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Research Article

Comparison of Analytical Solution of DGLAP Equations for $F_2^{NS}(x,t)$ at Small x by Two Methods

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The DGLAP equation for the nonsinglet structure function $F_2^{NS}(x,t)$ at LO is solved analytically at low x by converting it into a partial differential equation in two variables: Bjorken x and t ($t = \ln(Q^2/\Lambda^2)$ and then solved by two methods: Lagrange's auxiliary method and the method of characteristics. The two solutions are then compared with the available data on the structure function. The relative merits of the two solutions are discussed calculating the chi-square with the used data set.

1. Introduction

The structure function of the nucleon has played a pivotal role in our understanding of the internal composition of the proton and the neutron. Experimentally, structure functions are extracted from the cross-section measurement in deep inelastic scattering experiments, and theoretically in QCD, they have a simple interpretation in terms of the quark and gluon momentum distribution, a set of universal functions known as parton distribution functions (PDFs). In DIS kinematics, the PDFs are expressed in terms of two DIS variables: the Bjorken x and Q^2, the four momentum transferred squared. These functions are not calculable in QCD but their evolution in Q^2 is predicted by a set of integrodifferential equations known as DGLAP equations [1–4]. The standard and most widely used practice of studying these structure functions is through the numerical solution of these equations [5, 6]. The x dependence of the parton structure function at some initial scale Q_0^2, which is small but not so small to break down the perturbative picture, is assumed on various physical grounds, and the parameters are then obtained from a global fit of DIS data. It has been shown by numerous analyses that these equations are in good agreement with the DIS data over a wide kinematic region in x and Q^2. They can explain data down to very small value of $Q^2 \sim 1\,\mathrm{GeV}^2$ traditionally explained by soft processes and also for very small value of $x \ll 1$, where these equations are not expected to work. These give credibility to the DGLAP approach as a proper one to study the parton distribution functions. However, apart from the numerical solution, there is the alternative approach of studying analytically these equations at small x and there are many analytical solutions available in the literature [7–10], and the present authors have also pursued such an approach with reasonable phenomenological success [11–15]. The analytical approach, though not possible to carry out to higher order in (x, Q^2) space due to the complex nature of the splitting functions involved, is intuitive one in the sense that the solutions obtained allow us to visualize their dependence on the variables. In this paper, we study some analytical solutions of the nonsinglet structure functions that is, the flavour dependent contributions to the structure functions considering the corresponding DGLAP evolution equations. As is well known, the nonsinglet structure functions, in DIS plays an important role for precise description of the quark densities; it is comparatively easy because it is not coupled to the singlet and the gluon and can be regarded as a starting ground for the analysis of the other structure functions. We convert the LO DGLAP equation which is an integrodifferential equation into a partial differential equation in the two variables (x, Q^2) by a Taylor series expansion valid to be at low x. The resulting equation is then solved analytically by two different methods: Lagrange's auxiliary method [16] and the method of characteristics [17, 18].

In earlier work, the DGLAP equation was solved approximately by using either Lagrange's method [11–13] or by the method of characteristics [14, 15]. Besides that, the levels of approximation were also different. The aim of the paper is to make a detailed comparison of the predictions of the two methods with two different levels of approximations. We have highlighted above how the present work differs from the earlier works.

The paper is organized as follows: in Section 2 we give the formalism, Section 3 is devoted to discussion of the solutions by comparison with the available data, and in Section 4 we give our conclusion.

2. Formalism

The nonsinglet flavour dependent contribution is defined as

$$q^{\mathrm{NS}}\left(x, Q^2\right) = \sum_i \left(q_i\left(x, Q^2\right) - \bar{q}_i\left(x, Q^2\right)\right), \tag{1}$$

where q_i is the density of quark of ith flavour and the sum run over all quark flavours. The corresponding nonsinglet structure function is the charged weighed sum of these densities multiplied by x. Since the gluon and the singlet quark are flavour independent, the nonsinglet contribution evolves independently in the DGLAP approach and is given by [21]

$$\frac{\partial q^{\mathrm{NS}}\left(x, Q^2\right)}{\partial t} = \frac{\alpha_s\left(Q^2\right)}{2\pi} P_{qq}^{\mathrm{NS}} \otimes q^{\mathrm{NS}}\left(x', Q^2\right), \tag{2}$$

where $t = \ln(Q^2/\Lambda^2)$ and P_{qq}^{NS} are the splitting functions which give the probability of radiating a parton with momentum fraction x from a parton with higher momentum fraction x'. The symbol $\otimes$ stands for the convolution integral with respect to the first variable x defined as

$$(f \otimes g)(x) = \int_x^1 \frac{dy}{y} f(y)\, g\left(\frac{x}{y}\right). \tag{3}$$

Using the explicit form of the splitting function $P_{qq}^{\mathrm{NS}} = C_F((1+z^2)/(1-z))_+$ from [21] the evolution equation for the nonsinglet structure function can be written as

$$\begin{aligned} \frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial t} &= \frac{A_f}{t}\left[\{3 + 4\ln(1-x)\} F_2^{\mathrm{NS}}(x,t)\right. \\ &\quad + 2\int_x^1 \frac{dz}{1-z}\left\{\left(1+z^2\right) F_2^{\mathrm{NS}}\left(\frac{x}{z}, t\right)\right. \\ &\quad \left.\left. -2F_2^{\mathrm{NS}}(x,t)\right\}\right]. \end{aligned} \tag{4}$$

Here $A_f = 4/3\beta_0$ and $\beta_0 = 11 - (2/3)n_f$ are QCD beta function at LO.

To simplify and reduce the integrodifferential equation to a partial differential equation, we introduce a variable $u = 1 - z$ and expand the argument x/z as

$$\frac{x}{z} = \frac{x}{1-u} = x\sum_{k=0}^{\infty} u^k = x + x\sum_{k=1}^{\infty} u^k. \tag{5}$$

Using (5) we expand $F^{\mathrm{NS}}(x/z)$ in Taylor series as

$$\begin{aligned} F_2^{\mathrm{NS}}\left(\frac{x}{z}, t\right) &= F_2^{\mathrm{NS}}(x,t) + x\sum_{k=1}^{\infty} u^k \frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial x} \\ &\quad + \frac{\left(x\sum_{k=1}^{\infty} u^k\right)^2}{2!} \frac{\partial^2 F_2^{\mathrm{NS}}(x,t)}{\partial x^2} + \cdots. \end{aligned} \tag{6}$$

The series (6) is convergent [11–15], and hence at small x, we can approximate these by

$$F_2^{\mathrm{NS}}\left(\frac{x}{z}, t\right) = F_2^{\mathrm{NS}}(x,t) + x\sum_{k=1}^{\infty} u^k \frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial x}. \tag{7}$$

Now putting (7) in (4) we get

$$\begin{aligned} &\frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial t} \\ &\quad = \frac{A_f}{t}\left[\{3 + 4\ln(1-x)\} F_2^{\mathrm{NS}}(x,t)\right. \\ &\qquad + 2\int_x^1 \frac{dz}{1-z}\left(z^2 - 1\right) F_2^{\mathrm{NS}}(x,t) \\ &\qquad \left. +2\int_x^1 \frac{dz}{1-z}\left(1+z^2\right)\left(x\sum_{k=1}^{\infty} u^k\right) \frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial x}\right]. \end{aligned} \tag{8}$$

Carrying out the integration in z, we can write (8) as

$$\begin{aligned} &\frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial t} - \frac{A_f x}{t} \frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial x}\left[2\ln\left(\frac{1}{x}\right) + \left(1 - x^2\right)\right] \\ &\quad = \frac{A_f}{t}\left[3 + 4\ln(1-x) + (x-1)(x+3)\right] F_2^{\mathrm{NS}}(x,t). \end{aligned} \tag{9}$$

Equation (9) is a partial differential equation for the nonsinglet structure function $F_2^{\mathrm{NS}}(x,t)$ with respect to the variables x and t. Beyond its traditional use in t evolution, it gives also x evolution at small x. There are various methods for solving the partial differential equation in two variables. We here adopt the two different methods as mentioned in Section 1.

While performing the integration in z and neglecting terms $\mathcal{O}(x^2)$ and higher, we can also express (8) as

$$\begin{aligned} &\frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial t} - \frac{8A_f}{3} \frac{x}{t} \frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial x} \\ &\quad = \frac{A_f\{4\ln(1-x) + 2x\}}{t} F_2^{\mathrm{NS}}(x,t). \end{aligned} \tag{10}$$

This is what we get by considering

$$x\sum_{k=1}^{\infty}u^{k}=xu=x(1-z), \quad \text{considering } k=1 \text{ only} \tag{11}$$

during integration.

We solve both the PDEs (9) and (10) with the two formalisms described here, the Lagrange's method and method of characteristics. Though both these PDEs are obtained from the same equation (4), the levels of approximations are different for both (9) and (10). We continue our discussion for comparing the formalisms with the solutions derived by solving (9) and (10).

2.1. Solution by the Method of Characteristics. Being a differential equation in two variables, it requires two boundary conditions for its solution. But usually we have only one given boundary condition which is the nonperturbative x distribution of the function at some initial scale Q_0^2. So, the solutions obtained are not unique, but they give only a range of solutions. This limitation can be avoided by adopting the method of characteristics [17, 18]. From the theory of differential equation, we know that most of the important properties of the solution of (9) depend on the principal part of the equation, that is, the left hand side in (9). But this part is actually a total derivative along the solution of the characteristic equation

$$\frac{dx(t)}{dt}=-\frac{A_f}{t}\left[2x\ln\left(\frac{1}{x}\right)+x\right] \tag{12}$$

which gives the characteristic curve of (9). That is, along the characteristic curve, which is a solution of (12), the partial differential equation becomes an ordinary differential equation, and then we can solve it with only one boundary condition.

On using (12) in (9), the left hand side becomes an ordinary derivative with respect to t, and the equation becomes an ordinary differential equation:

$$\frac{dF_2^{\mathrm{NS}}(x(t),t)}{dt}=c^{\mathrm{NS}}(x(t),t)\,F_2^{\mathrm{NS}}(x(t),t), \tag{13}$$

where

$$c^{\mathrm{NS}}(x(t),t)=\frac{A_f\{4\ln(1-x(t))+(x-1)(x+3)\}}{t}. \tag{14}$$

If the characteristic curve passes through a point $(\bar{x},\bar{t})$, that is, $x(\bar{t})=\bar{x}$ in the $x-t$ space, we get the solution of (12) to be

$$\ln\ln\left(\frac{\bar{x}}{x(t)}\right)^{3\beta_0/8}=\ln\left(\frac{\bar{t}}{t}\right). \tag{15}$$

If the characteristic curve cuts the initial curve $t=t_0$ at a point $x(t_0)=\tau$, then (15) gives

$$\ln\ln\left(\frac{\bar{x}}{\tau}\right)^{3\beta_0/8}=\ln\frac{\bar{t}}{t_0}, \tag{16}$$

so that,

$$\tau=\bar{x}\exp\left(\frac{t_0}{\bar{t}}\right)^{8/3\beta_0}. \tag{17}$$

Since $(\bar{x},\bar{t})$ is any point on the $x-t$ space, therefore dropping the bars over x and t, the equation of the characteristic is

$$x(t)=\tau\exp\left(\frac{t}{t_0}\right)^{8/3\beta_0}. \tag{18}$$

Integrating (13) over t from t_0 to t along the characteristic curve $x(t)$ (18), we get the solution for the nonsinglet structure function $F_2^{\mathrm{NS}}(x,t)$ as

$$F_2^{\mathrm{NS}}(x,t)=F_2^{\mathrm{NS}}(\tau)\exp\left[\frac{8}{3\beta_0}\left\{2\ln\left(\frac{t}{t_0}\right)\ln x-x\ln\left(\frac{t}{t_0}\right)\right\}\right]. \tag{19}$$

Equation (19) is the analytical solution for the nonsinglet structure function of (9) within the present formalism. We can also express it as

$$F_2^{\mathrm{NS}}(x,t)=F_2^{\mathrm{NS}}(\tau)\left(\frac{t}{t_0}\right)^{\alpha}, \quad \alpha=\frac{8}{3\beta_0}\{2\ln x-x\}. \tag{20}$$

Using the same formalism for the PDE (10), we get the characteristic equation as

$$\frac{dx(t)}{dt}=-\frac{8A_f}{3}\frac{x}{t}. \tag{21}$$

Substituting (21) in (10), we get an ordinary differential equation with respect to t,

$$\frac{dF_2^{\mathrm{NS}}(x(t),t)}{dt}=c^{\mathrm{NS}}(x(t),t)\,F_2^{\mathrm{NS}}(x(t),t), \tag{22}$$

where

$$c^{\mathrm{NS}}(x(t),t)=\frac{A_f\{4\ln(1-x(t))+2x(t)\}}{t}. \tag{23}$$

Solving (22) along the characteristic curve, in a similar way we get a different form of solution for the nonsinglet structure function $F_2^{\mathrm{NS}}(x,t)$ as

$$F_2^{\mathrm{NS}}(x,t)=F_2^{\mathrm{NS}}(\tau)\exp\left[\frac{3}{4A_f}x\left(\left(\frac{t}{t_0}\right)^{8A_f/3}-1\right)-\frac{3}{2A_f}\left\{\sum_{k=1}^{\infty}\frac{x^k}{k^2}\left(\left(\frac{t}{t_0}\right)^{8A_f k/3}-1\right)\right\}\right] \tag{24}$$

which can be expressed also as

$$F_2^{\mathrm{NS}}(x,t)=F_2^{\mathrm{NS}}(\tau)\left(\frac{t}{t_0}\right)^{\beta}, \tag{25}$$

where

$$\beta=\frac{1}{\ln(t/t_0)}\left[\frac{3}{4A_f}x\left(\left(\frac{t}{t_0}\right)^{8A_f/3}-1\right)\right.$$
$$\left.-\frac{3}{2A_f}\left\{\sum_{k=1}^{\infty}\frac{x^k}{k^2}\left(\left(\frac{t}{t_0}\right)^{8A_fk/3}-1\right)\right\}\right]. \tag{26}$$

Equations (19) and (24) are the two analytical solutions of (9) and (10), which are obtained from the same evolution equation (4) with different level of approximations.

2.2. Solution by the Lagrange's Auxiliary Method. To solve (9) by the Lagrange's auxiliary method [16], we write the equation in the form

$$Q(x,t)\frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial t}+P(x,t)\frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial x}=R\left(x,t,F_2^{\mathrm{NS}}\right), \tag{27}$$

where

$$P(x,t)=-A_f x\left[2\ln\left(\frac{1}{x}\right)+\left(1-x^2\right)\right], \tag{28}$$

$$Q(x,t)=t, \tag{29}$$

$$R\left(x,t,F_2^{\mathrm{NS}}\right)=R'(x)F_2^{\mathrm{NS}}(x,t) \tag{30}$$

with

$$R'(x)=A_f\left[3+4\ln(1-x)+(x-1)(x+3)\right]. \tag{31}$$

The general solution of (27) is obtained by solving the following auxiliary system of ordinary differential equations

$$\frac{dx}{P(x)}=\frac{dt}{Q(t)}=\frac{dF_2^{\mathrm{NS}}(x,t)}{R\left(x,t,F_2^{\mathrm{NS}}(x,t)\right)}. \tag{32}$$

If $u(x,t,F_2^{\mathrm{NS}})=C_1$ and $v(x,t,F_2^{\mathrm{NS}})=C_2$ are the two independent solutions of (30), then in general, the solution of (27) is

$$F(u,v)=0, \tag{33}$$

where F is an arbitrary function of u and v.

Solving the auxiliary system, we get

$$u\left(x,t,F_2^{\mathrm{NS}}\right)=tX^{\mathrm{NS}}(x),$$
$$v\left(x,t,F_2^{\mathrm{NS}}\right)=F_2^{\mathrm{NS}}(x,t)\,Y^{\mathrm{NS}}(x). \tag{34}$$

The functions $X^{\mathrm{NS}}(x)$ and $Y^{\mathrm{NS}}(x)$ are defined as

$$X^{\mathrm{NS}}(x)=\exp\left[-\int\frac{dx}{P(x)}\right], \tag{35}$$

$$Y^{\mathrm{NS}}(x)=\exp\left[-\int\frac{R'(x)}{P(x)}dx\right]. \tag{36}$$

The explicit analytical form of $X^{\mathrm{NS}}(x)$ in the leading $(1/x)$ approximation is [22, 23]

$$X^{\mathrm{NS}}(x)=\exp\left[-\frac{1}{2}\log\left|\log x\right|\right]. \tag{37}$$

In this approach we try to find a specific solution that satisfies some physical conditions on the structure function. Such a solution can be extracted from the combination of u and v linear in F_2^{NS},

$$u+\alpha v=\beta, \tag{38}$$

where α and β are two quantities to be determined from the boundary conditions on F_2^{NS}. The two physically plausible boundary conditions are

$$F_2^{\mathrm{NS}}(x,t)=F_2^{\mathrm{NS}}(x,t_0)\quad\text{at some low } t=t_0, \tag{39}$$

$$F_2^{\mathrm{NS}}(1,t)=0\quad\text{for any } t. \tag{40}$$

The first one corresponds to a nonperturbative input at some low momentum scale, and the second one corresponds to large x behaviour of any structure function at any momentum transfer consistent with constituent quark counting rules. These two boundary conditions leads us to

$$t_0X^{\mathrm{NS}}(x)+\alpha F_2^{\mathrm{NS}}(x,t_0)\,Y^{\mathrm{NS}}(x)=\beta, \tag{41}$$

$$tX^{\mathrm{NS}}(1)=\beta. \tag{42}$$

The term $Y^{\mathrm{NS}}(1)$ does not appear in (42) due to (40). With these boundary conditions, the solution of (27) takes the form

$$F_2^{\mathrm{NS}}(x,t)=F_2^{\mathrm{NS}}(x,t_0)\left(\frac{t}{t_0}\right)\frac{\left[X^{\mathrm{NS}}(x)-X^{\mathrm{NS}}(1)\right]}{\left[X^{\mathrm{NS}}(x)-(t/t_0)\,X^{\mathrm{NS}}(1)\right]}. \tag{43}$$

From (37) we know

$$X^{\mathrm{NS}}(1)\approx 0. \tag{44}$$

Hence, we get

$$F_2^{\mathrm{NS}}(x,t)=F_2^{\mathrm{NS}}(x,t_0)\left(\frac{t}{t_0}\right). \tag{45}$$

This gives the t evolution for $F_2^{\mathrm{NS}}(x,t)$, nonsinglet structure function in LO at small x.

In a similar way when we solve (10) by the Lagrange's method we write that equation in a form

$$Q(x,t)\frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial t}+P(x,t)\frac{\partial F_2^{\mathrm{NS}}(x,t)}{\partial x}=R\left(x,t,F_2^{\mathrm{NS}}\right), \tag{46}$$

where

$$P(x,t)=-\frac{8A_f}{3}x,$$
$$Q(x,t)=t, \tag{47}$$
$$R\left(x,t,F_2^{\mathrm{NS}}\right)=R'(x)F_2^{\mathrm{NS}}(x,t)$$

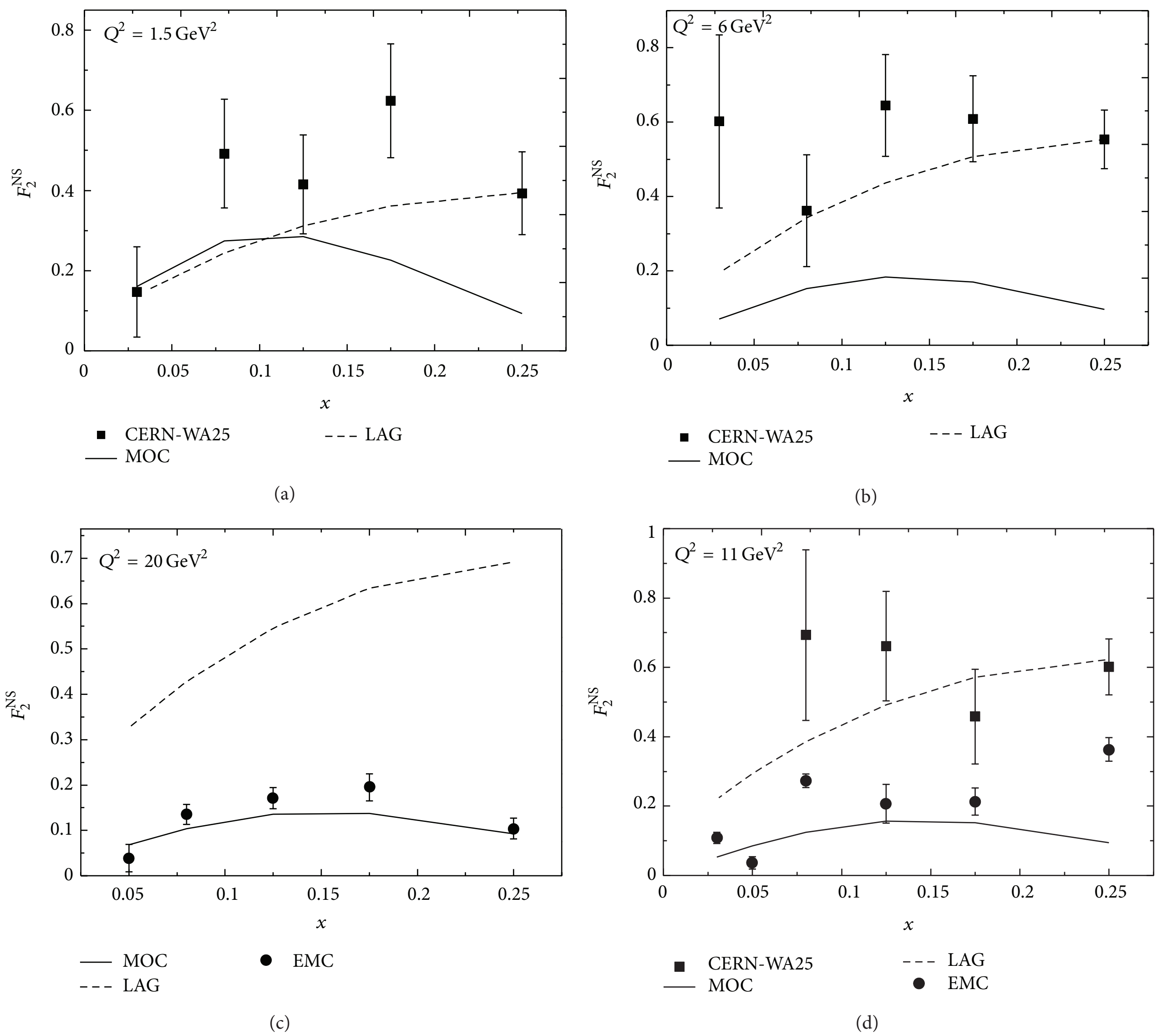

FIGURE 1: Nonsinglet structure function $F_2^{\mathrm{NS}}(x, Q^2)$ as a function of x at different fixed Q^2 according to (19) and (45). Data from [19, 20].

with

$$R'(x) = A_f\,[4\ln(1-x) + 2x]\,. \tag{48}$$

Solution of this ODE (46) then leads us to a solution for the nonsinglet structure function F_2^{NS} as given below

$$F_2^{\mathrm{NS}}(x,t) = F_2^{\mathrm{NS}}(x,t_0)\left(\frac{t}{t_0}\right)\frac{\left[X^{\mathrm{NS}}(1) - X^{\mathrm{NS}}(x)\right]}{\left[(t/t_0)\,X^{\mathrm{NS}}(1) - X^{\mathrm{NS}}(x)\right]}. \tag{49}$$

Here, $X^{\mathrm{NS}}(x)$ as defined by (35) using $P(x,t) = -8A_f x/3$ gives

$$X^{\mathrm{NS}}(x) = \exp\left[\frac{75}{32}\ln(x)\right]. \tag{50}$$

But in this case,

$$X^{\mathrm{NS}}(1) = 1. \tag{51}$$

Defining $h(x,t)$ as

$$h(x,t) = \frac{\left[X^{\mathrm{NS}}(1) - X^{\mathrm{NS}}(x)\right]}{\left[(t/t_0)\,X^{\mathrm{NS}}(1) - X^{\mathrm{NS}}(x)\right]}. \tag{52}$$

Equation (49) becomes,

$$F_2^{\mathrm{NS}}(x,t) = F_2^{\mathrm{NS}}(x,t_0)\left(\frac{t}{t_0}\right)h(x,t) \tag{53}$$

with $h(x,t) \le 1$ for $t \ge t_0$. Here $h(x,t)$ measures deviation of (53) from solution given by (45).

We note that the apparent absence of $\log x$ dependence in the solution (53) is due to algebraic cancellation and boundary condition (40).

Equations (45) and (53) are the solutions of (9) and (10). They are obtained from the same evolution equation (4) as noted earlier. In the next section, we consider their

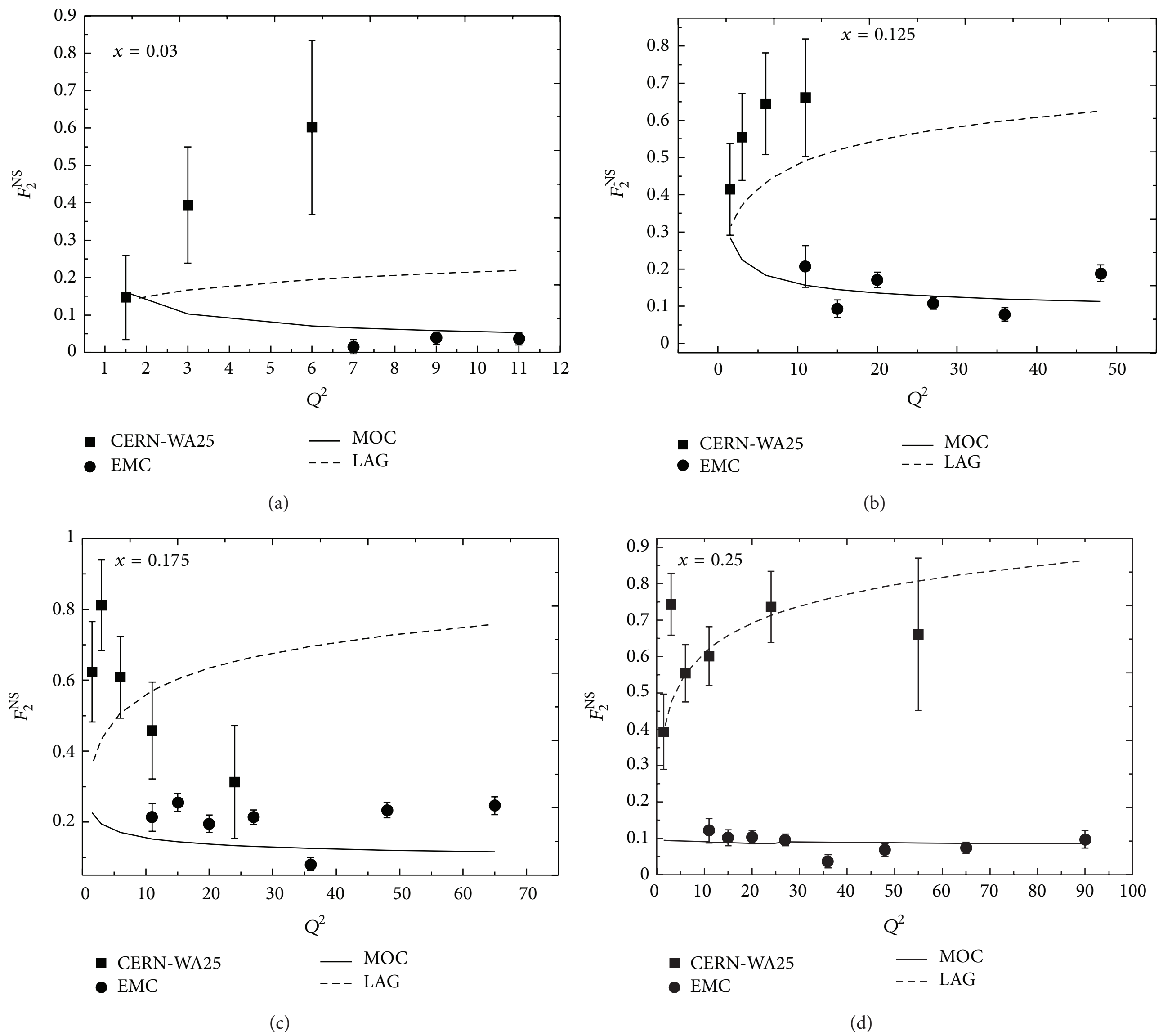

FIGURE 2: Nonsinglet structure function $F_2^{NS}(x, Q^2)$ as a function of Q^2 at different fixed x according to (19) and (45). Data from [19, 20].

phenomenological utility with respect to each other vis-a-vis the available experimental data. Then we perform a chi-square test to test their compatibility with the data.

3. Results and Discussion

We have obtained two sets of analytical solutions of the same DGLAP evolution equation by applying two different methods, having different levels of approximations. The question we have to address is: which solution is valid in which kinematic region? As well as which set of solution is more compatible with data and why? We test the validity of the solutions by comparing them directly with the available data on the nonsinglet structure function. We also compare our solutions with the MSTW 08 numerical solutions to test the analytical methods. Experiments usually publish the data only for the proton and the neutron structure functions F_2^p and F_2^n. So the nonsinglet structure function data is to be extracted from these data using the formula $F_2^{NS} = 3(F_2^p - F_2^n)$. While deriving the data for the nonsinglet, the statistical and systematic errors of the individual data are added in quadrature, so that maximum possible errors come out. For our analysis we use the data from [19, 20, 24]. To evolve our solutions, we use the MRST2004 [25] input and MSTW2008 [26] input for two different representations of our solutions.

For comparison at first we take the solutions (19) and (45). In Figure 1, we plot the solutions given by (19) and (45) as functions of x at some representative fixed Q^2, where the data in the said references are given. Though we explore the range of $0.03 \le x \le 0.25$ and $1.5\,\mathrm{GeV}^2 \le Q^2 \le 55\,\mathrm{GeV}^2$ for CERN-WA25 experiment and $0.03 \le x \le 0.25$ and $7\,\mathrm{GeV}^2 \le Q^2 \le 90\,\mathrm{GeV}^2$ for EMC collaboration data, due to

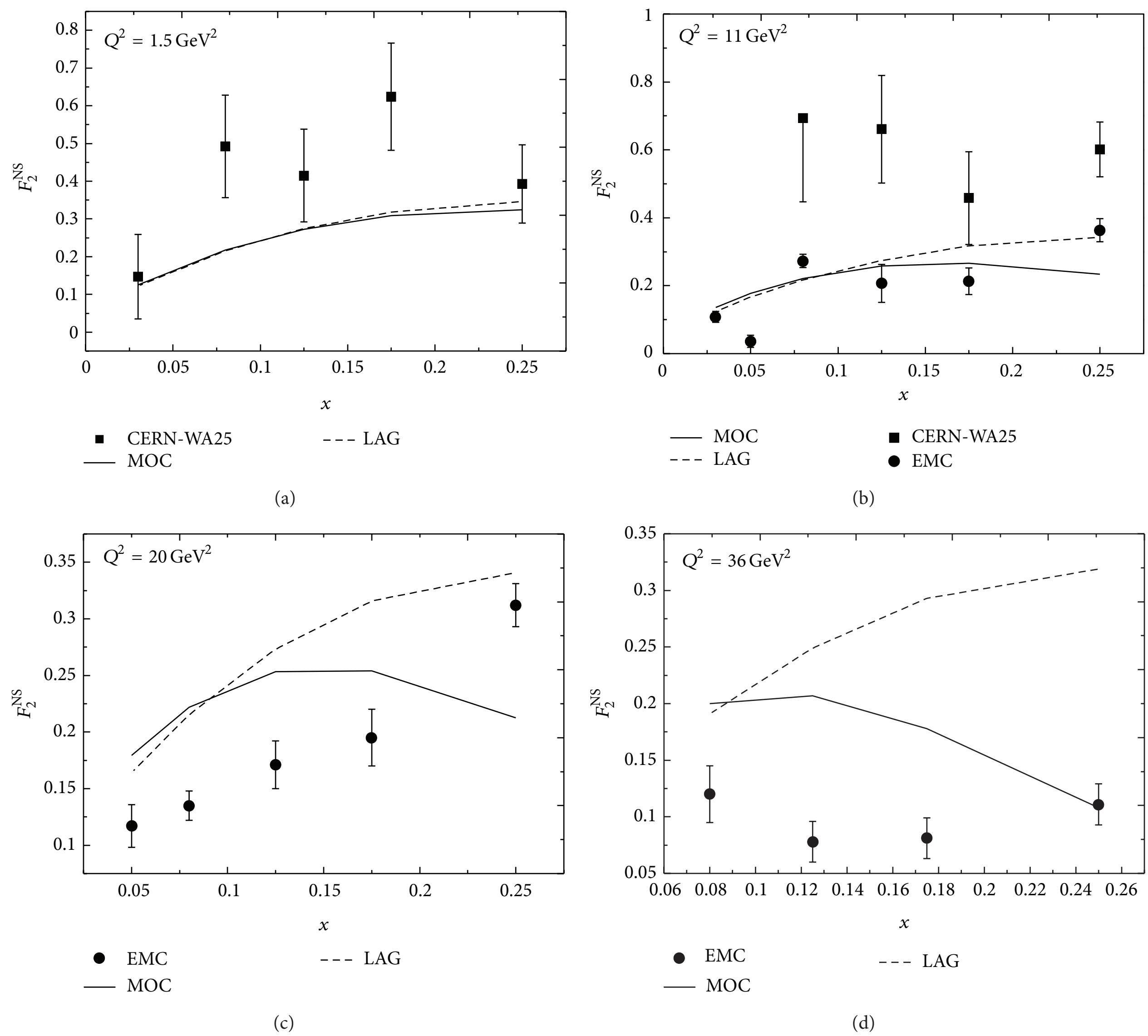

FIGURE 3: Nonsinglet structure function $F_2^{\mathrm{NS}}(x, Q^2)$ as a function of x at different fixed Q^2 according to (24) and (53). Data from [19, 20].

very few experimental data points available, we plot here the data within the range $1.5\,\mathrm{GeV}^2 \le Q^2 \le 20\,\mathrm{GeV}^2$ only. In the given range, we see that in the small Q^2 region $1\,\mathrm{GeV}^2 \le Q^2 \le 10\,\mathrm{GeV}^2$ the solution by the Lagrange's method (see (45)) gives a better description of the experimental data than the solution by the method of characteristics (see (19)), whereas in the larger Q^2 region $11\,\mathrm{GeV}^2 \le Q^2 \le 20\,\mathrm{GeV}^2$ the opposite is true.

In Figure 2, we plot the solutions (i.e., (19) and (45)) as functions of Q^2 at different fixed x and compare them with the same set of data. Here also we notice the same pattern consistent with the observation in the first graph. The solution by the Lagrange's method (45) is good for low values of $Q^2 \le 10\,\mathrm{GeV}^2$, particularly for the data from the CERN-WA25 [19], whereas the second solution by the method of characteristics (19) represents correctly the data which fall comparatively on the high $Q^2 \ge 10\,\mathrm{GeV}^2$ side, more particularly the data from EMC [20]. Comparing both the graphs we see that this observation, that is, validity of the solution (45) in the low $Q^2 \le 10\,\mathrm{GeV}^2$ region and that of (19) in the region $Q^2 \ge 10\,\mathrm{GeV}^2$, is true for all values of x explored here.

In Figure 3, we plot the solutions given by (24) and (53) as functions of x at some fixed Q^2 values with the same set of data. In the given range of data we see that both our analytical solutions are no more compatible with the experimental data beyond $Q^2 \ge 11\,\mathrm{GeV}^2$ for the entire x range $0.03 \le x \le 0.25$ explored here. Similarly in Figure 4, the solutions (24) and (53) are plotted as function of Q^2 against some fixed x values. From Figure 4 also we see that our analytical solutions (24) and (53) do not follow the trend of data beyond $x \ge 0.175$ for the entire Q^2 range explored.

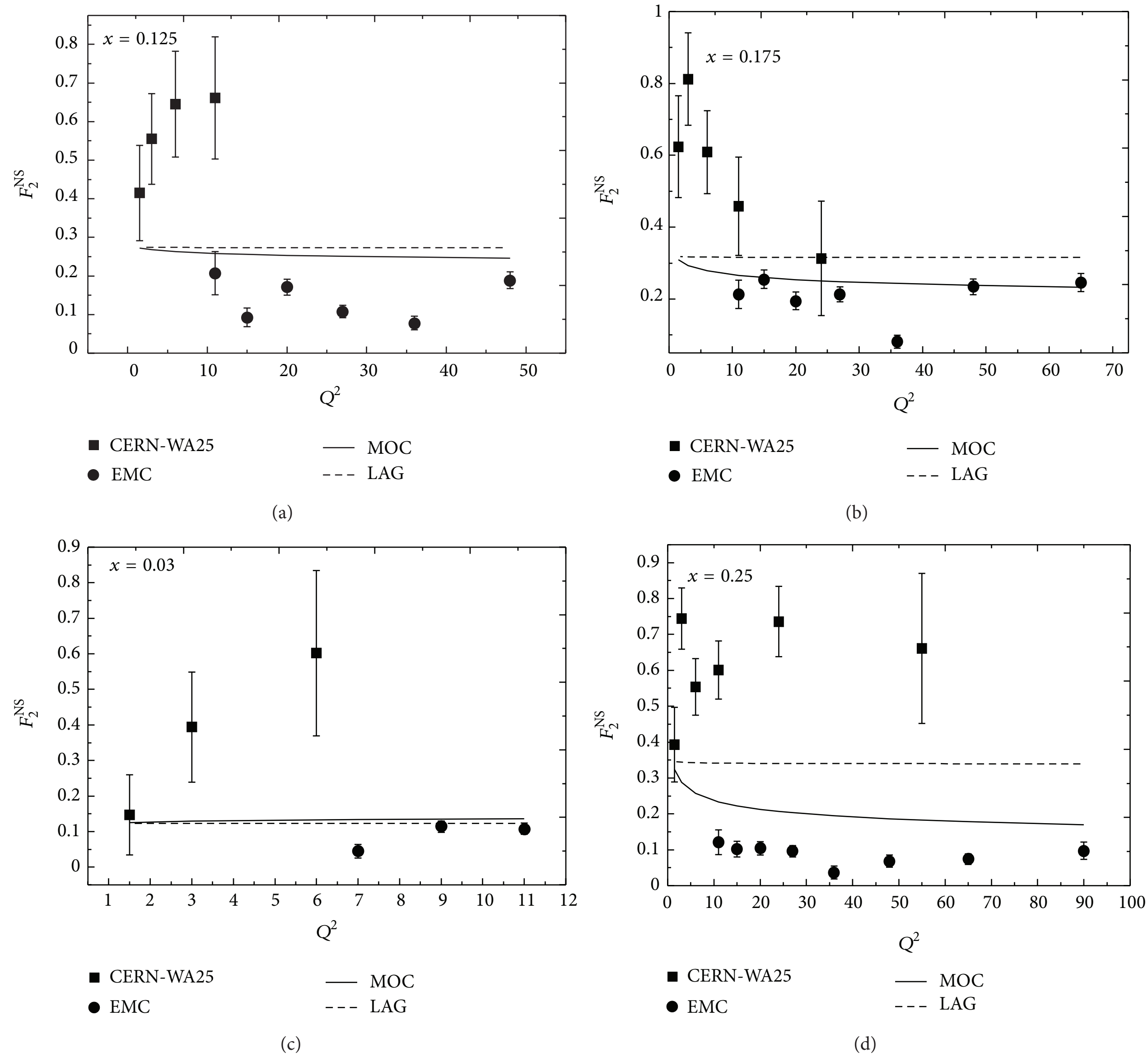

Figure 4: Nonsinglet structure function $F_2^{NS}(x, Q^2)$ as function of Q^2 at different fixed x according to (24) and (53). Data from [19, 20].

In Figures 5 and 6, we plot our set of solutions (19), (45), (24), and (53), respectively, with the BCDMS data where we explore a relatively high x range, $0.07 \le x \le 0.75$, and the corresponding Q^2 range is also high $13\,\mathrm{GeV}^2 \le Q^2 \le 63\,\mathrm{GeV}^2$. Here we use the recent MSTW08 input to evolve our solutions and we compare our solutions with the numerical solution by MSTW08 [27]. We compare our solutions as a function of x for different Q^2 values. From the figure we observe that though our both sets of solutions follow the general trend of data and agree with the numerical solutions towards low x range, as we approach the high x range our solutions overshoot both data and the numerical solutions.

From the above observations we can conclude that for both the set of analytical solutions (19), (45), (24), and (53) obtained from the two PDEs (9) and (10), towards low x they converge to the same limit, that is, they predict the same behaviour at low value of x. In case of the solutions obtained by Lagrange's method given by (45) and (53), while (45) shows logarithmic growth with the increasing Q^2 values, the other solution given by (53) remains almost constant for increasing Q^2 for fixed x values. The solutions by method of characteristics (19) and (24), also show very slow growth with increasing Q^2 for fixed x values.

We note that (10) is less accurate than (9), since the infinite sum in (5) is approximated by only two terms as shown in (11) to derive it and so are the solutions (24) and (53). It is also to be noted that (9) and (10) can be considered equivalent to the accuracy for $\mathcal{O}(x^2)$. Similarly the solutions (19) and (45) are also equivalent to the accuracy for $\mathcal{O}(x^2)$. But for the solutions (24) and (53), due to the neglect of few terms in (10) they do

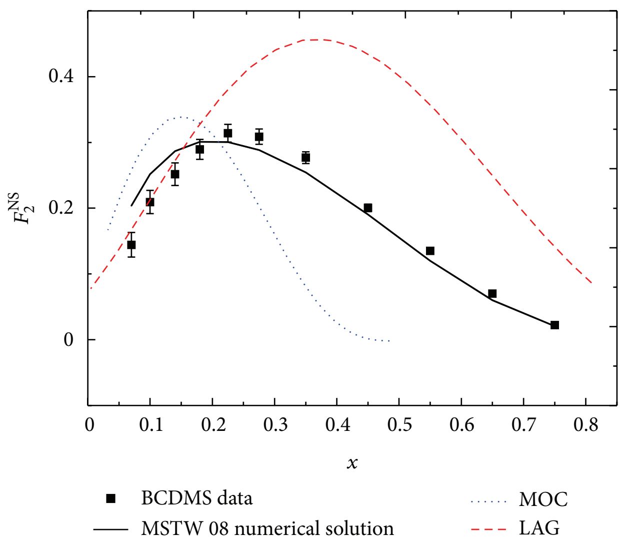

FIGURE 5: Nonsinglet structure function $F_2^{NS}(x, Q^2)$ as a function of x at different Q^2 values according to (19) and (45). Data from [24].

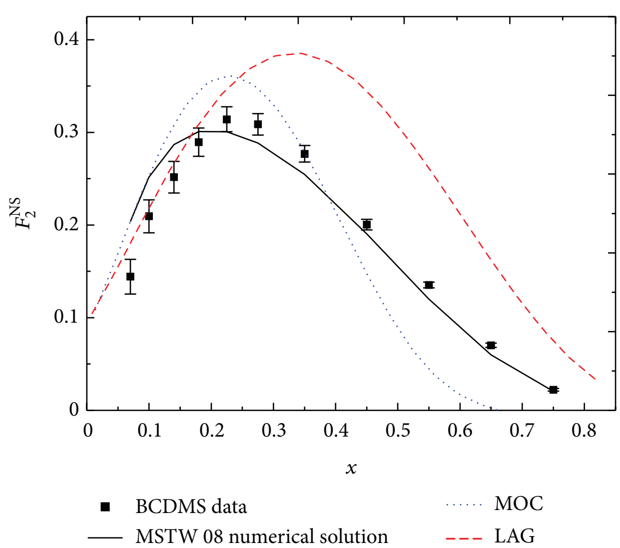

FIGURE 6: Nonsinglet structure function $F_2^{NS}(x, Q^2)$ as a function of x at different Q^2 values according to (24) and (53). Data from [24].

not require to be so. The solutions (19) and (24) appear to be different. The reason of difference is due to the fact that the solution (24) was obtained after neglecting a few terms $\mathcal{O}(x^2)$ in (10). Similar is the case for (45) and (53).

For a quantitative estimate of the goodness of fit between the solutions of the two analytical methods with the experimental data, we also do an χ^2 testing using the formula $\chi^2 = \sum_i ((X_{th} - X_{ex})/\sigma_i)^2$, where the theoretical (X_{th}) and experimental (X_{ex}) values are for the same ith data point with estimated uncertainty σ_i. Usually if χ^2/d.o.f is not much larger than one, the theoretical calculations are considered as being statistically consistent with data. In Table 1. we show that the χ^2/d.o.f values are consistent with the above discussion for the first set of solutions given by (19) and (45). The numbers

TABLE 1: χ^2/d.o.f. for (19) and (45).

Method	Inputs	CERN-WA25	EMC collaboration
Method of characteristics	MRST2004	9.69	1.48
Lagrange's method	MRST2004	1.47	10.86

TABLE 2: χ^2/d.o.f. for (24) and (53).

Method	Inputs	CERN-WA25	EMC collaboration
Method of characteristics	MRST2004	5.52	4.23
Lagrange's method	MRST2004	4.04	7.06

of degrees of freedom for the experiments CERN WA25 and EMC are 20 and 32, respectively.

We also compare (24) and (53), the two solutions of (10), obtained by method of characteristics and Lagrange's method, respectively. χ^2/d.o.f. values for the two solutions obtained by solving the PDE (10) are shown in Table 2. The χ^2 analysis also supports our above discussion.

Let us discuss the compatibility or otherwise of the present work with those of López and Ynduráin [28] and Martin [29]. For small x, Lopez-Yndurain and Martin-like analysis leads to the following behaviour of nonsinglet structure function:

$$F_2^{NS}(x,t) \sim x^{\lambda}\left(\frac{t}{t_0}\right)^{-d_{NS}(1-\lambda)}, \tag{54}$$

where $\lambda < 1$ and d_{NS} is anomalous dimension given as

$$d_{NS}(n) = \frac{\Gamma_{NS}(n)}{2\beta_0}. \tag{55}$$

This is to be compared with (19), (24), (45), and (53) above. Using the standard MRST [25] PDF,

$$F_2^{NS}(x,t_0) \sim x^{\lambda}, \tag{56}$$

where $\lambda = 0.5$. Thus except for the factor x^{λ}, which originates from the input [25], the present work differs from that of López and Ynduráin [28] and Martin [29]. In a sense it is close to the work of Vovk et al. [30], where the effective λ is x dependent contrary to the expectations of López, Ynduráin and Martin.

4. Conclusion

The Taylor approximated DGLAP equation for the nonsinglet structure function, which turns out to be a partial differential equation in two variables, is solved analytically by two different methods: the Lagrange's auxiliary method and the method of characteristics. However, further approximations in the partial differential equation give us a rather unsatisfactory result. The solutions are valid vis-a-vis the data and numerical solution in two different region of x and Q^2: the Lagrange's method solution for the lower x and Q^2 range and the method of characteristics solution for the low x and comparatively high Q^2 range. Considering the solutions together, they are valid in a wide range of Q^2 as discussed. This demonstrated that two powerful methods of solving differential equations can be applied in the DGLAP framework to obtain analytical solutions. Results of these methods to the polarized structure function $g_1^{\mathrm{NS}}(x,t)$ have been reported elsewhere [31].

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