Holomorphic higher-derivative terms in supersymmetric effective actions

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Abstract

We review recent results on the manifestly supersymmetric form of higher-derivative terms in the low energy effective actions of $N=1$ and 2 supersymmetric gauge theories in four dimensions. There is a rich structure of these higher-derivative terms. Many are holomorphic and therefore non-perturbatively calculable by the same approach as used in Seiberg-Witten theory. The algebraic structure of these higher-derivative terms in superspace is intricate and largely unexplored. Their classification even at the 4-derivative order is not complete.

1 Introduction

A common picture of the form of the derivative expansion for the low energy effective action, of, say, $N = 2$ superQCD on its Coulomb branch is $L \sim \int d^4\theta\, F(W) + \int d^4\theta d^4\overline{\theta}\, J(W, \overline{W}) + \ldots$, where W is the U(1) chiral field strength superfield in $N = 2$ superspace, F is the holomorphic 2-derivative term, calculable by Seiberg-Witten theory [1], and J is a non-holomorphic 4-derivative term, and therefore not exactly calculable by those techniques. Similar expressions are often quoted for the effective actions on other branches of $N = 2$ superQCD moduli spaces and for $N = 1$ superQCD effective actions as well.

The basic purpose of this talk is to point out that the derivative expansion is much more intricate than the above would suggest, and to review some recent results towards a systematic understanding of supersymmetric higher-derivative terms.

In section 2, we review derivative counting in SUSY effective actions. Section 3 describes some systematic results for 3-derivative terms in $N = 1$ superspace, while section 4 reviews some known 3- and 4-derivative terms in $N = 2$ effective actions. In sections 5 and 6 we turn to some global issues on the moduli space: electric-magnetic duality transformations of higher-derivative terms on Coulomb branches in section 5, and generalizations of global but not local F-terms in section 6. Sections 2 and 4 are largely a review of [2, 3], while sections 3, 5, and 6 report new results, described in more detail in [4, 5].

2 Derivative counting in SUSY effective actions

Denote the derivative-dimension by square brackets, so

$$[\partial/\partial x^\mu] := 1. \tag{1}$$

Then compatibility with the supersymmetry algebra implies that Grassmann coordinates should be assigned derivative-dimension $-1/2$, giving

$$[\partial/\partial\theta^\alpha] = 1/2 \text{ and } [\textstyle\int d\theta^\alpha \approx D_\alpha] = 1/2. \tag{2}$$

To be suitable for effective actions on a moduli space coordinatized by the vevs of scalar fields ϕ, we must assign derivative dimensions to the fields as

$$[\phi] = 0\ , \quad [\psi_\alpha] = 1/2\ , \quad [F_{\mu\nu}] = 1 \quad ([A_\mu] = 0), \tag{3}$$

where ψ_α are spin-1/2 fermions and A_μ and $F_{\mu\nu}$ are gauge potential and field strength. This assignment is compatible with gauge invariance ($[\partial_\mu] = [D_\mu]$) of the low energy action since generically the only massless fields on the moduli space are U(1) gauge fields and neutral scalars and spinors. Thus this counting works only at generic vacua: it cannot be applied to IR effective actions at special submanifolds where charged massless fields or non-abelian gauge fields persist.

For $N = 1$ superfields, we must then assign derivative dimensions

$$[\Phi] = 0\ , \quad [W_\alpha] = 1/2\ , \quad [V] = -1\ , \tag{4}$$

where Φ is a scalar chiral superfield, W_α is the field strength spinor chiral superfield, and V is the vector potential real superfield. All auxiliary fields have derivative dimensions consistent with the supersymmetry algebra. Familiar $N = 1$ superspace terms in the effective Lagrangian are

$$L \sim \int d^2\theta\, W(\Phi) + \int d^2\theta\, \tau(\Phi) W_\alpha W^\alpha + \int d^2\theta d^2\bar\theta\, K(\Phi, \overline{\Phi}) + \ldots \tag{5}$$

Derivative counting shows that the τ- and K-terms are 2-derivative (kinetic) terms, while the W-term (the superpotential) is a 1-derivative term. This last is in contradiction to the fact that the superpotential term gives rise to 0-derivative scalar potentials through integrating out auxiliary fields. This illustrates the fact that this assignment of derivative dimensions is only valid for the IR effective action, where only the *massless fields* on the moduli space appear, and where $W(\Phi) \equiv 0$ by definition.

With this derivative counting, there are a finite number of terms at each order in the derivative expansion with coefficients which are functions on the moduli space. This follows since space-time derivatives and superderivatives as well as all other gauge invariant superfields have positive derivative dimension. The moduli-dependent coefficient functions, like $\tau(\Phi)$ and $K(\Phi, \overline{\Phi})$ encode all the observables of the low energy effective action, and computing their dependence on the microscopic (UV) parameters defining the theory is the ultimate goal of developing a supersymmetric effective action formalism. The implicit assumption is made that all gauge invariant terms in the low energy effective action can be written using only the field strength superfield W_α, and not the potential superfield V. The validity of this assumption of the non-existence of *Chern-Simons-like* superspace terms [3] is an open question, though examples of such terms globally on the moduli space are known (see section 6 below).

For $N = 2$ superfields in harmonic superspace (see *e.g.* [6] for a review), derivative dimensions are assigned as

$$[q^+] = 0\ , \quad [W] = 0\ , \quad [V^{++}] = -1\ , \tag{6}$$

where q^+ is the scalar analytic superfield describing a hypermultiplet, W is the scalar chiral superfield describing the field strength multiplet, and V^{++} is the analytic superfield describing the vector potential multiplet. In harmonic superspace, the harmonic coordinates $u^\pm$ on the internal S^2 carry zero derivative-dimension. Familiar examples of low-derivative terms in $N = 2$ effective actions are

$$L \sim \int d^4\theta\, F(W) + \int du\, d^4\theta^+\, L(q^+, \tilde{q}^+, u^\pm) + \int d^4\theta d^4\overline{\theta}\, J(W, \overline{W}) + \ldots$$

where $\int d^4\theta^+ \approx (D^-)^4$ is the analytic superspace measure, $\int d^4\theta$ is the chiral superspace measure, and $\int du$ is the harmonic S^2 invariant measure. Derivative counting shows that F and L are 2-derivative (generalized kinetic) terms, including for example component terms like $g(\phi)(\partial\phi)^2$, $h(\phi)\overline{\psi}\partial\!\!\!/\psi$, $\tau(\phi)\mathrm{tr}(F^2)$, *etc.*, while J is a 4-derivative term, containing *e.g.*, $\mathrm{tr}(F^4)$, $(\overline{\psi}\psi)^4$, *etc.*

As in the $N = 1$ case, the assumption of no Chern-Simons-like superspace terms leads to a well-defined derivative expansion. One potentially troubling point is that since the derivative dimension vanishes for the harmonic coordinates $u^\pm$ and their derivatives, it would seem that at each order in the derivative expansion an infinite series of harmonic derivatives should be considered (thus rendering the derivative expansion practically useless in harmonic superspace). However, it was shown in [2] that there exists a finite bound on the number of harmonic derivatives that need be included at each order in the derivative expansion: upon eliminating auxiliary fields, terms with more than this number of derivatives are not independent.

3 3-derivative terms in N=1 effective actions

Armed with this derivative-counting calculus it is easy to write a set of manifestly $N = 1$ supersymmetric 3-derivative terms which are inequivalent under integration by parts:

$$\begin{aligned}
L_{3a} &= \int d^2\theta\, \partial_\mu\Phi^i\partial^\mu\Phi^j\; F_{ij}(\Phi) \;+\; \text{c.c.} \\
L_{3b} &= \int d^2\theta\, (W^a \cdot W^b)(W^c \cdot W^d)\; G_{abcd}(\Phi) \;+\; \text{c.c.} \\
L_{3c} &= \int d^4\theta\, (W^a \cdot W^b)\; H_{ab}(\Phi, \overline{\Phi}) \;+\; \text{c.c.} \\
L_{3d} &= \int d^4\theta\, (W^a \cdot D\Phi^i)\; I_{ai}(\Phi, \overline{\Phi}) \;+\; \text{c.c.} \\
L_{3e} &= \int d^4\theta\, (D\Phi^i \cdot D\Phi^j)\; J_{ij}(\Phi, \overline{\Phi}) \;+\; \text{c.c.}
\end{aligned} \tag{7}$$

Here the i, j and $a, b, \ldots$ subscripts denote the different species of Φ and W_α chiral multiplets, and the dot product denotes spinor contraction.

The L_{3a} term is chiral (cannot be written as an integral over the full superspace) as long as $F_{i[j,k]} \neq 0$, where $X_{,k} := \partial X/\partial\Phi^k$. Otherwise, the L_{3a} term is a special case of the L_{3e} term (up to global issues on the moduli space; see section 6 below). Since the coefficients of the chiral L_{3a} and L_{3b} terms depend holomorphically on the Φ's, their functional forms are tightly restricted by the usual non-renormalization theorems. These are thus terms which in specific models may be exactly computed using Seiberg's arguments (see, *e.g.*, [7] for a review). Note that the L_{3a} (L_{3b}) holomorphic 3-derivative terms require at least two massless chiral (vector) multiplets to be non-zero, respectively.

In (7) we simply presented a list of manifestly supersymmetric 3-derivative terms. How do we know that that list is all of them? For example, perhaps there exists some complicated 2-derivative function of superfields, $f(W, \overline{W}, \Phi, \overline{\Phi}, D, \overline{D})$, which is chiral in a non-obvious way by virtue of the supersymmetry algebra and the Bianchi identity satisfied by W_α. If so, then $\int d^2\theta\, f$ would be another 3-derivative term.

It turns out that a systematic search for all such terms by the straight forward method of listing all possible terms and checking their supersymmetry variations is already lengthy for 3-derivative terms, and is prohibitively complicated for 4-derivative terms. The result of the 3-derivative search is that the above list is complete modulo global issues to be discussed below [4].

The algebraic structure underlying the classification of supersymmetric terms in low energy effective actions is a kind of differential complex with the superderivatives, D_α, $\overline{D}_{\dot\alpha}$, playing the role of the exterior derivatives and the superfields and their space-time derivatives playing the role of coordinate functions on some manifold. This point of view may be helpful in finding more efficient ways of classifying higher-derivative supersymmetry invariants [4].

4 3- and 4-derivative terms in N=2 effective actions

A similar list was been proposed for $N = 2$ effective actions in [2]. As might be expected, since $N = 2$ supersymmetry is more restrictive, there are fewer terms, and a list of both 3- and 4-derivative terms can be found:

$$\begin{aligned}
L_3 &= \int du\, (D^-)^2(\overline{D}^-)^2(D^+)^2\; F(q^+, W) \\
L_{4a} &= \int du\, (D^-)^2(\overline{D}^-)^2(D^+)^2\, (D^+W^i)(D^+W^j)\; G_{ij}(q^+, W) \\
L_{4b} &= \int du\, (D^-)^2(\overline{D}^-)^2(D^+)^2\, (\overline{D}^- q_a^+)(\overline{D}^- q_b^+)\; H^{ab}(q^+, W) \\
L_{4c} &= \int du\, (D^-)^2(\overline{D}^-)^2(D^+)^2(\overline{D}^+)^2\; J(q^+, q^-, W, \overline{W})
\end{aligned} \tag{8}$$

(For simplicity, the $u^\pm$ derivatives on the harmonic S^2 have been suppressed.) The Grassmann measures have been written as superderivatives for clarity. Note that the L_3, L_{4a}, and L_{4b} terms are integrals over 3/4 of superspace, the union of its chiral and the analytic halves. (The other three unions, *e.g.*, anti-chiral plus analytic, *etc.*, are also allowed terms.)

There are a few restrictions on the terms in (8) which should be mentioned:

• $L_3 \not\equiv 0$ only for mixed branches (*i.e.*, F must depend on both q^+ and W).

• $L_{4a} \neq L_{4c}$ only if $G_{i[j,k]} \neq 0$, which implies that the holomorphic L_{4a} can exist only on branches of the moduli space where the number of vector multiplets is greater than or equal to 2.

• $L_{4a} \equiv 0$ on pure Higgs branches (*i.e.*, if G depends only on q^+'s).

• $L_{4a} \equiv \int d^4\theta\, (\partial_\mu W^i)(\partial^\mu W^j) G_{ij}(W)$ on pure Coulomb branches (*i.e.*, if G depends only on W's).

- $L_{4b} \equiv 0$ on pure Coulomb branches.
- $L_{4b} \equiv \int du\, d^4\theta^+ (\partial_\mu q_a^+)(\partial^\mu q_b^+) H^{ab}(q^+)$ on pure Higgs branches.

There are (infinitely) many examples of asymptotically free $N = 2$ superQCD theories with appropriate moduli spaces so that non-vanishing L_3, L_{4a}, and L_{4b} terms could exist. Note further that the coefficient functions, F, G, and H, of these 3/4-superspace terms are holomorphic, and so are amenable to exact calculation using the technique of [1]. The basic idea behind this technique is analytic continuation from perturbative boundary data on the moduli space. One ingredient in this program, a boundary datum, has been calculated in [8] as the one-loop contribution to F in certain $N = 2$ superQCD theories. To perform the analytic continuation from such data, the global properties of F, G, and H on the moduli space must be determined. In particular, F, G, and H need not be single-valued functions on the moduli space, but may have non-trivial monodromies around singularities. The determination of these allowed monodromies is the topic of the next section.

Before turning to that topic, there is the question of whether the list of 3- and 4-derivative $N = 2$ terms in (8) is complete. A systematic search seems difficult, but in this case there is an indirect argument which implies that there must be another 4-derivative term. The argument is as follows: In the SU(2) $N = 4$ SYM theory there is a 4-derivative Wess-Zumino term in the effective action on the moduli space [9]. The contribution to the equations of motion coming from this term is covariant under the $N = 4$ SO(6) R-symmetry, treating the six real scalars on the moduli space symmetrically. However in [2] the $L_{4a,b,c}$ four-derivative terms were shown to contribute a Wess-Zumino term in which the hypermultiplet and vector multiplet scalars appear asymmetrically, and so cannot reproduce the known $N = 4$ Wess-Zumino term. Thus another $N = 2$ 4-derivative term must exist.

5 Electric-magnetic duality

For simplicity, concentrate on a pure Coulomb branch. In this case only the L_{4a} and L_{4c} terms occur:

$$\begin{aligned} L_{4a} &= \int d^4\theta\, (\partial_\mu W^i)(\partial^\mu W^j)\, G_{ij}(W), \qquad G_{i[j,k]} \neq 0, \\ L_{4c} &= \int d^4\theta\, d^4\overline{\theta}\ J(W, \overline{W}). \end{aligned}$$

We want to determine how G and J are allowed to change upon traversing a closed path on the Coulomb branch. The allowed changes correspond to the possible different low energy actions which describe equivalent physics.

On the Coulomb branch, the only non-trivial such equivalence is that of electric-magnetic duality, under which two U(1) theories with field strengths $F_{\mu\nu}$ and $G_{\mu\nu}$ are equivalent, $(1/e^2)\mathrm{tr}(F^2) \equiv e^2\mathrm{tr}(G^2)$, if $F_{\mu\nu}$ couples with strength e $(1/e)$ to massive electric (magnetic) sources, while $G_{\mu\nu}$ couples with the inverse strengths. This, together with the angularity of the theta angle, gives rise to the familiar $\mathrm{Sp}(2n, \mathbb{Z})$ discrete group of equivalences of the $\mathrm{U}(1)^n$ theory under which the allowed monodromies of the gauge coupling function $\tau_{ij}(W)$ are $\tau \to \tau_D = (A\tau + B)/(C\tau + D)$ for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2n, \mathbb{Z})$.

The above describes the action of electric-magnetic duality on the two-derivative term in the effective action. This is extended to higher-derivative terms in [3], where a very simple answer in $N = 2$ (harmonic) superspace is found: the electric-magnetic dual action is the Legendre transform with respect to chiral field strength superfield W,

$$L_D(W_D, \overline{W}_D) = L(W, \overline{W}) - \int d^4\theta\, W_D W + \text{c.c.}\Big|_{W_D = \delta L/\delta W}.$$

This is similar to the answer in non-supersymmetric non-linear electrodynamics [10], where the Legendre transform is with respect to the field strength. However, one new point is that when L includes higher-derivative terms, the Legendre transform is not well-defined by itself, but must be specified by further data in a way consistent with the derivative expansion of low energy effective actions [3].

Performing the Legendre transform appropriately gives that the original IR action, $L = \int d^4\theta\, [F + (\partial W)^2 G] + \int d^8\theta\, J$, transforms to the equivalent (dual) IR action, L_D, with $F \to F_D = F - W^i W_{Di}$, while the G and J terms remain invariant, $J \to J_D = J$ and $G_{ij} \to G_D^{ij} = F^{,ik} G_{k\ell} F^{,\ell j}$. Here the right hand sides are evaluated at $W^i = w^i(W_D)$ such that $F_{,i}(w) = W_{Di}$. Also $F^{,ij} \equiv (F_{,ij})^{-1}$ as a matrix on i, j, is the Jacobian for the change of variables from W to W_D. Thus the above transformation rule for G means that in local coordinates on the Coulomb branch, G is single-valued.

This simple result for the electric-magnetic duality monodromies only holds because the G and J terms are the leading higher-derivative corrections to the 2-derivative term on the Coulomb branch. On mixed branches, however, where the 3-derivative term is the leading correction, the electric-magnetic duality monodromies are more complicated, as the four derivative terms can shift by expressions involving the coefficient functions the three-derivative term [3].

6 Topological issues

There are other ways that the global structure of the moduli space can enter into the existence and determination of higher-derivative terms. The first way is through the existence of terms which can be written as non-chiral terms locally on the moduli space, but not globally on the moduli space. This obstruction means that globally they can only be written as integrals over a fraction of superspace, *i.e.*, as chiral terms or 3/4-superspace terms. A special set of such globally chiral terms was discussed in [11].

Global chiral terms and holomorphic de Rham cohomology. A systematic search among 3-derivative $N = 1$ terms [4] finds many such terms. Two examples which illustrate them are

$$\begin{aligned} \int d^4\theta\ \alpha W^2 &= \int d^2\theta\, d\bar{\theta}_{\dot\alpha}\ a\, W^2 \overline{D}^{\dot\alpha} \overline{\Phi}, \\ \int d^4\theta\ \alpha W D\Phi &= \int d^2\theta\ \left[a\,(WD\Phi)(\overline{D}^2\overline{\Phi}) + a'\,(WD\Phi)(\overline{D}\overline{\Phi})^2 - 2a\,(W\partial\Phi)(\overline{D}\overline{\Phi}) \right]. \end{aligned} \tag{9}$$

Here a and α are functions of the scalar chiral superfields Φ and $\overline{\Phi}$ related by

$$a = \alpha', \tag{10}$$

and the prime denotes a derivative with respect to $\overline{\Phi}$. The non-chiral 3-derivative terms on the left can be rewritten as the 3/4-superspace or chiral terms on the right when

(10) is satisfied. But if there is an obstruction to finding an a globally on the moduli space solving $a = a'$, then the right sides cannot be rewritten as the non-chiral left sides globally on the moduli space. This obstruction is measured by a holomorphic de Rham cohomology class of the moduli space.

A similar example occurs for the four-derivative $N = 2$ terms. The chiral $L_{4a} = \int d^4\theta\, \partial_\mu W^i \partial^\mu W^j\, G_{ij}(W)$ can be re-written using the Bianchi identity for W as the non-chiral term $-8 \int d^8\theta\, W^i \overline{W}^j\, K_{ij}(W)$ term when

$$G_{ij}(W) = \partial_{(j}[W^k K_{i)k}(W)] \tag{11}$$

for some K_{ik}. With two or more vector multiplets, (11) can fail to be integrable even locally, $G_{i[j,k]} \neq 0$, so locally holomorphic L_{4a} terms exist, as was pointed out above. But (11) can also fail to be integrable globally: L_{4a} defines a holomorphic quadratic form on Coulomb branch, $\mathcal{G} \equiv G(W)\,(dW)^2$, and the obstruction to solving (11) is again measured by the holomorphic de Rham cohomology on the Coulomb branch, $\mathcal{G} \sim \mathcal{G} + \partial\Gamma$.

Chern-Simons-like terms and Dolbeault cohomology A different way in which the topology of moduli space can play a role is through the existence of obstructions to certain local gauge-invariant terms being written in a manifestly gauge invariant way globally [3].

A simple example is the non-chiral $L_{4c} = \int d^8\theta\, J(W, \overline{W})$ term in $N = 2$ superspace. Any such term with $J(W, \overline{W}) = B^i(W, \overline{W})\, \overline{W}_i$, can be rewritten as

$$\int du\, d^8\theta\, V^{--i}\, D^+\big(A^i_j(W, \overline{W}) D^+ W^j\big) + \text{c.c.} \tag{12}$$

where $A^i_j = \partial_j B^i$. Note that (12) is a Chern-Simons-like term: it uses the potential superfield V^{--}. (Note that this does not imply that it gives rise to a Chern-Simons term in *components*—we are four dimensions, after all, where Chern-Simons terms do not exist.) Some algebra shows that (12) is non-vanishing whenever

$$\partial_{[k} A^i_{j]} = 0, \tag{13}$$

which is the local integrability condition to rewrite (12) as $\int d^8\theta\, J$.

But J (or the B^i) may fail to exist globally on the Coulomb branch. The Chern-Simons-like term defines a set of (1,0)-forms on the Coulomb branch, $\mathcal{A}^i \equiv A^i_j(W, \overline{W})\, dW^j$, so (12,13) implies that non-trivial Chern-Simons-like terms are classified by Dolbeault cohomology classes on the Coulomb branch.

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