}_H]$ of h_1 and h_2. This is exactly the product in $\mathbb{R}^f[G_f,\widetilde{\omega}_G]\hat{\otimes}\mathbb{R}^f[H_f,\widetilde{\omega}_H]$. Therefore Ψ, a priori a natural super vector space isomorphism, is also a natural superalgebra isomorphism. The final step is to get rid of the dependence on the choice of cocycle representative, which can be done using the invariant section construction as in Remark 4.2.30. □

The real Clifford algebras $C\ell_{\ell,k}$ are a twisted group algebra of a fermionic group with underlying bosonic group $\mathbb{Z}_2^{\ell+k}$; compare [421].

Corollary 4.2.43 (Albuquerque–Majid [420])**.** *With $E_{\ell,k}$ as in Definition 4.2.19, there is a canonical isomorphism of superalgebras*

$$\mathbb{R}^f[E_{\ell,k}] \xrightarrow{\cong} C\ell_{\ell,k}. \tag{4.2.44}$$

Proof. This follows by combining Lemma 4.2.26, Example 4.2.33, , and Proposition 4.2.39. ☐

See also Ichikawa–Tachikawa [422, Lemma 2.7].

Explicitly, $E_{\ell,k}$ is the extraspecial 2-group

$$\langle e_1, \dots, e_{\ell+k}, (-1)^F : e_1^2 = \dots e_\ell^2 = 1, e_{\ell+1}^2 = \dots e_{\ell+k}^2 = (-1)^F,$$
$$e_ie_j = (-1)^F e_je_i,\ i \neq j, (-1)^F e_i = e_i(-1)^F, ((-1)^F)^2 = 1\rangle,$$

the subset of $Cl_{\ell,k}$ generated by the set S as a group. Note that θ agrees with the grading on $C\ell_{\ell,k}$.

Remark 4.2.45. Recall from Remark 4.2.17 that the interesting E_∞-structure on $B\mathbb{Z}_2 \times B^2\mathbb{Z}_2$ can be interpreted algebraically using the Picard 1-groupoid of real superlines. The maximal Picard 2-groupoid $\mathrm{Pic}(\mathsf{sAlg}_\mathbb{R})$ of the symmetric monoidal Morita bicategory of real superalgebras $\mathsf{sAlg}_\mathbb{R}$ also inherits an E_∞-structure. It is equivalent to the more complicated E_∞-space $\pi_{\leq 2}\,\mathrm{Pic}(KO) \cong \mathbb{Z}_8 \times B\mathbb{Z}_2 \times B^2\mathbb{Z}_2$ [423, Theorem 5.27]. Here $\mathbb{Z}_8 = \pi_0\,\mathrm{Pic}(\mathsf{sAlg}_\mathbb{R})$ is the super Brauer group (also called Brauer-Wall group) of the real numbers [320]. The connected cover of this space recovers $BO/B\mathrm{Spin}$, corresponding to the fact that the automorphisms of the trivial algebra $\mathbb{R}$ are given by $\mathsf{sLine}_\mathbb{R}$. See [424, Section 5.2] for a definition of Brauer spectrum over a ring.

Remark 4.2.46. Using the diagonal map $X \to X \times X$ and the E_∞-space $\pi_{\leq 2}\,\mathrm{Pic}(KO)$, we obtain an E_∞-structure on the set of maps $X \to \mathbb{Z}_8 \times B\mathbb{Z}_2 \times B^2\mathbb{Z}_2$. The induced group structure on

$$\mathbb{Z}/8 \times H^1(X;\mathbb{Z}/2) \times H^2(X;\mathbb{Z}/2) \tag{4.2.47}$$

agrees with the (real) graded Brauer group of a space introduced in [415].

4.2.2.4 The Clifford module quotient group

In this section we will define a sequence of groups $\mathrm{CMQ}_n(A)$ for $n \in \mathbb{N}$ associated to a real superalgebra A. The importance of this group is that for some simple $\mathbb{Z}/2$-graded C^*-algebras, $\mathrm{CMQ}_n(A)$ is isomorphic to the K-theory of A in degree n. In some cases of interest we have $\mathrm{CMQ}_n(A) \ncong K_n(A)$, in which case we need more sophisticated descriptions of $K_n(A)$.

Let A be a real superalgebra and let Mod_A be the category of finitely generated projective modules over A, where we forget the grading on A. The set $\pi_0(\mathsf{Mod}_A)$ of isomorphism classes is a monoid under $\oplus$. Following the approach of Atiyah–Bott–Shapiro [3], we define

Definition 4.2.48. The *Clifford module quotient* group of A is the quotient of monoids

$$\mathrm{CMQ}_0(A) := \frac{\pi_0(\mathsf{Mod}_{A \hat{\otimes} Cl_{+1}})}{\pi_0(\mathsf{Mod}_{A \hat{\otimes} Cl_{1,1}})}. \tag{4.2.49}$$

More generally, $\mathrm{CMQ}_n(A)$ is defined as $\mathrm{CMQ}_0(Cl_{-n} \hat{\otimes} A)$.

Note that $A\hat{\otimes}Cl_{+1}$-modules are equivalent to graded A-modules. Even though $\pi_0(\mathsf{Mod}_A)$ is not a group, it can be shown that $CMQ_n(A)$ is a group.

Remark 4.2.50. For infinite-dimensional trivially graded algebras A, Definition 4.2.48 gives K-theory in degree zero. Indeed, in that case an $A\hat{\otimes}C\ell_{+1}$-module is nothing but a pair (M_1, M_2) of A-modules. It extends to a $A\hat{\otimes}C\ell_{1,1}$-module if and only if $M_1 \cong M_2$. Therefore the map

$$(M_1, M_2) \longmapsto M_1 - M_2 \tag{4.2.51}$$

defines an isomorphism from $CMQ_0(A)$ to the Grothendieck group of the monoid $\pi_0(\mathsf{Mod}_A)$ as desired.

Remark 4.2.52. If A is a finite-dimensional C^*-superalgebra, then $\mathrm{CMQ}_n(A)$ models the K-theory of A for all $n \in \mathbb{N}$. However, for infinite-dimensional algebras these groups do not match. For example, if $A = C(S^1, \mathbb{C})$ with trivial grading, a short computation shows that $\mathrm{CMQ}_1(A) = 0$; see [425, Example 3.61]. On the other hand, we have

$$K_1(C(S^1)) = K^1(S^1) = K^1(\mathrm{pt}) \oplus \tilde{K}^1(S^1) \cong \mathbb{Z}, \tag{4.2.53}$$

where $K_n(A)$ denotes K-theory of the (ungraded) C^*-algebra A in degree n.

Remark 4.2.54. If A is finite-dimensional and semisimple, Mod_A is the category of finite-dimensional modules over A. Indeed, finitely generated modules are exactly the finite-dimensional modules. These are automatically projective because A is semisimple.

We will mainly be interested in the case where the algebra is a real Clifford algebra $B = C\ell_{\ell,k}$ or a complex Clifford algebra $B = \mathbb{C}\ell_k$. In that case, Equation (4.2.49) recovers Atiyah–Bott–Shapiro's group A_k for $B = C\ell_{0,k}$ and similarly A_k^c for $B = \mathbb{C}\ell_k$. It follows from [3, Theorem 11.5] that

$$K_p(C\ell_{\ell,k}) \cong KO_{p-\ell+k}(*) \quad K_p(\mathbb{C}\ell_k) \cong KU_{p-k}(*). \tag{4.2.55}$$

4.2.2.5 The *K*-theory of fermionic group algebras

The classification of free fermion phases protected by a fermionic group G_f is in terms of the K-theory of its fermionic group algebra. In the case that G_f is finite, this K-theory is given by $\mathrm{CMQ}(\mathbb{R}^f[G_f])$. For readers more familiar with twisted equivariant K-theory, the following proposition might be useful.

Proposition 4.2.56. *Given a finite fermionic group G_f, the $\tau = (\theta, \omega)$-twisted equivariant KO-theory of a point of degree p is given by the K-theory of the $\mathbb{Z}/2$-graded real algebra $\mathbb{R}^f[G_f]\hat{\otimes}Cl_{-p}$.*

See e.g. [426]

Next we want to know about functoriality of this K-theory in morphisms of fermionic groups. The main issue is that K-theory is usually only covariant in algebra homomorphisms, while we want to have maps in the other direction. Luckily, homomorphisms $A \to B$ of algebras such that B is finitely generated over A, which is our situation of interest, do induce wrong-way maps.

Corollary 4.2.57. *Taking the K-theory of a fermionic group defines a contravariant functor* $\mathsf{FermGrp}^{finite} \to \mathsf{Ab}$.

Proof. Since the fermionic group algebra construction gives a covariant functor from $\mathsf{FermGrp}^{\text{finite}}$ to finite-dimensional superalgebras, it suffices to construct a contravariant functor from that to Ab. Let $f\colon A \to B$ be a map of finite-dimensional superalgebras. Recall that the pullback module f^*M is defined to be M as a set with the B-action $b \cdot m := f(b)m$. If M is a finitely generated projective B-module, the pullback module f^*M is a finitely generated projective A-module. Given B-modules M_1, M_2, there is an isomorphism $f^*(M_1 \oplus M_2) \cong f^*(M_1) \oplus f^*(M_2)$. It also follows that if the B-module M extends to a $Cl_{+1} \otimes B$-module, then the A-module f^*M extends to a $Cl_{+1} \otimes A$-module. We thus obtain a homomorphism $K_0(B) \to K_0(A)$. ☐

The following proposition follows by [3]:

Proposition 4.2.58. *If* $G_f = E_{\ell,k}$, *then*

$$K_p(\mathbb{R}^f[E_{\ell,k}]) \cong K_p(Cl_{\ell,k}) \cong KO_{p-\ell+k}(\text{pt}). \tag{4.2.59}$$

In particular, $(1,1)$-periodicity [3] takes the form

$$K_p(Cl_{\ell,k}) \cong K_p(Cl_{\ell-1,k-1}). \tag{4.2.60}$$

Example 4.2.61. Consider the fermionic group $E_{1,1}$ generated by e, f, and $(-1)^F$ with

$$e^2 = 1,\ f^2 = (-1)^F,\ ((-1)^F)^2 = 1,\ ef = (-1)^F fe,\ e(-1)^F = (-1)^F e,\ f(-1)^F = (-1)^F f. \tag{4.2.62}$$

This group is abstractly isomorphic to the symmetry group of a square with rotation f and reflection e, but as a fermionic group e and f are time-reversing, so the fermionic group algebra is $\mathbb{R}^f[E_{1,1}] \cong Cl_{1,1} \cong M_{1|1}(\mathbb{R})$, the algebra of 2×2 real matrices with off diagonal matrices in odd grading. This algebra is Morita equivalent to $\mathbb{R}$ and thus $K_p(\mathbb{R}^f[E_{1,1}]) \cong K_p(\mathbb{R}) \cong KO_p(\text{pt})$, corresponding to class D, so this is a fermionic group which on the free fermionic side is indistinguishable from class D. Following [6], we will call this D′. ◇

More generally, we can define the primed version of any symmetry group:

Definition 4.2.63. Let G_f be a fermionic group. Define $G'_f := G_f \hat{\times} E_{1,1}$

This construction allows us to study *primed symmetry types* in Section 4.2.5.

Note that there is a morphism $G_f \to G'_f$ in $\mathsf{FermGrp}$ given by inclusion in the first factor.

Proposition 4.2.64. *The map* $K(\mathbb{R}^f[G'_f]) \to K(\mathbb{R}^f[G_f])$ *induced by the inclusion* $G_f \to G'_f$ *is zero.*

Proof. In terms of algebras, this corresponds to the inclusion into the first factor $i\colon \mathbb{R}^f[G_f] \to \mathbb{R}^f[G_f] \otimes C\ell_{1,1}$, where we used $\mathbb{R}^f[G'_f] \cong \mathbb{R}^f[G_f] \otimes C\ell_{1,1}$. If M is a $\mathbb{R}^f[G'_f]$-module, the induced $\mathbb{R}^f[G_f]$-module i^*M is given by forgetting the second tensor factor. This $\mathbb{R}^f[G_f]$-module extends to a $\mathbb{R}^f[G_f] \otimes Cl_{-1}$-module by construction and so $i^*M = 0 \in K(\mathbb{R}^f[G_f])$. ☐

To accommodate for the complex symmetry classes, we will also consider the K-theory of the complexified fermionic group algebra, $K(\mathbb{R}^f[G_f] \otimes_{\mathbb{R}} \mathbb{C})$. In particular, for $G_f = E_{\ell,k}$ we obtain the K-theory of complex Clifford algebras

$$K(\mathbb{R}^f[E_{\ell,k}] \otimes_{\mathbb{R}} \mathbb{C}) = K(C\ell_{\ell,k} \otimes_{\mathbb{R}} \mathbb{C}) \cong KU_{k-\ell}(\mathrm{pt}), \tag{4.2.65}$$

by Equation (4.2.55). Over $\mathbb{C}$ the K-theory becomes 2-periodic, related to the fact that

$$C\ell_{1,0} \otimes_{\mathbb{R}} \mathbb{C} \cong C\ell_{1,0} \otimes_{\mathbb{R}} \mathbb{C} \quad \text{and} \quad C\ell_{2,0} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{End}(\mathbb{C}^{1|1}). \tag{4.2.66}$$

Tensoring with the complex numbers can be realized at the fermionic group level by extending the group $\mathbb{Z}/2^F$ of norm one scalars in $\mathbb{R}$ to the group $U(1)$ of norm one scalars in $\mathbb{C}$. Namely, given a fermionic group G_f, we can consider $G_f \hat{\times} U(1)$, which fits into a short exact sequence

$$1 \to U(1) \to G_f \hat{\times} U(1) \to G_b \to 1. \tag{4.2.67}$$

This short exact sequence corresponds to the element in $H^3(BG_b; \mathbb{Z})$ that is the Bockstein of $\omega \in H^2(BG_b; \mathbb{Z}/2)$.

Remark 4.2.68. Physically, we can think of $G_f \hat{\times} U(1)$ as 'G_f plus an additional $U(1)$-charge Q'. Even though we will not discuss fermionic group C^*-algebras of infinite fermionic groups here, one can think of modules over $\mathbb{R}^f[G_f] \otimes_{\mathbb{R}} \mathbb{C}$ as 'representations of $G_f \hat{\times} U(1)$ of unit charge.' See [325].

Remark 4.2.69. One could consider generalizing Equation (4.2.67) to arbitrary central extensions

$$1 \to U(1) \to K \to G_b \to 1, \tag{4.2.70}$$

equipped with a homomorphism $\theta\colon G_b \to \mathbb{Z}/2$. In other words, now the element of $H^3(BG_b; \mathbb{Z})$ need not be in the image of $H^2(BG_b; \mathbb{Z}/2) \to H^3(BG_b; \mathbb{Z})$.

Even more generally, it can be physically relevant to look at non-central extensions, where θ does not need to be induced by a homomorphism $\phi\colon K \to \mathbb{Z}/2$ satisfying $zk = kz^{\phi(h)}$ for all $k \in H_f$ and $z \in U(1) \subseteq K$. Physically speaking, these groups can contain symmetries that anticommute with the charge operator. Such symmetries are called charge conjugation operators in high energy physics and are related to particle-hole symmetries in condensed matter [427]. This group-theoretic structure has been applied to SPT phases in [294], who call it an *extended QM symmetry class*.

4.2.2.6 Fermionic group twists of spin bordism

The data of a fermionic group G_f also defines a variant of the notion of spin structure. This variant is the tangential structure present on spacetime in a fermionic field theory with G_f-symmetry. We first go over the generalities, then specialize to several examples that will be helpful in our analysis of the Bott spiral.

Definition 4.2.71. Let X be a space, $a \in H^1(X; \mathbb{Z}/2)$, and $b \in H^2(X; \mathbb{Z}/2)$. An (X, a, b)-*twisted spin structure* on a vector bundle $V \to M$ is data of a map $f\colon M \to X$ and trivializations of the classes $w_1(V) - f^*(a)$ and $w_2(V) + w_1(V)^2 - f^*(b)$. We consider two (X, a, b)-twisted spin structures to be equivalent if they lie in the same path component of such data.

This notion of twisted spin structure was studied by B.L. Wang [428, Definition 8.2] in the special case $a = 0$.

Example 4.2.72 (Vector bundle twists)**.** Suppose there is a vector bundle $E \to X$ with $w_1(E) = a$ and $w_2(E) = b$. Then the Whitney sum formula implies an (X, a, b)-twisted spin structure on $V \to M$ is equivalent to the data of the map $f\colon M \to X$ and a spin structure on $V \oplus f^*(E)$. In this case, we will call (X, a, b)-twisted spin bordism a *vector bundle twist* of spin bordism. See MacAlpine [429] and Stolz [419, §2.9] for related but different notions. ◇

Remark 4.2.73. Given (X, a, b), it is not always possible to find a vector bundle $E \to X$ with $w_1(E) = a$ and $w_2(E) = b$: see the discussion in [141, 430–433]. This is typically not a problem in practice: all twisted spin structures appearing in [1, 96, 98, 102, 145, 434] can be realized as vector bundle twists, and the same is true for all twists appearing in this paper. See [123, 435] for an example of a twisted spin structure that cannot be realized as a vector bundle twist, applied to an anomaly cancellation question in supergravity.

Example 4.2.74. Let G_f be a fermionic group, and, as in Definition 4.2.2, let $G_b \coloneqq G_f/\mathbb{Z}_2^F$ and (θ, ω) be the associated twist. Then the data (BG_b, θ, ω) defines a notion of twisted spin structure which we call a *G_f-twisted spin structure.* ◇

Definition 4.2.75 (Stolz [419, §2.6])**.** Given a fermionic group G_f and an integer $d \geq 0$, define the Lie group

$$H_d(G_f) \coloneqq (\mathrm{Pin}_d^+ \mathbin{\hat{\times}} G_f)_{ev}, \tag{4.2.76}$$

where "$(-)_{ev}$" means to take $\theta^{-1}(0)$, the even subgroup. The double cover $\mathrm{Pin}_d^+ \to \mathrm{O}_d$ induces a map $H_d(G_f) \to \mathrm{O}_d$ given by killing the G_f factor of the fermionic tensor product.

Taking the colimit, we also allow $d = \infty$, defining a topological group $H(G_f)$ with a map to O; apply the classifying space functor and we obtain a tangential structure in the sense of Lashof [112], which we call an *$H(G_f)$-structure.*

Proposition 4.2.77 (Stehouwer [319])**.** *Let G_f be a fermionic group, $G_b \coloneqq G_f/\mathbb{Z}_2^F$, and (θ, ω) be the associated twist. The following are equivalent data on a vector bundle $V \to M$.*

1. *A (BG_b, θ, ω)-twisted spin structure.*
2. *An $H(G_f)$-structure.*

Stehouwer does not state the result in exactly this way, but it follows from [319, Proposition 14], the definition of the pullback, and the Whitney sum formula. The maps $H_d(G_f) \to \mathrm{O}_d$ commute with the inclusions $H_d(G_f) \to H_{d+1}(G_f)$ (induced by $\mathrm{Pin}_d^+ \hookrightarrow \mathrm{Pin}_{d+1}^+$) and $\mathrm{O}_d \hookrightarrow \mathrm{O}_{d+1}$, so there is a notion of bordism of manifolds with G_f-twisted spin structures on their tangent bundles. A standard application of the Pontrjagin-Thom theorem identifies the corresponding bordism groups with the homotopy groups of the Thom spectrum $MTH(G_f)$.

One can check using the Whitney sum formula that if M is a spin manifold and N has a G_f-twisted spin structure, then $M \times N$ has a canonical G_f-twisted spin structure. This refines to define the structure of an $MT\mathrm{Spin}$-module spectrum on $MTH(G_f)$, and we obtain a spectrum-level lift of Proposition 4.2.77:

Proposition 4.2.78 (Shearing [436, Lemma 10.18])**.** *Suppose there is a rank-r virtual vector bundle $V \to BG_b$ with $w_1(V) = \theta$ and $w_2(V) = \omega$. Then there is an equivalence of $MT\mathrm{Spin}$-modules, natural in the data of G_f and V,*

$$MTH(G_f) \xrightarrow{\simeq} MT\mathrm{Spin} \wedge (BG_b)^{V-r}. \tag{4.2.79}$$

Typically the vector bundle V is not unique. Furthermore, the homotopy type of $MT\mathrm{Spin} \wedge (BG_b)^{V-r}$ depends only on $w_1(V)$ and $w_2(V)$. See [102, Theorem 1.39].

Example 4.2.80. Consider the two fermionic groups $G_f = \mathrm{Pin}_1^{\pm}$; for both of these groups, $G_b \cong \mathbb{Z}/2$ and θ is nontrivial, but $\omega = 0$ for Pin_1^+ and $\omega = \theta^2$ for Pin_1^-. [437, Proposition 1.26] constructs isomorphisms $H_d(\mathrm{Pin}_1^{\pm}) \cong \mathrm{Pin}_d^{\mp}$ for all d including $d = \infty$.

Let $\sigma \to B\mathbb{Z}/2$ denote the tautological line bundle. Then $w_1(\sigma) \neq 0$ and $w_2(\sigma) = 0$, so by Proposition 4.2.78, Pin_1^+-twisted spin bordism groups are the homotopy groups of $MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{\sigma-1}$. The identification $H(\mathrm{Pin}_1^+) \cong \mathrm{Pin}^-$, together with Proposition 4.2.77, implies the following equivalence of $MT\mathrm{Spin}$-modules, first recorded by Peterson [142, §7]:

$$MT\mathrm{Pin}^- \xrightarrow{\simeq} MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{\sigma-1}. \tag{4.2.81a}$$

The Whitney sum formula implies that if V is either of the virtual bundles 3σ or $-\sigma$, then $w_1(V) \neq 0$ and $w_2(V) = w_1(V)^2$, so the Stiefel-Whitney classes of V match the associated twist of Pin_1^-. Thus, following a similar line of reasoning as we did for Pin_1^+, we deduce that Pin_1^--twisted spin structures are the homotopy groups of

$$MT\mathrm{Pin}^+ \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{1-\sigma} \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{3\sigma-3}. \tag{4.2.81b}$$

This is a theorem of Stolz [143, §8]. ◇

The most important examples in this paper are the fermionic groups defined in terms of the extraspecial 2-groups in Definition 4.2.19. The following theorem will play an important role.

Theorem 4.2.82. *The functor* **Thom**: $\mathsf{FermGrp} \to \mathsf{Mod}_{MT\mathrm{Spin}}$ *sending* $G_f \mapsto MTH(G_f)$ *is symmetric monoidal with respect to the fermionic tensor product and* $\wedge_{MT\mathrm{Spin}}$*.*

Remark 4.2.83. In this thesis, we only require the statement for vector bundle twists: the category whose objects are fermionic groups G_f together with a virtual vector bundle $V \to BG_b$ and data of identifications $w_1(V) = \theta$ and $w_2(V) = \omega$. In this case, the above statement is not hard to show from Proposition 4.2.78. See [12] for the general proof.

Definition 4.2.84. For $k, \ell \geq 0$, recall the fermionic group $E_{\ell,k}$ from Definition 4.2.19 and let $ME_{\ell,k} := ((B\mathbb{Z}/2)^{\sigma-1})^{\wedge \ell} \wedge ((B\mathbb{Z}/2)^{1-\sigma})^{\wedge k}$.[25] By Proposition 4.2.77, an $MTH(E_{\ell,k})$-structure on a manifold M is equivalent to a spin structure on the virtual bundle $TM + L_1 + \cdots + L_\ell - L_{\ell+1} - \cdots - L_{\ell+k}$ for a collection of line bundles L_i over M. We will call such a manifold a *spin-(ℓ, k) manifold.*

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[25] The notation ME is meant to represent a Thom spectrum ("M") for an elementary abelian group ("E").

Lemma 4.2.85. *There is an* $MT\mathrm{Spin}$*-module equivalence*

$$MTH(E_{\ell,k}) \xrightarrow{\simeq} MT\mathrm{Spin} \wedge ME_{\ell,k}. \tag{4.2.86}$$

Proof. This follows by combining Example 4.2.80 and Theorem 4.2.82. ☐

We elaborate on the case $\ell = k = 1$, which we began in Example 4.2.61 and which will be a running example.

Definition 4.2.87 (Kaidi–Parra-Martinez–Tachikawa [438, §6.1])**.** There is an isomorphism $H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[x, y]$ with $|x| = |y| = 1$ specified by asking that, under the canonical isomorphisms

$$H^1(B\mathbb{Z}/2 \times B\mathbb{Z}/2; \mathbb{Z}/2) \xrightarrow{\cong} \mathrm{Hom}(\pi_1(B\mathbb{Z}/2 \times B\mathbb{Z}/2), \mathbb{Z}/2) \xrightarrow{\cong} \mathrm{Hom}(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2), \tag{4.2.88}$$

$x(1, 0) = 1$, $x(0, 1) = 0$, $y(1, 0) = 0$, and $y(0, 1) = 1$. A *dpin structure* on a vector bundle V is a $(B\mathbb{Z}/2 \times B\mathbb{Z}/2, x, xy)$-twisted spin structure.

Lemma 4.2.89. *There is a natural homotopy equivalence between the spaces of dpin structures and* $E_{1,1}$*-twisted spin structures on any vector bundle* V*.*

Proof. We have $(E_{1,1})_b \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. By Lemma 4.2.11, the associated twist is $(x+y, x^2+xy)$, so $E_{1,1}$-twisted spin structures are $(B\mathbb{Z}/2 \times B\mathbb{Z}/2, x + y, x^2 + xy)$-twisted spin structures.

There is an automorphism φ of $\mathbb{Z}/2 \times \mathbb{Z}/2$ sending $(1, 0) \mapsto (0, 1)$ and $(0, 1) \mapsto (1, 1)$, and whose value on the remaining elements is uniquely specified by the stipulation that φ is a group homomorphism. It follows from our construction of x and y that $\varphi^*(x) = x + y$ and $\varphi^*(y) = x$, so if we pull back the twisting data defining a dpin structure along φ, we get the twisting data of an $E_{1,1}$-twisted spin structure. Thus, precomposing the map to $B\mathbb{Z}/2 \times B\mathbb{Z}/2$ by $B\varphi$ or $B(\varphi^{-1})$ turns dpin structures into $E_{1,1}$-twisted spin structures and vice versa. ☐

We will refer to spin-$(1, 1)$ structures as dpin structures throughout the rest of this paper, leaving the automorphism φ implicit.

Remark 4.2.90. In Corollary 4.2.43, we showed $\mathbb{R}^f[E_{1,1}] \cong C\ell_{1,1}$, which is Morita equivalent to $\mathbb{R}$. Thus the K-theory we associated with this fermionic group coincides with that of the trivial fermionic group: $K(C\ell_{1,1}) \simeq KO$. However, this is *not* true for twisted spin bordism—see Table 4.2. Looking at Ω_4, the bordism groups do not even agree rationally!

This disparity will be a running theme. When we define models in Section 4.2.3 for the abelian groups of free fermion phases and SPT phases, in terms of K-theory and spin bordism respectively, we will show that the groups of free fermion phases are an invariant of the Morita class of $\mathbb{R}[G_f]$, but the groups of SPTs are not. This "Morita variance" is an important aspect of the definition of a free-to-interacting map.

Remark 4.2.91. Exactly the same thing happens for $E_{k,k}$, whose group algebra $C\ell_{k,k}$ (Corollary 4.2.43) is Morita trivial, but whose spin-(k, k) bordism groups in general depend on k (see Proposition 4.2.262). All of these different notions of twisted spin bordism in some sense correspond to class D free fermionic phases.

d	Ω_d^{Spin}	Ω_d^{DPin}
0	$\mathbb{Z}$	$\mathbb{Z}/2$
1	$\mathbb{Z}/2$	$\mathbb{Z}/2$
2	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$
3	0	$\mathbb{Z}/8$
4	$\mathbb{Z}$	$(\mathbb{Z}/2)^2$
5	0	0
6	0	$(\mathbb{Z}/2)^2$

Table 4.2: Comparing low-dimensional spin and dpin bordism groups [438, Theorem F.1].

Remark 4.2.92 (The complex case). To define twisted spin^c bordism associated to a fermionic group, we use the map of spectra $MT\mathrm{Spin} \to MT\mathrm{Spin}^c$, which refines the fact that spin structures induce spin^c structures.[26] This map gives $MT\mathrm{Spin}^c$ the structure of an E_∞-$MT\mathrm{Spin}$-algebra, guaranteeing that the constructions in this section are natural. The necessary modifications are:

1. Swap Pin_d^c for Pin_d^+ in Definition 4.2.75.

2. Note that both $\mathrm{Pin}_1^\pm$ define the same notion of twisted spin^c bordism, namely pin^c bordism—the Lie groups $\mathrm{Pin}_d^\pm \times_{\{\pm 1\}} \mathrm{U}_1$ are isomorphic, with the isomorphism commuting with the structure map to O_d.

   In terms of group algebras: $\mathbb{R}[\mathrm{Pin}_1^\pm] \cong C\ell_{\pm 1}$, and the complexifications of these two superalgebras are isomorphic: if $e^2 = 1$, then $(ie)^2 = -1$. We will see more algebraic consequences of this isomorphism in §4.2.7.3.

Under the hood, there is a more general notion of twisted spin^c structure defined in terms of a map to $B\mathrm{O}/B\mathrm{Spin}^c \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3)$ (see [123, 439, 440]), and the twisting data (θ, ω) associated to a fermionic group only manifests through θ and the integral Bockstein of ω. So different choices of ω with the same Bockstein lead to equivalent notions of twisted spin^c bordism; this is what happened for $\mathrm{Pin}_1^\pm$. Up to homotopy, $BG_b \to B\mathrm{O}/B\mathrm{Spin}^c$ is equivalent to a group homomorphism $\theta\colon G_b \to \mathbb{Z}/2$ and an isomorphism class of an extension of G_b by $U(1)$; compare with Remark 4.2.69.

Similarly to Remark 4.2.45, there is a larger E_∞-space $\pi_{\leq 3}\,\mathrm{Pic}(KU) \cong \mathbb{Z}/2 \times B\mathbb{Z}/2 \times B^3\mathbb{Z}$ of which the connected cover is $B\mathrm{O}/B\mathrm{Spin}^c$. Analogously to Remark 4.2.46, the induced group structure on

$$\mathbb{Z}/2 \times H^1(X; \mathbb{Z}/2) \times H^3(X; \mathbb{Z}) \tag{4.2.93}$$

is the complex graded Brauer group of X as defined by [415].

4.2.3 Classification of SPT phases

In this section, we discuss the classifications of both free and interacting SPTs, in preparation for defining maps between them in Section 4.2.4. Our discussion is more detailed than in

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[26] In terms of obstructions, if $w_2 = 0$, then $W_3 = \beta(w_2) = 0$ as well.

Section 4.1.2. We first introduce free fermion Hamiltonians and their K-theoretic classification in Section 4.2.3.1, then discuss invertible field theories as a model for interacting SPT phases in Section 4.2.3.2.

4.2.3.1 Free fermion phases and K-theory

In the influential paper [7], Kitaev proposed a classification of gapped phases of non-interacting fermions using K-theory, building from work of [8, 441, 442]. Kitaev considered two complex and eight real symmetry classes of systems, together forming the "tenfold way." The symmetry groups of the tenfold way are closely related to Clifford algebras, and the real classes relate to the symmetry groups $E_{\ell,k}$ we introduced in Section 4.2.2.1. Indeed, $E_{\ell,k}$ is the group with $\ell + k$ Clifford algebra generators, of which ℓ square to 1 and k square to -1 for integers $\ell, k \geq 0$. The reason that this doubly infinite family effectively reduces to a family of eight is related to the isomorphisms of real superalgebras

$$C\ell_{1,1} \cong \mathrm{End}(\mathbb{R}^{1|1}) \quad \text{and} \quad C\ell_{4,0} \cong \mathrm{End}(\mathbb{H}^{1|1}) \cong C\ell_{0,4}, \tag{4.2.94}$$

which give Morita trivializations of $C\ell_{0,8}$, $C\ell_{8,0}$ and $C\ell_{1,1}$. This is one instantiation of Bott periodicity [3].

Kitaev's proposal has been generalized to many settings, such as disordered phases and the Bogoliubov-de-Gennes framework [441, 443, 444]. Generalizations to arbitrary symmetries employ various related formalisms for K-theory: twisted equivariant K-theory of topological spaces [294], K-theory of topological groupoids [324], van Daele K-theory of $\mathbb{Z}/2$-graded C^*-algebras [445], and Karoubi K-theory of $\mathbb{Z}/2$-graded C^*-algebras [446]. We employ the formalism developed by the fourth named author, who in [325] argues the following.

Ansatz 4.2.95. In spatial dimension d, free fermion SPT phases with finite fermionic symmetry G_f are classified by $KO_{2-d}(\mathbb{R}^f[G_f])$.

Proposition 4.2.56 allows readers more familiar with twisted equivariant K-theory to reformulate the above ansatz in that language.

We will spend some time developing a physical argument for this ansatz. First, we require a suitable topological space of gapped free Hamiltonians with finite fermionic symmetry G_f. To that end, consider a Hilbert space $\mathcal{H}$ together with a self-adjoint operator $H \colon \mathcal{H} \to \mathcal{H}$. We would like to say that H is "free" if it can be written quadratically in terms of creation and annihilation operators of some sort of single particle states. For simplicity, we will first assume spatial dimension zero, so that $\mathcal{H}$ is a finite-dimensional super $\mathbb{C}\ell_s$-module, where $s = 2\mathrm{dim}V$. A free Hamiltonian is of the form

$$H = \sum_i iH^{ij} e_i e_j, \tag{4.2.96}$$

where e_i is the element of $\mathbb{C}\ell_s$ corresponding to the ith basis vector, and which we view as a Majorana operator. Assuming H is self-adjoint and normal-ordered, it is equivalent—via the equation $iH_{ij} = \langle e_i, he_j \rangle$—to a skew-adjoint operator h on $\mathbb{R}^s$, which we will call a *Bogoliubov-de Gennes (BdG) Hamiltonian.* We say the BdG Hamiltonian h is *gapped* if zero is not in its spectrum.[27] Once we have passed to BdG Hamiltonians, the data of $\mathcal{H}$ is redundant because $\mathbb{C}\ell_s$ has a single irreducible supermodule. We can therefore forget about the multi-particle state space $\mathcal{H}$ and view the space of gapped BdG Hamiltonians as acting on the finite-dimensional real vector space $\mathbb{R}^s$.[28] The deformation class of the Hamiltonian is preserved under *spectral flattening*, where nonzero eigenvalues are sent to ± 1, and so the space of these BdG Hamiltonians deformation retracts via $h \mapsto h/|h| = \tilde{h}$ to the space of $\tilde{h}$ on $\mathbb{R}^s$ such that $\tilde{h}^2 = -\mathrm{id}_{\mathbb{R}^s}$; i.e. the space of complex structures on $\mathbb{R}^s$. The space of such structures is equivalent to the symmetric space $O(s)/U(s/2)$, which, after stabilization, gives the classifying space for KO^{-2}. See [447, Section 4] and [7, "Classification Principles"] for comparable discussions.

This is the basic case, but we would also like to incorporate symmetries and nonzero dimensions. For the former, we generalize $\mathbb{R}^s$ to an arbitrary (ungraded) module M over the fermionic group algebra $\mathbb{R}^f[G_f]$ for a finite fermionic group G_f. Since this algebra encodes a symmetry, its elements should commute with the Hamiltonian H on $\mathcal{H}$. However, odd elements of $\mathbb{R}^f[G_f]$ should be physically interpreted as time-reversing symmetries and so anticommute with i. Given that the correspondence between h and H involves multiplication by i, we thus obtain

Conclusion 1. The space of G_f-symmetric gapped BdG Hamiltonians on an ungraded $\mathbb{R}^f[G_f]$-module M is homotopy equivalent to the space of complex structures $h\colon M \to M$ such that

$$ah = (-1)^{|a|} ha \tag{4.2.97}$$

for all $a \in \mathbb{R}^f[G_f]$.

Meanwhile, to get to positive dimensions, we use Bloch theory and generalize M to a Real vector bundle $M \to \bar{S}^d$ of Bloch states over the Brillouin zone $\bar{S}^d$. Note that because are not interested in weak phases in this chapter, we assume the Brillouin zone is an (equivariant) sphere instead of a torus—see Section 4.1.2 for a comparison with weak phases. By translation invariance, h decomposes into Bloch Hamiltonians for separate momenta, which mathematically means it is a vector bundle automorphism of $M \to \bar{S}^d$. Elements of the symmetry algebra are all real, and so they automatically act by Real bundle morphisms.

Conclusion 2. A G_f-symmetric state space of spatial dimension d consists of

1. A Real vector bundle $M \to \bar{S}^d$ of Bloch states;
2. a collection of Real vector bundle morphisms $a\colon M \to M$ for $a \in \mathbb{R}^f[G_f]$ composing according to the algebra in $\mathbb{R}^f[G_f]$.

The space of G_f-symmetric (flattened) gapped BdG Hamiltonians in spatial dimension n on a fixed G_f-symmetric state space M is a Real vector bundle morphism $h\colon M \to M$ such that $h^2 = -\mathrm{id}_M$ and

$$ah = (-1)^{|a|} ha \tag{4.2.98}$$

for all $a \in \mathbb{R}^f[G_f]$.

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[27] Since the spectrum of the operator is closed, this implies that there is a gap between the ground state and the next energy level, which persists once we second-quantize.

[28] Note that s is necessarily even, as odd-dimensional vector spaces do not have skew-adjoint automorphisms.

To get to the classification of SPTs from here, we must stabilize the spaces of BdG Hamiltonians by completing under the stacking operation of direct sum.[29] The necessity of stabilization was argued originally by Kitaev [7], who defined two gapped phases to be equivalent if they are equivalent after adding some number of trivial bands. We will stabilize through the model of Karoubi triples in a similar way to [446, Section 8], and for simplicity, we will assume the $\mathbb{R}^f[G_f]$-module to be finitely generated projective. This condition can be weakened to a compactness condition on the difference between the two complex structures; see [324, Theorem 4.20].

Definition 4.2.99. A *Karoubi triple* consists of a G_f-symmetric state space M of dimension d such that the fibers of M are finitely generated projective $\mathbb{R}^f[G_f]$-modules, together with two G_f-symmetric gapped BdG Hamiltonians h_1 and h_2. The space of Karoubi triples is a monoid under the stacking operation

$$(M, h_1, h_2) \oplus (M', h_1', h_2') = (M \oplus M', h_1 \oplus h_1', h_2 \oplus h_2'). \tag{4.2.100}$$

A Karoubi triple [448, Definition 2.13] is *elementary* if h_1 and h_2 are homotopy equivalent in the space of G_f-symmetric gapped BdG Hamiltonians. Two Karoubi triples $(E, h_1, h_2), (E', h_1', h_2')$ are *isomorphic* if there exists a Real vector bundle isomorphism $T \colon E \to E'$ such that $Ta = aT$ for $a \in \mathbb{R}^f[G_f]$ and $Th_i = h_i'T$ for $i = 1, 2$.

Definition 4.2.101. The group of *free fermion SPT phases* is the set of Karoubi triples modulo the equivalence relation that $T_1 \sim T_2$ if there exists an elementary triple T such that $T_1 \oplus T$ and $T_2 \oplus T$ are isomorphic, equipped with the group operation $\oplus$.

Remark 4.2.102. The group of free SPT phases we have defined agrees with the K-theory of the Banach functor $\mathsf{Mod}_{A \hat{\otimes} Cl_{-2}} \to \mathsf{Mod}_{A \hat{\otimes} Cl_{-1}}$ in the sense of [449, p. II.2.13].

Remark 4.2.103. With our definitions, the group of free G_f-protected SPT phases has generators pairs of Hamiltonians, a point stressed in [450]. However, in practice there is often a canonical 'trivial Hamiltonian' available. For example, suppose the space M of Majorana operators comes with a pairing into annihilation and creation operators a_i and $a_i^\dagger$. Then M comes with a canonical complex structure h_{triv}, which maps a_i to $-ia_i$ and $a_i^\dagger$ to $ia_i^\dagger$. This complex structure often respects the G_f-symmetry, such as in the case where G_f is induced by a representation on the one particle space.[30] In this situation, a gapped G_f-symmetric BdG Hamiltonian $h \colon M \to M$ induces a K-theory class using the Karoubi triple (M, h_{triv}, h); see [446, Remark 8.7].

The following will be shown in [325], providing motivation for Ansatz 4.2.95.

Theorem 4.2.104. *The group of free fermion SPT phases is isomorphic to* $K_{2-d}(\mathbb{R}^f[G_f])$.

Note that since G_f is finite, the abstract isomorphism class of the group of SPT phases can be easily computed using methods explained in Section 4.2.2.5.

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[29] Recall that the operation of stacking is given by tensor product on the multi particle state space $\mathcal{H}$. On $\mathbb{R}^s$, however, stacking is instead given by $\oplus$ because $C\ell(\mathbb{R}^{s_1} \oplus \mathbb{R}^{s_2}) \cong C\ell(\mathbb{R}^{s_1}) \hat{\otimes} C\ell(\mathbb{R}^{s_2})$ and $\bigwedge(V_1 \oplus V_2) \cong \bigwedge V_1 \otimes \bigwedge V_2$. Similarly for M.

[30] This is not the case for particle-hole symmetries.

Example 4.2.105. Since the fermionic group algebra of $E_{\ell,k}$ is $C\ell_{\ell,k}$, $E_{\ell,k}$-protected phases are classified by $K_{2-d}(C\ell_{\ell,k})$. By Morita invariance of K-theory and Equation (4.2.94), this group only depends on $\ell - k \pmod 8$. More specifically, Corollary 4.2.43 tells us that this group is isomorphic to $K_{2-d}(C\ell_{\ell,k}) \cong KO_{2-d-\ell+k}(\mathrm{pt})$. This recovers the eightfold periodicity of free phases in the eight real classes of the tenfold way. ◇

However, following the isomorphisms appropriately to figure out which element corresponds to a given pair of Hamiltonians can be nontrivial in practice.

Remark 4.2.106. There is a correspondence between $C\ell_{-1}\hat{\otimes}A$-modules and graded $C\ell_{-2}\hat{\otimes}A$-modules given by

$$C\ell_{-1}\hat{\otimes}A \xrightarrow{-\hat{\otimes}_{\mathbb{R}}\mathbb{R}^{1|1}} C\ell_{1,1}\hat{\otimes}C\ell_{-1}\hat{\otimes}A = C\ell_{+1}\hat{\otimes}C\ell_{-2}\hat{\otimes}A. \tag{4.2.107}$$

In the first arrow we used that $C\ell_{1,1} \cong M_{1|1}(\mathbb{R})$ is Morita trivial and the Morita trivialization is given by $\mathbb{R}^{1|1}$ as an $(M_{1|1}(\mathbb{R}), \mathbb{R})$-bimodule. This gives an isomorphism of groups

$$\frac{\pi_0\mathsf{Mod}_{A\hat{\otimes}Cl_{-1}}}{\pi_0\mathsf{Mod}_{A\hat{\otimes}Cl_{-2}}} \cong \frac{\pi_0\mathsf{Mod}_{A\hat{\otimes}Cl_{-2}\hat{\otimes}Cl_{+1}}}{\pi_0\mathsf{Mod}_{A\hat{\otimes}Cl_{-2}\hat{\otimes}Cl_{1,1}}} = \mathrm{CMQ}_2(A).$$

In the special case where A is finite-dimensional, $\mathrm{CMQ}(A)$ agrees with $K(A)$, and so this gives a special case of 4.2.104.

Conclusion 3. Let $h\colon M \to M$ be a G_f-symmetric gapped BdG Hamiltonian, which we assume without loss of generality to be flattened. Let $N = M \oplus M$ be the graded $C\ell_{-2} \hat{\otimes} \mathbb{R}^f[G_f]$-module with Clifford actions

$$f_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; f_2 = \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix}, \tag{4.2.108}$$

action of $g \in \mathbb{R}^f[G_f]$ given by

$$\begin{pmatrix} g & 0 \\ 0 & (-1)^{\theta(g)}g, \end{pmatrix} \tag{4.2.109}$$

and grading operator

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.2.110}$$

Then N represents the class in $K_2(\mathbb{R}^f[G_f])$ associated to h.

Remark 4.2.111. Let A be a finite-dimensional semisimple superalgebra. A class in $CMQ_{2-d}(A)$ is represented by a finite-dimensional $C\ell_{-2}\hat{\otimes}\, C\ell_{+d}\hat{\otimes}A$-module M. Using the above conclusion, we can realize such a class as a zero-dimensional symmetric Hamiltonian $h\colon M \to M$ with $C\ell_d \otimes A$-symmetry. Moreover, this class is trivial if and only if this $C\ell_d \otimes A$-symmetry extends to a $C\ell_{d,1} \otimes A$-symmetry. This perspective plays a prominent role in [7].

Example 4.2.112. Class D superconductors have no symmetries when viewed as a system of neutrally charged fermions. Theorem 4.2.104 therefore says that the classification is given by $K_{2-d}(\mathbb{R})$. In particular, in dimension 2 the classification is $K_0(\mathbb{R}) \cong KO^0 \cong \mathbb{Z}$, while in dimension 1 the classification is $K_1(\mathbb{R}) \cong KO^{-1} \cong \mathbb{Z}/2$, generated by the Majorana chain [95]. ◇

Example 4.2.113. Class BDI superconductors admit a time-reversal symmetry T that squares to 1, corresponding to the fermionic group $G_f = \mathrm{Pin}^+(1) = E_{1,0}$. The symmetry algebra is thus $\mathbb{R}^f[E_{1,0}] \cong C\ell_{+1}$, leading to the classification $K_{2-d}(C\ell_{+1}) = K_{1-d}(\mathbb{R})$. In spatial dimension one, the generator of the group $KO_0 \cong \mathbb{Z}$ of class BDI phases corresponds to the time-reversal symmetric Majorana chain.[31] This model generates the $\mathbb{Z}$ of possible phases under stacking—any number of stacked copies of the chain is not adiabatically connected to the trivial phase via symmetric, quadratic terms [104]. ◇

4.2.3.2 Interacting SPT phases and invertible field theories

In the physics literature, the codomain of a free-to-interacting map is the space of all gapped invertible topological phases of matter of a given dimension and symmetry type. Providing a rigorous mathematical construction of this space is a difficult open question, but it is conjectured that the passage from a gapped invertible topological phase of matter to its low-energy limit, which is a reflection-positive invertible field theory (IFT) (see Section 2.5.2), preserves its deformation class. Reflection-positive IFTs have been classified in work of Freed–Hopkins [1] and Grady [76], as discussed in Section 2.4.2.

For convenience, we re-state Theorem 2.4.21 down here:

Theorem 4.2.114 (Freed–Hopkins [1, Theorem 1.1], Grady [76, Theorem 1])**.** *There is a natural isomorphism from the abelian group of deformation classes of n-dimensional reflection-positive IFTs on manifolds with ξ-structure to $[MT\xi, \Sigma^{n+1}I_{\mathbb{Z}}]$.*

Notation 4.2.115. We will often denote the group $[MT\xi, \Sigma^{n+1}I_{\mathbb{Z}}]$ by $\mho_\xi^{n+1}$.

As we discussed in §4.2.2.6, for a fermionic theory with a symmetry given by a fermionic group G_f, the tangential structure of the corresponding IFTs is the structure $H(G_f)$ constructed in [419, §2.6] (here Definition 4.2.75).

Example 4.2.116 (2d pin$^-$ IFTs)**.** Let $G_f = \mathrm{Pin}_1^+$, so that the spacetime tangential structure is pin$^-$, as we established in Example 4.2.80. To classify reflection-positive pin$^-$ IFTs, by Theorem 4.1.24 we should compute $[MT\mathrm{Pin}^-, \Sigma^{n+1}I_{\mathbb{Z}}]$, which by (4.1.23) can be deduced from $\pi_*(MT\mathrm{Pin}^-)$, the bordism groups of manifolds with pin$^-$ structures on their tangent bundles. These bordism groups are finite [192, Theorem 5.1], so $\mathrm{Hom}(\Omega_n^{\mathrm{Pin}^-}, \mathbb{Z}) = 0$, which by (4.1.23) implies that the group of n-dimensional reflection-positive pin$^-$ IFTs is canonically isomorphic to $\mathrm{Ext}(\Omega_n^{\mathrm{Pin}^-}, \mathbb{Z})$. For any finite abelian group A, $\mathrm{Ext}(A, \mathbb{Z})$ is noncanonically isomorphic to A, but we would like a canonical description. The "exponential" short exact sequence

$$1 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \longrightarrow 1 \tag{4.2.117}$$

induces a long exact sequence on Ext, and since $\mathrm{Hom}(A, \mathbb{C}) = 0$ and $\mathrm{Ext}(A, \mathbb{C}) = 0$ for a finite abelian group A, this long exact sequence simplifies to a canonical isomorphism

$$\mathrm{Hom}(\Omega_n^{\mathrm{Pin}^-}, \mathbb{C}^\times) \xrightarrow{\cong} \mathrm{Ext}(\Omega_n^{\mathrm{Pin}^-}, \mathbb{Z}). \tag{4.2.118}$$

---

[31] Note that while this has the same underlying Hamiltonian as Example 4.2.112, it is different as an SPT because time-reversing deformations are prohibited.

Thus n-dimensional reflection-positive pin$^-$ IFTs α are classified by group homomorphisms $\Omega_n^{\mathrm{Pin}^-} \to \mathbb{C}^\times$; the homomorphism associated to α is in fact the partition function of α on closed pin$^-$ manifolds (see [1, §5.3]).

Now we specialize to $n = 2$. Anderson–Brown–Peterson [192, Theorem 5.1] showed $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$,[32] so the group of 2d pin$^-$ reflection-positive IFTs is also isomorphic to $\mathbb{Z}/8$. The theory α_{ABK} whose partition function is the *Arf–Brown–Kervaire invariant* [110, 451] is a generator of this group [55, Section 5]. This IFT and its role as the low-energy field theory of the time-reversal symmetric Majorana chain of Example 4.2.113 is discussed in detail in [55]. See [332, 452, 453] for more general results on 2d pin$^-$ TFTs. ◇

For a comparison between the various formalisms we have discussed at this point for the tenfold way symmetry types, see Table 4.3.

s	AZ class	G_f	$\mathbb{R}^f[G_f]$	$H^c(s) = H(G_f)$
0	A	$\mathrm{U}(1)$	$\mathbb{C}$	$H^c(0) = \mathrm{Spin}^c$
1	AIII	$\mathrm{U}(1) \times \mathbb{Z}_2$	$\mathbb{C}\ell_1$	$H^c(1) = \mathrm{Pin}^c$

s	AZ class	G_f	$\mathbb{R}^f[G_f]$	$H(s) = H(G_f)$
-3	CII	$\mathrm{Pin}^-(3)$	$C\ell_{-3}$	$H(-3) = \mathrm{Pin}^{h-} := \mathrm{Pin}^- \times_{\{\pm1\}} \mathrm{SU}(2)$
-2	AII	$\mathrm{Pin}^-(2)$	$C\ell_{-2}$	$H(-2) = \mathrm{Pin}^{\tilde{c}+} := \mathrm{Pin}^+ \ltimes_{\{\pm1\}} \mathrm{U}(1)$
-1	DIII	$\mathrm{Pin}^-(1)$	$C\ell_{-1}$	$H(-1) = \mathrm{Pin}^+$
0	D	$\mathrm{Spin}(1)$	$\mathbb{R}$	$H(0) = \mathrm{Spin}$
1	BDI	$\mathrm{Pin}^+(1)$	$C\ell_1$	$H(1) = \mathrm{Pin}^-$
2	AI	$\mathrm{Pin}^+(2)$	$C\ell_2$	$H(2) = \mathrm{Pin}^{\tilde{c}-} := \mathrm{Pin}^- \ltimes_{\{\pm1\}} \mathrm{U}(1)$
3	CI	$\mathrm{Pin}^+(3)$	$C\ell_3$	$H(3) = \mathrm{Pin}^{h+} := \mathrm{Pin}^+ \times_{\{\pm1\}} \mathrm{SU}(2)$
4	C	$\mathrm{Spin}(3)$	$C\ell_4$	$H(4) = \mathrm{Spin}^h := \mathrm{Spin} \times_{\{\pm1\}} \mathrm{SU}(2)$

Table 4.3: Here we give another version of Table 4.1, this time emphasizing the role of the fermionic group G_f (column 3) from which are produced compatible superalgebras $\mathbb{R}^f[G_f]$ (column 4) and twisted spin structures $H(G_f)$ (column 5).

4.2.4 Defining the free-to-interacting map

In this section, we discuss *free-to-interacting maps*, which relate the group of free fermion SPT phases with the group of interacting SPT phases for a specified symmetry type. Whereas in Section 4.1.2.4 we black-boxed the generalized Atiyah–Bott–Shapiro orientation, here we discuss it in detail and generalize it to handle new symmetry types, including spin-(ℓ, k) bordism.

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[32] The result in [192] is quite general; see [252, Footnote 6] for how to extract the case $n = 2$, as well as Giambalvo [193, Theorem 3.4(b)], Kirby–Taylor [110, Lemma 3.6], and Campbell [96, Theorem 6.4] for additional calculations of $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$.

4.2.4.1 Classical Atiyah–Bott–Shapiro orientation

The heart of the free-to-interacting map is a natural transformation between spin bordism and KO-theory called the Atiyah–Bott–Shapiro (ABS) orientation. In the untwisted case, Atiyah–Bott–Shapiro [3] provided formulas for the Thom class in KO-theory (resp. K-theory) of a spin (resp. spin^c) vector bundle. The specification of Thom classes defines an orientation, which gives a map of spectra $MT\mathrm{Spin} \to ko$ [367, 454] (resp. $MT\mathrm{Spin}^c \to ku$). In [1], Freed–Hopkins generalized the ABS map to twists of spin bordism corresponding to the symmetries of the tenfold way as recorded in Tables 4.1 and 4.3. We recall their construction for these ten kinds of twists. Then, we explain our generalization for twists by $E_{\ell,k}$.

In this section, we prefer a more topological perspective, gearing up for our homotopical computations in Section 4.2.7. In the following, let X be a locally compact Hausdorff space. Taking X finite, $A = C_0(X)$, and regarding A as a purely even superalgebra recovers the setting of Section 4.2.2.4.

Definition 4.2.119. Let $V \to X$ be a real Euclidean vector bundle over X, so that each fiber is equipped with a metric. The *Clifford bundle* $C\ell(V)$ of V is the algebra bundle whose fiber at x is the Clifford algebra $C\ell(V_x)$.

Definition 4.2.120. A *graded Clifford module* of $V \to X$ is a $\mathbb{Z}/2$-graded real vector bundle $E \to X$ equipped with actions $V \otimes_{\mathbb{R}} E^0 \to E^1$ and $V \otimes_{\mathbb{R}} E^1 \to E^0$ such that for each $v \in V$ and $e \in E$,

$$v(v(e)) = -||v||^2 e. \tag{4.2.121}$$

(The norm on V is determined by the quadratic form.)

Example 4.2.122. Let $V \to X$ be a vector bundle with a quadratic form. The exterior power bundle $\Lambda^\bullet V$ admits the structure of a $\mathbb{Z}/2$-graded $C\ell(V)$-module: its graded pieces are the even and odd powers $E^0 = \Lambda^{\mathrm{ev}} V$, $E^1 = \Lambda^{\mathrm{odd}} V$ and the action of V is given by

$$\begin{aligned} V \otimes_{\mathbb{R}} \Lambda^\bullet(V) &\longrightarrow \Lambda^\bullet V & (4.2.123a)\\ v \otimes e &\longmapsto v \wedge e - v \neg e. \end{aligned}$$

◇

In (4.2.49), we defined the Clifford module quotient of an algebra by considering graded modules modulo those that extend to the action of one more Clifford generator. For the vector bundle setting, let $V \to X$ be a k-dimensional vector bundle with quadratic form Q_V and let E be a graded $C\ell(V)$-module. Equip $V \oplus 1$ with the quadratic form $Q_V \oplus \mathrm{id}_1$. Then if E extends to a $C\ell(V \oplus 1)$ module, each fiber E_x admits the action of $C\ell(V_x)\hat{\otimes} C\ell_{+1}$, by Lemma 4.2.26.[33]

Definition 4.2.124. Let V and $V \oplus 1$ be as above, and let $\mathsf{Mod}_{C\ell(V)}$ be the Grothendieck group of graded $C\ell(V)$-modules under $\oplus$. The Clifford module quotient[34] is

$$CMQ^0(V) = \mathrm{coker}\left(\mathsf{Mod}_{C\ell(V\oplus 1)} \to \mathsf{Mod}_{C\ell(V)}\right), \tag{4.2.125}$$

---

[33] To check the sign, recall that for a vector space W with quadratic form Q, $C\ell(W)$ is the quotient of the tensor algebra by the ideal $w \cdot w - Q(w) \cdot 1$.

[34] which Atiyah–Bott–Shapiro write as $A(V)$

where the map of groups is induced by the inclusion $V \hookrightarrow V \oplus 1$. More generally, define $CMQ^{-n}(X)$ as $CMQ^0(V \oplus \underline{\mathbb{R}}^n)$, where the trivial bundle $\underline{\mathbb{R}}^n$ is equipped with the standard quadratic form.

In the complex case, there is an analogous group for $\mathbb{C}\ell(V)$-modules.

As noted in Remark 4.2.52, the Clifford module quotient group does not in general give a model for K-theory. Instead, $CMQ^0(V)$ is in general a subgroup of $\widetilde{KO}^0(V)$, which we can equivalently write as $\widetilde{KO}^0(X^V)$.[35] There is a homomorphism $CMQ^0(V) \to \widetilde{KO}^0(X^V)$, which Atiyah–Bott–Shapiro call the Euler characteristic map.

Definition 4.2.126. Let $V \to X$ be a real vector bundle, let $D(V)$ and $S(V)$ be the disk bundle and sphere bundle, resp., and let $\pi\colon D(V) \to X$ be the projection. The Euler characteristic map makes the assignment

$$\chi_V\colon CMQ^0(V) \to \widetilde{KO}^0(X^V) \tag{4.2.127}$$

$$E = E^0 \oplus E^1 \longmapsto [\pi^* E^1, \pi^* E^0; \sigma] \tag{4.2.128}$$

where σ is given by multiplication by minus the vector in the base. That is, $\sigma|_{v \in V}\colon e \mapsto -v \cdot e$, where $e \in C\ell(V)|_v$.

See e.g. [3, Sections 8, 9] for more details. The Euler characteristic map for complex K-theory is analogous.

Theorem 4.2.129 ([3, Section 11, 12])**.** *Let V be a k-dimensional complex vector bundle. Then V admits a spin^c structure. Let $\Lambda^\bullet V$ be the graded Clifford module of Example 4.2.122 and let $\beta_{\mathbb{C}} \in K^{-2}$ be the complex Bott class. The image $\chi_V(\Lambda^\bullet V) \cdot \beta_{\mathbb{C}}^{-k} \in \widetilde{K}^{2k}(X^V)$ is the K-theory Thom class of V.*

The complex ABS orientation uses the above construction for the particular case of the tangent bundle of a spin^c manifold.

Example 4.2.130 ([3, Section 11])**.** Let M be a $2k$-dimensional spin^c manifold. The Thom class of its tangent bundle is $\chi_{TM}(\Lambda^\bullet(TM))$. Let us compare with the geometric perspective, where the spin^c structure allows us to write the tangent bundle of M as an associated bundle $TM \cong P_{\mathrm{Spin}^c} \times_{\mathrm{Spin}^c(2k)} \mathbb{R}^{2k}$, where P_{Spin^c} is a principal $\mathrm{Spin}^c(2k)$ bundle. Take the Clifford module $\Lambda^\bullet(\mathbb{C}^k)$ and form the Clifford module $P_{\mathrm{Spin}^c} \times_{\mathrm{Spin}^c(2k)} \Lambda^\bullet(\mathbb{C}^k)$, using the action of $\mathrm{Spin}^c(2k) \subset \mathbb{C}\ell_{2k}$. The KO-theory Thom class of M is equivalently given by $\chi(P_{\mathrm{Spin}^c} \times_{\mathrm{Spin}^c(2k)} \Lambda^\bullet(\mathbb{C}^k))$. See [3, Proposition 11.6]. ◇

For real KO-theory, the construction is analogous.

Example 4.2.131 ([3, §11])**.** Let $V \to X$ be an $8k$-dimensional spin vector bundle, and let P_{Spin} be the principal Spin_{8k} bundle such that $V \cong P_{\mathrm{Spin}} \times_{\mathrm{Spin}_{8k}} \mathbb{R}^{8k}$. The KO-theory Thom class of V is the class

$$\chi_V(P_{\mathrm{Spin}} \times_{\mathrm{Spin}_{8k}} \lambda^k) \cdot \beta^{-k} \in \widetilde{KO}^{8k}(X^V) \tag{4.2.132}$$

---

[35] We are implicit about using compact supports.

where λ^k is the generator of A_{8k} (i.e. k-copies of the irreducible $C\ell_8$ module) and $\beta \in KO^{-8}$ is the real Bott class. See [3] Cor. (6.6).

For bundles of dimensions not divisible by 8, one can stabilize to a bundle of dimension $8k$ and then apply the suspension isomorphism as needed. ◇

Let us continue with the spin^c case because it is simpler. To specify the orientation map $M\mathrm{Spin}^c \to K$, it suffices to provide compatible maps $M\mathrm{Spin}^c_n \to K_n$ for each n. Note that $M\mathrm{Spin}^c_n = \mathrm{Th}(V_n^{\mathrm{Spin}^c})$, where $V_n^{\mathrm{Spin}^c}$ is the pullback of the universal spin^c bundle along the map $\mathrm{Spin}^c(n) \to \mathrm{Spin}^c$, while K_n is a classifying space for the functor K^{-n}. The classifying map of Example 4.2.130 applied to the tautological spin bundle gives such a map:

$$\chi(\Lambda^\bullet(V_n^{\mathrm{Spin}^c})) \in \tilde{K}^0(\mathrm{Th}(V_n^{\mathrm{Spin}^c})) \quad \rightsquigarrow \quad M\mathrm{Spin}^c_n \to KO_n. \tag{4.2.133}$$

See e.g. [455] for more details on the map of spectra.

4.2.4.2 Twisted ABS maps

In [1], Freed–Hopkins define twisted ABS maps for a set of ten symmetry types corresponding to the tenfold way. We start off by describing their constructions for the real symmetry types $H(s)$ with the parameter $|s| \leq 3$.

Definition 4.2.134. Let $V \to BO_s$ be the universal vector bundle. Using the Clifford module of Example 4.2.122, define

$$\lambda_V \coloneqq \chi_V([\Lambda^{\mathrm{ev}}(V), \Lambda^{\mathrm{odd}}(V); \sigma_V]), \tag{4.2.135}$$

where σ_V is the map sending $(v, w) \in V \times \Lambda^\bullet V$ to $(v, v \wedge w + v \lrcorner w)$ in the other graded piece of $V \times \Lambda^\bullet V$.

Remark 4.2.136. If V were complex, the class λ_V described above would give the K-theory Thom class of V (see Theorem 4.2.129). Any complex vector bundle is also spin^c, and thus oriented for K-theory. In real KO-theory, the universal bundles over BO_s are not spin, and thus do not have Thom isomorphisms.

Note that the classifying map of λ_V runs from $BO_s^V \to KO$. We may desuspend s times to arrange it to start from a rank-zero Thom spectrum:

$$\Sigma^{-s}\lambda_V \colon BO_s^{V-s} \to \Sigma^{-s}KO. \tag{4.2.137}$$

Freed–Hopkins use this class to define the twisted ABS maps for $s = 1, 2, 3$.

Definition 4.2.138 ([1, Proposition 10.27])**.** Let $s = 1$, 2, or 3, and let $V \to BO_s$ be the tautological bundle. The twisted ABS map for $MTH(s)$ is the composition

$$MTH(s) \simeq MT\mathrm{Spin} \wedge BO_s^{V-s} \xrightarrow{ABS_0 \wedge \Sigma^{-s}\lambda_V} KO \wedge \Sigma^{-s}KO \to \Sigma^{-s}KO. \tag{4.2.139}$$

For a comparison to index-theoretic definitions, see Prop. 10.27 in [1]. Our cases of interest are most related to $s = \pm 1$. For these, see also the more recent discussion in [456].

Example 4.2.140. For $s = 1$, the twisted ABS map is

$$ABS_{s=+1}\colon MT\mathrm{Pin}^- \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{\sigma-1} \xrightarrow{ABS_0\wedge\Sigma^{-1}\lambda_\sigma} \Sigma^{-1}KO. \tag{4.2.141}$$

◇

For $s < 0$, there is a different construction. In the following, for V the universal bundle over BO_s, we will use sm_V to denote the Smith map

$$\mathrm{sm}_V\colon BO_s^{-V} \to (BO_s)_+. \tag{4.2.142}$$

In what follows, denote by $\underline{1}$ the classifying map for the trivial line bundle, in this case over $BO_{|s|}$.

Definition 4.2.143 ([1, Proposition 10.24])**.** For $s = -1$, -2, or -3, the twisted ABS map for $MTH(s)$ is the composition

$$MTH(s) \simeq MT\mathrm{Spin} \wedge BO_{|s|}^{|s|-V} \xrightarrow{ABS_0\wedge\Sigma^{|s|}\mathrm{sm}_V} KO \wedge \Sigma^{|s|}(BO_{|s|})_+ \xrightarrow{\mathrm{id}\wedge\underline{1}} \Sigma^{|s|}KO. \tag{4.2.144}$$

Example 4.2.145. For $s = -1$, the twisted ABS map is

$$ABS_{s=-1}\colon MT\mathrm{Pin}^+ \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{1-\sigma} \xrightarrow{ABS_0\wedge\Sigma\mathrm{sm}_\sigma} \Sigma KO \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{id}\wedge\underline{1}} \Sigma KO. \tag{4.2.146}$$

◇

Remark 4.2.147*.* Note that these definitions are asymmetric: the generalized ABS maps for $s > 0$ and $s < 0$ are different. In the pin$^+$ case, if we instead use the equivalent identification[36] $MT\mathrm{Pin}^+ \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{3\sigma-3}$, we could have defined a map

$$MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{3\sigma-3} \xrightarrow{ABS_0\wedge\Sigma^{-3}\lambda_{3\sigma}} \Sigma^{-3}KO \tag{4.2.148}$$

but this does not agree with the index of the relevant Dirac operator.

Example 4.2.149. The Klein bottle K has four pin$^+$ structures; two of them are bounding and two are nonbounding, and either of the two nonbounding ones generates $\Omega_2^{\mathrm{Pin}^+} \cong \mathbb{Z}/2$ [110, Proposition 3.9]. In this example, we compute $\mathrm{ABS}_1(K, \mathfrak{p})$ for each pin$^+$ structure $\mathfrak{p}$ of K.

The Klein bottle is a circle bundle over the circle, explicitly described as the quotient of $S^1 \times [0,1]$ by the reflection $(z,0) \sim (\overline{z},1)$. If t denotes the coordinate on $[0,1]$, then the vector field ∂_t on $S^1 \times [0,1]$ descends to K and is nonvanishing, so $TK \cong \sigma \oplus \underline{\mathbb{R}}$ for a real line bundle $\sigma \to K$, which is the pullback of the Möbius bundle on the base S^1; thus $w_1(K)$ is the pullback of the top-degree class on the base, meaning that the inclusion $j\colon S^1 \to K$ of any fiber is a representative of the Poincaré dual of $w_1(K)$. The vector field ∂_t also trivializes the normal bundle to j, so any choice of pin$^+$ structure $\mathfrak{p}$ on K induces a pin$^+$ structure on the fiber, hence (after choosing once and for all an orientation) a spin structure $\mathfrak{s}_\mathfrak{p}$.

The assignment $(K,\mathfrak{p}) \mapsto (S^1, \mathfrak{s}_\mathfrak{p})$ is part of the data of the Smith homomorphism $\mathrm{sm}_\sigma\colon \Omega_2^{\mathrm{Pin}^+} \to \Omega_1^{\mathrm{Spin}}(B\mathbb{Z}/2)$—there is also a principal $\mathbb{Z}/2$-bundle. In other words, the composition

$$\Omega_2^{\mathrm{Pin}^+} \xrightarrow{\mathrm{sm}_\sigma} \Omega_1^{\mathrm{Spin}}(B\mathbb{Z}/2) \xrightarrow{c} \Omega_1^{\mathrm{Spin}}, \tag{4.2.150}$$

---

[36] Since 4σ is spin, we may suspend $B\mathbb{Z}/2$ by $4\sigma - 4$ as many times as we would like without changing the twist of spin bordism.

in which the map c forgets the principal $\mathbb{Z}/2$-bundle, sends $(K, \mathfrak{p}) \mapsto (S^1, \mathfrak{s}_\mathfrak{p})$. Kirby–Taylor (*ibid.*) show that $\mathfrak{s}_\mathfrak{p}$ is nonbounding if and only if $\mathfrak{p}$ is: (4.2.150) is an isomorphism $\mathbb{Z}/2 \to \mathbb{Z}/2$.

The last step in the computation of the twisted ABS map is to apply ABS_0, which is an isomorphism $\Omega_1^{\mathrm{Spin}} \to KO_1$. That is:

- If $\mathfrak{p}$ is one of the two nonbounding pin$^+$ structures on K, $\mathrm{ABS}_1(K, \mathfrak{p}) = \mathrm{ABS}_0(S^1_{nb}) = \eta$, the nonzero element of $KO_1 \cong \mathbb{Z}/2$.
- If $\mathfrak{p}$ is one of the two bounding structures on K, $\mathrm{ABS}_1(K, \mathfrak{p}) = \mathrm{ABS}_0(S^1_b) = 0$.

Since $\Omega_3^{\mathrm{Pin}^+}$ is torsion [193], this also implies that the Anderson dual of ABS_1 is an isomorphism in degree 2, as originally shown by Freed–Hopkins [1, Corollary 9.83]. ◇

Example 4.2.151. Consider the case where $n = 4$ and $s = -2$, where the ABS map is from the bordism of manifolds with pin$^{\tilde{c}+}$ structures to KO_2. We found in Appendix B (for the purposes of Section 4.1) that the map

$$\Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \cong (\mathbb{Z}/2)^2 \xrightarrow{ABS_{-2}} KO_2 \cong \mathbb{Z}/2 \tag{4.2.152}$$

sends the bordism generators $\mathbb{R}P^4$ and $\mathbb{C}P^2$ with appropriate pin$^{\tilde{c}+}$ structures to $0 \in KO_2$, while the class of $\mathbb{C}P^1 \times \mathbb{C}P^1$ with pin$^{\tilde{c}+}$ structure induced from its complex structure is sent to the generator of KO_2, which is dual to $\eta^2 \in KO^{-2}$. ◇

4.2.4.3 Generalization for Spin-(ℓ, k)

We now generalize the ABS orientation to emanate from $MT\mathrm{Spin}$ smashed with the Thom spectrum $ME_{\ell,k}$. This will be the main ingredient in the generalization of the free-to-interacting map for primed Altland–Zirnbauer classes (Section 4.2.5.1). Our generalization combines the techniques for the pin$^+$ and pin$^-$ cases above.

Definition 4.2.153. The twisted ABS map $ABS_{(\ell,k)}$ is given by

$$\begin{tikzcd}
MT\mathrm{Spin} \wedge (B\mathbb{Z}/2^{\sigma-1})^{\wedge \ell} \wedge (B\mathbb{Z}/2^{1-\sigma})^{\wedge k} \arrow[d, "ABS_0 \wedge (\Sigma^{-1}\lambda_\sigma)^{\wedge \ell} \wedge (\Sigma \mathrm{sm}_\sigma)^{\wedge k}"] \\
(\Sigma^{-1} KO)^{\wedge \ell} \wedge (\Sigma (B\mathbb{Z}/2)_+)^{\wedge k} \arrow[d, "\mathrm{id}^{\wedge \ell} \wedge (\Sigma \underline{1})^{\wedge k}"] \\
\Sigma^{-\ell} KO \wedge \Sigma^k KO \arrow[d, "\mu"] \\
\Sigma^{k-\ell} KO.
\end{tikzcd} \tag{4.2.154}$$

Remark 4.2.155. We refer to this as a twisted ABS *map* rather than a twisted ABS *orientation* because an orientation should be defined from a multiplicative cohomology theory. Our spectrum $MT\mathrm{Spin} \wedge ME_{\ell,k}$ is not a ring spectrum in general, but only an $MT\mathrm{Spin}$-module spectrum.

Example 4.2.156. Return to the case of $ME_{1,1} \simeq MT\mathrm{DPin}$. Then $ABS_{(1,1)}$ is the composition

$$\begin{tikzcd}
MT\mathrm{DPin} \arrow[r, phantom, "\simeq"] & MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{\sigma-1} \wedge (B\mathbb{Z}/2)^{1-\sigma} \arrow[d, "ABS_0 \wedge \Sigma^{-1}\lambda_\sigma \wedge \Sigma \mathrm{sm}_\sigma"] \\
& \Sigma^{-1}KO \wedge \Sigma(B\mathbb{Z}/2)_+ \arrow[d, "\mathrm{id} \wedge \Sigma \underline{1}"] \\
& \Sigma^{-1}KO \wedge \Sigma KO \arrow[d, "\mu"] \\
& KO. \quad \diamond
\end{tikzcd} \tag{4.2.157}$$

On homotopy, we get a map

$$ABS_{1,1} \colon \Omega_d^{\mathrm{DPin}} \longrightarrow KO_d. \tag{4.2.158}$$

Remark 4.2.159. For the physical applications in this paper, we are most interested in twists by (external sums of) the bundle $\sigma \to B\mathbb{Z}/2$. However, our twisted ABS map generalizes readily to any twist arising from a representation of a finite group, as in Example 4.2.5. Let G_b be a finite group, $\rho \colon G_b \to O_n$ be a k-dimensional G_b-representation, and G_f be the fermionic group formed as the pin$^+$ twist by ρ. That is, G_f has the twist data $\theta = w_1(\rho)$ and $\omega = w_2(\rho)$.

Now, let $V \to BG_b$ be the associated vector bundle to the representation ρ. Using the functor **Thom** of Theorem 4.2.82, we can produce a tangential structure $MTH(G_f)$ with a shearing isomorphism $MTH(G_f) \simeq MT\mathrm{Spin} \wedge BG_b^{V-k}$. Form the KO-theory class $\lambda_V = \chi(\Lambda^\bullet V) \in \widetilde{KO}^0(BG_b^V)$ as in Example 4.2.122. We can define a generalized ABS map for manifolds with $H(G_f)$ structures by the following formula, generalizing Definition 4.2.138:

$$MTH(G_f) \simeq MT\mathrm{Spin} \wedge BG_b^{V-k} \xrightarrow{ABS \wedge \Sigma^{-k}\lambda_V} \Sigma^{-k}KO. \tag{4.2.160}$$

Similarly, for a virtual representation associated to $-V$, we may generalize Definition 4.2.143. Let $G_f^\perp$ now denote the image under **Thom** of the twist by $-V$. Then $MTH(G_f^\perp) \simeq MT\mathrm{Spin} \wedge BG_b^{k-V}$, we can define the following twisted ABS map:

$$MTH(G_f^\perp) \simeq MT\mathrm{Spin} \wedge BG_b^{k-V} \xrightarrow{ABS \wedge \Sigma^k \mathrm{sm}_V} \Sigma^k(BG_b)_+ \xrightarrow{\mathrm{id} \wedge \underline{1}} \Sigma^k KO. \tag{4.2.161}$$

Note that when $G_f = E_{\ell,k}$, we had the isomorphism Equation (4.2.59), which reduced the free classification to a non-equivariant KO-theory group. In general, we should not expect our free classification to be given by a single shift of real or complex K-theory, and so more care is needed for the ABS map.

4.2.4.4 Free-to-Interacting maps from ABS maps

To get from ABS maps to maps between free fermion theories and invertible field theories, we Anderson-dualize. Recall that the Anderson dual to the sphere spectrum is the spectrum $I_\mathbb{Z}$ defined as the fiber of the *exponential map* $I_\mathbb{C} \to I_{\mathbb{C}^\times}$. By Theorem 4.1.24, invertible field theories are classified by Anderson-dual ξ-bordism: maps from $MT\xi$ to a shift of $I_\mathbb{Z}$.

Meanwhile, free theories are classified by K-theory, as discussed in Section 4.2.3.1 for the case of finite group superalgebra symmetries. This classification may be recast in terms of Anderson-dual K-theory using the Anderson self-duality of K-theory:

Theorem 4.2.162 ([337, Theorem 4.16])**.** *There are equivalences* $I_{\mathbb{Z}}KO \simeq \Sigma^4 KO$ *and* $I_{\mathbb{Z}}KU \cong KU$ *of KO, resp. KU-modules.*

For proofs, see Anderson's unpublished notes [337], or any of Yosimura [75, Theorem 4], Freed–Moore–Segal [346, Proposition B.11], Ricka [348, Corollary 5.8], or Hebestreit–Land–Nikolaus [349, Example 2.8]. (Some of these proofs only show an equivalence of spectra.) See Heard–Stojanoska [347, Theorem 8.1] for Anderson self-duality for KR-theory.

From self-duality of KU and KO, we may deduce self-duality of K-theory for finite group superalgebras. We consider the real case; the complex case is analogous.

Corollary 4.2.163. *Let* G_f *be a finite fermionic group. There is a KO-module equivalence* $I_{\mathbb{Z}}K(\mathbb{R}^f[G_f]) \simeq \Sigma^4 K(\mathbb{R}^f[G_f])$.

Example 4.2.164. In our case of interest for the Bott spiral, free theories in dimension d with $C\ell_{\ell,k}$-symmetry are classified by the group $K_{2-d}(C\ell_{\ell,k}) \cong KO^{d+\ell-k-2}(\mathrm{pt})$ (see Example 4.2.105). There is an identification $I_{\mathbb{Z}}\Sigma^{d+\ell-k-2}KO \cong \Sigma^{d+\ell-k+2}KO$. ◇

We may express Anderson self-duality as a map of spectra called the Pfaffian.

Definition 4.2.165. Choose a generator of $\pi_{-4}KO \cong \mathbb{Z}$ and consider a representing map $S^{-4} \to KO$. The real Pfaffian is the map $\mathrm{Pfaff}\colon KO \to \Sigma^4 I_{\mathbb{Z}}$ obtained by applying the universal property of the Anderson dual (4.1.23) to this representing map.

See [1, Section 9.2.5] for physical interpretations of the Pfaffian.

The free-to-interacting maps proposed by Freed–Hopkins for the tenfold way are the following.

Definition 4.2.166 (Freed–Hopkins [1, Conjecture 9.70])**.** Let $\mathrm{ABS}_s\colon MTH(s) \to \Sigma^s KO$ be the twisted ABS map of (4.2.143) or (4.2.138), depending on the sign of s. The free-to-interacting map for symmetry type $H(s)$ is the dualized map

$$\Sigma^s KO \simeq I_{\mathbb{Z}}\Sigma^{s+4}KO \xrightarrow{I_{\mathbb{Z}}ABS_s} \Sigma^4 I_{\mathbb{Z}}MTH(s). \tag{4.2.167}$$

Equivalently, for a fixed degree, the free-to-interacting map is the map sending a free theory $x \in KO^{d+s-2}(\mathrm{pt})$ to the class of the composition

$$MTH(s) \xrightarrow{ABS_s \wedge x} \Sigma^{-s}KO \wedge \Sigma^{d+s-2}KO \xrightarrow{\text{mult}} \Sigma^{d-2}KO \xrightarrow{\text{Pfaff}} \Sigma^{d+2}I_{\mathbb{Z}} \tag{4.2.168}$$

in $[MTH(s), \Sigma^{d+2}I_{\mathbb{Z}}]$.

Example 4.2.169. Consider the time-reversal symmetric Majorana chain, which was the first example in the physics literature of an interacting classification breakdown. As discussed in Example 4.2.113, it generates a $\mathbb{Z}$'s worth of phases when considered as a free phase. However, Fidkowski–Kitaev showed that when quartic terms are allowed in the Hamiltonian, eight copies of the Majorana chain can be adiabatically connected to the trivial phase [104]. Hence the physical computation witnesses a classification breakdown from $\mathbb{Z}$ to $\mathbb{Z}/8$.

Mathematically, this map is modeled by the case $s = 1$ of Definition 4.2.166 on homotopy in degree $2 - d = 1$:

$$KO^0 \cong \mathbb{Z} \xrightarrow{I_{\mathbb{Z}}ABS_{+1}} \mho^2_{\mathrm{Pin}^-} \cong \mathbb{Z}/8. \tag{4.2.170}$$

Freed–Hopkins computed [1, Corollary 9.83] that this map is a reduction modulo eight, reproducing the physical computation. The low-energy TQFT describing the Majorana chain is the Arf–Brown–Kervaire IFT of Example 4.2.116, and the deformation class of this IFT generates the group $\mho^2_{\mathrm{Pin}^-}$. ◇

Equipped with the generalized orientation (4.2.154), we may now define the free-to-interacting map for symmetry types in the Bott spiral and beyond.

Definition 4.2.171. The free-to-interacting map for $E_{\ell,k}$ twists is

$$F2I_{(\ell,k)} \coloneqq \Sigma^{k-\ell}KO \simeq I_{\mathbb{Z}}\Sigma^{k-\ell+4}KO \xrightarrow{I_{\mathbb{Z}}ABS_{(\ell,k)}} I_{\mathbb{Z}}\Sigma^4 MT\mathrm{Spin} \wedge ME_{(\ell,k)} \tag{4.2.172}$$

Equivalently, let $x \in KO^{d+s-2}(\mathrm{pt})$ represent a free theory in spatial dimension d of symmetry type s, and let ℓ, k be such that $\ell - k = s$. The free-to-interacting map is the map

$$F2I_{(\ell,k)} \colon KO^{d+\ell-k-2}(\mathrm{pt}) \to [MT\mathrm{Spin} \wedge ME_{\ell,k}, \Sigma^{d+2}I_{\mathbb{Z}}] \tag{4.2.173}$$

such that $F2I_{(\ell,k)}(x)$ is given by the class of the composition

$$\begin{tikzcd}
MT\mathrm{Spin} \wedge (B\mathbb{Z}/2^{\sigma-1})^{\wedge l} \wedge (B\mathbb{Z}/2^{1-\sigma})^{\wedge k} \arrow[r, phantom, "\simeq"] & MT\mathrm{Spin} \wedge (B\mathbb{Z}/2^{\sigma-1})^{\wedge l} \wedge (B\mathbb{Z}/2^{1-\sigma})^{\wedge k} \wedge \mathbb{S} \arrow[d, "ABS_{(l,k)} \wedge x"] \\
 & \Sigma^{k-l}KO \wedge \Sigma^{d+l-k-2} \arrow[d, "\mathrm{mult}"] \\
 & \Sigma^{d-2}KO \arrow[d, "\mathrm{Pfaff}"] \\
 & \Sigma^{d+2}I_{\mathbb{Z}}.
\end{tikzcd} \tag{4.2.174}$$

Remark 4.2.175. Later, we will compute these free-to-interacting maps. This boils down to computing the maps $\mathrm{ABS}_{\ell,k}$ from (4.2.154). Since these are MTSpin-module maps of the form $MT\mathrm{Spin} \wedge ME_{\ell,k} \to \Sigma^{k-\ell}KO$, where the domain's MTSpin-module structure is through extension of scalars, $\mathrm{ABS}_{\ell,k}$ as a map of MTSpin-modules is determined by the map of spectra $\Phi_{\ell,k} \colon ME_{\ell,k} \to \Sigma^{k-\ell}KO$ defined by the restricting $\mathrm{ABS}_{\ell,k}$ to $ME_{\ell,k}$. See Section 4.2.7.

4.2.5 Modeling the Bott spiral

In the following subsections, we apply the approach from Sections 4.2.2 and 4.2.3 to describe the free and interacting phases of models in *primed* Altland–Zirnbauer classes (Section 4.2.5.1), as well as construct a mathematical dimensional reduction map for both the free (Definition 4.2.184) and interacting (Definition 4.2.199) theories, using ideas from Chapter 3.

4.2.5.1 Primed Altland–Zirnbauer classes

In previous sections, we explained how to classify free G_f-protected phases via Ansatz 4.2.95 as well as interacting $H(G_f)$-protected phases via Theorem 4.1.24, and in Table 4.3, we reviewed the fermionic groups that encoded the symmetry types for the tenfold way. To study the Bott spiral, we need a generalization of the tenfold way symmetry types: in addition to the ten Altland–Zirnbauer classes specified in Table 4.1, Queiroz–Khalaf–Stern make use of *primed* classes, generalizing from definitions of class D′ put forward in [287, 457]. We will first provide a physical description of these classes, then give our mathematical model, which uses the fermionic groups $E_{\ell,k}$ of Definition 4.2.19.

Consider a topological phase in the tenfold way corresponding to an *integer* invariant ν. For example, in two dimensions in class D, we consider a system with invariant $\nu \in KO^0 \cong \mathbb{Z}$, and view ν as counting (with sign) the number of chiral edge modes. Stack this with a phase in the same class but with the opposite invariant, $-\nu$. The composite system hosts a time reversal symmetry $\mathcal{T}$ that reverses the direction of the chiral modes and which squares to -1. Considered only with this additional $\mathcal{T}$ symmetry, the system is in the trivial phase, since the invariants ν and $-\nu$ just cancel—in the case of starting with class D, this would be the trivial class DIII system. However, the composite system also has a unitary symmetry $\mathcal{R}$, which reflects the chiral modes and which squares to -1. See [6, 287, 457], where the last author calls the class of the system DIII+$\mathcal{R}$. Considered with these two additional symmetries $\mathcal{T}$ and $\mathcal{R}$, the resulting system is in a primed symmetry class—for example, D′ if we began with two copies of class D superconductors—and has a nontrivial topological invariant ν.

Overall, two $\mathbb{Z}/2$ symmetries have been added to the original system, one unitary and one anti-unitary, but the free classification remains unchanged. Motivated by this, we propose the following definition.

Definition 4.2.176. Let G_f be the symmetry group associated to a certain Altland–Zirnbauer class, as specified in Table 4.1. The *primed class* associated to it is described by the fermionic group $G'_f = G_f \hat{\times} E_{1,1}$ of Definition 4.2.63.

Remark 4.2.177. While the $C\ell_{1,1}$ periodicity Proposition 4.2.58 of KO-theory ensures that the free classifications of primed and unprimed phases will be isomorphic, the same is not true for interacting classifications. The tangential structure $H(G_f \hat{\times} E_{1,1})$ is measurably different from $H(G_f)$: as explained in Remark 4.2.90, the corresponding bordism groups and, dually, the groups of IFTs, are *not* the same. For a more in-depth discussion of the phenomenon of "Morita variance" in interacting SPTs, see [319].

Example 4.2.178. Class D, sometimes called the case of "no symmetry," corresponds to the trivial fermionic group and to the tangential structure $H(*) = \mathrm{Spin}$. We model class D′ using the fermionic group $E_{1,1}$. In Lemma 4.2.89, we showed that the corresponding tangential structure is equivalent to the dpin structure of [438]. Free class D′ phases in dimension $d = 2$ are thus classified by $KO_0(C\ell_{1,1}) \cong KO^0 \cong \mathbb{Z}$, while interacting phases are classified by $\mho_2^{\mathrm{DPin}} \cong (\mathbb{Z}/2)^2$. ◇

Remark 4.2.179. Queiroz–Khalaf–Stern restrict their approach to integral free phase classifications, avoiding cases where the initial KO-group is instead $\mathbb{Z}/2$. For the same reason,

AZ class	D′	AI′	C′	AII′	A′
(ℓ, k)	$(1,1)$	$(3,1)$	$(5,1)$	$(1,3)$	$(1,1)$

AZ class	BDI′	CI′	CII′	DIII′	AIII′
(ℓ, k)	$(2,1)$	$(4,1)$	$(1,4)$	$(1,2)$	$(2,1)$

Table 4.4: For concreteness, we provide a table, listing the more-physically-interesting half of the primed Altland–Zirnbauer classes first. Note also that the table is not redundant—the rightmost entriess are complex classes.

Queiroz–Khalaf–Stern consider primed phases only in *even* spatial dimensions—physically, primed phases in even dimensions are notable because they are *non-chiral*.

Since our mathematical definition of primed phases works for any input class and dimension, we will consider all cases going forward. As we compute in Section 4.2.7, the cases beginning with a $\mathbb{Z}/2$ classification do *not* feature an interesting breakdown under interactions.

4.2.5.2 Image of the free-to-interacting map for primed phases

Recall from Example 4.2.169 that the interacting classification of class BDI phases is $\mathbb{Z}/8$. Fidkowski–Kitaev found this by devising an explicit quartic interaction term in the system with eight copies of the time-reversal-symmetric Majorana chain [104, Equation 8] that allowed the system to be adiabatically connected to the trivial system. Queiroz–Khalaf–Stern generalized this to an interaction term for any primed or unprimed Altland–Zirnbauer class [6, Equation 9] in zero dimensions, then extended to higher-dimensional phases using dimensional reduction. They discovered that the interacting classification of these free models was not periodic, but had a more complex relationship with the dimension of the system [6]. They studied three cases: where the symmetry class of the corresponding $d = 1$ system was BDI, CII, or AIII.

In our homotopy-theoretic model, the interacting classification of these free systems is given by the image of the free-to-interacting map of Definition 4.2.171 for $(\ell, k) = (1, 1)$, and of the analogous free-to-interacting maps for other nine primed symmetry types from Definition 4.2.176.

Our detailed computation can be found in Section 4.2.7, where the full result for class D′ is stated in Theorem 4.2.266 and for other primed phases in the later subsections.

Example 4.2.180. Continuing Example 4.2.178, the free-to-interacting map

$$KO_0(C\ell_{1,1}) \longrightarrow \mho^3_{\mathrm{DPin}} \tag{4.2.181}$$

$$\mathbb{Z} \longrightarrow \mathbb{Z}/8 \tag{4.2.182}$$

is a reduction mod 8. ◇

Remark 4.2.183. Queiroz–Khalaf–Stern's results do not fully characterize the interacting classification of phases, but only the image of our free-to-interacting map. The cokernel of that map includes intrinsically-interacting phases, which are not our focus in this work and which we also do not fully compute. However, we argue in Remark 4.2.228 that the pieces that our computation omits, which we term *bosonic summands*, do not correspond to interesting fermionic theories.

4.2.5.3 Dimensional reduction

Dimensional reduction is a common technique in physics, from compactification in the context of string theories to defect creation processes in condensed matter. Following [6, 8, 457, 458], we model dimensional reduction for free fermion theories as an isomorphism derived from the suspension isomorphism and $(1,1)$-periodicity in KO-theory. On the interacting side, it is not obvious how to model dimensional reduction as a map of IFTs. Informed by the ability of Smith homomorphisms to describe defect anomaly matching for high energy theories (Chapter 3), we propose a modified Smith homomorphism to model an interacting version of the dimensional reduction process of [6]. In Section 4.2.6, we prove that our dimensional reduction map of IFTs fits into a commutative diagram with the free dimensional reduction map and two free-to-interacting maps.

4.2.5.3.1 The free dimensional reduction map Physically, dimensional reduction is achieved by adding a symmetry-breaking mass term to the starting Hamiltonian, then enforcing boundary conditions, similar to the defect creation process discussed in Definition 3.7.1. We leave the details to the physics references [6, 8, 457, 458]. In our framework, the isomorphism representing this process takes the following form:

Definition 4.2.184. Let A be a finite superalgebra representing the symmetry of the system. The *free dimensional reduction map*

$$KO_{2-d}(A\hat{\otimes}C\ell_{+1}) \xrightarrow{\cong} KO_{2-(d-1)}(A). \tag{4.2.185}$$

This map is just the identity $\mathrm{id}\colon KO \to KO$ on underlying spectra, but on the level of homotopy classes it gives an equivalence between models in two different degrees. In terms of Karoubi's model for K-theory, the free dimensional reduction map is described explicitly in [448, p. III.5.9]. Similar formulas have appeared in the physics literature; see e.g. [458, (A3)].

We use an analogous formula in the complex case.

Example 4.2.186. The complex Bott spiral starts from the zero-dimensional class A′ insulator, which is dimensionally-reduced from the one-dimensional class AIII insulator, whose generator corresponds to the low-energy theory of the Su–Schrieffer–Heeger lattice model [105]. Mathematically, we model this by the isomorphism

$$K_0(C\ell_{1,1}) \xrightarrow{\cong} K_1(C\ell_{-1}) \tag{4.2.187}$$
$$\text{0d class A}' \longmapsto \text{1d class AIII}. \tag{4.2.188}$$

By definition, a 0d class A′ state space is given by a $\mathbb{C}\ell_{1,1}$-module. Recalling that $\mathbb{C}\ell_2 \cong \mathbb{C}\ell_{1,1}$, this is equivalently a $\mathbb{C}\ell_2$-module, i.e. a complex vector space V with two anticommuting operators $\epsilon_1, \epsilon_2\colon V \to V$ such that $\epsilon_1^2 = \epsilon_2^2 = \mathrm{id}_V$. The Morita equivalence between class A′ and class A realizing the isomorphism $K^{-2}(\mathrm{pt}) \cong K^0(\mathrm{pt})$ is induced by the isomorphism

$\mathbb{C}\ell_{1,1} \cong \mathrm{End}(\mathbb{C}^{1|1})$ of superalgebras. We choose to represent this isomorphism by the Pauli matrices

$$\epsilon_1 \longmapsto \sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \epsilon_2 \longmapsto \sigma^Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.2.189}$$

which are indeed odd anticommuting operators on $\mathbb{C}^{1|1}$.

A 0d class A system is given by a finite-dimensional complex vector space W and a gapped Hamiltonian $h\colon W \to W$ with corresponding grading $W = W_{E>0} \oplus W_{E<0}$ in terms of states below and states above the gap. It is straightforward to see that an isomorphism $K^0(\mathrm{pt}) \cong \mathbb{Z}$ is given by

$$[W, h_1, h_2] \longmapsto (\mathrm{dim}W_{E_1>0} - \mathrm{dim}W_{E_1<0}) - (\mathrm{dim}W_{E_2>0} - \mathrm{dim}W_{E_2<0}). \tag{4.2.190}$$

In particular, a generator of $K^0(\mathrm{pt})$ is given by the positive Hamiltonian h_1 and the negative Hamiltonian h_2 on $W = \mathbb{C}$. The class A Hamiltonian h corresponds to the class A' Hamiltonian

$$\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \tag{4.2.191}$$

on the $\mathbb{C}\ell_2$-module $V := W \oplus \Pi W$. Note that this Hamiltonian instead graded commutes with the $\mathbb{C}\ell_2$-action.

Given a 0d class A′ Hamiltonian $h'\colon V \to V$, we have to construct a 1d class AIII Hamiltonian. For this, we may take the trivial vector bundle $V \times S^1$ over the Brillouin zone S^1 with the Bloch Hamiltonian $H_k = \epsilon_2 \cos k + h' \sin k$. ◇

Next we can discuss the real cases.

Example 4.2.192. We return to our running example, where the dimensional reduction map runs from class D′ in two dimensions to class BDI in one dimension. Plugging in the symmetry algebra $A = C\ell_{-1}$ and the dimension $d = 1$ yields the isomorphism

$$KO_0(C\ell_{1,1}) \xrightarrow{\cong} KO_1(C\ell_{-1}) \tag{4.2.193}$$

$$\text{2d class D}' \longmapsto \text{1d class BDI}. \tag{4.2.194}$$

◇

Remark 4.2.195. Note that the isomorphism of Definition 4.2.184 is *not* simply "forgetting the prime." That is, the map on K-theory induced by the inclusion of the symmetry group G_f into $G_f' = G_f \hat{\otimes} C\ell_{1,1}$ is not this isomorphism, but rather the zero map, as we showed in Proposition 4.2.64.

4.2.5.3.2 The interacting dimensional reduction map Though Queiroz–Khalaf–Stern [6] only define a dimensional reduction map for free fermion phases, it is natural to wonder whether their construction can be generalized to interacting phases. We will show that, at least using our models for the spectra classifying free and interacting phases, there *is* an analogue of the dimensional reduction map for invertible field theories, and the two maps are compatible, in the sense we state and prove in Theorem 4.2.210. Our approach uses a modified Smith homomorphism, addressing the question of Hason–Komargodski–Thorngren [265, Section 5] about how the Bott spiral and the defect anomaly map (Section 3.8.2) are related.

Our approach is to break one of the $\mathbb{Z}/2$-symmetries and then take a defect map. Let sm_σ denote the Smith map, and let $c_X\colon \Sigma^\infty_+ X \to \mathbb{S}$ be the crush map for a space X, which is defined to be Σ^∞_+ of the unique map $X \to \mathrm{pt}$. When X is clear from context, we will write c instead of c_X.

Definition 4.2.196. Let $\varphi^{0,1}_{\mathrm{int}}\colon ME_{0,1} \to \Sigma\mathbb{S}$ denote the composition

$$\varphi^{0,1}_{\mathrm{int}}\colon ME_{0,1} \simeq (B\mathbb{Z}/2)^{1-\sigma} \xrightarrow{\mathrm{sm}_\sigma} \Sigma(B\mathbb{Z}/2)_+ \xrightarrow{\Sigma c} \Sigma\mathbb{S}. \tag{4.2.197}$$

Then, for $\ell \geq 0$ and $k \geq 1$, let $\varphi^{\ell,k}_{\mathrm{int}}\colon ME_{\ell,k} \to \Sigma ME_{\ell,k-1}$ be the map $\mathrm{id}_{ME_{\ell,k-1}} \wedge \varphi^{0,1}_{\mathrm{int}}$, using the canonical homotopy equivalence $ME_{\ell,k} \wedge ME_{\ell',k'} \simeq ME_{\ell+\ell',k+k'}$. We will refer to the maps $\varphi^{\ell,k}_{\mathrm{int}}$ as *twisted projection* maps.

Remark 4.2.198. The map $\varphi^{0,1}_{\mathrm{int}}$ from Definition 4.2.196, interpreted as a twisted stable cohomotopy class $[\varphi^{0,1}_{\mathrm{int}}] \in \pi^1_s((B\mathrm{O}_1)^{1-\sigma})$, coincides with Becker's universal twisted Euler class [116, §13].[37]

Definition 4.2.199. For a spectrum E, let $\mathrm{sp}^{\ell,k}_{\mathrm{int}}(E)\colon \Sigma I_{\mathbb{Z}}(E \wedge ME_{\ell,k-1}) \to I_{\mathbb{Z}}(E \wedge ME_{\ell,k})$ be $I_{\mathbb{Z}}(\mathrm{id}_E \wedge \varphi^{\ell,k}_{\mathrm{int}})$. When $E = MT\mathrm{Spin}$ or $MT\mathrm{Spin}^c$, we will write $\mathrm{sp}^{\ell,k}_{\mathrm{int}} \coloneqq \mathrm{sp}^{\ell,k}_{\mathrm{int}}(E)$ and call $\mathrm{sp}^{\ell,k}_{\mathrm{int}}$ the *Bott spiral map*; the choice of $MT\mathrm{Spin}$ versus $MT\mathrm{Spin}^c$ will be clear from context.

The maps $\mathrm{sp}^{\ell,k}_{\mathrm{int}}$ model interacting dimensional reduction of SPT phases. We use the more general maps $\mathrm{sp}^{\ell,k}_{\mathrm{int}}(E)$ chiefly for $E = ko$ and ku; they will be a useful intermediate step for our computation of $\mathrm{sp}^{\ell,k}_{\mathrm{int}}$ in Section 4.2.7.1 and Section 4.2.7.3. The assignment $E \mapsto \mathrm{sp}^{\ell,k}_{\mathrm{int}}(E)$ is contravariant.

Remark 4.2.200. For an explicit comparison of φ_{int} with sm_σ, consider the following commutative diagram.

$$\begin{tikzcd}
(B\mathbb{Z}/2^{\sigma-1})^{\wedge l} \wedge (B\mathbb{Z}/2^{-\sigma+1})^{\wedge k} \arrow[d, "\mathrm{id}^{\wedge l+k-1} \wedge \mathrm{sm}_\sigma"] & (B\mathbb{Z}/2)^{\wedge(l-k)(\sigma-1)} \arrow[l, "\Delta"'] \arrow[d, "\mathrm{sm}_\sigma"] \\
(B\mathbb{Z}/2^{\sigma-1})^{\wedge l} \wedge (B\mathbb{Z}/2^{-\sigma+1})^{\wedge k-1} \wedge (B\mathbb{Z}/2)_+ \arrow[d, "\simeq"] & \Sigma(B\mathbb{Z}/2)^{\wedge(l-k+1)(\sigma-1)} \arrow[l, "\Delta"'] \arrow[dd, "\mathrm{id}"] \\
(B\mathbb{Z}/2^{\sigma-1})^{\wedge l} \wedge (B\mathbb{Z}/2^{-\sigma+1})^{\wedge k-1} \wedge (\mathbb{S} \wedge B\mathbb{Z}/2) \arrow[d, "\mathrm{id}^{\wedge l} \wedge \mathrm{proj}_1"] & \\
\Sigma(B\mathbb{Z}/2^{\sigma-1})^{\wedge l} \wedge (B\mathbb{Z}/2^{-\sigma+1})^{\wedge k-1} & \Sigma(B\mathbb{Z}/2)^{\wedge(l-k+1)(\sigma-1)}. \arrow[l, "\Delta"']
\end{tikzcd} \tag{4.2.201}$$

The "diagonal" maps Δ are defined as follows.

---

[37] See [459, §2], [460, §II.4], and [9, Definition 4.2] for additional, equivalent definitions of this class.

- Given a map of spaces $f\colon X \to Y$ and a vector bundle $V \to Y$, there is a map of Thom spectra $X^{f^*V} \to Y^V$ (either by doing an explicit construction with Thom spaces, or by abstract nonsense with twists).
- For each of these three Δ maps, the map of spaces is the diagonal map $B\mathbb{Z}/2 \hookrightarrow (B\mathbb{Z}/2)^{\times m}$ for different values of m (green and red: $m = k + l$, blue: $m = k + l - 1$)
- The smash product of Thom spectra is the Thom spectrum of the external direct sum of vector bundles: $X^V \wedge Y^W = (X \times Y)^{V \boxplus W}$.
  - Therefore the codomain of the green Δ is the Thom spectrum of the vector bundle
    $$\sigma_1 \boxplus \sigma_2 \boxplus \cdots \boxplus \sigma_l \boxplus (-\sigma_{l+1}) \boxplus \cdots \boxplus (-\sigma_{k+l}) \longrightarrow (B\mathbb{Z}/2)^{k+l}, \qquad (4.2.202)$$
    where σ_i denotes the tautological bundle over the ith copy of $B\mathbb{Z}/2$.
  - For the red Δ, the codomain is the Thom spectrum of the virtual bundle
    $$\sigma_1 \boxplus \sigma_2 \boxplus \cdots \boxplus \sigma_l \boxplus (-\sigma_{l+1}) \boxplus \cdots \boxplus (-\sigma_{k+l-1}) \boxplus 0 \longrightarrow (B\mathbb{Z}/2)^{k+l}. \qquad (4.2.203)$$
  - For the blue Δ, the codomain is the Thom spectrum of the virtual bundle
    $$\sigma_1 \boxplus \sigma_2 \boxplus \cdots \boxplus \sigma_l \boxplus (-\sigma_{l+1}) \boxplus \cdots \boxplus (-\sigma_{k+l-1}) \longrightarrow (B\mathbb{Z}/2)^{k+l-1}. \qquad (4.2.204)$$

Example 4.2.205. We return to the running example between symmetry types dpin and pin$^-$. Consider the following map of spectra:

$$\begin{tikzcd}
B\mathbb{Z}/2^{\sigma-1} \wedge B\mathbb{Z}/2^{-\sigma+1} \arrow[r, "\mathrm{id}\wedge\mathrm{sm}_\sigma"] & B\mathbb{Z}/2^{\sigma-1} \wedge \Sigma(B\mathbb{Z}/2)_+ \arrow[r, "\simeq"] & \Sigma B\mathbb{Z}/2^{\sigma-1} \wedge (\mathbb{S} \vee B\mathbb{Z}/2) \arrow[d, "\mathrm{id}\wedge\mathrm{proj}_1"] \\
 & & \Sigma B\mathbb{Z}/2^{\sigma-1}.
\end{tikzcd} \qquad (4.2.206)$$

Smash with $MT\mathrm{Spin}$ to get

$$MT\mathrm{Spin} \wedge B\mathbb{Z}/2^{\sigma-1} \wedge B\mathbb{Z}/2^{-\sigma+1} \simeq MT\mathrm{DPin} \to MT\mathrm{Spin} \wedge \Sigma B\mathbb{Z}/2^{\sigma-1} \simeq \Sigma MT\mathrm{Pin}^-. \qquad (4.2.207)$$

Map into $\Sigma^{d+2} I_{\mathbb{Z}}$ to get the interacting Bott spiral map

$$\mho^{d-1}_{\mathrm{Pin}^-} = [MT\mathrm{Pin}^-, \Sigma^{d+1} I_{\mathbb{Z}}] = [\Sigma MT\mathrm{Pin}^-, \Sigma^{d+2} I_{\mathbb{Z}}] \to [MT\mathrm{DPin}, \Sigma^{d+2} I_{\mathbb{Z}}] = \mho^d_{\mathrm{DPin}}. \qquad (4.2.208)$$

Now, specialize to $d = 3$. The map on bordism is given by

$$\begin{tikzcd}
\Omega_3^{\mathrm{DPin}} \cong \mathbb{Z}/8 \arrow[r] & \mathbb{Z}/8 \cong \Omega_2^{\mathrm{Pin}^-} \\
(\mathbb{R}P^3, L_1, L_2) \arrow[r, mapsto] & (\mathbb{R}P^2, L_1)
\end{tikzcd}$$

where the line bundles are such that $T\mathbb{R}P^3 + L_1 - L_2$ is spin and such that $T\mathbb{R}P^2 + L_1$ is spin. Recall that $T\mathbb{R}P^n = (n+1)\sigma - 1$ to see that this requires one to take $L_1 = L_2 = \sigma$.

On field theories (see Section 4.2.7), this map dualizes to

$$I^3_{\mathbb{Z}}(MT\mathrm{Pin}^-) \cong \mathbb{Z}/8 \to \mathbb{Z}/8 \cong I^4_{\mathbb{Z}}(MT\mathrm{DPin}). \tag{4.2.209}$$

The generator of the former group is the interacting time reversal invariant Majorana chain. Under this map, it is sent to the two-dimensional interacting D′ superconductor. ◇

4.2.6 Interactions commute with dimensional reduction

Physically, we expect that the process of incorporating interactions should commute with dimensional reduction. Mathematically, we phrase this as a commutative diagram between the free-to-interacting maps and dimensional reduction maps:

Theorem 4.2.210. *The diagram*

$$\begin{tikzcd}
\Sigma^{2-d-\ell+k}KO \arrow[r, "\mathrm{F2I}_{\ell,k}"] \arrow[d, "\mathrm{id}"'] & \Sigma^{d+2}I_{\mathbb{Z}}(MT\mathrm{Spin}\wedge ME_{\ell,k}) \arrow[d, "\Sigma^{d+2}I_{\mathbb{Z}}(\mathrm{id}\wedge\varphi^{\ell,k}_{int})"] \\
\Sigma^{2-d-\ell+k}KO \arrow[r, "\mathrm{F2I}_{\ell,k+1}"'] & \Sigma^{d+3}I_{\mathbb{Z}}(MT\mathrm{Spin}\wedge ME_{\ell,k+1})
\end{tikzcd} \tag{4.2.211}$$

commutes.

In Equation (4.2.211), the left vertical map is the free dimensional reduction map, which is just the identity on spectra id: $KO \to KO$. The right vertical map is the interacting dimensional reduction map $\varphi^{\ell,k}_{\mathrm{int}}\colon ME_{\ell,k} \to \Sigma ME_{\ell,k-1}$ (4.2.196), and the horizontal maps are the free-to-interacting maps for the symmetry types (ℓ,k) and $(\ell,k+1)$, resp.

We will prove this by working backwards through a series of lemmas simplifying (4.2.211) into simpler commutative diagrams.

Lemma 4.2.212. *Recall the map* $\mathrm{ABS}_{\ell,k}\colon MT\mathrm{Spin}\wedge ME_{\ell,k} \to \Sigma^{k-\ell}KO$ *defined in* (4.2.154). *Data witnessing the commutativity of the diagram of* $MT\mathrm{Spin}$*-modules*

$$\begin{tikzcd}
MT\mathrm{Spin}\wedge ME_{\ell,k+1} \arrow[dd, "\mathrm{id}\wedge\varphi^{l,k}_{\mathrm{int}}"'] \arrow[dr, "\mathrm{ABS}_{\ell,k+1}"] & \\
& \Sigma^{k+1-\ell}KO \\
MT\mathrm{Spin}\wedge\Sigma ME_{\ell,k} \arrow[ur, "\Sigma\mathrm{ABS}_{\ell,k}"'] &
\end{tikzcd} \tag{4.2.213}$$

induces commutativity data for (4.2.211)*, i.e. proving Theorem 4.2.210.*

Proof. Data witnessing commutativity of (4.2.211) is equivalent to the data of a homotopy $\Sigma^{d+2}I_{\mathbb{Z}}(MT\mathrm{Spin}\wedge\varphi^{\ell,k}_{\mathrm{int}}) \circ \mathrm{F2I}_{\ell,k} \simeq \mathrm{F2I}_{\ell,k+1}$. Using the definitions of the free-to-interacting maps in (4.2.174), we see that given $x \in KO^{2-d-\ell+k}(\mathrm{pt}) = [\mathbb{S}, \Sigma^{d+\ell-k-2}KO]$, in the diagram

$$\begin{tikzcd}
MT\mathrm{Spin} \wedge ME_{\ell,k+1} \arrow[d, "\mathrm{id}\wedge\varphi_{\mathrm{int}}^{\ell,k}"'] \arrow[dr, "\mathrm{ABS}_{\ell,k+1}\wedge x"] & & & \\
(4.2.213)\wedge x & \Sigma^{k+1-\ell}KO \wedge \Sigma^{d+\ell-k-2}KO \arrow[r, "\mu"] & \Sigma^{d-1}KO \arrow[r, "\mathrm{Pfaff}"] & \Sigma^{d+3}I_{\mathbb{Z}} \\
MT\mathrm{Spin} \wedge \Sigma ME_{\ell,k} \arrow[ur, "\Sigma\mathrm{ABS}_{\ell,k}\wedge x"'] & & &
\end{tikzcd} \tag{4.2.214}$$

the composition $\mathrm{Pfaff} \circ \mu \circ \mathrm{ABS}_{\ell,k+1} \wedge x$ of the arrows along the top is $\mathrm{F2I}_{\ell,k+1}(x)$ and the composition $\mathrm{Pfaff} \circ \mu \circ \Sigma\mathrm{ABS}_{\ell,k} \wedge x$ of the arrows along the bottom is $\mathrm{F2I}_{\ell,k}(x)$. Therefore data witnessing commutativity of (4.2.213) induces data witnessing commutativity of (4.2.214), i.e. providing a homotopy $\Sigma^{d+2}I_{\mathbb{Z}}(MT\mathrm{Spin} \wedge \varphi_{\mathrm{int}}^{\ell,k}) \circ \mathrm{F2I}_{\ell,k} \simeq \mathrm{F2I}_{\ell,k+1}$ as desired. ☐

Lemma 4.2.215. *Data witnessing the commutativity of*

$$\begin{tikzcd}
ME_{\ell,k+1} = ((B\mathbb{Z}/2)^{\sigma-1})^{\wedge\ell} \wedge ((B\mathbb{Z}/2)^{1-\sigma})^{\wedge(k+1)} \arrow[r, "\lambda_\sigma^{\wedge\ell}\wedge\mathrm{sm}_\sigma^{\wedge(k+1)}"'] & (\Sigma^{-1}KO)^{\wedge\ell} \wedge (\Sigma(B\mathbb{Z}/2))_+^{\wedge(k+1)} \arrow[d, "\mathrm{id}^{\wedge(\ell+k)}\wedge\Sigma\mathrm{sm}_\sigma"', "\mathrm{id}^{\wedge\ell}\wedge(\Sigma\underline{1})^{\wedge(k+1)}"] & \\
((B\mathbb{Z}/2)^{\sigma-1})^{\wedge\ell} \wedge ((B\mathbb{Z}/2)^{1-\sigma})^{\wedge k} \wedge \Sigma(B\mathbb{Z}/2)_+ & (\Sigma^{-1}KO)^{\wedge\ell} \wedge (\Sigma KO)^{\wedge(k+1)} \arrow[d, "\simeq"'] \arrow[dr, "\mathrm{mult.}"] & \\
\Sigma((B\mathbb{Z}/2)^{\sigma-1})^{\wedge\ell} \wedge ((B\mathbb{Z}/2)^{1-\sigma})^{\wedge k} \wedge (B\mathbb{Z}/2)_+ \arrow[d, "\mathrm{id}^{\wedge(\ell+k)}\wedge c"'] & \Sigma(\Sigma^{-1}KO)^{\wedge\ell} \wedge (\Sigma KO)^{\wedge k} \arrow[r, "\mathrm{mult.}"'] & \Sigma^{k+1-\ell}KO \\
\Sigma ME_{\ell,k} = \Sigma((B\mathbb{Z}/2)^{\sigma-1})^{\wedge\ell} \wedge ((B\mathbb{Z}/2)^{1-\sigma})^{\wedge k} \arrow[r, "\Sigma\lambda_\sigma^{\wedge\ell}\wedge\mathrm{sm}_\sigma^{\wedge k}"'] & \Sigma(\Sigma^{-1}KO)^{\ell} \wedge (\Sigma(B\mathbb{Z}/2)_+)^{\wedge k} \arrow[u, "\mathrm{id}^{\wedge\ell}\wedge(\Sigma\underline{1})^{\wedge k}"'] &
\end{tikzcd} \tag{4.2.216}$$

induces data witnessing the commutativity of (4.2.213), *i.e. proving Theorem 4.2.210. Here* $c\colon \Sigma_+^\infty X \to \mathbb{S}$ *is the infinite suspension of the crush map* $X \to \mathrm{pt}$ *for a space* X.

Proof. Match the data in (4.2.216) with the data in (4.2.213):

1. The composition of the downward arrows on the left side of (4.2.216), which are all colored red, is $\varphi_{\mathrm{int}}^{\ell,k}$ by the definition in (4.2.196). Therefore smashing with $MT\mathrm{Spin}$ gives the left side map of (4.2.213).

2. By the definition in (4.2.154), the composition of the maps from the upper left of the diagram to the lower right along the top of the diagram, consisting of the green arrows, is the map $\Phi_{\ell,k+1}$ from Remark 4.2.175. Thus smashing that composition with $MT\mathrm{Spin}$, then composing with the multiplication $KO \wedge KO \to KO$, produces the map $\mathrm{ABS}_{\ell,k+1}$ in (4.2.213).

3. The bottommost maps in (4.2.216), colored blue, can be smashed with $MT\mathrm{Spin}$ then composed to obtain $\Phi_{\ell,k}$ and thus also $\Sigma\mathrm{ABS}_{\ell,k}$ in the same way, again using the definition in (4.2.154). ☐

Lemma 4.2.217. *Data witnessing the commutativity of*

$$\begin{tikzcd}
\Sigma^\infty_+ B\mathbb{Z}/2 \arrow[r, "\underline{1}"] \arrow[d, "c"'] & KO \\
\mathbb{S} \arrow[ur, "\mathbf{1}"'] &
\end{tikzcd} \tag{4.2.218}$$

induces data witnessing commutativity of (4.2.216). *Here* $\mathbf{1}\colon \mathbb{S} \to KO$ *is the unit map.*

Proof. The diagram (4.2.216) can be constructed by taking the smash product of several smaller diagrams. Specifically, take ℓ copies of

$$\begin{tikzcd}
(B\mathbb{Z}/2)^{\sigma-1} \arrow[r, "\lambda_\sigma"] \arrow[d, "\mathrm{id}"'] & \Sigma^{-1}KO \arrow[d, "\mathrm{id}"] \\
(B\mathbb{Z}/2)^{\sigma-1} \arrow[r, "\lambda_\sigma"'] & \Sigma^{-1}KO,
\end{tikzcd} \tag{4.2.219a}$$

k copies of

$$\begin{tikzcd}
(B\mathbb{Z}/2)^{1-\sigma} \arrow[r, "\mathrm{sm}_\sigma"] \arrow[dd, "\mathrm{id}"'] & \Sigma(B\mathbb{Z}/2)_+ \arrow[dr, "\Sigma\underline{1}"] & \\
& & \Sigma KO, \\
(B\mathbb{Z}/2)^{1-\sigma} \arrow[r, "\mathrm{sm}_\sigma"'] & \Sigma(B\mathbb{Z}/2)_+ \arrow[ur, "\Sigma\underline{1}"'] &
\end{tikzcd} \tag{4.2.219b}$$

and one copy of sm_σ composed with the suspension of (4.2.218), then postcompose with the multiplication map on KO. The diagrams (4.2.219a) and (4.2.219b) trivially commute, so all we need to establish commutativity of (4.2.216) is commutativity of (4.2.218). ☐

Lemma 4.2.220. (4.2.218) *commutes.*

As discussed above, this suffices to prove Theorem 4.2.210.

Proof. The map $\underline{1}\colon \Sigma^\infty_+(B\mathbb{Z}/2) \to KO$ is the KO-class of the trivial bundle. That $\underline{1}$ factors through c is the fact that the trivial bundle on $B\mathbb{Z}/2$ pulls back from the point. Because the trivial bundle tensored with itself is trivial, the map $\mathbb{S} \to KO$ induced by the trivial bundle on a point is a homomorphism of ring spectra, hence must be $\mathbf{1}$. ☐

4.2.7 Computations

Now we explicitly compute our free-to-interacting maps, modulo a piece that is trivial in a sense we make precise in Definition 4.2.227.

4.2.7.1 The general story over *ko*

Our main results in this section are:

1. Theorem 4.2.266, computing the free-to-interacting map

$$KO^{d-2+\ell-k} \to (I_{\mathbb{Z}}MT\mathrm{Spin})^{d+2}(ME_{\ell,k}),$$

and

2. Theorem 4.2.306, computing the Bott spiral map

$$\mathrm{sp}^{\ell,k}_{\mathrm{int}}\colon (I_{\mathbb{Z}}MT\mathrm{Spin})^{d+2}(ME_{\ell,k}) \to (I_{\mathbb{Z}}MT\mathrm{Spin})^{d+3}(ME_{\ell,k+1}).$$

Our results are founded on a simple description of $H^*(ME_{\ell,k};\mathbb{Z}/2)$ as an $\mathcal{A}(1)$-module that is $(4,1)$-periodic in (ℓ,k), again modulo a trivial piece. This is a twisted version of a continuing program of understanding $H^*(BV;\mathbb{Z}/2)$, $ku_*(BV)$, and $ko_*(BV)$ for V an elementary abelian 2-group, including work of [96, 283, 461–472], and we build on their computations.

Recall that there is a ring isomorphism

$$ko_* \cong \mathbb{Z}[\eta, a, \beta]/(\eta^3, 2\eta, \eta a, a^2-4\beta) \quad |\eta|=1, |a|=4, |\beta|=8. \tag{4.2.221}$$

Here β is the real *Bott class.* For any space or spectrum X, $KO_*(X) \cong ko_*(X)[\beta^{-1}]$, and more generally, for any ko-module M, there is a natural isomorphism

$$\pi_*(M)[\beta^{-1}] \xrightarrow{\cong} \pi_*(KO \wedge_{ko} M). \tag{4.2.222}$$

The Eilenberg-MacLane spectrum $H\mathbb{Z}/2$ has a unique ko-algebra structure via the action map $ko \to H\mathbb{Z} \to H\mathbb{Z}/2$, which is the composition of 0-truncation and reduction mod 2 (see for example [473, Remark 2.17]). Whenever we refer to $H\mathbb{Z}/2$ as a ko-module, we always mean the module structure induced by this algebra structure. Since β acts trivially on $H\mathbb{Z}/2$ for degree reasons, $(H\mathbb{Z}/2)[\beta^{-1}] \simeq 0$.

Definition 4.2.223. Let M be a bounded below ko-module of finite type, so that there is a ko-module splitting

$$M \simeq \overline{M} \vee \bigvee_i \Sigma^{k_i} H\mathbb{Z}/2 \tag{4.2.224}$$

over some (necessary countable) indexing set i, such that $\overline{M}$ has no $\Sigma^m H\mathbb{Z}/2$ summands. This specifies $\overline{M}$ uniquely up to ko-module equivalence. We say M is *of elementary abelian (EA) type* if

1. β acts injectively on $\pi_*(\overline{M})$, and
2. there is an $m \in \mathbb{Z}$ and a KO_*-module isomorphism

$$\pi_*(M)[\beta^{-1}] \xrightarrow{\cong} \widetilde{KO}_{*+m}(\mathbb{RP}^\infty). \tag{4.2.225}$$

Remark 4.2.226. For convenience we recall $\widetilde{KO}_*(\mathbb{RP}^\infty)$ from [474, Theorem 2] (see also [475, §4]). Let μ_{2^∞} denote the abelian group of all n^{th} roots of unity as n ranges over all powers of 2; this group is sometimes denoted $\mathbb{Z}/2^\infty$. Then $\widetilde{KO}_m(\mathbb{RP}^\infty)$ vanishes for $m = 0, 4, 5, 6 \bmod 8$, is isomorphic to $\mathbb{Z}/2$ in degrees $1, 2 \bmod 8$, and is isomorphic to μ_{2^∞} in degrees $3, 7 \bmod 8$. The action of $\eta\colon \widetilde{KO}_n(\mathbb{RP}^\infty) \to \widetilde{KO}_{n+1}(\mathbb{RP}^\infty)$ is injective in degrees $1, 2 \bmod 8$ and vanishes in all other degrees; in particular it hits $-1 \in \mu_{2^\infty}$ in $\widetilde{KO}_3(\mathbb{RP}^\infty)$. The action of a is uniquely determined by the relation $a^2 = 4\beta$ and the requirement that β acts invertibly.

Definition 4.2.227. Let M be a *ko*-module of EA-type. We say a $\mathbb{Z}$-module summand $N \subset \pi_*(M)$ is

1. a *long summand* if the image of N under the map $\psi\colon \pi_*(M) \to \pi_*(M)[\beta^{-1}]$ is nonzero and contained in a μ_{2^∞} summand,
2. a *short summand* if the image of N under ψ is nonzero and contained in a $\mathbb{Z}/2$ summand, and
3. a *bosonic summand* if $\psi(N) = 0$.

For each n, $\pi_n(M)$ admits a direct-sum decomposition where each summand is long, short, or bosonic.

Remark 4.2.228 (Bosonic Summands). Above, we foreshadowed that we will give a *mostly* complete description of the interacting Bott spiral map and the free-to-interacting map. "Mostly" here means that $ko \wedge ME_{\ell,k}$ is a *ko*-module of EA-type (Proposition 4.2.260) and we will be ignoring bosonic summands.

Bosonic summands are so-called because the twisted spin IFTs dual to them are trivial, in the sense that they do not depend on the twisted spin structure, and therefore the corresponding phases are fermionic only in a trivial sense. See [102, §3.3].

Remark 4.2.229. There is an important caveat to the name "bosonic summand:" there are IFTs dual to classes in non-bosonic summands that also do not depend on a choice of twisted spin structure. These theories are multiples or dimensional reductions[38] of other IFTs which do depend on a choice of twisted spin structure; hence, if we are working with a direct-sum decomposition or studying the Ω_*^{Spin}-module structure on groups of IFTs for the purpose of computations, these IFTs should not be included in bosonic summands.

An illustrative example occurs in two-dimensional pin$^-$ bordism, which was discussed in Example 4.2.116. There is a homotopy equivalence $MT\mathrm{Pin}^- \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{\sigma-1}$ [142, §7], and after base-changing from $MT\mathrm{Spin}$ to ko, we consider the *ko*-module $ko \wedge (B\mathbb{Z}/2)^{\sigma-1}$. In Example 4.2.250 we show this *ko*-module is of EA-type, and that its homotopy groups have long and short summands, but no bosonic summands. In particular, $ko_2((B\mathbb{Z}/2)^{\sigma-1}) \cong$

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[38] Keeping track of the tangential structure in a general dimensional reduction can be technical: see Schommer-Pries [361, §9] for a comprehensive analysis. We only need the dimensional reduction of a twisted spin theory on a spin manifold; in this case things are simpler, because the product of a spin manifold and an (X, V)-twisted spin manifold has a canonical (X, V)-twisted spin structure. Moreover, Yamashita–Yonekura [362, §7.3] show that for invertible theories, dimensional reduction of this sort is a formal consequence of the MTSpin-module structure on the spectra classifying twisted spin IFTs. See also Tachikawa–Yamashita [366, §2.2.6].

$\mathbb{Z}/8$ [110, 192], matching the group of deformation classes of 2d reflection-positive IFTs of pin$^-$ manifolds.[39]

There are two pin$^-$ structures on $\mathbb{RP}^2$, and their Arf–Brown–Kervaire invariants are $e^{\pm i\pi/4} \in \mu_8$ [110, §3], so α_{ABK} depends on the choice of pin$^-$ structure. The same argument applies to any power of α_{ABK} except for $\alpha_{ABK}^{\otimes 4n}$—indeed, for any closed pin$^-$ 2-manifold Σ [110, Lemma 3.6],

$$\mathrm{ABK}(\Sigma)^4 = (-1)^{\chi(\Sigma)} \in \{\pm 1\}. \tag{4.2.230}$$

So $\alpha_{\mathrm{ABK}}^{\otimes 4}$ is bosonic!

"Surprisingly bosonic" IFTs such as $\alpha_{ABK}^{\otimes 4}$ occur whenever there is a class in the E_∞-page of the Adams spectral sequence in filtration 0 (i.e. along the x-axis in the standard drawing convention); see [477, §8.4] or [102, §3.3]. This often occurs in long summands, as can be seen in the Ext calculations in Examples 4.2.241, 4.2.244, 4.2.247, and 4.2.250, drawn in the right-hand sides of Figures 4.1, 4.2, 4.3, and 4.4. For example, four-dimensional pin$^+$ bordism is isomorphic to $\mathbb{Z}/16$ [193, §2] and eight times any generating IFT α_η does not depend on the pin$^+$ structure. One consequence of this appears in work of Barkeshli–Hsin–Kobayashi [478] on the higher-group symmetry of the fermionic toric code: where one might expect a $\mathbb{Z}/16$ symmetry defined by stacking with α_η before gauging the $\mathbb{Z}/2$ symmetry, the lack of dependence of $\alpha_\eta^{\otimes 8}$ on the pin$^+$ structure reduces the symmetry group to $\mathbb{Z}/8$. See (*ibid.*, §2.1.1).

Next we will give several examples of *ko*-modules of EA-type. Each is of the form $ko \wedge X$ for some X; we will also compute $H^*(X;\mathbb{Z}/2)$ as a module over the Steenrod subalgebra $\mathcal{A}(1) \coloneqq \langle \mathrm{Sq}^1, \mathrm{Sq}^2\rangle$. This allows us to display the second page of the Adams spectral sequence. In each of these cases the spectral sequence collapses at E_2 without any extension problems, furnishing a proof that $ko \wedge X$ is in fact of EA-type.

In the examples below, we will briefly use the Adams spectral sequence in the form constructed by Baker–Lazarev [479]; we recommend Beaudry–Campbell [98] for additional background. Let $\mathcal{A}(1) \coloneqq \langle \mathrm{Sq}^1, \mathrm{Sq}^2\rangle$. Baker [480, Theorem 5.1] shows that $\mathcal{A}(1)$ is exactly the subalgebra of the Steenrod algebra consisting of maps $H\mathbb{Z}/2 \to \Sigma^k H\mathbb{Z}/2$ that are *ko*-module maps. That is, if for a *ko*-module M we let

$$H^*_{ko}(M) \coloneqq \pi_{-*}\mathrm{Map}_{ko}(M, H\mathbb{Z}/2), \tag{4.2.231}$$

we have $H^*_{ko}(H\mathbb{Z}/2) \cong \mathcal{A}(1)$.

The Baker–Lazarev Adams spectral sequence for finite type *ko*-modules M and M' has signature [479]

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*_{ko}(M), H^*_{ko}(M')) \Longrightarrow \pi_{t-s}\mathrm{Map}_{ko}(M', M)^\wedge_2. \tag{4.2.232}$$

Usually one plugs in $M' = ko$, so this spectral sequence computes the 2-completed homotopy groups of M: using the natural homotopy equivalence $\mathrm{Map}_R(R \wedge X, Y) \simeq \mathrm{Map}_{\mathbb{S}}(X, Y)$ with $R = ko$, $Y = M$, and $X = \mathbb{S}$,

$$\pi_*\mathrm{Map}_{ko}(ko, M)^\wedge_2 \cong \pi_*\mathrm{Map}_{\mathbb{S}}(\mathbb{S}, M)^\wedge_2 = \pi_*(M)^\wedge_2. \tag{4.2.233}$$

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[39] The group of 2d pin$^-$ IFTs, without requiring reflection positivity data, is isomorphic to $\mathbb{C}^\times \times \mathbb{Z}/4$. See [476, (5.1)] and [55, Examples 4.17 and 4.22].

This spectral sequence is additive in the sense that a ko-module splitting $M \simeq M_1 \vee M_2$ induces a direct-sum splitting on H^*_{ko} and on Ext, and in fact a splitting of the entire Adams spectral sequence into the respective Adams spectral sequences for M_1 and M_2.

If $M = ko \wedge X$ for a spectrum X, then by definition $H^*_{ko}(M) \cong H^*(X; \mathbb{Z}/2)$ and the above spectral sequence reduces to the usual $\mathcal{A}(1)$-based ko-Adams spectral sequence:

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*(X;\mathbb{Z}/2), \mathbb{Z}/2) \Longrightarrow \pi_*(ko \wedge X)^\wedge_2 = ko_*(X)^\wedge_2. \tag{4.2.234}$$

Equation (4.2.234) is the version one more commonly encounters, often proven via a change-of-rings theorem, such as in [98]. We will need a few ko-modules which do not factor as $ko \wedge X$, so must use the more general (4.2.232).

Throughout this subsection, $\mathrm{Ext}(N)$ means $\mathrm{Ext}_{\mathcal{A}(1)}(N, \mathbb{Z}/2)$ if not otherwise specified.

Let $\mathbb{E} \coloneqq \mathrm{Ext}(\mathbb{Z}/2)$. This is the Adams E_2-page for ko, and the E_∞-ring structure on ko induces a commutative ring structure on $\mathbb{E}$, which was computed by Liulevicius [481, Theorem 3]:

$$\mathbb{E} \coloneqq \mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2) \cong \mathbb{Z}/2[h_0, h_1, v, w]/(h_0h_1, h_1^3, vh_1, h_0^2w - v^2) \tag{4.2.235}$$

with $h_0 \in \mathrm{Ext}^{1,1}$, $h_1 \in \mathrm{Ext}^{1,2}$, $v \in \mathrm{Ext}^{3,7}$, and $w \in \mathrm{Ext}^{4,12}$. Thus $\mathbb{E}$ acts on the E_2-page of the Baker–Lazarev Adams spectral sequence of any ko-module; this action commutes with all differentials, and is useful for solving extension problems: h_0 lifts to multiplication by 2, h_1 to η, v to a, and w to β.

Theorem 4.2.236 (Margolis' theorem for ko-modules)**.** *Let M be a bounded below ko-module of finite type. Any isomorphism of $\mathcal{A}(1)$-modules*

$$\overline{\varphi}\colon H^*_{ko}(M) \xrightarrow{\cong} \overline{N} \oplus F, \tag{4.2.237a}$$

where F is a free $\mathcal{A}(1)$-module, lifts to a ko-module splitting: there is a ko-module N with $H^*_{ko}(N) \cong \overline{N}$ and a ko-module equivalence

$$\varphi\colon N \vee \bigvee_i \Sigma^{n_i} H\mathbb{Z}/2 \xrightarrow{\simeq} M \tag{4.2.237b}$$

such that $H^*_{ko}(\bigvee \Sigma^{n_i} H\mathbb{Z}/2) \cong F$ and the pullback map of φ on H^*_{ko} is equal to $\overline{\varphi}$.

Proof. Since M has finite type, the rank of F is countable, so we may induct starting with the lowest-degree summand in F, possible because M is bounded below. Therefore in the rest of the proof we will assume F is rank one: $F \cong \Sigma^m \mathcal{A}(1)$.

It suffices to produce ko-module maps $f\colon \Sigma^m H\mathbb{Z}/2 \to M$ and $g\colon M \to \Sigma^m H\mathbb{Z}/2$ with $g \circ f \simeq \mathrm{id}$. To do this, run two instances of the Baker–Lazarev Adams spectral sequence (4.2.232): one for $\mathrm{Map}_{ko}(M, \Sigma^m H\mathbb{Z}/2)$ and one for $\mathrm{Map}_{ko}(\Sigma^m H\mathbb{Z}/2, M)$. Since $H^*_{ko}(H\mathbb{Z}/2) \cong \mathcal{A}(1)$, as we noted above, the two E_2-pages are

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*_{ko}(M), \Sigma^m\mathcal{A}(1)) \qquad \text{and} \qquad \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\Sigma^m\mathcal{A}(1), H^*_{ko}(M)). \tag{4.2.238}$$

Margolis proved that $\mathcal{A}(1)$ is both projective and injective as a graded $\mathcal{A}(1)$-module [482, Theorem 12.5, Proposition 12.8, Theorem 12.9] ("graded" is important: see [483]), so both

E_2-pages in (4.2.238) vanish for $s > 0$, and for $s = 0$, they coincide with graded Hom. This implies these Adams spectral sequences collapse for degree reasons, and that any $\mathcal{A}(1)$-module homomorphism $H^*_{ko}(M) \to \Sigma^m \mathcal{A}(1)$ (resp. $\Sigma^m \mathcal{A}(1) \to H^*_{ko}(M)$) lifts to a ko-module morphism $M \to \Sigma^m H\mathbb{Z}/2$ (resp. $\Sigma^m H\mathbb{Z}/2 \to M$) and this lift can be chosen to preserve composition up to ko-module homotopy equivalence. Thus it suffices to exhibit maps f and g with $g \circ f \simeq \mathrm{id}$ at the level of $\mathcal{A}(1)$-modules, where they are the maps splitting the rank-one free summand off of $H^*_{ko}(M)$. □

Remark 4.2.239. A *stable isomorphism* of $\mathcal{A}(1)$-modules is an $\mathcal{A}(1)$-module map $f\colon M \to N$ whose kernel and cokernel are free over $\mathcal{A}(1)$. This automatically upgrades to f being the direct sum of an isomorphism and the zero map between two free $\mathcal{A}(1)$-modules.

Theorem 4.2.236 thus implies that a stable isomorphism on H^*_{ko} affects bosonic summands, but leaves everything we are actually interested in alone.

Remark 4.2.240. Margolis' original theorem [484] is essentially the same result but with $\mathbb{S}$ in place of ko and $H^*(-;\mathbb{Z}/2)$ in place of H^*_{ko}. Using this, one can use a change-of-rings theorem to prove a weaker version of Theorem 4.2.236: one requires $M \simeq ko \wedge X$, and the splitting is only as spectra, not ko-modules. See [102, Theorem 3.22].

Example 4.2.241. It will be little surprise to the reader that $ko \wedge \mathbb{RP}^\infty$ is of EA-type. For degree reasons we will find it slightly more convenient to work with $ME_{1,0} := (B\mathbb{Z}/2)^{\sigma-1}$, which is equivalent to $\Sigma^{-1}\mathbb{RP}^\infty$ (see, e.g., [187, Lemma 2.6.5]).

Let $N_1 := H^*((B\mathbb{Z}/2)^{\sigma-1};\mathbb{Z}/2)$. We draw this module in Figure 4.1, left. $\mathrm{Ext}_{\mathcal{A}(1)}(N_1)$ was computed by Gitler–Mahowald–Milgram [485, §2] to be the $\mathbb{E}$-module

$$\mathrm{Ext}(N_1) \cong \mathbb{E}\{\alpha, a_1, a_2, a_3, \dots\}/(h_0\alpha, v\alpha, h_1^2\alpha = h_0^2 a_1, h_1 a_i, v a_i = h_0^3 a_{i+1}, h_0^4 a_1, h_1^2 w\alpha = h_0^6 a_3) \tag{4.2.242}$$

where $\alpha \in \mathrm{Ext}^{0,0}$ and $a_i \in \mathrm{Ext}^{0,4i-1}$. We draw this in Figure 4.1, right; all differentials vanish either for degree reasons or because they must commute with h_0 or h_1, and there are no extension problems left after taking the $\mathbb{E}$-module structure into account. Therefore we learn that as ko_*-modules,

$$\widetilde{ko}_*(ME_{1,0}) \cong ko_*\{\overline{\alpha}, \overline{a}_i\}/(2\overline{\alpha}, a\overline{\alpha}, \eta^2\overline{\alpha} = 4\overline{a}_1, \eta\overline{a}_i, a\overline{a}_i = 8\overline{a}_{i+1}, 16\overline{a}_1, \eta^2\beta\overline{\alpha} = 64\overline{a}_3), \tag{4.2.243}$$

where $|\overline{\alpha}| = 0$ and $|\overline{a}_i| = 4i - 1$. These relations imply $ko \wedge ME_{1,0}$ is of EA-type, and that $\overline{a}_i$ generates a cyclic summand of order 2^{3+4j} (if $i = 2j+1$) or 2^{4j} (if $i = 2j$); these are the long summands. The classes $\beta^n\overline{\alpha}$ and $\eta\beta^n\overline{\alpha}$, $n \geq 0$, generate short summands. There are no bosonic summands.

Thus the same conclusions are true for $\widetilde{ko}_*(\mathbb{RP}^\infty)$ with the degrees shifted upwards by 1. This recovers preexisting computations of $ko_*(\mathbb{RP}^\infty)$: Mahowald [486, Lemma 7.3] computes the groups $\widetilde{ko}_*(\mathbb{RP}^\infty)$, and Bruner–Mira–Stanley–Snaith [472, §3.1] give the ko_*-module structure. ◇

Example 4.2.244. Let A_4 denote the alternating group on four letters. Then $H^*(BA_4;\mathbb{Z}/2) \cong \mathbb{Z}/2[u, v, w]/(u^3 + v^2 + w^2 + vw)$ with $|u| = 2$ and $|v| = |w| = 3$ [487, Theorem 2.6]. The Steenrod squares of the generators are $\mathrm{Sq}(u) = u + v + w + u^2$, $\mathrm{Sq}(v) = v + u^2 + uw + v^2$,

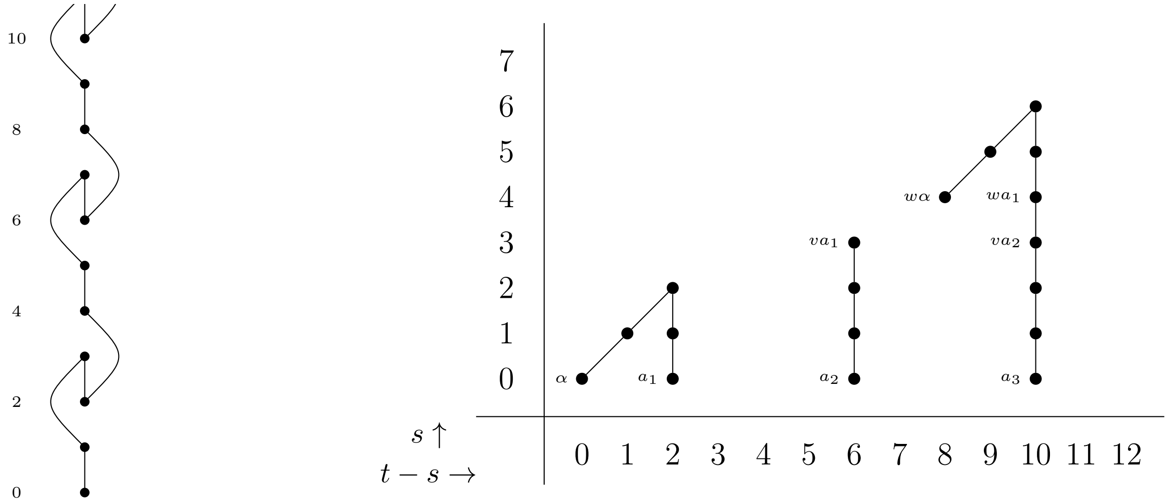

Figure 4.1: Left: the $\mathcal{A}(1)$-module $N_1 := H^*((B\mathbb{Z}/2)^{\sigma-1};\mathbb{Z}/2)$. This is also $\widetilde{H}^*(\mathbb{RP}^\infty;\mathbb{Z}/2)$ shifted down in degree by 1. Right: $\mathrm{Ext}(N_1)$, the E_2-page of the Adams spectral sequence for $ko\wedge(B\mathbb{Z}/2)^{\sigma-1}$. All differentials and extension questions are trivial. We discuss N_1 and its Adams spectral sequence in Example 4.2.241.

and $\mathrm{Sq}(w) = w + u^2 + uv + v^2$.[40] One can therefore compute that as an $\mathcal{A}(1)$-module, $\widetilde{H}^*(BA_4;\mathbb{Z}/2)$ is a direct sum of countably many free $\mathcal{A}(1)$-modules whose lowest-degree elements are in degrees 6, 8, 12, 12,. . . , along with exactly one indecomposable summand S; we define $N_2 := \Sigma^{-2}S$.[41] We draw a picture of N_2 in Figure 4.2, left. Yu [462, Theorem 3.1] showed that $\mathrm{Ext}(N_2)$ is isomorphic as $\mathbb{E}$-modules to

$$\frac{\mathbb{E}\{\beta,\beta',b_1,b_2,b_3,\dots\}}{\substack{(h_0\beta,\, h_1^2\beta,\, v\beta,\, w\beta=h_1\beta',\, h_0b_1=h_1\beta,\, h_1b_i \text{ for all } i,\\ h_0^3b_2,\, vb_1,\, h_1w\beta=h_0^5b_3,\, wb_1=h_0^5b_3,\, vb_i=h_0^3b_{i+1} \text{ for } i\geq 2)}}, \tag{4.2.245}$$

where $\beta \in \mathrm{Ext}^{0,0}$, $\beta' \in \mathrm{Ext}^{3,10}$, and $b_i \in \mathrm{Ext}^{0,1+4i}$.

We draw the Adams chart for $\mathrm{Ext}(N_2)$ in Figure 4.2, right. Because $H^*(BA_4;\mathbb{Z}/2)$ is stably isomorphic to $\Sigma^2 N_2$, Margolis' theorem for ko-modules (Theorem 4.2.236) implies that there is a ko-module M_2 with $H^*_{ko}(M_2) \cong N_2$ and that (the 2-completion of) $ko \wedge BA_4$ is ko-module equivalent to a wedge sum of $\Sigma^2 M_2$ and a sum of shifts of $H\mathbb{Z}/2$. Thus $\mathrm{Ext}(N_2)$ is the E_2-page of the Adams spectral sequence computing $\pi_*(M_2)$, which is a summand of $\widetilde{ko}_{*+2}(BA_4)$.

Similarly to what we saw in Example 4.2.241 for N_1, all differentials for vanish because they are $\mathbb{E}$-linear, and the $\mathbb{E}$-action determines all extensions, so there is a ko_*-module isomorphism

$$\pi_*(M_2) \cong ko_*\{\overline{\beta},\overline{\beta}',\bar{b}_1,\bar{b}_2,\bar{b}_3,\dots\}/\mathcal{R}_2, \tag{4.2.246a}$$

where $|\overline{\beta}| = 0$, $|\overline{\beta}'| = 7$, and $|\bar{b}_i| = 4i+1$, and the ideal $\mathcal{R}_2$ of relations is

$$\mathcal{R}_2 = (2\overline{\beta}, \eta^2\overline{\beta}, a\overline{\beta}, \beta\overline{\beta} = \eta\overline{\beta}', 2\bar{b}_1 = \eta\overline{\beta}, \eta\bar{b}_i \text{ for all } i, 8\bar{b}_2, a\bar{b}_1, \eta\beta\overline{\beta} = 32\bar{b}_3, \beta\bar{b}_1 = 32\bar{b}_3, a\bar{b}_i = 8\bar{b}_{i+1}) \tag{4.2.246b}$$

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[40] Mitchell-Priddy do not explicitly give the Steenrod squares, but they can be worked out from their proof. They are also given in [102, Proposition 5.1].

[41] N_2 is called $\Sigma^{-2}P_2$ in [466, Definition 4.4] and R_5 in [98, 102].

for $i \geq 2$. Inverting β, one sees that M is a ko-module of EA-type; $\pi_*(M)$ has a long summand of order 2^{4j+2} in degree $8j+1$, a long summand of order 2^{4j+3} in degree $8j+5$, and short summands in degrees $7, 0 \bmod 8$. There are no bosonic summands.

As discussed above, $ko \wedge BA_4 \simeq \Sigma^{-2}M_2 \vee F$, where F is a sum of shifts of $H\mathbb{Z}/2$, so $ko \wedge BA_4$ is also of EA-type. It has long summands of order 2^{4j+2} in degree $8j+3$ and order 2^{4j+3} in degree $8j+7$ and short summands in degrees $1, 2 \bmod 8$, except not in degree 1. In addition, the $H\mathbb{Z}/2$ summands contribute bosonic summands to $\widetilde{ko}_*(BA_4)$. See Bruner–Greenlees [283, §7.7.E] for prior work on $ko_*(BA_4)$. ◇

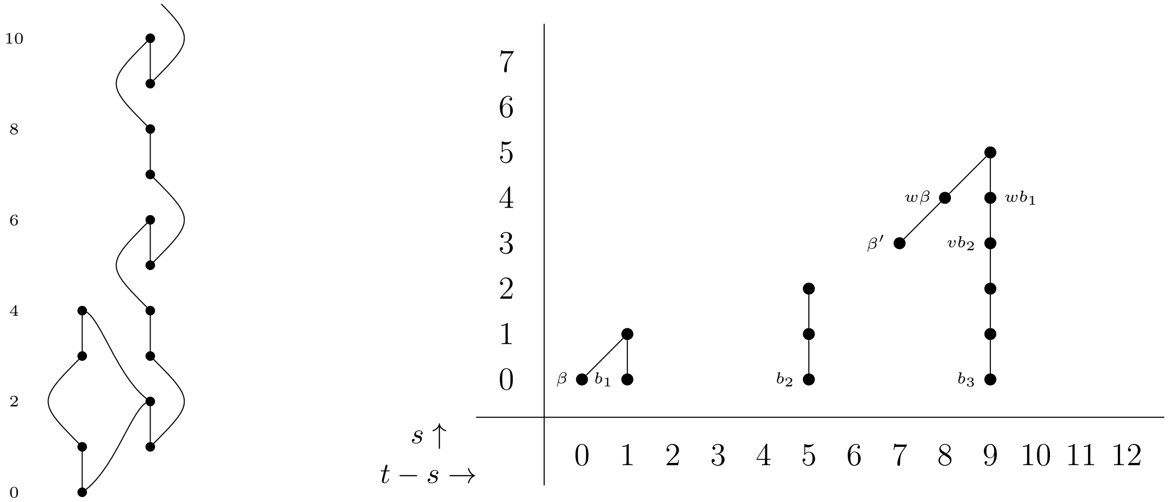

Figure 4.2: Left: the $\mathcal{A}(1)$-module N_2, the unique non-free summand of $\Sigma^{-2}\widetilde{H}^*(BA_4;\mathbb{Z}/2)$. Right: $\mathrm{Ext}(N_2)$, the E_2-page of the Adams spectral sequence for a summand of $ko \wedge \Sigma^{-2}BA_4$. All differentials and extension questions are trivial. We discuss N_2 and its Adams spectral sequence in Example 4.2.244.

Example 4.2.247. Using the Künneth theorem and the description of $H^*(BA_4;\mathbb{Z}/2)$ from Example 4.2.244, one can show that as an $\mathcal{A}(1)$-module, $\widetilde{H}^*(BA_4 \wedge B\mathbb{Z}/2;\mathbb{Z}/2)$ is a direct sum of a free $\mathcal{A}(1)$-module of countably infinite rank and exactly one indecomposable summand S'; we let $N_3 := \Sigma^{-3}S'$. We draw a picture of N_3 in Figure 4.3, left. Margolis' theorem for ko-modules (Theorem 4.2.236) thus tells us that there is a connective ko-module M_3 with $H^*_{ko}(M_3) \cong N_3$ such that $ko \wedge BA_4 \wedge B\mathbb{Z}/2$ splits as a ko-module as a sum of $\Sigma^3 M_3$ and a sum of shifts of $H\mathbb{Z}/2$.

Yu [462, Theorem 3.1] exhibits an $\mathbb{E}$-module isomorphism

$$\mathrm{Ext}(N_3) \cong \frac{\mathbb{E}\{\gamma, \gamma', c_1, c_2, \dots\}}{(h_0\gamma, h_1\gamma, v\gamma, w\gamma = h_1^2\gamma' = h_0^4 c_2, h_0\gamma', v\gamma', h_0^2 c_1, wc_1 = h_0^4 c_3, h_1 c_i, vc_i = h_0^3 c_{i+1})} \tag{4.2.248}$$

with $\gamma \in \mathrm{Ext}^{0,0}$, $\gamma' \in \mathrm{Ext}^{2,8}$, and $c_i \in \mathrm{Ext}^{0,4i}$. We draw a picture of this $\mathbb{E}$-module in Figure 4.3, right; it is the E_2-page of the Baker–Lazarev Adams spectral sequence computing $\pi_*(M_3)^\wedge_2$.

In a pattern that is probably increasingly clear, all differentials vanish because they commute with the h_0- and h_1-actions, and there are no hidden extensions, so we obtain a

ko_*-module isomorphism

$$\pi_*(M_3) \cong \frac{ko_*\{\overline{\gamma}, \overline{\gamma}', \overline{c}_1, \overline{c}_2, \dots\}}{(2\overline{\gamma}, \eta\overline{\gamma}, a\overline{\gamma}, \beta\overline{\gamma} = \eta^2\overline{\gamma}' = 16\overline{c}_2, 2\overline{\gamma}', a\overline{\gamma}', 4\overline{c}_1, \beta\overline{c}_1 = 16\overline{c}_3, \eta\overline{c}_i, a\overline{c}_i = 8\overline{c}_{i+1})}, \tag{4.2.249}$$

with $|\overline{\gamma}| = 0$, $|\overline{\gamma}'| = 6$, and $|\overline{c}_i| = 4i$. Inverting β, we see M_3 is of EA-type; its homotopy groups have no bosonic summands and have short summands in degrees $2, 3 \bmod 8$. There is a long summand of the form $\mathbb{Z}/2^{4j+1}$ in degree $8j$ and another of the form $\mathbb{Z}/2^{4j+2}$ in degree $8j+4$ for all $j \geq 0$. Thus $ko \wedge BA_4 \wedge B\mathbb{Z}/2$ is also of EA-type, and the short and long summands in it homotopy can be worked out similarly to what we did for $ko \wedge BA_4$ in Example 4.2.244; there are also bosonic summands. ◇

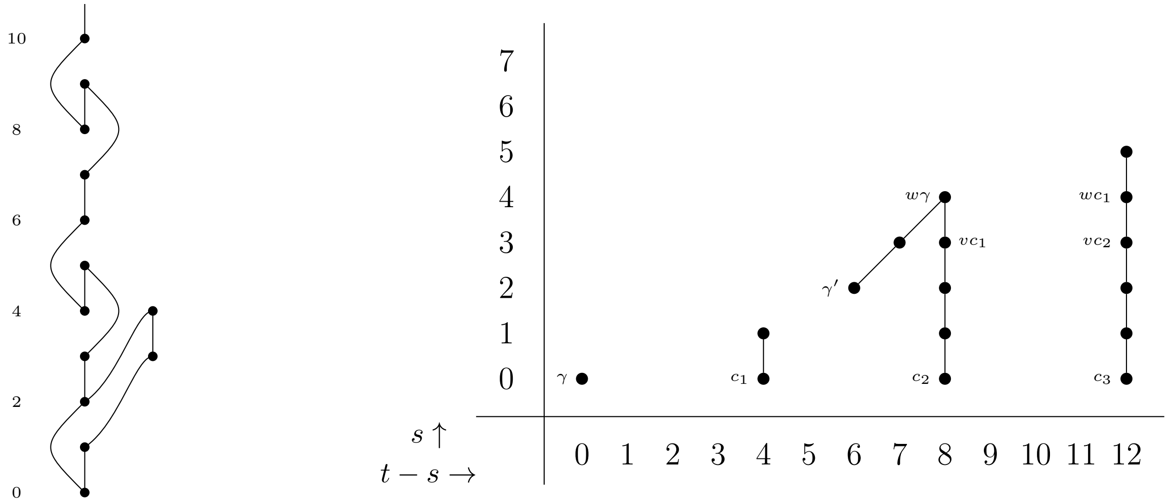

Figure 4.3: Left: the $\mathcal{A}(1)$-module N_3, the unique non-free summand of $\Sigma^{-3}\widetilde{H}^*(BA_4 \wedge B\mathbb{Z}/2; \mathbb{Z}/2)$. Right: $\mathrm{Ext}(N_3)$, the E_2-page of the Baker–Lazarev Adams spectral sequence for a ko-module summand M of $\Sigma^{-3}ko \wedge BA_4 \wedge B\mathbb{Z}/2$. All differentials and extension questions are trivial. See Example 4.2.247 for more information on N_3, M, and this Adams spectral sequence.

Example 4.2.250. Let $N_0 := H^*((B\mathbb{Z}/2)^{1-\sigma}; \mathbb{Z}/2)$. We draw this module in Figure 4.4, left. $\mathrm{Ext}_{\mathcal{A}(1)}(N_0)$ was computed by Gitler–Mahowald–Milgram [485, §2] to be the $\mathbb{E}$-module

$$\mathrm{Ext}(N_0) \cong \frac{\mathbb{E}\{\delta, \delta', d_1, d_2, d_3, \dots\}}{(h_0\delta, h_1\delta, h_0\delta', h_1^2\delta' = v\delta = h_0^3 d_1, w\delta = h_0^4 d_2, h_1 d_i, v d_i = h_0^3 d_{i+1})}, \tag{4.2.251}$$

where $\delta \in \mathrm{Ext}^{0,0}$, $\delta' \in \mathrm{Ext}^{1,3}$, and $d_i \in \mathrm{Ext}^{0,4i}$. We draw this in Figure 4.4, right. All differentials and extension problems are trivial, as in previous examples; therefore we learn that as ko_*-modules,

$$\widetilde{ko}_*((B\mathbb{Z}/2)^{1-\sigma}) \cong ko_*\{\overline{\delta}, \overline{\delta}', \overline{d}_i\}/(2\overline{\delta}, \eta\overline{\delta}, 2\overline{\delta}', \eta^2\overline{\delta}' = a\overline{\delta} = 8\overline{d}_1, \beta\overline{\delta} = 16\overline{d}_2, \eta\overline{d}_i, a\overline{d}_i = 8\overline{d}_{i+1}), \tag{4.2.252}$$

where $|\overline{\delta}| = 0$, $|\overline{\delta}'| = 2$, and $|\overline{d}_i| = 4i$. These relations imply $ko \wedge (B\mathbb{Z}/2)^{1-\sigma}$ is of EA-type: for $k \geq 0$, $\beta^k\overline{\delta}'$ and $\eta\beta^k\overline{\delta}'$ generate short summands in degrees $8k+2$, resp. $8k+3$, and $\overline{d}_i$ generates a long summand isomorphic to $\mathbb{Z}/2^{4i}$ (i odd) or $\mathbb{Z}/2^{4i-3}$ (i even) in degree $4i$. There are no bosonic summands.

This Adams spectral sequence computation is far from new: Giambalvo [193] and Kirby–Taylor [488] study the Adams spectral sequence for $MT\mathrm{Spin}\wedge(B\mathbb{Z}/2)^{1-\sigma}$, and Campbell [96, Example 6.6] studies it for $ko\wedge(B\mathbb{Z}/2)^{1-\sigma}$. ◇

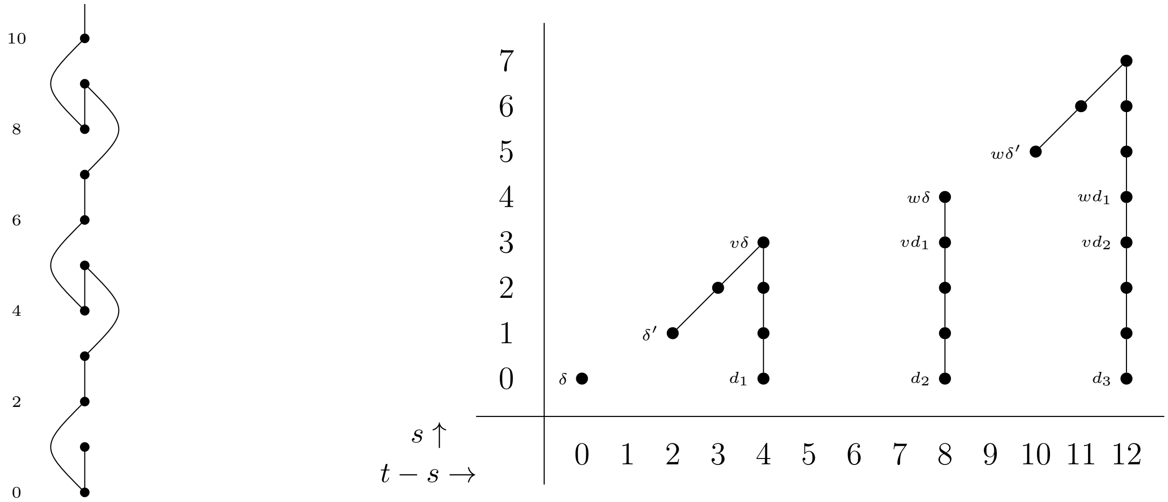

Figure 4.4: Left: the $\mathcal{A}(1)$-module $N_0 \coloneqq H^*((B\mathbb{Z}/2)^{1-\sigma};\mathbb{Z}/2)$. Right: $\mathrm{Ext}(N_0)$, the E_2-page of the Adams spectral sequence for $ko\wedge(B\mathbb{Z}/2)^{1-\sigma}$. All differentials and extension questions are trivial. See Example 4.2.250 for more information.

Lemma 4.2.253. *Let X be a space or spectrum of finite type such that as $\mathcal{A}(1)$-modules, $H^*(X;\mathbb{Z}/2)\cong\Sigma^nN_i\oplus F$ where $0\le i\le 3$ and F is a free $\mathcal{A}(1)$-module. Then $ko\wedge X$ is of EA-type.*

Proof. First assume $F=0$. The arguments we gave in Examples 4.2.241, 4.2.244, 4.2.247, and 4.2.250 showing those ko-modules are finite type used nothing more about them than the $\mathcal{A}(1)$-module structure on their mod 2 cohomology, so the proofs generalize to X. If $F\not\simeq 0$, Theorem 4.2.236 implies $ko\wedge X\simeq\overline{M}\vee E$, where E is a sum of shifts of $H\mathbb{Z}/2$ and $H^*_{ko}(\overline{M})\cong\Sigma^nN_i$ for some i, so the lemma follows by reducing to the previous case. ☐

Remark 4.2.254. Not all ko-modules of EA-type look like the ones in these examples. Let C denote the cofiber of the map $B\mathrm{Pin}_2^-\hookrightarrow B\mathrm{Spin}_3$ sending a rank-2 pin$^-$ vector bundle V to $V\oplus\mathrm{Det}(V)$. Bayen–Bruner [474, Corollary 3] show that $ko\wedge C$ is a ko-module of EA-type, but its cohomology looks nothing like the N_i, and its Adams spectral sequence has both differentials and hidden extensions (*ibid.*, Theorem 4).

Lemma 4.2.255 (Yu [462, Lemma 2.5])**.** *There are stable isomorphisms of $\mathcal{A}(1)$-modules (see Remark 4.2.239) $N_0\otimes N_i\simeq\Sigma N_i$, $N_1\otimes N_1\simeq N_2$, $N_1\otimes N_2\simeq N_3$, and $N_1\otimes N_3\simeq\Sigma^4N_4$.*

Remark 4.2.256. Lemma 4.2.255 establishes a fourfold periodicity in $N_1^{\otimes n}$, and therefore a fourfold periodicity in the ko-homology and spin bordism of $B(\mathbb{Z}/2)^{\wedge n}$ modulo bosonic summands. There is another fourfold periodicity over $MT\mathrm{Spin}$ and ko, namely that of iterated Smith homomorphisms for the same vector bundle [9, Examples 6.14 and 7.8]. These two periodicities are related: the diagonal map $B\mathbb{Z}/2\to(B\mathbb{Z}/2)^{\wedge n}$ induces a map of Thom spectra $\Delta\colon ko\wedge(B\mathbb{Z}/2)^{(\ell-k)(\sigma-1)}\to ko\wedge ME_{\ell,k}$. The map Δ intertwines the Smith map on one side with the interacting Bott spiral map on the other.

Example 4.2.257. Let V be an elementary abelian 2-group. For spaces X and Y, there is a natural splitting

$$\Sigma^\infty(X\times Y)\simeq\Sigma^\infty X\vee\Sigma^\infty Y\vee\Sigma^\infty(X\wedge Y);\tag{4.2.258}$$

applying this iteratively to the $B\mathbb{Z}/2$ pieces of BV, one deduces a splitting[42]

$$\Sigma^\infty BV\simeq\bigvee_{k=1}^{n}\bigvee_{i=1}^{\binom{n}{k}}\Sigma^\infty((B\mathbb{Z}/2)^{\wedge k}).\tag{4.2.259}$$

We showed $ko\wedge B\mathbb{Z}/2$ is of EA-type in Example 4.2.241; then Lemmas 4.2.253 and 4.2.255 allow us to inductively prove that $ko\wedge(B\mathbb{Z}/2)^{\wedge k}$ is also of EA-type. Thus we conclude that $ko\wedge BV$ splits as a sum of ko-modules of EA-type. This is the reason for the name "EA-type:" EA stands for Elementary Abelian.

Mitchell–Priddy [489, Theorem A] produce a stable splitting of $\Sigma^\infty BV$ finer than the one in (4.2.259); the splitting of $ko\wedge BV$ induced by their work is coarser than the one obtained by using the methods in this section and tracking the bosonic summands. It would be interesting to compare these two splittings, which would generalize work of Bayen [490, Proposition 3.4.4, §3.6.3] applicable to the case $n=2$; see also [464, Theorem 4.4].

◇

Proposition 4.2.260. *For all $\ell,k\geq 0$, $ko\wedge ME_{\ell,k}$ is of EA-type, and there is a stable isomorphism*

$$H^*(ME_{\ell,k};\mathbb{Z}/2)\simeq\Sigma^{k+4\sigma}N_{\ell\bmod 4},\tag{4.2.261}$$

where $\ell=4\sigma+t$ for $s\geq 0$ and $0\leq t<4$.

Proof. By Lemma 4.2.253, the second part of this proposition implies the first part. The Künneth theorem tells us that since $ME_{\ell,k}$ was defined as a smash product of ℓ copies of $(B\mathbb{Z}/2)^{\sigma-1}$ and k copies of $(B\mathbb{Z}/2)^{1-\sigma}$, $H^*(ME_{\ell,k};\mathbb{Z}/2)$ is isomorphic to a tensor product of ℓ copies of N_1 and k copies of N_0, so it suffices to identify such a tensor product modulo free summands, which can be done by iteratively applying Lemma 4.2.255. ☐

Proposition 4.2.262. *For all $n\geq 0$, there are isomorphisms*

$$ko_n(ME_{\ell,k})/(\text{bosonic summands})\cong\begin{cases}\mathbb{Z}/2^{4+4\sigma-\ell}, & n=8\sigma+k-\ell+3\\ \mathbb{Z}/2^{5+\sigma-\ell}, & n=8\sigma+k-\ell+7\\ \mathbb{Z}/2, & n=8\sigma+k-\ell+1\text{ or }n=8\sigma+k-\ell+2\\ 0, & \text{otherwise.}\end{cases}\tag{4.2.263}$$

If $a\leq 0$, the group $\mathbb{Z}/2^a$ is to be interpreted as the zero group.

Proof. By Margolis' theorem for ko-modules (Theorem 4.2.236), free $\mathcal{A}(1)$-module summands in cohomology split off as $H\mathbb{Z}/2$ summands of a ko-module, which in this case are bosonic summands. Therefore we may ignore those summands and just focus on N_0, N_1, N_2, or N_3

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[42] The analogous splitting with $B\mathbb{Z}/2$ replaced with $\mathbb{T}$ is the *James splitting* used in the computation of groups of weak topological phases of free fermions. See [294, Theorem 11.8].

by Proposition 4.2.260. Run the Adams spectral sequence for these modules; as worked out in Examples 4.2.241, 4.2.244, 4.2.247, and 4.2.250, in all four cases the spectral sequence collapses without extension problems, leading to the *ko*-homology groups in the proposition statement. □

To discuss the free-to-interacting map, we want to apply Anderson duality. This modifies the discussion of short, long, and bosonic summands slightly. Recall the "exponential fiber sequence":

$$H\mathbb{C} \longrightarrow I_{\mathbb{C}^\times} \xrightarrow{\delta} \Sigma I_{\mathbb{Z}}. \tag{4.2.264}$$

Lemma 4.2.265. *If M is a ko-module of EA-type, the map $\delta_M \colon I_{\mathbb{C}^\times}M \to \Sigma I_{\mathbb{Z}}M$ is a ko-module equivalence.*

Proof. If we can show $H\mathbb{C}_*(M) = \pi_*(M) \otimes \mathbb{C}$ vanishes, then we can plug that into (4.2.264) to deduce δ_M is an isomorphism on homotopy groups, hence an equivalence. Since $\pi_*(M)$ is finitely generated in each degree, it suffices to show that each homotopy group is torsion. The $H\mathbb{Z}/2$ summands contribute $\mathbb{Z}/2$ summands to homotopy, which are torsion, and the rest of $\pi_*(M)$ injects into $\pi_*(M)[\beta^{-1}]$, which by definition of EA-type is torsion, so $\pi_*(M)$ is torsion and we conclude. □

The universal property of $I_{\mathbb{C}^\times}$, namely that $\pi_{-n}(I_{\mathbb{C}^\times}X) \cong \mathrm{Hom}(\pi_n(X), \mathbb{C}^\times)$ implies that a direct-sum decomposition of $\pi_n(X)$ induces a dual direct-sum decomposition of $(I_{\mathbb{C}^\times}X)^n$. Thus if X is a *ko*-module of EA-type, we also get a dual direct-sum decomposition of $(I_{\mathbb{Z}}X)^{n+1}$ by Lemma 4.2.265. We refer to the duals of long, short, and bosonic summands as long, short, and bosonic summands respectively, as it will never be ambiguous whether we are working with an EA-type *ko*-module or its Anderson dual.

Now we can state our first main computational result.

Theorem 4.2.266. *On homotopy groups, the free-to-interacting map $F2I_{\ell,k} \colon \Sigma^{k-\ell-3}KO \to \Sigma I_{\mathbb{Z}}(ko \wedge ME_{\ell,k})$ has the following explicit description.*

Long summands, part 1 *For $n = 8\sigma + k - \ell + 3$, $\pi_n(F2I_{\ell,k})$ is a surjective map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2^{4+4\sigma+\ell}$ onto the unique long summand in this degree, with kernel $(4 + 4\sigma + \ell)\mathbb{Z}$.*

Long summands, part 2 *For $n = 8\sigma + k - \ell + 7$, $\pi_n(F2I_{\ell,k})$ is a surjective map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2^{5+4\sigma+\ell}$ onto the unique long summand in this degree, with kernel $(5 + 4\sigma + \ell)\mathbb{Z}$.*

Short summands *For $n \equiv k - \ell + 1 \bmod 8$ or $n \equiv k - \ell + 2 \bmod 8$, $\pi_n(F2I_{\ell,k})$ is an isomorphism $\mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2$ onto the unique short summand in this degree.*

In all other degrees the domain of the free-to-interacting map is zero; thus the image of the free-to-interacting map does not intersect the bosonic summands nontrivially.

We also solve the *ko*-theoretic version of the interacting Bott spiral map modulo bosonic summands. Recall the definition of $\mathrm{sp}^{\ell,k}_{\mathrm{int}}(ko)$ from Definition 4.2.199.

Theorem 4.2.267. *Modulo bosonic summands, the map $\mathrm{sp}^{\ell,k}_{\mathrm{int}}(ko) \colon I_{\mathbb{Z}}(ko \wedge ME_{\ell,k}) \to \Sigma I_{\mathbb{Z}}(ko \wedge ME_{\ell,k+1})$ is a π_*-isomorphism.*

We prove Theorem 4.2.266 in a series of lemmas (Lemmas 4.2.268, 4.2.271, and 4.2.273).

Lemma 4.2.268 (Freed–Hopkins [1, §10])**.** *Theorem 4.2.266 is true for* $k = 1$ *and* $\ell = 0$*, and for* $k = 0$ *and* $\ell = 1$*.*

Proof. As is clear by the definition, our free-to-interacting maps for the cases $k = 1$, $\ell = 0$, resp. $k = 0$, $\ell = 1$ coincide with Freed–Hopkins' free-to-interacting maps in their cases $s = -1$, resp. $s = 1$, so it suffices to observe that the description we gave in our theorem statement coincides with their calculation in these cases. ☐

Lemma 4.2.269. *The map* $\varphi_{\mathrm{int}}^{\ell,k}\colon \Sigma ME_{\ell,k-1} \to ME_{\ell,k}$ *we defined in Definition 4.2.196 induces an injective map on mod 2 cohomology.*

Proof. We defined $\varphi_{\mathrm{int}}^{\ell,k}$ in Definition 4.2.196 by smashing $ME_{\ell,k-1}$ with the map $(B\mathbb{Z}/2)^{1-\sigma} \to \Sigma\mathbb{S}$ which is the composition of the Smith homomorphism $\mathrm{sm}_\sigma\colon (B\mathbb{Z}/2)^{1-\sigma} \to \Sigma(B\mathbb{Z}/2)_+$ and the crush map $c\colon \Sigma(B\mathbb{Z}/2)_+ \to \Sigma\mathbb{S}$. The pullback $c^*\colon H^*(\Sigma\mathbb{S};\mathbb{Z}/2) \to H^*(\Sigma(B\mathbb{Z}/2)_+;\mathbb{Z}/2)$ is injective, the inclusion of the degree-1 part, and since we are over a field, this remains true after tensoring with $ME_{\ell,k-1}$. For the second piece of $\varphi_{\mathrm{int}}^{\ell,k}$, namely the Smith homomorphism, we make use of the fiber sequence [9, (7.1)]

$$\mathbb{S} \longrightarrow (B\mathbb{Z}/2)^{1-\sigma} \xrightarrow{\mathrm{sm}_\sigma} \Sigma(B\mathbb{Z}/2)_+. \tag{4.2.270}$$

Using the induced long exact sequence in mod 2 cohomology[43] and the fact that $H^*(\mathbb{S};\mathbb{Z}/2) \cong \mathbb{Z}/2$ concentrated in degree 0, we see that sm_σ is injective on mod 2 cohomology. Thus after tensoring with $H^*(ME_{\ell,k-1};\mathbb{Z}/2)$, the map is still injective. ☐

See Figure 4.5 for a proof of Lemma 4.2.269.

Lemma 4.2.271. *Theorem 4.2.266 is true for* $\ell = 1$ *and* k *arbitrary.*

Proof. Induct on k; the case $k = 1$ is Lemma 4.2.268. In Theorem 4.2.210, we proved that the the Bott spiral maps commute with the free-to-interacting map. Therefore it suffices to show that $\mathrm{sp}_{\mathrm{int}}^{\ell,k}(ko)\colon I_{\mathbb{Z}}(ko \wedge ME_{\ell,k-1}) \to \Sigma I_{\mathbb{Z}}(ko \wedge ME_{\ell,k})$ is a π_*-isomorphism modulo bosonic summands. By the universal property of $I_{\mathbb{Z}}$, this is equivalent to asking that $\mathrm{id}_{ko} \wedge \varphi_{\mathrm{int}}^{\ell,k}\colon ko \wedge ME_{\ell,k} \to \Sigma ko \wedge ME_{\ell,k-1}$ is a π_*-isomorphism modulo bosonic summands, and this is what we prove.

In Proposition 4.2.260 we showed that $H^*(ME_{\ell,k};\mathbb{Z}/2) \cong F \oplus \Sigma^{k+4s}N_{\ell \bmod 4}$, where $s = \lfloor \ell/4 \rfloor$ and F is a free summand. Thus the non-free summands in the cohomologies of $ME_{\ell,k}$ and $\Sigma ME_{\ell,k-1}$ are graded isomorphic. By Lemma 4.2.269, $(\varphi_{\mathrm{int}}^{\ell,k})^*\colon H^*(ME_{\ell,k};\mathbb{Z}/2) \to H^*(\Sigma ME_{\ell,k-1};\mathbb{Z}/2)$ is injective. Thus the $\Sigma^{k+4s}N_{\ell \bmod 4}$ summand in the domain maps isomorphically onto its image, but the only way to do that when the codomain is $\Sigma^{k+4s}N_{\ell \bmod 4}$ plus a free $\mathcal{A}(1)$-module is to map isomorphically to the first summand. Taking Ext groups, this implies that the map of Adams spectral sequences induced by $\mathrm{id}_{ko} \wedge \varphi_{\mathrm{int}}^{\ell,k}$ is an isomorphism except maybe for the pieces coming from the free summands. By Margolis' theorem, those free summands do not have nontrivial differentials or extension problems, and pass to bosonic summands in ko-homology. Thus we could conclude that, modulo bosonic summands, $\mathrm{id}_{ko} \wedge \varphi_{\mathrm{int}}^{\ell,k}$ is a π_*-isomorphism. ☐

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[43] After applying the Thom isomorphism to $H^*((B\mathbb{Z}/2)^{1-\sigma};\mathbb{Z}/2)$, this long exact sequence can be identified with the usual Gysin sequence for the tautological line bundle over $B\mathbb{Z}/2$. See [9, Remark 5.11].

Remark 4.2.272. Another way to show that $\mathrm{sp}^{\ell,k}_{\mathrm{int}}(ko)$ is an isomorphism modulo bosonic summands is to identify the fiber F of $\mathrm{id}_{ko} \wedge \varphi^{\ell,k}_{\mathrm{int}}$. It is possible to compute that F is, as a ko-module, equivalent to a sum of shifts of $H\mathbb{Z}/2$, and so in particular the Bott class acts trivially on F. This implies $F \wedge_{ko} KO \simeq F[\beta^{-1}]$ is trivial (see, e.g., (4.2.222)), so after base-changing to KO, the Bott spiral map is an isomorphism. Since base-changing to KO is injective on homotopy groups for ko-modules of EA-type, including both the domain and the codomain of $\mathrm{id}_{ko} \wedge \varphi^{\ell,k}_{\mathrm{int}}$, that map is injective on homotopy groups, which is enough to recover the consequences for the free-to-interacting map that prove Lemma 4.2.271. We will use a similar strategy to prove Lemma 4.2.273.

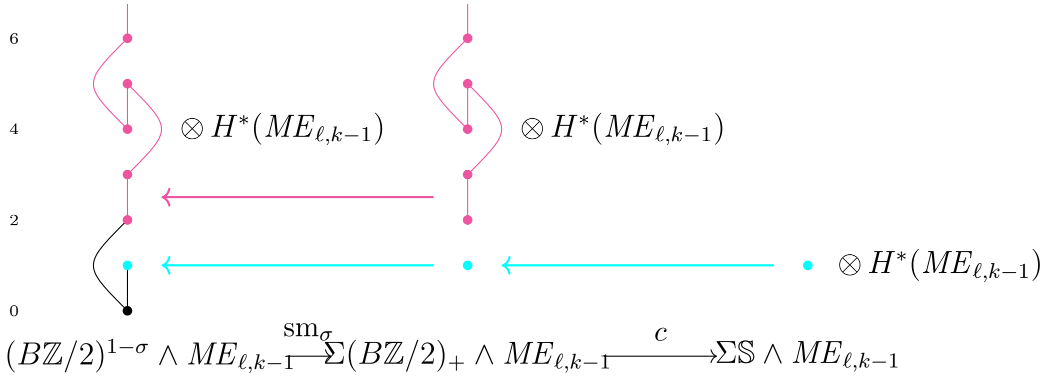


Figure 4.5: The effect of the interacting Bott spiral map $\varphi^{k,\ell}_{\mathrm{int}}$ on mod 2 cohomology. The map $\varphi^{1,0}_{\mathrm{int}}$ factors as the Smith map sm_σ composed with the crush map c; then $\varphi^{k,\ell}_{\mathrm{int}}$ is $\varphi^{1,0}_{\mathrm{int}}$ smashed with the identity on $ME_{\ell,k-1}$. We use this map on cohomology in the proof of Lemma 4.2.271.

Lemma 4.2.273. *Theorem 4.2.266 is true for all values of k and ℓ.*

We will prove this with another inductive argument, this time inducting on ℓ. To do this, we need a map relating $ko \wedge ME_{\ell,k}$ and $ko \wedge ME_{\ell-1,k}$. Thus, we will first construct and study this map, then return to Lemma 4.2.273.

Proposition 4.2.274. *The Thom spectrum $(B\mathbb{Z}/2)^{1-\sigma}$ admits the structure of a CW-spectrum with one k-cell for each $k \geq 0$, and such that the attaching map of the 2-cell to the 1-cell is trivial.*

Proof sketch. This can be directly checked by showing that the attaching map of the 2-cell to the 1-cell in $(\mathbb{RP}^n)^{1-\sigma}$ is trivial for all $n \geq 2$. If $\lambda(n)$ is the order of σ in $\widetilde{KO}^0(\mathbb{RP}^n)$, then $(\mathbb{RP}^n)^{1-\sigma} \simeq (\mathbb{RP}^n)^{(\lambda(n)-1)(\sigma-1)}$, so one can finish the proof by checking that the attaching map from the 2-cell to the 1-cell is trivial in the Thom spectrum of $(\lambda(n)-1)(\sigma-1) \to \mathbb{RP}^n$, and Adams [491, Theorem 7.4] calculates $\lambda(n)$ for all n. ☐

Corollary 4.2.275. *The k-cells of the above CW-structure on $(B\mathbb{Z}/2)^{1-\sigma}$ for $k \neq 1$, with the same attaching maps, define a CW-spectrum F with an "inclusion map" $F \to (B\mathbb{Z}/2)^{1-\sigma}$ whose cofiber is the 1-cell $\Sigma\mathbb{S}$.*

In essence, we want to "forget to include" the 1-cell of $(B\mathbb{Z}/2)^{1-\sigma}$ to define F. This works if and only if no 2-cell is nontrivially attached to the 1-cell, which Proposition 4.2.274 verifies. The following lemma appears in Bruner [464, §2], who writes that it is probably originally due to Davis or Mahowald, and that he learned it from Stolz.

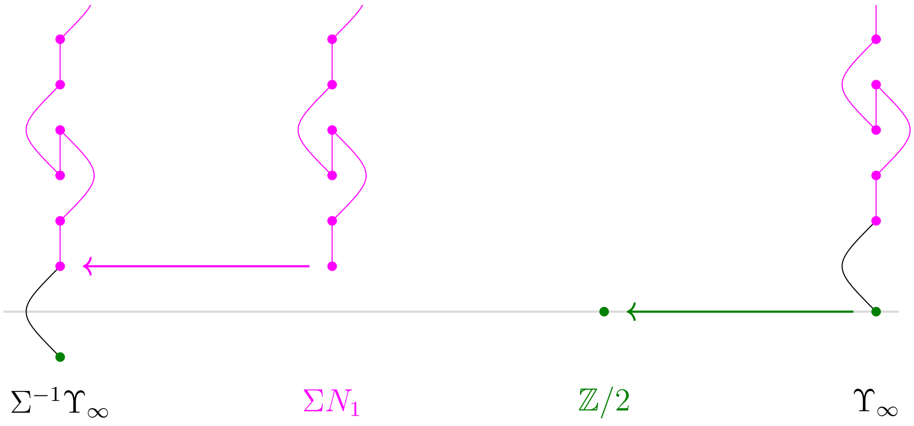
Lemma 4.2.276. *With F as in Corollary 4.2.275, the fiber of the map $\varsigma\colon \mathbb{S} \to F$ defined by including the 0-cell is a nontrivial map $\upsilon\colon \Sigma(B\mathbb{Z}/2)^{\sigma-1} \to \mathbb{S}$.*

We draw this fiber sequence in Figure 4.6.

Definition 4.2.277. Let Υ_∞ denote the *infinite seagull*, the $\mathcal{A}(1)$-module defined to be the submodule of $\Sigma^{-3}\widetilde{H}^*(\mathbb{RP}^\infty;\mathbb{Z}/2)$ of elements in degrees $d \geq 0$, $d \neq 1$.[44] Though a priori this is only a $\mathbb{Z}/2$-vector space, it is closed under Sq^1 and Sq^2, so is in fact an $\mathcal{A}(1)$-module.

The name "infinite seagull" and the seagull-like notation Υ_∞ are due to Adamyk [493, Definition 2.5]. See Figure 4.6, right, for a depiction of Υ_∞.

Lemma 4.2.278 (Bruner [464, Notation 2.1])**.** *There is an isomorphism $H^*(F;\mathbb{Z}/2) \cong \Upsilon_\infty$.*[45,46]



$$\begin{tikzcd}
\Sigma^{-1}F \arrow[r] & \Sigma(B\mathbb{Z}/2)^{\sigma-1} \arrow[r, "\upsilon"] & \mathbb{S} \arrow[r, "\varsigma"] & F
\end{tikzcd}$$

Figure 4.6: A picture of the fiber sequence including the maps ς and υ from [464, §2], introduced in the text in Lemma 4.2.276. We also label the $\mathcal{A}(1)$-modules appearing as the mod 2 cohomology of the spectra in this fiber sequence, including the infinite seagull Υ_∞ from Definition 4.2.277.

Definition 4.2.279. Recall the map $\lambda_\sigma\colon (B\mathbb{Z}/2)^{\sigma-1} \to \Sigma^{-1}KO$ from Definition 4.2.134. Because $(B\mathbb{Z}/2)^{\sigma-1}$ is (-2)-connected, λ_σ factors through the (-2)-connected cover $\Sigma^{-1}ko \to \Sigma^{-1}KO$ as a map $\lambda_\sigma'\colon (B\mathbb{Z}/2)^{\sigma-1} \to \Sigma^{-1}ko$. Smash λ_σ' with id_{ko} to define a ko-module spectrum map $\lambda_\sigma''\colon ko \wedge (B\mathbb{Z}/2)^{\sigma-1} \to \Sigma^{-1}ko$.

---

[44] This $\mathcal{A}(1)$-module is called $P/F_{-1,1}$ in [492], R in [283, §A.9] and [466, Definition 4.1], P in [96, Figure 5.7], and M_∞ in [98, Example 4.4.2].

[45] Bruner [464, Notation 2.1] writes that in the notation of [492], $H^*(F;\mathbb{Z}/2) \cong \Sigma F_{-1,1}$. This appears to be a typo: $\Sigma F_{-1,1}$ should be replaced with $P/F_{-1,1}$, as in Footnote 44.

[46] Since Υ_∞ is not an unstable $\mathcal{A}(1)$-module, it cannot be the $\mathcal{A}(1)$-module structure on the cohomology of any space. But if it were, that space would be a complete seagull space.

We believe the following lemma is well-known, but we were unable to find a proof in the literature.

Lemma 4.2.280. *$\widetilde{ko}^0(B\mathbb{Z}/2)^\wedge_2 \cong \mathbb{Z}_2$.*

Here $\mathbb{Z}_2$ denotes the 2-adic integers, not the integers mod 2.

Proof sketch. Apply the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\mathbb{Z}/2, \widetilde{H}^*(B\mathbb{Z}/2;\mathbb{Z}/2)) \Longrightarrow \pi_{t-s}\mathrm{Map}_{\mathbb{S}}(B\mathbb{Z}/2, ko)^\wedge_2. \tag{4.2.281}$$

Bruner–Greenlees [283, Figure A.9.16] compute the E_2-page of (4.2.281) and observe (*ibid.*, §A.9) that it collapses, implying the lemma statement.[47] ☐

Proposition 4.2.282. *As ko-module maps, the map λ''_σ in Definition 4.2.279 is equivalent to an odd multiple of* $\mathrm{id}_{ko} \wedge \Sigma^{-1}\upsilon$ *after* 2*-completing.*

Proof. Using the Smith isomorphism from Equation (3.6.1) and the equivalence $\mathrm{Map}_R(R \wedge A, B) \simeq \mathrm{Map}_{\mathbb{S}}(A, B)$,

$$\pi_0(\mathrm{Map}_{ko}(ko \wedge (B\mathbb{Z}/2)^{\sigma-1}, \Sigma^{-1}ko)^\wedge_2 \cong \widetilde{ko}^0(B\mathbb{Z}/2), \tag{4.2.283}$$

which by Lemma 4.2.280 is isomorphic to $\mathbb{Z}_2$. We will show that both λ''_σ and $\mathrm{id}_{ko} \wedge \Sigma^{-1}\upsilon$ define odd elements of this $\mathbb{Z}_2$ after 2-completion. Odd numbers can be inverted in the 2-adics, so each map is an odd multiple of the other. Given a map $f\colon ko \wedge (B\mathbb{Z}/2)^{\sigma-1} \to \Sigma^{-1}ko$, consider its effect on π_0, where it induces a map $\pi_0 f\colon \mathbb{Z}/2 \to \mathbb{Z}/2$. If $f \simeq 2g$, then $\pi_0 f = 2\pi_0 g$, so it must be the zero map on $\mathbb{Z}/2$. We will show that for $f = \lambda''_\sigma$ and $f = \mathrm{id}_{ko} \wedge \Sigma^{-1}\upsilon$, $\pi_0 f \neq 0$, so that both of these maps are odd.

For λ''_σ, precompose with the Atiyah–Bott–Shapiro map $MT\mathrm{Spin} \to ko$ and postcompose with $\Sigma^{-1}ko \to \Sigma^{-1}KO$, as in Example 4.2.140, to obtain a map $MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{\sigma-1} \to \Sigma^{-1}KO$ which is precisely Freed–Hopkins' twisted ABS map [1, §10]. Freed–Hopkins also show (*ibid.*, Corollary 9.85) that this map is an isomorphism $\mathbb{Z}/2 \to \mathbb{Z}/2$ in degree 0,[48] so the middle piece λ''_σ cannot be the zero map on π_0.

The cofiber of $\mathrm{id}_{ko} \wedge \Sigma^{-1}\upsilon$ is $ko \wedge \Sigma^{-1}F$. Mahowald [486, Lemma 7.2] shows that, 2-locally,[49]

$$ko \wedge F \simeq \bigvee_{n\geq 0} \Sigma^{4n} H\mathbb{Z}. \tag{4.2.284}$$

Plug this into the long exact sequence of homotopy groups induced by the fiber sequence $ko \wedge (B\mathbb{Z}/2)^{1-\sigma} \to \Sigma^{-1}ko \to ko \wedge F$; exactness forces $\pi_0(\mathrm{id}_{ko} \wedge \Sigma^{-1}\upsilon)$ to be surjective, hence an isomorphism $\mathbb{Z}/2 \to \mathbb{Z}/2$. ☐

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[47] As this is a half-plane spectral sequence, we must be careful about convergence. Fortunately, since ko is of finite type, Boardman [494, Theorem 15.6] shows that the Adams spectral sequence (4.2.281) converges conditionally. Thus, since this spectral sequence collapses, it also converges in the usual sense (*ibid.*, §7).

[48] Freed–Hopkins only report the effect of the Anderson dual of their map, but since the homotopy groups of $MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{\sigma-1}$ are torsion [192], this implies their original map is also an isomorphism in degree 0, using the universal property of the Anderson dual.

[49] This is an equivalence of spectra, but Hill [495, Lemma 3.3] showed that the two sides are *not* equivalent as ko-modules, where we give $H\mathbb{Z}$ the ko-algebra structure arising from the truncation map $ko \to H\mathbb{Z}$ of ring spectra. Instead, we hypothesize that $ko \wedge F$ is ko-module equivalent to the connective cover of the L-theory spectrum of $\mathbb{R}$, which is also known to split as $H\mathbb{Z} \vee \Sigma^4 H\mathbb{Z} \vee \Sigma^8 H\mathbb{Z} \vee \cdots$ as spectra but not ko-modules. See [349, 496].

Proposition 4.2.285. *The diagram*

$$\begin{tikzcd}
& I_{\mathbb{Z}}(ko \wedge ME_{\ell,k}) \\
\Sigma^{k-\ell}KO \arrow[ur, "F2I_{\ell,k}"] \arrow[dr, "\Sigma^{-1}F2I_{\ell-1,k}"'] & \\
& \Sigma^{-1}I_{\mathbb{Z}}(ko \wedge ME_{\ell-1,k}) \arrow[uu, "I_{\mathbb{Z}}(\mathrm{id}_{ko\wedge ME_{\ell-1,k}}\wedge\Sigma^{-1}\upsilon)"']
\end{tikzcd} \tag{4.2.286}$$

commutes after 2*-completion up to homotopy and multiplication by an odd integer (which is a unit in the* 2*-adic integers).*

The diagram (4.2.286) will play a key role in the proof of Lemma 4.2.273 completely analogous to the role of (4.2.211) in the proof of Lemma 4.2.271.

Proof. The proof is completely analogous to the proof of Theorem 4.2.210: expand out, then simplify the diagram in the same way as in Lemmas 4.2.212, 4.2.215, and 4.2.217, but replacing "data inducing commutativity" with "data inducing commutativity after 2-completion, up to homotopy and multiplication by an odd integer." Also, keep the factors of ko instead of getting rid of them. At the end of this simplification process, we have learned that if the diagram

$$\begin{tikzcd}
ko \wedge (B\mathbb{Z}/2)^{\sigma-1} \arrow[r, "\lambda''_\sigma"] \arrow[dr, "\mathrm{id}_{ko}\wedge\Sigma^{-1}\upsilon"'] & \Sigma^{-1}ko \arrow[d, "\mathrm{id}"] \\
& \Sigma^{-1}ko
\end{tikzcd} \tag{4.2.287}$$

commutes after 2-completion, up to homotopy and multiplication by an odd integer, then we have finished the proof. But that is exactly what we proved in Proposition 4.2.282. ☐

Lemma 4.2.288. *For $k, \ell \geq 1$, $H^*(F \wedge ME_{\ell-1,k}; \mathbb{Z}/2)$ is a free $\mathcal{A}(1)$-module.*

Proof. Let $Q_0 := \mathrm{Sq}^1$ and $Q_1 := \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$. Then for $i = 0, 1$, the Adem relations imply $Q_i^2 = 0$, so one may regard any $\mathcal{A}(1)$-module N as a cochain complex with differential Q_i. The homology of this complex is called the *Margolis homology* $H_*(M; Q_i)$ [482, 497]. We say a bounded-below $\mathcal{A}(1)$-module M is *Q_i-local* if $H_*(M; Q_i) \neq 0$ and $H_*(M; Q_{1-i}) = 0$.

We need three key facts about Margolis homology.

1. If M is a bounded-below $\mathcal{A}(1)$-module such that $H_*(M; Q_0) = 0$ and $H_*(M; Q_1) = 0$, then M is free. This is a special case of a result of Adams-Margolis [497, Theorem 4.2]; they attribute this special case to unpublished work of Wall.
2. N_0 is Q_1-local [466, Proposition 4.2].
3. If a bounded-below $\mathcal{A}(1)$-module M is Q_1-local, then $H_*(\Upsilon_\infty \otimes M; Q_i)$ vanishes for $i = 0, 1$ [466, Proof of Theorem 4.3].

By Example 4.2.250 and Lemma 4.2.278, $H^*(F \wedge (B\mathbb{Z}/2)^{1-\sigma}; \mathbb{Z}/2) \cong \Upsilon_\infty \otimes N_0$. Since N_0 is bounded below and Q_1-local, the Q_0- and Q_1-Margolis homology of $\Upsilon_\infty \otimes N_0$ both vanish, so $\Upsilon_\infty \otimes N_0$ is a free $\mathcal{A}(1)$-module.

To finish, use the Künneth formula on $F \wedge ME_{\ell-1,k} \simeq (F \wedge ME_{\ell-1,k-1}) \wedge (B\mathbb{Z}/2)^{1-\sigma}$ to show that $H^*(F \wedge ME_{\ell-1,k}; \mathbb{Z}/2) \cong H^*(ME_{\ell-1,k-1}; \mathbb{Z}/2) \otimes (\Upsilon_\infty \otimes N_0)$. Since the second term in the tensor product is free, the entire tensor product is a free module. ☐

Proof of Lemma 4.2.273. Let $\chi \coloneqq \mathrm{id}_{ko \wedge ME_{\ell-1,k}} \wedge \Sigma^{-1}\upsilon \colon \Sigma^{-1}ko \wedge ME_{\ell-1,k} \to ko \wedge ME_{\ell,k}$ We will show that the map $I_\mathbb{Z}\chi$ in (4.2.286) is a π_*-isomorphism modulo bosonic summands after 2-completion. With this in hand, the same strategy from the proof of Lemma 4.2.271, but this time using the commutativity result from Proposition 4.2.285, implies the conclusion of the lemma, albeit only after 2-completion and multiplication by an odd integer. Since the homotopy groups of $I_\mathbb{Z}(ko \wedge ME_{\ell,k})$ are 2-torsion, this suffices to prove the lemma.

Now back to $I_\mathbb{Z}\chi$. The domain and codomain of this map are the Anderson duals of $\Sigma^{-1}ko \wedge ME_{\ell-1,k}$, resp. $ko \wedge ME_{\ell,k}$, which have torsion homotopy groups by Proposition 4.2.262. Therefore by Lemma 4.2.265 it suffices to prove $I_{\mathbb{C}^\times}\chi$ is a π_*-isomorphism modulo bosonic summands after 2-completion, and by the universal property of $I_{\mathbb{C}^\times}$, it suffices to prove the same for χ.

Since the domain and codomain of χ, namely $ko \wedge ME_{\ell,k}$, resp. $\Sigma^{-1}ko \wedge ME_{\ell-1,k}$, are ko-modules of EA-type, the vertical maps in the diagram

$$\begin{tikzcd}
ko \wedge ME_{\ell,k} \arrow[r, "\chi"] \arrow[d, "\beta^{-1}"'] & \Sigma^{-1}ko \wedge ME_{\ell-1,k} \arrow[d, "\beta^{-1}"] \\
KO \wedge ME_{\ell,k} \arrow[r, "\widetilde{\chi} \coloneqq \chi[\beta^{-1}]"] & \Sigma^{-1}KO \wedge ME_{\ell-1,k}
\end{tikzcd} \tag{4.2.289}$$

are injective on homotopy groups, so it suffices to prove the map $\widetilde{\chi} \colon KO \wedge ME_{\ell,k} \to \Sigma^{-1}KO \wedge ME_{\ell-1,k}$, i.e. the map induced from χ after base changing to KO, is an equivalence after 2-completion. The cofiber of $\widetilde{\chi}$ is $\mathrm{cofib}(\chi)[\beta^{-1}]$; since χ is obtained by smashing $\Sigma^{-1}\upsilon$ with $\mathrm{id}_{ko \wedge ME_{\ell-1,k}}$, Lemma 4.2.276 implies the cofiber of χ is $ko \wedge F \wedge ME_{\ell-1,k}$. By Lemma 4.2.288, $H^*(F \wedge ME_{\ell-1,k}; \mathbb{Z}/2)$ is a free $\mathcal{A}(1)$-module, so Margolis' theorem for ko-modules (Theorem 4.2.236) implies that after 2-completion $ko \wedge F \wedge ME_{\ell-1,k}$ is a sum of shifts of $H\mathbb{Z}/2$ – in particular, the Bott class β acts by 0 on this ko-module. Thus, $\mathrm{cofib}(\chi)[\beta^{-1}] \simeq 0$, but this was $\mathrm{cofib}(\widetilde{\chi})$, so $\widetilde{\chi}$ is an equivalence after 2-completion, and as noted above this suffices. ☐

Proof of Theorem 4.2.267. We have a commutative diagram

$$\begin{tikzcd}
 & I_\mathbb{Z}(ko \wedge ME_{\ell,k+1}) \arrow[dd, "\mathrm{sp}^{\ell,k}_{\mathrm{int}}(ko)"] \\
\Sigma^{k-\ell+1}KO \arrow[ur, "F2I"] \arrow[dr, "F2I"'] & \\
 & \Sigma I_\mathbb{Z}(ko \wedge ME_{\ell,k}).
\end{tikzcd} \tag{4.2.290}$$

Theorem 4.2.266 proves surjectivity of the free-to-interacting maps in this diagram (modulo bosonic summands), and we know that, modulo bosonic summands, the domain and codomain of this Bott spiral map are abstractly isomorphic, so commutativity of (4.2.290) forces $\mathrm{sp}^{\ell,k}_{\mathrm{int}}(ko)$ to be an isomorphism modulo bosonic summands. ☐

4.2.7.2 The Bott spiral starting with $MT\mathrm{Spin}$ and $MT\mathrm{Pin}^{\pm}$

In this subsection we lift from theorems for ko-modules, where the story is simpler, to theorems for MTSpin-modules, which are the actual spectra classifying bordism and invertible field theories of interest. The lift essentially amounts to Anderson–Brown–Peterson's splitting of MTSpin and the description of the Atiyah–Bott–Shapiro orientation as the projection onto the first summand of this splitting.

Our main theorems in this section are Corollary 4.2.305 and Theorem 4.2.306. Essentially they say that the description of the free-to-interacting map is the same as in Theorem 4.2.266 and that the Bott spiral map is an isomorphism modulo bosonic summands. However, to precisely state these theorems we must first introduce some definitions, so we do that, then state and prove our main theorems.

Our primary tool for lifting ko-theoretic statements to MTSpin-level statements is the seminal work of Anderson–Brown–Peterson, as we mentioned up in Example 2.2.45:

Theorem 4.2.291 (Anderson–Brown–Peterson [367]). *There are numbers $a_i, b_j, c_k \in \mathbb{N}$ and a 2-local homotopy equivalence*

$$MT\mathrm{Spin} \xrightarrow{\simeq} \bigvee_{i\geq 0} \Sigma^{a_i} ko \vee \bigvee_{j\geq 0} \Sigma^{b_j} \tau_{\geq 2} ko \vee \bigvee_{k\geq 0} \Sigma^{c_k} H\mathbb{Z}/2, \tag{4.2.292}$$

where $\tau_{\geq 2} ko$ denotes the 1-connective cover of ko. The degrees of the first few shifts are

$$\begin{aligned} (a_i) &= (0, 8, 16, 16, 24, 24, 24, 24, \dots) \\ (b_j) &= (8, 16, 16, 24, 24, 24, 24, \dots) \\ (c_k) &= (20, 22, 24, 26, 26, \dots). \end{aligned} \tag{4.2.293}$$

Moreover, the projection $\mathrm{proj}_{a_1} : MT\mathrm{Spin} \to ko$ onto the first factor is equivalent to the Atiyah–Bott–Shapiro map.

For any spectrum X, this induces a wedge-sum decomposition of $MT\mathrm{Spin} \wedge X$ which we call the *ABP decomposition*:

$$MT\mathrm{Spin} \wedge X \xrightarrow{\simeq} \bigvee_{i\geq 0} \Sigma^{a_i} ko \wedge X \vee \bigvee_{j\geq 0} \Sigma^{b_j} \tau_{\geq 2} ko \wedge X \vee \bigvee_{k\geq 0} \Sigma^{c_k} H\mathbb{Z}/2 \wedge X. \tag{4.2.294}$$

Remark 4.2.295. One can verify using the defining property of $I_{\mathbb{Z}}$ that the Anderson dual of a wedge sum is the product of the Anderson duals of the summands. Therefore (4.2.294) induces a direct product decomposition of $I_{\mathbb{Z}}(MT\mathrm{Spin} \wedge X)$.

The definition of a bosonic summand (i.e. the pieces we will ignore) is a little delicate in this setting, ultimately because Theorem 4.2.291 is not an MTSpin-module equivalence. We would like to define bosonic summands by comparing with ko-theory, but to do that we would need a map of ring spectra $ko \to MT\mathrm{Spin}$, which Stolz [498, §7] showed does not exist. Fortunately, Stolz (*ibid.*, §1) introduces a weaker criterion that is good enough for us.

Recall that the isomorphism $\Omega_0^{\mathrm{O}} \xrightarrow{\cong} \mathbb{Z}/2$ counting the number of points mod 2 lifts to an E_∞-ring map $p\colon M\mathrm{O} \to H\mathbb{Z}/2$. Mahowald [499, p. 2.6] constructed a section $s_{\mathrm{O}}\colon H\mathbb{Z}/2 \to M\mathrm{O}$ of p as homotopy-commutative ring spectra, which Hopkins observed refines to an E_3-ring map (see [500, §1]).[50]

Definition 4.2.296. A section $s\colon ko \to MT\mathrm{Spin}$ of the Atiyah–Bott–Shapiro map is a *weak algebra section* if

1. the induced map $s_*\colon H_*(ko;\mathbb{Z}/2) \to H_*(MT\mathrm{Spin};\mathbb{Z}/2)$ is an algebra homomorphism; and

2. the following diagram commutes up to homotopy:

$$\begin{tikzcd} ko \arrow[r, "s"] \arrow[d, "\tau_0"] & MT\mathrm{Spin} \arrow[d, "\phi"] \\ H\mathbb{Z}/2 \arrow[r, "s_{\mathrm{O}}"] & M\mathrm{O} \end{tikzcd} \tag{4.2.297}$$

   Here ϕ is induced by the forgetful map $B\mathrm{Spin} \to B\mathrm{O}$ and τ_0 is the Postnikov 0-truncation map.

Proposition 4.2.298 (Stolz [498, §6])**.** *Weak algebra sections of the Atiyah–Bott–Shapiro orientation exist, and all of them induce equal maps on mod* 2 *homology.*

Without loss of generality, choose such a map s.

Definition 4.2.299 (Stolz [498, §1])**.** A *homology ko-module* is a spectrum Y together with a map $\mu_Y\colon ko \wedge Y \to Y$ of spectra such that the induced map on homology

$$(\mu_Y)_*\colon H_*(ko;\mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H_*(Y;\mathbb{Z}/2) \longrightarrow H_*(Y;\mathbb{Z}/2) \tag{4.2.300}$$

defines the structure of an $H_*(ko;\mathbb{Z}/2)$-module on $H_*(Y;\mathbb{Z}/2)$.

Remark 4.2.301.

1. Any *ko*-module, including *ko*, $\tau_{\geq 2}ko$, and $H\mathbb{Z}/2$, is a homology *ko*-module in a canonical way.

2. Any $MT\mathrm{Spin}$-module Y with action map $\mu_Y^s\colon MT\mathrm{Spin} \wedge Y \to Y$ has a canonical homotopy *ko*-module structure with $\mu_Y \coloneqq \mu_Y^s \circ (s \wedge \mathrm{id}_Y)\colon ko \wedge Y \to Y$ (*ibid.*, (1.2)). Moreover, if the $MT\mathrm{Spin}$-module structure arises from a *ko*-module structure through the ABS map, this homotopy *ko*-module structure agrees with the one from (1).

3. The Anderson–Brown–Peterson splitting in Theorem 4.2.291 is a splitting of homology *ko*-modules, as is implicit in (*ibid.*, §6).

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[50] See [500–510] for additional proofs and generalizations of Mahowald's result.

Definition 4.2.302. Let M be an $MT\mathrm{Spin}$-module. A *bosonic summand* of M is an $H\mathbb{Z}/2$-summand of M as homotopy ko-modules.

If M acquired its $MT\mathrm{Spin}$-module structure from a ko-module structure of EA type via the ABS map, Definitions 4.2.227 and 4.2.302 coincide.

Remark 4.2.303. In (4.2.294), each $H\mathbb{Z}/2 \wedge X$ piece splits as a sum of shifts of $H\mathbb{Z}/2$ indexed by a basis of $H_*(X;\mathbb{Z}/2)$. Each of these $H\mathbb{Z}/2$ summands is by definition a bosonic summand.

Bosonic summands of M are dual to bosonic summands of $I_\mathbb{Z}M$, as one can verify using the compatibility of Anderson duality with wedge sums.

Lemma 4.2.304. *For all k and ℓ, the twisted Atiyah–Bott–Shapiro map $MT\mathrm{Spin} \wedge ME_{\ell,k} \to \Sigma^{k-\ell}KO$ from Equation* (4.2.154) *factors as the composition of* $\mathrm{proj}_{a_1} \wedge \mathrm{id}_{ME_{\ell,k}}\colon MT\mathrm{Spin} \wedge ME_{\ell,k} \to ko \wedge ME_{\ell,k}$ *and the map $ko \wedge ME_{\ell.k} \to \Sigma^{\ell-k}KO$ induced from $ABS_{(\ell,k)}$.*

Proof. This is true because the twisted Atiyah–Bott–Shapiro map is the smash product of the usual Atiyah–Bott–Shapiro map and a map $ME_{\ell,k} \to \Sigma^{k-\ell}KO$, and in Theorem 4.2.291 we saw that the usual Atiyah–Bott–Shapiro map factors through proj_{a_1}. ☐

Corollary 4.2.305.

1. *The image of the free-to-interacting map $\Sigma^{k-\ell}KO \to I_\mathbb{Z}(MT\mathrm{Spin} \wedge ME_{\ell,k})$ on homotopy groups is a subgroup of the group $(\mathrm{Dir}_{k,\ell})_*$ of IFTs whose partition functions vanish on all spin-(ℓ,k) manifolds whose image under the projection $\mathrm{proj}_{a_1}\colon \Omega_*^{\mathrm{Spin}}(ME_{\ell,k}) \to ko_*(ME_{\ell,k})$ vanishes.*

2. $(\mathrm{Dir}_{k,\ell})_* \cong \pi_{-*}(I_\mathbb{Z}ko)$, *and the image of the free-to-interacting map inside these groups is exactly as given in Theorem 4.2.266.*

Theorem 4.2.306. *The interacting Bott spiral map*

$$\mathrm{sp}_{\mathrm{int}}^{k,\ell}\colon I_\mathbb{Z}(MT\mathrm{Spin} \wedge ME_{\ell,k+1}) \to \Sigma I_\mathbb{Z}(MT\mathrm{Spin} \wedge ME_{\ell,k})$$

is an isomorphism modulo bosonic summands.

Proof of Theorem 4.2.306. In Definitions 4.2.196 and 4.2.199, we defined the interacting Bott spiral map by defining a map $ME_{\ell,k} \to \Sigma ME_{\ell,k-1}$, smashing with $MT\mathrm{Spin}$, and applying Anderson duality. Therefore, as we discussed above, we can compute the effect of the Bott spiral map by checking what it does on each of the three kinds of summands in the ABP decomposition of $I_\mathbb{Z}(MT\mathrm{Spin} \wedge -)$.

1. For the $\Sigma^{a_i}ko$ summands, we get the map $\mathrm{sp}_{\mathrm{int}}^{\ell,k}(ko)$, which we proved is an isomorphism modulo bosonic summands in Theorem 4.2.267.

2. Next the $\Sigma^{b_j}\tau_{\geq 2}ko$ summands. Hopkins [511][51] constructs a spectrum $\mathbf{J}$[52] such that $ko \wedge \mathbf{J} \simeq \tau_{\geq 2}ko$ as ko-modules. Therefore on these summands the interacting Bott spiral map has the form

$$\mathrm{Map}(\Sigma^{b_j}\mathbf{J}, I_\mathbb{Z}(ko \wedge ME_{\ell,k+1})) \longrightarrow \mathrm{Map}(\Sigma^{b_j}\mathbf{J}, I_\mathbb{Z}(\Sigma ko \wedge ME_{\ell,k})). \tag{4.2.307}$$

---

[51] Hopkins' thesis is difficult to find online; Baker [512, Corollary 4.2] also provides a proof.

[52] In fact Hopkins shows there are exactly two such homotopy classes of spectra, distinguished by whether Sq^4 acts nontrivially on their mod 2 cohomology, and they are Spanier-Whitehead dual to each other.

That is, we can think of it as a natural transformation of cohomology theories, evaluated on the spectrum $\Sigma^{b_j}\mathbf{J}$. We can compute the effect of this map using the Atiyah-Hirzebruch spectral sequence; if we replace $\Sigma^{b_j}\mathbf{J}$ with pt to calculate the E_2-page, we obtain $\mathrm{sp}^{k,\ell}_{\mathrm{int}}(ko)$, which we already know to be an isomorphism modulo bosonic summands. Therefore the same is true of the E_2-page, and therefore also of the E_∞-page, and therefore also of the interacting Bott spiral map on this summand, as always modulo bosonic summands.

3. The $\Sigma^{c_k}H\mathbb{Z}/2$ summands are bosonic (see Remark 4.2.303), so we ignore them. □

We address the cases of the Bott spiral starting form $MT\mathrm{Spin}^h$, $MT\mathrm{Pin}^{h\pm}$, and $MT\mathrm{Pin}^{\tilde{c}\pm}$ in the upcoming article [12].

4.2.7.3 The Bott spiral starting with $MT\mathrm{Spin}^c$ and $MT\mathrm{Pin}^c$

The story over $\mathbb{C}$, i.e. using KU and $MT\mathrm{Spin}^c$, is similar but easier. In this section we compute the F2I and interacting Bott spiral maps in the complex case, again modulo bosonic summands.

The most striking change is that our two indices ℓ and k behave identically!

Proposition 4.2.308. *Let R be one of $MT\mathrm{Spin}^c$, ku, or KU. Then there is a canonical R-module equivalence $R \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq R \wedge (B\mathbb{Z}/2)^{1-\sigma}$.*

We are not sure where Proposition 4.2.308 was first proven: it appears to be well-known but not written down. It follows from the fact that 2σ is complex, hence R-oriented.

Corollary 4.2.309. *Let $m := \ell + k$ and $ME_m := ME_{m,0}$. With R as in Proposition 4.2.308, there is a canonical R-module isomorphism $R \wedge ME_{\ell,k} \xrightarrow{\cong} R \wedge ME_m$.*

We will thus only keep track of the index m in this subsection.

There is an isomorphism

$$ku_* \cong \mathbb{Z}[\beta], \qquad |\beta| = 2, \tag{4.2.310}$$

where β is the Bott class. As for real K-theory, β implements Bott periodicity: there is a natural isomorphism $K_*(X) \cong ku_*(X)[\beta^{-1}]$, and for any ku-module M, $KU \wedge_{ku} M \simeq M[\beta^{-1}]$.

There is a ku-module structure on $H\mathbb{Z}/2$ defined analogously to the ko-module structure.

Remark 4.2.311 (Wilson [513, Proposition 1.2]). The reduced K-theory of $\mathbb{RP}^\infty$ vanishes in even degrees and is μ_{2^∞} in odd degrees.

Definition 4.2.312. We define ku-modules of EA-type exactly as in Definition 4.2.223, but with ku and KU in place of ko, resp. KO. Because $\widetilde{K}^*(\mathbb{RP}^\infty)$ is simpler than $\widetilde{KO}^*(\mathbb{RP}^\infty)$, our analogue of Definition 4.2.227 is simpler: we define bosonic and long summands in the same way, mutatis mutandis.

There is no analogue of short summands in the complex setting.

Recall the stable cohomology operations $Q_0 := \mathrm{Sq}^1$ and $Q_1 := \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$ we introduced in the proof of Lemma 4.2.288, and let $\mathcal{E}(1) := \langle Q_0, Q_1 \rangle \subset \mathcal{A}$. Baker [480, Theorem 5.1] shows that $\mathcal{E}(1)$ is the subalgebra of $\mathcal{A}$ of ku-module maps $H\mathbb{Z}/2 \to \Sigma^k H\mathbb{Z}/2$, so if we define H_{ku} analogously to H_{ko}, there is a Baker–Lazarev Adams spectral sequence for a ku-module M:

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(H_{ku}^*(M), \mathbb{Z}/2) \Longrightarrow \pi_*(M)_2^\wedge. \tag{4.2.313a}$$

And, just as in (4.2.234), if $M = ku \wedge X$, this spectral sequence simplifies to one more commonly constructed using the change-of-rings theorem:

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(H^*(X;\mathbb{Z}/2), \mathbb{Z}/2) \Longrightarrow ku_*(X)_2^\wedge. \tag{4.2.313b}$$

The algebra $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2)$ is the Adams E_2-page for ku, and it acts on the Adams E_2-page for any ku-module. Differentials commute with this action. There is an isomorphism [98, Example 4.5.6]

$$\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2) \xrightarrow{\cong} \mathbb{Z}/2[h_0, v_1], \tag{4.2.314}$$

with $h_0 \in \mathrm{Ext}^{1,1}$ and $v_1 \in \mathrm{Ext}^{1,3}$. The class h_0 lifts to multiplication by 2, and v_1 lifts to multiplication by β. The analogue of Margolis' theorem (Theorem 4.2.236) in this setting is true; see Bruner–Greenlees [465, §2.1].

Example 4.2.315. The prototypical ku-module of EA-type is $ku \wedge \mathbb{RP}^\infty$. Just like in Example 4.2.241, we will work with $ku \wedge ME_1 \simeq \Sigma^{-1}(ku \wedge \mathbb{RP}^\infty)$. To run the Adams spectral sequence, we need $\mathrm{Ext}_{\mathcal{E}(1)}(N_1)$, which was computed by [102, Proposition 4.48][53] to be

$$\mathrm{Ext}_{\mathcal{E}(1)}(N_1) \cong \mathbb{Z}/2[h_0, v_1]\{e_0, e_1, \ldots\}/(h_0 e_0, v_1 e_i = h_0 e_{i+1}) \tag{4.2.316}$$

with $e_i \in \mathrm{Ext}^{0,2i}$. We draw this in Figure 4.7, right. All differentials vanish for degree reasons, and all extension questions are solved by the h_0- and v_1-actions, so we learn that

$$\widetilde{ku}_*(ME_1) \cong \mathbb{Z}\{\overline{e}_i : i \geq 0\}/(2\overline{e}_0, \beta\overline{e}_i = 2\overline{e}_{i+1}) \tag{4.2.317}$$

with $|\overline{e}_i| = 2i$. Therefore $\widetilde{ku}_{2k+1}(ME_s) = 0$ and $\widetilde{ku}_{2k}(ME_s) \cong \mathbb{Z}/2^k$ generated by $\overline{e}_k$; this is a long summand. Inverting β, one concludes $ku \wedge ME_1$ is EA-type.

Just as for ko, there are no bosonic summands, and we can obtain the same results for $\mathbb{RP}^\infty$ by shifting degrees by 1. The groups $ku_*(\mathbb{RP}^\infty)$ were first computed by Hashimoto [514, Theorem 3.1]. ◇

Proposition 4.2.318. *For $m \geq 1$, $ku \wedge ME_m$ is of EA-type, and there is a stable isomorphism of $\mathcal{E}(1)$-modules $H^*(ME_m;\mathbb{Z}/2) \simeq \Sigma^m N_1$.*

Proof sketch. The proof is identical to that of Proposition 4.2.260, except using the stable equivalence $N_1 \otimes N_1 \simeq \Sigma N_1$ of $\mathcal{E}(1)$-modules due to Ossa [461, §3]. We also need the ku-module analogue of Lemma 4.2.253, that if $H^*(X;\mathbb{Z}/2) \simeq \Sigma^n N_1$ as $\mathcal{E}(1)$-modules then $ku \wedge X$ is finite type, but this can be proven in a similar way using the analysis of Example 4.2.315, which used nothing about ME_1 save for the $\mathcal{E}(1)$-module structure on its cohomology. ☐

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[53] This source computes $\mathrm{Ext}_{\mathcal{E}(1)}(N_0)$, but $N_0 \cong N_1$ as $\mathcal{E}(1)$-modules, which is yet another avatar of the fact that $\mathrm{Pin}^+ \times_{\{\pm 1\}} \mathrm{U}_1$ and $\mathrm{Pin}^- \times_{\{\pm 1\}} \mathrm{U}_1$ define the same symmetry type Pin^c.

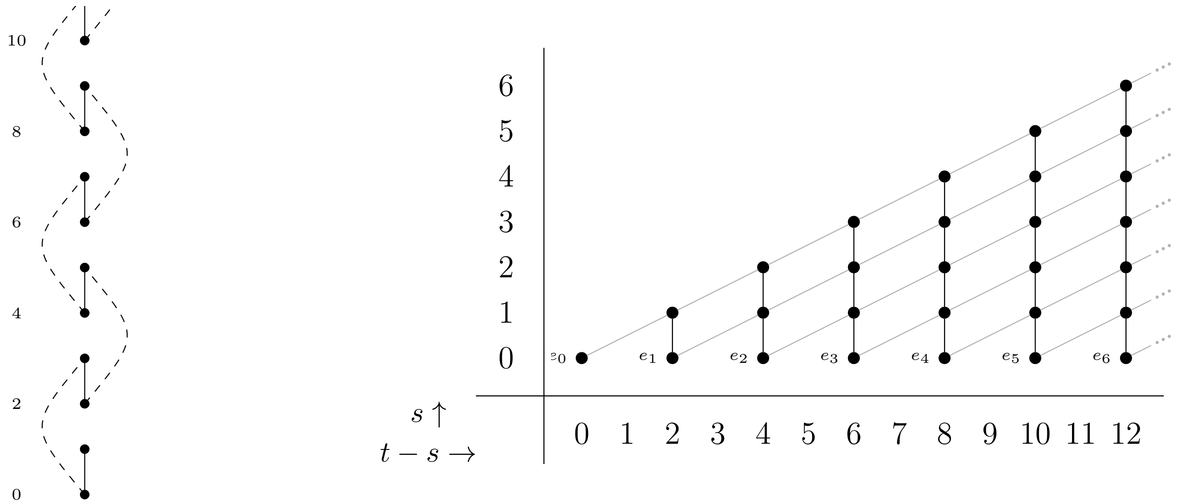

Figure 4.7: Left: the $\mathcal{E}(1)$-module structure on N_1; straight lines are Q_0-actions and dashed curved lines are Q_1-actions. Right: $\mathrm{Ext}_{\mathcal{E}(1)}(N_1)$, the E_2-page of the Adams spectral sequence for $ku \wedge (B\mathbb{Z}/2)^{\sigma-1}$. Gray diagonal lines denote v_1-actions. All differentials and extension questions are trivial. We discuss this Adams spectral sequence in Example 4.2.315. This figure adapted from [102, Figure 6, right].

Remark 4.2.319. Comparing $N_1^{\otimes n}$ as an $\mathcal{E}(1)$-module versus as an $\mathcal{A}(1)$-module, one discovers that the $\mathcal{A}(1)$-modules N_0, N_1, N_2, and N_3 are all stably equivalent as $\mathcal{E}(1)$-modules up to a shift. This can be verified directly (c.f. [462, Chapter 2] or [102, §4.4.4, §5.2.4]).

Proposition 4.2.320. *There is an isomorphism*

$$ku_n(ME_m)/(\text{bosonic summands}) \cong \begin{cases} \mathbb{Z}/2^{k+1}, & n-m=2k \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.321}$$

If $a \le 0$, $\mathbb{Z}/2^a$ is to be interpreted as the zero group.

Proof. Just as in the proof of Proposition 4.2.262, use Margolis' theorem to reduce to studying the Ext of the non-free summand of $H^*(ME_m;\mathbb{Z}/2)$. By Proposition 4.2.318, this amounts to just studying N_1 up to a shift, which we did in Example 4.2.315. ☐

Before we describe the image of the free-to-interacting map, we recall Anderson–Brown–Peterson's splitting of $MT\mathrm{Spin}^c$ from Example 2.2.48. It is a little simpler than the $MT\mathrm{Spin}$ version from Theorem 4.2.291.

Theorem 4.2.322 (Anderson–Brown–Peterson)**.** *There are numbers $a_i, b_j \in \mathbb{N}$ and a 2-local homotopy equivalence*

$$MT\mathrm{Spin}^c \xrightarrow{\simeq} \bigvee_{i\ge 0} \Sigma^{a_i} ku \vee \bigvee_{j \ge 0} \Sigma^{b_j} H\mathbb{Z}/2. \tag{4.2.323}$$

The degrees of the first few shifts are

$$\begin{aligned} (s_i) &= (0,4,8,8,12,12,12,16,16,16,16,16,\dots) \\ (t_j) &= (10,14,18,18,18,20,22,22,22,22,22,\dots) \end{aligned} \tag{4.2.324}$$

The projection $\mathrm{proj}_{s_1} : MT\mathrm{Spin}^c \to ku$ *onto the first summand is equivalent to the Atiyah–Bott–Shapiro map.*

See Abdallah–Salch [515] for discussion of the degrees (t_j).

Remark 4.2.325. The real Atiyah–Bott–Shapiro map is 7-connected, which means that in physics applications to IFTs of (spatial) dimension 6 and below, one can essentially ignore the difference between $MT\mathrm{Spin}$ and ko. However, because $s_2 = 4$ in the complex ABP splitting, the complex Atiyah–Bott–Shapiro map is only 3-connected, and therefore using ku instead of $MT\mathrm{Spin}^c$ misses IFTs beginning in dimension 3. This includes, for example, the example studied by Wang–Senthil in [516, §III.E].

We define a bosonic summand of an $MT\mathrm{Spin}^c$-module to be a bosonic summand of the underlying $MT\mathrm{Spin}$-module.[54]

Theorem 4.2.326. *The image of the free-to-interacting map* $\Sigma^{-m}KU \to I_{\mathbb{Z}}(MT\mathrm{Spin}^c \wedge ME_m)$ *is contained in the subgroup* $(I_{\mathbb{Z}}(ku \wedge ME_m))^* \subset (I_{\mathbb{Z}}(MT\mathrm{Spin}^c \wedge ME_m))^*$ *corresponding to the first factor in the Atiyah–Bott–Shapiro decomposition. Inside that subgroup, the free-to-interacting map is a surjection* $\mathbb{Z} \to \mathbb{Z}/2^{k+1}$ *onto the unique long summand in degrees* n *with* $n - m = 2k$ *and the zero map in degrees* $n - m = 2k + 1$*; the bosonic summands are not in the image.*

Proof sketch. The proof is quite similar to the proofs of Theorem 4.2.266 and Corollary 4.2.305. The only significant difference is a simplification: when studying the image of the F2I map $\Sigma^{-m}KU \to I_{\mathbb{Z}}(ku \wedge ME_m)$, instead of a two-step induction on k, then ℓ, we only need to induct on m. Therefore this proof includes analogues of Lemmas 4.2.268 and 4.2.271, but we do not need to adapt Lemma 4.2.273 or the arguments supporting it. ☐

Theorem 4.2.327. *The interacting Bott spiral map*

$$\mathrm{sp}_{\mathrm{int}}^{k,\ell} \colon I_{\mathbb{Z}}(MT\mathrm{Spin}^c \wedge ME_{m+1}) \to \Sigma I_{\mathbb{Z}}(MT\mathrm{Spin}^c \wedge ME_m)$$

is an isomorphism modulo bosonic summands.

The argument is completely analogous similar to Theorems 4.2.267 and 4.2.306.

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[54] One could try to generalize Stolz' analysis in [498] to maps $ku \to MT\mathrm{Spin}^c$ splitting the Atiyah–Bott–Shapiro map, and therefore obtain an principle narrower definition of a bosonic summand and therefore a sharper result on the Bott spiral. We felt this would digress too far from the present discussion.

Appendix A

An Example Long Exact Sequence in Bordism

This appendix is modified from [9, Appendix A], which is the subject of Chapter 3.

Here, we explicitly describe the Smith long exact sequence of bordism groups and work through an example. As in Section 3.4, let (X, ξ) be a stable tangential structure, let V be a virtual bundle over X, and let W be a real vector bundle over X of rank r. The corresponding long exact sequence of bordism groups of Corollary 3.7.6 is

$$\cdots \to \Omega_k^\xi(S_X(W)^{p^*V}) \xrightarrow{p} \Omega_k^\xi(X^V) \xrightarrow{\mathrm{sm}_W} \Omega_{k-r}^\xi(X^{V+W-r}) \xrightarrow{\delta} \Omega_{k-1}^\xi(S_X(W)^{p^*V}) \to \cdots \tag{A.0.1}$$

Here, $p\colon S_X(W) \to X$ is the projection, sm_W is the Smith homomorphism, and δ is the connecting map. In this section, we will be explicit about taking the pullback p^*V of V to $S_X(W)$.

Starting from the left, $\Omega_k^\xi(S_X(W)^{p^*V})$ is the bordism group of k-manifolds M equipped with a map $f\colon M \to S_X(W)$ together with a ξ-structure on $TM \oplus f^*(p^*V)$. Next, $\Omega_k^\xi(X^V)$ is the bordism group of k-manifolds equipped with a map to X with the analogous twisted ξ-structure, and $\Omega_{k-r}^\xi(X^{V+W-r})$ is the bordism group of $(k-r)$-manifolds N equipped with a map g to X with a ξ-structure on $TM \oplus g^*V \oplus g^*W$. Note that the Smith homomorphism lowers the dimension by r and twists the tangential structure condition by W.

Now we describe each map at the level of manifolds.

1. p: Let M be a closed k-manifold equipped with a map $h\colon M \to S_X(W)$ such that $TM \oplus h^*V$ has a ξ-structure, so that M represents a bordism class in $\Omega_k^\xi(S_X(W)^{p^*V})$. The image of M under p is represented by the same manifold M with an (X, V)-twisted ξ-structure given by the composition with the projection. That is, equip M with the map $M \xrightarrow{h} S_X(W) \xrightarrow{p} X$.

2. sm_W: Now let M be a closed k-manifold equipped with a map $f\colon M \to X$ such that $TM \oplus f^*V$ has a ξ-structure. Let $s\colon M \to W$ be a generic section, which is transverse to the zero section s_0. Then, the intersection $N \coloneqq s(M) \pitchfork s_0(M)$ is a $(k-r)$-dimensional manifold. Let g be the composite $g\colon N \hookrightarrow M \xrightarrow{f} X$. Since the normal bundle ν to N satisfies $\nu \cong f^*W|_N = g^*W$, $TM|_N \cong TN \oplus \nu \cong TN \oplus g^*W$, and hence N carries an

$(X, V+W)$-twisted ξ-structure coming from the (X, V)-twisted ξ-structure on M. We have $\mathrm{sm}_W \colon M \mapsto N$.

3. δ: This is the connecting map in the long exact sequence. Start with a closed $k-r$-manifold N with an $(X, V+W)$-twisted ξ-structure given by, as above, $g \colon N \to X$ and a ξ-structure on $TN \oplus g^*V \oplus g^*W$. Consider the sphere bundle $S_N(g^*W)$ of W restricted to N: it has a map to $S_X(W)$ given by inclusion.

   We claim that $S_N(g^*W)$ is the image under δ of N, but it remains to show that $S_N(g^*W)$ has the appropriate tangential structure. This will be a corollary of a general splitting result of tangent bundles of sphere bundles.

   Lemma A.0.2. *For any vector bundle $\pi \colon V \to B$, there is an isomorphism of vector bundles, canonical up to a contractible space of choices,*

   $$TS(V) \oplus \underline{\mathbb{R}} \xrightarrow{\cong} \pi^*(TB) \oplus \pi^*(V). \tag{A.0.3}$$

   Proof. Choose a metric and connection on V; both of these are contractible choices. For any fiber bundle $\pi \colon E \to B$ of smooth manifolds, the choice of connection splits TE as a direct sum of the horizontal subbundle, which is isomorphic to $\pi^*(TB)$, and the vertical tangent bundle $T_vE = \ker(\pi_*)$, which when pulled back to a fiber is the tangent bundle of that fiber.

   Let ν be the normal bundle of $S(V) \hookrightarrow V$. Then there is a canonical isomorphism $T_vS(V) \oplus \nu \cong \pi^*(V)$, which is a parametrized version of the standard isomorphism $TS^n \oplus \nu_{S^n \hookrightarrow \mathbb{R}^{n+1}} \cong \underline{\mathbb{R}}^{n+1}$. Combining this with the previous paragraph,

   $$TS(V) \oplus \nu \cong \pi^*(TB) \oplus T_vS(V) \oplus \nu \cong \pi^*(TB) \oplus \pi^*(V), \tag{A.0.4}$$

   and the fiberwise outward unit normal vector field trivializes ν. ☐

   If we now analyze the vertical and horizontal pieces of the tangent bundle to $S_N(g^*W)$, we find that $T(S_N(g^*W)) \oplus \underline{\mathbb{R}} \cong p^*TN \oplus p^*g^*W$. Then, we can pull back the relationship describing the tangential structure of N to see that $p^*TN \oplus p^*g^*W \oplus p^*g^*V$ over $S_N(g^*W)$ has a ξ-structure. So, $T(S_N(g^*W)) \oplus \underline{\mathbb{R}} \oplus p^*g^*V$ has a ξ-structure, and thus $S(g^*W)$ has a $(S_X(W), p^*V)$-twisted ξ-structure.

Let us now go through the long exact sequence of bordism groups for the Smith map 3.6.38. In this case, the Smith homomorphism is a map

$$\mathrm{sm}_{2\sigma} \colon \Omega_k^{\mathrm{Pin}^-} \longrightarrow \Omega_{k-2}^{\mathrm{Pin}^+} \tag{A.0.5}$$

between the bordism group of k-dimensional pin$^-$ manifolds to the bordism group of $(k-2)$-dimensional pin$^+$ manifolds, described by sending a pin$^-$ manifold M to any closed submanifold N whose homology class is Poincaré dual to $w_1(M)^2$. Alternatively, in view of Definition 3.2.8, we could define $\mathrm{sm}_{2\sigma}$ by choosing a section s of the pullback of 2σ to M transverse to the zero section, then letting N be the zero locus of s. Recall from Example 3.5.23 that a pin$^-$ structure is a trivialization of $w_1(M)^2 + w_2(M)$, while a pin$^+$

structure on M is equivalent to a trivialization of $w_2(M)$. Equivalently, a pin$^-$ manifold M admits a spin structure on $TM \oplus \det(M)$, while a pin$^+$ manifold M admits a spin structure on $TM \oplus 3\det(M)$. These conditions mean that if N is Poincaré dual to $w_1(M)^2$ inside a pin$^-$ manifold M, then N acquires a pin$^+$ structure.

The third set of groups in this long exact sequence corresponds to the homotopy groups of the fiber, $MT\mathrm{Spin} \wedge \Sigma^{-1}\mathbb{R}P^2$. By Pontrjagin-Thom, these are the groups $\widetilde{\Omega}^{\mathrm{Spin}}_{*+1}(\mathbb{RP}^2)$: bordism groups of spin manifolds X equipped with maps $f\colon X \to \mathbb{RP}^2$, modulo the subgroup for which f is null-homotopic. Equivalently, we may consider the twisted bordism groups $\Omega^{\mathrm{Spin}}_*(\mathbb{RP}^1, \sigma)$. Elements of this group are represented by manifolds N with maps $f\colon N \to \mathbb{RP}^1$ such that $TN \oplus f^*\sigma$ is spin.

We next describe the other two maps that appear alongside $\mathrm{sm}_{2\sigma}$ in the bordism long exact sequence and provide several lemmas that help us understand the geometry.

Definition A.0.6. Define a map $p\colon \Omega^{\mathrm{Spin}}_*(\mathbb{RP}^1, \sigma) \to \Omega^{\mathrm{Pin}^-}_*$ by sending $(N, f\colon N \to \mathbb{RP}^1)$ to N.

Lemma A.0.7. *If N has an $(\mathbb{RP}^1, \sigma)$-twisted spin structure, then N has a canonical pin$^-$ structure (so the map p lands in pin$^-$ bordism as claimed).*

Proof. The orientation of $TN \oplus f^*\sigma$ is equivalent data to an isomorphism $\mathrm{Det}(TN) \xrightarrow{\cong} f^*\sigma$, so we obtain a spin structure on $TN \oplus \mathrm{Det}(TN)$, i.e. a pin$^-$ structure. $\square$

The third map in the long exact sequence is the connecting map $\delta\colon \Omega^{\mathrm{Pin}^+}_* \to \Omega^{\mathrm{Spin}}_{*+1}(\mathbb{RP}^1, \sigma)$. The map δ sends a pin$^+$ manifold M to the total space of the sphere bundle $S(2\mathrm{Det}(TM))$.[1] The key to understanding δ is showing that $S(2\mathrm{Det}(TM))$ has a $(\mathbb{RP}^1, \sigma)$-twisted spin structure; in particular, we must cook up a map to $\mathbb{RP}^1$.

Definition A.0.8. Given a pin$^+$ manifold M, choose a metric on $\mathrm{Det}(TM)$ (a contractible choice); then, given $x \in M$ and $p, q \in \sigma_x$ with $\sqrt{|p^2| + |q^2|} = 1$, so that $(x, p, q) \in S(2\mathrm{Det}(TM))$, the two sections of $\pi^*(2\mathrm{Det}(TM))$

$$\begin{aligned} (x, p, q) &\longmapsto (p, q) \\ (x, p, q) &\longmapsto (-q, p) \end{aligned} \tag{A.0.9}$$

are everywhere linearly independent, so $\pi^*(2\mathrm{Det}(TM))$ is canonically trivial. This allows us to define a map $\varphi_M\colon S(2\mathrm{Det}(TM)) \to \mathbb{RP}^1$: given $(x, p, q) \in S(2\mathrm{Det}(TM))$ as above, $(p, q) \in (\pi^*(2\mathrm{Det}(TM)))_{(x,p,q)}$, which is canonically identified with $\mathbb{R}^2$, send (p, q) to its image $[p : q] \in \mathbb{RP}^1$ (using that p and q are never both 0).

Definition A.0.10. Let $\delta\colon \Omega^{\mathrm{Pin}^+}_* \to \Omega^{\mathrm{Spin}}_{*+1}(\mathbb{RP}^1, \sigma)$ be the map sending $M \mapsto (S(2\mathrm{Det}(TM)), \varphi_M)$, where φ_M is defined above in Definition A.0.8.

If $\sigma \to \mathbb{RP}^1$ is the Möbius bundle, then $\varphi_M^*(\sigma) = \pi^*(\mathrm{Det}(TM))$.

Lemma A.0.11. *If M is pin$^+$, $(S(2\mathrm{Det}(TM)), \varphi_M)$ has a canonical $(\mathbb{RP}^1, \sigma)$-twisted spin structure, up to a contractible space of choices, so that δ lands in $\Omega^{\mathrm{Spin}}_{*+1}(\mathbb{RP}^1, \sigma)$ as claimed.*

---

[1] Note that $S(2\mathrm{Det}(TM)) \simeq S(g^*(2\sigma))$.

Proof. Plugging in $V = 2\mathrm{Det}(TM)$ to Lemma A.0.2, we learn

$$TS(2\mathrm{Det}(TM)) \oplus \underline{\mathbb{R}} \cong \pi^*(TM) \oplus 2\pi^*(\mathrm{Det}(TM)). \tag{A.0.12a}$$

Since $\varphi_M^*(\sigma) \cong \pi^*(\mathrm{Det}(TM))$,

$$TS(2\mathrm{Det}(TM)) \oplus \varphi_M^*(\sigma) \oplus \underline{\mathbb{R}} \cong \pi^*(TM) \oplus 3\pi^*(\mathrm{Det}(TM)). \tag{A.0.12b}$$

Since M is pin$^+$, the right-hand-side of (A.0.12b) is spin, so the left-hand side is too; by two-out-of-three, this means $TS(2\mathrm{Det}(TM)) \oplus \varphi_M^*(\sigma)$ is also spin. ☐

The maps $\mathrm{sm}_{2\sigma}$, p, and δ assemble into a long exact sequence in bordism, as we will draw in Figure A.2. To write out this long exact sequence, we need to know the relevant bordism groups in low dimensions. Giambalvo [193, §2, §3] computes $\Omega_k^{\mathrm{Pin}^+}$ for $k \leq 12$, more than good enough for us, and gives generating manifolds in all degrees we need except $k = 2, 3$ (though see [196] for a correction); the rest were given by Kirby-Taylor [110, Proposition 3.9, Theorem 5.1]. Anderson-Brown-Peterson [192, Theorem 5.1] computed pin$^-$ bordism groups, with generating manifolds again described by Giambalvo [193, Theorem 3.4] and Kirby-Taylor [110, Theorem 2.1]. However, the twisted spin bordism of $\mathbb{RP}^1$ is less well-documented, so we calculate it here, using another Smith homomorphism.

Lemma A.0.13. *There is an abelian group A of order 4 such that*

$$\Omega_k^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong \begin{cases} \mathbb{Z}/2, & k = 0, 1, 3, 4 \\ A, & k = 2 \\ 0, & k = 5. \end{cases} \tag{A.0.14}$$

Proof. We may start the computation of $\Omega^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ using the observation of Kirby and Taylor [110] that the degree two map

$$\mathbb{S} \xrightarrow{\cdot 2} \mathbb{S} \longrightarrow \Sigma_+^{\infty-1}\mathbb{RP}^2 \tag{A.0.15}$$

of Example 3.6.13 induces multiplication by two on spin bordism. Taking the spin bordism long exact sequence of A.0.15 and inputting the spin bordism of a point, we may deduce the groups $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ in low dimensions, up to one ambiguity, as indicated in Figure A.1. ☐

Remark A.0.16. To address the question as to whether A is isomorphic to $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, one could appeal to geometric arguments or an Adams spectral sequence calculation, but it turns out that the Smith long exact sequence that we will study in Figure A.2 provides a cleaner argument that $A \cong \mathbb{Z}/4$.

We will provide some explicit descriptions of the interesting maps in this sequence using knowledge of the generators of each bordism group, which for pin$^+$ and pin$^-$ may be found in [110]. For the twisted spin bordism of $\mathbb{RP}^1$, we use what we learned in Lemma A.0.13.

(a) $* = 0$: The group $\Omega_0^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong \mathbb{Z}/2$ is generated by the class of the point equipped with the inclusion i into $\mathbb{RP}^1$. The condition of $T\mathrm{pt} \oplus i^*\sigma$ being spin is satisfied since $i^*\sigma$ is trivial. The map f forgets i, so sends this generator to the point with its pin$^-$ structure, which is a generator of $\Omega_0^{\mathrm{Pin}^-} \cong \mathbb{Z}/2$.

$$\begin{tikzcd}
* & \Omega_*^{\mathrm{Spin}} & \Omega_*^{\mathrm{Spin}} & \Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \\
5 & 0 & 0 & 0 \\
4 & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \\
3 & 0 & 0 & \mathbb{Z}/2 \arrow[dll, rounded corners] \\
2 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & A \arrow[dll, rounded corners] \\
1 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[dll, rounded corners] \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2
\end{tikzcd}$$

Figure A.1: Long exact sequence in spin bordism partially determining $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$

(b) $* = 1$: Consider the circle with spin structure induced from its Lie group framing, denoted S^1_{nb}, equipped with the degree two map $\phi\colon S^1 \to S^1 \simeq \mathbb{RP}^1$. If $x \in H^1(\mathbb{RP}^1; \mathbb{Z}/2)$ is the generator, we have

$$w(TS^1 \oplus \phi^*\sigma) = w(TS^1)\phi^* w(\sigma) = (1)(1 + 2\phi^*(x)) = 1, \tag{A.0.17}$$

so (S^1_{nb}, ϕ) has an $(\mathbb{RP}^1, \sigma)$-twisted spin structure. The map p forgets ϕ, so sends the bordism class of (S^1_{nb}, ϕ) to S^1_{nb}, which generates $\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}/2$ [110, Theorem 2.1].

(c) $* = 2$ (part 1): Exactness of the Smith long exact sequence at $\Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong A$ implies that A maps injectively to $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$, so $A \cong \mathbb{Z}/4$, and we have resolved the extension problem from Lemma A.0.13.

The Klein bottle K is an S^1-bundle over $\mathbb{RP}^1$, with the monodromy of the fiber S^1 around the base given by reflection. Therefore $K = S(\sigma \oplus \underline{\mathbb{R}})$ as S^1-bundles over $\mathbb{RP}^1$. Let $\pi\colon K \to \mathbb{RP}^1$ be the bundle map; then Lemma A.0.2 defines an isomorphism $TK \oplus \underline{\mathbb{R}} \cong \pi^*(\sigma) \oplus \underline{\mathbb{R}}^2$ (using the Lie group trivialization of $T\mathbb{RP}^1$). The Möbius bundle σ represents the nonzero class in $[\mathbb{RP}^1, B\mathrm{O}] = \pi_1(B\mathrm{O}) \cong \mathbb{Z}/2$, so 2σ is trivializable,[2] and in particular spin, meaning that (K, π) admits an $(\mathbb{RP}^1, \sigma)$-twisted spin structure (in fact, it admits 4).

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[2] To make this argument carefully, one must know that addition in $[S^1, B\mathrm{O}]$ corresponds to direct sum of vector bundles. A priori this is not true—addition in $[S^1, X]$ is built from the pinch map $S^1 \to S^1 \vee S^1$. That this coincides with the group structure on $[S^1, B\mathrm{O}]$ arising from direct sum of virtual vector bundles depends on the Eckmann-Hilton argument.

$$\begin{tikzcd}
* & \Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma) & \Omega_*^{\mathrm{Pin}^-} & \Omega_{*-2}^{\mathrm{Pin}^+} \\
6 & 0 & \mathbb{Z}/16 \arrow[r, "\text{(g)}"] & \mathbb{Z}/16 \\
5 & 0 & 0 & \mathbb{Z}/2 \arrow[dll, "\text{(f)}"'] \\
4 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \arrow[dll, "\text{(e)}"'] \\
3 & \mathbb{Z}/2 & 0 & 0 \\
2 & \mathbb{Z}/4 \arrow[r, "\text{(c)}"] & \mathbb{Z}/8 \arrow[r, "\text{(d)}"] & \mathbb{Z}/2 \\
1 & \mathbb{Z}/2 \arrow[r, "\text{(b)}"] & \mathbb{Z}/2 & 0 \\
0 & \mathbb{Z}/2 \arrow[r, "\text{(a)}"] & \mathbb{Z}/2 & 0
\end{tikzcd}$$

Figure A.2: Bordism Long Exact Sequence for $\mathrm{Pin}^- \rightsquigarrow \mathrm{Pin}^+$

That (K,π) generates $\Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)$ depends on which of the four $(\mathbb{RP}^1,\sigma)$-twisted spin structures one chooses. Specifically, each $(\mathbb{RP}^1,\sigma)$-twisted spin structure restricts to a spin structure on the fiber S^1, and we need this to be the spin structure on S^1 induced by the Lie group framing. Two of the four $(\mathbb{RP}^1,\sigma)$-twisted spin structures satisfy this. To then see that either of these two Klein bottles generates, one can play with the Smith long exact sequence from Example 3.6.13

$$\cdots \longrightarrow \Omega_k^{\mathrm{Spin}} \xrightarrow{\cdot 2} \Omega_k^{\mathrm{Spin}} \longrightarrow \Omega_k^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma) \xrightarrow{\mathrm{sm}_\sigma} \Omega_{k-1}^{\mathrm{Spin}} \longrightarrow \cdots \tag{A.0.18}$$

to see that $\mathrm{sm}_\sigma\colon \Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma) \to \Omega_1^{\mathrm{Spin}}$ is the unique surjective map $\mathbb{Z}/4 \to \mathbb{Z}/2$; the Poincaré dual to $w_1(\sigma)$ is represented by the fiber S^1 in K, which we chose to have the Lie group spin structure, so $\mathrm{sm}_\sigma(K,\pi) = S^1_{nb}$, which generates Ω_1^{Spin}, implying (K,π) generates $\Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)$.

Now take $f(K,\pi)$, which amounts to forgetting π and finding the pin$^-$ bordism class of K. The Arf-Brown-Kervaire invariant is a complete invariant $\Omega_2^{\mathrm{Pin}^-} \xrightarrow{\cong} \mathbb{Z}/8$ [110, 451], so it suffices to compute this invariant on K, as has been explicitly worked out in [453, §II.D]. Our choice of the nonbounding spin structure on the fiber implies that the Arf-Brown-Kervaire map $\Omega_2^{\mathrm{Pin}^-} \xrightarrow{\cong} \mathbb{Z}/8$ sends $[K] \mapsto \pm 2$, so $f\colon \mathbb{Z}/4 \to \mathbb{Z}/8$ sends $1 \mapsto 2$, as required by exactness.

(d) $* = 2$ (part 2): There are two pin$^-$ structures on $\mathbb{RP}^2$, and both are generators of $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ [110, §3]. Pick either of these pin$^-$ structures; the class $w_2(\sigma) \in H^2(\mathbb{RP}^2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ is a generator, and the Smith homomorphism $\Omega_2^{\mathrm{Pin}^-} \to \Omega_0^{\mathrm{Pin}^+}$ maps the input $\mathbb{RP}^2$ to the Poincaré dual of $w_2(2\sigma)$. The class $PD(w_2(2\sigma))$ is $1 \in H_0(\mathbb{RP}^2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ and is represented by a single pin$^+$ point. The class of the point also corresponds to the zero-dimensional intersection of the zero section and a generic section of 2σ.

(e) $* = 4 \to 3$: $\Omega_2^{\mathrm{Pin}^+} \cong \mathbb{Z}/2$ is generated by the Klein bottle K, where as before we need the nonbounding spin structure on the S^1 fiber of K. The connecting map δ sends K to $S(2\mathrm{Det}(K))$; we saw above in part (c) that $\mathrm{Det}(K) \cong \sigma$ and 2σ is trivialized over K, so $S(2\mathrm{Det}(K)) \cong S^1 \times K$.

Tracking the (twisted) spin structures through this argument, one sees that we obtain the nonbounding spin structure on S^1, so $g(K) = [S^1_{nb} \times K] \in \Omega_3^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong \mathbb{Z}/2$, and $[S^1_{nb} \times K]$ is indeed the generator.[3]

(f) $* = 5 \to 4$: $\Omega_3^{\mathrm{Pin}^+} \cong \mathbb{Z}/2$ is generated by $S^1_{\mathrm{nb}} \times K$ [110, §5], and $\Omega_4^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong \mathbb{Z}/2$ is generated by $S^1_{nb} \times S^1_{nb} \times K$, with the map to $\mathbb{RP}^1$ induced from the fiber bundle $K \to \mathbb{RP}^1$ from part (c).[4] Thus the story is the same as in (e), crossed with S^1_{nb}.

(g) $* = 6$: The group $\Omega_6^{\mathrm{Pin}^-}$ is generated by $\mathbb{RP}^6$ with either of its two pin$^+$ structures, while $\Omega_4^{\mathrm{Pin}^+}$ is generated by $\mathbb{RP}^4$ with either of its two pin$^-$ structures. Since the normal bundle to $\mathbb{RP}^4$ inside $\mathbb{RP}^6$ is indeed the restriction of 2σ, $\mathbb{RP}^4$ represents the Poincaré dual homology class to $e(2\sigma)$ and is the image of the Smith homomorphism applied to $\mathbb{RP}^6$.

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[3] Another way to see this is that because the connecting morphism in the Smith long exact sequence is obtained from a map of spectra by taking homotopy groups, the connecting morphism commutes with the $\pi_*(\mathbb{S})$-actions on $\Omega_*^{\mathrm{Pin}^+}$ and $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$. The Pontrjagin-Thom theorem identifies this $\pi_*(\mathbb{S})$-action on bordism groups with taking products with stably framed manifolds; focusing specifically on the nonzero element of $\pi_1(\mathbb{S})$, which is represented by the bordism class of S^1_{nb}. Thus, since $\times S^1_{nb} \colon \Omega_2^{\mathrm{Pin}^+} \to \Omega_3^{\mathrm{Pin}^+}$ is an isomorphism [110, §5] and the Smith maps $\Omega_{k-2}^{\mathrm{Pin}^+} \to \Omega_k^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ are isomorphisms for $k = 3, 4$ as we saw in the long exact sequence, then $\times S^1_{nb} \colon \Omega_3^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \to \Omega_4^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ is also an isomorphism.

[4] Another choice of generator is the K3 surface with trivial map to $\mathbb{RP}^1$, as follows from (A.0.18). The complicated topology of the K3 surface makes this generator harder to work with explicitly.

Appendix B

Calculation of the twisted ABS map $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \to \mathbb{Z}_2$

Our goal in this appendix is to explicitly calculate Freed-Hopkins' twisted Atiyah-Bott-Shapiro map in dimension 4 in class $s = -2$. We use this calculation in Example 4.1.59.

Freed-Hopkins' original calculation of this free-to-interacting map in [1, §10] is purely homotopy-theoretical, coming from an Adams spectral sequence computation. We make a more concrete and less technical calculation, which additionally results in an explicit understanding of the manifold generators of the relevant bordism groups. Specifically, we use the long exact sequence associated to a *Smith homomorphism*, following a general theory worked out in [517]. See [197, 252, 436, 518–521] for more examples of calculations applying this technique.

Remark B.0.1. Freed-Hopkins' original definition of the twisted Atiyah-Bott-Shapiro map is index-theoretic, as we review in §4.1.2.4.1. It therefore seems reasonable that there should be a description of the map $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \to KO_2 \cong \mathbb{Z}_2$ as a mod 2 index of the Dirac operator from Example 4.1.44. We would be interested in learning whether it is possible to prove Theorem B.0.9 by calculating this mod 2 index on a generating set for $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}}$.

Definition B.0.2 (Hason-Komargodski-Thorngren [265, §4.1])**.** Let $V \to X$ be a virtual vector bundle. An (X, V)*-twisted spin structure* on a vector bundle $E \to M$ is data of a map $f\colon M \to X$ and a spin structure on $E \oplus f^*(V)$.

Given a fermionic group G_f, there is often a virtual vector bundle $V \to BG_b$ such that the tangential structure associated to G as defined in §4.1.2.3.2 is equivalent to a (BG_b, V)-twisted spin structure. This occurs for all fermionic groups we consider in this paper; that it is not true in general follows from work of Gunawardena-Kahn-Thomas [141]. We will need the following three examples.

Lemma B.0.3 (Freed-Hopkins [1, §10])**.**

1. *$\mathrm{Pin}^{\tilde{c}+}$ structures are naturally equivalent to $(B\mathrm{O}(2), -V)$-twisted spin structures, where $V \to B\mathrm{O}(2)$ is the tautological bundle.*

2. *Pin^{+} structures are naturally equivalent to $(B\mathrm{O}(1), -\sigma)$-twisted spin structures, where $\sigma \to B\mathrm{O}(1)$ is the tautological bundle.*

3. *Spin^c structures are naturally equivalent to $(B\mathrm{U}(1), -L)$-twisted spin structures, where $L \to B\mathrm{U}(1)$ is the tautological complex line bundle.*

Remark B.0.4 (Alternate characterizations). Stolz [143, §6] showed that pin$^+$ structures are naturally equivalent to $(B\mathrm{O}(1), 3\sigma)$-twisted spin structures, and it is implicit in Stong [218, Chapter XI] that spin^c structures are $(B\mathrm{U}(1), L)$-twisted spin structures.

The pullback of $V \to B\mathrm{O}(2)$ along the standard inclusion $B\mathrm{O}(1) \to B\mathrm{O}(2)$ is isomorphic to $\sigma \oplus \underline{\mathbb{R}}$, which means that a pin$^+$ structure on a vector bundle $E \to X$ induces a pin$^{\tilde{c}+}$ structure: a spin structure on $E - f^*\sigma$, where f is some map $X \to B\mathrm{O}(1)$, is equivalent data to a spin structure on $E - f^*\sigma \oplus \underline{\mathbb{R}}$.

Similarly, the pullback of $V \to B\mathrm{O}(2)$ along the map $B\mathrm{U}(1) \to B\mathrm{O}(2)$ induced by the inclusion $\mathrm{U}(1) \cong \mathrm{SO}(2) \hookrightarrow \mathrm{O}(2)$ is L. Thus, analogously to the way a pin$^+$ structure defines a pin$^{\tilde{c}+}$ structure, a spin^c structure also induces a pin$^{\tilde{c}+}$ structure. In particular, complex manifolds have canonical pin$^{\tilde{c}+}$ structures induced from their canonical spin^c structures.

Definition B.0.5. It follows from Lemma B.0.3, part (3), that a spin^c structure on a manifold M is equivalent data to a complex line bundle $L \to M$ and a spin structure on $TM \oplus L$. If M is an almost complex manifold, this is equivalent to the condition $c_1(TM \oplus L) \bmod 2 = 0$, i.e. $c_1(M) \equiv c_1(L) \bmod 2$ by the Whitney sum formula. Since $c_1(M) = c_1(\mathrm{Det}_{\mathbb{C}}(TM))$, we can also use the determinant bundle to characterize spin^c structures.

There is a canonical isomorphism $H^2(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}$ sending $\mathcal{O}(m) \mapsto m$, and $\mathrm{Det}_{\mathbb{C}}(T\mathbb{CP}^n) \cong \mathcal{O}(-(n+1))$, so a spin^c structure on $\mathbb{CP}^n$ is equivalent data to an integer m such that $m \equiv n+1 \bmod 2$: then $TM \oplus \mathcal{O}(m)$ admits a spin structure, and its spin structures are a torsor over $H^1(\mathbb{CP}^n; \mathbb{Z}/2) = 0$, so this spin structure is unique. We let $(\mathbb{CP}^n, m)$ denote the spin^c manifold $\mathbb{CP}^n$ with the spin^c structure defined by $\mathcal{O}(m)$ in this way.

Thus the spin^c structure on $\mathbb{CP}^n$ induced by its complex structure is $(\mathbb{CP}^n, -(n+1))$. If we refer to $\mathbb{CP}^n$ as a spin^c manifold without clarifying, we mean this structure.

Lemma B.0.6 (Freed-Hopkins [1, Theorem 9.87])**.** *There is an isomorphism $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \cong \mathbb{Z}_2^3$.*

Proposition B.0.7. *The bordism classes of the following three manifolds are linearly independent in $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}}$, and therefore form a basis.*

1. *$\mathbb{RP}^4$, with pin$^{\tilde{c}+}$ structure induced from either of its two pin$^+$ structures.*
2. *$\mathbb{CP}^2$, with pin$^{\tilde{c}+}$ structure induced from the spin^c structure $(\mathbb{CP}^2, -1)$.*
3. *$\mathbb{CP}^1 \times \mathbb{CP}^1$, with pin$^{\tilde{c}+}$ structure induced from its complex structure.*

Proof. The fact that the bordism classes of $\mathbb{RP}^4$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ are linearly independent in $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}}$ is shown in [519, Proposition A.25]. Thus it suffices to find a bordism invariant $\xi\colon \Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \to \mathbb{Z}_2$ which vanishes on $\mathbb{RP}^4$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$, but is nonzero on $(\mathbb{CP}^2, -1)$.

Given a pin$^{\tilde{c}+}$ manifold X, let $E \to X$ denote the rank-2 vector bundle associated to the pin$^{\tilde{c}+}$ structure: by Lemma B.0.3, part (1), a pin$^{\tilde{c}+}$ structure is a $(B\mathrm{O}(2), -V)$-twisted spin structure, so we have a map $f\colon X \to B\mathrm{O}(2)$, and $E := f^*(V)$. By a standard argument due to Pontryagin [522] (see Milnor-Stasheff [523, Theorem 4.9]), $\xi\colon (X, E) \mapsto \int_X w_2(E)^2$ is a bordism invariant $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \to \mathbb{Z}_2$.

If the $\mathrm{pin}^{\tilde{c}+}$ structure on X is induced from a pin^+ structure, then as discussed above the pullback map of E factors through $B\mathrm{O}(1)$ and therefore $E \cong L \oplus \underline{\mathbb{R}}$ for some real line bundle L. Thus in this case $w_2(E) = 0$, so $\xi(\mathbb{RP}^4) = 0$.

To show $\xi(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$, we use that since the $\mathrm{pin}^{\tilde{c}+}$ structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is induced from its complex structure,

$$\begin{aligned}\xi(\mathbb{CP}^1 \times \mathbb{CP}^1) &= \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} w_2(\mathrm{Det}_{\mathbb{C}}(T(\mathbb{CP}^1 \times \mathbb{CP}^1)) \\ &= \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} c_1(\mathrm{Det}_{\mathbb{C}}(T(\mathbb{CP}^1 \times \mathbb{CP}^1)) \bmod 2 \\ &= \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} c_1(\mathbb{CP}^1 \times \mathbb{CP}^1) \bmod 2.\end{aligned} \tag{B.0.8}$$

Since $\mathbb{CP}^1 \cong S^2$ has a spin structure, so does $\mathbb{CP}^1 \times \mathbb{CP}^1$, and therefore its first Chern class is even, so $\xi(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$.

For $(\mathbb{CP}^2, -1)$, $E = \mathcal{O}(-1)$, which has odd Chern class, so $w_2(E) \neq 0$. Since $H^*(\mathbb{CP}^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[a]/(a^2)$ with $|a| = 2$, then as soon as we know $w_2(E) \neq 0$ we see $w_2(E)^2$ is the unique nonzero element of $H^4(\mathbb{CP}^2; \mathbb{Z}_2)$, so $\xi(\mathbb{CP}^2, -1) = 1$. ☐

Now that we know a set of generators, we can state the main theorem of this appendix, which is the calculation of the twisted Atiyah-Bott-Shapiro map on these generators.

Theorem B.0.9. *The twisted Atiyah-Bott-Shapiro map* $\mathrm{ABS}_{-2}\colon \Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \to KO_2 \cong \mathbb{Z}_2$ *sends* $[\mathbb{RP}^4] \mapsto 0$, $[\mathbb{CP}^2, -1] \mapsto 0$, *and* $[\mathbb{CP}^1 \times \mathbb{CP}^1] \mapsto 1$.

The key fact that enables us to get at ABS_{-2} is:

Proposition B.0.10 (Freed-Hopkins [1, Proposition 10.27])**.** *For* $-3 \leq s \leq -1$, *the twisted Atiyah-Bott-Shapiro map* ABS_s *factors as*

$$\Omega_n^{H(s)} \xrightarrow{\mathrm{sm}_V} \Omega_{n+s}^{\mathrm{Spin}}(B\mathrm{O}(-s)) \xrightarrow{c} \Omega_{n+s}^{\mathrm{Spin}} \xrightarrow{\mathrm{ABS}_0} KO_{n+s}, \tag{B.0.11}$$

where $V \to B\mathrm{O}(-s)$ *is the tautological bundle,* sm_V *is the* Smith homomorphism *defined by taking a manifold representative of the Poincaré dual of the Euler class of* V*, and* c *is the map forgetting the data of the map to* $B\mathrm{O}(-s)$*.*

Remark B.0.12. Freed-Hopkins do not define their map in exactly this way. Instead of sm_V, they use the map defined by the zero section of the tautological bundle; see [517, Proposition 3.17] for a proof identifying this with the Smith homomorphism. Instead of $\mathrm{ABS}_0 \circ c$, they tensor ABS_0 with a map to KO-theory corresponding to the trivial line bundle on $B\mathrm{O}(-s)$, but the trivial line bundle pulls back from the point so we may use the forgetful map c.

Remark B.0.13. The Euler class mentioned in Proposition B.0.10 is not the usual Euler class, but an analogue in the spin cobordism generalized cohomology theory. This Euler class has subtle behavior and can be tricky to calculate: see [517, Appendix B]. For this reason, we will for the most part calculate the Smith homomorphism indirectly in this section.

The Smith homomorphisms in Proposition B.0.10 can be fit into long exact sequences which often can be explicitly computed. Focusing again on $s = -2$, we have:

Proposition B.0.14. *There is a long exact sequence*

$$\cdots \to \Omega_4^{\mathrm{Pin}^+} \xrightarrow{i} \Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \xrightarrow{\mathrm{sm}_V} \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \xrightarrow{\delta} \Omega_3^{\mathrm{Pin}^+} \to \cdots \tag{B.0.15}$$

where i is the map on bordism corresponding to the induced pin$^{\tilde{c}+}$ structure on a pin$^+$ manifold described above and δ applied to the bordism class of a spin manifold Σ and a rank-2 vector bundle $E \to \Sigma$ is the bordism class of the sphere bundle $S(E)$ with a certain pin$^+$ structure.

Proof. Let $E \to X$ be a virtual vector bundle and $F \to X$ be a vector bundle of rank r. Let $p\colon S(F) \to X$ be the sphere bundle of F. Introduce the following three tangential structures:

1. a ξ-structure is a $(S(F), p^*(E))$-twisted spin structure,
2. a η-structure is an (X, E)-twisted spin structure, and
3. a ζ-structure is an $(X, E \oplus F)$-twisted spin structure.

Then [517, Corollary 5.8] there is a long exact sequence

$$\cdots \to \Omega_n^{\xi} \xrightarrow{p_*} \Omega_n^{\eta} \xrightarrow{\mathrm{sm}_F} \Omega_{n-r}^{\zeta} \xrightarrow{\delta} \Omega_{n-1}^{\xi} \to \cdots, \tag{B.0.16}$$

called the *Smith long exact sequence*, where sm_F is the Smith homomorphism associated to F and δ is induced by taking the sphere bundle of the pullback of F with a certain ξ-structure.

Let $X = B\mathrm{O}(2)$ and $V \to B\mathrm{O}(2)$ denote the tautological bundle. Then let $E = -V$ and $F = V$, so that a ζ-structure is a spin structure with a map to $B\mathrm{O}(2)$ and, by Lemma B.0.3, a η-structure is equivalent to a pin$^{\tilde{c}+}$ structure.

There is a homotopy equivalence $S(V) \simeq B\mathrm{O}(1)$ such that the bundle map $p\colon S(V) \to B\mathrm{O}(2)$ is identified with $i\colon B\mathrm{O}(1) \to B\mathrm{O}(2)$,[1] so a ξ-structure is a $(B\mathrm{O}(1), -i^*(V))$-twisted spin structure; as noted above, this is equivalent to a $(B\mathrm{O}(1), -\sigma)$-twisted spin structure and therefore by Lemma B.0.3 a pin$^+$ structure. This finishes the identification of this Smith long exact sequence with the one in the theorem statement. ☐

Corollary B.0.17. *For any closed pin$^+$ 4-manifold X, $\mathrm{ABS}_{-2}(X) = 0$. In particular, $\mathrm{ABS}_{-2}(\mathbb{RP}^4) = 0$.*

Proof. Exactness of (B.0.15) implies $\mathrm{sm}_V \circ i = 0$, so $\mathrm{sm}_V(X) = 0$; Proposition B.0.10 tells us that ABS_{-2} factors through sm_V, so $\mathrm{ABS}_{-2}(X) = 0$ too. ☐

That's one-third of Theorem B.0.9 right there!

For $(\mathbb{CP}^2, -1)$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ we have to perform a more detailed analysis: $\mathbb{CP}^2$ has no pin$^+$ structure (as that combined with an orientation would be a spin structure, but $\mathbb{CP}^2$ is not spin), and though $\mathbb{CP}^1 \times \mathbb{CP}^1$ has a pin$^+$ structure, that structure does not induce the pin$^{\tilde{c}+}$ structure we use in this section.

Definition B.0.18. Define maps $\varphi_1, \varphi_2, \varphi_3\colon \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \to \mathbb{Z}_2$ as follows on a closed spin 2-manifold Σ with rank-2 vector bundle $E \to \Sigma$.

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[1] This is a standard result; one non-original reference is [517, Example 7.57].

1. $\varphi_1 = \mathrm{ABS}_0 \circ c$, as in Proposition B.0.10.

2. φ_2 is the composition

$$\Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \xrightarrow{\det} \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(1)) \xrightarrow{\mathrm{sm}_\sigma} \Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}_2, \tag{B.0.19}$$

where det is induced from the determinant map $\mathrm{O}(2) \to \mathrm{O}(1)$, sm_σ is the Smith homomorphism introduced by Anderson-Brown-Peterson [192], which takes a manifold representative of the Poincaré dual of the Euler class of the principal $\mathrm{O}(1)$-bundle, and the isomorphism $\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}_2$ was established by (*ibid.*, Theorem 5.1).

3. $\varphi_3(\Sigma, E) = \int_\Sigma w_2(E)$.

Proposition B.0.20. *The following map is an isomorphism:*

$$\boldsymbol{\varphi} \coloneqq (\varphi_1, \varphi_2, \varphi_3)\colon \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \xrightarrow{\cong} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2. \tag{B.0.21}$$

The bordism classes of the following manifolds form the basis for $\Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2))$ dual to $(\varphi_1, \varphi_2, \varphi_3)$.

- $(S^1_{nb} \times S^1_{nb}, \underline{\mathbb{R}}^2)$*, where S^1_{nb} refers to the nonbounding spin structure on the circle. $\boldsymbol{\varphi}(S^1_{nb} \times S^1_{nb}, \underline{\mathbb{R}}^2) = (1, 0, 0)$.*
- $(S^1_{nb} \times S^1_b, \sigma_R \oplus \underline{\mathbb{R}})$*, where S^1_b refers to the bounding spin structure on the circle and $\sigma_R \to S^1_{nb} \times S^1_b$*[2] *is the pullback of the Möbius bundle $\sigma \to S^1$ by the projection onto the second factor of $S^1_{nb} \times S^1_b$. $\boldsymbol{\varphi}(S^1_{nb} \times S^1_b, \sigma_R \oplus \underline{\mathbb{R}}) = (0, 1, 0)$.*
- $(\mathbb{CP}^1, \mathcal{O}(1))$*: $\boldsymbol{\varphi}(\mathbb{CP}^1, \mathcal{O}(1)) = (0, 0, 1)$.*

Proof. Mitchell-Priddy [524, Theorem C] show that, modulo odd-primary torsion, for any generalized homology theory h_*, there is a natural map $\psi_1\colon h_*(B\mathrm{O}(2)) \to h_*(B\mathrm{SO}(3))$ and an isomorphism

$$(c, \psi_1, \psi_2, \det)\colon h_n(B\mathrm{O}(2)) \xrightarrow{\cong} h_n(\mathrm{pt}) \oplus \widetilde{h}_n(B\mathrm{SO}(3)) \oplus h_n(L(2)) \oplus \widetilde{h}_n(B\mathrm{O}(1)) \tag{B.0.22}$$

for a certain spectrum $L(2)$ and map $\psi_2\colon B\mathrm{O}(2) \to L(2)$. Bayen [490, §3.5.3, §3.6.3] shows $\Omega_k^{\mathrm{Spin}}(L(2))$ vanishes in degrees 3 and below, so we will not need to worry about this factor. Wan-Wang [253, §5.5.3] show $\widetilde{\Omega}_2^{\mathrm{Spin}}(B\mathrm{SO}(3))) \cong \mathbb{Z}_2$, and Anderson-Brown-Peterson [192] show $\widetilde{\Omega}_2^{\mathrm{Spin}}(B\mathrm{O}(1)) \cong \mathbb{Z}_2$. The additional hypothesis on odd-primary torsion can be removed: Randal-Williams [525, §5.1] shows that the odd-primary torsion in $\widetilde{h}_*(B\mathrm{O}(2))$ coincides with that of a 3-connected space, meaning there can be none in degree 2. Thus we have the abstract isomorphism (B.0.21) and the fact that φ_1 and φ_2 are linearly independent, but we still need to address φ_3. The integral of a stable characteristic class is a bordism invariant by an argument of Pontryagin [522] (see also Milnor-Stasheff [523, Theorem 4.9]), so φ_3 indeed defines a map $\Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \to \mathbb{Z}_2$; we need to show this map is linearly independent of φ_1 and φ_2. To do so, we will calculate $\boldsymbol{\varphi}$ on the three surfaces in the theorem statement.

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[2] The "R" in σ_R refers to the **R**ight-hand factor of S^1.

- φ_2 and φ_3 by definition vanish on trivial bundles, so $\boldsymbol{\varphi}(S^1_{nb} \times S^1_{nb}, \underline{\mathbb{R}}^2) = (?, 0, 0)$; for the value of φ_1 observe that $\mathrm{ABS}_0 \circ c$ is the Arf invariant, which equals 1 on $S^1_{nb} \times S^1_{nb}$.

- For $(S^1_{nb} \times S^1_b, \sigma_R \oplus \underline{\mathbb{R}})$, we have $S^1_{nb} \times S^1_b = \partial(S^1_{nb} \times D^2)$, so c kills this manifold and therefore φ_1 does too. For φ_2, $\mathrm{Det}(\sigma_R \oplus \underline{\mathbb{R}}) \cong \sigma_R$. This bundle is trivializable when restricted to $S^1_{nb} \times \{x\} \subset S^1_{nb} \times S^1_b$ for any $x \in S^1_b$, which means S^1_{nb} is Poincaré dual to the Euler class of σ_R and therefore $\mathrm{sm}_\sigma(S^1_{nb} \times S^1_b, \sigma_R) = S^1_{nb}$, whose class in $\Omega_1^{\mathrm{Pin}^-}$ is nonzero [110, Theorem 2.1]. Thus $\varphi_2(S^1_{nb} \times S^1_b, \sigma_R \oplus \underline{\mathbb{R}}) = 1$. For φ_3, $w_2(\sigma_R \oplus \underline{\mathbb{R}}) = w_2(\sigma_R) = 0$, because σ_R is a line bundle.

- Finally, $(\mathbb{CP}^1, \mathcal{O}(1))$: since $\mathbb{CP}^1 \cong S^2$ is simply connected, it has a unique spin structure, which bounds D^3 and therefore has trivial Arf invariant, so $\varphi_1(\mathbb{CP}^1, \mathcal{O}(1)) = 0$. Since $\mathcal{O}(1))$ is complex, it is oriented, so its real determinant bundle vanishes, and therefore $\varphi_2(\mathbb{CP}^1, \mathcal{O}(1)) = 0$. Since $c_1(\mathcal{O}(1)) \mapsto 1$ under the isomorphism $H^2(\mathbb{CP}^1; \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$ defined by the orientation induced by the complex structure, $w_2(\mathcal{O}(1)) = c_1(\mathcal{O}(1)) \bmod 2$ is the nonzero element of $H^2(\mathbb{CP}^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and therefore $\int_{\mathbb{CP}^1} w_2(\mathcal{O}(1)) = 1$.

Thus we have shown that the bordism classes of these three surfaces are linearly independent, and dual to the three invariants in $\boldsymbol{\varphi}$, as promised. ☐

Recall the map δ from Proposition B.0.14.

Lemma B.0.23. *There is a (necessarily unique) isomorphism* $q\colon \Omega_3^{\mathrm{Pin}^+} \xrightarrow{\cong} \mathbb{Z}_2$*, and the composition* $q \circ \delta\colon \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \to \mathbb{Z}_2$ *equals* φ_2*.*

Proof. The calculation $\Omega_3^{\mathrm{Pin}^+} \cong \mathbb{Z}_2$ is due to Giambalvo [193, §2]. To identify $q \circ \delta = \varphi_2$, it suffices by Proposition B.0.20 to show $\delta(S^1_{nb} \times S^1_{nb}, \underline{\mathbb{R}}^2) = 0$, $\delta(S^1_{nb} \times S^1_b, \sigma_R \oplus \underline{\mathbb{R}}) = 1$, and $\delta(\mathbb{CP}^1, \mathcal{O}(1)) = 0$.

First observe that sm_V is not surjective: its domain and codomain are both sets of size 8 (Lemma B.0.6, resp. Proposition B.0.20) but $\mathrm{sm}_V(\mathbb{RP}^4) = 0$ (Corollary B.0.17). Since sm_V is not surjective, exactness of (B.0.15) implies $\delta \neq 0$. Therefore if we can show $\delta(S^1_{nb} \times S^1_{nb}, \underline{\mathbb{R}}^2) = 0$ and $\delta(\mathbb{CP}^1, \mathcal{O}(1)) = 0$, then it must follow that $\delta(S^1_{nb} \times S^1_b, \sigma_R \oplus \underline{\mathbb{R}}) = 1$.

For $S^1_{nb} \times S^1_{nb}$, the vector bundle is trivial, so the total space of its sphere bundle is T^3 with some pin$^+$ structure. Since T^3 is orientable, this pin$^+$ structure is induced from a spin structure, but $\Omega_3^{\mathrm{Spin}} = 0$ [278] so T^3 bounds some compact spin 4-manifold X. This is also a pin$^+$ null-bordism of T^3, so $\delta(S^1_{nb} \times S^1_{nb}, \underline{\mathbb{R}}^2) = 0$.

For $\mathbb{CP}^1$, the sphere bundle map $S(\mathcal{O}(1)) \to \mathbb{CP}^1$ is one definition of the Hopf fibration, so the total space is S^3. The argument in the previous paragraph shows that any pin$^+$ structure on any closed, orientable 3-manifold is null-bordant, so $\delta(\mathbb{CP}^1, \mathcal{O}(1)) = 0$. ☐

Thus, by exactness of (B.0.15), $\varphi_2 \circ \mathrm{sm}_V = 0$.

Lemma B.0.24. $\boldsymbol{\varphi} \circ \mathrm{sm}_V(\mathbb{CP}^1 \times \mathbb{CP}^1) = (1, 0, 0)$.

Proof. By Lemma B.0.23, $\varphi_2 \circ \mathrm{sm}_V = 0$.

To compute $\varphi_3 \circ \mathrm{sm}_V(\mathbb{CP}^1 \times \mathbb{CP}^1)$, let $i\colon \Sigma \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ be a manifold representative for the Poincaré dual of the Euler class of the principal $\mathrm{O}(2)$-bundle $V \to \mathbb{CP}^1 \times \mathbb{CP}^1$ associated to the pin$^{\tilde{c}+}$ structure. If $\nu \to \Sigma$ denotes the normal bundle to the embedding

i, then there is an isomorphism $i^*(V) \cong \nu$ and $\mathrm{sm}_V(\mathbb{CP}^1 \times \mathbb{CP}^1)$ is by definition the class of $(\Sigma, i^*(V))$ in $\Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2))$. Apply the Whitney sum formula to the decomposition $T\Sigma \oplus \nu \cong i^*(T(\mathbb{CP}^1 \times \mathbb{CP}^1))$ to deduce

$$w_1(\Sigma) + w_1(\nu) = i^*(w_1(\mathbb{CP}^1 \times \mathbb{CP}^1)) \tag{B.0.25a}$$

$$w_2(\Sigma) + w_1(\Sigma)w_1(\nu) + w_2(\nu) = i^*(w_2(\mathbb{CP}^1 \times \mathbb{CP}^1)). \tag{B.0.25b}$$

Since $\mathbb{CP}^1 \cong S^2$, it has a spin structure, so $\mathbb{CP}^1 \times \mathbb{CP}^1$ does as well, and therefore $w_i(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$ for $i = 1, 2$. Thus (B.0.25a) simplifies to $w_1(\Sigma) = w_1(\nu)$, and so (B.0.25b) simplifies to $w_2(\Sigma) + w_1(\Sigma)^2 + w_2(\nu) = 0$. The Wu formula implies $w_2(\Sigma) + w_1(\Sigma)^2 = 0$ because the Wu class $v_2 = w_2 + w_1^2$ vanishes on closed 2-manifolds such as Σ, so we have calculated that $w_2(\nu) = 0$ and therefore

$$\varphi_3(\mathbb{CP}^1 \times \mathbb{CP}^1) \coloneqq \int_\Sigma w_2(\nu) = 0. \tag{B.0.26}$$

Since $\varphi_i \circ \mathrm{sm}_V(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$ for $i = 2, 3$, to show $\varphi_1 \circ \mathrm{sm}_V(\mathbb{CP}^1 \times \mathbb{CP}^1) = 1$ is the same as showing $\boldsymbol{\varphi}(\mathbb{CP}^1 \times \mathbb{CP}^1) \neq 0$. Since $\boldsymbol{\varphi}$ is an isomorphism (Proposition B.0.20), this is the same as showing $\mathrm{sm}_V(\mathbb{CP}^1 \times \mathbb{CP}^1) \neq 0$, which by exactness of (B.0.15) is equivalent to showing $[\mathbb{CP}^1 \times \mathbb{CP}^1] \notin \mathrm{Im}(i)$. The domain of i, $\Omega_4^{\mathrm{Pin}^+}$, is a cyclic group [193, §2] and by construction $[\mathbb{RP}^4] \in \mathrm{Im}(i)$, so since the bordism classes of $\mathbb{RP}^4$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ are linearly independent (Proposition B.0.7), $[\mathbb{CP}^1 \times \mathbb{CP}^1]$ cannot also be in $\mathrm{Im}(i)$. Thus $\varphi_1 \circ \mathrm{sm}_V(\mathbb{CP}^1 \times \mathbb{CP}^1) = 1$. ☐

Lemma B.0.27. *$\varphi_1 \circ \mathrm{sm}_V(\mathbb{CP}^2, -1) = 0$.*

Proof. One characterization of $\mathrm{sm}_V(\mathbb{CP}^2, -1)$ is that it is the bordism class of the zero set of any section of $V \to \mathbb{CP}^2$ which is transverse to the zero section [517, Definition 3.7]. For the $\mathrm{pin}^{\tilde{c}+}$ structure $(\mathbb{CP}^2, -1)$, $V = \mathcal{O}(-1)$, and the zero set of any such section is isotopic to the standard embedding $\mathbb{CP}^1 \to \mathbb{CP}^2$. Thus $\mathrm{sm}_V(\mathbb{CP}^2, -1) \in \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2))$ is the bordism class of $\mathbb{CP}^1$ with some spin structure and some rank-2 vector bundle. The map φ_1 forgets the vector bundle, so $\varphi_1 \circ \mathrm{sm}_V(\mathbb{CP}^2, -1) \in \Omega_2^{\mathrm{Spin}}$ is the bordism class of $\mathbb{CP}^1 \cong S^2$ with some spin structure. Since S^2 is simply connected, it has a unique spin structure, which therefore is the spin structure appearing at the boundary $S^2 \cong \partial D^3$, where D^3 is given its canonical (also unique) spin structure. Therefore $[\mathbb{CP}^1] = 0$ in Ω_2^{Spin} and therefore $\varphi_1 \circ \mathrm{sm}_V(\mathbb{CP}^2, -1) = 0$. ☐

Remark B.0.28. It is possible to show $\boldsymbol{\varphi} \circ \mathrm{sm}_V(\mathbb{CP}^2, -1) = (0, 0, 1)$ similarly to the proof of Lemma B.0.24.

Since the first component of $\boldsymbol{\varphi} \circ \mathrm{sm}_V$ is ABS_{-2}, Corollary B.0.17 and Lemmas B.0.24 and B.0.27 finish the proof of Theorem B.0.9.

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