INVERSION METHOD AND ITS APPLICATIONS

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Abstract:

The inversion method, which is the generalization of the method of the free energy, and the effective potential (or the action), is explained and applied to several problems. These include strong coupling *QED* and the confining parameter in *QCD*.

Introduction In various fields of physics, we encounter the situation where the ground state which is realized in nature is not that of the naive perturbative one. It is realized after the "condensation" of the objects which behave as normal particles in the perturbative phase. The attractive interaction between normal particles is the driving force of the condensation. The usual way to study the phenomenon is to introduce the free energy and find its minimum. For the field theory of zero temperature, the effective potential or the action plays the role. Let us recall the way how they are defined.

First the Lagrangian L of the system is changed into $L_J = L+JO$ where J is the artificial source and the operator O is chosen to break the symmetry of L so that the order parameter $<O>\equiv\phi$ calculated by L_J is non-zero even in the perturbation theory. Calculate the vacuum action functional $W[J]$ in the theory governed by L_J and the effective action $\Gamma[\phi]$ is defined, through the Legendre transformation, as

$$\Gamma[\phi] = W[J]-J\partial W/\partial J, \qquad \partial W/\partial J = \phi. \tag{1}$$

The relation $\phi = \partial W/\partial J$ is inverted to express J as the function of ϕ. The stationarity condition $\partial\Gamma/\partial\phi = -J = 0$ assures the recovery of the original theory at this point.

Inversion Method We can generalize the above procedure to the case where ϕ is not written as the expectation value of some operator. Change L to L_J where $L_{J=0}=L$, which is the only requirements for L_J. It need not be $L+JO$. Then the order parameter ϕ is calculated in perturbative series;

$$\phi = \sum_{n=0}^{\infty}(g^2)^n h_n(J) \tag{2}$$

where g is the coupling strength and $h_n(J)$ is calculable diagrammatically. The modified Lagrangian L_J is so chosen as to get the non-zero series (2). Now we invert (2) to get

$$J = \sum_{m=0}^{\infty}(g^2)^m f_m(\phi). \tag{3}$$

The co-efficient function $f_m(\phi)$ for $m \leq N$ are obtainable from $h_n(J)$ with $n \leq N$. The solution to $J=0$ is looked for in the inverted form (3). We can get the non-perturbative solution, if it exists at all, by this method besides the trivial solution $\phi=0$.

Illustration The ladder Schwinger-Dyson equation is derived by this method as an example. Take the Lagrangian L_{QED} of the Quantum Electrodynamics (QED) and consider the electron self-energy function $S_F(p)$. Here $g=e$, the charge of the electron , and the source term is introduced by changing the action of QED by adding $\int d^4pJ(p)\overline{\psi}(-p)\psi(p)$ where ψ is the electron field. The original series (2) up to e^2 is

$$S_F(p) = S_0^J(p) - e^2 S_0^J(p)\int d^4q[\gamma_\mu S_0^J(p+q)\gamma_\nu D_0^{\mu\nu}(q)]S_0^J(p), \tag{4}$$

where $D_0^{\mu\nu}$ is the photon propagator and $(S_0^J)^{-1}(p) = S_0^{-1}(p) - iJ(p)$ is the inverse of the free electron propagator in the presence of J. The lowest inversion ($e^2=0$) gives us $J(p) = iS_F^{-1}(p) - iS_0^{-1}(p)$ so that the inverted series up to e^2 is

$$J(p) = iS_F^{-1}(p) - iS_0^{-1}(p) - ie^2\int d^4q\gamma_\mu S_F(p+q)\gamma_\nu D^{\mu\nu}(q), \tag{5}$$

which is just the ladder Schwinger-Dyson equation if we set $J=0$.

Strong Coupling QED[2] Consider the gauge invariant order parameter $\phi=<\overline{\psi}(x)\psi(x)>$ and the term $J\int d^4x\overline{\psi}(x)\psi(x)$ is added to the action of QED. The vacuum action function $W[J]$ is calculated. It has the form

$$W[J] = -iTr\ln(S_0^J)^{-1} + (i/2)Tr\ln D_0^{-1} - i\sum_{n=1}^{\infty}(e^2)^n W_J^{(n)}, \qquad (6)$$

where $(S_0^J)^{-1} = p+J$ and $W_J^{(n)}$ is the vacuum graphs of the order $(e^2)^n$ in the presence of J. We calculate ϕ by the formula $\phi = \partial W/\partial J/\Omega$ where Ω is the space-time volume. The term up to e^2 is calculated below. It involves the two loop diagram having one photon propagator and is evaluated by opening the photon propagator, which is nothing but the vacuum polarization graph with mass insertion of the electron. This produces the gauge invariant results. The original series is obtained in this way which is given for small J as;

$$\phi = (J/4\pi^2)[\Lambda_f^2+(3\alpha/2\pi)\{1-(J^2/\Lambda_p^2)(\ln J^2/\Lambda_p^2)^2\}] + O(J^3\ln J) \qquad (7)$$

where $\Lambda_f(\Lambda_p)$ is the electron (photon) momentum cut-off. The inverted series is

$$J = (4\pi^2\phi/\Lambda_f^2)\{1-\frac{\alpha}{\alpha_c}+24\pi^3\alpha(\phi^2/\Lambda_f^6)(\ln\frac{16\pi^2\phi^2}{\Lambda_f^4\Lambda_p^2})^2\}, \qquad (8)$$

where α=e$^2/4\pi$ and $\alpha_c=2\pi/3\eta$, $\eta=\Lambda_p/\Lambda_f$. This is noting but the negative of the derivative of the effective potential. Equation (8) has the same form, except for the $(\ln\phi^2)^2$ term, as the Landau theory of the phase transition. We conclude; for $\alpha>\alpha_c$ the chiral condensation ($\phi\neq0$) is realized. Our theory predicts the mean field type behavior near $\alpha=\alpha_c$; $\phi\sim(\alpha-\alpha_c)^{1/2}/\ln(\alpha-\alpha_c)$.

<u>QCD and string tension</u> For Quantum Chromodynamics (QCD), the expected non- perturbative solution behaves near g=0 as $\phi^{1/\delta}\sim\mu\exp(1/2b_0g^2)$, where all the quantities are the renormalized ones and μ is the subtraction point. The index $\delta>0$ is the dimension of ϕ in mass unit

and $\beta(g)=b_0g^3+b_1g^5+\cdots$. We change the variable from ϕ to $t\equiv g^2\ln\phi^{1/\delta}/\mu$ which is of the order unity near the solution. The source J is assumed to have the dimension of mass and consider the following inverted series by extracting the factor $\phi^{1/\delta}$. This factor is always present since $\phi=0$ is one of the solution to $J=0$;

$$J = \phi^{1/\delta}\sum_{n=0}^{\infty}(g^2)^n f_n(t)\equiv\phi^{1/\delta}f(t,g^2). \tag{9}$$

Before calculating $f_n(t)$ explicitly, the renormalization group equation tells us much about the form of $f_n(t)$.

In order to see this, we choose J in such a way that J is independent of the subtraction parameter μ: $\mu\frac{dJ}{d\mu}=0$. This can always be done by multiplying a suitable factor in front of J. By noting that when applied to the right hand side of (9),

$$\mu\frac{d}{d\mu} = \mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\mu\frac{d\phi^{1/\delta}}{d\mu}\frac{\partial}{\partial\phi^{1/\delta}},$$

and by requiring that the each coefficient of $(g^2)^n$ vanishes, we get the set of ordinary differential equations for $f_n(t)$. The first member of this set is

$$(2b_0t-1)f_0{}'(t)+\frac{\gamma_1}{\delta}f_0(t) = 0, \tag{10}$$

where $f_0{}'\equiv\frac{df_0}{dt}$ and

$$\frac{1}{\phi}\mu\frac{d\phi}{d\mu} = \gamma(g) = \gamma_1g^2+\gamma_2g^4+\cdots. \tag{11}$$

The solution is

$$f_0(t) = C\left(t-\frac{1}{2b_0}\right)^{-\frac{\gamma_1}{2\delta b_0}} \tag{12}$$

where C is the integration constant. Since $b_0<0$, $\delta>0$, the non-trivial solution to $J=0$ is present if $\gamma_1>0$.

The sign $\gamma_1>0$ has the physical meaning. For the two quark field $\phi\sim qq$ or $\overline{q}q$ for instance, the first term of the anomalous dimension is calculated by the one gluon exchange diagram between two quarks or antiquark which determines whether the force acting between two fermions is attractive or repulsive. The correspondence is indeed[1)]

$\gamma_1>0 \leftrightarrow$ *attractive*,

$\gamma_1<0 \leftrightarrow$ *repulsive*,

therefore our conclusion is that ϕ condenses as long as we have the attractive force between two particles. The condition $\gamma_1>0$ for the condensation of ϕ can be used as a generalized criterion in the case where ϕ is not written as the product of two fields.

Now we know that the correct non-perturbative value of ϕ is

$$\phi^{1/\delta} = \mu\exp\int^{g}\frac{1-\gamma(x)/\delta}{\beta(x)}dx. \tag{13}$$

Thus the variable t has the expansion

$$t = g^2\ln\frac{\phi^{1/\delta}}{\mu} = \frac{1}{2b_0}-\frac{1}{2b_0}\{\frac{\gamma_1}{\delta}+\frac{b_1}{b_0}\}g^2\ln g^{-2}$$
$$+ Cg^2+dg^4+\cdots, \tag{14}$$

where C, d etc. are some constants. The solution (12) to the lowest truncation reproduces the first term of the expansion (14). In order to discuss the higher truncation systematically and most conveniently, we define $\overline{f}(t,g^2)$ as

$$\overline{f}(t,g) = K(g)^{-1}f(t,g)^{1/\eta}, \tag{15}$$

where

$$K(g) \equiv \lim_{g_0\to 0} g_0^2\exp-\int_{g_0}^{g}\frac{dx\gamma(x)}{\delta\eta\beta(x)} \tag{16}$$

$$= g^2\{1+(\frac{\gamma_2}{\gamma_1}-\frac{b_1}{b_0})g^2+\cdots\}, \tag{17}$$

$$\eta = -\frac{\gamma_1}{2\delta b_0}. \tag{18}$$

Remember that $K(g)$ has the Taylor expansion about g=0. The merit of using $\mathcal{T}(t,g)$ is that it satisfies a simple equation;

$$(\mu\frac{\partial}{\partial\mu}+\hat{\beta}(g)\frac{\partial}{\partial g})\mathcal{T}(t,g) = 0, \tag{19}$$

where ϕ is fixed in this equation and

$$\hat{\beta}(g) = \frac{\beta(g)}{1-\gamma(g)/\delta} = b_0g^3+\hat{b}_1g^5+\cdots, \tag{20}$$

$$\hat{b}_1 = b_1 + \frac{\gamma_1 b_0}{\delta}. \tag{21}$$

The function $\mathcal{T}$ has the expansion

$$\mathcal{T}(t,g^2) = \frac{W_{-1}(t)}{g^2}+W_0(t)+g^2W_1(t)+.... \tag{22}$$

and $W_l(t)$ ($l\geq-1$) satisfies

$$(1-2b_0t)W_l{}'-2b_0lW_l-2\hat{b}_1[tW_{l-1}{}'+(l-1)W_{l-1}] = 0, \tag{23}$$

where $W_{-2}\equiv0$. The solution up to W_0 is given as

$$\mathcal{T}(t,g^2) = \frac{C_{-1}}{g^2}(t-\frac{1}{2b_0}) - \frac{C_{-1}\hat{b}_1}{2b_0^2}\ln(1-2b_0t) + C_0 \tag{24}$$

with C_{-1} and C_0 being the integration constants which are calculated through the series (9) diagrammatically. The zero of (24) is slightly modified compared with the lowest value $\frac{1}{2b_0}$, and it behaves for small g^2 as

$$t = \frac{1}{2b_0} - \frac{\hat{b}_1}{2b_0^2}g^2\ln g^{-2} + 0(g^2\ln\ln g^{-2}) \tag{25}$$

so that we have recovered the second term of the correct expansion (14) but the term $O(g^2 \ln\ln g^{-2})$ should be $O(g^2)$. The discrepancy is of course due to the truncation (24) and we have for $\phi^{1/\delta}$, the solution to $\overline{J}=0$, the scale non-invariant behavior under the variation of μ

$$\phi^{1/\delta}(g^2)^\eta \propto (\ln g^{-2})^{\frac{\hat{b}_1}{2b_0^2}}. \tag{26}$$

The right hand side should be constant for the correct solution since the left hand side is the scale invariant quantity.

We can improve the situation by taking into more terms of W_l. This can most conveniently be done by deriving the partial differential equation for $K(t,g^2)\equiv\frac{-2b_0}{C_{-1}}\times(W_0+g^2W_1+g^4W_2+\cdots)$. By multiplying $(g^2)^l$ to (23) and summing up from $l=-1$, we get

$$[\{(1+e^2)s-e^2\}\frac{\partial}{\partial s}+(1+e^2)e^2\frac{\partial}{\partial e^2}]\widetilde{K}(s,e^2)=1 \tag{27}$$

where $e^2=g^2\frac{\hat{b}_1}{b_0}$, $\widetilde{K}(s,e^2)=\frac{b_0}{\hat{b}_1}K(t,g^2)$ and $s=1-2b_0t$. We sum up the term $\frac{(lns)^m}{s^l}$ with $1\leq m\leq l$ appearing in W_l which are the most singular terms in W_l. This can be accomplished by looking for the solution to (27) with $1+e^2$ replaced by unity. So that we solve

$$\{(s-e^2)\frac{\partial}{\partial s}+e^2\frac{\partial}{\partial e^2}\}\widetilde{K}_0(s,e^2)=1 \tag{28}$$

and get the solution in the implicit form,

$$\widetilde{K}_0=\ln(s+e^2\widetilde{K}_0) \tag{29}$$

$$=\ln(s+e^2\ln(s+e^2\ln(s+e^2\ln\cdots))). \tag{30}$$

The lowest solution $\widetilde{K}_0=\ln s$ reproduces (24). The next truncation gives $\widetilde{K}_0=\ln(s+e^2\ln s)$ which leads to the solution

$$t=\frac{1}{2b_0}-\frac{\hat{b}_1}{2b_0^2}g^2\ln g^{-2}+O(g^2\ln\ln\ln g^{-2}) \tag{31}$$

and for $\phi^{1/\delta}$

$$\phi^{1/\delta}(g^2)^{\eta} \propto (\ln\ln g^{-2})^{\frac{\hat{b}_1}{2b_0^2}}. \tag{32}$$

The situation is improved but still it is not scale invariant.

In order to obtain the scale invariant formula for the solution ϕ, we have to use the exact solution $\widetilde{K}_0$ given by (29). Therefore we solve (29) and the equation

$$\frac{s}{g^2} + \frac{\hat{b}_1}{b_0}\tilde{K}_0 + \widetilde{C}_0 = 0 \tag{33}$$

simultaneously where $\widetilde{C}_0 = -2b_0 C_0 / C_{-1}$. By inserting $s = \exp\widetilde{K}_0 - e^2\widetilde{K}_0$ obtained from (29) into (33), $\widetilde{K}_0$ is given by

$$\widetilde{K}_0 = \ln(-g^2\widetilde{C}_0) \tag{34}$$

which leads to

$$\begin{aligned} s &= 1-2b_0 t = 1-2b_0 g^2 \ln\frac{\phi^{1/\delta}}{\mu} \\ &= -g^2\widetilde{C}_0 - \frac{\hat{b}_1}{b_0}\ln(-g^2\widetilde{C}_0) \end{aligned} \tag{35}$$

so that

$$\phi^{1/\delta} = \mu\exp\{\frac{1}{2b_0 g^2} + \frac{\widetilde{C}_0}{2b_0} + \frac{\hat{b}_1}{2b_0^2}\ln(-g^2\widetilde{C}_0)\}. \tag{36}$$

The scale parameter μ is eliminated using the QCD scale parameter Λ_{QCD};

$$\Lambda_{QCD} = \mu\exp\{\frac{1}{2b_0 g^2} - \frac{b_1}{2b_0^2}\ln(\frac{-b_0 g^2}{1+\frac{b_1}{b_0}g^2})\} \tag{37}$$

which is the expression assuming $\beta(g) = b_0 g^3 + b_1 g^5$. From (36) and (37) we finally get, by taking the limit $g \to 0$,

$$\phi^{1/\delta}(g^2)^\eta = D\Lambda_{QCD}, \tag{38}$$

where D is the numerical constant given by

$$D = \exp\{\frac{\tilde{C}_0}{2b_0}+\frac{b_1}{2b_0^2}\ln(\frac{\tilde{C}_0}{b_0})+\frac{\gamma_1}{2\delta b_0}\ln(-\tilde{C}_0)\} \tag{39}$$

$$= \exp\{\frac{\tilde{C}_0}{2b_0}+\frac{\hat{b}_1}{2b_0^2}\ln(\frac{\tilde{C}_0}{b_0})+\frac{\gamma_1}{2\delta b_0}\ln(-b_0)\}. \tag{39'}$$

The above two expressions are equivalent. These formulas suggest that the solution exists as long as $\tilde{C}_0<0$. The numerical evaluation of $\tilde{C}_0$ for the case of

$$\phi=<\overline{\psi}\psi>,\quad \frac{1}{g^2}\times(\text{string tension}),\qquad <G_{\mu\nu}^2>$$

in QCD is under way but the sign of $\tilde{C}_0$ for these quantities is indeed negative.

Finally the formula for the energy density, more exactly the difference of the energy density ΔE between the normal and the condensed vacuum, is given below. ΔE has no anomalous dimension so it needs a separate discussion. The source should be introduced in such a way that the energy is lowered, i.e. $\Delta E<0$. Let us write $\Delta E=-aJ^4(a>0)$ then the inverted series has the form, with $\varepsilon\equiv-\Delta E/a$ and $t=g^2\ln\frac{\varepsilon^{1/4}}{\mu}$,

$$J = \varepsilon^{1/4}\{1+g^2f_1(t)+g^4f_2(t)+\cdots\}. \tag{40}$$

Define $W(t,g^2)$ by

$$W(t,g^2) = \{g^2f_1(t)+g^4f_2(t)+\cdots\}^{-1}$$
$$\equiv \frac{W_{-1}(t)}{g^2}+W_0(t)+g^2W_1(t)+\cdots, \tag{41}$$

then by the same arguments as before, we have up to W_0

$$W(t,g^2) = \frac{\overline{C}_{-1}}{g^2}(t-\frac{1}{2b_0}) - \frac{\overline{C}_{-1}b_1}{2b_0^2}\ln(1-2b_0t)+\overline{C}_0.$$

The scale invariant formula for $\varepsilon^{1/4}$ is now

$$\varepsilon^{1/4} = \overline{D}\Lambda_{QCD},$$

$$\overline{D} = \exp[-\frac{1+\overline{C}_0}{\overline{C}_{-1}} + \frac{b_1}{2b_0^2}\ln\{\frac{2(1+\overline{C}_0)}{-\overline{C}_{-1}}\}].$$

For the real solution to exist $(1+\overline{C}_0)/\overline{C}_{-1}<0$ should be satisfied. The calculation of $\overline{C}_0$ requires two loop vacuum diagrams under the presence of the suitable source which is not yet carried out.

References

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2) M. Ukita, M. Komachiya and R. Fukuda, to appear in Int. J. Mod. Phys. A