Oscillating Universe in Hořava-Lifshitz Gravity

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Abstract

We study the dynamics of isotropic and homogeneous universes in the generalized Hořava-Lifshitz gravity, and classify all possible evolutions of vacuum spacetime. In the case without the detailed balance condition, we find a variety of phase structures of vacuum spacetimes depending on the coupling constants as well as the spatial curvature K and a cosmological constant Λ. A bounce universe solution is obtained for $\Lambda > 0, K = \pm 1$ or $\Lambda = 0, K = -1$, while an oscillation spacetime is found for $\Lambda \geq 0, K = 1$, or $\Lambda < 0, K = \pm 1$.

1 Introduction

Recently Hořava proposed a power-counting renormalizable theory of gravity [1], which has attracted much attention over the past year. In Hořava's theory, Lorentz symmetry is broken and it exhibits a Lifshitz-like anisotropic scaling in the ultraviolet (UV), $t \to \ell^z t, \vec{x} \to \ell \vec{x}$, with the dynamical critical exponent $z = 3$. (For this reason the theory is called Hořava-Lifshitz (HL) gravity.) Anisotropic scaling leads higher curvature terms in action with arbitrary coupling constants. In the original HL gravity, the so-called detailed balance condition is assumed, which limit these coupling constants. However, this condition can be loosened. In application to cosmology, a possibility of big bang initial singularity avoidance is indicated, which is led by higher curvature terms in the action come from anisotropic scaling. We study the dynamics of isotropic and homogeneous universe without any matters in the generalized HL gravity, and discuss the conditions for non-singular solutions i.e. bouncing and oscillating solutions.

2 Hořava-Lifshitz gravity and the coupling constants

The basic variables in HL gravity are the lapse function, N, the shift vector, N_i, and the spatial metric, g_{ij}. These variables are subject to the action [1, 2]

$$S = \frac{1}{2\kappa^2} \int dt d^3x \sqrt{g} N \left(\mathscr{L}_K - \mathscr{V}_{\rm HL}[g_{ij}] \right), \tag{1}$$

where $\kappa^2 = 1/M_{\rm PL}^2$ and the kinetic term is given by

$$\mathscr{L}_K = \mathcal{K}_{ij}\mathcal{K}^{ij} - \lambda \mathcal{K}^2 \tag{2}$$

with

$$\mathcal{K}_{ij} := \frac{1}{2N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) \tag{3}$$

being the extrinsic curvature. The potential term $\mathscr{V}_{\rm HL}$ will be defined shortly. In general relativity we have $\lambda = 1$, only for which the kinetic term is invariant under general coordinate transformations. In HL

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gravity, however, Lorentz symmetry is broken in exchange for renormalizability and the symmetry of the theory is invariance under the foliation-preserving diffeomorphism transformations,

$$t \to \bar{t}(t), \quad x^i \to \bar{x}^i(t, x^j). \tag{4}$$

As implied by the symmetry (4) it is most natural to consider the projectable version of HL gravity, for which the lapse function is dependent only on t: $N = N(t)$ [1]. Since the Hamiltonian constraint is derived from the variation with respect to the lapse function, in the projectable version of the theory the resultant constraint equation is not imposed locally at each point in space, but rather is an integration over the whole space. In the cosmological setting, the projectability condition results in an additional dust-like component in the Friedmann equation [see Eq. (7) below] [3].

The most general form of the potential $\mathcal{V}_{\rm HL}$ is given by [2]

$$\begin{aligned} \mathcal{V}_{\rm HL} &= 2\Lambda + g_1\mathcal{R} + \kappa^2\left(g_2\mathcal{R}^2 + g_3\mathcal{R}^i_j\mathcal{R}^j_i\right) + \kappa^3 g_4\epsilon^{ijk}\mathcal{R}_{i\ell}\nabla_j\mathcal{R}^\ell_k \\ &\quad + \kappa^4\left(g_5\mathcal{R}^3 + g_6\mathcal{R}\,\mathcal{R}^i_j\mathcal{R}^j_i + g_7\mathcal{R}^i_j\mathcal{R}^j_k\mathcal{R}^k_i + g_8\mathcal{R}\Delta\mathcal{R} + g_9\nabla_i\mathcal{R}_{jk}\nabla^i\mathcal{R}^{jk}\right), \end{aligned} \tag{5}$$

where Λ is a cosmological constant, $\mathcal{R}^i_j$ and $\mathcal{R}$ are the Ricci and scalar curvatures of the 3-metric g_{ij}, respectively, and g_i's $(i = 1, ..., 9)$ are the dimensionless coupling constants. Requiring a stability of flat spacetime, we impose following conditions [see [4]]: $g_1 < 0, \quad g_9 > 0$. By a suitable rescaling of time, we then set $g_1 = -1$. In what follows, we adopt the unit of $\kappa^2 = 1 (M_{\rm PL}^2 = 1)$ for brevity.

3 FLRW universe in Hořava-Lifshitz gravity

We discuss an isotropic and homogeneous vacuum universe in Hořava-Lifshitz gravity. Note that such a vacuum spacetime is not realized in general relativity.

Assuming a FLRW spacetime, which metric is given by

$$ds^2 = -dt^2 + a^2\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right), \tag{6}$$

with $K = 0$ or ±1. We find the Friedmann equation as

$$H^2 + \frac{2}{(3\lambda-1)}\frac{K}{a^2} = \frac{2}{3(3\lambda-1)}\left[\Lambda + \frac{g_{\rm d}}{a^3} + \frac{g_{\rm r}}{a^4} + \frac{g_{\rm s}}{a^6}\right], \tag{7}$$

where $H = \dot{a}/a$,

$$\begin{aligned} g_{\rm d} &:= 8C\,. \\ g_{\rm r} &:= 6(g_3 + 3g_2)K^2\,, \\ g_{\rm s} &:= 12(9g_5 + 3g_6 + g_7)K^3\,. \end{aligned} \tag{8}$$

A constant C may appear from the projectability condition and could be "dark matter"[2]. For a flat universe ($K = 0$), the higher curvature terms do not give any contribution, and then the dynamics is almost trivial. Hence, in this paper, we discuss only non-flat universe ($K = \pm1$). Note that imposing detailed balance condition, $g_r > 0$ and $g_r = 0$.

If $\lambda = 1$, we find a usual Friedmann equation for an isotropic and homogeneous universe in GR with a cosmological constant, dust, radiation and stiff matter. If $g_{\rm d}, g_{\rm r}$, and $g_{\rm s}$ are non-negative, such a spacetime gives a conventional FLRW universe model. However, since those coefficients come from higher curvature terms, their positivity is not guaranteed. Rather some of them could be negative. As a result, we find an unconventional cosmological scenario, which we shall discuss here. In what follows, we assume that $\lambda > 1/3$, but do not fix it to be unity.

Here we assume $C = 0$ just for simplicity. The Friedmann equation is written as

$$\frac{1}{2}\dot{a}^2 + \mathcal{U}(a) = 0\,, \tag{9}$$

where

$$\mathcal{U}(a) = \frac{1}{3\lambda - 1}\left[K - \frac{\Lambda}{3}a^2 - \frac{g_{\rm r}}{3a^2} - \frac{g_{\rm s}}{3a^4}\right] . \tag{10}$$

Since the scale factor a changes as a particle with zero energy in this "potential" $\mathcal{U}$, the condition $\mathcal{U}(a) \leq 0$ gives the possible range of a when the universe evolves. So we can classify the "motion" of the universe by the signs of K and Λ, and by the values of $g_{\rm r}$ and $g_{\rm s}$. For the case of $\Lambda \neq 0$, introducing the curvature scale ℓ which is defined by

$$\frac{\Lambda}{3} = \frac{\epsilon}{\ell^2} , \tag{11}$$

where $\epsilon = \pm 1$, we can rescale the variables and rewrite the "potential" $\mathcal{U}$ by the rescaled variables as

$$\mathcal{U}(a) = \frac{1}{3\lambda - 1}\left[K - \epsilon\tilde{a}^2 - \frac{\tilde{g}_{\rm r}}{3\tilde{a}^2} - \frac{\tilde{g}_{\rm s}}{3\tilde{a}^4}\right] , \tag{12}$$

where $\tilde{a} = a/\ell$, $\tilde{g}_{\rm r} = g_{\rm r}/\ell^2$, and $\tilde{g}_{\rm s} = g_{\rm s}/\ell^4$. Using this potential and variables, we can discuss the fate of the universe without specifying the value of Λ.

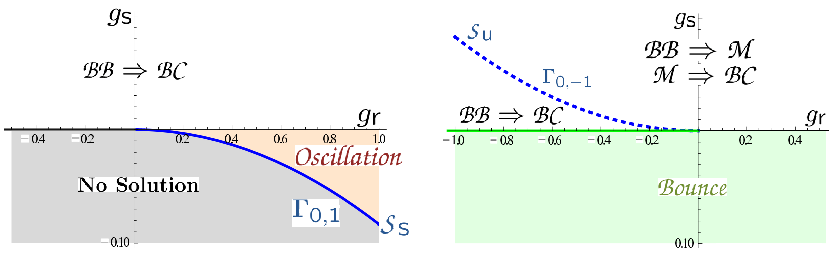
A static universe will appear if we find a solution $a = a_S (> 0)$ which satisfies $\mathcal{U}(a_S) = 0$ and $\mathcal{U}'(a_S) = 0$. If $\Lambda \neq 0$ ($\epsilon = \pm 1$), it happens if there is a relation between $\tilde{g}_{\rm r}$ and $\tilde{g}_{\rm s}$, which is defined by

$$\tilde{g}_{\rm s} \;\; = \;\; \tilde{g}_{\rm s}^{[\epsilon,K](\pm)}(\tilde{g}_{\rm r}) := \frac{1}{9\epsilon^2}\left[2K - 3\epsilon K\tilde{g}_{\rm r} \pm 2(1 - \epsilon\tilde{g}_{\rm r})^{3/2}\right] . \tag{13}$$

This gives the curve $\Gamma_{\epsilon,K(\pm)}$ on the $\tilde{g}_{\rm r}$-$\tilde{g}_{\rm s}$ plane, which gives the boundary between two different phases of spacetime. If $\Lambda = 0$, we find

$$g_{\rm s} = -\frac{K}{12}g_{\rm r}^2 , \tag{14}$$

which is found from (13) in the limit of $\epsilon = 0$. The corresponding curve on the $g_{\rm r}$-$g_{\rm s}$ plane is denoted by $\Gamma_{0,K}$. We summarize our result in Figure 3 to 3. In these figures, we denote $\mathcal{BB}$ is big bang, $\mathcal{BC}$ is big crunch, $d\mathcal{S}$ is de Sitter universe, $\mathcal{M}$ is Milne universe, *bounce* is bouncing universe, *oscillation* is oscillating universe and $\mathcal{S}$ is static universe. There are two types of static universes: one is stable ($\mathcal{S}_{\rm s}$) and the other is unstable ($\mathcal{S}_{\rm u}$).

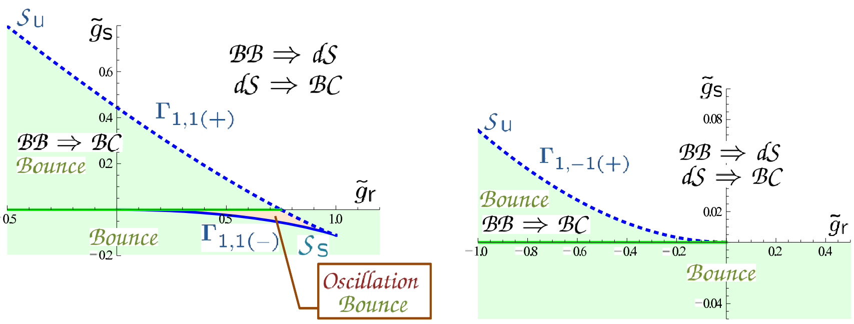


(a) $K = 1$ (b) $K = -1$

Figure 1: Phase diagram of spacetimes for $\Lambda = 0$. The oscillating universe is found only for the case of $K = 1$. The stable and unstable static universes ($\mathcal{S}_{\rm s}$ and $\mathcal{S}_{\rm u}$) exist on the boundary $\Gamma_{0,1}$ and $\Gamma_{0,-1}$, respectively.

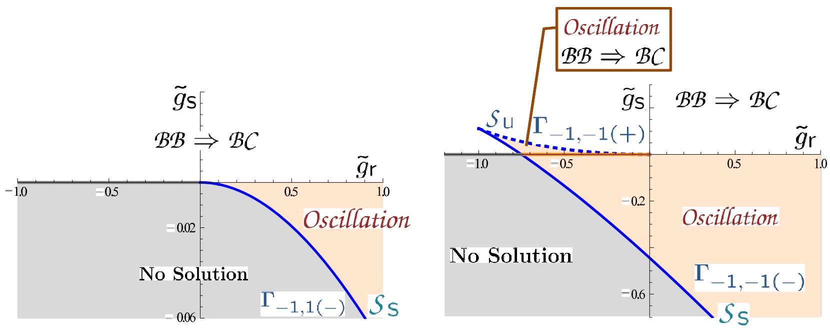
4 Conclusion

We have study the classification of vacuum FLRW universe in generalized HL gravity. As we have evaluated, the oscillation period and amplitude are expected to be the Planck scale or the scale ℓ defined



(a) $K = 1$ (b) $K = -1$

Figure 2: Phase diagram of spacetimes for $\Lambda > 0$. The oscillating universe is found only for the case of $K = 1$. The static universes ($\mathcal{S}_{\rm s}$ and $\mathcal{S}_{\rm u}$) exist on the boundaries $\Gamma_{1,K(\pm)}$.



(a) $K = 1$ (b) $K = -1$

Figure 3: Phase diagram of spacetimes for $\Lambda < 0$. The oscillating universe is found for both $K = \pm 1$. The static universe exists on the boundary $\Gamma_{-1,1(-)}$ ($K = 1$) and on $\Gamma_{-1,-1(\pm)}$ ($K = -1$).

by a cosmological constant Λ, unless the coupling constants are unnaturally large. Hence it cannot be a cyclic universe, which period is macroscopic such as the age of the universe. To find more realistic universe, we discuss further in [4]. We also have another extension of the present FLRW spacetime to anisotropic one. It may be interesting and important not only to study the dynamics of Bianchi spacetime but also to analyze the stability of the FLRW universe against anisotropic perturbations.

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