Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B 965 (2021) 115355
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Yang-Baxter R-operators for osp superalgebras

A.P. Isaev [a,b,d], D. Karakhanyan [a,c], R. Kirschner [e,*]

[a] *Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Russia*
[b] *Physics Faculty, M.V. Lomonosov State University, Moscow, Russia*
[c] *Yerevan Physics Institute, 2 Alikhanyan br., 0036 Yerevan, Armenia*
[d] *St. Petersburg Department of Steklov Mathematical Institute of RAS, Fontanka 27, 191023 St. Petersburg, Russia*
[e] *Institut für Theoretische Physik, Universität Leipzig, PF 100 920, D-04009 Leipzig, Germany*

Received 10 December 2020; accepted 21 February 2021
Available online 25 February 2021
Editor: Hubert Saleur

Abstract

We study Yang-Baxter equations with orthosymplectic supersymmetry. We extend a new approach of the construction of the spinor and metaplectic $\hat{\mathcal{R}}$-operators with orthogonal and symplectic symmetries to the supersymmetric case of orthosymplectic symmetry. In this approach the orthosymplectic $\hat{\mathcal{R}}$-operator is given by the ratio of two operator valued Euler Gamma-functions. We illustrate this approach by calculating such $\hat{\mathcal{R}}$ operators in explicit form for special cases of the $osp(n|2m)$ algebra, in particular for a few low-rank cases. We also propose a novel, simpler and more elegant, derivation of the Shankar-Witten type formula for the osp invariant $\hat{\mathcal{R}}$-operator and demonstrate the equivalence of the previous approach to the new one in the general case of the $\hat{\mathcal{R}}$-operator invariant under the action of the $osp(n|2m)$ algebra.

1. Introduction

The similarities between the orthosymplectic supergroups $OSp(N|M)$ (here $M = 2m$ is an even number) and their orthogonal $SO(N)$ and symplectic $Sp(M)$ bosonic subgroups can be traced back to the existence of invariant metrics in the (super)spaces $\mathcal{V}_{(N|M)}$, $\mathcal{V}_N$ and $\mathcal{V}_M$ of their

* Corresponding author.
E-mail addresses: isaevap@theor.jinr.ru (A.P. Isaev), karakhan@yerphi.am (D. Karakhanyan), Roland.Kirschner@itp.uni-leipzig.de (R. Kirschner).

https://doi.org/10.1016/j.nuclphysb.2021.115355
0550-3213/

defining representations. These similarities lead to the consideration of the supergroup OSp and its superalgebra osp in full analogy with the unified treatment (see e.g. [1]) of the groups SO, Sp and their Lie algebras. Moreover these similarities are inherited in the study of solutions of the Yang-Baxter equations that possess such symmetries.

In the present paper, we continue our study [2] of the solutions of the Yang-Baxter equations symmetric with respect to ortho-symplectic groups. We start with the graded RLL-relations with the R-matrix in the defining representation $R \in \mathrm{End}(\mathcal{V}_{(N|M)} \otimes \mathcal{V}_{(N|M)})$ and find the L-operator, $L(u) \in \mathrm{End}(\mathcal{V}_{(N|M)}) \otimes \mathcal{A}$, where $\mathcal{A}$ is a super-oscillator algebra invariant under the action of the $OSp(N|M)$ group. Then this L operator allows one (via another type of RLL relations) to define a richer and more complicated family of solutions of the Yang-Baxter equations, namely the $\hat{\mathcal{R}}$-operators, which take values in the tensor product $\mathcal{A} \otimes \mathcal{A}$ and are expressed as an expansion over the invariants in $\mathcal{A} \otimes \mathcal{A}$. The orthogonal and symplectic groups are embedded in the ortho-symplectic super-group OSp, and the $\hat{\mathcal{R}}$-operators invariant under the $so(N)$ and $sp(M)$ algebras can be obtained from the OSp-invariant $\hat{\mathcal{R}}$-operator as special cases. In the orthogonal case the algebra $\mathcal{A}$ is the N-dimensional Clifford algebra and the operator $\hat{\mathcal{R}}$ is called the spinor R-matrix. In the symplectic case the algebra $\mathcal{A}$ is the oscillator algebra and $\hat{\mathcal{R}}$ is called the metaplectic R-operator.

The standard approach to the problem of finding the spinor (so-invariant) $\hat{\mathcal{R}}$-operator was developed in [3], [4] and is based on the expansion of the $\hat{\mathcal{R}}$-operator over the invariants I_k realized in the spaces $\mathcal{A} \otimes \mathcal{A}$. Here the factors $\mathcal{A}$ are the Clifford algebras with the generators $(c^a)^\alpha_\beta$, where α, β and a are respectively spinor and vector indices. Then the invariants I_k are given by the contraction of the antisymmetrized products of $c_1^{(a_1} \dots c_1^{a_k)} \in \mathcal{A} \otimes I$ and $c_2^{(b_1} \dots c_2^{b_k)} \in I \otimes \mathcal{A}$ with the invariant metrics $\varepsilon_{a_i b_i}$. In that approach we obtain the spinor $\hat{\mathcal{R}}$-operator as a sum over invariants I_k with the coefficients r_k which obey recurrence relations. Analogous formulae of the Shankar-Witten (SW) type for the $\hat{\mathcal{R}}$-operators were deduced for the symplectic case in [1] and then were generalized for the ortho-symplectic case in [2]. Note that we cannot consider these expressions for the $\hat{\mathcal{R}}$-operators as quite satisfactory, since they do not provide closed formulas for the considered $\hat{\mathcal{R}}$-operators. For example, in the symplectic and ortho-symplectic cases, the sum over I_k is infinite.

On the other hand, it is known that an analogous $\hat{\mathcal{R}}$-operator invariant under the $s\ell(2)$ algebra can be represented (see [5], [6]) in a compact form of the ratio of two operator-valued Euler Gamma-functions. Surprisingly, as it was shown in a recent paper [7], the so and sp invariant $\hat{\mathcal{R}}$-operators (for special Clifford and oscillator representations of so and sp) are also represented in the Faddeev-Tarasov-Takhtajan (FTT) form of the ratio of two operator-valued Euler Gamma-functions.

In the present paper, we generalize the results of [7] to the supersymmetric case and show that the osp invariant $\hat{\mathcal{R}}$-operator can also be represented in the FTT form. This is the main result of our paper. The natural conjecture is that the osp-invariant SW type $\hat{\mathcal{R}}$-operator given as a sum over invariants I_k is equal to the osp-invariant FTT type $\hat{\mathcal{R}}$-operator given by the ratio of two Gamma-functions. This conjecture is based on the fact that both $\hat{\mathcal{R}}$-operators are solutions of the same system of finite-difference equations which arise from the RLL relations.

A complete proof of this conjecture is still missing. In the present paper we propose another simpler and more elegant derivation of the SW type formula for the osp invariant $\hat{\mathcal{R}}$-operator. This new derivation supports the conjecture of the equivalence of the SW and FTT expressions for the $\hat{\mathcal{R}}$-operators. Indeed, in the previous derivation, the role of invariant, "colorless", elements in $\mathcal{A} \otimes \mathcal{A}$ is played by the operators I_k. In the new derivation, we prove that the invariants I_k are polynomials of one invariant $I_1 \sim z$ only and rewrite the RLL relation itself into a "colorless" form from the very beginning in terms of a system of finite-difference equations in the variable z.

We relate this new system of equations to both the SW and the FTT expressions for the $\hat{\mathcal{R}}$-operator. On one hand, the FTT type $\hat{\mathcal{R}}$-operator is its solution. On the other hand we show that the expansion of the SW type $\hat{\mathcal{R}}$-operator over I_k satisfies this system of finite-difference equations as well.

The paper is organized as follows. In Section 2, we recall some basic facts of the linear algebra on the superspace $\mathcal{V}_{(N|M)}$ with N bosonic and M fermionic coordinates and briefly formulate the theory of supergroups $OSp(N|M)$ and their Lie superalgebras $osp(N|M)$. In this section we fix our notation and conventions. In Section 3, we define the osp-invariant solution of the Yang-Baxter equation as an image of a special element of the Brauer algebra in the tensor representation in super-spaces $\mathcal{V}_{(N|M)}^{\otimes r}$. Section 4 is devoted to the formulation of the graded RLL relations. In this Section, we find a special L-operator that solves the RLL relations in the case of the osp algebra and introduce (see also [2]) the notion of the linear evaluation of the Yangian $\mathcal{Y}(osp)$. In Section 5 we define the super-oscillator algebra $\mathcal{A}$ and describe the super-oscillator representation for the linear evaluation of the Yangian $\mathcal{Y}(osp)$. In particular, we define the set of Osp invariant operators I_k in $\mathcal{A} \otimes \mathcal{A}$ and their generating function.

In terms of these invariant operators we construct in Section 6 the osp invariant $\hat{\mathcal{R}}$-operators in the super-oscillator representation. We find two forms for such $\hat{\mathcal{R}}$-operator. One of these forms represents the $\hat{\mathcal{R}}$-operator as a ratio of Euler Gamma-functions. For the $s\ell(2)$ case this type solution was first obtained in [5] (see also [6]) and we call these solutions the FTT type $\hat{\mathcal{R}}$-operators. Another form of the osp invariant $\hat{\mathcal{R}}$-operator in the super-oscillator representation generalizes the SW solution [3] of the spinor-spinor so-invariant $\hat{\mathcal{R}}$-operator. This solution (see eqs. (6.6) and (6.8)) for the osp-invariant $\hat{\mathcal{R}}$-matrix in the super-oscillator representation was first obtained in our paper [2] by using the methods developed in [4], [8] and [1]. In [2] we have generalized formulas for the so-type R-matrices (in the Clifford algebra representation) obtained in [3], [8] (see also [9], [10], [11], [12], [1]). In [2] we have also generalized the formulae for sp-type R-matrices (in the oscillator, or metaplectic, representation of the Lie algebra sp), which were deduced in [1]. It has been shown in [2] that all these so- and sp-invariant R-matrices are obtained from (6.6), (6.8) by restriction to the corresponding bosonic Lie subalgebras of osp.

In Section 7 the result for the FTT type R operator is studied in detail in particular cases of $osp(N|M)$. The arguments of the Gamma-functions involve the invariant operator $z \sim I_1$ which decomposes into a bosonic and a fermionic part. The finite spectral decomposition of the fermionic part is considered and used to decompose the R operator with respect to the corresponding projection operators.

In Section 8 we present the new and more direct derivation of the solutions (6.6) and (6.8). Two Appendices are devoted to the proofs of the statements made in the main body of the paper.

2. The ortho-symplectic supergroup and its Lie superalgebra

Consider (see, e.g., [13], [2]) a superspace $\mathcal{V}_{(N|M)}$ with graded coordinates z^a $(a = 1, \dots, N+M)$. The grading $\text{grad}(z^a)$ of the coordinate z^a will be denoted as $[a] = 0, 1\,(\text{mod}2)$. If the coordinate z^a is even then $[a] = 0\,(\text{mod}2)$, and if the coordinate z^a is odd then $[a] = 1\,(\text{mod}2)$. It means that the coordinates z^a and w^b of two supervectors $z, w \in \mathcal{V}_{(N|M)}$ commute as follows

$$z^a\, w^b = (-1)^{[a][b]}\, w^b\, z^a \, . \tag{2.1}$$

Let the superspace $\mathcal{V}_{(N|M)}$ be endowed with a bilinear form

$$(z \cdot w) \equiv \varepsilon_{ab} z^a w^b = z^a w_a = z_b w_a \bar{\varepsilon}^{ab} , \quad (z \cdot w) = \epsilon (w \cdot z) , \tag{2.2}$$

which is symmetric for $\epsilon = +1$ and skewsymmetric for $\epsilon = -1$. In eq. (2.2) we define $w_a \equiv \varepsilon_{ab} w^b$, where, in accordance with the last relation in (2.2), the super-metric ε_{ab} and inverse super-metric $\bar{\varepsilon}^{ab}$ have the properties

$$\varepsilon_{ab} \bar{\varepsilon}^{bd} = \bar{\varepsilon}^{db} \varepsilon_{ba} = \delta^d_a , \quad \varepsilon_{ab} = \epsilon (-1)^{[a][b]} \varepsilon_{ba} \;\Leftrightarrow\; \bar{\varepsilon}^{ab} = \epsilon (-1)^{[a][b]} \bar{\varepsilon}^{ba} . \tag{2.3}$$

We stress that the super-metric ε_{ab} is an even matrix in the sense that $\varepsilon_{ab} \neq 0$ iff $[a] + [b] = 0$ (mod2):

$$\varepsilon_{ab} = (-1)^{[a]+[b]} \varepsilon_{ab}. \tag{2.4}$$

In other words the supermatrix ε_{ab} is block-diagonal and its non-diagonal blocks vanish. Using (2.4), the properties (2.3) can be written as

$$\varepsilon_{ab} = \epsilon (-1)^{[a]} \varepsilon_{ba} = \epsilon (-1)^{[b]} \varepsilon_{ba} , \quad \bar{\varepsilon}^{ab} = \epsilon (-1)^{[a]} \bar{\varepsilon}^{ba} = \epsilon (-1)^{[b]} \bar{\varepsilon}^{ba} . \tag{2.5}$$

Further, we use the following agreement on raising and lowering indices for super-tensor components

$$z^{\dots c}{}_a{}^{d \dots} = \varepsilon_{ab}\, z^{\dots c\, b\, d \dots} , \quad z_{\dots c}{}^a{}_{d \dots} = \bar{\varepsilon}^{ab}\, z_{\dots cbd \dots} . \tag{2.6}$$

According to this rule, we have $\varepsilon^{ab} = \bar{\varepsilon}^{ac} \bar{\varepsilon}^{bd} \varepsilon_{cd} = \bar{\varepsilon}^{ba}$ and the metric tensor with the upper indices ε^{ab} does not coincide with the inverse matrix $\bar{\varepsilon}^{ab}$. Further, we use only the inverse matrix $\bar{\varepsilon}^{ab}$ and never the metric tensor ε^{ab}.

Consider a linear transformation in $\mathcal{V}_{(N|M)}$

$$z^a \to z'^a = U^a_{\ b} z^b , \tag{2.7}$$

which preserves the grading of the coordinates $\mathrm{grad}(z'^a) = \mathrm{grad}(z^a)$. For the elements $U^a_{\ b}$ of the supermatrix U from (2.7) we have $\mathrm{grad}(U^a_{\ b}) = [a] + [b]$. The ortho-symplectic group OSp is defined as the set of supermatrices U which preserve the bilinear form (2.2) under the transformations (2.7)

$$(-1)^{[c]([b]+[d])} \varepsilon_{ab} U^a_{\ c} U^b_{\ d} = \varepsilon_{cd} \;\Rightarrow\; (-1)^{[c]([b]+[d])} U^a_{\ c} U^b_{\ d} \bar{\varepsilon}^{cd} = \bar{\varepsilon}^{ab} . \tag{2.8}$$

Now we write the relations (2.8) in the coordinate-free form as

$$\varepsilon_{\langle 12}\, U_1 (-)^{12} U_2 (-)^{12} = \varepsilon_{\langle 12} \;\Leftrightarrow\; U_1 (-)^{12} U_2 (-)^{12}\, \bar{\varepsilon}^{12\rangle} = \bar{\varepsilon}^{12\rangle} , \tag{2.9}$$

where the concise matrix notation is used

$$\begin{gathered} \bar{\varepsilon}^{12\rangle} \in \mathcal{V}_{(N|M)} \otimes \mathcal{V}_{(N|M)} , \quad U_1 = U \otimes I , \quad U_2 = I \otimes U , \\ ((-)^{12})^{a_1 a_2}_{\ \ b_1 b_2} = (-1)^{[a_1][a_2]} \delta^{a_1}_{\ b_1} \delta^{a_2}_{\ b_2} , \quad (-)^{12} \in \mathrm{End}(\mathcal{V}_{(N|M)} \otimes \mathcal{V}_{(N|M)}) . \end{gathered} \tag{2.10}$$

Here $\otimes$ denotes the graded tensor product:

$$(I \otimes B)(A \otimes I) = (-1)^{[A][B]} (A \otimes B) , \quad (A \otimes I)(I \otimes B) = (A \otimes B) ,$$

and $[A] := \mathrm{grad}(A)$, $[B] := \mathrm{grad}(B)$.

Consider the elements $U \in OSp$ which are close to unity I: $U = I + tA + \dots$. Here the parameter t is small and dots denote terms of order t^2, t^3, etc. In this case, the defining relations (2.8) give conditions for the supermatrices A:

$$(-1)^{[c]([b]+[d])}\varepsilon_{ab}(\delta^a_{\ c}A^b_{\ d} + A^a_{\ c}\delta^b_d) = \big((-1)^{[c]+[c][d]}\varepsilon_{cb}A^b_{\ d} + \varepsilon_{ad}\,A^a_{\ c}\big) = 0 \;\;\Rightarrow \\ A_{cd} = -\epsilon(-1)^{[c][d]+[c]+[d]}A_{dc}\,. \tag{2.11}$$

The coordinate free form of relations (2.11) is directly deduced from equalities (2.9):

$$\varepsilon_{\langle 12}(A_1 + (-)^{12}A_2(-)^{12}) = 0 \quad\Leftrightarrow\quad (A_1 + (-)^{12}A_2(-)^{12})\bar{\varepsilon}^{12\rangle} = 0\,. \tag{2.12}$$

The vector space of super-matrices A, which satisfy (2.11), (2.12), forms the Lie superalgebra osp of the supergroup OSp.

Any such matrix A can be represented as

$$A^a_{\ c} = E^a_{\ c} - (-1)^{[c]+[c][d]}\varepsilon_{cb}E^b_{\ d}\bar{\varepsilon}^{da} \tag{2.13}$$

where $||E^a_{\ c}||$ is an arbitrary matrix. Let $\{e_g^{\ f}\}$ be the matrix units, i.e., matrices with the components $(e_g^{\ f})^b_{\ d} = \delta^f_d\,\delta^b_g$. If we substitute $E = e^f_{\ g} = \bar{\varepsilon}^{fg'}\varepsilon_{gf'}e_{g'}^{\ f'}$ in (2.13), then we obtain the basis elements $\{\widetilde{G}^f_{\ g}\}$ in the space osp of matrices (2.12):

$$(\widetilde{G}^f_{\ g})^a_{\ c} \equiv (e^f_{\ g})^a_{\ c} - (-1)^{[c]+[c][d]}\varepsilon_{cb}(e^f_{\ g})^b_{\ d}\bar{\varepsilon}^{da} = \bar{\varepsilon}^{fa}\varepsilon_{gc} - \epsilon(-1)^{[c][a]}\delta^f_c\,\delta^a_g\,. \tag{2.14}$$

Now any super-matrix $A \in osp$ which satisfies (2.11), (2.12) can be expanded over the basis (2.14)

$$A^a_{\ c} = a^g_{\ f}(\widetilde{G}^f_{\ g})^a_{\ c}\,, \tag{2.15}$$

where $a^g_{\ f}$ are the components of the super-matrix. Since the elements $(\widetilde{G}^f_{\ g})^a_{\ c}$ are even, i.e., $(\widetilde{G}^f_{\ g})^a_{\ c} \neq 0$ iff $[f]+[g]+[a]+[c] = 0$ (mod2), then from the condition $\mathrm{grad}(A^a_{\ c}) = [a]+[c]$ we obtain that $\mathrm{grad}(a^g_{\ f}) = [g]+[f]$. It means that the usual commutator appears as a super-commutator for the basis elements $\widetilde{G}^f_{\ g}$:

$$[A, B]^a_{\ c} = [a^g_{\ f}(\widetilde{G}^f_{\ g}),\; b^n_{\ k}(\widetilde{G}^k_{\ n})]^a_{\ c} = a^g_{\ f}b^n_{\ k}\,\big([\widetilde{G}^f_{\ g},\; \widetilde{G}^k_{\ n}]_\pm\big)^a_{\ c}\,,$$

where in the component form the super-commutator is

$$\big([\widetilde{G}^{a_1}_{\ b_1},\; \widetilde{G}^{a_2}_{\ b_2}]_\pm\big)^{a_3}_{\ c_3} \equiv (\widetilde{G}^{a_1}_{\ b_1})^{a_3}_{\ b_3}(\widetilde{G}^{a_2}_{\ b_2})^{b_3}_{\ c_3} - (-1)^{([a_1]+[b_1])([a_2]+[b_2])}(\widetilde{G}^{a_2}_{\ b_2})^{a_3}_{\ b_3}(\widetilde{G}^{a_1}_{\ b_1})^{b_3}_{\ c_3}\,. \tag{2.16}$$

We notice that the elements of the matrices $\widetilde{G}^a_{\ b}$ are numbers. However, the super-commutator (2.16) is written for $\widetilde{G}^a_{\ b}$ as for the graded elements with $\deg(\widetilde{G}^a_{\ b}) = [a]+[b]$.

Now we substitute the explicit representation (2.14) in the right-hand side of (2.16) and deduce the defining relations for the basis elements of the superalgebra osp:

$$(-1)^{[b_1][a_2]}\cdot[\widetilde{G}^{a_1}_{\ b_1},\; \widetilde{G}^{a_2}_{\ b_2}]_\pm = -(-1)^{[a_1][a_2]}\,\bar{\varepsilon}^{a_1a_2}\,\widetilde{G}_{b_1b_2} + \epsilon\,\delta^{a_2}_{b_1}\,\widetilde{G}^{a_1}_{\ b_2} + \\ +(-1)^{[a_1][a_2]}\,\varepsilon_{b_1b_2}\,\widetilde{G}^{a_2a_1} - \epsilon(-1)^{[a_1]([b_1]+[a_2])}\,\delta^{a_1}_{b_2}\,\widetilde{G}^{a_2}_{\ b_1}\,, \tag{2.17}$$

where we have omitted the matrix indices. Below we use the standard component-free form of notation, where we substitute $(\widetilde{G}^{a_i}_{\ b_i})^{a_k}_{\ b_k} \to \widetilde{G}_{ik}$ (here i and k are numbers $1, 2, 3$ of two super-spaces $\mathcal{V}_{(N|M)}$ in $\mathcal{V}^{\otimes 3}_{(N|M)}$). In this notation, taking into account (2.16), the relation (2.17) is written as

$$[(-)^{12}\widetilde{G}_{13}(-)^{12}\,,\; \widetilde{G}_{23}] = [\epsilon\mathcal{P}_{12} - \mathcal{K}_{12}\,,\; \widetilde{G}_{23}]\,, \tag{2.18}$$

where we introduce two matrices $\mathcal{K}, \mathcal{P} \in \text{End}(\mathcal{V}^{\otimes 2}_{(N|M)})$:

$$\mathcal{K}^{a_1 a_2}_{b_1 b_2} = \bar{\varepsilon}^{a_1 a_2} \varepsilon_{b_1 b_2} \, , \quad \mathcal{P}^{a_1 a_2}_{b_1 b_2} = (-1)^{[a_1][a_2]} \delta^{a_1}_{b_2} \delta^{a_2}_{b_1} \, . \tag{2.19}$$

The matrix $\mathcal{P}$ is called superpermutation since it permutes super-spaces, e.g., using this matrix one can write (2.1) as $\mathcal{P}^{ab}_{cd} w^c z^d = z^a w^b$. Note that the generators (2.14) of the Lie super-algebra osp can be expressed in terms of $\mathcal{P}$ and $\mathcal{K}$ as

$$\widetilde{G} = \mathcal{K} - \epsilon \, \mathcal{P} \, , \tag{2.20}$$

and after substituting (2.20) into (2.18) can be written (2.18) in the form

$$[(-)^{12} \widetilde{G}_{13} (-)^{12} \, , \; \widetilde{G}_{23}] + [\widetilde{G}_{12} \, , \; \widetilde{G}_{23}] = 0 \, . \tag{2.21}$$

One can explicitly check the relation (2.21) by making use of the identities for the operators $\mathcal{P}$ and $\mathcal{K}$ presented in Appendix **A**.

Note that conditions (2.12) for the osp generators $A^a{}_c = (\widetilde{G}^f{}_g)^a{}_c$, given in (2.14) and (2.20), can be written as

$$\mathcal{K}_{12} (\widetilde{G}_{31} + (-)^{12} \widetilde{G}_{32} (-)^{12}) = 0 \, , \quad (\widetilde{G}_{31} + (-)^{12} \widetilde{G}_{32} (-)^{12}) \mathcal{K}_{12} = 0 \, . \tag{2.22}$$

One can verify that these conditions are equivalent to

$$\mathcal{K}_{12} ((-)^{12} \widetilde{G}_{13} (-)^{12} + \widetilde{G}_{23}) = 0 \, , \quad ((-)^{12} \widetilde{G}_{13} (-)^{12} + \widetilde{G}_{23}) \mathcal{K}_{12} = 0 \, . \tag{2.23}$$

Using (2.23) and the commutation relations of super-permutation $\mathcal{P}$ and generators $\widetilde{G}$ (see appendix **A**)

$$\mathcal{P}_{12} (-)^{12} \widetilde{G}_{13} (-)^{12} = \widetilde{G}_{23} \mathcal{P}_{12} \, , \quad (-)^{12} \widetilde{G}_{13} (-)^{12} \mathcal{P}_{12} = \mathcal{P}_{12} \widetilde{G}_{23} \, , \tag{2.24}$$

we write (2.18) as

$$[(-)^{12} \widetilde{G}_{13} (-)^{12} \, , \; \widetilde{G}_{23}] = [\epsilon \mathcal{P}_{12} - \mathcal{K}_{12} \, , \; (-)^{12} \widetilde{G}_{13} (-)^{12}] \, . \tag{2.25}$$

It means that the defining relations (2.17) can be written in many equivalent forms. At the end of this section we note that the matrix (2.20) is the split Casimir operator for the Lie superalgebra osp in the defining representation.

3. The OSp-invariant R-matrix and the graded Yang–Baxter equation

Consider the three OSp invariant operators in $\mathcal{V}^{\otimes 2}_{(N|M)}$: the identity operator **1**, the superpermutation operator $\mathcal{P}$ and metric operator $\mathcal{K}$. According to definition (2.19), the superpermutation $\mathcal{P}_{12}$ is a product of the usual permutation P_{12} and the sign factor $(-)^{12}$,

$$\mathcal{P}_{12} = (-)^{12} P_{12} \, , \qquad \text{or in components} \qquad \mathcal{P}^{a_1 a_2}_{b_1 b_2} = (-1)^{[a_1][a_2]} \delta^{a_1}_{b_2} \delta^{a_2}_{b_1} , \tag{3.1}$$

while the operator $\mathcal{K}_{12}$ is defined as

$$\mathcal{K}_{12} = \bar{\varepsilon}^{12\rangle} \varepsilon_{\langle 12} \, , \qquad \text{or in components} \qquad \mathcal{K}^{a_1 a_2}_{b_1 b_2} = \bar{\varepsilon}^{a_1 a_2} \varepsilon_{b_1 b_2} . \tag{3.2}$$

Their OSp invariance means that (see (2.9))

$$\begin{aligned} U_1 (-)^{12} U_2 (-)^{12} \mathcal{K}_{12} &= \mathcal{K}_{12} U_1 (-)^{12} U_2 (-)^{12} \, , \\ U_1 (-)^{12} U_2 (-)^{12} \mathcal{P}_{12} &= \mathcal{P}_{12} U_1 (-)^{12} U_2 (-)^{12} \, . \end{aligned} \tag{3.3}$$

In particular, it follows from these relations that the comultiplication for the supermatrices $U \in Osp(N|M)$ has the graded form $\Delta(U)_{12} = U_1(-)^{12}U_2(-)^{12}$. In fact this comultiplication follows from the transformation (2.7) applied to the second rank tensor $z^{a_1} \cdot z^{a_2}$.

Using the operators $\mathcal{P}, \mathcal{K}$ one can construct a set of operators $\{s_i, e_i | i = 1, \dots, n-1\}$ in $\mathcal{V}^{\otimes n}_{(N|M)}$:

$$s_i = \epsilon \mathcal{P}_{i,i+1} \equiv \epsilon I^{\otimes(i-1)} \otimes \mathcal{P} \otimes I^{\otimes(n-i-1)}, \quad e_i = \mathcal{K}_{i,i+1} \equiv I^{\otimes(i-1)} \otimes \mathcal{K} \otimes I^{\otimes(n-i-1)}, \tag{3.4}$$

which define the matrix representation T of the Brauer algebra $B_n(\omega)$ [16], [17] with the parameter

$$\omega = \varepsilon_{cd}\, \bar{\varepsilon}^{cd} = \epsilon(N - M) . \tag{3.5}$$

Recall that here N and M are the numbers of even and odd coordinates, respectively. Indeed, one can check directly (see Appendix **A**) that the operators (3.4) satisfy the defining relations for the generators of the Brauer algebra $B_n(\omega)$

$$\begin{array}{l} s_i^2 = 1 , \quad e_i^2 = \omega e_i , \quad s_i\, e_i = e_i\, s_i = e_i , \quad i = 1, ..., n-1, \\ s_i s_j = s_j s_i , \quad e_i e_j = e_j e_i , \quad s_i e_j = e_j s_i , \quad |i - j| > 1, \end{array} \tag{3.6}$$

$$\begin{array}{l} s_i\, s_{i+1}\, s_i = s_{i+1}\, s_i\, s_{i+1} , \quad e_i\, e_{i+1}\, e_i = e_i , \quad e_{i+1}\, e_i\, e_{i+1} = e_{i+1} , \\ s_i\, e_{i+1}\, e_i = s_{i+1}\, e_i , \quad e_{i+1}\, e_i\, s_{i+1} = e_{i+1}\, s_i , \quad i = 1, ..., n-2 . \end{array} \tag{3.7}$$

(here $\hat{1}$ is the unit element in $B_n(\omega)$). Note that this presentation of the Brauer algebra is obtained in the special limit $q \to 1$ from the BMW algebra presentation [15] and it is used in many investigations (see, e.g., [18], [19], [20], [21]).

We stress that the matrix representation T (3.4) of the generators $s_i, e_i \in B_n(\omega)$ acts in the space $\mathcal{V}^{\otimes n}_{(N|M)}$.

For what follows, we need the following statement (see, e.g., [19] and [2]).

Proposition 1. *The element*

$$\hat{\rho}_i(u) = u(u + \beta)\, s_i - (u + \beta)\, \hat{1} + u\, e_i \;\; \in \;\; B_n(\omega) , \tag{3.8}$$

where u is a spectral parameter and $\beta = 1 - \frac{\omega}{2}$, satisfies the Yang-Baxter equation

$$\hat{\rho}_i(u)\hat{\rho}_{i+1}(u + v)\hat{\rho}_i(v) = \hat{\rho}_{i+1}(v)\hat{\rho}_i(u + v)\hat{\rho}_{i+1}(u) , \tag{3.9}$$

and the unitarity condition $\hat{\rho}_i(u)\hat{\rho}_i(-u) = (u^2 - 1)(u^2 - \beta^2)\hat{1}$.

The matrix representation T (3.4) of the element (3.8) is

$$\hat{R}(u) \equiv \epsilon\, T(\hat{\rho}(u)) = u(u + \beta)\mathcal{P} - \epsilon(u + \beta)\mathbf{1} + \epsilon u \mathcal{K} . \tag{3.10}$$

Here we suppress index i for simplicity. It follows from (3.9) that $\hat{R}(u)$ satisfies the braid version of the Yang–Baxter equation

$$\hat{R}_{12}(u - v)\hat{R}_{23}(u)\hat{R}_{12}(v) = \hat{R}_{23}(v)\hat{R}_{12}(u)\hat{R}_{23}(u - v). \tag{3.11}$$

Thus, in the supersymmetric case the braid version (3.11) of the Yang–Baxter equation is the same as in the non supersymmetric case. Further we use the following R-matrix

$$R(u) = \mathcal{P}\hat{R}(u) = (u - \frac{\omega}{2} + 1)(u\mathbf{1} - \epsilon\mathcal{P}) + u\mathcal{K}$$
$$= u(u+\beta)\mathbf{1} - \epsilon(u+\beta)\mathcal{P} + u\mathcal{K}\,, \tag{3.12}$$

which is the image of the elements [19]:

$$\rho_i(u) = u(u+\beta)\mathbf{1} - (u+\beta)\,s_i + u\,e_i \;\in\; B_n(\omega)\,.$$

Proposition 2. *The standard R-matrix $R(u) = \mathcal{P}\hat{R}(u)$, which was defined in (3.12), satisfies the graded version of the Yang–Baxter equation [22]*

$$R_{12}(u-v)(-)^{12}R_{13}(u)(-)^{12}R_{23}(v) = R_{23}(v)(-)^{12}R_{13}(u)(-)^{12}R_{12}(u-v). \tag{3.13}$$

Proof. The matrix $R \in \mathrm{End}(\mathcal{V}^{\otimes 2}_{(N|M)})$ is an even matrix since we have $R^{i_1 i_2}_{j_1 j_2} \neq 0$ iff $[i_1]+[i_2]+[j_1]+[j_2] = 0$ (mod2). This follows from the explicit form (3.12) of the operator $R(u)$. Therefore, for arbitrary k we have the identity

$$R_{ij}(-)^{ik}(-)^{jk} = (-)^{ik}(-)^{jk}R_{ij}\,, \tag{3.14}$$

where i, j and k are numbers of super-spaces $\mathcal{V}_{(N|M)}$ in the product $\mathcal{V}^{\otimes n}_{(N|M)}$ and the operator $(-)^{ik}$ is defined in (2.10). Substituting $\hat{R}_{ij}(u) = \mathcal{P}_{ij}R_{ij}(u) = (-)^{ij}P_{ij}R_{ij}(u)$ into (3.11) and moving all usual permutations P_{ij} to the left we write (3.11) with the help of (3.14) in the form (3.13). □

Remark 1. We stress that the sign operators $(-)^{12}$ in (3.13) can be substituted by the operators $(-)^{23}$ by means of manipulations similar to (3.14). Moreover, if $R_{ij}(u)$ solves the Yang-Baxter equation (3.13), then the twisted R-matrix $(-)^{ij}R_{ij}(u)(-)^{ij}$ is also a solution of (3.13).

Remark 2. Eqs. (3.10), (3.12) give unified forms for solutions of the Yang-Baxter equations (3.11), (3.13) which are invariant under the action of all Lie (super)groups SO, Sp and OSp. Recall that for the SO case the R-matrix (3.12) was found in [24] and for the Sp case it was indicated in [25]. For the OSp case such R-matrices were considered in many papers (see, e.g., [23], [14], [26]).

4. Graded RLL-relation and the linear evaluation of Yangian $\mathcal{Y}(osp)$

We start with the following graded form of the RLL-relation (see, e.g., [26] and references therein)

$$R_{12}(u-v)L_1(u)(-)^{12}L_2(v)(-)^{12} = (-)^{12}L_2(v)(-)^{12}L_1(u)R_{12}(u-v)\,, \tag{4.1}$$

where the R-matrix is given in (3.12). This graded form of the RLL relations is also motivated by the invariance conditions (3.3). It is known (see, e.g., [2], [14] and references therein) that eqs. (4.1) with the R-matrix (3.12) are defining relations for the super-Yangian $\mathcal{Y}(osp)$. In [2] we proved the following statement.

Proposition 3. *The L-operator*

$$L^a{}_b(u) = (u+\alpha)\,\mathbf{1}\delta^a_b + G^a{}_b\,, \tag{4.2}$$

where α is an arbitrary constant, solves the RLL-relation (4.1) iff $G^a{}_b$ is a traceless matrix of generators of the Lie superalgebra osp, i.e., it satisfies equations (cf. (2.22))

$$\mathcal{K}_{12}\Big\{G_1 + (-)^{12}G_2(-)^{12}\Big\} = 0 = \Big\{G_1 + (-)^{12}G_2(-)^{12}\Big\}\mathcal{K}_{12}\,, \tag{4.3}$$

defining relations for osp-algebra (cf. (2.18))

$$G_1(-)^{12}G_2(-)^{12} - (-)^{12}G_2(-)^{12}G_1 = [\mathcal{K}_{12} - \epsilon\mathcal{P}_{12},\ G_1]\,, \tag{4.4}$$

and in addition obeys the quadratic characteristic identity

$$G^2 + \beta\, G - \frac{\epsilon}{\omega}\,\mathrm{str}\Big(G^2\Big)\,\mathbf{1} = 0\,, \tag{4.5}$$

where as usual $\beta = 1 - \omega/2$.

The L-operator (4.2), where the elements G^a_b satisfy the conditions (4.3), (4.6) and (4.5), is called *the linear evaluation* of the Yangian $\mathcal{Y}(osp)$.

Remark 3. The relations (4.4) are written after the exchange $1 \leftrightarrow 2$ in the form

$$(-)^{12}G_1(-)^{12}G_2 - G_2(-)^{12}G_1(-)^{12} = [\epsilon\mathcal{P}_{12} - \widetilde{\mathcal{K}}_{12},\ G_2]\,, \tag{4.6}$$

where $\widetilde{\mathcal{K}}_{12} = \mathcal{K}_{21} = (-)^{12}\mathcal{K}_{12}(-)^{12}$, or $\widetilde{\mathcal{K}}^{a_1a_2}_{b_1b_2} = \bar{\varepsilon}^{a_2a_1}\varepsilon_{b_2b_1}$. Now we are able to compare the defining relations (4.4), (4.6) with (2.18), (2.25), where the elements G^a_b are represented as matrices $\widetilde{G}^a_b$ acting in the super-space $\mathcal{V}_{(\mathcal{N}|\mathcal{M})}$, namely, the commutation relations (4.6) turn into the commutation relations (2.18) after the change of the definition of the supermetric $\varepsilon_{ab} \to \varepsilon_{ba} = \epsilon(-1)^{[a]}\varepsilon_{ab}$ (see also the discussion in Remark 5 below).

Remark 4. The conditions (4.3) for the generators of osp read in component form (cf. (2.11)):

$$G_{ab} + \epsilon(-1)^{[a][b]+[a]+[b]}G_{ba} = 0\,, \qquad G_{ab} \equiv \varepsilon_{ac}\,G^c_b\,. \tag{4.7}$$

In particular, it follows from (4.3), (4.7) that the matrix G is traceless

$$0 = \mathcal{K}_{12}\Big(G_1 + (-)^{12}G_2(-)^{12}\Big)\mathcal{K}_{12} = 2(\varepsilon_{ab}\,G^a{}_c\,\bar{\varepsilon}^{cb})\,\mathcal{K}_{12} = 2\,\epsilon\,\mathrm{str}(G)\,\mathcal{K}_{12}\,.$$

Remark 5. The characteristic identity (4.5) is equivalent to the equation

$$\mathcal{K}_{12}\Big(\beta G_1 + G_1(-)^{12}G_2(-)^{12}\Big) = \Big(\beta G_1 + (-)^{12}G_2(-)^{12}G_1\Big)\mathcal{K}_{12}\,,$$

provided that the relations (4.3) and (4.6) are satisfied.

5. Super-oscillator representation for linear evaluation of $\mathcal{Y}(osp)$

In this section we intend to construct an explicit representation of $\mathcal{Y}(osp)$ in which the generators of $osp \subset \mathcal{Y}(osp)$ satisfy the quadratic characteristic equation (4.5). We follow the approach of [2] and introduce a generalized algebra $\mathcal{A}$ of super-oscillators that consists of both bosonic and fermionic oscillators simultaneously.

Consider the super-oscillators c^a ($a = 1, 2, \ldots, N + M$) as generators of an associative algebra $\mathcal{A}$ with the defining relation

$$[c^a, c^b]_\epsilon \equiv c^a c^b + \epsilon(-1)^{[a][b]}c^b c^a = \bar{\varepsilon}^{ab}, \tag{5.1}$$

where the matrix $\bar{\varepsilon}^{ab}$ is defined in (2.3) and (2.5). In view of (2.1), for $\epsilon = -1$, the super-oscillators c^a with $[a] = 0$ (mod2) are bosonic and with $[a] = 1$ (mod2) are fermionic. For $\epsilon = +1$ the statistics of the super-oscillators c^a is unusual and we will discuss this in more detail in Remark **8** at the end of this section. Nevertheless, we assume the grading to be standard $\mathrm{grad}(c^a) = [a]$ in both cases $\epsilon = \pm 1$ and therefore the defining relations (5.1) are invariant under the action $c^a \to c'^a = U^a_{\ c} c^c$ of the super-group OSp with the elements $U \in Osp$ (see [2]).

With the help of convention (2.6) for lowering indices one can write relations (5.1) in the equivalent forms

$$[c_a, c_b]_\epsilon \equiv c_a c_b + \epsilon(-1)^{[a][b]} c_b c_a = \varepsilon_{ba} \;\; \Leftrightarrow \;\; c_a c^b + \epsilon(-1)^{[a][b]} c^b c_a = \delta^b_a \, . \tag{5.2}$$

The super-oscillators c^a satisfy the following contraction identities:

$$c^a c_a = \bar{\varepsilon}^{ab} \varepsilon_{ad} c_b c^d = \epsilon(-1)^{[a]} c_a c^a \, , \quad c_a c^a = \bar{\varepsilon}^{ab} \varepsilon_{ad} c^d c_b = \epsilon(-1)^{[a]} c^a c_a \, .$$

So, we have

$$\begin{gathered} c^a c_a = \tfrac{1}{2} \bar{\varepsilon}^{ab} (c_b c_a + \epsilon(-1)^{[a]} c_a c_b) = \tfrac{1}{2} \bar{\varepsilon}^{ab} \varepsilon_{ab} = \tfrac{\omega}{2} \, , \\ c_a c^a = \tfrac{1}{2} \bar{\varepsilon}^{ab} (c_a c_b + \epsilon(-1)^{[a]} c_b c_a) = \tfrac{1}{2} \bar{\varepsilon}^{ab} \varepsilon_{ba} = \tfrac{D}{2} \, , \quad D \equiv N + M \, . \end{gathered} \tag{5.3}$$

Further we need the super-symmetrized product of two super-oscillators:

$$c^{(a} c^{b)} := \frac{1}{2} \big(c^a c^b - \epsilon(-1)^{[a][b]} c^b c^a \big) = -\epsilon(-1)^{[a][b]} c^{(b} c^{a)} \in \mathcal{A} \, , \tag{5.4}$$

and define the operators

$$F^{ab} \equiv \epsilon(-1)^{[b]} c^{(a} c^{b)} \, , \qquad F^a_{\ b} = \varepsilon_{bc} F^{ac} \, . \tag{5.5}$$

In [2] we have proved the following statement.

Proposition 4. *The operators $F^{ab} \in \mathcal{A}$ defined in (5.5) are traceless and possess the symmetry property (4.3), (4.7):*

$$\mathrm{str}(F) = (-1)^{[a]} F^a_{\ a} = 0 \, , \quad F^{ab} = -\epsilon(-1)^{[a][b]+[a]+[b]} F^{ba} . \tag{5.6}$$

In addition they satisfy the supercommutation relations (4.6) for the generators of osp

$$(-)^{12} F_1 (-)^{12} F_2 - F_2 (-)^{12} F_1 (-)^{12} = [\epsilon \mathcal{P}_{12} - \widetilde{\mathcal{K}}_{12}, \; F_2] \, , \tag{5.7}$$

and obey the quadratic characteristic identity (4.5):

$$F^a_{\ b} F^b_{\ c} + \beta F^a_{\ c} - \frac{\epsilon}{\omega} \mathrm{str}(F^2) \delta^a_c = 0, \tag{5.8}$$

where $\beta = 1 - \omega/2$.

Thus, the elements $F^a_{\ b} = \epsilon \, \varepsilon_{bd} (-1)^{[b]} c^{(a} c^{d)} \in \mathcal{A}$ given in (5.5) form a set of traceless generators of osp which satisfy all conditions of Proposition **3** and it means that the following statement holds.

Proposition 5. *The L-operator (4.2) in the super-oscillator representation (5.1):*

$$L^a_{\ b}(u) = (u + \alpha - \frac{1}{2}) \delta^a_b + \epsilon(-1)^{[b]} c^a c_b \equiv (u + \alpha - \frac{1}{2}) \delta^a_b + B^a_{\ b} \, , \tag{5.9}$$

where we introduce for convenience $B^a_b \equiv F^a_b + \frac{1}{2}\delta^a_b = \epsilon(-1)^{[b]}c^a c_b$*, obey the RLL equation (4.1) which in the component form is given by*

$$\begin{aligned}&(-1)^{[c_1]([b_2]+[c_2])} R^{a_1a_2}_{b_1b_2}(u-v) L^{b_1}_{c_1}(u) L^{b_2}_{c_2}(v)\\ &\quad = (-1)^{[a_1]([a_2]+[b_2])} L^{a_2}_{b_2}(v) L^{a_1}_{b_1}(u) R^{b_1b_2}_{c_1c_2}(u-v)\,,\end{aligned} \tag{5.10}$$

and the R*-matrix (3.12) is*

$$R^{a_1a_2}_{b_1b_2}(u) = u(u+\beta)\delta^{a_1}_{b_1}\delta^{a_2}_{b_2} - \epsilon(u+\beta)(-1)^{[a_1][a_2]}\delta^{a_1}_{b_2}\delta^{a_2}_{b_1} + u\bar{\varepsilon}^{a_1a_2}\varepsilon_{b_1b_2}\,.$$

Remark 6. The Quadratic Casimir operator C_2 of the superalgebra $osp(N|M)$ in the differential representation (5.5) is equal to the fixed number

$$C_2 = (-1)^{[a]} F^a_{\ b} F^b_{\ a} = \frac{\epsilon}{4}\omega(\omega-1)\,. \tag{5.11}$$

It means that this realization (5.5) corresponds to a limited class of representations of the superalgebra $osp(N|M)$. This fact reflects the general statement of [27] that not all representations of simple Lie algebras $\mathfrak{g}$ of B, C and D types are the representations of the corresponding Yangians $Y(\mathfrak{g})$.

Remark 7. For $\epsilon = -1$ and even $M = 2m$ the super-oscillator algebra (5.1) is represented in terms of m copies of the bosonic Heisenberg algebras $c^j = x^j$, $c^{m+j} = \partial^j$, $j = 1, \dots, m$, and N fermionic oscillators $c^{2m+\alpha} = b^\alpha$, $\alpha = 1, 2, \dots, N$, with the (anti)commutation relations

$$[x^i,\, \partial^j] = -\delta^{ij}\,, \quad [b^\alpha, b^\beta]_+ := b^\alpha\, b^\beta + b^\beta\, b^\alpha = 2\delta^{\alpha\beta}\,, \quad [x^i,\, b^\alpha] = 0 = [\partial^i,\, b^\alpha]\,, \tag{5.12}$$

which are equivalent to (5.1) with the choice of the metric $\bar{\varepsilon}^{ab}$ as $(M+N)\times(M+N)$ matrix

$$\bar{\varepsilon}^{ab} = \begin{pmatrix} 0 & -I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 2\,I_N \end{pmatrix} \;\Rightarrow\; \varepsilon_{ab} = \begin{pmatrix} 0 & I_m & 0 \\ -I_m & 0 & 0 \\ 0 & 0 & \frac{1}{2}I_N \end{pmatrix}. \tag{5.13}$$

The fermionic variables b^β with the commutation relations (5.12) generate the N-dimensional Clifford algebra. Let N be an even number $N = 2n$. In this case, one can introduce the longest element $b^{(N+1)} = (i)^n\, b^1 b^2 \cdots b^N$ which anticommutes with all generators b^α and possesses $(b^{(N+1)})^2 = 1$. Then, for $\epsilon = +1$ and even numbers $M = 2m$, $N = 2n$, one can realize the super-oscillator algebra (5.1) (with the metric (5.13)) in terms of the generators

$$c^j = x^j \cdot b^{(N+1)}\,, \quad c^{m+j} = \partial^j \cdot b^{(N+1)} \quad (j = 1, \dots, m)\,, \quad c^{2m+\alpha} = b^\alpha \quad (\alpha = 1, 2, \dots, N)\,, \tag{5.14}$$

where the operators x^i, ∂^j and b^α satisfy (5.12). Note that the super-oscillator algebra (5.1) for $\epsilon = +1$ has an unusual property that generators c^a and c^b with gradings $[a] = 0$ and $[b] = 1$ anticommute, which is not usual feature of bosons and fermions in field theories.

The implementation (5.12) of algebra (5.1) suggests the rules of Hermitian conjugation for the generators c^a

$$\begin{aligned}&(c^j)^\dagger = c^j\,, \quad (c^{m+j})^\dagger = -c^{m+j}\,, \quad j = 1, \dots, m\,,\\ &\qquad (c^{2m+\alpha})^\dagger = c^{2m+\alpha}\,, \quad \alpha = 1, 2, \dots, N\,,\end{aligned} \tag{5.15}$$

which follow from the commonly used properties of the Heisenberg and Clifford algebras: $(x^j)^\dagger = x^j$, $(\partial^j)^\dagger = -\partial^j$, $(b^\alpha)^\dagger = b^\alpha$, $(b^{(N+1)})^\dagger = b^{(N+1)}$. We shall apply the rules (5.15) below.

Remark 8. Consider the graded tensor product $\mathcal{A} \otimes \mathcal{A}$ and denote the generators of the first and second factors in $\mathcal{A} \otimes \mathcal{A}$ respectively as c_1^a and c_2^a. Since $\otimes$ is the graded tensor product, we have (cf. (5.1))

$$[c_1^a,\ c_2^b]_\epsilon \equiv c_1^a c_2^b + \epsilon(-1)^{ab} c_2^b c_1^a = 0\,. \tag{5.16}$$

Any element of $\mathcal{A} \otimes \mathcal{A}$ can be written as a polynomial $f(c_1^a, c_2^b)$ and its condition of invariance under the action of the group Osp is written as

$$\Big[A_{ba}(F_1^{ab} + F_2^{ab}),\ f(c_1^a, c_2^b)\Big] = 0,$$

where (see (5.5))

$$F_1^{ab} \equiv \epsilon(-1)^b c_1^{(a} c_1^{b)}\,, \quad F_2^{ab} \equiv \epsilon(-1)^b c_2^{(a} c_2^{b)} \tag{5.17}$$

are the generators of the osp algebras and A_{ba} are the super-parameters (with $\mathrm{grad}(A_{ba}) = [a] + [b]$). In the case of an even function f, when $\mathrm{grad}(f) = 0$, this invariance condition is equivalent to

$$\Big[(F_1^{ab} + F_2^{ab}),\ f(c_1^a, c_2^b)\Big] = 0\,. \tag{5.18}$$

Now we introduce the super-symmetrized product $c^{(a_1} \cdots c^{a_k)}$ of any number of super-oscillators, which generalizes the super-symmetrized product of two super-oscillators (5.4). The general definition and properties of such super-symmetrized products are given in Appendix **B**. In [2] we have proved the following statement.

Proposition 6. *The elements*

$$I_k = \varepsilon_{a_1 b_1} \ldots \varepsilon_{a_k b_k}\, c_1^{(a_1} \cdots c_1^{a_k)} c_2^{(b_k} \cdots c_2^{b_1)} \in \mathcal{A} \otimes \mathcal{A}\,, \quad k = 1, 2, \ldots\,, \tag{5.19}$$

are invariant under the action (2.7) of the supergroup OSp: $c^a \;\to\; U^a{}_b\, c^b$. It means that the elements (5.19) are invariant under the action of the Lie superalgebra osp and satisfy the invariance condition (5.18):

$$\Big[\varepsilon_{a_1 b_1} \ldots \varepsilon_{a_k b_k} c_1^{(a_1} \cdots c_1^{a_k)} c_2^{(b_k} \cdots c_2^{b_1)},\, F_1^{ab} + F_2^{ab}\Big] = 0\,, \tag{5.20}$$

where F_1^{ab} and F_2^{ab} are the generators (5.17) of the Lie super-algebra osp (see Proposition **4***).*

It turns out that the invariants (5.19) are not functionally independent. Indeed, we have the following statement.

Proposition 7. *The invariants (5.19) satisfy the recurrence relation*

$$I_k\, I_1 = I_{k+1} + \frac{k}{4}\big((k-1) - \omega\big) I_{k-1}\,, \qquad \omega = \epsilon(N - M)\,, \tag{5.21}$$

where $I_0 = 1$ and $I_1 = \varepsilon_{ab} c_1^a\, c_2^b = c_1^a\, c_{2a}$. In the representations (5.12), (5.14) and (5.13) Hermitian conjugations of invariant elements (5.19) are

$$I_{2k}^\dagger = I_{2k}\,, \qquad I_{2k+1}^\dagger = -I_{2k+1}\,. \tag{5.22}$$

Proof. The derivation of the recurrence relation (5.21) is given in Appendix **B**. To prove (5.22), it is useful to define the invariants

$$\widetilde{I}_m = \sigma^m \, I_m \,, \tag{5.23}$$

where $\sigma^2 = -1$, i.e. $\sigma = \pm i$. Then, the recurrence relation (5.21) for new invariants $\widetilde{I}_k$ has the form:

$$\widetilde{I}_{k+1} = z\widetilde{I}_k + \frac{k}{4}(k-1-\omega)\widetilde{I}_{k-1}, \qquad \omega = \epsilon(N-M)\,, \tag{5.24}$$

where $\widetilde{I}_0 = 1$ and we introduce the operator

$$z := \widetilde{I}_1 = \sigma I_1 = \sigma\varepsilon_{ab}c_1^a c_2^b = \sigma(c_1\cdot c_2) = -\sigma\varepsilon_{ba}c_2^b c_1^a = -\sigma(c_2\cdot c_1), \tag{5.25}$$

which is Hermitian $z^\dagger = z$ in the representations (5.12), (5.14) and (5.13). One can prove the latter statement by making use of the rules (5.15) and commutation relations (5.16). In view of the recurrence relation (5.24) and initial conditions $\widetilde{I}_0 = 1$ and $\widetilde{I}_1 = z$ all invariant operators $\widetilde{I}_k$ are k-th order polynomials (with real coefficients) of the Hermitian operator z. Therefore all $\widetilde{I}_k$ are the Hermitian operators $\widetilde{I}_k^\dagger = \widetilde{I}_k$, and therefore, taking into account (5.23) and $\sigma^* = -\sigma$, we deduce (5.22). □

Now we introduce a generating function of the Hermitian invariant operators $\widetilde{I}_k$:

$$F(x|z) = \sum_{k=0}^{\infty} \widetilde{I}_k \frac{x^k}{k!}\,. \tag{5.26}$$

Since the invariants $\widetilde{I}_k$ are polynomials in z, the generating function (5.26) depends on x and z only.

Proposition 8. *The generating function (5.26) is equal to*

$$F(x|z) = \left(1-\frac{x}{2}\right)^{\frac{\omega}{2}-z}\left(1+\frac{x}{2}\right)^{\frac{\omega}{2}+z}. \tag{5.27}$$

Proof. Using the recurrence relation (5.24) we obtain:

$$\sum_{k=0}^{\infty}\widetilde{I}_{k+1}\frac{x^k}{k!} = z\sum_{k=0}^{\infty}\widetilde{I}_k\frac{x^k}{k!} + \frac{1}{4}\sum_{k=2}^{\infty}\widetilde{I}_{k-1}\frac{x^k}{(k-2)!} - \frac{\omega}{4}\sum_{k=1}^{\infty}\widetilde{I}_{k-1}\frac{x^k}{(k-1)!}. \tag{5.28}$$

Now changing the summation indices and using (5.26) one deduces:

$$F_x(x|z) = zF(x|z) + \frac{x^2}{4}F_x(x|z) - \frac{x\omega}{4}F(x|z)\,, \tag{5.29}$$

where $F_x(x|z) \equiv \partial_x F(x|z) = \sum_{k=0}^{\infty}\widetilde{I}_{k+1}\frac{x^k}{k!}$. The general solution to this ordinary differential equation is given in (5.27) up to an arbitrary constant factor c. The invariants $\widetilde{I}_k$ are extracted from the generating function (5.26) using the formula

$$\widetilde{I}_k(z) = \partial_x^k F(0|z) = c\,\partial_x^k\,(1-\frac{x}{2})^{\frac{\omega}{2}-z}(1+\frac{x}{2})^{\frac{\omega}{2}+z}\Big|_{x=0}, \tag{5.30}$$

from which we fix the constant $c = F(0|z) = \widetilde{I}_0 = 1$. □

6. The construction of the R-operator in the super-oscillator representation

Let T be the defining representation of the Yangian $Y(osp)$. In the previous section we have considered the RLL-relation (4.1) and (5.10) that intertwines L-operators $||L^a{}_b(u)|| \in T(Y(osp)) \otimes \mathcal{A}$ (given in (5.9)) by means of the R-matrix (3.12) in the defining representation, i.e., $R(u) \in T(Y(osp)) \otimes T(Y(osp))$. In other words, the R-matrix in the RLL-relations (4.1) and (5.10) acts in the space $\mathcal{V}^{\otimes 2}_{(N|M)}$, where $\mathcal{V}_{(N|M)}$ is the space of the defining representation T of $Y(osp(N|M))$.

There is another type of RLL-relations which intertwines the L-operators (5.9) by means of the R-matrix in the super-oscillator representation, i.e., $\hat{\mathcal{R}}(u) \in \mathcal{A} \otimes \mathcal{A}$, where $\otimes$ is the graded tensor product. In components, this type of RLL relations has the form

$$\hat{\mathcal{R}}_{12}(u) L_1{}^a_b(u+v) L_2{}^b_c(v) = L_1{}^a_b(v) L_2{}^b_c(u+v) \hat{\mathcal{R}}_{12}(u), \tag{6.1}$$

or after substitution of the L-operator (5.9) we have

$$\begin{aligned} &\hat{\mathcal{R}}_{12}(u)\big((u+v)\delta^a_b + \epsilon(-1)^b c^a_1 c_{1b}\big)\big(v\delta^b_c + \epsilon(-1)^c c^b_2 c_{2c}\big) = \\ &= \big(v\delta^a_b + \epsilon(-1)^b c^a_1 c_{1b}\big)\big((u+v)\delta^b_c + \epsilon(-1)^c c^b_2 c_{2c}\big)\hat{\mathcal{R}}_{12}(u). \end{aligned} \tag{6.2}$$

Here for simplicity we fix $\alpha = 1/2$ in the definition of the L-operators and associate the first and second factors in $\mathcal{A} \otimes \mathcal{A}$, respectively, with the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ generated by the elements c^a_1 and c^b_2 such that $[c^a_1, c^b_2]_\epsilon = 0$ (see (5.16)).

The RLL relation (6.2) is quadratic with respect to the parameter v. The terms proportional to v^2 are canceled, the terms proportional to v give

$$\hat{\mathcal{R}}_{12}(u)(c^a_1 c_{1c} + c^a_2 c_{2c}) = (c^a_1 c_{1c} + c^a_2 c_{2c})\hat{\mathcal{R}}_{12}(u), \tag{6.3}$$

while the terms independent of v are

$$\hat{\mathcal{R}}_{12}(u)\big(u\delta^a_b + \epsilon(-1)^b c^a_1 c_{1b}\big)(-1)^c c^b_2 c_{2c} = (-1)^b c^a_1 c_{1b}\big(u\delta^b_c + \epsilon(-1)^c c^b_2 c_{2c}\big)\hat{\mathcal{R}}_{12}(u). \tag{6.4}$$

6.1. The Shankar-Witten form of the R operator

The relations (6.3) are nothing but the invariance conditions (5.18) with respect to the adjoint action of osp

$$\Big[\hat{\mathcal{R}}(u), F^{ab}_1 + F^{ab}_2\Big] = 0\,. \tag{6.5}$$

It means that one can search for the $\hat{\mathcal{R}}(u)$-operator as a sum of osp-invariants (5.19)

$$\hat{\mathcal{R}}_{12}(u) = \sum_k \frac{r_k(u)}{k!} I_k = \sum_k \frac{r_k(u)}{k!} \varepsilon_{\vec{a},\vec{b}}\; c_1^{(a_1\ldots a_k)} c_2^{(b_k\ldots b_1)}\,, \tag{6.6}$$

where we use the concise notation

$$\varepsilon_{\vec{a},\vec{b}} = \varepsilon_{a_1 b_1}\ldots\varepsilon_{a_k b_k}\,, \quad c_1^{(a_1\ldots a_k)} := c_1^{(a_1}\cdots c_1^{a_k)}\,, \quad c_2^{(b_k\ldots b_1)} := c_2^{(b_k}\cdots c_2^{b_1)}\,.$$

Inserting this ansatz into the condition (6.4), we obtain (see [2]) the recurrence relation for $r_k(u)$

$$r_{k+2}(u) = \frac{4(u-k)}{k+2+u-\omega} r_k(u)\,, \tag{6.7}$$

which is solved in terms of the Γ-functions:

$$
\begin{aligned}
r_{2m}(u) &= (-4)^m \frac{\Gamma(m-\frac{u}{2})}{\Gamma(m+1+\frac{u-\omega}{2})} A(u), \\
r_{2m+1}(u) &= (-4)^m \frac{\Gamma(m-\frac{u-1}{2})}{\Gamma(m+1+\frac{u-\omega+1}{2})} B(u) ,
\end{aligned}
\tag{6.8}
$$

where the parameter $\omega = \epsilon(N-M)$ was defined in (3.5) and $A(u), B(u)$ are arbitrary functions of u. Substituting (6.8) in (6.6) gives the expression for the osp-invariant R-matrix which intertwines two L operators in (6.1).

The methods used in [2] (for derivation of (6.6) and (6.8)) require the introduction of additional auxiliary variables and are technically quite nontrivial and cumbersome. Below in this paper, in Section **8**, we give a simpler and more elegant derivation of conditions (6.7). This derivation is based on an application of the generating function (5.27) for the invariants $\widetilde{I}_k$, where the explicit form (5.27) is obtained by means of the recurrence relation (5.21).

6.2. The Faddeev-Takhtajan-Tarasov type R operator

There is another form of R operators which intertwines the L operators in the RLL equations (6.2) and are expressed as a ratio of Euler Gamma-functions. For the $s\ell(2)$ case this type of solutions for R operator was first obtained in [5] (see also [6] and [28]). The generalization to the $s\ell(N)$ case (for a wide class of representations of $s\ell(N)$) was given in [29]. For orthogonal and symplectic algebras (and a very special class of their representations) analogous solutions of (6.2) were recently obtained in [7]. Below we generalize the results of [7] and find the solutions for the super-oscillator Faddeev-Takhtajan-Tarasov type R-operator in the case of osp Lie superalgebras.

Proposition 9. *The R operator intertwining the super-oscillator L operators in the RLL equations (6.1), (6.2) obeys the finite-difference equation*

$$
\hat{\mathcal{R}}_{12}(u|z+1)\,(z-u) = \hat{\mathcal{R}}_{12}(u|z-1)\,(z+u) , \tag{6.9}
$$

where $z = \sigma\, c_1^a\, c_{2a}$ and $\sigma^2 = -1$. The solution of this functional equation is given by the ratio of the Euler Gamma-functions

$$
\hat{\mathcal{R}}_{12}(u|z) = r(u,z) \frac{\Gamma\big(\frac{1}{2}(z+1+u)\big)}{\Gamma\big(\frac{1}{2}(z+1-u)\big)}, \tag{6.10}
$$

where $r(u,z)$ is an arbitrary periodic function $r(u,z+2) = r(u,z)$ which normalizes the solution.

Proof. Taking into account the experience related to the orthogonal and symplectic cases (see [7]), we will look for a solution to the first equation (6.3) as

$$
\hat{\mathcal{R}}_{12}(u) = \hat{\mathcal{R}}_{12}(u|z), \qquad z = \sigma c_1^a c_{2a} = \epsilon\sigma(-1)^a c_{1a} c_2^a = -\sigma c_2^a c_{1a} , \tag{6.11}
$$

where σ is a numerical constant to be defined. In the last chain of equalities we have used (5.16). In other words, the operator $\hat{\mathcal{R}}_{12}(u)$ acting in $\mathcal{V}_1 \otimes \mathcal{V}_2$ is given by a function of an invariant z bilinear in super-oscillators c_1^a and c_2^a. Note that in the orthogonal and symplectic cases [7] the conventional invariants I_k (5.19) are in one-to-one correspondence with polynomials of z of the order k. In the super-symmetric case of the algebras osp we prove this fact in Appendix **B** (see eq. (B.9) and comment after this equation). To justify the ansatz (6.11), we recall that the super-oscillators belonging to different factors in $\mathcal{A}\otimes\mathcal{A}$ and acting in different auxiliary spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ commute according to (5.16)

$$c_1^a c_2^b = -\epsilon(-1)^{ab} c_2^b c_1^a, \tag{6.12}$$

so we have

$$\begin{aligned} zc_1^b &= \sigma c_1^a c_{2a} c_1^b = -\epsilon(-1)^{ab}\sigma c_1^a c_1^b c_{2a} = (-1)^{ab+1}\sigma\big(\epsilon\bar{\varepsilon}^{ab} - (-1)^{ab} c_1^b c_1^a\big)c_{2a} \\ &= c_1^b z - \sigma c_2^b, \end{aligned} \tag{6.13}$$

$$zc_2^b = \sigma c_1^a c_{2a} c_2^b = c_1^a \sigma(\delta_a^b - \epsilon(-1)^{ab} c_2^b c_2^a) = \sigma c_1^b - \sigma\epsilon(-1)^{ab} c_1^a c_2^b c_{2a} = \sigma c_1^b + c_2^b z. \tag{6.14}$$

Combining these relations we obtain

$$z\,(c_1^a c_{1b} + c_2^a c_{2b}) = (c_1^a c_{1b} + c_2^a c_{2b})\,z, \tag{6.15}$$

i.e. z commutes with the sum $c_1^a c_{1b} + c_2^a c_{2b}$, and hence an arbitrary function $\hat{\mathcal{R}}_{12}(u|z)$ depending on z satisfies the invariance conditions (6.3) and (6.5).

Let us introduce

$$c_\pm^b := (c_1^b \pm \sigma c_2^b)\,, \tag{6.16}$$

and consider a linear combination of (6.13) and (6.14)

$$zc_\pm^b \equiv z(c_1^b \pm \sigma c_2^b) = c_\pm^b z \pm \sigma^2(c_1^b \mp \sigma^{-1} c_2^b) = c_\pm^b (z \mp 1), \tag{6.17}$$

where the last equation is obtained under the choice

$$\sigma^2 = -1 \quad \Rightarrow \quad \sigma = \sqrt{-1} = \begin{cases} i\,, \\ -i\,. \end{cases} \tag{6.18}$$

Taking into account (6.17), we have

$$\hat{\mathcal{R}}_{12}(u|z)\,c_\pm^b = c_\pm^b\,\hat{\mathcal{R}}_{12}(u|z\mp 1)\,, \qquad c_\pm^b \hat{\mathcal{R}}_{12}(u|z) = \hat{\mathcal{R}}_{12}(u|z\pm 1)c_\pm^b\,. \tag{6.19}$$

Then multiplying (6.4) by $c_\pm^d \varepsilon_{da}$ (or by $c_\mp^d \varepsilon_{da}$) from the left and by $c_\pm^c$ from the right and contracting oscillator vector indices, one obtains four independent scalar relations. Two of them are

$$\begin{aligned} &c_\pm^d \varepsilon_{da}\hat{\mathcal{R}}_{12}(u|z)\big(u\delta_b^a + \epsilon(-1)^b c_1^a c_{1b}\big)(-1)^c c_2^b c_{2c} c_\pm^c = \\ &= c_\pm^d \varepsilon_{da}(-1)^b c_1^a c_{1b}\big(u\delta_c^b + \epsilon(-1)^c c_2^b c_{2c}\big)\hat{\mathcal{R}}_{12}(u|z)c_\pm^c\,. \end{aligned} \tag{6.20}$$

Applying (6.19), (5.3), the definition (6.11) of z and

$$\begin{aligned} &c_\pm^d c_{2d} = \sigma(-z \pm \tfrac{\omega}{2}),\;\; c_\pm^d c_{1d} = \tfrac{\omega}{2} \mp z,\;\; (-1)^c c_{2c} c_\pm^c = \epsilon\sigma(z \pm \tfrac{\omega}{2}), \\ &(-1)^c c_{1c} c_\pm^c = \epsilon(\tfrac{\omega}{2} \pm z), \end{aligned}$$

these two relations (6.20) turn to be functional equations on $\hat{\mathcal{R}}_{12}(u|z)$:

$$\begin{aligned} &\hat{\mathcal{R}}_{12}(u|z\pm 1)c_\pm^d \varepsilon_{da}\big(u\delta_b^a + \epsilon(-1)^b c_1^a c_{1b}\big)(-1)^c c_2^b c_{2c} c_\pm^c = \\ &= c_\pm^d \varepsilon_{da}(-1)^b c_1^a c_{1b}\big(u\delta_c^b + \epsilon(-1)^c c_2^b c_{2c}\big)c_\pm^c \hat{\mathcal{R}}_{12}(u|z\mp 1)\,, \end{aligned} \quad \Rightarrow$$

$$\hat{\mathcal{R}}_{12}(u|z\pm 1)\epsilon(u\mp z)\big(z^2-\frac{\omega^2}{4}\big)=\epsilon(-u\mp z)\big(z^2-\frac{\omega^2}{4}\big)\hat{\mathcal{R}}_{12}(u|z\mp 1)\,.$$

Canceling the common factor $\epsilon\big(z^2-\frac{\omega^2}{4}\big)$ in both sides we obtain a pair of equations

$$\hat{\mathcal{R}}_{12}(u|z\pm 1)\;(u\mp z)=(-u\mp z)\;\hat{\mathcal{R}}_{12}(u|z\mp 1)\,, \tag{6.21}$$

which are equivalent for both choices of signs to the one equation (6.9). In a similar fashion the other pair of relations gives identities

$$\begin{aligned}
&c^d_{\mp}\varepsilon_{da}\hat{\mathcal{R}}_{12}(u|z)\big(u\delta^a_b+\epsilon(-1)^b c^a_1 c_{1b}\big)(-1)^c c^b_2 c_{2c} c^c_{\pm}= \\
&=c^d_{\mp}\varepsilon_{da}(-1)^b c^a_1 c_{1b}\big(u\delta^b_c+\epsilon(-1)^c c^b_2 c_{2c}\big)\hat{\mathcal{R}}_{12}(u|z)c^c_{\pm}, \quad\Rightarrow \\
&\hat{\mathcal{R}}_{12}(u|z\mp 1)c^d_{\mp}\varepsilon_{da}\big(u\delta^a_b+\epsilon(-1)^b c^a_1 c_{1b}\big)(-1)^c c^b_2 c_{2c} c^c_{\pm}= \\
&=c^d_{\mp}\varepsilon_{da}(-1)^b c^a_1 c_{1b}\big(u\delta^b_c+\epsilon(-1)^c c^b_2 c_{2c}\big)c^c_{\pm}\hat{\mathcal{R}}_{12}(u|z\mp 1)), \quad\Rightarrow \\
&\hat{\mathcal{R}}_{12}(u|z\mp 1)\epsilon(u\pm z)\big(z\pm\frac{\omega}{2}\big)^2=\epsilon(u\pm z)\big(z\pm\frac{\omega}{2}\big)^2\hat{\mathcal{R}}_{12}(u|z\mp 1),
\end{aligned} \tag{6.22}$$

which are satisfied automatically. Finally, the solution of the functional equations (6.9), (6.21) can be found immediately and is given in (6.10) by the ratio of the Euler Gamma-functions. □

We see that the scalar projections (6.20) and (6.22) of the RLL relation are exactly the same as in the non-supersymmetric case [7], i.e. no signs related to grading appear. Moreover, we stress that the functional equation (6.9) is independent of the parameter ϵ, which distinguishes the cases of the algebras $osp(N|M)$ and $osp(M|N)$.

Remark 9. We have two choices (6.18) of the parameter σ and therefore we have two versions of the solution (6.10)

$$\hat{\mathcal{R}}^{(\pm)}_{12}(u|z)=r^{(\pm)}(u,z)\,\frac{\Gamma\big(\frac{1}{2}(\pm i c^a_1 c_{2a}+1+u)\big)}{\Gamma\big(\frac{1}{2}(\pm i c^a_1 c_{2a}+1-u)\big)}. \tag{6.23}$$

In view of the identity $\Gamma(1-x)\Gamma(x)=\pi/\sin(\pi x)$, these two versions are equivalent to each other up to a special choice of the normalization functions $r^{(\pm)}(u,z)$. So one can consider only one of the solutions (6.23).

7. The $\hat{\mathcal{R}}$ operator in special cases of $osp(N|2m)$

In this section, we work out the explicit form of the solution (6.10) in a few particular cases.

7.1. The case of $osp(M|N)=osp(1|2)$

In this case, we have $N=2$ and $M=1$, and the superalgebra $osp(1|2)$ is described by the bosonic oscillator $c^1\equiv a^\dagger, c^2\equiv a$ (in the holomorphic representation we have $c^1\equiv x, c^2\equiv\partial$) and by one fermionic variable $c^3\equiv b$ with the commutation relations (5.1):

$$[x,\;\partial]=-1\,,\qquad \{b,b\}=2\,,\qquad [x,\;b]=0=[\partial,\;b]\,, \tag{7.1}$$

where $\{b,b'\}\equiv b\cdot b'+b'\cdot b$ denotes the anticommutator. To obtain (7.1) from (5.1) and (5.2), we fix there $\epsilon=-1$ and specify the metric matrix as

$$\bar{\varepsilon}^{ab} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \Rightarrow \quad \varepsilon_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

Note that the fermionic variable b can be understood in the matrix representation as a single Pauli matrix (say τ^3). To define the operator z in (6.11) we need two copies of super-oscillator algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with the generators $c_1^a = (x_1, \partial_1, b_1)$ and $c_2^a = (x_2, \partial_2, b_2)$ which act in two different spaces $\mathcal{V}_1$ and $\mathcal{V}_2$. Then, the invariant operator z in (6.11) looks like

$$z = \sigma c_1^a c_{2a} = \sigma \varepsilon_{ab} c_1^a c_2^b = \sigma (x_1 \partial_2 - x_2 \partial_1) + \frac{\sigma}{2} b_1 b_2 \equiv \mathsf{x} + \mathsf{b}, \tag{7.2}$$
$$\mathsf{x} = \sigma (x_1 \partial_2 - x_2 \partial_1), \qquad \mathsf{b} = \frac{\sigma}{2}\, b_1 b_2, \qquad \sigma = \pm i,$$

where b_i satisfy $b_i^2 = 1$ in view of (7.1) and anticommute $b_1 b_2 = -b_2 b_1$ in order to ensure (6.12). The characteristic equation for the fermionic part of z:

$$\mathsf{b}^2 = \frac{\sigma^2}{4} b_1 b_2 b_1 b_2 = -\frac{1}{4} b_1 b_2 b_1 b_2 = \frac{1}{4} (b_1)^2 (b_2)^2 = \frac{1}{4}, \tag{7.3}$$

(here we took into account (7.1)) allows one to introduce the projection operators:

$$P_{\pm\frac{1}{2}} = \frac{1}{2} \pm \mathsf{b}, \qquad P_i \cdot P_j = \delta_{ij} P_i, \quad i, j = \pm\frac{1}{2}, \qquad P_{+\frac{1}{2}} + P_{-\frac{1}{2}} = 1. \tag{7.4}$$

Now any function f of b can be decomposed in these projectors

$$\mathsf{b} \cdot P_{\pm\frac{1}{2}} = \pm\frac{1}{2} P_{\pm\frac{1}{2}} \quad \Rightarrow \quad f(\mathsf{b}) = f(\mathsf{b}) \cdot (P_{+\frac{1}{2}} + P_{-\frac{1}{2}}) = f\big(1/2\big)\, P_{+\frac{1}{2}} + f\big(-1/2\big)\, P_{-\frac{1}{2}}. \tag{7.5}$$

Accordingly, the R-operator (6.10) can also be decomposed as:

$$\hat{\mathcal{R}}_{12}(u|z) = \hat{\mathcal{R}}_{12}(u|\mathsf{x} + \mathsf{b}) = \big(\frac{1}{2} + \mathsf{b}\big) \cdot \hat{\mathcal{R}}_{12}\big(u|\mathsf{x} + \frac{1}{2}\big) + \big(\frac{1}{2} - \mathsf{b}\big) \cdot \hat{\mathcal{R}}_{12}\big(u|\mathsf{x} - \frac{1}{2}\big), \tag{7.6}$$

and finally we have

$$\hat{\mathcal{R}}_{12}^{osp(1|2)}(u|z) = r_+(u, \mathsf{x}) \frac{\Gamma\big(\frac{1}{2}(\mathsf{x} + \frac{3}{2} + u)\big)}{\Gamma\big(\frac{1}{2}(\mathsf{x} + \frac{3}{2} - u)\big)} \cdot P_{+\frac{1}{2}} + r_-(u, \mathsf{x}) \frac{\Gamma\big(\frac{1}{2}(\mathsf{x} + \frac{1}{2} + u)\big)}{\Gamma\big(\frac{1}{2}(\mathsf{x} + \frac{1}{2} - u)\big)} \cdot P_{-\frac{1}{2}}, \tag{7.7}$$

where $r_\pm(u, \mathsf{x}) = r(u, \mathsf{x} \pm \frac{1}{2})$ are periodic functions in x, i.e., the general $osp(1|2)$-invariant R-operators consist of two independent terms acting on two invariant subspaces, corresponding to eigenvalues $\pm\frac{1}{2}$ of the fermionic part $\mathsf{b} \equiv \sigma b_1 b_2$ of the invariant operator z. The coefficients in the expansion (7.7) in projectors $P_{\pm\frac{1}{2}}$ are the functions of the bosonic part $\mathsf{x} = \sigma(x_1 \partial_2 - x_2 \partial_1)$ of the invariant operator z. These coefficients are nothing but the R-operators for the bosonic subalgebra $s\ell(2) \simeq sp(2) \subset osp(1|2)$.

7.2. *The case of* $osp(2|2)$

In this case, we have two bosonic $c^1 = x$, $c^2 = \partial$ and two fermionic $c^3 = b^1$, $c^4 = b^2$, oscillators which we realize using even and odd variables with the commutation relations (5.1):

$$[x,\, \partial] = -1, \qquad \{b^\alpha, b^\beta\} = 2\delta^{\alpha\beta}, \qquad [x,\, b^\alpha] = 0 = [\partial,\, b^\alpha]. \tag{7.8}$$

Here again we fix $\epsilon = -1$ and

$$\bar{\varepsilon}^{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \Rightarrow \quad \varepsilon_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} .$$

We introduce two super-oscillator algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with the generators $\{c_1^a\}$ and $\{c_2^a\}$, respectively. The invariant operator (6.11) is

$$z = \sigma\, \varepsilon_{ab} c_1^a c_2^b = \sigma\,(x_1\partial_2 - x_2\partial_1) + \frac{\sigma}{2}\, b_1^\alpha\, b_2^\alpha \equiv \mathsf{x}+\mathsf{b}, \tag{7.9}$$

where $\sigma = \pm i$.

The fermionic oscillators $b_i^\alpha \in \mathcal{A}_i$ with commutation relations (7.8) and (5.16) generate the 4-dimensional Clifford algebra. It is well known (see, e.g., [30]) that the generators of this Clifford algebra can be realized in terms the Pauli matrices τ^α:

$$b_1^1 = \tau^1 \otimes I_2\,, \quad b_1^2 = \tau^2 \otimes I_2\,, \quad b_2^1 = \tau^3 \otimes \tau^1\,, \quad b_2^2 = \tau^3 \otimes \tau^2\,,$$

where I_2 is the unit (2×2) matrix.

The characteristic equation for the fermionic part $\mathsf{b} = \frac{\sigma}{2}\, b_1^\alpha\, b_2^\alpha$ of the operator (7.9) is

$$\mathsf{b}(\mathsf{b}^2 - 1) = 0. \tag{7.10}$$

The invariant subspaces spanned by the eigenvectors corresponding to eigenvalues $0, \pm 1$ of b are extracted by the projectors:

$$P_0 = 1 - \mathsf{b}^2, \quad P_{+1} = \frac{1}{2}(\mathsf{b}^2 + \mathsf{b}), \quad P_{-1} = \frac{1}{2}(\mathsf{b}^2 - \mathsf{b}), \qquad P_0 + P_{+1} + P_{-1} = 1\,. \tag{7.11}$$

The R-operator is decomposed as follows:

$$\begin{gathered} \hat{\mathcal{R}}_{12}(u|z) = \hat{\mathcal{R}}_{12}(u|\mathsf{x}+\mathsf{b}) = \\ = \hat{\mathcal{R}}_{12}(u|\mathsf{x}+\mathsf{b})(P_0 + P_{+1} + P_{-1}) = \sum_{\ell=0,\pm 1} \hat{\mathcal{R}}_{12}(u|\mathsf{x}+\ell)\, P_\ell . \end{gathered} \tag{7.12}$$

Then (6.10) implies that the spinor-spinor R-operator invariant with respect to $osp(2|2)$ supersymmetry has the form

$$\hat{\mathcal{R}}_{12}^{osp(2|2)}(u|z) = \sum_{\ell=0,\pm 1} r(u|\mathsf{x}+\ell)\frac{\Gamma\big(\frac{1}{2}(\mathsf{x}+\ell+1+u)\big)}{\Gamma\big(\frac{1}{2}(\mathsf{x}+\ell+1-u)\big)} P_\ell, \qquad r(u|z+2) = r(u|z). \tag{7.13}$$

Note that in view of the periodicity condition $r(u|\mathsf{x}-1) = r(u|\mathsf{x}+1)$ one can rewrite (7.13) as follows:

$$\begin{aligned} \hat{\mathcal{R}}_{12}^{osp(2|2)}(u|z) &= r(u|\mathsf{x})\frac{\Gamma\big(\frac{1}{2}(\mathsf{x}+1+u)\big)}{\Gamma\big(\frac{1}{2}(\mathsf{x}+1-u)\big)}(1-\mathsf{b}^2) \\ &\quad + \frac{1}{2} r(u|\mathsf{x}+1)\frac{\Gamma\big(\frac{1}{2}(\mathsf{x}+u)\big)}{\Gamma\big(\frac{1}{2}(\mathsf{x}-u)+1\big)}(\mathsf{x}\mathsf{b}^2 + u\mathsf{b}). \end{aligned} \tag{7.14}$$

In the pure bosonic case of the orthogonal algebras $so(2k)$, the general solution for the $\hat{\mathcal{R}}$-operator splits into two independent solutions corresponding to two nonequivalent chiral left and right representations (see [8], [1], [7]). This does not happen here in the super-symmetric case, where the even and odd functions of b are not separated, due to the dependence on the bosonic operator x in the coefficients $r(u|\mathsf{x}+\ell)$ which mixes the chiral representations of $so(2k)$.

7.3. The case of $osp(n|2)$

This case is a generalization of the examples considered in the previous subsections (for $n=1$ and $n=2$ we respectively reproduce the results for the $osp(1|2)$ and $osp(2|2)$ algebras). Consider the super-oscillator algebra $\mathcal{A}$ with two bosonic $c^1 = x$, $c^2 = \partial$ and n fermionic generators $c^{2+\alpha} = b^\alpha$ ($\alpha = 1, ..., n$) with the commutation relations (5.1):

$$[x,\ \partial] = -1\,, \qquad \{b^\alpha, b^\beta\} = 2\delta^{\alpha\beta}\,, \qquad [x,\ b^\alpha] = 0 = [\partial,\ b^\alpha]\,, \tag{7.15}$$

where the fermionic elements b^α are the generators of the n-dimensional Clifford algebra. This corresponds to the choice of the parameter $\epsilon = -1$ and the metric tensor in the block form

$$\bar{\varepsilon}^{ab} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2\,I_n \end{pmatrix} \;\Rightarrow\; \varepsilon_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} I_n \end{pmatrix} \tag{7.16}$$

where I_n stands for the $n \times n$ unit matrix. The invariant operator $z \in \mathcal{A} \otimes \mathcal{A}$ is

$$z = \sigma\,\varepsilon_{ab} c_1^a c_2^b = \sigma\,(x_1 \partial_2 - x_2 \partial_1) + \frac{\sigma}{2}\, b_1^\alpha b_2^\alpha \equiv \mathsf{x} + \mathsf{b}\,, \tag{7.17}$$

where $\{c_1^a\}$ and $\{c_2^a\}$ are the generators of the first and second factor in $\mathcal{A} \otimes \mathcal{A}$, and the cross-commutation relations of $\{c_1^a\}$ and $\{c_2^a\}$ are given in (5.16). The characteristic equation for the fermionic part $\mathsf{b} = \frac{\sigma}{2}\, b_1^\alpha b_2^\alpha$ of the invariant z has the order $n+1$ (cf. (7.3) and (7.10)):

$$\prod_{m=0}^{n} \big(\mathsf{b} - m + \frac{n}{2}\big) = 0\,. \tag{7.18}$$

One can prove (7.18) by noticing that the operator b is represented as $\mathsf{b} = \bar{z}^\alpha z^\alpha - n/2$, where (see (6.16)) $\bar{z}^\alpha \equiv \frac{1}{2}(b_1^\alpha - \sigma b_2^\alpha) = c_-^{2+\alpha}$ and $z^\alpha \equiv \frac{1}{2}(b_1^\alpha + \sigma b_2^\alpha) = c_+^{2+\alpha}$ are respectively the creation and annihilation fermionic operators in the Fock space $\mathcal{F}$ which is created from the vacuum $|0\rangle$: $z^\alpha|0\rangle = 0$ ($\forall\alpha$). Then the operator in the left-hand side of (7.18) is equal to zero since it is zero on all basis vectors $\bar{z}^{\alpha_1} \cdots \bar{z}^{\alpha_m}|0\rangle \in \mathcal{F}$ (here $1 \le \alpha_1 < ... < \alpha_m \le n$ and $m \le n$) which are the eigenvectors of b with eigenvalues $(m - \frac{n}{2})$.

The projectors P_ℓ on invariant subspaces in $\mathcal{F}$ spanned by the eigenvectors of b corresponding to eigenvalues $(m - \frac{n}{2}) \equiv \ell$, where $\ell = -\frac{n}{2}, -\frac{n}{2}+1, ..., \frac{n}{2}$, are immediately obtained from (7.18):

$$P_\ell = \prod_{\substack{m \neq \ell \\ m=-n/2}}^{n/2} \frac{\mathsf{b} - m}{\ell - m}\,, \qquad \mathsf{b} \cdot P_\ell = \ell\, P_\ell\,, \qquad \sum_{\ell=-n/2}^{n/2} P_\ell = 1\,. \tag{7.19}$$

The case of even $n = 2k$

We see that eigenvalues of b are integer (or half-integer) when the number n is even (or odd). Thus, for the case $osp(n|2) = osp(2k|2)$, when $n = 2k$ is even, the expansion of the solution (6.10) goes over integer eigenvalues

$$\hat{\mathcal{R}}_{12}^{osp(2k|2)}(u|z) = \hat{\mathcal{R}}_{12}^{osp(2k|2)}(u|\mathsf{x} + \mathsf{b}) = \sum_{\ell=-k}^{k} \hat{\mathcal{R}}_{12}^{osp(2k|2)}(u|\mathsf{x} + \ell)\, P_\ell, \tag{7.20}$$

and it implies

$$\hat{\mathcal{R}}_{12}^{osp(2k|2)}(u|z) = \sum_{\ell=-k}^{k} r(u|\mathsf{x}+\ell)\frac{\Gamma\big(\frac{1}{2}(\mathsf{x}+\ell+1+u)\big)}{\Gamma\big(\frac{1}{2}(\mathsf{x}+\ell+1-u)\big)}P_\ell, \qquad r(u|z+2)=r(u|z). \tag{7.21}$$

The case of odd $n=(2k+1)$

For the case $osp(n|2)=osp(2k+1|2)$, when $n=2k+1$ is odd, the expansion of the solution (6.10) goes over half-integer eigenvalues of b: $-\frac{2k+1}{2}, -\frac{2k-1}{2}, \dots, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots, \frac{2k+1}{2}$, and we have the expansion

$$\hat{\mathcal{R}}_{12}^{osp(2k+1|2)}(u|z) = \hat{\mathcal{R}}_{12}^{osp(2k+1|2)}(u|\mathsf{x}+\mathsf{b}) = \sum_{\ell=-k-\frac{1}{2}}^{k+\frac{1}{2}} \hat{\mathcal{R}}_{12}^{osp(2k+1|2)}(u|\mathsf{x}+\ell)P_\ell, \tag{7.22}$$

which for solution (6.10) implies

$$\hat{\mathcal{R}}_{12}^{osp(2k+1|2)}(u|z) = \sum_{\ell=-k-\frac{1}{2}}^{k+\frac{1}{2}} r(u|\mathsf{x}+\ell)\frac{\Gamma\big(\frac{1}{2}(\mathsf{x}+\ell+1+u)\big)}{\Gamma\big(\frac{1}{2}(\mathsf{x}+\ell+1-u)\big)}P_\ell\,, \tag{7.23}$$

where the periodic function $r(u|z+2)=r(u|z)$ normalizes the solution.

7.4. The case of $osp(n|2m)$

We consider the $osp(n|2m)$ invariant super-oscillator algebra which is realized in terms of m pairs of the bosonic oscillators $c^j = x^j$, $c^{m+j}=\partial^j$, $j=1,\dots,m$ and n fermionic oscillators $c^{2m+\alpha}=b^\alpha$, $\alpha=1,2,\dots,n$, with the commutation relations (5.12) deduced from (5.1) with the choice of the parameter $\epsilon=-1$ and metric tensor (5.13). In this case, the invariant operator $z\in\mathcal{A}\otimes\mathcal{A}$ defined in (6.11) is

$$z=\sigma\,\varepsilon_{ab}c_1^a c_2^b = \sigma\sum_{j=1}^{m}(x_1^j\partial_2^j - x_2^j\partial_1^j) + \frac{\sigma}{2}\sum_{\alpha=1}^{n} b_1^\alpha b_2^\alpha \equiv \mathsf{x}+\mathsf{b}. \tag{7.24}$$

Here the operator b is the same as in the previous examples of Section **7.3**. Thus, the R operator (6.10) in the case of the algebra $osp(n|2m)$ is expanded over the projection operators P_ℓ like in the case of $osp(n|2)$, and the final expression for $\hat{\mathcal{R}}_{12}^{osp(2m|n)}(u|z)$ will be given by (7.21) or (7.23):

$$\hat{\mathcal{R}}_{12}^{osp(n|2m)}(u|z) = \sum_{\ell\in\Omega_n} r(u|\mathsf{x}+\ell)\frac{\Gamma\big(\frac{1}{2}(\mathsf{x}+\ell+1+u)\big)}{\Gamma\big(\frac{1}{2}(\mathsf{x}+\ell+1-u)\big)}P_\ell\,, \tag{7.25}$$

where

$$\mathsf{x}=\sigma\sum_{j=1}^{m}(x_1^j\partial_2^j - x_2^j\partial_1^j)\,, \qquad r(u|z)=r(u|z+2)\,,$$

the projectors P_ℓ are defined in (7.19) and

$$\Omega_n = \begin{cases} \{-k,\ 1-k,\ \dots, k-1, k\}, & n=2k, \\ \{-k-\frac{1}{2},\ \dots, k+\frac{1}{2}\}, & n=2k+1 \end{cases} \qquad k\in\mathbb{N}\,. \tag{7.26}$$

8. The relation between two approaches

In this section, we give a more direct and elegant derivation of the R matrix solution (6.6), (6.7), that does not require the introduction of additional auxiliary variables (as it was done in [2]) and is based only on using the generating function[1] (5.26), (5.27) of the invariant operators $\widetilde{I}_k$. In addition, this derivation partially explains the relationship between the two types of solutions (6.6), (6.7) and (6.10) for the R operator.

Now we clarify the relation of the Shankar-Witten form of the R operator (6.6), (6.7) and Faddeev-Takhtajan-Tarasov type R operator given by the ratio of two Euler Gamma-functions in (6.10). First, we write (6.6) in the form

$$\hat{\mathcal{R}}(z) = \sum_{k=0}^{\infty} \frac{\widetilde{r}_k(u)}{k!} \widetilde{I}_k(z), \tag{8.1}$$

where $\widetilde{r}_k(u) = (-\sigma)^k\, r_k(u)$ and $\widetilde{I}_k = \sigma^k\, I_k$. Recall that $\widetilde{I}_k$ are the Hermitian invariants introduced in the proof of Proposition 7.

Proposition 10. *The R operator (8.1) obeys (6.4) or equivalently the finite-difference equation (6.9):*

$$W \equiv (z-u)\,\hat{\mathcal{R}}(z+1) - (z+u)\,\hat{\mathcal{R}}(z-1) = 0\,, \tag{8.2}$$

(which was used to find the second solution (6.10)) if the coefficients $\widetilde{r}_k(u)$ satisfy

$$\widetilde{r}_{k+2}(u) = -\frac{4(u-k)}{k+2+u-\omega}\,\widetilde{r}_k(u)\,, \tag{8.3}$$

that in terms of $r_k(u)$ is written as (6.7).

Proof. One can write (8.2) as

$$W = \sum_{k=0}^{\infty} \frac{\widetilde{r}_k(u)}{k!}\Big((z-u)\widetilde{I}_k(z+1) - (z+u)\widetilde{I}_k(z-1)\Big) = \tag{8.4}$$

$$= \sum_{k=0}^{\infty} \frac{\widetilde{r}_k(u)}{k!}\Big((z-u)\partial_x^k F(x|z+1) - (z+u)\partial_x^k F(x|z-1)\Big)_{x=0}\,.$$

We use the relation (5.29) in the form:

$$zF(x|z) = \left[(1-\frac{x^2}{4})\partial_x + \frac{\omega x}{4}\right] F(x|z),$$

and obtain

$$W = \sum_{k=0}^{\infty} \frac{\widetilde{r}_k(u)}{k!}\partial_x^k\Big((z+1-u-1)F(x|z+1) - (z-1+u+1)F(x|z-1)\Big)_{x=0} = \tag{8.5}$$

[1] However we stress that generating function (5.27) is obtained by using of the recurrence relation (5.24) while the latter is derived in the Appendix B by means of auxiliary variables.

$$= \sum_{k=0}^{\infty} \frac{\widetilde{r}_k(u)}{k!} \partial_x^k \Big([(1 - \frac{x^2}{4})\partial_x + \frac{\omega x}{4} - u - 1] F(x|z+1)$$

$$- [(1 - \frac{x^2}{4})\partial_x + \frac{\omega x}{4} + u + 1] F(x|z-1) \Big)_{x=0}.$$

Then we use the equations

$$F(x|z+1) = \frac{1 + \frac{x}{2}}{1 - \frac{x}{2}} F(x|z), \qquad F(x|z-1) = \frac{1 - \frac{x}{2}}{1 + \frac{x}{2}} F(x|z),$$

which follow from formula (5.27) for the generating function and write (8.5) in the form

$$W = \sum_{k=0}^{\infty} \frac{r_k(u)}{k!} \partial_x^k \left\{ (1 - \frac{x^2}{4}) x \partial_x - u - (u - \omega + 2)\frac{x^2}{4} \right\} \Big(\frac{2F(x|z)}{1 - \frac{x^2}{4}} \Big)_{x=0}. \tag{8.6}$$

Now we apply the identity $\partial_x^k \, x = x\partial_x^k + k\partial_x^{k-1}$ to move derivatives ∂_x in (8.6) to the right and obtain:

$$W = \sum_{k=0}^{\infty} \frac{\widetilde{r}_k(u)}{k!} \left[(k-u)\partial_x^k - \frac{k+u-\omega}{4} k(k-1)\partial_x^{k-2} \right] \Big(\frac{2F(x|z)}{1 - \frac{x^2}{4}} \Big)_{x=0}. \tag{8.7}$$

In the second term in square brackets we shift the summation parameter $k \to k+2$ and deduce

$$W = \sum_{k=0}^{\infty} \frac{1}{k!} \Big((k-u)\widetilde{r}_k(u) - \frac{k+2+u-\omega}{4} \widetilde{r}_{k+2}(u) \Big) \partial_x^k \Big(\frac{2F(x|z)}{1 - \frac{x^2}{4}} \Big)_{x=0} = 0.$$

The resulting expression vanishes due to (8.3). Thus, we prove that the finite-difference equation (8.2) is valid if the coefficients $\widetilde{r}_k(u)$ satisfy (8.3). □

Remark 10. We prove that both R operators (8.1), (8.3) and (6.10) satisfy the same equation (8.2) and indeed obey the RLL relations (6.2). It is worth also to note that the differential operator in the curly brackets in (8.6) coincides (up to change of variable $x = \sigma\lambda$) with the differential operator in curly brackets presented in formula (6.43) of our work [2]. This suggests to regard the generating function $F(x|z)(1 - \frac{x^2}{4})^{-1}$ as a coherent state in the super-oscillator space.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors would like to thank S. Derkachov for useful discussions and comments. A.P.I. acknowledges the support of the Russian Science Foundation, grant No. 19-11-00131. The work of D.K. was partially supported by the Armenian State Committee of Science grant 18T-132 and by the Regional Training Network on Theoretical Physics sponsored by VolkswagenStiftung Contract nr. 86-260.

Appendix A. Properties of operators $\mathcal{P}$, $\mathcal{K}$ and relations for matrix generators of Brauer algebra

We use here the concise matrix notation introduced in Sections 2, 3 (this convenient notation was proposed in [31]).

The matrices (2.19) satisfy the identities

$$\mathcal{P}_{12}=\mathcal{P}_{21}\,,\quad \mathcal{K}_{12}=(-)^{12}\mathcal{K}_{21}(-)^{12}\,,\quad (-)^{1}\mathcal{K}_{12}=(-)^{2}\mathcal{K}_{12}\,,\quad \mathcal{K}_{12}(-)^{1}=\mathcal{K}_{12}(-)^{2}\,,$$
$$\mathcal{P}_{12}\mathcal{P}_{12}=\mathbf{1}\,,\quad \mathcal{K}_{12}\mathcal{K}_{12}=\omega\mathcal{K}_{12}\,,\quad \mathcal{K}_{12}\mathcal{P}_{12}=\mathcal{P}_{12}\mathcal{K}_{12}=\epsilon\mathcal{K}_{12}, \tag{A.1}$$

where $\omega=\epsilon(N-M)$, the operator $(-)^{12}$ is defined in (2.10) and we introduce the matrix $(-)^{i}=(-1)^{[a_i]}\delta^{a_i}_{b_i}$ of super-trace in the i-th factor $\mathcal{V}_{(N|M)}$ of the product $\mathcal{V}^{\otimes 2}_{(N|M)}$. Then, by making use definitions (3.1) and (3.2), we have

$$(-)^{1}\mathcal{P}_{12}=\mathcal{P}_{12}(-)^{2}\,,\quad (-)^{23}\mathcal{P}_{13}=\mathcal{P}_{13}(-)^{12}\,,\quad \mathcal{P}_{13}(-)^{23}=(-)^{12}\mathcal{P}_{13}\,, \tag{A.2}$$
$$\begin{aligned}&\mathcal{P}_{12}\mathcal{P}_{23}=(-)^{12}\mathcal{P}_{13}(-)^{12}\mathcal{P}_{12}=\mathcal{P}_{23}(-)^{23}\mathcal{P}_{13}(-)^{23}=\mathcal{P}_{23}(-)^{12}\mathcal{P}_{13}(-)^{12}\,,\\ &\qquad \mathcal{P}_{12}\mathcal{K}_{13}=(-)^{12}\mathcal{K}_{23}(-)^{12}\mathcal{P}_{12}\,,\quad \mathcal{K}_{23}\mathcal{P}_{12}=\mathcal{P}_{12}(-)^{12}\mathcal{K}_{13}(-)^{12}\,,\end{aligned} \tag{A.3}$$
$$\epsilon\mathcal{K}_{12}\mathcal{P}_{31}=\mathcal{K}_{12}(-)^{12}\mathcal{K}_{32}(-)^{12}\,,\quad \epsilon\mathcal{P}_{31}\mathcal{K}_{12}=(-)^{12}\mathcal{K}_{32}(-)^{12}\mathcal{K}_{12}\,, \tag{A.4}$$
$$\mathcal{K}_{23}\mathcal{K}_{12}=\epsilon\,\mathcal{K}_{23}(-)^{12}\mathcal{P}_{13}(-)^{12}=\epsilon\,(-)^{23}\mathcal{P}_{13}(-)^{23}\mathcal{K}_{12}\,, \tag{A.5}$$
$$\mathcal{K}_{31}\mathcal{K}_{12}=\epsilon\,(-)^{12}\mathcal{P}_{32}(-)^{12}\mathcal{K}_{12}=\epsilon\,\mathcal{K}_{31}(-)^{13}\mathcal{P}_{32}(-)^{13}\,. \tag{A.6}$$

The mirror counterparts of identities (A.3) – (A.6) are also valid. The identities (A.2), (A.3) follow from the representation (3.1): $\mathcal{P}_{12}=(-)^{12}P_{12}=P_{12}(-)^{12}$, where P_{12} is the usual permutation operator. The identities (A.6) follow from the definitions (2.19), (3.1), (3.2) of the operators $\mathcal{P}$ and $\mathcal{K}$. We prove only the first equality in (A.6) since the other identities in (A.4), (A.5) and (A.6) can be proved in the same way. We denote the incoming matrix indices by a_1,a_2,a_3 and the outcoming indices by c_1,c_2,c_3 while summation indices are b_i and d_i. Then we have

$$\begin{aligned}(\mathcal{K}_{31}\mathcal{K}_{12})^{a_1a_2a_3}_{c_1c_2c_3}&=\bar{\varepsilon}^{a_3a_1}\varepsilon_{c_3b_1}\bar{\varepsilon}^{b_1a_2}\varepsilon_{c_1c_2}=\bar{\varepsilon}^{a_3a_1}\delta^{a_2}_{c_3}\varepsilon_{c_1c_2}=\delta^{a_2}_{c_3}\delta^{a_3}_{b_2}\epsilon(-)^{[a_1][b_2]}\bar{\varepsilon}^{a_1b_2}\varepsilon_{c_1c_2}=\\ &=\epsilon(-1)^{[a_2][a_3]}(\mathcal{P}_{23})^{a_2a_3}_{b_2c_3}(-1)^{[a_1][b_2]}(\mathcal{K}_{12})^{a_1b_2}_{c_1c_2}=\epsilon\left(\mathcal{P}_{23}(-)^{23}\,(-)^{12}\,\mathcal{K}_{12}\right)^{a_1a_2a_3}_{c_1c_2c_3}\,,\end{aligned}$$

and in view of the relation $(-)^{23}\mathcal{K}_{12}=(-)^{13}\mathcal{K}_{12}$ which follows from (2.4) and obvious identity $\mathcal{P}_{23}\,(-)^{13}=(-)^{12}\,\mathcal{P}_{23}$ we obtain the first equality in (A.6).

By means of the relations (A.3) – (A.6) one can immediately check eqs. (2.23), (2.24) and also deduce

$$\mathcal{P}_{12}\mathcal{P}_{23}\mathcal{P}_{12}=\mathcal{P}_{23}\mathcal{P}_{12}\mathcal{P}_{23}. \tag{A.7}$$
$$\mathcal{K}_{12}\mathcal{K}_{23}\mathcal{K}_{12}=\mathcal{K}_{12},\quad \mathcal{K}_{23}\mathcal{K}_{12}\mathcal{K}_{23}=\mathcal{K}_{23}, \tag{A.8}$$
$$\mathcal{P}_{12}\mathcal{K}_{23}\mathcal{K}_{12}=\mathcal{P}_{23}\mathcal{K}_{12}\,,\quad \mathcal{K}_{12}\mathcal{K}_{23}\mathcal{P}_{12}=\mathcal{K}_{12}\mathcal{P}_{23}, \tag{A.9}$$
$$\mathcal{P}_{23}\mathcal{K}_{12}\mathcal{K}_{23}=\mathcal{P}_{12}\mathcal{K}_{23}\,,\quad \mathcal{K}_{23}\mathcal{K}_{12}\mathcal{P}_{23}=\mathcal{K}_{23}\mathcal{P}_{12}\,. \tag{A.10}$$

The identity (A.7) follows from the relations in the first line of (A.3). We consider a few relations in (A.8) – (A.10) in detail. We start to prove the first relation in (A.8):

$$(\mathcal{K}_{12}\mathcal{K}_{23}\mathcal{K}_{12})^{a_1a_2a_3}_{c_1c_2c_3}=\bar{\varepsilon}^{a_1a_2}\varepsilon_{b_1b_2}\bar{\varepsilon}^{b_2a_3}\varepsilon_{d_2c_3}\bar{\varepsilon}^{b_1d_2}\varepsilon_{c_1c_2}=\bar{\varepsilon}^{a_1a_2}\delta^{a_3}_{b_1}\delta^{b_1}_{c_3}\varepsilon_{c_1c_2}=\mathcal{K}^{a_1a_2}_{c_1c_2}\delta^{a_3}_{c_3}\,.$$

The second relation in (A.8) can be proved in the same way. Then we prove the first equation in (A.9). For the left hand side of (A.9) one has:

$$(\mathcal{P}_{12}\mathcal{K}_{23}\mathcal{K}_{12})^{a_1a_2a_3}_{c_1c_2c_3} = (-1)^{[a_1][a_2]}\delta^{a_1}_{b_2}\delta^{a_2}_{b_1}\bar{\varepsilon}^{b_2a_3}\varepsilon_{d_2c_3}\bar{\varepsilon}^{b_1d_2}\varepsilon_{c_1c_2} = (-1)^{[a_1][a_2]}\bar{\varepsilon}^{a_1a_3}\delta^{a_2}_{c_3}\varepsilon_{c_1c_2} =$$
$$= \delta^{a_2}_{c_3}\delta^{a_3}_{b_2}(-1)^{[a_1][c_3]}\bar{\varepsilon}^{a_1b_2}\varepsilon_{c_1c_2} = \delta^{a_2}_{c_3}\delta^{a_3}_{b_2}(-1)^{[b_2][c_3]}\bar{\varepsilon}^{a_1b_2}\varepsilon_{c_1c_2} = (\mathcal{P}_{23}\mathcal{K}_{12})^{a_1a_2a_3}_{c_1c_2c_3},$$

and similarly one deduces other relations in (A.9) and (A.10). From the identities (A.7) – (A.10) we also deduce the following relations:

$$\mathcal{K}_{12}\mathcal{P}_{23}\mathcal{K}_{12} = \epsilon\mathcal{K}_{12}, \quad \mathcal{K}_{23}\mathcal{P}_{12}\mathcal{K}_{23} = \epsilon\mathcal{K}_{23}. \tag{A.11}$$

$$\mathcal{P}_{12}\mathcal{K}_{23}\mathcal{P}_{12} = \mathcal{P}_{23}\mathcal{K}_{12}\mathcal{P}_{23}. \tag{A.12}$$

$$\mathcal{P}_{12}\mathcal{P}_{23}\mathcal{K}_{12} = \mathcal{K}_{23}\mathcal{P}_{12}\mathcal{P}_{23}, \quad \mathcal{K}_{12}\mathcal{P}_{23}\mathcal{P}_{12} = \mathcal{P}_{23}\mathcal{P}_{12}\mathcal{K}_{23}. \tag{A.13}$$

At the end of this appendix, we stress that the identities (A.1), (A.7) – (A.10) are the images of the defining relations (3.6), (3.7) for the Brauer algebra in the representation (3.4). The R-matrix (3.10) is the image of the element (3.8) and the Yang-Baxter equation (3.11) is the image of the identity (3.9). Thus, it follows from Proposition **1** that the R-matrix (3.10) is a solution of the braided version of the Yang-Baxter equation (3.11).

Appendix B. Supersymmetrized products of super-oscillators

The product of k super-oscillators is transformed under the action (2.7) of the group Osp as follows:

$$c^{a_1}c^{a_2}\cdots c^{a_k} \;\to\; \Delta^{(k-1)}(U)^{a_1a_2\ldots a_k}_{b_1b_2\ldots b_k}c^{b_1}c^{b_2}\cdots c^{b_k}, \tag{B.1}$$

where $U \in Osp$, and the tensor product of k defining representations of the group Osp is given by the formula

$$\Delta^{(k-1)}(U)^{a_1a_2\ldots a_k}_{b_1b_2\ldots b_k} = U^{a_1}_{b_1}(-1)^{b_1a_2}U^{a_2}_{b_2}(-1)^{b_1b_2}\cdots(-1)^{(\sum\limits_{j=1}^{k-1}b_j)a_k}U^{a_k}_{b_k}(-1)^{(\sum\limits_{j=1}^{k-1}b_j)b_k},$$

or in the concise notation we have

$$\Delta^{(k-1)}(U)_{12\ldots k} = U_1(-)^{[1][2]}U_2(-)^{[1][2]}\cdots(-)^{[k]\sum\limits_{j=1}^{k-1}[j]}U_k(-)^{(\sum\limits_{j=1}^{k-1}[j])[k]}.$$

One can check that any element $X \in B_k(\omega)$ of the Brauer algebra (3.6), (3.7) in the representation (3.4) commutes with the action of the Osp supergroup

$$\Delta^{(k-1)}(U)_{12\ldots k}\cdot X = X\cdot\Delta^{(k-1)}(U)_{12\ldots k}. \tag{B.2}$$

Define the super-symmetrized product of two super-oscillators c^a, c^b as

$$c^{(a}c^{b)} \equiv \frac{1}{2}\left(c^ac^b - \epsilon(-1)^{[a][b]}c^bc^a\right) = \frac{1}{2}\left(c^ac^b - \epsilon\,\mathcal{P}^{ab}_{de}c^dc^e\right) = (A_2)^{ab}_{de}c^dc^e, \tag{B.3}$$

where $\mathcal{P}$ is the super-permutation matrix (3.1) and $(A_2)^{ab}_{de}$ is the antisymmetrizer $A_2 = \frac{1}{2}(1 - s_1)$ in the representation (3.4). The direct generalization of (B.3) to the super-symmetrized product of any number of super-oscillators is the following:

$$c^{(a_1}c^{a_2}\cdots c^{a_k)} = (A_k)^{a_1a_2\ldots a_k}_{b_1b_2\ldots b_k}c^{b_1}c^{b_2}\cdots c^{b_k}, \tag{B.4}$$

where A_k is the k-th rank antisymmetrizer in the representation (3.4). The antisymmetrizer A_k can be defined via the recurrence relations (see, e.g., [30])

$$A_k = \frac{1}{k!}\big(1 - s_{k-1} + ... + (-1)^{k-1} s_1 s_2 \cdots s_{k-1}\big) \cdots \big(1 - s_2 + s_1 s_2\big)\big(1 - s_1\big)\,, \tag{B.5}$$

where the generators s_n are taken in the representation (3.4). We stress that in view of the relation (B.2) the super-symmetrized product (B.4) is transformed under the action of Osp as a usual product (B.1).

Note that upon opening the parentheses, the element (B.5) equals the alternating sum of all $k!$ elements of the permutation group S_k. Using this fact one can give a more explicit formula for super-symmetrized product (B.4) of a higher number of super-oscillators

$$c^{(a_1} c^{a_2} \dots c^{a_k)} \equiv \frac{1}{k!} \sum_{\sigma \in S_k} (-\epsilon)^{p(\sigma)} (-1)^{\hat{\sigma}} c^{a_{\sigma_1}} \dots c^{a_{\sigma_k}} = \frac{1}{k!} \partial_\kappa^{a_1} \cdots \partial_\kappa^{a_k} (\kappa \cdot c)^k = \tag{B.6}$$

$$= \partial_\kappa^{a_1} \cdots \partial_\kappa^{a_k} \, \exp(\kappa_a \, c^a)|_{\kappa=0}\,,$$

where $p(\sigma) = 0, 1$ denotes the parity of the permutation σ. Here we introduce (see [2]) auxiliary super-vector κ_a such that the derivatives $\partial_\kappa^a = \frac{\partial}{\partial \kappa_a}$ satisfy

$$\partial_\kappa^{a_1} c^{a_2} = -\epsilon(-1)^{a_1 a_2} c^{a_2} \partial_\kappa^{a_1}\,, \quad \partial_\kappa^b \,(\kappa_a \, c^a) = c^b + (\kappa_a \, c^a)\, \partial_\kappa^b\,, \quad \partial_\kappa^a \partial_\kappa^b = -\epsilon(-1)^{[a][b]} \partial_\kappa^b \partial_\kappa^a\,,$$

and (B.6) holds due to the Leibniz rule.

Now we explain the notation $\hat{\sigma}$ in (B.6). Let $s_j \equiv \sigma_{j,j+1}$ be an elementary transposition of the j-th and $(j+1)$-st site. For the transposition s_j we define $\hat{s_j} = [a_j][a_{j+1}]$. Then for a general permutation $\sigma = s_{j_1} s_{j_2} \dots s_{j_{k-1}} s_{j_k} \in S_k$, we have

$$\hat{\sigma} = [a_{j_k}][a_{j_k+1}] + [a_{s_{j_k}(j_{k-1})}][a_{s_{j_k}(j_{k-1}+1)}] + \cdots + [a_{s_{j_2} \cdots s_{j_k}(j_1)}][a_{s_{j_2} \cdots s_{j_k}(j_1+1)}]. \tag{B.7}$$

As an example, according to the definition (B.6), we have the relation useful in practice:

$$c^{(a_1} \cdots c^{a_i} \cdots c^{a_j} \cdots c^{a_k)} = (-\epsilon)(-1)^{[a_i][a_j] + \sum\limits_{l=i+1}^{j-1} ([a_i]+[a_j])[a_l]} c^{(a_1} \cdots c^{a_j} \cdots c^{a_i} \cdots c^{a_k)}. \tag{B.8}$$

In eq. (5.19) we have defined the supersymmetric invariants I_m Using this definition, the representation (B.6) and the definition (6.11) of z, we obtain the recurrence relation

$$\begin{aligned} -\sigma\, I_m \cdot z = I_m\, I_1 = \varepsilon_{a_1 b_1} \dots \varepsilon_{a_m b_m} \partial_{\kappa_1}^{a_1} \dots \partial_{\kappa_1}^{a_m} \partial_{\kappa_2}^{b_m} \dots \partial_{\kappa_2}^{b_1} \varepsilon_{ab} (\partial_{\kappa_1}^a + \tfrac{1}{2}\epsilon(-1)^a \kappa_1^a)(\partial_{\kappa_2}^b + \\ + \tfrac{1}{2}\epsilon(-1)^b \kappa_2^b) e^{\kappa_1 \cdot c_1 + \kappa_2 \cdot c_2}|_{\kappa_i=0} = I_{m+1} + \tfrac{1}{4} \varepsilon_{a_1 b_1} \dots \varepsilon_{a_m b_m} \partial_{\kappa_1}^{a_1} \dots \partial_{\kappa_1}^{a_m} \partial_{\kappa_2}^{b_m} \dots \partial_{\kappa_2}^{b_1} \varepsilon_{ab} (-1)^a \times \\ \times \kappa_1^a \kappa_2^b e^{\kappa_1 \cdot c_1 + \kappa_2 \cdot c_2}|_{\kappa_i=0} = I_{m+1} - \tfrac{\epsilon}{4} \textstyle\sum_{i=1}^m \varepsilon_{a_1 b_1} \dots \varepsilon_{a_{i-1} b_{i-1}} \varepsilon_{a_{i+1} b_{i+1}} \dots \varepsilon_{a_m b_m} \partial_{\kappa_1}^{a_1} \dots \partial_{\kappa_1}^{a_{i-1}} \times \\ \times \partial_{\kappa_1}^{a_{i+1}} \dots \partial_{\kappa_1}^{a_m} \big(\tfrac{\omega}{\epsilon} - \epsilon \kappa_1^d \partial_{\kappa_1, d}\big) \partial_{\kappa_2}^{b_m} \dots \partial_{\kappa_2}^{b_{i+1}} \partial_{\kappa_2}^{b_{i-1}} \dots \partial_{\kappa_2}^{b_1} e^{\kappa_1 \cdot c_1 + \kappa_2 \cdot c_2}|_{\kappa_i=0} \\ = I_{m+1} + \tfrac{m}{4}\big((m-1) - \omega\big) I_{m-1}\,. \end{aligned} \tag{B.9}$$

For a proof of (B.9) we refer to the papers [1] (see analogous calculation in eq. (5.6) there) and [2]. Taking into account the initial conditions $I_0 = 1$ and $I_1 = -\sigma z$, we deduce from (B.9) that the invariants I_m are polynomials in z of the order m.

References

[1] A.P. Isaev, D. Karakhanyan, R. Kirschner, Orthogonal and symplectic Yangians and Yang-Baxter R-operators, Nucl. Phys. B 904 (2016) 124–147, arXiv:1511.06152v1.

[2] J. Fuksa, A.P. Isaev, D. Karakhanyan, R. Kirschner, Yangians and Yang–Baxter R-operators for ortho-symplectic superalgebras, Nucl. Phys. B 917 (2017) 44, arXiv:1612.04713 [math-ph].

[3] R. Shankar, E. Witten, The S-matrix of the kinks of the $(\bar{\psi}\psi)^2$ model, Nucl. Phys. B 141 (1978) 349–363.

[4] D. Chicherin, S. Derkachov, A.P. Isaev, Conformal group: R-matrix and star-triangle relation, J. High Energy Phys. 04 (2013) 020, arXiv:1206.4150 [math-ph].

[5] V.O. Tarasov, L.A. Takhtajan, L.D. Faddeev, Local Hamiltonians for integrable quantum models on a lattice, Theor. Math. Phys. 57 (1983) 163–181.

[6] L.D. Faddeev, How algebraic Bethe ansatz works for integrable model, in: A. Connes, K. Kawedzki, J. Zinn-Justin (Eds.), Quantum Symmetries/Symetries Quantiques, Proc. Les-Houches Summer School, LXIV, North-Holland, 1998, pp. 149–211, arXiv:hep-th/9605187.

[7] D. Karakhanyan, R. Kirschner, Spinorial RRR operator and algebraic Bethe ansatz, Nucl. Phys. B 951 (2020) 114905, arXiv:1911.08385 [math-ph].

[8] D. Chicherin, S. Derkachov, A.P. Isaev, Spinorial R-matrix, J. Phys. A 46 (48) (2013) 485201, arXiv:1303.4929 [math-ph].

[9] M. Karowski, H.J. Thun, Complete S-matrix of the $O(2N)$ Gross-Neveu model, Nucl. Phys. B 190 (FS3) (1981) 61–92.

[10] Al.B. Zamolodchikov, Factorizable Scattering in Asymptotically Free 2-dimensional Models of Quantum Field Theory, PhD Thesis, Dubna, 1979, unpublished.

[11] N.Yu. Reshetikhin, Algebraic Bethe-ansatz for $SO(N)$ invariant transfer-matrices, Zap. Nauč. Semin. LOMI 169 (1988) 122; J. Math. Sci. 54 (3) (1991) 940–951.

[12] E. Ogievetsky, P. Wiegmann, N. Reshetikhin, The principal chiral field in two-dimensions on classical Lie algebras: the Bethe ansatz solution and factorized theory of scattering, Nucl. Phys. B 280 (1987) 45.

[13] F.A. Berezin, Introduction to Algebra and Analisis with Anticommuting Variables, Moscow State University Press, Moscow, 1983.

[14] D. Arnaudon, J. Avan, N. Crampe, L. Frappat, E. Ragoucy, R-matrix presentation for (super)-Yangians Y(g), J. Math. Phys. 44 (2003) 302, arXiv:math.QA/0111325;
D. Arnaudon, J. Avan, N. Crampe, A. Doikou, L. Frappat, E. Ragoucy, Bethe ansatz equations and exact S matrices for the $osp(M|2n)$ open super spin chain, Nucl. Phys. B 687 (3) (2004) 257.

[15] J. Birman, H. Wenzl, Braids, link polynomials and a new algebra, Trans. Am. Math. Soc. 313 (1989) 249–273.

[16] R. Brauer, On algebras which are connected with the semisimple continuous groups, Ann. Math. (2) 38 (4) (1937) 857–872.

[17] H. Wenzl, On the structure of Brauer's centralizer algebras, Ann. Math. 128 (1988) 173–193.

[18] M. Nazarov, Young's orthogonal form for Brauer's centralizer algebra, J. Algebra 182 (1996) 664–693.

[19] A.P. Isaev, A.I. Molev, Fusion procedure for the Brauer algebra, Algebra Anal. 22 (3) (2010) 142–154.

[20] A.P. Isaev, A.I. Molev, O.V. Ogievetsky, A new fusion procedure for the Brauer algebra and evaluation homomorphisms, Int. Math. Res. Not. (2012) 2571–2606.

[21] A.P. Isaev, M.A. Podoinitsyn, D-dimensional spin projection operators for arbitrary type of symmetry via Brauer algebra idempotents, e-print, arXiv:2004.06096 [hep-th], 2020.

[22] P.P. Kulish, E.K. Sklyanin, On solutions of the Yang-Baxter equation, J. Sov. Math. 19 (5) (1982) 1596; Zap. Nauč. Semin. 95 (1980) 129.

[23] P.P. Kulish, Integrable graded magnets, J. Sov. Math. 35 (1986) 2648; Zap. Nauč. Semin. 145 (1985) 140;
P.P. Kulish, N.Yu. Reshetikhin, Universal R matrix of the quantum superalgebra Osp(2 | 1), Lett. Math. Phys. 18 (1989) 143.

[24] A.B. Zamolodchikov, Al.B. Zamolodchikov, Relativistic factorized S-matrix in two dimensions having O(N) isotopic symmetry, Nucl. Phys. B 133 (1978) 525.

[25] B. Berg, M. Karowski, P. Weisz, V. Kurak, Factorized $U(n)$ symmetric S-matrices in two dimensions, Nucl. Phys. B 134 (1) (1978) 125–132.

[26] A.P. Isaev, Quantum groups and Yang-Baxter equations, preprint MPIM (Bonn), MPI 2004-132, http://webdoc.sub.gwdg.de/ebook/serien/e/mpi_mathematik/2004/132.pdf.

[27] V.G. Drinfeld, Hopf algebras and the quantum Yang-Baxter equation, Dokl. Akad. Nauk SSSR 283 (5) (1985) 1060.

[28] S.E. Derkachov, Factorization of R-matrix. I, J. Math. Sci. 143 (1) (2007) 2773, arXiv:math.QA/0503396.

[29] S.E. Derkachov, A.N. Manashov, Factorization of R-matrix and Baxter Q-operators for generic $s\ell(N)$ spin chains, J. Phys. A, Math. Theor. 42 (2009) 075204, arXiv:0809.2050 [nlin.si].
[30] A.P. Isaev, V.A. Rubakov, Theory of Groups and Symmetries. Representations of Groups and Lie Algebras, Applications, World Scientific, 2020, 600 pp.
[31] L.D. Faddeev, N.Yu. Reshetikhin, L.A. Takhtajan, Quantization of Lie groups and Lie algebras, Algebra Anal. 1 (1) (1989) 178–206, (in Russian). English translation in: Leningr. Math. J. 1 (1) (1990) 193–225.