Faculté des Sciences

Field theoretic and gravitational aspects of Warped Conformal Field Theories

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Abstract

One of the main challenges in modern mathematical physics is to develop a unified framework that reconciles General Relativity and Quantum Field Theory. A promising approach is the holographic principle, which posits a duality between a d-dimensional gravity theory and a $(d-1)$-dimensional quantum field theory living on the boundary of the d-dimensional spacetime. Building on black hole thermodynamics and the seminal AdS/CFT correspondence proposed by Maldacena, various holographic dualities have been explored. This thesis focuses on a specific holographic correspondence: Warped Anti-de Sitter in three dimensions (WAdS_3) and Warped Conformal Field Theory in two dimensions (WCFT_2).

WAdS_3 spacetimes emerge in diverse contexts such as extremal black holes, string theory, and cold atoms. These spacetimes are solutions to extended theories of gravity, like topologically massive gravity, rather than to standard Einstein gravity. The dual field theories, WCFTs, are characterized by symmetries that form a Virasoro-Kač-Moody algebra, distinct from the symmetries of traditional conformal field theories.

In this thesis, we begin by exploring quantum energy conditions for WCFTs and the various holographic descriptions of entanglement entropy in these theories. Next, we determine the Hamiltonian reduction of Lower Spin Gravity and its connection to the geometric action on the coadjoint orbits of the Warped Virasoro group. Finally, we investigate the relationship between the quasinormal modes of warped black holes and their associated photon rings.

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Publications

This thesis is based on the following publications

[1] S. Detournay, D. Grumiller, M. Riegler and Q. Vandermiers, *Uniformization of Entanglement Entropy in Holographic Warped Conformal Field Theories*, 2006.16167

[2] S. Detournay and Q. Vandermiers, *Geometric Actions for Lower-Spin Gravity*, 2408.13198. Accepted for publication in JHEP

[3] S. Detournay, S. Kanuri, A. Lupsasca, P. Spindel, Q. Vandermiers and R. Wutte, *Photon Rings and Quasi-Normal Modes for generic Warped AdS_3 black holes*, . to appear

Contents

Introduction

Motivation

One of the main goals of modern mathematical physics is to produce a theory satisfying both General Relativity (GR) and Quantum Field Theory (QFT). Over the years, several connections between these two theories were made. The one of our interest is the holographic principle, proposed first by 't Hooft [4] and Susskind [5] in 1993. It states that a d dimensional gravity theory is dual to a $d-1$ QFT living on the boundary of the d dimensional spacetime. The name is inspired by holograms in optics where information contained in a two dimensional object is sufficient to reproduce an image in three dimensions. The holographic correspondence originates from black hole thermodynamics. In 1973, Bardeen, Carter and Hawking proposed that black holes are thermodynamic objects and satisfy an analogue of the four laws of thermodynamics [6]. In particular, a temperature and an entropy can be associated with them. This entropy was first suggested by Bekenstein [7] and confirmed by Hawking [8]. The Bekenstein-Hawking entropy of a black hole is shown to be proportional to the area A of its event horizon

$$S_{BH} = \frac{k_B c^3 A}{4G\hbar}\,, \tag{1}$$

where k_B is the Boltzmann constant, c the speed of light, $\hbar$ the Planck constant divided by 2π and G the Newton constant. In natural units, the entropy formula takes a more compact form $S = A/4G$. It is remarkable that this formula works for any black hole in any dimensions. Despite being extensive, the black hole entropy depends on the area of the black hole instead of its volume. This observation leads to the proposal of a holographic description of black holes.

In 1997, Maldacena proposed the first holographic model where the $\mathcal{N}=4$ super Yang-Mills theory in 3+1 dimensions is dual to type IIb superstrings in $\mathrm{AdS}_5 \times S^5$ [9]. This model lays the foundations for an important correspondence, the AdS/CFT correspondence, which even applies beyond the string theory framework. Indeed, the seminal work of Brown and Henneaux in 1986 showed that the asymptotic symmetries of AdS_3 – in essence, the symmetries of the classical phase space – are identical to those of a two-dimensional conformal field theory, a CFT_2 [10].

This novel perspective on gravity has led to the idea of many holographic correspondences, such as flat dualities [11] or Kerr/CFT [12]. This thesis will focus on a particular correspondence: (W)AdS$_3$/WCFT$_2$.

Warped AdS$_3$ (WAdS$_3$) are spacetimes emerging in various contexts such as the near-horizon region of extremal black holes [12–15], perfect fluid solutions in 2+1 dimensions [16], Godel spacetimes [17, 18], string theory [19, 20], cold atoms [21–24] and others [25, 26]. WAdS$_3$ appears not to be a solution to usual Einstein gravity but rather vacuum solutions of extended versions, such as three-dimensional topologically massive gravity (TMG) [27, 28] where they are expected to be stable, contrary to vacuum AdS in the framework of this theory [29]. In the same way that black hole solutions can be obtained from discrete quotients of AdS$_3$ [30–32], warped black holes can be constructed from quotients of WAdS$_3$ [33] and are also solutions to TMG [16, 34, 35].

The asymptotic structure of WAdS spacetimes has been studied and their asymptotic symmetry algebra was shown to consist of a Virasoro-Kač-Moody algebra [36–41]. This is the algebra of a specific class of field theories, known as warped conformal field theory (WCFT) [42].

A WCFT is a two-dimensional quantum field theory invariant under "warped conformal transformations". Parameterizing the theory with coordinates $t_\pm$, these are given by

$$t_+ \to f(t_+)\,, \qquad t_- \to t_- + g(t_+)\,, \tag{2}$$

where $f(t_+)$ and $g(t_+)$ are two arbitrary functions. The periodicity of $t_\pm$ depends on the ensemble[1] and the specific holographic model (see [43]). The symmetries (2) can be shown to arise assuming translation invariance and chiral scaling [44], analogously to the emergence of local conformal symmetry in unitary Poincaré-invariant two-dimensional QFTs with a global scaling symmetry and a discrete non-negative spectrum of scaling dimensions [45]. Reparametrizations of t_+ and coordinate-dependent translations of t_- are generated by a stress tensor $T(t_+)$ and a current density $P(t_+)$ respectively, whose modes span a Virasoro-Kač-Moody algebra with a global $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$ subalgebra.

The holographic duality relating WAdS$_3$ spaces and WCFTs has passed several tests, including the matching of Bekenstein–Hawking entropy [42], of greybody factors from correlation functions [46], and of one-loop determinants in the bulk from characters [43, 47]. The computation of WCFT entanglement entropy and its relation to a holographic equivalent were studied in [48–53]. Unlike AdS/CFT where the usual Ryu-Takayanagi prescription [54] is sufficient to reproduce the CFT entanglement entropy, the latter prescription needs to be enhanced for WCFTs and different holographic descriptions were proposed over the years [48, 50, 51].

Explicit examples of WCFTs are given in [55, 56] for bosonic WCFTs, in [57, 58] for fermionic ones, in [59, 60] with a link to Sachdev-Ye-Kitaev (SYK) models or

[1] A WCFT can be described in two different ensembles, called the canonical ensemble and the quadratic ensemble. Proper definitions will be given in Section 2.1. The nomenclature *ensemble* has nothing to do with ensembles in statistical physics.

in [61] for supersymmetric WCFT models. In this thesis, we will primarily focus on two holographic scenarios featuring WCFTs.

In [55], a new set of boundary conditions, different from the well-known Brown-Henneaux boundary conditions [10], were proposed for AdS_3 leading to a Virasoro-Kač-Moody asymptotic charge algebra. This holographic dual has the advantage to be part of Einstein gravity and contains empty AdS_3 and BTZ black holes [30]. Also notice that in [62], the most general set of boundary conditions for AdS_3 was introduced, showing that a variety of symmetry algebras differing from the conformal one could appear just in pure AdS_3 gravity by an appropriate choice of boundary conditions. Those new boundary conditions were able to encompass all previous boundary conditions [10, 63–65], including the Compère-Song-Strominger (CSS) boundary conditions [55].

The second holographic model is the so-called Lower Spin Gravity, a $SL(2,\mathbb{R}) \times U(1)$ Chern-Simons action. It was constructed in [57] by coupling WCFTs to a background geometry. The latter turns out not to be Riemannian but a Newton-Cartan structure in addition to a scaling symmetry. The metric interpretation of the Chern-Simons connections was able to match the different $WAdS_3$ spacetimes. The authors of [66] provided asymptotic symmetries for the model at finite temperature and showed that they reproduce the algebra of a WCFT. They also demonstrated that WAdS black holes are contained within this model.

Structure of the thesis

The thesis will be structured as follows. The first chapter will properly introduce the different $WAdS_3$ spacetimes and the black holes that can be constructed using quotients. We will see the two different descriptions of a warped black hole, in the canonical ensemble and the quadratic ensemble. For each ensemble, we will present the asymptotic boundary conditions and their related asymptotic symmetry charge algebra. Additionally, we will discuss the corresponding Bekenstein-Hawking entropy and the coordinate transformations allowing one to transition from one ensemble to the other.

The second chapter will provide the outlines of WCFTs. We will start from (2) and determine the symmetry algebra that matches the asymptotic charge algebra of the warped black hole in the canonical ensemble. Even if the theory is not Lorentz invariant, the presence of modular covariance of the partition function, up to an anomaly, will enable us to derive the asymptotic density of states and a warped Cardy formula. This entropy agrees with the black hole entropy of the warped black hole in the canonical ensemble. A non-local charge redefinition can transform the symmetry algebra of the canonical ensemble to the symmetry algebra of the quadratic ensemble, matching the asymptotic charge algebra of the warped black hole in the corresponding ensemble. This other ensemble has the particularity to have a charge dependant Kač-Moody level and to be more suitable for deriving thermodynamic quantities since, as for AdS/CFT, a relation between Rindler and inertial observers can be exhibited [42, 67]. Once again, the derived entropy will match the Bekenstein-Hawking entropy of the warped black hole in the corresponding ensemble. We will also compute the warped Virasoro characters [43], a generating function that counts the (weighted)[2] number of states at each level in a given representation of the algebra. In Chapter 4, we will find this particularly valuable when we evaluate the partition function of a specific theory dual to a WCFT, as the partition function can be expressed as a sum of characters. Following this, we will review the construction of the Lower Spin Gravity as described in [57, 66], which will also be use in Chapter 4.

The third chapter will present our first project, which consists in deriving Quantum Energy Conditions (QEC) for WCFTs. The stress-tensor $T_{\mu\nu}$ of a Quantum Field Theory (QFT) has an important impact on the geometry through Einstein's equations. Proofs of several gravity theorems, such as the black hole area law [68] and singularity theorems [69] depend on the positivity of an energy density called Null Energy Condition (NEC). However some quantum phenomena like the Casimir effect or Hawking radiation violate the NEC. The Quantum Null Energy Condition (QNEC) [70] is a local extension of NEC conjectured to account for quantum effects. The latter statement has been proven for CFTs [71] and our goal is to extend this to a simple holographic model involving a WCFT, AdS$_3$ with CSS boundary conditions [55]. We succeeded to present saturated QECs for WCFTS, and we will explain why we did not manage to prove unsaturated conditions.

The fourth chapter is dedicated to our second project. In the context of asymptotically AdS spacetimes [72], the Hamiltonian reduction of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons action on the boundaries of the manifold, carrying potential holonomies, was matched to the *geometric action*, an action constructed on the coadjoint orbits of the Virasoro group. The origin of this connection arises from the transformation laws of the expectation value of the dual stress-tensor under a conformal transformation. The latter transforms in the coadjoint representation of the Virasoro group. In [73], a similar correspondence was observed for asymptotically flat spacetimes. In this chapter, we successfully match the geometric action that can be constructed on coadjoint orbits of the Warped Virasoro group with the reduced Lower Spin Gravity action on the boundary of our manifold. Additionally, we establish a correspondence between the orbit representatives and the holonomies of our connections. As a preliminary step for this chapter, we begin by reviewing a similar problem in the context of AdS$_3$/CFT$_2$ to familiarize ourselves with different methods and expected results [72, 74]. We review the computation of the one-loop partition function [75] and compare it with the Virasoro characters [76]. We finally apply the same strategy to Lower Spin gravity, showing that the one-loop partition function allows to recover one of the two Warped Virasoro characters from [43], reviewed in Chapter 2. We conclude by speculating on how the second character could be obtained.

The fifth chapter will discuss quasinormal modes (QNMs) in a WAdS$_3$ black

[2] Due to the non-unitarity of the theory and the presence of states with negative norm.

hole background and their connection through the eikonal limit to the photon ring of the spacetime. The photon ring consists in the unstable bounded photon orbits of a black hole, and QNMs are considered as its vibration modes. They appeared in various contexts over the years, such as black hole perturbations [77], scattering of gravitational waves [78], or the gravitational collapse of a star into a black hole [79–81]. The computation of QNMs in $WAdS_3$ black hole spacetime was previously conducted in [82]. We have improved this computation by connecting the modes to the photon ring of the black hole. QNMs are solutions to the wave equation in a specified black hole background with dissipative boundary conditions and can be thought of, in the eikonal limit, as scalar perturbations of the black hole leaking out of its photon ring. In asymptotically flat spacetimes, the different boundary conditions at spatial infinity (finite flux, ougoing waves and Dirichlet conditions) are equivalent [83], but we will demonstrate that the situation is different in $WAdS_3$ black hole spacetimes. To select the most natural boundary conditions, we require to recover the modes originating from the outer photon ring. For this purpose, we conduct the same analysis near the outer photon ring using three different methods: the geometric optics approximation, the Penrose limit, and the near-ring region limit [84–98]. The boundary conditions that we impose at infinity differ from [82], leading to different results. Additionally, we compute the observable conformal symmetry of the photon ring as viewed by a distant observer, similar to the analysis done for Schwarzschild and Kerr black holes [83], and the self-dual warped black hole [98]. Finally, we take the near-horizon limit and the BTZ limit of our quasinormal modes to compare them with the modes obtained in the self-dual WAdS case [98] and the BTZ black hole respectively [99, 100].

The last chapter is devoted to summarising our results and to discussing the perspectives.

Chapter 1

Warped AdS Spacetime

As the name suggests, Warped Anti-de Sitter spacetimes (WAdS) are deformations of AdS spacetimes. A notable distinction is that WAdS spacetimes do not solve Einstein's equations but instead arise as solutions in other theories, such as the three-dimensional *Topologically Massive Gravity* (TMG) with a negative cosmological constant [27, 28]. This theory consists of the Einstein-Hilbert action plus a Chern-Simons term where the connection is the Levi-Civita connection:

$$S_{\text{TMG}} = \frac{1}{16\pi G}\int \mathrm{d}x^3\sqrt{-g}(R-2\Lambda)+\frac{1}{32\pi G\mu}\int \mathrm{d}x^3\sqrt{-g}\varepsilon^{\lambda\mu\nu}(\Gamma^{\rho}_{\lambda\sigma}\partial_\mu\Gamma^{\sigma}_{\rho\nu}+\frac{2}{3}\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho})\,, \tag{1.1}$$

where ε is the covariant Levi-Civita symbol $\sqrt{-g}\,\varepsilon^{012} = 1$, $\Lambda = -1/\ell^2$ the cosmological constant and μ the Chern-Simons coupling constant. The latter can be interpreted as the graviton mass because, as the name TMG indicates, the graviton is allowed to have a mass in this theory. The equations of motion of TMG are

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda\,g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0\,, \tag{1.2}$$

where $C_{\mu\nu}$ is the Cotton tensor

$$C_{\mu\nu} = \varepsilon_{\mu}{}^{\alpha\beta}\nabla_\alpha(R_{\beta\nu} - \frac{1}{4}R\,g_{\beta\nu})\,. \tag{1.3}$$

Every solution of Einstein's equations is also a solution of TMG's equations of motion. As a consequence, we can also work with AdS within the framework of this theory. However, in this scenario, the graviton carries negative energy, leading to the anticipation that AdS vacua are typically unstable [33]. Warped spacetimes can be constructed from AdS by deforming it in the following way:

$$\mathrm{d}s^2_{\text{WAdS}} = \mathrm{d}s^2_{\text{AdS}} - H^2\xi\otimes\xi\,, \tag{1.4}$$

where ξ is an AdS Killing vector of unit norm. Depending on the nature of the Killing vector, we can have a variety of WAdS spacetimes that we will describe shortly: timelike, spacelike, and null. AdS spacetimes possess an $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ global symmetry. By choosing a Killing vector belonging to one of these $SL(2,\mathbb{R})$ groups, this symmetry is broken to a $U(1)$, resulting in isometry group of WAdS being $SL(2,\mathbb{R})\times U(1)$.

Timelike WAdS

The timelike warped AdS spacetime, sometimes also called elliptic warped AdS because of the choice an elliptic element ξ in $SL(2,\mathbb{R})$, is given by

$$\mathrm{d}s^2 = \frac{\ell}{\nu^2+3}\left(\cosh^2\sigma\,\mathrm{d}u^2 + \mathrm{d}\sigma^2 - \frac{4\nu^2}{\nu^2+3}(\mathrm{d}\tau + \sinh\sigma\,\mathrm{d}u)^2\right). \tag{1.5}$$

with $\{u,\tau,\sigma\}\in\,]-\infty,+\infty[$. We have introduced a new parameter ν that is related to the Chern-Simons coupling constant μ as

$$\nu = \frac{\mu\ell}{3}. \tag{1.6}$$

When this parameter is set to one, we recover AdS_3 written as a kind of Hopf fibration over Euclidean AdS_2, where the real line plays the role of the fiber. Depending on the range of ν, the timelike warped AdS spacetimes split into two categories. If $\nu^2>1$, the coefficient $4\nu^2/(\nu^2+3)$ is greater than one, and the spacetime is called stretched. It was shown in [13,17] that these spacetimes carry closed timelike curves and are identified with the Godel spacetime. If $\nu^2<1$, the spacetime is called squashed and can be rewritten in global coordinates.

$$\frac{\mathrm{d}s^2}{\ell^2} = -\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r\left((\nu^2+3)r+4\right)} + 2\nu r\,\mathrm{d}t\,\mathrm{d}\theta + \frac{r}{4}\left(3(1-\nu^2)r+4\right)\mathrm{d}\theta^2. \tag{1.7}$$

where $\theta\in[0,2\pi[$ is an angle and the origin lays at $r=0$. The stretched case could also be written in global coordinates but would have global singularities [33].

Spacelike WAdS

For reasons that will become apparent later on, the spacelike warped solution is the one that interests us the most. It is given by the metric

$$\mathrm{d}s^2 = \frac{\ell}{\nu^2+3}\left(-\cosh^2\sigma\,\mathrm{d}\tau^2 + \mathrm{d}\sigma^2 + \frac{4\nu^2}{\nu^2+3}(\mathrm{d}u + \sinh\sigma\,\mathrm{d}\tau)^2\right), \tag{1.8}$$

and can also be called hyperbolic WAdS. Again, if $\nu^2>1$, the metric is called stretched, and if $\nu^2<1$, it is squashed. When $\nu=1$, we recover AdS_3 written once more as a Hopf fibration but this time over Lorentzian AdS_2.

Null WAdS

The null warped AdS spacetimes are solutions of TMG only for $\nu^2 = 1$. They can be obtained as a kind of Penrose limit with $\nu \to 1$ of the timelike or spacelike warped solution with $\nu \neq 1$ and are given by the metrics [33]

$$\frac{\mathrm{d}s^2}{\ell^2} = \frac{\mathrm{d}u^2 + \mathrm{d}x^+\mathrm{d}x^-}{u^2} \pm \frac{\mathrm{d}x^{-\,2}}{u^4}\,. \tag{1.9}$$

Usual AdS is recovered by dropping the last term.

1.1 Black Holes as quotient space

Aside from the different warped vacua, regular black hole solutions were found for $\nu > 1$ [16, 34, 35] and were shown in [33] to be discrete quotients by an element of $SL(2,\mathbb{R}) \times U(1)$ of WAdS_3. The same approach arises in AdS_3, where BTZ black holes are quotients of AdS by a discrete subgroup of the isometry group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ [30–32]. In this section, we review the quotients done for AdS. We can identify points $\mathcal{P}$ in AdS under the action of a Killing vector ξ

$$\mathcal{P} \sim e^{2\pi k \xi}\mathcal{P}\,, \tag{1.10}$$

where k is a positive integer. The vector ξ defines a one-parameter subgroup of the isometries of AdS. We then identify points belonging in the same orbits under the action of this subgroup to form the so-called quotient space. This identification preserves the local properties of the original spacetime. Therefore, the quotient space is locally AdS. In order to avoid closed timelike curves, the Killing vector must be spacelike, which is a necessary condition but not sufficient in general [31]. For AdS, this construction yields the BTZ black hole [30]

$$\mathrm{d}s^2_{\mathrm{BTZ}} = -\left(\frac{r^2}{\ell^2} - MG + \frac{J^2G^2}{4r^2}\right)\mathrm{d}t^2 + \left(\frac{r^2}{\ell^2} - MG + \frac{J^2G^2}{4r^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta - \frac{JG}{2r^2}\mathrm{d}t\right)^2\,. \tag{1.11}$$

The spacetime possesses two Killing horizons, that we will refer as the inner and outer horizons, at radii $r_\pm$. The mass and angular momentum appearing in (1.11) can be expressed in terms of these radii as

$$M = \frac{r_+^2 + r_-^2}{G\ell^2} \quad , \quad J = \frac{2}{G\ell}r_+r_-\,. \tag{1.12}$$

We can perform a similar identification (1.10) starting from a WAdS spacetime. Among the various types of WAdS spacetimes, only the spacelike WAdS spacetime with $\nu \geq 1$ leads to a quotient space that is not pathological. Throughout this thesis, this quotient space will be referred to as a warped black hole. This black hole has two different descriptions that will be reviewed in the following sections.

1.2 Warped Black Holes in the canonical ensemble

The first description will be denoted as Warped black hole in the canonical ensemble. The metric in this ensemble is given by

$$ds^2_{\text{CE}} = -N_{\text{CE}}(r)^2 \mathrm{d}t^2 + \frac{\ell^2}{4R_{\text{CE}}(r)^2 N_{\text{CE}}(r)^2}\mathrm{d}r^2 + R_{\text{CE}}(r)^2 \left(\mathrm{d}\theta - N^\theta_{\text{CE}}(r)\mathrm{d}t\right)^2 \tag{1.13}$$

where the functions appearing in the above metric are defined by

$$R_{\text{CE}}(r)^2 = \frac{r}{4}\left(3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+ r_-(\nu^2+3)}\right), \tag{1.14}$$

$$N_{\text{CE}}(r)^2 = \frac{1}{4R_{\text{CE}}(r)^2}(\nu^2+3)(r-r_+)(r-r_-), \tag{1.15}$$

$$N^\theta_{\text{CE}}(r) = \frac{1}{R_{\text{CE}}(r)^2}\left(\nu r - \frac{1}{2}\sqrt{r_+ r_-(\nu^2+3)}\right), \tag{1.16}$$

and the radii $r_\pm$ here denote the locations of the inner and outer Killing horizons of the black holes. They should not be confused with those appearing in the BTZ black holes and other metrics. If it is not specified, it will typically be clear which metric we are referring to. When $\nu = 1$, we recover the BTZ black hole in a rotating frame.

When r is taken very large in (1.13), the metric becomes

$$ds^2 = \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{(\nu^2+3)r^2} - 2\nu r\,\mathrm{d}t\,\mathrm{d}\theta + \frac{3(\nu^2-1)}{4}r^2\,\mathrm{d}\theta^2 . \tag{1.17}$$

This metric covers a patch of (1.8) when $\nu^2 > 1$, but with the coordinate θ that is no longer an angle. In this sense, it is convenient to say that the warped black hole (1.13) is asymptotically spacelike warped AdS, even though strictly speaking, it involves an unwrapping of a coordinate.

In [39, 41], a consistent set of boundary conditions was proposed, encompassing the warped black hole (1.13)[1]:

$$\begin{array}{lll} g_{tt} = 1 + O(r^{-1}) & g_{tr} = O(r^{-2}) & g_{t\theta} = -\nu r + O(r^0) \\ g_{rr} = \frac{1}{(\nu^2+3)r^2} + O(r^{-3}) & g_{r\theta} = O(r^{-1}) & g_{\theta\theta} = \frac{3(\nu^2-1)}{4}r^2 + O(r) \end{array} \tag{1.18}$$

Those boundary conditions are preserved under an asymptotic symmetry algebra generated by asymptotic Killing vectors

$$l_n = e^{in\theta}\left(\partial_\theta - inr\partial_r\right), \qquad p_n = e^{in\theta}\partial_t . \tag{1.19}$$

[1] It is important to precise that contrary to the AdS case, the vacuum (1.8) is not contained in these boundary conditions [29].

Each of these vectors is associated with a conserved charge denoted as L_n and P_n with $n \in \mathbb{Z}$. The asymptotic conserved charges satisfy to the algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m}\,, \\
[L_n, P_m] &= -mP_{m+n}\,, \\
[P_n, P_m] &= \frac{k}{2}n\delta_{n+m}\,.
\end{aligned} \tag{1.20}$$

This algebra precisely matches that of a Warped CFT, as we will demonstrate in (2.31). The central charges can be expressed in terms of the TMG parameters.

$$c = \frac{(5\nu^2+3)}{\nu(\nu^2+3)}\frac{\ell}{G}\,, \qquad k = -\frac{\nu^2+3}{6\nu}\frac{1}{\ell G}\,. \tag{1.21}$$

As for the AdS/CFT correspondence, this suggests a link between WAdS spacetimes and WCFT.

The laws of black holes thermodynamics can be verified for warped black holes. One can compute a mass M^{CE} and an angular momentum J^{CE} associated respectively with the Killing vectors ∂_t and ∂_θ [35]:

$$\begin{aligned}
M^{\text{CE}} &= \frac{\nu^2+3}{24\nu G\ell}\left((r_+ + r_-)\nu - \sqrt{r_+r_-(\nu^2+3)}\right) \\
J^{\text{CE}} &= \frac{\nu^2+3}{96\nu G\ell}\left[\left((r_+ + r_-)\nu - \sqrt{r_+r_-(\nu^2+3)}\right)^2 - \frac{5\nu^2+3}{4}(r_+ - r_-)^2\right]\,.
\end{aligned} \tag{1.22}$$

Those conserved charges correspond to the zero modes P_0 and $-L_0$ respectively. One can also derive a Wald entropy for the black hole (1.13) [33, 101–103]

$$S^{\text{CE}} = \frac{\pi}{24\nu G}\left((9\nu^2+3)r_+ - (\nu^2+3)r_- - 4\nu\sqrt{r_+r_-(\nu^2+3)}\right)\,. \tag{1.23}$$

All these quantities satisfy the first law of black hole thermodynamics

$$\mathrm{d}M^{\text{CE}} = T^{\text{CE}}\,\mathrm{d}S^{\text{CE}} + \Omega^{\text{CE}}\,\mathrm{d}J^{\text{CE}}\,, \tag{1.24}$$

where T^{CE} is the Hawking temperature and Ω^{CE} the angular velocity of the outer horizon r_+. Both are given by [35]

$$\begin{aligned}
T^{\text{CE}} &= \frac{\nu^2+3}{4\pi\ell}\frac{r_+ - r_-}{2\nu r_+ - \sqrt{r_+r_-(\nu^2+3)}}\,, \\
\Omega^{\text{CE}} &= \frac{2}{2\nu r_+ - \sqrt{r_+r_-(\nu^2+3)}}\,.
\end{aligned} \tag{1.25}$$

It concludes this introductory discussion about warped black holes in the canonical ensemble. However, it is not the only way to describe warped black holes. There exists another approach that leads to a different asymptotic algebra and different charges, which also represents a solution to the TMG equations (1.2).

1.3 Warped Black Holes in quadratic ensemble

As mentioned earlier, we will now describe another family of black hole spacetimes in this section. These black holes have the advantage of precisely reducing to BTZ when a certain limit is taken. The metric for a warped black hole in the quadratic ensemble is given by [104]

$$\mathrm{d}s^2_{\mathrm{QE}} = -N_{\mathrm{QE}}(\mathrm{r})^2\mathrm{dt}^2 + \frac{1-2H^2}{R_{\mathrm{QE}}(\mathrm{r})^2 N_{\mathrm{QE}}(\mathrm{r})^2}\mathrm{r}^2\mathrm{dr}^2 + R_{\mathrm{QE}}(\mathrm{r})^2\left(\mathrm{d}\varphi + N^{\varphi}_{\mathrm{QE}}(\mathrm{r})\mathrm{dt}\right)^2 \tag{1.26}$$

where

$$R_{\mathrm{QE}}(\mathrm{r})^2 = \mathrm{r}^2 - 2H^2\left(\frac{\mathrm{r}^2+\mathrm{r}_+\mathrm{r}_-}{\mathrm{r}_++\mathrm{r}_-}\right)^2 ; \tag{1.27}$$

$$N_{\mathrm{QE}}(\mathrm{r})^2 = \frac{(1-2H^2)}{R_{\mathrm{QE}}(\mathrm{r})^2L^2}(\mathrm{r}^2-\mathrm{r}_+^2)(\mathrm{r}^2-\mathrm{r}_-^2)\,; \tag{1.28}$$

$$N^{\Theta}(\mathrm{r}) = \frac{1}{R_{\mathrm{QE}}(\mathrm{r})^2L}\left((1-2H^2)\mathrm{r} + 2H^2\frac{(\mathrm{r}^2-\mathrm{r}_+^2)(\mathrm{r}^2-\mathrm{r}_-^2)}{(\mathrm{r}_++\mathrm{r}_-)}\right). \tag{1.29}$$

The constants $\mathrm{r}_\pm$ mark the location of the horizons and H^2 and L are related to the TMG parameters ν and ℓ as follows

$$H^2 = -\frac{3(\nu^2-1)}{2(\nu^2+3)}\,, \qquad L = \frac{2\ell}{\sqrt{\nu^2+3}}\,. \tag{1.30}$$

The sign of H^2 may seem intriguing, but it is necessary to avoid closed timelike curves.

This metric is a deformation of the BTZ black hole with AdS radius L because it can be written as

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\mathrm{BTZ}} - H^2\xi\otimes\xi \tag{1.31}$$

where ξ is a spacelike Killing vector given by

$$\xi = \frac{1}{\mathrm{r}_+-\mathrm{r}_-}(-L\partial_{\mathrm{t}} + \partial_\phi)\,. \tag{1.32}$$

H^2 denotes the deviation away from BTZ. When $H^2 = 0$, equivalent to $\nu = 1$, we recover a BTZ black hole with AdS radius $L = \ell$ in the usual coordinates (1.11).

This other family of black holes does not belong to the phase space generated by the asymptotic boundary conditions (1.18). In [105], a new set of boundary conditions was proposed that includes the warped black hole (1.26). As usual, these asymptotic boundary conditions are invariant under an asymptotic symmetry algebra generated by asymptotic Killing vectors. The latter are associated with charges satisfying the algebra

$$\begin{aligned}\left[\tilde{L}_n, \tilde{L}_m\right] &= (n-m)\tilde{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m}\,,\\ \left[\tilde{L}_n, \tilde{P}_m\right] &= -m\tilde{P}_{m+n} + m\tilde{P}_0\delta_{n+m}\,,\\ \left[\tilde{P}_n, \tilde{P}_m\right] &= -2n\tilde{P}_0\delta_{n+m}\,,\end{aligned} \tag{1.33}$$

where the central c can be expressed in terms of H and L

$$c = \frac{2(1-H^2)L}{\sqrt{1-H^2}G}\,. \tag{1.34}$$

This algebra differs from (1.20) primarily because the U(1) central charge is now charge dependent. The generators of (1.33) are related to those of (1.20) through

$$\begin{aligned} \tilde{L}_n &= L_n - \frac{2}{k}P_0P_n + \frac{1}{k}P_0^2\delta_n\,, \\ \tilde{P}_n &= \frac{2}{k}P_0P_n - \frac{1}{k}P_0^2\delta_n\,. \end{aligned} \tag{1.35}$$

The name *quadratic* ensemble comes from the fact that this transformation is quadratic.

In the same way that the charges are different from the canonical ensemble, the mass and angular momentum of the black hole also differ. They are now given by [104]

$$\begin{aligned} M^{\mathrm{QE}} &= \frac{(3-4H^2)(\mathrm{r}_+^2+\mathrm{r}_-^2) - 2\mathrm{r}_+\mathrm{r}_-}{24G^2L\sqrt{1-2H^2}}\,, \\ J^{\mathrm{QE}} &= \frac{\mathrm{r}_+^2+\mathrm{r}_-^2 - 2(3-4H^2)\mathrm{r}_+\mathrm{r}_-}{24GL\sqrt{1-2H^2}}\,, \end{aligned} \tag{1.36}$$

and they are related to the zero modes by

$$M^{\mathrm{QE}} = \frac{1}{L}(\tilde{L}_0 + \tilde{P}_0)\,, \qquad J^{\mathrm{QE}} = \tilde{L}_0 - \tilde{P}_0\,. \tag{1.37}$$

The Wald entropy is given by

$$S^{\mathrm{QE}} = \frac{\pi}{6G\sqrt{1-2H^2}}\Big((3-4H^2)\mathrm{r}_+ - \mathrm{r}_-\Big)\,. \tag{1.38}$$

In this ensemble, the first law of black hole thermodynamics can also be verified

$$\mathrm{d}M^{\mathrm{QE}} = T^{\mathrm{QE}}\,\mathrm{d}S^{\mathrm{QE}} + \Omega^{\mathrm{QE}}\,\mathrm{d}J^{\mathrm{QE}}\,, \tag{1.39}$$

with the Hawking temperature and the black hole angular velocity

$$T^{\mathrm{QE}} = \frac{\mathrm{r}_+^2 - \mathrm{r}_-^2}{2\pi L\mathrm{r}_+}\,, \qquad \Omega^{\mathrm{QE}} = -\frac{\mathrm{r}_-}{\mathrm{r}_+}\,. \tag{1.40}$$

As we will see in the next chapter, the quadratic ensemble has the advantage of being more suitable for analyzing thermal properties from a field theory point of view. The structure, being more comparable to a CFT, makes analogies and interpretations easier. In [105], they matched the entropy of the warped black hole in the quadratic ensemble to the Warped Cardy formula (2.83) using an appropriate vacuum.

As a final word of this chapter; we mention that the two ensembles are related through the following change of coordinates:

$$\begin{aligned} \mathrm{t} &= L\frac{t}{\mathbf{A}}\,, \\ \phi &= -\theta - \frac{t}{\mathbf{A}} \\ \mathrm{r}^2 &= \frac{\nu^2+3}{4\nu^2}\left(\mathbf{A}\nu r - \frac{3}{4}r_+r_-(\nu^2-1)\right)\,, \end{aligned} \tag{1.41}$$

where

$$\mathbf{A} = (r_+ + r_-)\nu - \sqrt{r_+r_-(\nu^2+3)} \propto M^{\text{CE}}\,. \tag{1.42}$$

Since the time is rescaled by mass, the transformation is state-dependent, similar to the relation between the generators of the two ensembles (1.35). It is interesting to note that using the transformation (1.41), the entropy of the two ensembles matches

$$S^{\text{CE}} = S^{\text{QE}}\,, \tag{1.43}$$

which was expected since the Wald entropy is invariant under diffeomorphism.

Chapter 2

Warped Conformal Field Theory

In this chapter, we will cover the fundamentals of the Warped Conformal Field Theory (WCFT), focusing specifically on its two-dimensional formulation, starting from its definition to the construction of a holographic dual through its coupling to geometric background.

2.1 Definition of WCFT

To properly define a Warped Conformal Field Theory (WCFT), we first need to introduce a broader class of field theories in two dimensions known as 2d Generalized Conformal Field Theory ($GCFT_2$) [1], which includes Conformal Field Theory (CFT) as a special case [57]. A $GCFT_2$ is a unitary local Quantum Field Theory characterized by at least three global symmetries. These symmetries consist of translations in both coordinates x and t, and rescaling for one of them, which we will choose as x without loss of generality. It's important to note that these coordinates are not necessarily associated with spatial and time dimensions. The conserved charges H, $\bar{H}$, and D generate the associated symmetries as

$$H : x \to x + \delta x\,, \quad \bar{H} : t \to t + \delta t\,, \quad D : x \to \lambda x\,. \tag{2.1}$$

As a consequence of locality, there are conserved currents J^μ, $\bar{J}^\mu$, and J^μ_D associated with the conserved charges H, $\bar{H}$, and D, respectively.

It is important to note that generic $GCFT_2$ theories are not generally Lorentz invariant. However, some theories achieve Lorentz invariance by incorporating additional symmetries. For example, CFT_2 includes an additional scaling symmetry, where $t \to \bar{\lambda} t$. In this case, the coordinates (x, t) represent light-cone variables.

The commutators for the three charges of a $GCFT_2$ are

$$i\,[D, H] = H\,, \quad i\left[H, \bar{H}\right] = 0\,, \quad i\left[D, \bar{H}\right] = 0\,. \tag{2.2}$$

[1] This should not be confused with Galilean CFT, which is also abbreviated as GCFT.

As in [45], we assume that the eigenvalues λ_i of D are discrete and non-negative, and there exists a complete basis of local operators Φ_i, which have no explicit dependence on the coordinates, with weight λ_i. This basis satisfies

$$i\,[H,\Phi_i] = \partial_x\Phi_i\,,\quad i\left[\bar{H},\Phi_i\right] = \partial_t\Phi_i\,,\quad i\,[D,\Phi_i] = x\partial_x\Phi_i + \lambda_i\Phi_i\,. \tag{2.3}$$

Translation and scaling invariances imply that the vacuum two-point function satisfies

$$\langle\Phi_i(x,t),\Phi_j(x',t')\rangle = \frac{f_{ij}(t-t')}{(x-x')^{\lambda_i+\lambda_j}}\,, \tag{2.4}$$

for some unknown functions f_{ij}.

It was shown in [44] that (J^x, J^t) and $(\bar{J}^x, \bar{J}^t)$ are local operators of weight $(1,2)$ and $(0,1)$ respectively. The currents have certain degrees of freedom. The commutation relations are still satisfied under shifts of the form $\pm\partial_\mu O(x,t)$, where O can potentially have explicit dependence on the coordinates x and t [44]

$$O(x,t) = \sum_i f_i(x,t)\Phi_i(x,t)\,. \tag{2.5}$$

It was shown in Appendix A of [44], using this shift ambiguity of currents, that for any charge Q commuting with H and $\bar{H}$, the corresponding conserved current J^μ_Q satisfies the following commutators:

$$i\left[H,J^\mu_Q\right] = \partial_x J^\mu_Q\,,\qquad i\left[\bar{H},J^\mu_Q\right] = \partial_t J^\mu_Q\,. \tag{2.6}$$

However, for the current associated to D, since D and H do not commute with each other, the commutator of H and J^μ_D requires an additional term proportional to J^μ:

$$i\,[H,J^\mu_D] = \partial_x J^\mu_D - J^\mu\,. \tag{2.7}$$

This implies that J^μ_D is not a local operator and must explicitly depend on x. Nevertheless, we can rewrite it as:

$$J^\mu_D = xJ^\mu + S^\mu_D\,, \tag{2.8}$$

where (S^x_D, S^t_D) are local operators of weight $(0,1)$.

The conservation of the currents J^μ and J^μ_D leads to the equation

$$J^x + \partial_x S^x_D + \partial_t S^t_D = 0\,. \tag{2.9}$$

Since S^x_D has weight 0, its vacuum two-point function with itself depends only on t, implying that $\partial_x S^x_D = 0$.

We will use the shift ambiguity (2.5) to redefine the current J^μ

$$J^x \to J^x + \partial_t S^t_D\,,\quad J^t \to J^t - \partial_x S^t_D\,, \tag{2.10}$$

such that

$$J^x = 0\,. \tag{2.11}$$

Then, according to the conservation of the current J^μ, J^t can only depend on x

$$J^t \equiv \mathcal{L}(x)\,. \tag{2.12}$$

It leads to the existence of an infinite family of charges

$$\mathcal{L}_\xi = \int \mathrm{d}x\, \xi(x)\mathcal{L}(x)\,, \tag{2.13}$$

where $\xi(x)$ is a smooth function. The charges $\mathcal{L}_\xi$ form a Virasoro algebra [44].

As $\bar{J}^x$ is also a local operator of weight 0, $\partial_x \bar{J}^x = 0$, and due to the conservation equation, $\partial_t \bar{J}^t = 0$. This implies that $\bar{J}^x$ depends only on t and $\bar{J}^t$ depends only on x. We can then define

$$\bar{J}^x = \bar{\mathcal{L}}(t)\,, \quad \bar{J}^t = \mathcal{P}(x)\,. \tag{2.14}$$

As before, we can construct infinite families of conserved charges

$$\bar{\mathcal{L}}_{\bar{\xi}} = \int \mathrm{d}t\, \bar{\xi}(t)\bar{\mathcal{L}}(t)\,, \quad \mathcal{P}_\xi = \int \mathrm{d}x\, \xi(x)\mathcal{P}(x)\,. \tag{2.15}$$

which will form another Virasoro algebra and a $U(1)$ Kač-Moody algebra for $\bar{\mathcal{L}}_{\bar{\xi}}$ and $\mathcal{P}_\xi$, respectively.

A CFT_2 is a particular case of this construction where the addition of Lorentz invariance implies $\mathcal{P}(x) = 0$, and the theory then has just two Virasoro algebras.

Another particular case of interest is when $\bar{\mathcal{L}}(t) = 0$. Such a theory is called a WCFT_2, resulting in the addition of another symmetry generated by the conserved charge $\bar{B}$:

$$\bar{B} : t \to t + vx\,, \tag{2.16}$$

where v is a real constant. This symmetry is called a generalized boost symmetry [57]. It implies that a WCFT is not Lorentz invariant since the coordinates x and t do not enjoy the same symmetries. The commutation relations with the previous charges are

$$i\left[H, \bar{B}\right] = -\bar{H}\,, \quad i\left[D, \bar{B}\right] = -\bar{B}\,, \quad i\left[\bar{H}, \bar{B}\right] = 0\,. \tag{2.17}$$

Like before, the non-vanishing commutators of $\bar{B}$ with H and D imply that the conserved current $J^\mu_{\bar{B}}$ associated with $\bar{B}$ is not a local operator, but it can still be rewritten as

$$J^\mu_{\bar{B}} = x\bar{J}^\mu + S^\mu_{\bar{B}} \tag{2.18}$$

where $S^\mu_{\bar{B}}$ is local operator. The conservation equations of $J^\mu_{\bar{B}}$ and $\bar{J}^\mu$ induce that

$$\bar{J}^x + \partial_x S^x_{\bar{B}} + \partial_t S^t_{\bar{B}} = 0\,. \tag{2.19}$$

It was argued in [44, 57] that unitarity and shift ambiguity lead to $S^\mu_{\bar{B}} = 0$. The result is then

$$\bar{J}^x = 0\,. \tag{2.20}$$

This constraint kills one of the Virasoro algebras. The remaining currents satisfy the algebra:

$$\begin{aligned} i\,[\mathcal{L}(x),\mathcal{L}(y)] &= \mathcal{L}'(y)\,\delta(x-y) - 2\,\mathcal{L}(y)\,\partial_x\delta(x-y) + \frac{c}{24\pi}\partial_x^3\delta(x-y)\,, \\ i\,[\mathcal{L}(x),\mathcal{P}(y)] &= \mathcal{P}'(y)\,\delta(x-y) - \mathcal{P}(y)\,\partial_x\delta(x-y)\,, \\ i\,[\mathcal{P}(x),\mathcal{P}(y)] &= -\frac{\kappa}{4\pi}\partial_x\delta(x-y)\,, \end{aligned} \tag{2.21}$$

while for the charges,

$$\begin{aligned} i\,[\mathcal{L}_\xi,\mathcal{L}_\chi] &= \mathcal{L}_{\xi'\chi-\xi\chi'} + \frac{c}{48\pi}\int \mathrm{d}x\,(\xi''\chi' - \xi'\chi'')\,, \\ i\,[\mathcal{L}_\xi,\mathcal{P}_\chi] &= \mathcal{P}_{-\xi\chi'}\,, \\ i\,[\mathcal{P}_\xi,\mathcal{P}_\chi] &= \frac{\kappa}{8\pi}\int \mathrm{d}x\,(\xi'\chi - \xi\chi')\,. \end{aligned} \tag{2.22}$$

This algebra is a semidirect product of a Virasoro algebra of central charge c and an affine $u(1)$ Kač-Moody current algebra with central extension κ.

2.1.1 Finite transformations

The finite transformations of a WCFT_2 can be written as

$$x \to f(x)\,, \qquad t \to t + g(x)\,. \tag{2.23}$$

The commutation relations (2.21) imply the following infinitesimal transformations of the current

$$\begin{aligned} \delta\mathcal{L} &= \epsilon\mathcal{L}' + 2\epsilon'\mathcal{L} + \sigma'\mathcal{P} + \frac{c}{12}\epsilon'''\,, \\ \delta\mathcal{P} &= \epsilon\mathcal{P}' + \epsilon'\mathcal{P} - \frac{k}{2}\sigma'\,, \end{aligned} \tag{2.24}$$

where we have defined

$$\delta = i[\mathcal{L}_\epsilon,\cdot] + i[\mathcal{P}_\sigma,\cdot]\,. \tag{2.25}$$

The functions ϵ and σ are the infinitesimal transformations of the coordinates

$$\delta x = \epsilon(x)\,, \qquad \delta t = \sigma(x)\,. \tag{2.26}$$

We can build the unique finite transformations that reduce to these infinitesimal versions and that compose appropriately [42]:

$$\begin{aligned} \mathcal{P}(x) \to& \frac{1}{f'}\left(\mathcal{P}(x) + \frac{k}{2}g'\right)\,, \\ \mathcal{L}(x) \to& \frac{1}{f'^2}\left(\mathcal{L}(x) - \frac{c}{12}\{f;x\} - g'\mathcal{P}(x) - \frac{k}{4}g'^2\right)\,, \end{aligned} \tag{2.27}$$

where we have introduced the Schwarzian derivative

$$\{f;x\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 . \tag{2.28}$$

For example, we can put our theory on a cylinder by considering the map

$$x = e^{i\phi} . \tag{2.29}$$

On this cylinder, the currents can be decomposed in modes by picking test functions

$$\xi_n = x^n = e^{in\phi} , \tag{2.30}$$

leading to the canonical WCFT algebra

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m} , \\ [L_n, P_m] &= -mP_{n+m} , \\ [P_n, P_m] &= \frac{k}{2}n\delta_{n+m} , \end{aligned} \tag{2.31}$$

with $L_n = i\mathcal{L}_{\xi_{n+1}}$ and $P_n = \mathcal{P}_{\chi_n}$. The zero modes are the angular momentum and the energy and generate the translations:

$$L_0 = Q[\partial_\phi] , \qquad P_0 = Q[\partial_t] . \tag{2.32}$$

We can then add an arbitrary *tilt* α

$$t \to t + 2\alpha\phi , \tag{2.33}$$

transforming the currents as

$$\begin{aligned} \mathcal{P}^\alpha(\phi) &= ix\mathcal{P}(x) - k\alpha , \\ \mathcal{L}^\alpha(\phi) &= -x^2\mathcal{L}(x) + \frac{c}{24} + 2i\alpha x\mathcal{P}(x) - k\alpha^2 . \end{aligned} \tag{2.34}$$

By defining modes on the cylinder as

$$P_n^\alpha = -\frac{1}{2\pi}\int \mathrm{d}\phi\, \mathcal{P}^\alpha(\phi)e^{in\phi} , \qquad L_n^\alpha = -\frac{1}{2\pi}\int \mathrm{d}\phi\, \mathcal{L}^\alpha(\phi)e^{in\phi} , \tag{2.35}$$

this corresponds, in terms of the original modes, to

$$\begin{aligned} P_n^\alpha &= P_n + k\alpha\, \delta_n , \\ L_n^\alpha &= L_n + 2\alpha P_n + \left(k\alpha^2 - \frac{c}{24}\right)\delta_n . \end{aligned} \tag{2.36}$$

We observe that the tilt, which is arbitrary, changes the zero modes on the cylinder. This situation is completely different from unitary CFT where the zero modes on the cylinder is uniquely determined from the zero modes on the plane and the central charges.

It was argued in [42] that for unitary representations of the WCFT algebra, namely

$$L_{-n} = L_n^\dagger\,, \qquad P_{-n} = P_n^\dagger\,, \tag{2.37}$$

the positivity of states $L_{-n}|p,h\rangle$, $P_{-n}|p,h\rangle$ and $P_0|p,h\rangle$ implies the restrictions

$$c > 0\,, \qquad k > 0\,, \qquad h \geq \frac{p^2}{k}\,, \qquad p \in \mathbb{R}\,, \tag{2.38}$$

where p and h are the eigenvalues of P_0 and L_0 of the primary states $|p,h\rangle$ defined as

$$\begin{aligned} P_n\,|p,h\rangle &= 0 \quad n>0\,, \qquad & L_n\,|p,h\rangle &= 0 \quad n>0\,, \\ P_0\,|p,h\rangle &= p\,|p,h\rangle\ , \qquad & L_0\,|p,h\rangle &= h\,|p,h\rangle\ . \end{aligned} \tag{2.39}$$

On the cylinder, it may happen that $P_0 \neq 0$ in the vacuum states. Again, they argued in [42] that

$$L_0 - \frac{P_0^2}{k} \geq -\frac{c}{24}\,, \qquad c > 1 \tag{2.40}$$

using the modes of the Sugawara-substracted stress tensor

$$L_n^{(s)} = L_n - \frac{1}{k}\sum_{m=-\infty}^{\infty} :P_m P_{n-m}: \quad , \tag{2.41}$$

where :: indicates normal ordering defined in the usual way [106]

$$:P_m P_{n-m}: = \sum_{m\leq -1} P_m P_{n-m} + \sum_{m>-1} P_{n-m} P_m\,. \tag{2.42}$$

This Sugawara basis has the convenient property to commute with P_n and to satisfy a Virasoro algebra of central charge $c-1$.

In the vacuum state, this bound is saturated and the charges can be parametrized as

$$P_0^v = q\,, \qquad L_0^v = \frac{q^2}{k} - \frac{c}{24}\,. \tag{2.43}$$

The last equation holds true even in non-unitary theories with imaginary eigenvalues of P_0, provided that the vacuum state can be associated with a unit operator.

However, unitary WCFTs do not correspond to holographic duals of gravity models. Generally, the latter possess a negative $U(1)$ level, leading to descendant states with negative norm. Despite the presence of these negative norm states, the modular bootstrap method can still be applied [43]. If the Virasoro-Kač-Moody primaries have positive norm, there exist at least two states with imaginary $U(1)$ charge p, one of them being the vacuum. In these holographic warped theories, the requirements

$$c > 1\,, \qquad h \geq \frac{p^2}{k}\,, \qquad \text{with } k < 0\,, \tag{2.44}$$

still apply. Depending on the real or imaginary nature of the charge of the primaries states, the $U(1)$ charges are either hermitian

$$P_n^\dagger = P_{-n}\,, \qquad \text{if}\, p \in \mathbb{R}\,, \tag{2.45}$$

or antihermitian

$$P_n^\dagger = -P_{-n}\,, \qquad \text{if}\, p \in i\mathbb{R}\,. \tag{2.46}$$

2.1.2 Asymptotic density of states and Warped Cardy formula

Similar to CFTs, the symmetries are sufficient to derive an entropy by counting the asymptotic density of states. To achieve this, we place our WCFT on a torus, which is a plane with coordinates (t, ϕ) subject to the following identifications

- Thermal circle: $(t, \phi) \sim (t + i\beta,\, \phi + i\beta\Omega)$,
- Angular circle: $(t, \phi) \sim (t,\, \phi + 2\pi)$,

where $\beta = 1/T$ is the inverse temperature and Ω is the angular velocity. It corresponds to a theory at finite temperature and angular potential.

Like for CFTs, we seek a transformation that exchanges the thermal circle and the angular circle. This transformation plays the same role as a modular transformation. From the ansatz

$$\phi' = \lambda\phi\,, \qquad t' = t + 2\gamma\phi \tag{2.47}$$

the new identifications are

- Thermal circle: $(t', \phi') \sim (t' + i\beta(1 + 2\gamma\Omega),\, \phi' + i\lambda\beta\Omega)$,
- Angular circle: $(t', \phi') \sim (t' + 4\pi\gamma,\, \phi' + 2\pi\lambda)$.

If we choose

$$\lambda = -\frac{2\pi i}{\beta\Omega}\,, \qquad \gamma = -\frac{1}{2\Omega}\,, \tag{2.48}$$

the circles are exhanged

- Thermal circle: $(t', \phi') \sim (t',\, \phi' + 2\pi)$,
- Angular circle: $(t', \phi') \sim (t' - i\beta',\, \phi' - i\beta'\Omega')$,

where

$$\beta' = -\frac{2\pi i}{\Omega}\,, \qquad \Omega' = \frac{2\pi i}{\beta}\,. \tag{2.49}$$

Under these warped transformations, the stress energy tensor and the current become

$$\mathcal{L}'(\phi') = \frac{1}{\lambda^2}\left(\mathcal{L}(\phi) - 2\gamma\mathcal{P}(\phi) - k\gamma^2\right)\,, \qquad \mathcal{P}'(\phi') = \frac{1}{\lambda}\left(\mathcal{P}(\phi) + k\gamma\right)\,. \tag{2.50}$$

Now that we have identified the desired transformation and observed its effect on the currents, we aim to compute the density of states through the partition function. The evolution operator on the thermal circle is given by

$$U = \exp\left\{ i\int_0^{i\beta} \mathrm{d}t\,(\mathcal{P}(\phi) + \Omega\mathcal{L}(\phi))\right\} = e^{-\beta P_0 - \beta\Omega L_0}\,. \tag{2.51}$$

In the canonical ensemble, the partition function at inverse temperature β and angular velocity Ω is the trace of this evolution operator

$$Z(\beta,\Omega) = \mathrm{Tr}\; e^{-\beta P_0 - \beta\Omega L_0}\,. \tag{2.52}$$

As the energy P_0 and the angular momentum L_0 are charges generating the translation, like in (2.32), they naturally implement the identification

$$(t,\phi) \sim (t + i\beta,\, \phi + i\beta\Omega)\,. \tag{2.53}$$

We observe that the operator

$$i\int_0^{i\beta\Omega} \mathrm{d}\phi\, T(\phi)\,, \tag{2.54}$$

defined by integrating over the thermal circle on the original torus, becomes the time evolution operator on the new torus

$$i\int_0^{i\beta\Omega} \mathrm{d}\phi\, T(\phi) = -\frac{\beta'}{2\pi}\int_0^{2\pi} \mathrm{d}\phi'\,(\mathcal{P}'(\phi') + \Omega'\mathcal{L}'(\phi')) + \frac{k\beta'}{4\Omega'}\,. \tag{2.55}$$

The presence of the last term on the right hand side of the last equation indicates the presence of an anomaly. Precisely,

$$Z(\beta,\Omega) = e^{\frac{k\beta'}{4\Omega'}}\, Z(\beta',\Omega') = e^{\frac{k\beta}{4\Omega}}\, Z(-\frac{2\pi i}{\Omega}, \frac{2\pi i}{\beta})\,. \tag{2.56}$$

We now consider the slowly rotating regime $\beta\Omega \to 0$. In this regime, the trace is projected onto the state of minimal L_0, denoted L_0^v, provided L_0 is bounded from below. P_0 then takes the value of the state with minimal L_0, denoted P_0^v. The bounded requirement is not needed for P_0 because for real value of the latter and β, the term containing P_0 is just a phase. In this slow rotating regime, the partition function can be approximated by

$$Z(\beta,\Omega) \approx \exp\left(\frac{2\pi i}{\Omega}P_0^v - \frac{4\pi^2}{\beta\Omega}L_0^v + \frac{k\beta}{4\Omega}\right)\,. \tag{2.57}$$

The entropy is derived from the partition function through the thermodynamic relation $S = (1 - \beta\partial_\beta - \Omega\partial_\Omega)\log Z(\beta,\Omega)$ giving

$$S = \frac{2\pi i}{\Omega}P_0^v - \frac{8\pi^2}{\beta\Omega}L_0^v\,. \tag{2.58}$$

To express the entropy in the microcanonical ensemble, we first rewrite the partition function as

$$Z(\beta,\Omega) = \int \mathrm{d}L_0\, \mathrm{d}P_0\, \rho(L_0,P_0)\, e^{-\beta P_0 - \beta\Omega L_0}\,, \tag{2.59}$$

where $\rho(L_0,P_0)$ is the density of states. By performing a Laplace transformation, we can express the density of states as

$$\rho(L_0,P_0) = \int \mathrm{d}\beta\, \mathrm{d}\Omega\, Z(\beta,\Omega)\, e^{\beta P_0 + \beta\Omega L_0}\,. \tag{2.60}$$

To compute this integral, we will use the saddle point approximation in the slow rotating regime

$$\rho(L_0,P_0) \approx Z(\tilde{\beta},\tilde{\Omega})\, e^{\tilde{\beta}P_0 + \tilde{\beta}\tilde{\Omega}L_0}\,, \tag{2.61}$$

where $Z(\tilde{\beta},\tilde{\Omega})$ is (2.57) and $\tilde{\beta}$, $\tilde{\Omega}$ are the solutions of

$$\partial_\beta\left(Z(\beta,\Omega)\, e^{\beta P_0 + \beta\Omega L_0}\right) = 0\,, \qquad \partial_\Omega\left(Z(\beta,\Omega)\, e^{\beta P_0 + \beta\Omega L_0}\right) = 0\,. \tag{2.62}$$

Then, the entropy is given by the logarithm of the density of states $S = \log\rho(L_0,P_0)$. To conclude, in the microcanonical ensemble, the entropy becomes

$$S = -\frac{4\pi i P_0 P_0^v}{k} + 4\pi\sqrt{-\left(L_0^v - \frac{(P_0^v)^2}{k}\right)\left(L_0 - \frac{P_0^2}{k}\right)}\,. \tag{2.63}$$

To go any further, we can implement the parametrization (2.43) and get

$$S = -4\pi i q\frac{P_0}{k} + 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{P_0^2}{k}\right)}\,. \tag{2.64}$$

This equation is called the Warped Cardy formula and is the analogue for WCFT of the Cardy formula. If the theory is unitary, P_0 is Hermitian and q should be zero. This implies that the first term of (2.64) is not present. However, if we work with a non-unitary theory, P_0 is allowed to be complex. Such situations occur for warped black holes in the canonical ensemble, where the vacuum charges are given by

$$L_0^v = -\frac{\ell}{24\nu G}\,, \qquad P_0^v = -\frac{i}{6G}\,. \tag{2.65}$$

The warped Cardy formula (2.64) matches the entropy given by the warped black hole (1.23). Therefore, the equation (2.64) represents an entropy despite the negative contributions in the partition function [42].

2.1.3 The quadratic ensemble

The Warped Cardy formula (2.64), derived in the previous section, differs from the usual Cardy formula for CFTs. However, there exists an ensemble for WCFTs where the two formulas appear similar. This ensemble is the quadratic ensemble, which satisfies the algebra:

$$\begin{aligned}
[\tilde{L}_n, \tilde{L}_m] &= (n-m)\tilde{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m}\,, \\
[\tilde{L}_n, \tilde{P}_m] &= -m\tilde{P}_{n+m} + m\tilde{P}_0\delta_{n+m}\,, \\
[\tilde{P}_n, \tilde{P}_m] &= 2n\tilde{P}_0\delta_{n+m}\,.
\end{aligned} \tag{2.66}$$

The main difference with the algebra (2.31) is that now the $U(1)$ is charge-dependent with the presence of $\tilde{P}_0$. As we saw in Chapter 1, the algebra (2.66) is related to (2.31) through the charges redefinition (1.35). In particular for the zero modes

$$\begin{aligned}
\tilde{L}_0 &= L_0 - \frac{P_0^2}{k}\,, \\
\tilde{P}_0 &= \frac{P_0^2}{k}\,.
\end{aligned} \tag{2.67}$$

This corresponds to a nonlocal reparametrization of the theory, i.e. [42]

$$x^+ = \frac{k}{2P_0}t + \phi\,, \quad x^- = \phi\,, \tag{2.68}$$

where $x^\pm$ are the new coordinates in the quadratic ensemble.

If we examine the algebra (2.66) for the modes where $n+m=0$, it corresponds to two copies of commuting Virasoro algebra with central charges c and 0:

$$\begin{aligned}
[\tilde{L}_n, \tilde{L}_{-n}] &= 2n\tilde{L}_0 + \frac{c}{12}n(n^2-1)\,, \\
[\tilde{L}_n, \tilde{P}_{-n}] &= 0\,, \\
[\tilde{P}_n, \tilde{P}_{-n}] &= 2n\tilde{P}_0\,.
\end{aligned} \tag{2.69}$$

On the other hand, if we consider only the modes where $n+m \neq 0$, they correspond to the modes of the usual Virasoro-Kač-Moody algebra:

$$\begin{aligned}
[\tilde{L}_n, \tilde{L}_m] &= (n-m)\tilde{L}_{n+m}\,, \\
[\tilde{L}_n, \tilde{P}_m] &= -m\tilde{P}_{n+m}\,, \\
[\tilde{P}_n, \tilde{P}_m] &= 0\,.
\end{aligned} \tag{2.70}$$

We can adapt the computation done in the previous section to this quadratic ensemble. Starting from a theory at finite temperature

$$Z(\beta_L, \beta_R) = \operatorname{Tr} e^{-\beta_L\tilde{P}_0 - \beta_R\tilde{L}_0}\,, \tag{2.71}$$

it was shown in [42] that we can again use the WCFT symmetries to determine a transformation that exchanges the thermal circle and the angular circle. For an infinitesimal tilt

$$\delta x^+ = -\frac{\delta\gamma}{2}x^-\,, \tag{2.72}$$

the zero modes transform like

$$\delta\tilde{L}_0 = 0\,, \qquad \delta\tilde{P}_0 = \tilde{P}_0\delta\gamma\,. \tag{2.73}$$

By considering in addition a rescaling of the cylinder

$$\phi \to \lambda\phi\,, \tag{2.74}$$

the charges associated to the zero modes and generating translations transform as [42]

$$\tilde{Q}[\partial_{+'}] = e^{\gamma}\tilde{Q}[\partial_+]\,, \qquad \tilde{Q}[\partial_{-'}] = \frac{\tilde{Q}[\partial_-]}{\lambda}\,. \tag{2.75}$$

We notice that under the tilt and the rescaling, all charges transforms independently, which, in the quadratic ensemble, mimics the behavior of CFTs.

As for the canonical ensemble, we put the theory on a torus and make the following identifications

- Thermal circle: $(x^+, x^-) \sim (x^+ + i\beta_L, x^- + i\beta_R)$,
- Angular circle: $(x^+, x^-) \sim (x^+ + 2\pi, x^- + 2\pi)$.

By picking the parameters of the transformation (2.75) such as

$$e^{-\gamma} = -\frac{2\pi i}{\beta_L}\,, \qquad \lambda = -\frac{2\pi i}{\beta_R}\,, \tag{2.76}$$

one can exchange the angular and thermal circles

- Thermal circle: $(x^{+\prime}, x^{-\prime}) \sim (x^{+\prime} + 2\pi, x^{-\prime} + 2\pi)$,
- Angular circle: $(x^{+\prime}, x^{-\prime}) \sim (x^{+\prime} - i\beta'_L, x^{-\prime} - i\beta'_R)$,

with

$$\beta'_L = \frac{4\pi^2}{\beta_L}\,, \qquad \beta'_R = \frac{4\pi^2}{\beta_R}\,. \tag{2.77}$$

This transformation provides the identification of the partition functions but this time, without the presence of an anomaly

$$Z(\beta_L, \beta_R) = Z(\beta'_L, \beta'_R) = Z(\frac{4\pi^2}{\beta_L}, \frac{4\pi^2}{\beta_R})\,. \tag{2.78}$$

Indeed, by two repeated transformations, the evolution operator changes as

$$e^{-\beta_L\tilde{P}_0 - \beta_R\tilde{L}_0} \quad \to \quad e^{2\pi i(\tilde{P}_0 + \tilde{L}_0)} \quad \to \quad \tilde{e}^{-\frac{4\pi^2}{\beta_L}\tilde{P}_0 - \frac{4\pi^2}{\beta_R}\tilde{L}_0}\,. \tag{2.79}$$

It takes exactly the same form as a 2d CFT. We then take the large temperature limit where the right hand side of the last equation is projected onto the vacuum state

$$Z(\beta_L, \beta_R) \approx \exp\left(-\frac{4\pi^2}{\beta_L}\tilde{P}_0^v - \frac{4\pi^2}{\beta_R}\tilde{L}_0^v\right)\,. \tag{2.80}$$

Using the thermodynamic relation

$$S = (1 - \beta_L \partial_{\beta_L} - \beta_R \partial_{\beta_R}) \log Z(\beta_L, \beta_R)\,, \tag{2.81}$$

we get the entropy in the quadratic ensemble

$$S = -\frac{8\pi^2}{\beta_L} \tilde{P}_0^v - \frac{8\pi^2}{\beta_R} \tilde{L}_0^v\,. \tag{2.82}$$

Once again, we go to the microcanonical ensemble to express the entropy from the density of states $\rho(L_0, P_0)$ and in terms of charges $\tilde{P}_0$ and $\tilde{L}_0$:

$$S = 4\pi \sqrt{-\tilde{P}_0^v \tilde{P}_0} + 4\pi \sqrt{-\tilde{L}_0^v \tilde{L}_0}\,. \tag{2.83}$$

By defining the quantities c_L and c_R through

$$\tilde{P}_0^v = -\frac{c_L}{24}\,, \qquad \tilde{L}_0^v = -\frac{c_R}{24}\,, \tag{2.84}$$

we can rewrite the entropy in a more CFT kind of way

$$S = 2\pi \sqrt{\frac{c_L}{6} \tilde{P}_0} + 2\pi \sqrt{\frac{c_R}{6} \tilde{L}_0}\,. \tag{2.85}$$

Although this expression resembles the Cardy formula derived in CFTs, the quantities c_L and c_R are not both fixed by the symmetries alone. They can be expressed in terms of the original charges in the vacuum (2.43)

$$c_L = -24 \frac{q^2}{k}\,, \qquad c_R = c\,. \tag{2.86}$$

While c_R is fixed by the central extension, c_L depends on the details of the theory. As mentioned in the previous chapter, the warped Cardy formula (2.83) matched the bulk entropy under an appropriate choice of vacuum [105].

2.1.4 Warped Virasoro characters

We have seen how the modular invariance, or more precisely covariance in our case, can be used to derive the Warped Cardy formula. As stated in [43], it can also be used to derive the warped characters. By placing ourselves in the canonical ensemble and defining

$$\tau = \frac{i\beta\Omega}{2\pi}\,, \qquad z = \frac{i\beta}{2\pi}\,, \tag{2.87}$$

we can rewrite the partition function as

$$Z(\tau, z) = \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}} y^{P_0}\right)\,, \tag{2.88}$$

with

$$q = e^{2\pi i \tau}\,, \qquad y = e^{2\pi i z}\,. \tag{2.89}$$

Written is such a way, we can define two transformations generating an $SL(2,\mathbb{Z})$ group [42]

$$S: \qquad \tau \to -\frac{1}{\tau}\,, \quad z \to \frac{z}{\tau}\,, \tag{2.90}$$

$$T: \qquad \tau \to \tau + 1\,. \tag{2.91}$$

The S-transformations correspond to modular transformations while the T-transformations represent the addition of the angular circle to the thermal circle. For instance, the S-transformations act on the partition function as

$$Z\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{-\frac{i\pi}{2}\frac{z^2}{\tau}} Z(\tau, z)\,. \tag{2.92}$$

This is a rephrasing of equation (2.56).

As with CFTs, the Hilbert space decomposes into a sum over highest-weight representations of primary states $|p,h\rangle$ (2.39), which implies a decomposition of the partition function into warped Virasoro characters

$$Z(\tau, z) = \sum_{h,p} d_{h,p}\, \chi_{h,p}(\tau, z)\,, \tag{2.93}$$

where the sum runs only over primary states and $d_{h,p}$ takes into account their degeneracy. We will follow the path done in [43] to compute the warped character of a primary state $|p,h\rangle$ (2.39).

First, we use the Sugawara basis (2.41) to compute the contribution from the Virasoro descendants to the partition function. The advantage of this basis is that it commutes with every P_n, allowing us to factorize the norm of mixed states admitting both Virasoro and $U(1)$ descendants. As a result, the warped Virasoro character is simply the product of both contributions. The Virasoro descendants are thus written as

$$\prod_{k=1}^{\infty} \left(L^{(s)}_{-k}\right)^{n_k} |p,h\rangle\,, \tag{2.94}$$

with n_k integer characterising the descendant. As the commutator between L_0 and the Sugawara basis is

$$[L_0, L^{(s)}_{-n}] = n L^{(s)}_{-n}\,, \tag{2.95}$$

the descendants (2.94) are proportional to

$$\prod_{k=1}^{\infty} \left(L^{(s)}_{-k}\right)^{n_k} |p,h\rangle \propto |p, h+N\rangle\,. \tag{2.96}$$

where $N = \sum_k n_k\, k$. The degeneracy of the states $|p,h+N\rangle$ corresponds to the number of partitions of the positive integer N, often denoted as $p(N)$. For instance, $p(4) = 5$ because there are five different ways to partition 4: 1+1+1+1, 1+1+2, 2+2,

$3+1$, and 4. Under the condition that the conformal weight h of the primary state and the central charge c satisfy (2.44), the contribution of the Virasoro descendants is

$$\sum_{N=1}^{\infty} p(N)\, q^N = \prod_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{nk} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}\,, \tag{2.97}$$

like for CFTs. When the primary state is the vacuum $|p,0\rangle$, the charge L_{-1} acts on the vacuum as

$$L_{-1}^{(s)}|p,0\rangle = 0\,, \tag{2.98}$$

implying a state of vanishing norm that we need to exclude from the partition function by starting the product at $n=2$. This can be summarized as

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n}\left(1-\delta_{\text{vac}}\, q\right)\,, \tag{2.99}$$

where $\delta_{\text{vac}}=1$ for the vacuum and 0 otherwise.

For the $U(1)$ descendants, we need to separate the discussion into two cases. If the charge p is real, the P_n are Hermitian and the $U(1)$ descendants have two distinct norms, after an appropriate normalization:

$$\left|\prod_{k=1}^{\infty} P_{-k}^{n_k}|p,h\rangle\right|^2 = \begin{cases} +1 & \text{if } \sum_k n_k \text{ is even}\,, \\ -1 & \text{if } \sum_k n_k \text{ is odd}\,. \end{cases} \tag{2.100}$$

The action of the P_{-n}s does not modify the charge p but changes the weight since

$$[L_0, P_{-n}] = nP_{-n}\,, \tag{2.101}$$

implying that the $U(1)$ descendants are proportional to the states of weight $h+N$

$$\prod_{k=1}^{\infty} P_{-k}^{n_k}|p,h\rangle \propto |p,h+N\rangle\,. \tag{2.102}$$

Unlike the Virasoro descendants, the degeneracy of the state $|p,h+N\rangle$ is not simply the partition function $p(N)$ because, due to the presence of negative norms in (2.100), we also need to take into account the parity of the number of integers used to build the integer N. We denote this function as $f(N)$. For example, $f(4)=1$ because $1+1+1+1$, $2+2$, and $3+1$ contain an even number of integers, contributing $+3$, but $1+1+2$ and 4 use an odd number of integers, contributing -2. The total sum is $3+(-2)=1$. The contribution of the $U(1)$ descendants to the partition function is then[2]

$$\sum_{N=1}^{\infty} f(N)\, q^N = \prod_{n=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k q^{nk} = \prod_{n=1}^{\infty} \frac{1}{1+q^n}\,. \tag{2.103}$$

[2] A rigorous proof of the computation of the generating function of $f(N)$ can be done by adapting the proof done for the integer partition function $p(N)$ in Section 3.3 of [107].

Since no P_{-n} annihilates the vacuum, this result does not change when the primary state is the vacuum state.

The second possibility for the primary state is to possess a imaginary charge p. In this situation, the P_ns are antihermitians but their corresponding $U(1)$ descendants always have a positive norm. Their contribution to the partition will then take the same form as for the Virasoro descendants, without the discussion about the vacuum,

$$\sum_{N=1}^{\infty} p(N)\, q^N = \prod_{n=1}^{\infty} \frac{1}{1-q^n}\,, \tag{2.104}$$

In summary, the warped Virasoro characters are

$$\chi_{h,p} = q^{h-\frac{c}{24}} y^p \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}}\,(1-\delta_{\text{vac}}\, q) \qquad \text{for } p \in \mathbb{R}\,, \tag{2.105}$$

$$\chi_{h,p} = q^{h-\frac{c}{24}} y^p \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2}\,(1-\delta_{\text{vac}}\, q) \quad \text{for } p \in i\mathbb{R}\,. \tag{2.106}$$

2.2 Warped Geometry

Since WCFTs do not possess Lorentz invariance, they do not couple to Riemannian geometry, which is a theory of curved spaces that are locally Lorentz invariant. In this section, we will construct a geometry that carries the symmetries of a WCFT, known as Warped Conformal Geometry (WCG). Initially, we will set aside the scaling symmetry and focus solely on the boost symmetry. This construction will lead us to Warped Geometry (WG), which is the warped analogue of Riemannian geometry. Later, the addition of the scaling generator will lead to WCG. We will consider the cases for $d = 2$ and $d > 2$, starting with the former. For now, we will work in flat space and address curved space later.

2.2.1 Warped Geometry in $d = 2$

In flat space, the warped symmetries in some coordinates $x^a = (x, t)$ are:

$$x^a \to \Lambda^a{}_b x^b\,, \quad x^a \to \lambda^a{}_b x^b\,, \quad x^a \to x^a + \delta^a\,, \tag{2.107}$$

where

$$\Lambda^a{}_b = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \tag{2.108}$$

is a generalized boost transformation: $x \to x$ and $t \to t + vx$,

$$\lambda^a{}_b = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \tag{2.109}$$

is the scaling symmetry but for only one of the coordinates: $x \to \lambda x$ and $t \to t$, and δ^a stands in for translation invariance. We adopt the coordinate notation from [57], where (x, t) should not initially be understood as space and time, although in many contexts it will correspond to them.

As previously mentioned, we will set scaling invariance aside and focus on boost symmetry to define WG. Similar to Euclidean symmetry, we will use vectors to construct any geometric invariant, under the constraint of translation symmetry. Under boost symmetry, vectors must transform as

$$\bar{V}^a \to \Lambda^a{}_b \bar{V}^b \,. \tag{2.110}$$

The analog of the metric is a symmetric tensor with two lower indices invariant under the action of the boost symmetry

$$g_{ab} = \Lambda^c{}_a \, g_{cd} \, \Lambda^d{}_b \,. \tag{2.111}$$

Such a tensor is

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \,, \tag{2.112}$$

and we directly observe that the metric is degenerate because there exists a vector $\bar{q}^a$ such that

$$g_{ab}\bar{q}^b = 0 \,. \tag{2.113}$$

This vector is

$$\bar{q}^a = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,, \tag{2.114}$$

and is a boost invariant vector

$$\bar{q}^a = \Lambda^a{}_b \bar{q}^b \,. \tag{2.115}$$

The degeneracy of the metric prevents us from using it to lower and raise indices. However, we can still define a invariant scalar product as

$$\bar{U} \cdot \bar{V} = \bar{U}^a g_{ab} \bar{V}^b = \bar{U}^x \bar{V}^x \,, \tag{2.116}$$

and a invariant one form q_a

$$q_a = \begin{pmatrix} 1 & 0 \end{pmatrix} \,, \tag{2.117}$$

satisfying

$$q_a = \Lambda^b{}_a q_b \,. \tag{2.118}$$

The metric is the tensor product

$$g_{ab} = q_a q_b \,. \tag{2.119}$$

As stated before, the metric cannot be used to raise or lower indices. Nevertheless, we can define an antisymmetric tensor h_{ab} as

$$q_a \equiv h_{ab} \, \bar{q}^b \,. \tag{2.120}$$

It implies that

$$h_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.121}$$

This tensor is invariant under the boost transformation, non-degenerate and possesses an inverse h^{ab} with $h^{ab}h_{bc} = \delta^a{}_c$. Despite the metric being degenerate and non-invertible, it is possible to define a upper index metric $\bar{g}^{ab}$ as the tensor product of two vector $\bar{q}$

$$\bar{g}^{ab} = \bar{q}^a \bar{q}^b = h^{ac} g_{cd} h^{bd} . \tag{2.122}$$

The two-form h is then the natural way of rising and lowering indices

$$V_a = h_{ab} \bar{V}^b . \tag{2.123}$$

We see that the bar notation was introduced for the vectors to emphasize that they are not linked with their dual one-form using the traditional metric but with the two-form h. As a consequence, the product

$$V_a \bar{V}^a = \bar{V}^a h_{ab} \bar{V}^b = 0 , \tag{2.124}$$

because of the antisymmetry of h and is not a good definition of a norm. In any case, we can define a norm using the scalar product

$$||\bar{U}||^2 = \bar{U} \cdot \bar{U} . \tag{2.125}$$

However, it is pointless to talk about angle in the usual way for the reason that

$$\frac{\bar{U} \cdot \bar{V}}{||\bar{U}|| \, ||\bar{V}||} = 1 . \tag{2.126}$$

This detail is important to be mentioned since the scaling symmetry that we will add later on is known to preserve angles. A cross product can also be defined using the antisymmetric tensor h

$$\bar{U} \times \bar{V} = \bar{U}^a h_{ab} \bar{V}^b = \bar{U}^x \bar{V}^t - \bar{U}^t \bar{V}^x . \tag{2.127}$$

Now, using this other product, we can define an "angle":

$$\theta(\bar{U}, \bar{V}) = \frac{\bar{U} \times \bar{V}}{||\bar{U}|| \, ||\bar{V}||} . \tag{2.128}$$

Under a boost transformation, this angle transforms as $\theta \to \theta + v$ like normal angles under euclidean rotations. Under the scaling transformation $x^a \to \lambda^a{}_b x^b$, the different vectors and tensors defined above transform as

$$\begin{aligned} \lambda^a{}_b \bar{q}^b = \bar{q}^a \quad &, \quad \lambda^a{}_b q_a = \lambda q_b , \\ \lambda^a{}_b \lambda^c{}_d \, g_{ac} = \lambda^2 g_{bd} \quad &, \quad \lambda^a{}_b \lambda^c{}_d \, h_{ac} = \lambda h_{bd} . \end{aligned} \tag{2.129}$$

2.2.2 Warped Geometry in $d > 2$

There are two different ways to extend this framework to higher dimensions. One approach is to introduce additional coordinates like t, each enjoying its own distinct boost symmetry. The other method involves adding more coordinates like x while preserving a single boost symmetry for t. The approach considered in [57] follows the second method in order to construct a suitable holographic dual for these theories. Therefore, the coordinates in d dimensions are

$$x^a = \begin{pmatrix} x^I \\ t \end{pmatrix} , \tag{2.130}$$

where lower case indices a run from 1 to d and upper case indices I run from 1 to $d-1$. The global symmetries possessed by a WG in $d > 2$ are

- Translations: $x^a \to x^a + \delta^a$;
- Boosts: $x^I \to x^I \,,\; t \to t + v_I\, x^I$;
- Dilatations: $x^I \to \lambda\, x^I \,,\; t \to t$;
- Rotations: $x^I \to M^I{}_J\, x^J \,,\; t \to t$, with $M^I{}_J \in SO(d-1)$.

As in the previous section, we will initially omit the scaling symmetries and focus on the implications of the other three. The existence of invariant tensors $\bar{q}^a$ and g_{ab} is ensured by the boost symmetries, similarly to $d = 2$. However, the lower case tensor q_a needs to be adjusted

$$q_a \to q_a^I \,, \tag{2.131}$$

and is not anymore an invariant tensor since it transforms under rotations as

$$q_a^I \to M^I{}_J\, q_a^J \,. \tag{2.132}$$

The metric can be expressed from these one-forms

$$g_{ab} = q_a^I\, \delta_{IJ}\, q_b^J \,. \tag{2.133}$$

This metric can be used to defined a norm, as for the case $d = 2$,

$$||\bar{U}||^2 = \bar{U}^a g_{ab} \bar{U}^b \,. \tag{2.134}$$

We can also defined a d-form h_d using the totally antisymmetric euclidean invariant tensor $\varepsilon_{I_1 \ldots I_{d-1}}$

$$h_{a_1 \ldots a_d}\, \bar{q}^{a_d} = \varepsilon_{I_1 \ldots I_{d-1}}\, q_{a_1}^{I_1} \ldots q_{a_{d-1}}^{I_{d-1}} \,, \tag{2.135}$$

that provides the WG with a volume form to integrate over this space

$$I(\phi) = \int h_d \wedge \phi \,, \tag{2.136}$$

where ϕ is a scalar function.

2.2.3 Adding curvature to Warped Geometry

For now, we have only worked in flat space. The next logical step is to extend this discussion to curved space. To achieve this, we will use the standard approach: at each point on our curved manifold, the tangent space geometry will correspond to the flat space results from the previous sections. We will separate the discussion into two parts for simplicity: the case $d = 2$ and the case $d > 2$.

Let us start with the $d = 2$ case. We define an revertible map τ^a_μ from spacetime vectors in the manifold to tangent Warped Geometry variables

$$\tau^a_\mu \,:\, \partial_\mu \to \tau^a_\mu \partial_a \,. \tag{2.137}$$

It allows us to construct spacetime tensors from the WG tensors:

$$\begin{aligned} A_\mu = q_a \tau^a_\mu \quad &, \quad \bar{A}^\mu = \bar{q}^a \tau^\mu_a \,, \\ G_{\mu\nu} = \tau^a_\mu \tau^b_\nu g_{ab} \quad &, \quad H_{\mu\nu} = \tau^a_\mu \tau^b_\nu h_{ab} \,. \end{aligned} \tag{2.138}$$

These spacetime tensors are all invariant under the local boost transformations $\tau^a_\mu \to \Lambda^a{}_b, \tau^b_\mu$ because of the invariance of the flat tensors q_a, $\bar{q}^a$, g_{ab}, and h_{ab} under such transformations. The local boost symmetry can be viewed as a gauge symmetry induced in the manifold. We can define the full covariant derivative in the base manifold

$$D = \partial + \omega + \Gamma \,, \tag{2.139}$$

where ω is the "spin" connection one form, because of the gauge symmetry, and Γ is the affine connection associated with diffeomorphism. For example, the covariant derivative of the map τ^a_μ is

$$D_\mu \tau^a_\nu = \partial_\mu \tau^a_\nu + \omega^a{}_{b\mu} \tau^b_\nu - \Gamma^\rho_{\mu\nu} \tau^a_\rho \,. \tag{2.140}$$

Under a local boost transformation of the form

$$\Lambda^a{}_b = \delta^a{}_b + v \bar{q}^a q_b = \delta^a{}_b + v \bar{B}^a{}_b = \left(e^{v\bar{B}}\right)^a{}_b \,, \tag{2.141}$$

where we introduced the boost generator $\bar{B}^a{}_b = \bar{q}^a q_b$, the "spin" connection transforms as

$$\left(e^{-v\bar{B}}\right)^a{}_c \,\omega^c{}_{d\mu} \,\left(e^{v\bar{B}}\right)^d{}_b \to \omega^a{}_{b\mu} - \partial_\mu v \,\bar{B}^a{}_b \,. \tag{2.142}$$

as a gauge field transforms under a gauge symmetry. Therefore, we can express the "spin" connection in terms of the generator of the boost symmetry

$$\omega^a{}_{b\mu} = \omega_\mu \bar{B}^a{}_b \,, \tag{2.143}$$

where ω_μ transforms under a local boost transformation like

$$\omega_\mu \to \omega_\mu - \partial_\mu v \,. \tag{2.144}$$

By imposing that the covariant derivative preserves the metric, or in terms of the vielbein

$$D_\mu \tau^a_\nu = 0\,, \tag{2.145}$$

we can express the affine connection in terms of τ^a_μ and ω_μ

$$\Gamma^\rho_{\mu\nu} = \tau^\rho_a \partial_\mu \tau^a_\nu + \bar{A}^\rho A_\nu \omega_\mu\,. \tag{2.146}$$

Now we introduce the torsion and curvature two-forms with the aime of specify the geometry

$$R^a{}_b = \mathrm{d}\omega^a{}_b \quad , \quad T^a = \mathrm{d}\tau^a + \omega^a{}_b \wedge \tau^b\,, \tag{2.147}$$

and by using the equation (2.143), we can rewrite them as

$$R^a{}_b = \bar{q}^a q_b\, \mathrm{d}\omega \quad , \quad T^a = \mathrm{d}\tau^a + \bar{q}^a \omega \wedge A\,. \tag{2.148}$$

In Riemannian geometry, the next step is usually to require a vanishing torsion condition in order to fully express the spin connection in terms of the vieldbein and obtain a unique expression for the affine connection. However, for our Warped Geometry, if we make the same assumption, it was argued in [57] that it does not lead to a complete expression of the affine connection in terms of the vielbein τ^a_μ. In order to achieve this, warped geometry requires an additional structure called a scaling structure. This name arises because the scaling symmetry $\lambda^a{}_b$ defines two preferred axes in the tangent space: one associated with a coordinate that scales and one that does not. The scaling structure $J^a{}_b$ is defined by the relation

$$J^a{}_b J^b{}_c = J^a{}_c\,, \tag{2.149}$$

and possesses exactly one 0 eigenvalue with eigenvector $\bar{q}^a$ and one -1 eigenvalue with an eigenvector that we will note q^a. It then takes the form

$$J^a{}_b = -\frac{q^a q_b}{q^c q_c}\,. \tag{2.150}$$

In the (x, t) coordinates, the scaling structure would be

$$J^a{}_b = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\,, \tag{2.151}$$

and

$$q^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,. \tag{2.152}$$

This tensor is nothing else than the generator of dilatations

$$\lambda^a{}_b = \delta^a{}_b - \delta\lambda\, J^a{}_b\,, \tag{2.153}$$

for the infinitesimal transformation $\lambda = 1 + \delta\lambda$.

The analogue of the vanishing condition of the torsion in Riemannian geometry to fix the "spin" connection is the requirement that the covariant derivative maps weight -1 vectors to weight -1 vectors and weight 0 vectors to weight 0 vectors. It implies that the scaling structure is covariantly constant or, in terms of q^a,

$$D_\mu q^a = \partial_\mu q^a + q_b q^b \omega_\mu \bar{q}^a = 0\,. \tag{2.154}$$

Two constraints can be derived from this equation. The first one by multiplying the latter equation by q_a,

$$q_a \partial_\mu q^a = \frac{1}{2}\partial_\mu (q_a q^a) = 0\,. \tag{2.155}$$

So the vector q^a can be normalized such that $q_a q^a = 1$. In addition, we can define the vector $\bar{q}_a$ such that

$$\bar{q}_a q^a = 0 \quad \text{and} \quad \bar{q}_a \bar{q}^a = 1\,. \tag{2.156}$$

In the (x,t) coordinates, it would take the form

$$\bar{q}_a = \begin{pmatrix} 0 & 1 \end{pmatrix}\,. \tag{2.157}$$

The second constraint is an expression for ω_μ,

$$\omega_\mu = -\bar{q}_a \partial_\mu q^a\,. \tag{2.158}$$

It implies that in the (x,t) coordinate system, the spin connection vanishes as $\omega_\mu = 0$ and, after a local boost transformation, $\omega_\mu = -\partial_\mu$. Therefore, $\mathrm{d}\omega = 0$ in this coordinate system, which implies that a covariantly conserved scaling structure automatically imposes a vanishing curvature condition in any coordinate system.

$$R^a{}_b = 0\,. \tag{2.159}$$

The preferred basis q^a and $\bar{q}^a$ allows us to decompose the vielbein like

$$\tau^a_\mu = A_\mu q^a + \bar{A}_\mu \bar{q}^a\,, \tag{2.160}$$

and write the affine connection as

$$\Gamma^\rho_{\mu\nu} = A^\rho \partial_\mu A_\nu + \bar{A}^\rho \partial_\mu \bar{A}_\nu\,. \tag{2.161}$$

The tensor two-form can then be expressed as

$$T^a = \mathrm{d}\tau^a - \mathrm{d}q^a \wedge A = q^a \mathrm{d}A + \bar{q}^a \mathrm{d}\bar{A} \tag{2.162}$$

and decomposes in two parts

$$T = \mathrm{d}A\,, \quad \bar{T} = \mathrm{d}\bar{A}\,. \tag{2.163}$$

To summarize, our geometry in $d = 2$ has vanishing curvature and non-trivial torsion. It is known in the literature as Weitzenböck geometry [108].

In higher dimensions, we construct the spacetime tensors from the WG tensors in $d > 2$ using the invertible map τ_μ^a:

$$A_\mu^I = \tau_\mu^a q_a^I \quad , \quad \bar{A}^\mu = \tau_a^\mu \bar{q}^a \, , \qquad\qquad (2.164)$$
$$G_{\mu\nu} = A_\mu^I \, \delta_{IJ} \, A_\nu^J \quad , \quad H_{\mu_1 \ldots \mu_d} = \tau_{\mu_1}^{a_1} \ldots \tau_{\mu_d}^{a_d} \, h_{a_1 \ldots a_d} \, .$$

We also need the introduction of a scaling structure

$$J^a{}_b = -q_I^a q_b^I \, , \qquad\qquad (2.165)$$

that possesses $(d-1)$ eigenvalues -1 with eigenvectors q_I^a and one vanishing eigenvalue with eigenvector $\bar{q}^a$. From those q_I^a, we construct A_I^μ

$$A_I^\mu = \tau_a^\mu q_I^a \, . \qquad\qquad (2.166)$$

We then define the vector $\bar{A}_\mu$ such that

$$\bar{A}_\mu \bar{A}^\mu = 1 \ , \ \bar{A}_\mu A_I^\mu = 0 \, . \qquad\qquad (2.167)$$

The presence of the rotation symmetries involves an additional $SO(d-1)$ spin connection $\Omega^I{}_{J\mu}$ in the covariant derivative

$$D = \partial + \omega + \Omega + \Gamma \, . \qquad\qquad (2.168)$$

A local boost transformations can be expressed in terms of the boost generator $\left(\bar{B}^I\right)^a{}_b$:

$$\Lambda^a{}_b = \left(e^{v_I \bar{B}^I}\right)^a{}_b \, . \qquad\qquad (2.169)$$

Like in the case $d = 2$, we can describe the "spin" connection with the generators of the boost transformations

$$\omega^a{}_{b\mu} = \omega_{I\,\mu} \left(\bar{B}^I\right)^a{}_b \, , \qquad\qquad (2.170)$$

and under the requirement that the scaling structure is covariantly constant, we find, as in the case $d = 2$, that there exists a frame for which the 'spin' connection for each boost generator $\bar{B}^I$ must vanish. After local boost transformations, $\mathrm{d}\omega_I = 0$. Then, all curvatures associated with the 'spin' connection also vanish in any coordinate system. This implies that the only non-vanishing equations for the torsions and the Riemannian curvature are those associated with the $SO(d-1)$ symmetry. By decomposing the vielbein in the preferred basis q_I^a and $\bar{q}^a$

$$\tau_\mu^a = A_\mu^I q_I^a + \bar{A}_\mu \bar{q}^a \, , \qquad\qquad (2.171)$$

the remaining equations are

$$R^I{}_J = \mathrm{d}\Omega^I{}_J + \Omega^I{}_K \wedge \Omega^K{}_J \ , \quad T^I = \mathrm{d}A^I + \Omega^I{}_J \wedge A^J \ , \quad \bar{T} = \mathrm{d}\bar{A} \, . \qquad (2.172)$$

Once again, we have non-vanishing torsions, but in $d > 2$, not all curvatures are null, only those related to local boost transformations. This geometry can be viewed as a combination of Riemannian geometry, if we demand $T^I = 0$, in the $(d-1)$ subspace described by x^I, and Weitzenböck geometry concerning the symmetries relating I and the non-scaling direction.

2.3 Lower Spin Gravity: a holographic dual for WCFTs

Now that we have described our curved Warped Geometry, our goal in this section is to build a bulk holographic dual to WCFTs. There already exist holographic setups, such as the massive vector model [109] or TMG [27, 28]. However, those theories involve more bulk fields than required by the symmetry. In this section, we show how the authors of [57] build a bulk description that relies solely on the symmetries of the boundary WCFT, called Lower Spin Gravity.

The extra space-like dimension that we need in the bulk can be viewed as a radial direction. Thus, the holographic theory resides in the bulk, while the WCFT lives on the boundary. We primarily focus on a 3d holographic dual, meaning the boundary field theory is a WCFT_2. Next, we need to determine how the symmetries act on the extra dimension. Is it a boosted coordinate like t, or a scaling coordinate like x^I? Since we expect this extra dimension to contain information about the RG flow of the theory, we will consider this holographic direction as a scaling coordinate. This is why, in the previous sections discussing dimensions greater than 2, we retained only one boosted coordinate.

On one hand, we have the geometric pieces of our theory: the remaining curvatures $R^I{}_J$ and the torsions T^I and $\bar{T}$. In order to determine the spin connection $\Omega^I{}_J$, we impose that a vanishing condition for the torsions T^I:

$$T^I = 0\,. \tag{2.173}$$

Since we have already precisely determined the "spin" connection $\omega^a{}_b$, such a condition is not needed for $\bar{T}$.

On the other hand, we have the fields A^I and $\bar{A}$. The choice of equations of motion will be dictated by the most general covariant set of equations that we can assemble to leading order in derivative and without the introduction of extra fields. The existence of the $SO(2)$ antisymmetric tensor ε_{IJ} allows us to write

$$\begin{aligned} &T^I = 0\,;\\ &R^{IJ} + c\, A^I \wedge A^J + a\,\bar{T}\varepsilon^{IJ} = 0\,;\\ &\bar{T} + b\,\varepsilon_{IJ} A^I \wedge A^J + d\,\varepsilon_{IJ} R^{IJ} = 0\,. \end{aligned} \tag{2.174}$$

We can take linear combinations of those equations and if $1 - 2ad \neq 0$, which is automatically the case otherwise it would violate the assumption of vielbein invertibility [57], we can redefined the constants c and b (or set a and d to zero) and rewrite the equations like

$$\begin{aligned} &T^I = 0\,;\\ &R^{IJ} + c\, A^I \wedge A^J = 0\,;\\ &\bar{T} + b\;\varepsilon_{IJ}\, A^I \wedge A^J = 0\,. \end{aligned} \tag{2.175}$$

or in components:

$$\begin{aligned} \mathrm{d}A^1 - \Omega \wedge A^2 = 0 \quad &, \quad \mathrm{d}A^2 + \Omega \wedge A^1 \,, \\ \mathrm{d}\Omega - c\, A^1 \wedge A^2 = 0 \quad &, \quad \mathrm{d}\bar{A} + 2b\, A^1 \wedge A^2 = 0\,. \end{aligned} \tag{2.176}$$

where we defined $\Omega \equiv {\Omega^2}_1$. The constant c is expected to determine the central charge of the theory, like for AdS/CFT, and to be linked to the cosmological constant [57]. As a consequence, we fixed c to be postive. We will see later that the constant b is related to the warping parameter. Those equations can be derived from the action

$$\begin{aligned} S = \frac{kc}{8\pi} \int \Big(& A^1 \wedge \mathrm{d}A^1 + A^2 \wedge \mathrm{d}A^2 + 2\,\Omega \wedge A^1 \wedge A^2 \\ & + \frac{\alpha b^2 - 1}{c} \Omega \wedge \mathrm{d}\Omega + \alpha b\, \Omega \wedge \mathrm{d}\bar{A} + \frac{\alpha c}{4} \bar{A} \wedge \mathrm{d}\bar{A} \Big) \end{aligned} \tag{2.177}$$

where α is a free parameter and $\frac{kc}{8\pi}$ is for the moment an overall normalization which will become clearer later. This action can be brought to a more familiar form by the field redefinitions:

$$\bar{A} \equiv \sqrt{\left|\frac{8}{kc^2\alpha}\right|}\bar{B} - \frac{2b}{c} B^3\,, \quad A^1 \equiv \frac{B^1}{\sqrt{c}}\,, \quad A^2 \equiv \frac{B^2}{\sqrt{c}}\,, \quad \Omega \equiv B^3\,. \tag{2.178}$$

giving

$$S = \frac{k}{8\pi} \int \left(B^1 \wedge \mathrm{d}B^1 + B^2 \wedge \mathrm{d}B^2 - B^3 \wedge \mathrm{d}B^3 + 2\, B^1 \wedge B^2 \wedge B^3 \right) + \frac{\kappa}{8\pi} \int \bar{B} \wedge \mathrm{d}\bar{B}\,, \tag{2.179}$$

which is nothing else than an $SL(2,\mathbb{R}) \times U(1)$ Chern-Simons action

$$S = \frac{k}{4\pi} \int \mathrm{Tr} \left[B \wedge \mathrm{d}B + \frac{2}{3} B \wedge B \wedge B \right] + \frac{\kappa}{8\pi} \int \bar{B} \wedge \mathrm{d}\bar{B}\,, \tag{2.180}$$

where k is a continuous parameter determining the central charge of the Virasoro algebra, κ a discrete parameter resulting from the sign of α, describing the sign of the level of the $U(1)$ Kac-Moody algebra. $B = B^l L_l$ is the $SL(2,\mathbb{R})$ connection with L_l being the $SL(2,\mathbb{R})$ generators in the basis

$$L_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad L_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad L_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \tag{2.181}$$

and $\bar{B}$ is the $U(1)$ connection. This holographic dual of a WCFT_2 is called Lower Spin Gravity.

2.3.1 Boundary conditions and asymptotic symmetries

Now that we have formulated equations of motion governing the overall dynamics, the next crucial step is to incorporate boundary conditions. Our objective is to characterize WAdS_3 spacetimes and spacelike warped AdS_3 black holes using Lower Spin Gravity, ensuring that these solutions adhere to appropriate boundary conditions. These spacetimes are commonly parameterized by coordinates (ρ, t, φ), where ρ signifies the holographic direction, t serves as both time and a boosted boundary coordinate, and φ represents an angular and scaling coordinate at the boundary. The WCFT at the boundary is then invariant under the transformation

$$\varphi \to f(\varphi)\,, \quad t \to t + g(\varphi)\,. \tag{2.182}$$

Until the end of this section, we will use those coordinates.

Since the Lower Spin Gravity is described by the action (2.180), the equations of motion for the connections B and $\bar{B}$ are

$$\mathrm{d}B + B \wedge B = 0\,, \quad \mathrm{d}\bar{B} = 0\,. \tag{2.183}$$

It implies that the connections are flat. By making a gauge choice to fix the radial dependence of the gauge field B and $\bar{B}$, we can write the solutions of those equations of motion like

$$\begin{aligned} B(\rho, t, \varphi) &= \beta^{-1}(\rho)\Big(b(t,\varphi) + \mathrm{d}\Big)\beta(\rho)\,, \\ \bar{B}(\rho, t, \varphi) &= \bar{b}(t,\varphi)\,, \end{aligned} \tag{2.184}$$

with

$$\begin{aligned} b(t,\varphi) &= b_\varphi(t,\varphi)\,\mathrm{d}\varphi + b_t(t,\varphi)\,\mathrm{d}t\,, \\ \bar{b}(t,\varphi) &= \bar{b}_\varphi(t,\varphi)\,\mathrm{d}\varphi + \bar{b}_t(t,\varphi)\,\mathrm{d}t\,, \end{aligned} \tag{2.185}$$

where

$$\beta(\rho) = e^{\rho L_0} \tag{2.186}$$

is a group element of the $SL(2,\mathbb{R})$ algebra. This choice for β has no impact for the computation of asymptotic symmetries but will be relevant when we will make a metric interpretation of the connections.

In [66], the following boundary conditions were proposed

$$\begin{aligned} b_\varphi &= L_1 - \mathfrak{L}\,L_{-1} \quad , \quad b_t = \mu\,b_\varphi - \mu'\,L_0 + \frac{\mu''}{2}L_{-1}\,, \\ \bar{b}_\varphi &= \frac{4\pi}{\kappa}\mathcal{P} \quad , \quad \bar{b}_t = \mu\,\bar{b}_\varphi + \nu\,, \end{aligned} \tag{2.187}$$

where

$$\mathfrak{L} = \frac{2\pi}{k}\left(\mathcal{L} - \frac{2\pi}{\kappa}\mathcal{P}^2\right)\,, \tag{2.188}$$

and the prime denotes a derivative with respect to φ. The functions $\mathcal{L}$ and $\mathcal{P}$ appearing in the boundary conditions depend on t and φ and we can interpreter them as functions characterizing the physical state. Hence, μ and ν, also functions of t and φ, can be viewed as chemical potentials and are therefore fixed, i.e. $\delta\mu = 0 = \delta\nu$. The equations of motion determine the time evolution of the state-dependent functions:

$$\begin{aligned}\partial_t\mathcal{L} &= \mu\mathcal{L}' + 2\mathcal{L}\mu' - \frac{k}{4\pi}\mu''' + \mathcal{P}\nu' \,,\\ \partial_t\mathcal{P} &= \mu\mathcal{P}' + \mathcal{P}\mu' + \frac{\kappa}{4\pi}\nu' \,.\end{aligned} \tag{2.189}$$

The boundary conditions (2.187) are preserved by the following gauge transformations:

$$\delta_\varepsilon B_\mu = \partial_\mu\varepsilon + [B_\mu, \varepsilon]\,, \qquad \delta_{\bar\varepsilon}\bar{B}_\mu = \partial_\mu\bar\varepsilon\,, \tag{2.190}$$

where

$$\begin{aligned}\varepsilon(t,\varphi) &= \beta^{-1}\left(\epsilon\,(L_1 - \mathfrak{L}\,L_{-1}) - \epsilon' L_0 + \frac{\epsilon''}{2}L_{-1}\right)\beta\,,\\ \bar\varepsilon(t,\varphi) &= \sigma + \frac{4\pi}{\kappa}\mathcal{P}\,\epsilon\,.\end{aligned} \tag{2.191}$$

The gauge parameters ϵ and σ are functions on the boundary coordinates (t,φ) and have to satisfy

$$\partial_t\epsilon = \mu\epsilon'\,, \qquad \partial_t\sigma = -\frac{4\pi}{\kappa}\mu(\epsilon\mathcal{P})' - \epsilon\nu'\,. \tag{2.192}$$

This leads to the infinitesimal transformation behavior of the function $\mathcal{L}$ and $\mathcal{P}$:

$$\begin{aligned}\delta\mathcal{L} &= \epsilon\mathcal{L}' + 2\mathcal{L}\epsilon' + \mathcal{P}\sigma' - \frac{k}{4\pi}\epsilon'''\,,\\ \delta\mathcal{P} &= \epsilon\mathcal{P}' + \mathcal{P}\epsilon' + \frac{\kappa}{4\pi}\sigma'\,.\end{aligned} \tag{2.193}$$

Following [66, 110], the variation of the canonical boundary charge is given by

$$\delta Q[\varepsilon] + \delta Q[\bar\varepsilon] = \frac{k}{2\pi}\int \mathrm{d}\varphi\,\mathrm{Tr}(\varepsilon\,\delta B_\varphi) + \frac{\kappa}{4\pi}\int \mathrm{d}\varphi\,\mathrm{Tr}(\bar\varepsilon\,\delta\bar{B}_\varphi) \tag{2.194}$$

$$= \int \mathrm{d}\varphi(\delta\mathcal{L}\,\epsilon + \delta\mathcal{P}\,\sigma). \tag{2.195}$$

Those charges are integrables

$$Q = \int \mathrm{d}\varphi(\mathcal{L}\epsilon + \mathcal{P}\sigma) \tag{2.196}$$

and we can relate $\mathcal{L}$ and $\mathcal{P}$ to the currents of a WCFT (2.12, 2.14). Indeed, one can check that they satisfy the algebra (2.21). As the angle φ is 2π periodic, we can make a Fourier mode expansion for $\mathcal{P}$, $\mathcal{L}$ and the Dirac delta function $\delta(\varphi - \bar\varphi)$:

$$\mathcal{L} = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} L_n e^{-in\varphi}\,, \qquad \mathcal{P} = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} P_n e^{-in\varphi}\,, \qquad \delta(\varphi-\bar\varphi) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-in(\varphi-\bar\varphi)}\,, \tag{2.197}$$

to get the following algebras:

$$\begin{aligned}
[L_n, L_m] &= (n-m)\, L_{n+m} + \frac{c}{12}\, n^3\, \delta_{n+m}\ , \\
[L_n, P_m] &= -m\, P_{n+m}\,, \\
[P_n, P_m] &= \frac{\kappa}{2}\, n\, \delta_{n+m}\,,
\end{aligned} \tag{2.198}$$

with $c = 6k$. We recognize the algebra of a WCFT as expected from a holographic dual.

2.3.2 Metric interpretation

We will see in this section that we can establish a connection between the connections B and $\bar{B}$ of the action (2.180) and WAdS metrics. The boundary conditions (2.187) are valid for any functions $\mu(t,\varphi)$ and $\nu(t,\varphi)$. However, to relate the connections B and $\bar{B}$ to the metrics that interest us, it is sufficient to set $\mu = 0$ and $\nu = 1$. In this case, the connections become

$$\begin{aligned}
B^0 &= \mathrm{d}\rho\,, \qquad & B^+ &= e^{\rho}\mathrm{d}\varphi\,, \\
B^- &= e^{-\rho}\,\mathfrak{L}\,\mathrm{d}\varphi\,, \qquad & \bar{B} &= \mathrm{d}t + \frac{4\pi}{\kappa}\mathcal{P}\,\mathrm{d}\varphi\,.
\end{aligned} \tag{2.199}$$

We can now relate the fields B and $\bar{B}$ to the fields A^I, $\bar{A}$, and Ω using a more general identification than (2.178). For this, let us define three linearly independent vectors in $SL(2,\mathbb{R})$, $(\zeta_0^l, \zeta_1^l, \zeta_2^l)$, and their inverse vectors $(\hat{\zeta}_l^0, \hat{\zeta}_l^1, \hat{\zeta}_l^2)$ satisfying

$$\hat{\zeta}_l^I\, \zeta_J^l = \delta^I{}_J \tag{2.200}$$

for $I, J = 0, 1, 2$. The choice of these vectors will determine whether we are dealing with spacelike, timelike, or null warped AdS_3. Now we build the more general identification

$$A^0 \equiv \bar{A} = \sqrt{\left|\frac{8}{kc^2\alpha}\right|}\bar{B} - \frac{2b}{c}\hat{\zeta}_l^0 B^l\,, \quad A^1 \equiv \frac{\hat{\zeta}_l^1 B^l}{\sqrt{c}}\,, \quad A^2 \equiv \frac{\hat{\zeta}_l^2 B^l}{\sqrt{c}}\,, \quad \Omega \equiv \hat{\zeta}_l^0 B^l\,. \tag{2.201}$$

Notice that the extra parameters appearing here are not fully physical because they are absent from the action (2.180). This is analogous to the AdS radius in AdS/CFT, where the exact numerical value is not physical; only its unit matters. Since $\bar{B}$ has only trivial commutators with the $SL(2,\mathbb{R})$ generators, the natural metric on $SL(2,\mathbb{R})$ induces a metric on the fields A^I given by

$$M_{IJ} = \zeta_I^l\, g_{lk}\, \zeta_J^k \tag{2.202}$$

where g_{lk} is the natural metric on $SL(2,\mathbb{R})$ (see (A.5) in Appendix A). More explicitly, $g_{00} = \frac{1}{2}$, $g_{+-} = g_{-+} = -1$, and the other components are zero. Finally, it is possible to form an $SL(2,\mathbb{R})\times U(1)$ invariant quadratic form that we will interpret as the line element

$$\mathrm{d}s^2 = A^I M_{IJ} A^J\,. \tag{2.203}$$

Let us now reproduce the different Warped AdS spacetimes and the Warped black hole. Depending on which ζ_0 we pick, the geometry will turn out to be either spacelike WAdS for a hyperbolic generator of $SL(2,\mathbb{R})$, null WAdS for a parabolic generator, or timelike WAdS for an elliptic generator. We choose the following notation for our vectors: $\zeta=(+,0,-)$ and the parameters b, c, and α as

$$b^2=\frac{\nu^2}{2\ell^2}\,,\qquad c=\frac{\nu^2+3}{2\ell^2}\,,\qquad \alpha=\frac{8}{kb^2}\,. \tag{2.204}$$

Timelike WAdS. We pick $\zeta_0=(1,0,1)$ and complete the basis with $\zeta_1=(1,0,-1)$ and $\zeta_2=(0,1,0)$. Introducing the coordinates $r=\rho+\log 2$ and $\phi=-\varphi$ and by setting the currents to

$$\mathcal{L}=\frac{k}{8\pi}\,,\qquad \mathcal{P}=0\,, \tag{2.205}$$

we end up with

$$\mathrm{d}s^2=\frac{\ell^2}{\nu^2+3}\left(\mathrm{d}r^2+\cosh^2(r)\,\mathrm{d}\phi-\frac{4\nu^2}{\nu^2+3}(\mathrm{d}t+\sinh(r)\,\mathrm{d}\phi)^2\right)\,, \tag{2.206}$$

which corresponds to the time-like Warped AdS spacetime (1.5).

Null WAdS We pick $\zeta_0=(0,0,1)$ and complete the basis with $\zeta_1=(1,0,1)$ and $\zeta_2=(0,1,0)$. By setting the currents to zero

$$\mathcal{L}=0=\mathcal{P}\,, \tag{2.207}$$

ν to 1 as null WAdS only exist for this value of the wrapping parameter and making the following change of coordinates

$$u=e^{-\rho/2}\,,\qquad \varphi=-\frac{1}{\sqrt{2}}x^-\,,\qquad t=2^{3/2}x^+\,, \tag{2.208}$$

the final metric is

$$\frac{\mathrm{d}s^2}{\ell^2}=\frac{\mathrm{d}u^2}{u^2}+\frac{\mathrm{d}x^+\,\mathrm{d}x^-}{u^2}-\left(\frac{\mathrm{d}x^-}{u^2}\right)^2\,. \tag{2.209}$$

We recover the Null WAdS metric (1.9).

Spacelike WAdS We pick $\zeta_0 = (1, 0, -1)$ and complete the basis with $\zeta_1 = (1, 0, 1)$ and $\zeta_2 = (0, 1, 0)$. We are making the same change of coordinates as for the time-like case but this time we set the currents to

$$\mathcal{L} = -\frac{k}{8\pi}\,, \qquad \mathcal{P} = 0\,. \tag{2.210}$$

The end result is

$$\mathrm{d}s^2 = \frac{\ell^2}{\nu^2+3}\left(\mathrm{d}r^2 - \cosh^2(r)\,\mathrm{d}\phi + \frac{4\nu^2}{\nu^2+3}(\mathrm{d}t + \sinh(r)\,\mathrm{d}\phi)^2\right)\,, \tag{2.211}$$

the space-like Warped AdS metric (1.8).

Warped Black Hole As spacelike WAdS is the only warped spacetime that can carry non-pathological black holes, we will use the same vectors ζ_I. However we will perform a different change of coordinates:

$$\rho = 2\log\left(\frac{\ell}{2}\sqrt{c}(\sqrt{r-r_+} + \sqrt{r-r_-})\right)\,, \quad \varphi = -\phi\,, \quad t = \frac{\ell c}{\sqrt{2}b}\tau\,, \tag{2.212}$$

and set the currents to

$$\mathfrak{L} = \frac{\ell^4 c^2}{16}(r_+ - r_-)^2\,, \qquad \mathcal{P} = -\sqrt{\frac{k\alpha}{2}}\frac{\kappa}{16\pi}\ell^2 bc\left(r_+ + r_- - \sqrt{\frac{c\,r_+ r_-}{b^2}}\right)\,. \tag{2.213}$$

The final metric is

$$\begin{aligned}\frac{\mathrm{d}s^2}{\ell^2} = &\,\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{(\nu^2+3)(r-r_+)(r-r_-)} + \left(2\nu r - \sqrt{r_+ r_-(\nu^2+3)}\right)\mathrm{d}\tau\,\mathrm{d}\phi \\ &+ \frac{r}{4}\left(3(\nu^2-1)\,r + (\nu^2+3)(r_+ - r_-) - 4\nu\sqrt{r_+ r_-(\nu^2+3)}\right)\mathrm{d}\phi^2\,.\end{aligned} \tag{2.214}$$

It is the warped black hole in the canonical ensemble (1.13).

Chapter 3

Uniformization of entanglement entropy in holographic warped conformal field theories

The stress-tensor $T_{\mu\nu}$ of a Quantum Field Theory (QFT) dictates, through Einstein's equations, constraints on the geometry arising semi-classically when coupling gravity to matter described by this QFT. Various energy conditions on $T_{\mu\nu}$ can be formulated, expressing, for instance, the positivity of energy density (Weak Energy Condition) or the causal propagation of energy flow (Dominant Energy Condition). A weaker energy condition is the Null Energy Condition (NEC).

$$T_{\mu\nu}k^\mu k^\nu \geq 0 \qquad \forall k^\mu \mid k^\mu k_\mu = 0 \tag{3.1}$$

The proofs of the black hole area law [68] or singularity theorems [69] crucially rely on the NEC. This condition, however, is violated quantum-mechanically, e.g. in the Casimir effect or by Hawking radiation. Instead, quantum mechanically QFTs typically satisfy non-local conditions such as the Averaged NEC (ANEC) (see e.g. [111, 112] for recent proofs and refs. therein), which states that negative energy fluxes along null directions are compensated by positive energy fluxes (with "quantum interest" [113]).

The Quantum Null Energy Condition (QNEC) [70] is a local energy condition conjectured to extend the NEC to the quantum regime, and has attracted a lot of attention in recent years [114–121], including proofs for free QFTs [122], for holographic Conformal Field Theories (CFTs) [123], then for general CFTs [71], and shown to hold universally for generic QFTs under the same assumptions required for the averaged NEC [124]. For two-dimensional CFTs (CFT_2), QNEC reads [70, 125]

$$2\pi \langle T_{\mu\nu}k^\mu k^\nu\rangle \geq S'' + \frac{6}{c}\, S'^{\,2} \qquad \forall k^\mu \mid k^\mu k_\mu = 0\,, \tag{3.2}$$

where c is the central charge of the CFT, $\langle T_{\mu\nu}k^\mu k^\nu\rangle$ the expectation values of the null projections of the stress tensor for a given state, and S is the entanglement entropy (EE) for an arbitrary interval of this state; prime denotes variations of EE with respect to null deformations in the null direction defined by k^μ of one of the endpoints of the entangling region.

In the context of AdS_3/CFT_2, it was shown that QNEC saturates not only for the vacuum, for states dual to particles on AdS_3 or BTZ black holes, or for any state that is a Virasoro descendant thereof [120], but also for all states dual to Bañados geometries [126], some of which describe systems far from thermal equilibrium [121]. This was done by exploiting the fact that all Bañados geometries are locally AdS_3 and using a uniformization map between Poincaré AdS_3 and the Bañados geometries [127].

Following a similar strategy, a Quantum Energy Condition was derived recently [128] for a class of non-Lorentz invariant holographic theories with BMS_3 symmetries, through a uniformization map between Minkowski space and the flat version of Bañados geometries [129], yielding inequalities involving the supertranslation and superrotation fields instead of the CFT stress-tensor.

Importantly for this work, universal expressions for EE in WCFTs were derived holographically [48, 50, 51].

In this chapter, we take the first steps towards extending these results to another class of non-relativistic theories, Warped Conformal Field Theories (WCFTs) [42]. We first review the results for AdS_3 gravity with Brown–Henneaux boundary conditions and the derivation of the saturated version of (3.2) for Bañados geometries. We then turn to a simple holographic model for WCFTs, consisting in pure Einstein gravity in $2+1$ dimensions with a negative cosmological constant and chiral/Compère–Song–Strominger (CSS) boundary conditions [55]. The role of the Bañados geometries there is played by a gauge-fixed and on-shell version of the CSS boundary conditions, referred to as CSS geometries. We determine a uniformization map that allows us to derive EE for states of a WCFT dual to these geometries. We express components of the holographic stress tensor in a form reminiscent of the saturated form of QNEC (3.2). Finally, we present an unsuccessful attempt to derive unsaturated equations for the energy conditions based on the strategy applied in [120] for QNEC (3.2).

3.1 Saturated QNEC for holographic CFT_2

In this section we review a holographic derivation of the saturated QNEC for CFT_2. In AdS_3 gravity with Brown–Henneaux boundary conditions, the most general vacuum solution in Fefferman-Graham gauge is the Bañados metric [126] (see also [127, 130])

$$\frac{\mathrm{d}s^2}{\ell^2} = \frac{\mathrm{d}z^2 - \mathrm{d}x_+\,\mathrm{d}x_-}{z^2} + L_+\,\mathrm{d}x_+^2 + L_-\,\mathrm{d}x_-^2 - z^2 L_+ L_-\,\mathrm{d}x_+\,\mathrm{d}x_-\,, \tag{3.3}$$

where $L_\pm = L_\pm(x_\pm)$ and ℓ is the AdS radius.

The expectation values of the stress tensor are related to the functions $L_\pm$ present in the Bañados metric (3.3) by

$$2\pi \langle T_{\pm\pm}\rangle = \frac{c}{6} L_\pm \,, \tag{3.4}$$

where c is the Brown–Henneaux central charge [10]. The Poincaré patch is just a special case where $L_\pm = 0$ in (3.3). As the Bañados metric is locally AdS$_3$, there is a mapping for the Poincaré patch to (3.3) [131, 132]:

$$x_P^\pm = \int \frac{\mathrm{d}x_\pm}{\psi^{\pm 2}} - \frac{z^2 \psi^{\mp\prime}}{\psi^\pm \psi^\mp (1 - z^2/z_h^2)} \,, \tag{3.5a}$$

$$z_P = \frac{z}{\psi^+ \psi^- (1 - z^2/z_h^2)} \,, \tag{3.5b}$$

where the functions $\psi^\pm(x_\pm)$ satisfy Hill's equation:

$$\psi^{\pm\prime\prime} - L_\pm \psi^\pm = 0 \tag{3.6}$$

and z_h is one of the Killing horizons of the Bañados metric (3.3). Expressing the two independent solutions of Hill's equation by $\psi^\pm_{1,2}$, it is convenient to normalize them as

$$\psi_1^\pm \psi_2^{\pm\prime} - \psi_1^{\pm\prime} \psi_2^\pm = \pm 1 \,. \tag{3.7}$$

From this diffeomorphism, we can find the holographic EE of the Bañados metric starting from the one of the Poincaré patch [54]:

$$S_{PP} = \frac{c}{3} \ln \frac{l}{\epsilon} = \frac{c}{6} \ln \left(\frac{(x_1^+ - x_2^+)(x_1^- - x_2^-)}{\epsilon^2} \right) \,, \tag{3.8}$$

where $x^\pm_{1,2}$ are the boundary points of the entangling interval l. It is sufficient to know the near boundary behaviour of (3.5) to compute the holographic EE. Close to the boundary, one has the conformal transformation

$$x_P^\pm = \frac{\psi_1^\pm}{\psi_2^\pm} \,, \qquad z_p = \frac{z}{\psi_2^+ \psi_2^-} \,, \tag{3.9}$$

and one finds [132]

$$S_{\text{HEE}} = \frac{c}{6} \ln \frac{l^+(x_1^+, x_2^+) l^-(x_1^-, x_2^-)}{\epsilon^2} =: S_+ + S_- \,, \tag{3.10}$$

with

$$l^\pm(x_1^\pm, x_2^\pm) = \psi_1^\pm(x_1^\pm)\psi_2^\pm(x_2^\pm) - \psi_1^\pm(x_2^\pm)\psi_2^\pm(x_1^\pm) \,. \tag{3.11}$$

We are now in position to show that the Bañados geometries saturate QNEC [121]. Defining the 'vertex function'

$$V := \exp\left(\frac{6}{c} S\right) = \frac{l^+(x_1^+, x_2^+)\, l^-(x_1^-, x_2^-)}{\epsilon^2} \,, \tag{3.12}$$

it is straightforward to show that it satisfies Hill's equation (3.6),

$$V'' = L_{\pm} V \,. \tag{3.13}$$

On the other hand, the definition (3.12) implies

$$\frac{V''}{V} = \frac{6}{c}\left(S'' + \frac{6}{c} S'^2\right). \tag{3.14}$$

Now, using the relation between the stress tensor and the functions $L_{\pm}$, proves that for the Bañados metric the QNEC inequality (3.2) saturates.

$$2\pi \left\langle T_{\pm\pm} \right\rangle = S''_{\pm} + \frac{6}{c}\, S'^2_{\pm} \,, \tag{3.15}$$

3.2 Holographic WCFT model

We take as our holographic model AdS_3 gravity (again with AdS radius ℓ) with CSS boundary conditions [55]. The counterpart of the Bañados metric is given by

$$\begin{aligned}\frac{\mathrm{d}s^2}{\ell^2} = \frac{\mathrm{d}z^2}{z^2} &- \left(\frac{1}{z^2} + \frac{2\Delta}{k} P' + \frac{\Delta}{k^2} L z^2\right) \mathrm{d}t_+ \, \mathrm{d}t_- + \frac{\Delta}{k}\, \mathrm{d}t_-^2 \\ &+ \left(\frac{P'}{z^2} + \frac{1}{k}(L + \Delta\ P'^2) + \frac{\Delta}{k^2} L\ P'\ z^2\right) \mathrm{d}t_+^2 \,,\end{aligned} \tag{3.16}$$

where $k = \ell/4G$, Δ is a constant and the functions $P' =: \partial_+ P$ and L depend on t_+ only. If the latter functions vanish, we recover an extremal BTZ black hole with $\ell M - J = 0$ and $\ell M + J = \Delta$ [55].

The CSS boundary conditions in (z, t_+, t_-)-coordinates [55] read

$$\begin{aligned} g_{zz} &= \frac{\ell^2}{z^2} + \mathcal{O}(1)\,, \\ g_{z\pm} &= \mathcal{O}(z)\,, \\ g_{+-} &= -\frac{\ell^2}{2z^2} + \mathcal{O}(1)\,, \\ g_{++} &= \frac{P'\ell^2}{z^2} + \mathcal{O}(1)\,, \\ g_{--} &= 4G\ell\Delta + \mathcal{O}(z)\,, \end{aligned} \tag{3.17}$$

where the boundary is located at $z = 0$. The points $(z, t^+, t^-) \sim (z, t^+ + 2\pi, t^- - 2\pi)$ are identified. In Fefferman-Graham coordinates the boundary conditions (3.17) read

$$\mathrm{d}s^2 = \frac{\ell^2 \mathrm{d}z^2}{z^2} + \frac{\ell^2}{z^2}\left(g^{(0)}_{ab} + z^2 g^{(2)}_{ab} + \mathcal{O}(z^3)\right) \mathrm{d}x^a \mathrm{d}x^b \,, \tag{3.18a}$$

where $x^a = (t^+, t^-)$ and $g^{(0)}_{ab}$ and $g^{(2)}_{ab}$ read in $\{+,-\}$-coordinates

$$g^{(0)}_{--} = 0\,, \qquad g^{(0)}_{++} = P'\,, \qquad g^{(0)}_{-+} = g^{(0)}_{+-} = -\frac{1}{2}\,, \tag{3.18b}$$

$$g^{(2)}_{--} = \frac{4G}{\ell}\Delta\,. \tag{3.18c}$$

We will assume in the following that $\Delta > 0$. The asymptotic boundary is at $z = 0$; to map it to $r = +\infty$, we use from now on $r = 1/z$ as radial coordinate.

Asymptotically the CSS metric (3.16) has a Virasoro–Kač–Moody algebra symmetry that acts at the boundary as

$$\begin{aligned} t_+ &\to f(t_+)\,, \\ t_- &\to t_- + g(t_+)\,, \end{aligned} \tag{3.19}$$

and is generated infinitesimally by the asymptotic Killing vectors

$$\begin{aligned} \xi(\epsilon) &= \epsilon(t_+)\partial_+ - \frac{r}{2}\epsilon'(t_+)\partial_r + \text{subleading}\,, \\ \eta(\sigma) &= \sigma(t_+)\partial_- + \text{subleading}\,, \end{aligned} \tag{3.20}$$

The corresponding charges generating the Virasoro–Kač–Moody algebra are given by

$$\begin{aligned} Q_\epsilon &= \frac{1}{2\pi}\int \mathrm{d}\phi\; \epsilon(t_+)\left(L - \Delta P'^2\right)\,, \\ Q_\sigma &= \frac{1}{2\pi}\int \mathrm{d}\phi\; \sigma(t_+)\left(\Delta + 2\Delta P'\right)\,. \end{aligned} \tag{3.21}$$

By setting $\epsilon = e^{int_+}$, $\sigma = e^{int_+}$ and defining $L_n = Q_\epsilon$, $P_n = Q_\sigma$, we can write the algebra given by the Dirac bracket

$$\begin{aligned} i[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n^3\delta_{n+m}\,, \\ i[L_n, P_m] &= -mP_{n+m}\,, \\ i[P_n, P_m] &= -2P_0 n\,\delta_{n+m}\,, \end{aligned} \tag{3.22}$$

with

$$c = \frac{3\ell}{2G}\,, \qquad P_0 = \Delta\,. \tag{3.23}$$

We recognize the algebra of a WCFT in the quadratic ensemble (2.66) where the Kač-Moody level is charge-dependant.

3.3 Energy-momentum-tensor from well-defined variational principle

3.3.1 Variational principle

The variation of the action after addition of the Gibbons-Hawking term and standard counter term reads [55]

$$\delta S_0 = \frac{\ell}{16\pi G}\int \mathrm{d}^2x\sqrt{-g_{(0)}}\left(g_{(2)}^{ab} - g_{(0)}^{kl}g_{(2)kl}g^{(0)ab}\right)\delta g_{(0)ab}\,. \tag{3.24}$$

Here, indices are raised and lowered with the inverse metric

$$g_{(0)}^{ab} = \begin{pmatrix} 0 & -2 \\ -2 & -4P' \end{pmatrix}\,. \tag{3.25}$$

In our case, this reduces to

$$\delta S_0 = \frac{\ell}{16\pi G}\int \mathrm{d}t^+\mathrm{d}t^-\sqrt{-g_{(0)}}g_{(2)}^{ab}\delta g_{(0)ab}\,. \tag{3.26}$$

However, the variation of the action does not vanish due to the fact that $\delta g_{(0)++} \neq 0$. In particular, the variation of the action gives

$$\delta S_0 = \frac{\Delta}{2\pi}\int \mathrm{d}t^+\mathrm{d}t^-\delta P'\,. \tag{3.27}$$

Here $k = \ell/(4G)$. This can be remedied by addition of a boundary term [55]

$$S_1 = \frac{\Delta}{4\pi}\int \mathrm{d}t^+\mathrm{d}t^-\sqrt{-g_{(0)}}g_{(0)}^{--} = -\frac{\Delta}{2\pi}\int \mathrm{d}t^+\mathrm{d}t^- P'\,. \tag{3.28}$$

In this way, since $\delta\Delta = 0$

$$\delta S = \delta S_0 + \delta S_1 = 0\,. \tag{3.29}$$

The boundary term S_1 is not written in a covariant manner. To write (3.28) covariantly, we introduce the vector

$$k_a = \delta_a^-\,, \tag{3.30}$$

with which we can write (3.28) equivalently as

$$S_1 = \frac{\Delta}{4\pi}\int \mathrm{d}t^+\mathrm{d}t^-\sqrt{-g_{(0)}}g^{ab}k_a k_b\,. \tag{3.31}$$

3.3.2 Energy-momentum tensor

To find the energy momentum tensor, we must vary the entire action. The variation of S_0 gives (3.24). For the variation of S_1, we have

$$\begin{aligned}
\delta S_1 &= \frac{\Delta}{4\pi}\int \mathrm{d}t^+\mathrm{d}t^-\delta\left(\sqrt{-g_{(0)}}g_{(0)}^{--}\right) \\
&= \frac{\Delta}{4\pi}\int \mathrm{d}t^+\mathrm{d}t^-\sqrt{-g_{(0)}}\left(\delta g_{(0)}^{--} + \frac{1}{2}g_{(0)}^{ab}\delta g_{(0)ab}g_{(0)}^{--}\right) \\
&= \frac{\Delta}{4\pi}\int \mathrm{d}t^+\mathrm{d}t^-\sqrt{-g_{(0)}}\left(-g_{(0)}^{a-}g_{(0)}^{b-} + \frac{1}{2}g_{(0)}^{--}g_{(0)}^{ab}\right)\delta g_{(0)ab}\,.
\end{aligned} \tag{3.32}$$

From this, it follows that

$$\begin{aligned}\delta S &= \delta S_0 + \delta S_1 \\ &= \int \mathrm{d}t^+ \mathrm{d}t^- \sqrt{-g_{(0)}} \left(\frac{\ell}{16\pi G} g^{ab}_{(2)} - \frac{\Delta}{4\pi} g^{a-}_{(0)} g^{b-}_{(0)} + \frac{\Delta}{8\pi} g^{--}_{(0)} g^{ab}_{(0)} \right) \delta g_{(0)ab} \\ &= \frac{1}{2} \int \mathrm{d}t^+ \mathrm{d}t^- \sqrt{-g_{(0)}} T^{ab} \delta g_{(0)ab} \,,\end{aligned} \tag{3.33}$$

with

$$T^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{\pi}(L - \Delta P'^2) \end{pmatrix} . \tag{3.34}$$

We can also vary the action with respect to k_μ and obtain

$$\delta_k S = \frac{1}{2} \int \mathrm{d}t^+ \mathrm{d}t^- \sqrt{-g_{(0)}} \left(\frac{\Delta}{\pi} g^{ab}_{(0)} k_a \right) \delta k_b \equiv \frac{1}{2} \int \mathrm{d}t^+ \mathrm{d}t^- \sqrt{-g_{(0)}} J^a \delta k_a \,. \tag{3.35}$$

Here, we have defined J^a as

$$J^a = \frac{\Delta}{\pi} g^{ab}_{(0)} k_b = -\frac{2\Delta}{\pi} \begin{pmatrix} 1 \\ 2P' \end{pmatrix} . \tag{3.36}$$

The vector k_a can be used to project out the relevant components of T^{ab} and J^a

$$T^{ab} k_a k_b = 2 \frac{L - \Delta P'^2}{\pi} \,, \tag{3.37}$$

and

$$J^a k_a = -\frac{4\Delta}{\pi} P' \,. \tag{3.38}$$

3.4 Uniformized warped entanglement entropy

In this section, we derive the EE expressions for the family of metrics (3.16) after deriving a warped version of the uniformization procedure reviewed above.

3.4.1 Subleading terms

We first derive the explicit form of the infinitesimal diffeomorphism (3.20). The infinitesimal transformations leaving (3.16) invariant are of the form [129, 130]:

$$\chi^r = r\, \sigma(r) \,, \qquad \chi^a = \epsilon^a(x^b) - \ell^2 \partial_b \sigma \int_r^\infty \frac{\mathrm{d}r'}{r'} \gamma^{ab}(r', x^a) \,, \tag{3.39}$$

with

$$\mathrm{d}s^2 = \ell^2 \frac{\mathrm{d}r^2}{r^2} + \gamma_{ab}(r, x^c)\, \mathrm{d}x^a\, \mathrm{d}x^b \,, \tag{3.40}$$

and where ϵ is a conformal Killing vector at the $r = \infty$ boundary and σ is the Weyl factor of ϵ.

The explicit results

$$\begin{aligned}\chi^r &= -\frac{r}{2}\epsilon' \,,\\ \chi^+ &= \epsilon(t_+) + \frac{k\Delta\epsilon''}{2(k^2r^4 - L\Delta)}\,,\\ \chi^- &= \sigma(t_+) + \frac{k(kr^2 + P'\Delta)\epsilon''}{2(k^2r^4 - L\Delta)}\,,\end{aligned} \tag{3.41}$$

yield finite variations of the functions defining the physical state

$$\begin{aligned}\delta_\chi L &= 2\epsilon' L + \epsilon L' - \frac{k}{2}\epsilon''' \,,\\ \delta_\chi P' &= (\epsilon P' - \sigma)' \,.\end{aligned} \tag{3.42}$$

We recover the same infinitesimal transformation for L as in the Bañados metric. This is expected, since in both cases there is an underlying Virasoro symmetry. The transformation of P', however, is not governed by Virasoro symmetries; instead, it is governed by $\mathfrak{u}(1)$ Kač–Moody symmetries.

3.4.2 Uniformization from extremal BTZ

In the boundary conditions of [55], Δ is a fixed constant, and one cannot reach a metric (3.16) with $\Delta \neq 0$ from one with $\Delta = 0$, in particular Poincaré AdS_3.[1] Therefore, let us consider a CSS metric with $\Delta \neq 0$ and vanishing P' and L.

$$\frac{\mathrm{d}s_P^2}{\ell^2} = \frac{\mathrm{d}u^2 - \mathrm{d}y_+\,\mathrm{d}y_-}{u^2} + \frac{\Delta}{k}\,\mathrm{d}y_-^2 \,. \tag{3.43}$$

The change of coordinates between (3.43) and (3.16) is given by

$$y_+ = \int \frac{\mathrm{d}t_+}{\psi^2} - \frac{\Delta\psi' z^4}{\psi(k\psi^2 - \Delta\psi'^2 z^4)}\,, \tag{3.44}$$

$$y_- = t_- - C(t_+) - \sqrt{\frac{k}{\Delta}}\,\mathrm{artanh}\left(\sqrt{\frac{\Delta}{k}}\frac{\psi'}{\psi}z^2\right)\,, \tag{3.45}$$

$$u = \frac{\sqrt{k}z}{\sqrt{k\psi^2 - \Delta\psi'^2 z^4}}\,, \tag{3.46}$$

[1] This is related to the fact that the CSS boundary conditions are dual to a WCFT in the so-called quadratic ensemble, in which the level is $U(1)$ charge-dependent, the latter not being able to vary over phase space – here the zero mode $U(1)$ charge is given by Δ. There exist alternative boundary conditions yielding a constant level and a varying zero mode $U(1)$ charge, see Appendix of [55] or the AdS_3 limit of $WAdS_3$ boundary conditions of [39]. These are naturally dual to a WCFT in the canonical ensemble. See e.g. [43] for a discussion on the relation between the two ensembles.

where $C(t_+)$ and $\psi(t_+)$ are given by the warped analogue of Hill's equation.

$$P'(t_+) = C'(t_+)\,, \qquad \psi'' - \frac{L(t_+)}{k}\psi = 0\,. \tag{3.47}$$

3.4.3 Uniformized entanglement entropy

EE for a WCFT in a state dual to (3.43) is given by [48, 50, 51, 66][2]

$$S_{\text{EE}} = -\sqrt{\Delta k}(y_1^- - y_2^-) + k \ln\left[\sqrt{\frac{k}{\Delta}}\frac{y_1^+ - y_2^+}{2\epsilon^2}\right]\,, \tag{3.48}$$

where $y_i^\pm$ are the endpoints of the interval and ϵ a UV cut-off.

Performing the diffeomorphism (3.45) yields the EE of the WCFT state dual to the CSS metric

$$S_{\text{EE}} = S_P + S_L\,, \tag{3.49}$$

where we separated the entropy in a Kač-Moody part S_P and a Virasoro part S_L

$$\begin{aligned} S_P &= -\sqrt{\Delta k}(t_1^- - t_2^- - C(t_1^+) + C(t_2^+))\,, \\ S_L &= \frac{c}{6}\ln\left[\sqrt{\frac{k}{\Delta}}\frac{l^+(t_1^+, t_2^+)}{2\epsilon^2}\right]\,, \end{aligned} \tag{3.50}$$

and used the same normalization for ψ than in the CFT case and (3.11).

3.4.4 Entanglement entropy and warped conformal transformations

The expression for the uniformized warped entanglement entropy that we derived previously can also be used to understand the transformation behavior of the EE under finite and infinitesimal warped conformal transformations. To do this, we take (3.48) and perform a finite warped conformal transformation of the entangling intervals of the form (3.19). This yields

$$\begin{aligned} S_P &= -\sqrt{\Delta k}\left(y_1^- - y_2^- + g(y_1^+) - g(y_2^+)\right)\,, \\ S_L &= k\ln\left[\sqrt{\frac{k}{\Delta}}\frac{\left(f(y_1^+) - f(y_2^+)\right)}{2\epsilon^2\sqrt{f'(y_1^+)f'(y_2^+)}}\right]\,, \end{aligned} \tag{3.51}$$

where we have also taken into account the rescaling of the UV cutoff ϵ under this transformation as $\epsilon^2 \to \epsilon^2\sqrt{f'(y_1^+)f'(y_2^+)}$ that can be read of from the leading order term of (3.46).

[2] To recover the metric (3.43) from the section 5 in [66], one has to choose the parameters as $\alpha = 4/\Delta$, $b = 1/\sqrt{2}$, $c = 2$, makes the change of radial coordinate $e^{-\rho} = \sqrt{\Delta/k}z^2$ and sets the functions $\mathfrak{L}$ and $\mathcal{K}$ to zero.

Now, looking at infinitesimal warped conformal transformations of the form

$$y^+ \to y^+ + \varepsilon(y^+), \qquad y^- \to y^- + \sigma(y^+), \tag{3.52}$$

it is straightforward to work out the infintesimal transformation properties of S_P and S_L. These are given by

$$\delta S_P = S'_P \sigma, \qquad \delta S_L = S'_L \varepsilon - \frac{c}{12}\varepsilon', \tag{3.53}$$

where $c = 6k$. From this one can see that the $\mathfrak{u}(1)$ part of the entanglement entropy transforms like a weight-0 scalar and the $\mathfrak{sl}(2)$ part like an anomalous weight-0 scalar, which is not suprising given the underlying structures and symmetries of a WCFT.

3.4.5 WCFT saturation equations

The AdS_3 stress tensor [133–136] for the CSS boundary conditions reads [3]

$$\frac{12\pi}{c}\langle T_{ab}\rangle = g^{(2)}_{ab} - g^{kl}_{(0)} g^{(2)}_{kl} g^{(0)}_{ab} = \begin{pmatrix} \frac{L}{k} + \frac{\Delta}{k}P'^2 & -\frac{\Delta}{k}P' \\ -\frac{\Delta}{k}P' & \frac{\Delta}{k} \end{pmatrix}. \tag{3.54}$$

Using (3.49), its components are shown to satisfy

$$\begin{aligned} 2\pi\langle T_{++}\rangle &= S''_L + \frac{6}{c}(S'^2_L + S'^2_P)\,, \\ 2\pi\langle T_{+-}\rangle &= \frac{6}{c}S'_P\dot{S}_P\,, \\ 2\pi\langle T_{--}\rangle &= \frac{6}{c}\dot{S}^2_P\,, \end{aligned} \tag{3.55}$$

where *prime* denotes a derivation with respect to t_+ and the *dot* a derivation with respect to t_-. Another set of relations can be derived in terms of the currents responsible for the Virasoro-Kač-Moody charges (3.21). Defining

$$2\pi\langle T_L\rangle = L - \Delta P'^2 \quad , \quad 2\pi\langle T_P\rangle = \Delta + 2\Delta P'\,, \tag{3.56}$$

one has

$$\begin{aligned} 2\pi\langle T_L\rangle &= S''_L + \frac{6}{c}(S'^2_L - S'^2_P)\,, \\ 2\pi\langle T_P\rangle &= \frac{6}{c}(\dot{S}^2_P - 2S'_P\dot{S}_P)\,. \end{aligned} \tag{3.57}$$

[3] As argued in [57], the background geometry to which WCFTs couple is not Riemannian, but rather has Newton–Cartan structure. From this perspective, the so-defined stress tensor is not the most natural object to consider, but for now we take (3.54) as a useful way to repackage the WCFT currents generating the conserved charges.

These equalities are WCFT analogues of the QNEC saturation equations (3.15). If one defines a second constant boundary covector

$$\tilde{k}_a = \delta_a^+ \,, \tag{3.58}$$

then one obtains additionally

$$-\frac{1}{k}\dot{S}_P^2 = -\Delta = \frac{\pi}{2} J^a \tilde{k}_a \,. \tag{3.59}$$

To bring everything in a covariant form, we notice that

$$k^a \partial_a S_L = -2S_L' \,, \quad \tilde{k}^a \partial_a S_L = 0 \,, \quad k^a \partial_a S_P = 2\sqrt{\Delta k} P' \,, \quad \tilde{k}^a \partial_a S_P = 2\sqrt{\Delta k} \,. \tag{3.60}$$

Now, it is possible to write down fully covariant saturation equations:

$$\begin{aligned} 2\pi T^{ab} k_a k_b &= k^a k^b \partial_a \partial_b S_L + \frac{1}{k}\left((k^a \partial_a S_L)^2 - (k^a \partial_a S_P)^2 \right) . \\ \frac{\pi}{2} J^a k_a + \frac{\pi}{2} J^a \tilde{k}_a &= \frac{2}{k}\left(\tilde{k}^a \partial_a S_P \right) (k^a \partial_a S_P) - \frac{1}{k}\left(\tilde{k}^a \partial_a S_P \right)^2 , \end{aligned} \tag{3.61}$$

3.5 To an unsaturated version of the QEC for WCFT ?

After defining saturated equations, the natural next step is to prove their non-saturation in a precise context. As we have shown, every vacuum solution satisfying the CSS boundary conditions (3.16) will saturate the equations (3.61). Therefore, we need non-vacuum solutions satisfying the boundary conditions (3.17). In the AdS/CFT case, it was proposed in [120] to apply a shockwave in the bulk sourced by bulk matter. The strategy is to use the holographic correspondence to compute EE using the Ryu-Takayanagi (RT) prescription [54]. If the RT surface passes through the bulk matter, the QNEC is unsaturated, as long as one requires a certain bulk energy condition, weaker than the NEC but stronger than the ANEC, which can be derived from it by sending the endpoints of the interval to infinity. However, this strategy will face some difficulties for WCFT.

To derive the saturated equations (3.61), we separated the EE into two parts. These separations are quite natural from (3.49) and (3.51), but for a generic metric satisfying the CSS boundary conditions, it is not obvious. In AdS, the question does not arise since the dependence on both coordinates $x^\pm$ decouples from each other (3.10). For flat spacetime [128], the same issue appears, but it is possible to evade this question by using a flat limit from the AdS/CFT result. For WCFT, such a limit does not exist. One could suggest the limit $\nu \to 1$ mentioned in Chapter 1, but this limit allows us to transition from WAdS to AdS, not the other way around.

Furthermore, the RT recipe relies on the computation of extremal surfaces, which are geodesics in $2+1$ dimensions. For gravity duals of WCFT, a simple massive geodesic is not sufficient to express EE. In [52], an extension to massive spinning geodesics was proposed, while in [50], it was suggested that the spinless geodesic is not anchored at the boundary. One could also mention [51], where the world-line action of a massive charged particle was used, and [53] introduced the concept of swing surfaces. In Chern-Simons formalism, there also exists a method, using Wilson lines, to compute EE (see [137, 138] for AdS space holography, [139, 140] for flat holography or [66] for WAdS holography). One could then use a Chern-Simons formulation of CSS boundary conditions. The choice between these different holographic prescriptions of EE for WCFT provides a versatile toolbox for selecting the appropriate solution to our problem. However, it also lengthens the time required to explore each option and determine the most suitable one.

All these reasons explain why we did not succeed in proving non-saturated QNEC for WCFT. Nevertheless, this does not imply that the problem is unsolvable. Anyone interested in computing non-saturated QNEC for WCFT is encouraged to explore and utilize the toolbox described earlier.

Chapter 4

Coadjoint orbits of the Warped Virasoro group and $SL(2,\mathbb{R}) \times U(1)$ Chern-Simons reduction

In 1995, Coussaert, Henneaux, and van Driel [141] performed a reduction of the Chern-Simons action for AdS_3 under Brown-Henneaux boundary conditions [10]. Their results showed that the action can be expressed as two Wess-Zumino-Witten (WZW) chiral bosons [142–144]. A WZW model is a particular two-dimensional σ-model where the fields g live on a semi-simple Lie group manifold [145]. After implementing all the Brown-Henneaux boundary conditions in terms of the gauge connection, it reduces to a Liouville action on the asymptotic boundary. This action originally represented the equation of motion for the Liouville differential equation, which dates back to the 19th century and was introduced in the context of the uniformization theorem for Riemann surfaces [146–148]. In 1981, Polyakov [149] proposed that the Liouville differential equation describes the equation of motion for the quantum field theory encountered in string theory, particularly in relation to the transformation of the path integral measure under Weyl rescaling.

This all procedure is called the Hamiltonian reduction of the WZW model to Liouville theory [150–152]. To include Bañados-Teitelboim-Zanelli (BTZ) black holes [30] in this framework, one must allow the gauge field to carry holonomies [72, 75]. A fixed-time slice of a BTZ black hole has the topology of an infinite hollow cylinder. The actions on each boundary are coupled through their shared holonomy.

Although Liouville theory is a CFT, it does not serve as the quantum theory of AdS_3 because the derivation mentioned earlier is classical. The reduction from a WZW model to Liouville was performed at the quantum level and is known as Drinfeld-Sokolov Hamiltonian reduction [150, 153, 154]. This process reproduces the BTZ spectrum and the Brown-Henneaux central charge only within the semi-classical approximation. Consequently, Liouville theory can be regarded solely as an effective theory of the holographic dual CFT_2 [74].

In the 1980s, an alternative formulation of this reduced action was found using the symplectic 2-form, also known as the Kirillov-Kostant form, which can be defined on a coadjoint orbit of the central extension of the group of diffeomorphisms of the circle, the Virasoro group [152]. The orbit representatives are connected to the holonomies of the gauge fields. Asymptotically locally AdS_3 spacetimes solutions are Bañados geometries [155], parameterized by two functions representing the expectation values of the dual stress tensor. These expectations transform in the coadjoint representation of the Virasoro group [76], linking Bañados geometries to the coadjoint orbits of the Virasoro group [127, 154, 156, 157]. The orbit representatives match the global charges of the bulk geometry.

These constructions have been successfully adapted for other geometries, such as flat spacetimes [73, 158] and dS_3 spacetimes [159]. Other applications in higher spin physics and Carroll groups can be found in [160–163]. In this chapter, we demonstrate its applicability to warped spacetimes [33, 57, 66].

The reduced action in locally AdS_3 spacetimes is also related to a 1d effective theory through dimensional reduction of the action. This effective theory is known as Schwarzian theory [164], which has found applications in various contexts such as Sachdev-Ye-Kitaev (SYK) models [165, 166] and 2d Jackiw-Teitelboim (JT) dilaton gravity [167, 168]. From the dimensional reduction of the warped reduced action that we construct, a warped alternative called Warped Schwarzian theory [169] arises.

The partition function of the gravity dual of the geometric action was shown to be one-loop exact in the context of AdS_3 spacetimes [75] and flat spacetimes [73]. For these geometries, which exhibit a 2d conformal symmetry for AdS_3 and an $ISL(2,\mathbb{R})$ symmetry for BMS_3, the vacuum and its one-loop corrections suffice to derive the Virasoro character [170, 171] and the BMS character [172, 173]. We demonstrate that, through a path integral derivation, it is possible to partially recover the warped Virasoro characters [43].

The chapter will be structured as follows. The first section will explain the construction of the geometric action and its application to the Virasoro group. The second section will review well-known results found in the context of AdS/CFT. The tools mentioned in this first section will be applied to WCFTs in the final one. We will construct the coadjoint orbits, identify their orbit representatives, and develop the geometric action for the warped Virasoro group. Additionally, we will begin with a lower spin $SL(2,\mathbb{R}) \times U(1)$ Chern-Simons theory [57, 66], reviewed in Section 2.3 of this thesis, and perform a Hamiltonian reduction at the boundaries, taking into account the holonomies to establish correspondence between holonomies, global charges, and orbit representatives. Finally, we will briefly discuss the connection with Warped Schwarzian theory and conduct a one-loop computation of the partition function to derive the warped characters.

4.1 Geometric action on coadjoint orbits

As emphasized in the introduction of this chapter, various reduced actions can be formulated as a geometric action with an orbit representative. In this section, we will introduce the concept of geometric action and explain why it is meaningful to recover it at the boundary of our spacetime.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. We define the scalar product (hermitian product if the algebra is complex) as

$$\begin{aligned} \langle\ ,\ \rangle\ :\ \mathfrak{g}^*\times\mathfrak{g} &\to \mathbb{R} \quad (\text{or}\,\mathbb{C}\,\text{if hermitian product}) \\ (X,a) &\mapsto \langle X,a\rangle = X(a)\,. \end{aligned} \tag{4.1}$$

We define the adjoint representation as the action of the group G on the algebra $\mathfrak{g}$. If $g\in G$,

$$\begin{aligned} \mathrm{Ad}_g\ :\ \mathfrak{g} &\to \mathfrak{g} \\ a &\mapsto \mathrm{Ad}_g(a) = gag^{-1}\,. \end{aligned} \tag{4.2}$$

The infinitesimal version of (4.2) is the action of the algebra on itself

$$\begin{aligned} \mathrm{ad}_b\ :\ \mathfrak{g} &\to \mathfrak{g} \\ a &\mapsto \mathrm{ad}_b(a) = [b,a]\,, \end{aligned} \tag{4.3}$$

with $b\in\mathfrak{g}$. An action of the group G on the dual $\mathfrak{g}^*$ can also be defined and is called the coadjoint representation

$$\begin{aligned} \mathrm{Ad}^*_g\ :\ \mathfrak{g}^* &\to \mathfrak{g}^* \\ X &\mapsto \mathrm{Ad}^*_g(X)\,, \end{aligned} \tag{4.4}$$

where

$$\langle \mathrm{Ad}^*_g(X),a\rangle = \langle X,\mathrm{Ad}^{-1}_g(a)\rangle\,, \tag{4.5}$$

with $g\in G$, $X\in\mathfrak{g}^*$ and $a\in\mathfrak{g}$. Like for the adjoint action, there is an infinitesimal version of the coadjoint action

$$\begin{aligned} \mathrm{ad}^*_b\ :\ \mathfrak{g}^* &\to \mathfrak{g}^* \\ X &\mapsto \mathrm{ad}^*_b(X)\,, \end{aligned} \tag{4.6}$$

where

$$\langle \mathrm{ad}^*_b(X),a\rangle = \langle X,[a,b]\rangle\,, \tag{4.7}$$

with $X\in\mathfrak{g}^*$ and $a,b\in\mathfrak{g}$. For a chosen covector X_0, the coadjoint orbit $\mathcal{O}_{X_0}$ is the set of elements of $\mathfrak{g}^*$ reachable through the coadjoint action of an element $g\in G$ on X_0:

$$\mathcal{O}_{X_0} = \{X\in\mathfrak{g}^*\,|\,X=\mathrm{Ad}^*_g(X_0)\}\,. \tag{4.8}$$

The covector X_0 is called the orbit representative of $\mathcal{O}_{X_0}$. The orbit is isomorphic to the quotient G/H where H is the stabilizer subgroup, also called little group, defined as the set of elements in G leaving X_0 invariant under the adjoint action

$$H = \{g\in G\,|\,\mathrm{Ad}^*_g(X_0)=X_0\}\,. \tag{4.9}$$

All coadjoint orbits $\mathcal{O}_{X_0}$ are symplectic manifolds on which a symplectic structure can be defined. The dual space $\mathfrak{g}^*$ then foliates into symplectic leaves. We may think of these orbits as different phase spaces. As for classical systems where we can define Hamiltonian actions on the phase space, we can construct an action I, on the orbit, that is invariant under G. This action is called the geometric action.

The symplectic form, denoted as Kirillov-Kostant symplectic form [174,175], acts on tangent vectors to $\mathcal{O}_{X_0}$ on a point $\tilde{X}$. A tangent vector on $\mathcal{O}_{X_0}$ is a coadjoint vector that can be written as

$$\mathrm{ad}^*_{a_i}(X_0) = X_i \,. \tag{4.10}$$

We can then define a bilinear form acting on tangent vectors at a point $\tilde{X}$

$$\begin{aligned} \omega_{\tilde{X}} \,:\, \mathfrak{g}^* \times \mathfrak{g}^* &\to \mathbb{R} \quad (\text{or}\, \mathbb{C}) \\ (X_1, X_2) &\mapsto \omega_{\tilde{X}}(X_1, X_2) = \langle \tilde{X}, [a_1, a_2] \rangle \,. \end{aligned} \tag{4.11}$$

This bilinear form is nondegenerate and closed, as required for a symplectic form. Since it is closed, it is locally exact and one can find a 1-form α such that

$$\omega = \mathrm{d}\alpha \,. \tag{4.12}$$

It allows us to define an action I by integrating the 1-form α along a path γ on the orbit $\mathcal{O}_{X_0}$

$$I = \int_\gamma \alpha \,. \tag{4.13}$$

This action is the geometric action on a coadjoint orbit of representative X_0. As the orbit is isomorphic to G/H, it gives to the action an additional gauge symmetry. We can then add to the geometric action the integration of an element $L_0 \in H$ along the path without breaking the symmetry.

The groups we are going to work with are centrally extended $\tilde{G} = G \times \mathbb{R}^n$, where the number n depends on the dimension of the second cohomology space of G [176,177]. For the Virasoro group, $n = 1$, while for the Warped Virasoro group, $n = 3$ [178]. The algebra is also extended $\tilde{\mathfrak{g}} = \mathfrak{g} \times \mathbb{R}^n$. Let us see how the definitions above are impacted under the adding of a central extension for $n = 1$. The cases for $n > 1$ follow similarly. A group element of $\tilde{G}$ is now a pair (g, n) where $g \in G$ and n is a number. The group multiplication is defined as

$$(g_1, n_1) \cdot (g_2, n_2) = (g_1 \cdot g_2, n_1 + n_2 + C(g_1, g_2)) \tag{4.14}$$

The function $C(g_1, g_2)$ is the 2-cocycle of the group G[1] and satisfies the property

$$C(g, g^{-1}) = 0 \,. \tag{4.15}$$

Thus, the inverse element in $\tilde{G}$ is

$$(g, n)^{-1} = (g^{-1}, -n) \,. \tag{4.16}$$

[1]For the readers interesting in the connection bewteen cocycles, group extensions and the second cohomology group, see [179,180]

The coadjoint action of an element (g,n) of $\tilde{G}$ on an element (X,c) of $\tilde{\mathfrak{g}}^*$ is

$$\mathrm{Ad}^*_{(g,n)}(X,c) = \left(\mathrm{Ad}^*_g(X) + c\,\gamma(g), c\right) , \tag{4.17}$$

where $\mathrm{Ad}^*_g(X)$ is given by (4.5) anf $\gamma(g)$ is the 1-cocycle of the group G. We notice that the central element n acts trivially under the coadjoint action and that the central element c is left invariant.

To proceed further, we will specify the extended group we are working with and begin this section with the Virasoro group as originally presented in [152]. Here, the group G represents the group of diffeomorphisms of the circle $\mathrm{Diff}^+(S^1)$. The circle S^1 is parametrized by the real coordinated φ with the identification

$$\varphi \sim \varphi + 2\pi\,. \tag{4.18}$$

As mentioned earlier, the centrally extended group is $\hat{G} = G \times \mathbb{R}$. Elements of the algebra are pairs $(a(\varphi), \lambda)$ where $a(\varphi)\partial_\varphi$ is a vector field and λ a number. Elements of the dual are $(X(\varphi), c)$ where $X(\varphi)(\mathrm{d}\varphi)^2$ is a quadratic differential and c is dual to the center. The product $\langle\,,\rangle$ is defined by

$$\langle (X,c),(a,\lambda)\rangle = \oint \mathrm{d}\varphi\, X(\varphi)a(\varphi) + c\,\lambda\,. \tag{4.19}$$

Example of the Virasoro group

We will now continue the discussion by considering a well-known example: the Virasoro group and its algebra. This group corresponds to the central extension of the group of diffeomorphisms of the circle, $\mathrm{Diff}^+(S^1)$, and it has the corresponding algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m}\,. \tag{4.20}$$

This algebra appears in the context of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence and is present in the asymptotic charge algebra of the transformations preserving the AdS phase space under Brown-Henneaux boundary conditions [10] and in the symmetry algebra of the dual CFT. Later, we will observe that the geometric action constructed on the coadjoint orbits of the Virasoro group matches the reduced gravity action on the boundary of an asymptotically AdS_3 spacetime.

For the Virasoro group, the 2-cocycle is given by [176, 177]

$$C(g_1,g_2) = -\frac{1}{24\pi}\oint \mathrm{d}\varphi \log(g_1'\circ g_2)\frac{g_2''}{g_2'}\,, \tag{4.21}$$

and the 1-cocycle is

$$\gamma(g) = -\frac{1}{24\pi}\{g;\varphi\}\,, \tag{4.22}$$

where $\circ$ denotes function composition, $g\circ\varphi = f(\varphi)$, and $\{g;\varphi\}$ is the Schwarzian derivative (2.28).

To compute the symplectic form (4.11) on an orbit $\mathcal{O}_{X_0}$, we need the coadjoint action on X_0 and the commutator of the algebra. One may write the adjoint action of the group $\tilde{G}$ on its algebra $\tilde{\mathfrak{g}}$ as

$$\mathrm{Ad}_{(g,n)}(a,\lambda) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,(g,n)\cdot(e^{\epsilon a},\epsilon\lambda)\cdot(g,n)^{-1}\Big|_{\epsilon=0}\,, \tag{4.23}$$

and then computes the commutator (4.3)

$$\begin{aligned}[][(a_1,\lambda_1),(a_2,\lambda_2)] &= -\frac{\mathrm{d}}{\mathrm{d}\epsilon}\,\mathrm{Ad}_{(e^{\epsilon a_1},\epsilon\lambda_1)}(a_2,\lambda_2)\Big|_{\epsilon=0} \\ &= \left(a_1a_2' - a_1'a_2, \frac{1}{48\pi}\oint \mathrm{d}\varphi(a_1'''a_2 - a_1a_2''')\right).\end{aligned} \tag{4.24}$$

The coadjoint action of the Virasoro group now reads

$$\mathrm{Ad}^*_{(f,n)^{-1}}(X,c) = \left(f'^2X(g) - \frac{c}{24\pi}\{f;\varphi\},c\right). \tag{4.25}$$

By defining

$$(\tilde{X};c) = \mathrm{Ad}^*_{(f,n)^{-1}}(X,c)\,, \tag{4.26}$$

we observe that the orbit representative transforms as

$$\tilde{X}(\varphi) = f'^2X(f) - \frac{c}{24\pi}\{f;\varphi\}\,. \tag{4.27}$$

It is interesting to precise as a remark that this is also the transformation law of the stress tensor of a CFT_2.

For a large class of orbits, specifically those corresponding to highest-weight representations[2], it is possible to choose the representative X_0 such that it is constant. However, there exist others for which providing any tractable expression for X_0 is not feasible [76, 152]. These orbits will not be discussed further. For the remaining orbits, two different cases arise. If $X_0 \neq -\frac{c}{48\pi}n^2$ for $n\in\mathbb{N}$, the stabilizer is $U(1)$ and the orbits are isomorphic to $\mathrm{Diff}(S^1)/U(1)$. If the representative is $X_0 = -\frac{c}{48\pi}n^2$, the little group is significantly larger and extends to $SL^{(n)}(2,\mathbb{R})$ [76, 152].

With everything in hand, we are now ready to compute the Kirillov-Kostant symplectic form (4.11)

$$\begin{aligned}\omega_{12} &= \langle(\tilde{X},c),[(a_1,\lambda_1),(a_2,\lambda_2)]\rangle \\ &= -\oint \mathrm{d}\varphi\,\left(f'^2X_0(a_1a_2' - a_2a_1') + \frac{c}{24\pi}\,(a_1''a_2' + \{f,\varphi\}(a_1a_2' - a_2a_1'))\right)\end{aligned} \tag{4.28}$$

We can rewrite it as a form and using the transformation $\mathrm{d}f = f'\,\mathrm{d}a$ to end up with

$$\omega = -\oint \mathrm{d}\varphi\,\left(X_0\mathrm{d}f\wedge\mathrm{d}f' + \frac{c}{48\pi}\mathrm{d}\log f'\wedge(\mathrm{d}\log f')'\right). \tag{4.29}$$

[2] This statement is alluded to in Section (4.2.7), where Virasoro characters are computed for a broad set of orbit representatives.

By solving locally the equation $\omega = \mathrm{d}\alpha$ and by integrating over a path γ along the orbit, we find

$$I[f;X_0] = \int_\gamma \oint \mathrm{d}\varphi\Big(X_0 f'\mathrm{d}f + \frac{c}{48\pi}\frac{f''\mathrm{d}f'}{f'^2}\Big)\,. \tag{4.30}$$

We can parameterize the path by $\mathrm{d} = \mathrm{d}t\,\partial_t$ and add to the action an element L_0 of the stabilizer subgroup given by the zero mode of $\tilde{X}(\varphi)$:

$$L_0 = \oint \mathrm{d}\varphi\,\tilde{X}(\varphi) = \oint \mathrm{d}\varphi\left(f'^2 X_0 + \frac{c}{48\pi}\frac{f''^2}{f'^2}\right)\,, \tag{4.31}$$

where we performed a integration by parts for the last term. The final action is then

$$I = I[f;X_0] - \int \mathrm{d}t\,L_0 = \int \mathrm{d}t\,\mathrm{d}\varphi\Big(X_0 f'\partial_- f + \frac{c}{48\pi}\frac{f''\partial_- f'}{f'^2}\Big)\,. \tag{4.32}$$

with $\partial_- = \partial_t - \partial_\varphi$.

4.2 AdS$_3$ gravity and Virasoro symmetry

In this section, we review the connections between AdS$_3$ gravity, Chern-Simons actions, Liouville theory, Schwarzian mechanics, and geometric actions. We gather classic results on the AdS$_3$ gravity phase space, as well as more recent developments such as a one-loop computation of the partition function. This review will be beneficial in the next section when we attempt to conduct a similar analysis for WAdS$_3$ gravity. The curious reader that want to learn more about the Chern-Simons formalism for AdS$_2$ and the Hamiltonian reduction of the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ Chern-Simons action with Brown-Henneaux boundary conditions can look at [74, 181, 182].

4.2.1 Chern-Simons formalism for AdS$_3$

In [183], Achucarro and Townsend proposed a new way to describe three-dimensional Einstein gravity with a negative cosmological constant Λ by reformulating it as a Chern-Simons gauge theory with gauge group $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. They showed that the action is equivalent to the difference of chiral and anti-chiral Chern-Simons action:

$$S_E[A,\bar{A}] = S_{CS}[A] - S_{CS}[\bar{A}]\,. \tag{4.33}$$

The gauge field A ($\bar{A}$) is associated with the first (second)$SL(2,\mathbb{R})$ factor and the action $S_{CS}[A]$ is the Chern-Simons action

$$S_{CS}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left[A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right]\,, \tag{4.34}$$

where k is a coupling constant called level. The gauge fields A and $\bar{A}$ are connected to the triad e and the spin connection ω by $A = \omega + e/l$ and $\bar{A} = \omega - e/l$ using the existence of the length scale l of $\Lambda = -1/l^2$. Indeed up to a boundary term, one can show than the Chern-Simons action (4.33) is equivalent to the $3d$ Einstein-Hilbert's action with $k = \frac{1}{4\pi G}$

$$S_E[A, \bar{A}] = \frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g}(R + 2\Lambda)\mathrm{d}^3x - \frac{1}{16\pi G}\int_{\partial\mathcal{M}} \omega^a \wedge e_a \,. \tag{4.35}$$

This reinterpretation of Einstein gravity in $2+1$ dimensions as a Chern-Simons action was extended to arbitrary cosmological constant in [184, 185] and even to supergravity [183, 186] and higher-spins [187, 188].

4.2.2 Brown-Henneaux boundary conditions

In 1986, Brown and Henneaux proposed boundary conditions on the metric for three dimensional gravity with $\Lambda < 0$ [10]. In Fefferman-Graham coordinates where the metric is given by [74]

$$\mathrm{d}s^2 = \frac{l^2}{r^2}\mathrm{d}r^2 + \gamma_{ij}(r, x^k)\mathrm{d}x^i\mathrm{d}x^j \,, \tag{4.36}$$

with $i = 0, 1$ and the expansion, when $r \to \infty$, $\gamma_{ij}(r, x^k) = r^2 g^{(0)}_{ij}(x^k) + O(1)$, these boundary conditions read as

$$g^{(0)}_{ij}\mathrm{d}x^i\mathrm{d}x^j = -\mathrm{d}x^+\mathrm{d}x^- \,, \tag{4.37}$$

where $x^\pm = t \pm \varphi$ are the light-cone coordinates. They showed that the asymptotic charges algebra associated to the symmetries preserving these boundary conditions are two copies of the Virasoro algebra:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m} \,, \\ \left[L_m, \bar{L}_n\right] &= 0 \,, \\ \left[\bar{L}_m, \bar{L}_n\right] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m} \,, \end{aligned} \tag{4.38}$$

where c is the Brown-Henneaux central charges

$$c = \frac{3l}{2G} \,. \tag{4.39}$$

The most general solution satisfying the conditions (4.36)(4.37) (up to trivial diffeomorphism) is the Bañados metric [155]:

$$\mathrm{d}s^2 = \frac{l^2}{r^2}\mathrm{d}r^2 - \left(r\mathrm{d}x^+ - \frac{l^2}{r}L(x^-)\mathrm{d}x^-\right)\left(r\mathrm{d}x^- - \frac{l^2}{r}\bar{L}(x^+)\mathrm{d}x^+\right) \,, \tag{4.40}$$

where L and $\bar{L}$ are two single valued arbitrary functions. We can recover well-known metrics for specific values of L and $\bar{L}$. For example empty AdS_3 in global coordinates when $L=\bar{L}=-1/4$ or massless BTZ when $L=\bar{L}=0$. For generic BTZ, it corresponds to positive values of L , $\bar{L}$ related to the mass $M=(L+\bar{L})/(4G)$ and angular momentum $J=l(L-\bar{L})/(4G)$ of the black hole. These boundary conditions on the metric can be translated on boundary conditions on the connections A and $\bar{A}$ [141]:

$$A\sim\begin{pmatrix}\frac{\mathrm{d}r}{2r} & \frac{l}{r}\bar{L}(x^+)\mathrm{d}x^+\\ \frac{r}{l}\mathrm{d}x^+ & -\frac{\mathrm{d}r}{2r}\end{pmatrix}\,,\qquad \bar{A}\sim\begin{pmatrix}-\frac{\mathrm{d}r}{2r} & \frac{r}{l}\mathrm{d}x^-\\ \frac{l}{r}L(x^-)\mathrm{d}x^- & \frac{\mathrm{d}r}{2r}\end{pmatrix}\,. \tag{4.41}$$

These conditions can be separated in two sets of conditions:

(i) $A_-=0=\bar{A}_+$ or equivalently $A_t=A_\varphi$ and $\bar{A}_t=-\bar{A}_\varphi$;

(ii) $A_+=\frac{l}{r}\bar{L}(x^+)L_1+0L_0-\frac{r}{l}L_{-1}$ and $\bar{A}_-=\frac{r}{l}L_1+0L_0-\frac{l}{r}L(x^-)L_{-1}$ where L_a are generators of the $sl(2,\mathbb{R})$ algebra defined in (A.3).

Before proceeding, let us clarify one final point. The Brown-Henneaux boundary conditions do not lead to a well-defined action principle. First, let us explicitly introduce the coordinates r, t, and φ in our action (4.34):

$$S_{CS}[A]=\frac{k}{4\pi}\int_{\mathcal{M}}\mathrm{d}^3x\ \mathrm{Tr}\left[A_r\dot{A}_\varphi-A_\varphi\dot{A}_r+2A_tF_{\varphi r}\right]\,, \tag{4.42}$$

where $F=dA+A\wedge A$ is the curvature two-form associated to the connection A and we dropped a boundary that we will take care in a few steps. Now, varying the action (4.33), one obtains

$$\delta S_E=(\mathrm{EOM})+\frac{k}{2\pi}\int_{\partial\mathcal{M}}\mathrm{d}^2x\ \mathrm{Tr}\left[A_t\delta A_\varphi-\bar{A}_t\delta\bar{A}_\varphi\right]\,. \tag{4.43}$$

The last terms does not vanish under the conditions (4.41). To ensure this, we need to add the following surface term to the action (4.33)

$$I=-\frac{k}{4\pi}\int_{\partial\mathcal{M}}\mathrm{d}^2x\ \mathrm{Tr}\left[A_\varphi^2+\bar{A}_\varphi^2\right]\,, \tag{4.44}$$

such that the final action is

$$S[A,\bar{A}]\equiv S_E+I=S_{CS}[A]-S_{CS}[\bar{A}]-\frac{k}{4\pi}\int_{\partial\mathcal{M}}\mathrm{d}^2x\ \mathrm{Tr}\left[A_\varphi^2+\bar{A}_\varphi^2\right]\,. \tag{4.45}$$

4.2.3 From Chern-Simons action to WZW action

In [143, 144], it has been shown that the Chern-Simons action with the Brown-Henneaux boundary conditions reduces to a Wess-Zumino-Witten (WZW) action on the boundary. We will rederive their results. First, let us focus on the chiral part of the action (4.45). In the last term of (4.42), we observe that A_t plays the role of a Lagrange multiplier for the constraint $F_{r\varphi} = 0$. Therefore, by solving the constraint,

$$A_i = G^{-1}\partial_i G\,, \tag{4.46}$$

where $i = r, \varphi$ and G is a $SL(2,\mathbb{R})$ group element. Here, we have assumed no holonomies.[3] Plugging this into (4.45), we obtain for the chiral part

$$S[A] = \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\ \mathrm{Tr}[g^{-1}\partial_\varphi g\, g^{-1}\partial_- g] + \frac{k}{12\pi}\int_{\mathcal{M}} \mathrm{Tr}[(G^{-1}dG)^3]\,, \tag{4.47}$$

where g is the group element G evaluated on the boundary. This action represents a chiral Wess-Zumino-Witten action $S^R_{WZW}[g]$ and describes a right-moving group element. Indeed, the solution of the equation of motion $\partial_-(g^{-1}\partial_\varphi g) = 0$ is $g = f(t)k(x^+)$, which is the equation for a group element moving along the x^+ axis.

On the other hand, we can perform the same development for the anti-chiral part of (4.45) and get

$$S[\bar{A}] = \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\ \mathrm{Tr}[\bar{g}^{-1}\partial_\varphi \bar{g}\bar{g}^{-1}\partial_+\bar{g}] + \frac{k}{12\pi}\int_{\mathcal{M}} \mathrm{Tr}[(\bar{G}^{-1}d\bar{G})^3] = S^L_{WZW}[\bar{g}]\,, \tag{4.48}$$

where $\bar{g}$ is the group element $\bar{G}$ evaluated on the boundary. This time, the action describes a left-moving element. The Chern-Simons action reduced on the boundary is thus the difference between two chiral WZW actions

$$S[A,\bar{A}] = S^R_{WZW}[g] - S^L_{WZW}[\bar{g}]\,. \tag{4.49}$$

We can combine the left and right movers as $h \equiv g^{-1}\bar{g}$ and $H \equiv G^{-1}\bar{G}$ and define

$$\Pi \equiv -\bar{g}^{-1}\partial_\varphi g g^{-1}\bar{g} - \bar{g}^{-1}\partial_\varphi \bar{g}\,, \tag{4.50}$$

to end up with the standard (non-chiral) WZW action, after elimination of the auxiliary field Π using its equation of motion,

$$S[A,\bar{A}] = S_{WZW}[h] = \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\ \mathrm{Tr}[h^{-1}\partial_+ h h^{-1}\partial_- h] + \frac{k}{12\pi}\int_{\mathcal{M}} \mathrm{Tr}[(H^{-1}dH)^3] \tag{4.51}$$

4.2.4 From WZW action to Liouville action

It is important to clarify that we have not yet used all of the Brown-Henneaux boundary conditions (4.41). So far, we have only employed conditions (i) $A_- = \bar{A}_+ = 0$ to demonstrate that the Chern-Simons action reduces to a standard WZW action. Implementing the remaining conditions will lead us to a Liouville action [150–152].

[3] To account for holonomies, one would need to take $A_r = G^{-1}\partial_r G$ and $A_\varphi = G^{-1}\partial_\varphi G + G^{-1}KG$, where $K = K(t)$ is an element of the $sl(2,\mathbb{R})$ algebra that could depend on time. We will address this later in Section (4.2.6).

However, before proceeding, it will be useful to express (4.51) in a local form using a Gauss decomposition:

$$H=\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix}\begin{pmatrix}e^{\phi/2} & 0\\ 0 & e^{-\phi/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ Y & 1\end{pmatrix}, \tag{4.52}$$

where X, Y, ϕ depend on r, t, φ. This allows us to rewrite the action (4.51) as a two-dimensional integral

$$S_{WZW}[h]=\frac{k}{4\pi}\int_{\partial\mathcal{M}}\mathrm{d}^2x\left(\frac{1}{2}\partial_+\phi\partial_-\phi+2e^{-\phi}\partial_-X\partial_+Y\right), \tag{4.53}$$

where now X, Y, ϕ take values on the boundary. The last set of boundary conditions (ii) will take the following form in terms of the gauge field h:

$$\begin{aligned}{}[h^{-1}\partial_-h]^{(1)}&=\frac{1}{l}, \quad [h^{-1}\partial_-h]^{(0)}=0,\\ [-\partial_+hh^{-1}]^{(-1)}&=\frac{1}{l}, \quad [-\partial_+hh^{-1}]^{(0)}=0.\end{aligned} \tag{4.54}$$

The indices in parentheses are Lie algebra indices. It is interesting to notice that the left and right moving WZW currents are given by

$$J_a=h^{-1}\partial_a h, \quad \bar{J}=-\partial_a hh^{-1}. \tag{4.55}$$

Thus, the second set of boundary constraints (ii) has the effect to set the WZW currents to constant. In terms of the ϕ, X, Y fields, the constraints (4.54) are equivalent to the set

$$\begin{aligned}e^{-\phi}\partial_-X&=\frac{1}{l}, \quad e^{-\phi}\partial_+Y=-\frac{1}{l},\\ X&=2l\partial_+\phi, \quad Y=-2l\partial_-\phi.\end{aligned} \tag{4.56}$$

Before applying these constraints to the action, as for set (i) of the boundary conditions, we need to ensure a well-defined variational principle. It is achieved by adding a boundary term to the action (4.51):

$$S_{\text{impr}}=S_{WZW}[h]-\frac{k}{2\pi}\oint\mathrm{d}\varphi\left(e^{-\phi}(X\partial_+Y+Y\partial_-X)\right)\Big|_{t_1}^{t_2}. \tag{4.57}$$

After implementing the constraints, we end up with the Liouville action

$$S_{\text{impr}}=S_{\text{Liouville}}[\phi]=\frac{k}{4\pi}\int_{\partial\mathcal{M}}\mathrm{d}^2x\left(\frac{1}{2}\partial_+\phi\partial_-\phi+\frac{2}{l^2}e^{\phi}\right). \tag{4.58}$$

One might initially infer from this derivation that Liouville theory is the dual CFT of three-dimensional gravity with negative cosmological constant, but this is not actually the case. The reduction described here pertains to classical computation. To establish a legitimate dual theory, one must establish a correspondence at the quantum level. At the very least, Liouville theory serves as an effective theory in the holographic duality with CFT_2.

4.2.5 Link with Schwarzian action

In the SYK model and in nearly AdS_2, JT gravity gives rise to a model known as the Schwarzian theory, which is embedded within Liouville theory (4.58) through dimensional reduction [75, 164, 189, 190]. To observe this, we begin by performing a field redefinition $f'^2 = l^2 e^{\phi}$, where the prime denotes a derivative with respect to φ, and then project (4.58) onto the right (or left) moving sector:

$$S = \frac{k}{2\pi} \int_{\partial\mathcal{M}} \mathrm{d}^2x \left(\frac{\partial_- f' f''}{f'^2} - \partial_- f f' \right) . \tag{4.59}$$

The projection is not mandatory; otherwise, we would end up with two copies of the Schwarzian model. Now, we perform a Wick rotation to Euclidean time $t = iy$ and consider the time circle to be very small, $\Delta y \to 0$, with $k' = k\Delta y$ fixed. The result of this dimensional reduction is the Schwarzian action (up to boundary terms):

$$S = \frac{k'}{2\pi} \int \mathrm{d}\varphi \left(\frac{f''^2}{f'^2} + f'^2 \right) = -\frac{k}{\pi} \int \mathrm{d}\varphi \left\{ \tan\frac{f}{2}; \varphi \right\} , \tag{4.60}$$

where $\{f; x\}$ is the Schwarzian derivative (2.28).

4.2.6 Taking care of the holonomies

In the previous section, we neglected the potential holonomies present in our spacetime. However, they become crucial for describing black holes, as illustrated in Figure (4.1). To incorporate these holonomies, we assume our spacetime has two boundaries at spatial infinity, each corresponding to a disconnected region outside the eternal black hole. These asymptotically AdS_3 boundaries can be mapped onto the two boundaries of an annulus, denoted as Σ_o and Σ_i [72]. As previously discussed, we initially derived two chiral WZW actions which we later combined into a non-chiral one. Introducing holonomies complicates this approach, particularly when dealing with zero modes of the fields at the boundaries [72]. Therefore, it is more practical to refrain from combining them and instead focus on a single gauge field, such as A.

As before we want a well-defined variational principle for the action (4.42), we need to add two boundary terms, one for each boundary $\Sigma_{o,i}$:

$$I_{\Sigma_{o,i}} = -\frac{k}{4\pi} \int_{\Sigma_{o,i}} \mathrm{d}^2x \ \mathrm{Tr} A_\varphi^2 . \tag{4.61}$$

We impose the condition $A_- = 0$ in the outer boundary and $A_+ = 0$ in the inner one in order for the Hamiltonians on the respective boundaries to have the same sign. In that way, the time evolution on both sides runs in the same direction. Thus, the improved action for A is

$$S[A] = S_{CS}[A] + I_{\Sigma_i} + I_{\Sigma_o} . \tag{4.62}$$

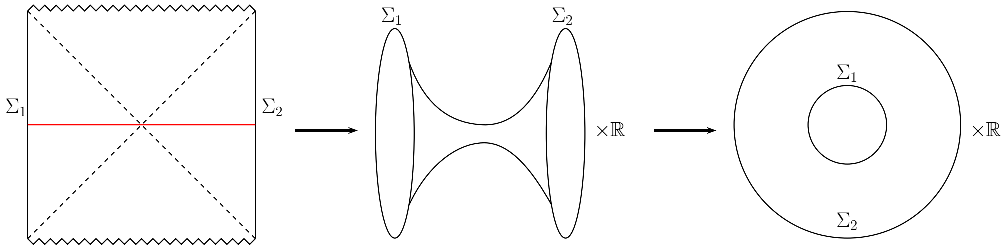

Figure 4.1: The topology of a fixed time slice of an eternal black hole is an infinite cylinder, which is equivalent to an annulus. In this way, the two asymptotic boundaries of AdS_3 are mapped to the two boundaries of the annulus. The figure is taken from [72].

Like in the case with no holonomy, we need to solve the constraint $F_{r\phi}=0$, giving the solutions

$$A_r = G^{-1}\partial_r G\,, \qquad A_\varphi = G^{-1}(\partial_\varphi + K(t))G\,. \tag{4.63}$$

the function $K(t)$ parameterizes the holonomy and takes value in the Lie algebra. By plugging these solutions in (4.62), we get the equivalent of (4.47) in the presence of holonomies

$$\begin{aligned}S[g_o,g_i,K(t)] = {}& \frac{k}{4\pi}\int_{\Sigma_o} \mathrm{d}^2x\,\mathrm{Tr}[g_o^{-1}\partial_\varphi g_o\, g_o\partial_- g_o + 2g_o^{-1}K\partial_- g_o - K^2] \\ &- \frac{k}{4\pi}\int_{\Sigma_i} \mathrm{d}^2x\,\mathrm{Tr}[g_i^{-1}\partial_\varphi g_i\, g_i\partial_+ g_i + 2g_i^{-1}K\partial_+ g_i + K^2] \\ &+ \frac{k}{12\pi}\int_{\mathcal{M}} \mathrm{Tr}[(G^{-1}dG)^3]\,,\end{aligned} \tag{4.64}$$

where we defined g_o and g_i as the boundary values of G on Σ_o and Σ_i respectively. Having two boundaries explains the presence of two boundary integrals in the action (4.64). The difference in sign between the latter terms arises from the distinction in boundary conditions between Σ_o and Σ_i.

Before proceeding further and imposing the remaining boundary conditions, specifically condition (ii), we need to discuss the holonomy $K(t)$. The action (4.64) provides its equation of motion at the outer boundary, with the computation at the inner boundary being analogous,

$$\partial_t K = [a, K]\,, \tag{4.65}$$

with

$$a = \frac{1}{2\pi}\oint \mathrm{d}\varphi\,\partial_- g_o\, g_o^{-1}\,. \tag{4.66}$$

Its solution is

$$K(t) = S(t)K(0)S(t)^{-1}\,, \qquad S(t) = \mathcal{T}\exp\int_0^t a(\tau)\mathrm{d}\tau\,, \tag{4.67}$$

with $\mathcal{T}$ the time-ordering operator. It shows that K does not leave its conjugacy class during time evolution. While the precise value of K is not physical, its conjugacy class is. As a consequence, we only need to specify a "canonical" element of either one of the three conjugacy classes of $SL(2,\mathbb{R})$.

Hyperbolic holonomy

Hyperbolic holonomies characterize non-extremal black holes. A canonical hyperbolic element can be chosen as

$$K(t) = k_h(t) L_0 \,, \qquad k_h(t) \neq 0 \,. \tag{4.68}$$

We then use the Gauss decomposition (A.11-A.12) to write the fields g_o and g_i as

$$g_o = e^{Y L_1} e^{\Phi L_0} e^{x L_{-1}} \,, \qquad g_i = e^{V L_{-1}} e^{\Psi L_0} e^{U L_1} \,. \tag{4.69}$$

The choice between the two different Gauss decompositions was done in order to impose a heighest-weight boundary condition on the outer boundary and a lowest-weight boundary condition on the inner one[4]. Plugging these decompositions in (4.64) gives

$$\begin{aligned} S = \frac{k}{4\pi} \int_{\Sigma_o} \mathrm{d}^2 x \left(\frac{1}{2} \partial_- \Phi (\Phi' + 2k_h) - 2 e^{\Phi} \partial_- X (Y' - k_h Y) - \frac{1}{2} k_h^2 \right) \\ - \frac{k}{4\pi} \int_{\Sigma_i} \mathrm{d}^2 x \left(\frac{1}{2} \partial_+ \Psi (\Psi' + 2k_h) - 2 e^{\Psi} \partial_+ U (V' + k_h V) + \frac{1}{2} k_h^2 \right) \,. \end{aligned} \tag{4.70}$$

We now add the set (ii) of boundary conditions adapted for the case of two boundary. On Σ_o, it reads

$$A_r = 0 \,, \qquad A_\varphi = L_1 - \mathcal{L}(t,\varphi) L_{-1} \,, \tag{4.71}$$

and on Σ_i,

$$A_r = 0 \,, \qquad A_\varphi = L_1 - \mathcal{M}(t,\varphi) L_{-1} \,. \tag{4.72}$$

The functions $\mathcal{L}$ and $\mathcal{M}$ are the analogues on each boundary of $\bar{L}(x^+)$ in (ii). In terms of the fields of the Gauss decomposition, one writes at the outer boundary

$$\begin{aligned} e^{\Phi}(Y' - k_h Y) &= 1 \,, \\ \Phi' + k_h &= -2X \,, \\ X' - X^2 &= -\mathcal{L} \,, \end{aligned} \tag{4.73}$$

and on the inner one

$$\begin{aligned} e^{-\Psi}(V' + k_h V) &= 1 \,, \\ \Psi' + k_h &= 2U \,, \\ U' - U^2 &= -\mathcal{M} \,. \end{aligned} \tag{4.74}$$

[4] This difference arises from the different conditions that we imposed on each boundary in order to have time evolution running in the same direction on both sides.

Solving the first two equations on each boundary will allow to express Y and X (V and U) in terms of Φ (Ψ) and k_h. The precise computation is done in Appendix C of [72]. The final action is then

$$S[k_h,\Phi,\Psi]=\frac{k}{4\pi}\int \mathrm{d}^2x\left(\frac{1}{2}\partial_-\Phi\Phi'-\frac{1}{2}\partial_+\Psi\Psi'+k_h(\partial_-\Phi-\partial_+\Psi)-k_h^2\right). \qquad (4.75)$$

This action represents two chiral bosons coupled to the holonomy. Although the kinetic terms for Φ and Ψ have opposite signs, their Hamiltonians have the same sign, ensuring time flows in the same direction for both boundaries. To complete the picture, one still needs to consider the other $SL(2,\mathbb{R})$ connection, $\bar{A}$. This will lead to a similar action as (4.75), resulting ultimately in four chiral bosons coupled through two different holonomies.

After a last effort, we can write (4.75) as the action of a free non-chiral boson on a cylinder by defining an inversible change of variables

$$\phi=\Phi-\Psi\,,\qquad \Pi_\phi=\Phi'+\Psi'+2k_h\,, \qquad (4.76)$$

and by elimating the conjugate momenta Π_ϕ using its own equation of motion

$$S[\phi]=\frac{k}{16\pi}\int \mathrm{d}^2x\left(\dot{\phi}^2-\phi'^2\right). \qquad (4.77)$$

It was shown in [191–193] that this final action can be classically mapped to the Liouville action (4.58) via a Bäcklund transformation. This implies that the effective action without holonomy and the one with hyperbolic holonomy, which includes the description of black holes, are equivalent. However, we will see later that this equivalence does not hold for other types of holonomies.

Before proceeding to consider other holonomies, let us backtrack a bit and redefine the field Y to explicitly highlight its connection with the geometric action. Boundary diffeomorphisms act as residual gauge symmetries that are compatible with the constraints of Hamiltonian reduction. These symmetries are parameterized by the L_1 factor in the Gauss decomposition, which makes it convenient to rewrite the action and constraints in terms of a field $f(t,\varphi)$ related to Y as follows

$$Y(t,\varphi)=e^{-k_h(t)(f(t,\varphi)-\varphi)}\,. \qquad (4.78)$$

The boundary conditions (4.73) become

$$\begin{aligned}e^{-\Phi}&=-k_h\,e^{-k_h(f-\varphi)}\,f'\,,\\ X&=\frac{f''}{2f'}-\frac{1}{2}k_hf'\,.\end{aligned} \qquad (4.79)$$

The function $\mathcal{L}$ is now expressed in terms of f as

$$\mathcal{L}=\frac{k_h}{4}f'^2-\frac{1}{2}\{f;\varphi\}\,, \qquad (4.80)$$

with $\{f;\varphi\}$ the Schwarzian derivative (2.28). Up to boundary terms, the reduced action (4.75) on the outer boundary only takes the form

$$S^{\Sigma_o}_{\text{bdy}} = \frac{k}{8\pi}\int_{\Sigma_o} \mathrm{d}^2x \left(\frac{\partial_- f' f''}{f'^2} + k_h(t)^2 \partial_- f\, f'\right) . \tag{4.81}$$

We recognize the geometric action of the Virasoro group (4.32) for a positive orbit representative.

Elliptic holonomy

A solution with an elliptic holonomy describes point particle sources and define conical singularities [160, 194, 195]. A canonical elliptic element can be chosen as

$$K(t) = \frac{k_e(t)}{2}\left(L_1 + L_{-1}\right) , \qquad k_e(t) \neq 0 . \tag{4.82}$$

Contrary to the hyperbolic case, we will use the Iwasawa decomposition (A.13) instead of the Gauss decompostion. We then write g_o and g_i as

$$\begin{aligned} g_o &= e^{\theta(t,\varphi)(L_1+L_{-1})}\, e^{\Phi(t,\varphi)L_0}\, e^{\eta(t,\varphi)L_{-1}} \\ g_i &= e^{\vartheta(t,\varphi)(L_1+L_{-1})}\, e^{\Psi(t,\varphi)L_0}\, e^{\nu(t,\varphi)L_1} \end{aligned} \tag{4.83}$$

All fields used in the Iwasawa decomposition are periodic in φ. Inserting all of those in (4.64) leads to

$$\begin{aligned} S = \frac{k}{4\pi}\int_{\Sigma_o} \mathrm{d}^2x &\left(\frac{1}{2}\partial_-\Phi\Phi' - 2\partial_-\theta(\theta' + k_e) - 2e^{\Phi}\partial_-\eta\left(\theta' + \frac{1}{2}k_e\right) + \frac{1}{2}k_e^2\right) \\ - \frac{k}{4\pi}\int_{\Sigma_i} \mathrm{d}^2x &\left(\frac{1}{2}\partial_+\Psi\Psi' - 2\partial_+\vartheta(\vartheta' + k_e) - 2e^{-\Psi}\partial_+\nu\left(\vartheta' + \frac{1}{2}k_e\right) - \frac{1}{2}k_e^2\right) . \end{aligned} \tag{4.84}$$

We can now rewrite the boundary condition (ii) in terms of the fields of the Iwasawa decomposition on the outer boundary

$$\begin{aligned} e^{\Phi}\left(\theta' + \frac{1}{2}k_e\right) = 1\,, \quad \eta = -\frac{1}{2}\Phi'\,, \\ \mathcal{L} = -e^{-2\Phi} + \frac{1}{4}(\Phi'^2 + 2\Phi'')\,, \end{aligned} \tag{4.85}$$

and on the inner boundary

$$\begin{aligned} e^{-\Psi}\left(\vartheta' + \frac{1}{2}k_e\right) = -1\,, \quad \nu = \frac{1}{2}\Psi'\,, \\ \mathcal{M} = -e^{2\Psi} + \frac{1}{4}(\Psi'^2 - 2\Psi'')\,. \end{aligned} \tag{4.86}$$

Using a last field redefinition to express the action in terms of a diffeomorphism of the circle $f(t,\varphi)$ with $f(t,\varphi+2\pi)=f(t,\varphi)+2\pi$:

$$\theta(t,\varphi)=\frac{k_e(t)}{2}\left(f(t,\varphi)-\varphi\right)\,, \tag{4.87}$$

and the constraints, the action on the outer boundary turns into

$$S^{\Sigma_o}_{\text{bdy}}=\frac{k}{8\pi}\int_{\Sigma_o}\mathrm{d}^2x\left(\frac{\partial_- f' f''}{f'^2}-k_h(t)^2\partial_- f\, f'\right)\,, \tag{4.88}$$

with the function $\mathcal{L}$ given by

$$\mathcal{L}=-\frac{k_e}{4}f'^2-\frac{1}{2}\{f;\varphi\}\,. \tag{4.89}$$

The difference with the hyperbolic case (4.81) is the relative sign of the representative term. One could obtain (4.81) through an analytic continuation $k_e=ik_h$. We can then interpret (4.88) as the action of a chiral boson as for the hyperbolic case but with purely imaginary zero modes.

Parabolic holonomy

The last case to analyze is parabolic holonomies. They describe extremal BTZ black holes. They can be parameterized by

$$K(t)=k_p(t)L_{-1}\,. \tag{4.90}$$

We will also use a Iwasawa decomposition but inverting the order of (4.83):

$$\begin{aligned} g_o&=e^{\eta(t,\varphi)L_{-1}}\,e^{\Phi(t,\varphi)L_0}\,e^{\theta(t,\varphi)(L_1+L_{-1})}\,,\\ g_i&=e^{\nu(t,\varphi)L_{1}}\,e^{\Psi(t,\varphi)L_0}\,e^{\vartheta(t,\varphi)(L_1+L_{-1})}\,. \end{aligned} \tag{4.91}$$

One again, by putting those in (4.64), the action separates into different boundary actions

$$\begin{aligned} S=&\frac{k}{4\pi}\int_{\Sigma_o}\mathrm{d}^2x\left(\frac{1}{2}\Phi'\partial_-\Phi-2\theta'\partial_-\theta-2e^{-\Phi}\partial_-\theta(\eta'+k_p)\right)\\ &-\frac{k}{4\pi}\int_{\Sigma_i}\mathrm{d}^2x\left(\frac{1}{2}\Psi'\partial_+\Psi-2\vartheta'\partial_-\vartheta-2e^{\Psi}\partial_-\vartheta(\nu'+k_p)\right) \end{aligned} \tag{4.92}$$

with as constraints on Σ_o coming from the boundary conditions

$$\Phi'=2(\theta'-1)\cot\theta\,,\qquad \eta'=-k_p+e^{\Phi}\Phi'\cot 2\theta\,, \tag{4.93}$$

and

$$\mathcal{L}=-\theta'-\frac{1}{2}\Phi'\cot\theta\,. \tag{4.94}$$

On Σ_i, we get

$$\Psi' = -2(\vartheta' + 1)\cot\vartheta\,, \qquad \nu' = -k_p - e^{-\Psi}\Psi'\cot 2\vartheta\,, \tag{4.95}$$

and

$$\mathcal{M} = \vartheta' - \frac{1}{2}\Psi'\cot\vartheta\,. \tag{4.96}$$

By making the following field redefinition

$$\cot\theta = -\frac{f''}{2f'} \tag{4.97}$$

and by using the constraints, one can show that the action on the outer boundary becomes

$$S^{\Sigma_o}_{\text{bdy}} = \frac{k}{8\pi}\int_{\Sigma_o} \mathrm{d}^2x \frac{\partial_- f' f''}{f'^2}\,, \tag{4.98}$$

with the function $\mathcal{L}$ given by

$$\mathcal{L} = -\frac{1}{2}\{f;\varphi\}\,. \tag{4.99}$$

One would get an equivalent action at the inner boundary. We notice that the dependence on the holonomy drops out of the action. This is the geometric action with a vanishing representative.

As a conclusion, we can combine the three different actions (4.81, 4.88, 4.98) into

$$S^{\Sigma_o}_{\text{bdy}} = \frac{k}{8\pi}\int_{\Sigma_o} \mathrm{d}^2x \left(\frac{\partial_- f' f''}{f'^2} + \alpha\partial_- f f'\right)\,, \tag{4.100}$$

with the function $\mathcal{L}(t,\varphi)$ given by, for the three different conjugacy classes,

$$\mathcal{L} = \frac{\alpha}{4}f'^2 - \frac{1}{2}\{f;\varphi\}\,, \tag{4.101}$$

with

Holonomy	$K(t)$	α
Hyperbolic	$k_h(t)L_0$	k_h^2
Elliptic	$\frac{1}{2}k_e(t)(L_1 + L_{-1})$	$-k_e^2$
Parabolic	$k_p(t)L_{-1}$	0

(4.102)

The equation of motions of this action imply that

$$\partial_-\mathcal{L} = 0\,, \tag{4.103}$$

indicating that $\mathcal{L}$ is a function of x^+ only. This result makes sense since the function $\mathcal{L}$ was originally defined as the analogue of $\bar{L}(x^+)$ in (ii).

Those actions are equivalent to the geometric action on the coadjoint orbit of the Virasoro algebra (4.32) with the correspondence between the holonomy α and the orbit representative X_0

$$X_0 = \frac{c}{48\pi}\alpha\,. \tag{4.104}$$

The hyperbolic (elliptic) holonomies represent positive (negative) representatives while the parabolic holonomys lead to vanishing representatives.

One can also establish a connection between the little group of an orbit representative and the residual gauge symmetry of the action. For cases where $\alpha \neq -n^2$, which includes hyperbolic and parabolic holonomies, the residual gauge transformation is $U(1)$, consistent with the little group. However, when $\alpha = -n^2$, which occurs only for elliptic holonomies, the residual gauge group is enhanced to $SL^{(n)}(2,\mathbb{R})$, aligning with the stabilizer group of the corresponding coadjoint orbit [72].

4.2.7 Virasoro characters from the one-loop partition function of the reduced action

In this section, we will review the computation of the Virasoro characters as outlined in Section 5 of [75]. Starting from the geometric actions on the coadjoint orbits of the Virasoro group, we have the action (4.100):

$$S^{\Sigma_o}_{\text{bdy}} = \frac{k}{8\pi}\int_{\Sigma_o} \mathrm{d}^2x \left(\frac{\partial_- f' f''}{f'^2} + \alpha \partial_- f f'\right) , \tag{4.105}$$

on the outer boundary where α encodes the different types of holonomies. In [75], they considered 2 distinct cases. First, $\alpha = -1$, corresponding to the vacuum and an elliptic holonomy with $\text{Diff}(S^1)/SL(2,\mathbb{R})$ symmetry. Second, $\alpha > -1$, where the residual symmetry group is always $\text{Diff}(S^1)/U(1)$. In both cases, the Hamiltonian of the action (4.105) is bounded from below when we expand the field around a critical value, allowing for a path integral derivation of the Virasoro character. Indeed, the Hamiltonian is related to L_0 (4.31)

$$H = L_0 = -\frac{c}{24\pi}\int \mathrm{d}\varphi \left(\{f;\varphi\} - \frac{\alpha}{2} f'^2\right) , \tag{4.106}$$

and possesses a unique critical value for $f = \varphi$ [75]. We can expand H around this critical point to get

$$H[f = \varphi + \sum_n f_n e^{in\varphi}] = 2\pi X_0 + \frac{c}{12}\sum_n n^2(n^2+\alpha)|f_n|^2 + O(f_n^3) . \tag{4.107}$$

The second term is always positive when $\alpha \geq -1$ with a sum beginning at $n = 1$ when this is a strict inequality and at $n = 2$ when $\alpha = -1$. Thus, in the case of a elliptic holonomy with $\alpha < -1$, it is expected that the path integral does not exist and the computations below are not valid. We will start by making no distinction between the two cases until it is required.

On the inner one, Σ_i, we have the equivalent action:

$$S^{\Sigma_i}_{\text{bdy}} = -\frac{k}{8\pi}\int_{\Sigma_i} \mathrm{d}^2x \left(\frac{\partial_+ f' f''}{f'^2} + \alpha(t) \partial_+ f f'\right) . \tag{4.108}$$

We will focus only on Σ_i to stick with [75], the computation on the other boundary being equivalent. First, we perform a Wick rotation $t = -iy$ to get the Euclidean action

$$S_E = -iS = \frac{c}{24\pi}\int_{\Sigma_i} \mathrm{d}^2x \left(\frac{\bar{\partial} f' f''}{f'^2} + \alpha(t)\bar{\partial} f f'\right), \tag{4.109}$$

where $\bar{\partial} = \partial_\varphi + i\partial_y$. The field f satisfies the following periodicity conditions:

$$\begin{aligned} f(\varphi + \Omega\beta, y + \beta) &= f(\varphi, y)\,, \\ f(\varphi + 2\pi, y) &= f(\varphi, y) + 2\pi\,. \end{aligned} \tag{4.110}$$

To get the Virasoro character, we need to compute the partition function. For a chiral CFT on the torus of module τ, it is given by

$$Z(\tau) = \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}}\right), \qquad q = e^{2\pi i\tau}, \tag{4.111}$$

where $c = 6k$ and may be decomposed into a sum of Virasoro characters

$$Z(\tau) = q^{-\frac{c}{24}} \prod_{n=2}^{\infty} \frac{1}{1-q^n} + \sum_h q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n}\,. \tag{4.112}$$

The first term represents the vacuum module while the other terms stands for a sum over Virasoro primaries of weight h.

For the gravity dual, we aim to compute the partition function

$$Z = \int \mathcal{D}f\, e^{iS} = \int \mathcal{D}f\, e^{-S_E}\,. \tag{4.113}$$

This path integral is in general very complex to evaluate. Typically, the partition function is expanded in terms of a dimensionless coupling constant, in our case k. Around a saddle point f_0, the path integral takes the form

$$\log Z \sim -kS^{(0)} + S^{(1)} + O(k^{-1})\,, \tag{4.114}$$

where $S^{(0)}$ is the classical on-shell action and $S^{(1)}$ represents the first-order correction to the saddle point action, corresponding to the one-loop contribution. We assume that k is large, equivalent to a large central charge, allowing us to neglect higher-order corrections. However, it has been argued in [75, 171] that for locally AdS_3 gravity, the computation is in fact one-loop exact. The one-loop contribution suffices to reproduce the entire partition function. A similar argument applies to BMS_3 as well [73, 173].

Thus, we are interested in solution of the equation of motion for f (4.109)

$$\frac{1}{f'}\bar{\partial}\left(\frac{\alpha}{2}f'^2 - \{f;\varphi\}\right) = 0\,, \tag{4.115}$$

consistent with the remaining boundary condition (4.101). Satisfying the previous periodicity conditions, the saddle point is

$$f_0 = \varphi - \Omega y\,, \tag{4.116}$$

giving the saddle point action

$$S_0 = -i\frac{k}{4}\alpha\beta(\Omega + i) = -i\frac{\pi k}{2}\alpha\tau\,, \tag{4.117}$$

where we define $\tau = \frac{1}{2\pi}(\beta\Omega + i\beta)$.

Now we expand the field f around its saddle point:

$$f(\varphi, y) = f_0 + \sum_{m,n}\frac{f_{m,n}}{(2\pi)^2}e^{\frac{2\pi i m y}{\beta}}e^{inf_0}\,, \tag{4.118}$$

and plug this into the Euclidean action, leaving us with

$$S_E = S_0 - \frac{ic}{96\pi^3}\sum_{m=-\infty}^{\infty}\sum_{n\neq 0}|f_{m,n}|^2 n(n^2+\alpha)(m-n\tau) + \mathcal{O}(f^3)\,, \tag{4.119}$$

where the sum over n excludes 0 for generic values of α and excludes $-1, 0, 1$ when $\alpha = -1$.

Now we can evaluate the one-loop partition function

$$Z_{\text{1-loop}} = \int \mathcal{D}f e^{-S_E} = \mathcal{N}q^{\frac{c\alpha}{24}}\prod_{m,n}\Big((m-\tau n)(n^3+\alpha n)\Big)^{-1/2}\,, \tag{4.120}$$

where we define $q = e^{2\pi i\tau}$.

To compute the product over m, we will employ the following technique. First, we differentiate with respect to τ the logarithm of the partition function. This allows us to perform a sum over m

$$\begin{aligned}
\partial_\tau \log Z_{\text{1-loop}} &= \frac{i\pi}{12}c\alpha + \frac{1}{2}\sum_{n\neq 0}\sum_{m=-\infty}^{\infty}\frac{n}{m-\tau n} \\
&= \frac{i\pi}{12}c\alpha - \frac{1}{2}\sum_{n\neq 0}\left(\frac{1}{\theta} + 2n\sum_{m=1}^{\infty}\frac{\tau n}{\tau^2 n^2 - m^2}\right) \\
&= \frac{i\pi}{12}c\alpha - \frac{1}{2}\pi\sum_{n\neq 0} n\cot(n\pi\tau) \\
&= \frac{i\pi}{12}c\alpha - \pi\sum_{n=1}^{\infty} n\cot(n\pi\tau)
\end{aligned}$$

or with a sum over n beginning at 2 when $\alpha = -1$. This sum diverges. One way to regularize it is to use the zeta function

$$\sum_{n=1}^{\infty} n\cot(n\pi\tau) \to \sum_{n=1}^{\infty} n(\cot(n\pi\tau) + i) - i\sum_{n=1}^{\infty} n\,. \tag{4.121}$$

The first sum converges for $\mathrm{Im}(\tau) > 0$, and the second is evaluated using zeta-function regularization. Now, after exponentiation and integration over τ, we obtain a partition function with only a sum over n, up to an overall normalization constant:

$$Z_{\text{1-loop}} = q^{\frac{c\alpha-1}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \,. \tag{4.122}$$

By comparing to the partition function (4.112), we see that the Hilbert space simply corresponds to a single Verma module of weight

$$h = \frac{(\alpha+1)c-1}{24} \,. \tag{4.123}$$

When $\alpha = -1$, this result is not valid because (4.120) contains a division by 0 if we allow n to be 1. Thus, we need to start the product at $n = 2$, and in this case, the derivative with respect to the modular parameter τ of the one-loop partition function is

$$\partial_\tau \log Z_{\text{1-loop}} = -\frac{i\pi}{12}c - \pi \sum_{n=2}^{\infty} n \cot(n\pi\tau) \,. \tag{4.124}$$

The zeta function regularisation adds only a sum starting at $n = 2$ and gives

$$\sum_{n=2}^{\infty} = \zeta(-1) - 1 = -\frac{13}{12} \,. \tag{4.125}$$

The one-loop partition function is then

$$Z_{\text{1-loop}} = q^{\frac{-c-13}{24}} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \,. \tag{4.126}$$

We obtain the vacuum module with central charge $c + 13$. This result is consistent with the computation performed in [171], where they calculate the one-loop partition function of thermal AdS_3 using the Heat Kernel method. Although the computation in this section is a one-loop calculation within the large c approximation, equation (4.126) is considered one-loop exact as argued in [75].

4.3 Lower Spin Gravity and Warped Virasoro Symmetry

We aim to extend the analysis from the previous section to WCFTs. To achieve this, we will examine one of the simplest holographic models of a WCFT, known as Lower Spin Gravity, as reviewed in Section (2.3) of this thesis. In this model, the Warped Virasoro group for $WAdS_3$ plays a role analogous to the Virasoro group in AdS_3 gravity. Our objective in this section will be to perform the Hamiltonian reduction of Lower Spin Gravity and establish connections with the dual theory on the boundary, such as the Warped Schwarzian action and the geometric action of the Warped Virasoro group.

We will begin by labeling the different coadjoint orbits with a representative, computing their corresponding little group, and constructing a geometric action on each of them.

4.3.1 Coadjoint representation

The warped conformal group is the centrally extended semi-direct product group of diffeomorphisms of the circle $\text{Diff}^+(S^1)$ with $C^\infty(S^1)$. In Appendix A of [178], the coadjoint representations of this group were studied, which we will summarize here for convenience and to fix conventions.

We take φ to be a coordinate along the circle and denote elements of $\text{Diff}^+(S^1)$ by $f(\varphi)$, satisfying $f'(\varphi)>0$ and $f(\varphi+2\pi)=f(\varphi)+2\pi$, and elements of $C^\infty(S^1)$ by $p(\varphi)$. The group multiplication of the semi-direct product group $G=\text{Diff}^+(S^1)\ltimes C^\infty(S^1)$ is

$$(f_1,p_1)\cdot(f_2,p_2)=(f_1\circ f_2, p_1+\sigma_{f_1}p_2)\,,\qquad \text{with: } \sigma_f p = p\circ f^{-1}\,. \tag{4.127}$$

where $\circ$ denotes functional composition, $f\circ\varphi=f(\varphi)$.

Since the second cohomology space of G is three-dimensional, the warped conformal group has three central extensions. They are defined by the three 2-cocycles in G. Denoting elements of the centrally extended group $\hat{G}=G\times\mathbb{R}^3$, by $(f,p;\lambda,\mu,\nu)$, group operation now reads

$$\begin{aligned}(f_1,p_1;\lambda_1,\mu_1,\nu_1)\cdot(f_2,p_2;\lambda_2,\mu_2,\nu_2)=\Big(&f_1\circ f_2, p_1+\sigma_{f_1}p_2;\lambda_1+\lambda_2+B(f_1,f_2),\\ &\mu_1+\mu_2+C(f_1,p_2),\nu_1+\nu_2+D(p_1,\sigma_{f_1}p_2)\Big)\,.\end{aligned} \tag{4.128}$$

Here B,C and D are real-valued 2-cocycles in the group. They are given explicitly by

$$\begin{aligned}B(f_1,f_2)&=-\frac{1}{48\pi}\oint \mathrm{d}\varphi\ \log(\partial_\varphi f_1\circ f_2)\partial_\varphi\log\partial_\varphi f_2\,,\\ C(f_1,p_2)&=-\frac{1}{2\pi}\oint \mathrm{d}\varphi\ p_2\partial_\varphi\log(\partial_\varphi f_1)\,,\\ D(f_1,\sigma_{f_1}p_2)&=-\frac{1}{4\pi}\oint \mathrm{d}\varphi\ p_1\partial_\varphi\sigma_{f_1}p_2\,.\end{aligned} \tag{4.129}$$

We recognize for the 2-cocycle $B(f_1,f_2)$, the Bott-Thurston cocycle of the Virasoro group (4.21).

The adjoint action of a group element $g_1=(f_1,p_1;\lambda_1,\mu_1,\nu_1)$ on an element of the algebra $(X_2,p_2;\lambda_2,\mu_2,\nu_2)$ is computed by

$$\text{Ad}_{g_1}(X_2,p_2;\lambda_2,\mu_2,\nu_2)=\frac{\mathrm{d}}{\mathrm{d}\epsilon}\,g_1\cdot(e^{\epsilon X_2},\epsilon p_2,\epsilon\lambda_2,\epsilon\mu_2,\epsilon\nu_2)\cdot g_1^{-1}\Big|_{\epsilon=0}\,, \tag{4.130}$$

which explicitly evaluates to

$$\begin{aligned}\mathrm{Ad}_{(f,p_1)}(X,p_2;\lambda,\mu,\nu) = \Big(&\mathrm{Ad}_f X, \sigma_f p_2 + \Sigma_{\mathrm{Ad}_f X} p_1; \lambda - \frac{1}{24\pi}\oint \mathrm{d}\varphi\, X(\varphi)\{f;\varphi\},\\ &\mu - \frac{1}{2\pi}\oint \mathrm{d}\varphi\, \left(p_2 \partial_\varphi \log f' - X(\varphi) f'^2 \partial_f^2 (p_1\circ f)\right),\\ &\nu - \frac{1}{2\pi}\oint \mathrm{d}\varphi\, \left(p_1 \partial_\varphi \sigma_f p_2 - \frac{1}{2} X(\varphi)[\partial_\varphi(p_1\circ f)]^2\right)\Big)\,. \end{aligned} \tag{4.131}$$

Here, the notation $\mathrm{Ad}_f X$ denotes the adjoint action of the Virasoro group on the components of a vector field $X(\varphi)\partial_\varphi$, i.e. the transformation of a vector field on the circle under diffeomorphisms: $\mathrm{Ad}_f X = f'(f^{-1})X(f^{-1})$.

The symbol Σ_X denotes the infinitesimal version of σ_f, and hence $\Sigma_X p_1 = X(\varphi)p_1'(\varphi)$. Finally, $\{f;\varphi\}$ denotes the Schwarzian derivative (2.28).

The adjoint representation of the algebra on itself gives the commutators

$$\begin{aligned}-\mathrm{ad}_{(X_1,p_1)}\left(X_2,p_2;\lambda_2,\mu_2,\nu_2\right) &= [(X_1,p_1;\lambda_1,\mu_1,\nu_1),(X_2,p_2;\lambda_2,\mu_2,\nu_2)] \\ &= \Big(X_1X_2' - X_1'X_2, X_1p_2' - p_1'X_2; \frac{1}{24\pi}\oint \mathrm{d}\varphi\, X_1'''X_2,\\ &\qquad \frac{1}{2\pi}\oint \mathrm{d}\varphi\, \left(X_1''p_2 - p_1''X_2\right), \frac{1}{2\pi}\oint \mathrm{d}\varphi\, p_1p_2'\Big)\,.\end{aligned} \tag{4.132}$$

From here we can read off the warped conformal algebra, by taking the generators to be $L_n = (e^{in\varphi},0;0,0,0), P_n = (0,e^{in\varphi};0,0,0)$ and $Z_1 = (0,0;1,0,0), Z_2 = (0,0;0,1,0)$ and $Z_3 = (0,0;0,0,1)$:

$$\begin{aligned} i[L_n,L_m] &= (n-m)L_{n+m} + \frac{Z_1}{12}n^3\delta_{n+m}\,,\\ i[L_n,P_m] &= -mP_{m+n} - in^2 Z_2\delta_{n+m}\,,\\ i[P_n,P_m] &= -mZ_3\delta_{n+m}\,. \end{aligned} \tag{4.133}$$

Elements of the dual space $\mathfrak{g}^*$ will be denoted by $(\mathcal{L},\mathcal{P};c,k,\kappa)$. We denote by $\langle,\rangle$ the pairing between the dual space and the algebra, i.e. $\langle,\rangle$ is a map from $\mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$. In our case, we will take

$$\langle(\mathcal{L},\mathcal{P};c,k,\kappa),(X,p;\lambda,\mu,\nu)\rangle = \oint \mathrm{d}\varphi\,(\mathcal{L}(\varphi)X(\varphi) + \mathcal{P}(\varphi)p(\varphi)) + c\lambda + k\mu + \kappa\nu\,. \tag{4.134}$$

We can now define the coadjoint action of the group on elements of the dual space as

$$\langle \mathrm{Ad}^*_{(f,p_1)^{-1}}(\mathcal{L},\mathcal{P};c,k,\kappa),(X,p_2;\lambda,\mu,\nu)\rangle = \langle(\mathcal{L},\mathcal{P};c,k,\kappa),\mathrm{Ad}_{(f,p_1)}(X,p_2;\lambda,\mu,\nu)\rangle\,. \tag{4.135}$$

By using (4.131) and (4.134) we obtain

$$\mathrm{Ad}^*_{(f,p_1)^{-1}}(\mathcal{L},\mathcal{P};c,k,\kappa) = (\tilde{\mathcal{L}},\tilde{\mathcal{P}};c,k,\kappa)\,, \tag{4.136}$$

with

$$\tilde{\mathcal{L}}(\varphi) = f'^2\left(\mathcal{L}(f) + \mathcal{P}(f)\partial_f(p\circ f) - \frac{c}{24\pi}\frac{\{f;\varphi\}}{f'^2} + \frac{k}{2\pi}\partial_f^2(p\circ f) + \frac{\kappa}{4\pi}[\partial_f(p\circ f)]^2\right),$$
$$\tilde{\mathcal{P}}(\varphi) = f'\left(\mathcal{P}(f) - \frac{k}{2\pi}\frac{f''}{f'^2} + \frac{\kappa}{2\pi}\partial_f(p\circ f)\right). \tag{4.137}$$

The notation here differs from the appendix A of [178], but this is due to the fact that there $\mathrm{Ad}^*_{(f,p)}(\mathcal{L},\mathcal{P};c,k,\kappa)(f(\varphi))$ is computed, whereas here we have given $\mathrm{Ad}^*_{(f,p)^{-1}}(\mathcal{L},\mathcal{P};c,k,\kappa)(\varphi)$. For completeness, we also list the other expression here, which is equivalent to (4.137):

$$\begin{aligned}\mathrm{Ad}^*_{(f,p)}\mathcal{L}(f(\varphi)) = \frac{1}{f'^2}\Big(&\mathcal{L}(\varphi) - \mathcal{P}(\varphi)(p\circ f)'(\varphi) + \frac{c}{24\pi}\{f;\varphi\} - \frac{k}{2\pi}(p\circ f)''(\varphi)\\ &+ \frac{\kappa}{4\pi}[(p\circ f)'(\varphi)]^2\Big),\\ \mathrm{Ad}^*_{(f,p)}\mathcal{P}(f(\varphi)) = \frac{1}{f'}\Big(&\mathcal{P}(\varphi) + \frac{k}{2\pi}\frac{f''}{f'} - \frac{\kappa}{2\pi}(p\circ f)'(\varphi)\Big).\end{aligned}$$

4.3.2 Coadjoint orbits

Here we classify the coadjoint orbits of the warped Virasoro group and their corresponding little group. The main point is that coadjoint orbits $\mathcal{O}_{(\mathcal{L}_0,\mathcal{P}_0)}$ are defined as all elements of the dual space $(\mathcal{L},\mathcal{P})$ which can be obtained from a fixed representative $(\mathcal{L}_0,\mathcal{P}_0)$ by the coadjoint action (4.137). The orbit is isomorphic to the symplectic manifold $\mathcal{O}\sim G/\mathcal{H}_{(\mathcal{L}_0,\mathcal{P}_0)}$, where G is the group manifold and $\mathcal{H}_{(\mathcal{L}_0,\mathcal{P}_0)}$ is the stabilizer subgroup of the orbit, consisting of all elements which leave $(\mathcal{L}_0,\mathcal{P}_0)$ invariant under the coadjoint action (4.137). From now on we will take $(\mathcal{L}_0,\mathcal{P}_0)$ to be constant. Through the reduction of the Chern-Simons theory with constant charges, we should be able to relate the orbit representatives $(\mathcal{L}_0,\mathcal{P}_0)$ to the Chern-Simons charges of the bulk solution. Generically, the infinitesimal coadjoint action (4.137) gives

$$\begin{aligned}\mathrm{ad}^*_{\epsilon_L,\epsilon_P}\mathcal{L}(\varphi) =&\epsilon_L\mathcal{L}' + 2\epsilon_L'\mathcal{L} - \frac{c}{24\pi}\epsilon_L''' + \epsilon_P'\mathcal{P} + \frac{k}{2\pi}\epsilon_P'',\\ \mathrm{ad}^*_{\epsilon_L,\epsilon_P}\mathcal{P}(\varphi) =&\epsilon_L'\mathcal{P} + \epsilon_L\mathcal{P}' - \frac{k}{2\pi}\epsilon_L'' + \frac{\kappa}{2\pi}\epsilon_P'.\end{aligned} \tag{4.138}$$

We will here focus on orbits with constant representatives, hence they as obtained from the coadjoint action of the algebra on constant $\mathcal{L}_0$ and $\mathcal{P}_0$, in which case the above formula simplifies to

$$\mathrm{ad}^*_{\epsilon_L,\epsilon_P}\mathcal{L}_0 =2\epsilon_L'\mathcal{L}_0 - \frac{c}{24\pi}\epsilon_L''' + \epsilon_P'\mathcal{P}_0 + \frac{k}{2\pi}\epsilon_P'', \tag{4.139}$$

$$\mathrm{ad}^*_{\epsilon_L,\epsilon_P}\mathcal{P}_0 =\epsilon_L'\mathcal{P}_0 - \frac{k}{2\pi}\epsilon_L'' + \frac{\kappa}{2\pi}\epsilon_P'. \tag{4.140}$$

As we will see below, whenever $\kappa \neq 0$, one can make a field redefinition such that $k = 0$, so we should distinguish two cases, $\kappa = 0$ and $k \neq 0$ or $\kappa \neq 0$ and $k = 0$. Let us focus first on the case where $\kappa = 0$.

• $\kappa = 0$ **and** $k \neq 0$

Let us first classify the orbits $\mathcal{O}_p$ for $p \in \mathfrak{g}^*$ under the σ-action of G. This amounts to solving the differential equation

$$\epsilon_L' \mathcal{P}_0 - \frac{k}{2\pi} \epsilon_L'' = 0 \,, \tag{4.141}$$

for ϵ_L a function on the S^1. The general solution is

$$\epsilon_L = a + b\, e^{\frac{2\pi \mathcal{P}_0}{k} \varphi} \,. \tag{4.142}$$

This is only well defined on the S^1 whenever $\mathcal{P}_0 = ikn/2\pi$ and hence there are two distinct cases

- $\mathcal{P}_0 \neq ikn/2\pi$: $b = 0$ and the little group G_p is $U(1)$
- $\mathcal{P}_0 = ikn/2\pi$: $b \neq 0$ and the little group G_p is two-dimensional: $i[L_0, L_n] = -nL_n$ for $L_n = e^{in\varphi}$ and $L_0 = 1$

In the first case, the remaining equation (4.139) gives

$$\epsilon_P' \mathcal{P}_0 + \frac{k}{2\pi} \epsilon_P'' = 0 \tag{4.143}$$

which is solved by $\epsilon_P = c + d e^{-\frac{2\pi \mathcal{P}_0}{k} \varphi}$. This once again gives only one solution on the circle, under the assumption that $\mathcal{P}_0 \neq ikn/2\pi$. Hence these orbits are characterized by a two-dimensional Abelian little group.

The second case is more interesting. In that case the remaining equation (4.139) gives

$$in\epsilon_P' + \epsilon_P'' = -\frac{2\pi i n}{k}(2\mathcal{L}_0 + \frac{c}{24\pi} n^2) e^{in\varphi} \equiv a_n e^{in\varphi} \,. \tag{4.144}$$

In this case, the solution reads

$$\epsilon_P = c + d e^{-in\varphi} - \frac{a_n}{2n^2} e^{in\varphi} \,. \tag{4.145}$$

Hence, *regardless* of the value of $\mathcal{L}_0$, this solution is well defined on the circle, however, only when $a_n = 0$, or, whenever

$$\mathcal{L}_0 = -\frac{c}{48\pi} n^2 \,, \tag{4.146}$$

does the algebra $[(\epsilon_L,\epsilon_P),(\epsilon_L,\epsilon_P)]$, understood as (4.132), close linearly. In that case, the algebra written in the form of the generators $L_0=(1,0)$, $L_n=(e^{in\varphi},0)$, $P_0=(0,1)$, and $P_{-n}=(0,e^{-in\varphi})$ reads:

$$\begin{aligned} i[L_0,L_n]&=-nL_n\,, \qquad & i[L_0,P_{-n}]&=nP_{-n}\,,\\ i[L_n,P_{-n}]&=nP_0\,, \qquad & i[P_0,L_n]&=0\,. \end{aligned} \tag{4.147}$$

In the case of $n=1$ this gives exactly the P_2^c algebra, or the central extension of the two dimensional Poincaré algebra.

To summarize, if $\mathcal{P}_0\neq ikn/2\pi$, the little group is $U(1)\times U(1)$ and for $\mathcal{P}_0=ikn/2\pi$, it necessarily implies that $\mathcal{L}_0=-\frac{c}{48\pi}n^2$ and the little group has the algebra (4.147).

• $\kappa\neq 0$

In this case, the analysis in slightly different, because the semi-direct product does not act on an Abelian group. It is always possible to set k to zero when $\kappa\neq 0$ [178]. Indeed, by making the redefinition

$$\epsilon_P\to\epsilon_P+\frac{k}{\kappa}\epsilon_L'\,, \tag{4.148}$$

the equation (4.140), set to zero in order to analyze the orbits of the σ action of G on $C^\infty(S^1)$, gives

$$\epsilon_P=a-\frac{2\pi\mathcal{P}_0}{\kappa}\epsilon_L\,. \tag{4.149}$$

Note that now there is no special value for $\mathcal{P}_0$. The gauge parameter ϵ_P is completely fixed in terms of ϵ_L, up to a constant a (which represents the $U(1)$ factor in the little group).

Plugging the above into (4.139), redefining

$$c_{eff}=c-12\frac{k^2}{\kappa}\,, \tag{4.150}$$

and setting this to zero gives

$$\epsilon_L'\mathfrak{L}_0-4\epsilon_L'''=0\,, \tag{4.151}$$

where

$$\mathfrak{L}_0=\frac{12\pi}{c_{eff}}\left(\mathcal{L}_0-\frac{\pi\mathcal{P}_0^2}{\kappa}\right)\,. \tag{4.152}$$

This is the usual differential equation which characterizes the little group of Virasoro coadjoint orbits. Hence there are two distinct cases

- $\mathfrak{L}_0\neq-\frac{n^2}{4}$. These are the generic orbits with $U(1)$ little group

- $\mathfrak{L}_0 = -\frac{n^2}{4}$. In this case the orbits little group is enhanced to the n-fold cover of $SL(2,\mathbb{R})$.

Combined with the previous $U(1)$ little group in (4.149), this implies that coadjoint orbits of $\kappa \neq 0$ warped Virasoro group are characterized by the value of the Sugawara shifted $\mathfrak{L}_0$. For generic values of this quantity there is a two-dimensional Abelian little group, whereas for $\mathfrak{L}_0 = -\frac{n^2}{4}$ the little group on the coadjoint orbits is $SL^{(n)}(2,\mathbb{R}) \times U(1)$.

4.3.3 Geometric actions

The coadjoint orbits form symplectic manifolds, with the symplectic structure given by the Konstant-Kirillov symplectic form ω. For the warped Virasoro group, this has been constructed in [169], which we follow here in our own notation:

$$\omega_{12} = \langle (\mathcal{L},\mathcal{P};c,k,\kappa), \mathrm{ad}_{(X_1,p_1)}\,(X_2,p_2;\lambda_2,\mu_2,\nu_2)\rangle\,. \tag{4.153}$$

See also [158] for the generic construction of the geometric actions for centrally extended semi-direct product groups, such as the warped Virasoro group. To consider the orbit $\mathcal{O}_{(\mathcal{L}_0,\mathcal{P}_0)}$, we take the dual space elements to be

$$\begin{aligned}
\mathcal{L}(\varphi) &= f'^2\left(\mathcal{L}_0 + \mathcal{P}_0\partial_f(p\circ f) - \frac{c}{24\pi}\frac{\{f;\varphi\}}{f'^2} + \frac{k}{2\pi}\partial_f^2(p\circ f) + \frac{\kappa}{4\pi}[\partial_f(p\circ f)]^2\right)\,,\\
\mathcal{P}(\varphi) &= f'\left(\mathcal{P}_0 - \frac{k}{2\pi}\frac{f''}{f'^2} + \frac{\kappa}{2\pi}\partial_f(p\circ f)\right)\,.
\end{aligned} \tag{4.154}$$

Using this, (4.153) reads explicitly

$$\begin{aligned}
\omega_{12} = &-\oint \mathrm{d}\varphi\,\left(f'^2(\mathcal{L}_0 + \partial_f(p\circ f)\mathcal{P}_0)\,(X_1X_2' - X_2X_1') + \mathcal{P}_0 f'\,(X_1p_2' - X_2p_1')\right)\\
&+\frac{c}{24\pi}\oint \mathrm{d}\varphi\,\left(X_1''X_2' + \{f,\varphi\}(X_1X_2' - X_2X_1')\right)\\
&+\frac{k}{2\pi}\oint \mathrm{d}\varphi\,\left(X_1'p_2' - X_2'p_1' - f'^2\partial_f^2(p\circ f)(X_1X_2' - X_2X_1') + (\log f')'(X_1p_2' - X_2p_1')\right)\\
&-\frac{\kappa}{4\pi}\oint \mathrm{d}\varphi\,\left(\mathbf{2}p_1p_2' + (p\circ f)'^2(X_1X_2' - X_2X_1') + 2(p\circ f)'(X_1p_2' - X_2p_1')\right)
\end{aligned} \tag{4.155}$$

In terms of differential forms, the above expression can be brought in the form

$$\begin{aligned}
\omega = &-\oint \mathrm{d}\varphi\,\left(\mathcal{L}_0 f'^2\mathrm{d}X\wedge\mathrm{d}X' + \mathcal{P}_0 f'\mathrm{d}X\wedge(\mathrm{d}p + (p\circ f)'\mathrm{d}X)'\right)\\
&-\frac{c}{48\pi}\oint \mathrm{d}\varphi\,\left(\mathrm{d}X'\wedge\mathrm{d}X'' - 2\{f;\varphi\}\mathrm{d}X\wedge\mathrm{d}X'\right)\\
&+\frac{k}{2\pi}\oint \mathrm{d}\varphi\,\left(\mathrm{d}X' + (\log f')'\mathrm{d}X\right)\wedge\left(\mathrm{d}p + (p\circ f)'\mathrm{d}X)'\right)\\
&-\frac{\kappa}{4\pi}\oint \mathrm{d}\varphi\,(\mathrm{d}p + (p\circ f)'\mathrm{d}X)\wedge(\mathrm{d}p + (p\circ f)'\mathrm{d}X)'
\end{aligned} \tag{4.156}$$

We now change variables to the finite transformations, by using that

$$\mathrm{d}f = f'\mathrm{d}X\,, \qquad \mathrm{d}g = \mathrm{d}p + \partial_\varphi(p\circ f)\mathrm{d}X\,, \tag{4.157}$$

where $g \equiv p\circ f$ does not only transform under $\mathrm{Diff}(S^1)$ but also under $C^\infty(S^1)$. The result simplifies even further by writing it as

$$\begin{aligned}\omega = -\oint \mathrm{d}\varphi \Big(&\mathcal{L}_0\mathrm{d}f\wedge\mathrm{d}f' + \mathcal{P}_0\mathrm{d}f\wedge\mathrm{d}g' \\ &+\frac{c}{48\pi}\mathrm{d}\log f'\wedge(\mathrm{d}\log f')' - \frac{k}{2\pi}\mathrm{d}\log f'\wedge\mathrm{d}g' + \frac{\kappa}{4\pi}\mathrm{d}g\wedge\mathrm{d}g'\Big).\end{aligned} \tag{4.158}$$

It is now strikingly clear that whenever $\kappa\neq 0$, the term proportional to k can be removed by redefining

$$g = \tilde{g} + \frac{k}{\kappa}\log f'\,. \tag{4.159}$$

The result is (dropping a total derivative term proportional to $\mathcal{P}_0$):

$$\omega = -\oint \mathrm{d}\varphi\,\Big(\mathcal{L}_0\mathrm{d}f\wedge\mathrm{d}f' + \mathcal{P}_0\mathrm{d}f\wedge\mathrm{d}g' + \frac{c_{\text{eff}}}{48\pi}\mathrm{d}\log f'\wedge(\mathrm{d}\log f')' + \frac{\kappa}{4\pi}\mathrm{d}g\wedge\mathrm{d}g'\Big). \tag{4.160}$$

where

$$c_{\text{eff}} = c - 12\frac{k^2}{\kappa}\,. \tag{4.161}$$

Now that we have the Konstant-Kirillov symplectic form for any orbit with constant representatives $(\mathcal{L}_0,\mathcal{P}_0)$, we can obtain the geometric action on this orbit. Since $\mathrm{d}\omega=0$, we can locally write $\omega=\mathrm{d}\alpha$. Then the kinetic part of the geometric action on the coadjoint orbit is the integral over the orbit of α, i.e.

$$\begin{aligned}I[f,g;\mathcal{L}_0,\mathcal{P}_0] &= \int_\gamma \alpha \\ &= \int_\gamma\oint\mathrm{d}\varphi\Big(\mathcal{L}_0 f'\mathrm{d}f + \mathcal{P}_0 f'\mathrm{d}g + \frac{c}{48\pi}\frac{f''\mathrm{d}f'}{f'^2} + \frac{k}{2\pi}\frac{f''}{f'}\mathrm{d}g + \frac{\kappa}{4\pi}g'\mathrm{d}g\Big)\,.\end{aligned} \tag{4.162}$$

Here γ is a path along the orbit. By parameterizing γ by a coordinate t on the orbit and using $\mathrm{d}=\mathrm{d}t\partial_t$ we find the two dimensional action

$$I = \int\mathrm{d}t\mathrm{d}\varphi\,\left(\mathcal{L}_0 f'\dot{f} + \frac{c_{\text{eff}}}{48\pi}\frac{f''\dot{f}'}{f'^2}\right) + \int\mathrm{d}t\mathrm{d}\varphi\,\left(\mathcal{P}_0 f'\dot{\tilde{g}} + \frac{k}{4\pi}\tilde{g}'\dot{\tilde{g}}\right)\,. \tag{4.163}$$

Here dots denote t-derivatives and we have used the redefinition (4.159) to remove the term proportional to k. The above action is expressed in terms of $\tilde{g}\circ f\circ\varphi$. By redefining $f\circ\varphi$ as $\tilde{\varphi}$ and switching variables as $\tilde{g}(\tilde{\varphi})=\Phi(\varphi)$, we may equivalently write the action above as:

$$I = \int\mathrm{d}t\mathrm{d}\varphi\,\left(\mathcal{L}_0 f'\dot{f} + \frac{c_{\text{eff}}}{48\pi}\frac{f''\dot{f}'}{f'^2} + \frac{\kappa}{4\pi}\Phi'\dot{\Phi} + \mathcal{P}_0\dot{\Phi}f'\right)\,. \tag{4.164}$$

The Hamiltonian for this action can be found as the action of L_0 and P_0, i.e. we can add $-\int \mathrm{d}t(L_0+P_0)$ to (4.163), as in section 4 of [169] and in (4.32) for the Virasoro group. This generalizes the warped Schwarzian action (4.5) of [169] to a generic orbit parameterized by $\mathcal{L}_0$ and $\mathcal{P}_0$. For further insights, we refer the reader to [158] for a discussion on invariant Hamiltonians for geometric actions. The key concept is that L_0 always belongs to the stabilizer subgroup of the orbit, thereby generating a gauge symmetry on the orbit. The generator of this gauge symmetry can then be included in the action as the Hamiltonian, preserving the gauge symmetry.

From (4.160), we can make the field redefinition $\tilde{g} \to \tilde{g} - \frac{2\pi}{\kappa}\mathcal{P}_0 f$ and get the following symplectic form:

$$\omega = -\oint \mathrm{d}\varphi \left((\mathcal{L}_0 - \frac{\pi}{\kappa}\mathcal{P}_0^2)\mathrm{d}f \wedge \mathrm{d}f' + \frac{c_{\text{eff}}}{48\pi}\mathrm{d}\log f' \wedge (\mathrm{d}\log f')' + \frac{\kappa}{4\pi}\mathrm{d}g \wedge \mathrm{d}g' \right). \tag{4.165}$$

leading to the action

$$I = \int \mathrm{d}t\mathrm{d}\varphi \left((\mathcal{L}_0 - \frac{\pi}{\kappa}\mathcal{P}_0^2) f'\dot{f} + \frac{c_{\text{eff}}}{48\pi}\frac{f''\dot{f}'}{f'^2} + \frac{\kappa}{4\pi}\Phi'\dot{\Phi} \right) . \tag{4.166}$$

But now the field Φ is no longer periodic around the circle. So we need to make a last field redefinition $\Phi \to \Phi + \frac{2\pi}{\kappa}\mathcal{P}_0\varphi$ to get the final geometric action after dropping some boundary terms:

$$I = \int \mathrm{d}t\mathrm{d}\varphi \left((\mathcal{L}_0 - \frac{\pi}{\kappa}\mathcal{P}_0^2) f'\dot{f} + \frac{c_{\text{eff}}}{48\pi}\frac{f''\dot{f}'}{f'^2} + \frac{\kappa}{4\pi}\Phi'\dot{\Phi} + \mathcal{P}_0\dot{\Phi} \right) . \tag{4.167}$$

Adding the Hamiltonian

$$H = \mu L_0 + \nu P_0 \, , \tag{4.168}$$

where μ and ν are chemical potentials and L_0 and P_0 are the zero modes of $\mathcal{L}(\varphi)$ and $\mathcal{P}(\varphi)$ respectively:

$$L_0 = \int \mathrm{d}\varphi \left((\mathcal{L}_0 - \frac{\pi}{\kappa}\mathcal{P}_0^2) f'^2 + \frac{c_{\text{eff}}}{48\pi}\frac{f''^2}{f'^2} + \frac{\kappa}{4\pi}\Phi'^2 + \frac{\pi}{\kappa}\mathcal{P}_0^2 \right) , \quad P_0 = \int \mathrm{d}\varphi \mathcal{P}_0 \, , \tag{4.169}$$

this leads to the action

$$\begin{aligned} S &= I - \int \mathrm{d}t \, H \\ &= \int \mathrm{d}t\mathrm{d}\varphi \left((\mathcal{L}_0 - \frac{\pi}{\kappa}\mathcal{P}_0^2) f'\partial_- f + \frac{c_{\text{eff}}}{48\pi}\frac{f''\partial_- f'}{f'^2} + \frac{\kappa}{4\pi}\Phi'\partial_-\Phi + \mathcal{P}_0\dot{\Phi} - \mu\frac{\pi}{\kappa}\mathcal{P}_0^2 - \nu\mathcal{P}_0 \right) . \end{aligned} \tag{4.170}$$

This action represents the geometric action on the coadjoint orbits of the warped Virasoro group with orbit representatives $(\mathcal{L}_0, \mathcal{P}_0)$. We expect it to match the reduction of the Chern-Simons theory with constant charges that we will compute in the next section.

4.3.4 Chern-Simons reduction

We start with a $SL(2,\mathbb{R})\times U(1)$ Chern-Simons action:

$$S_{CS}=\frac{k}{4\pi}\int_{\mathcal{M}}\left\langle A\wedge \mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\right\rangle+\frac{\kappa}{4\pi}\int_{\mathcal{M}}\langle C\wedge \mathrm{d}C\rangle\,, \tag{4.171}$$

written in Hamiltonian form:

$$S_{CS}=\frac{k}{4\pi}\int_{\mathcal{M}}\left\langle A_\varphi\dot{A}_r-A_r\dot{A}_\varphi+2A_tF_{r\varphi}\right\rangle \mathrm{d}^3x+\frac{\kappa}{2\pi}\int_{\mathcal{M}}(C_\varphi\dot{C}_r+C_t\tilde{F}_{r\varphi})\,\mathrm{d}^3x\,, \tag{4.172}$$

where $F=\mathrm{d}A+A\wedge A$ and $\tilde{F}=\mathrm{d}C$. We suppose that the topology of our manifold $\mathcal{M}$ is a annulus times $\mathbb{R}$, such that there are two boundaries at $r=r_i$ and $r=r_o$, respectively for the inner and the outer ones. On these boundaries, we impose the following boundary conditions for μ and ν constant:

$$\begin{aligned}
&\text{At } r=r_o:\begin{cases}A_\varphi=L_1-\mathfrak{L}L_{-1}\\ A_t=\mu A_\varphi\end{cases}\text{ and }\begin{cases}C_\varphi=\frac{2\pi}{\kappa}\mathcal{P}\\ C_t=\mu C_\varphi+\nu\end{cases}\\
&\text{At } r=r_i:\begin{cases}A_\varphi=L_{-1}-\tilde{\mathfrak{L}}L_1\\ A_t=-\tilde{\mu}A_\varphi\end{cases}\text{ and }\begin{cases}C_\varphi=\frac{2\pi}{\kappa}\tilde{\mathcal{P}}\\ C_t=-\tilde{\mu}C_\varphi-\tilde{\nu}\end{cases}
\end{aligned} \tag{4.173}$$

with $\mathfrak{L}=\frac{2\pi}{k}(\mathcal{L}-\frac{\pi}{\kappa}\mathcal{P}^2)$ and $\tilde{\mathfrak{L}}=\frac{2\pi}{k}(\tilde{\mathcal{L}}-\frac{\pi}{\kappa}\tilde{\mathcal{P}}^2)$. The equations of motion fix the time evolution of $\mathcal{L}$ and $\mathcal{K}$ for constant chemical potentials:

$$\partial_t\mathcal{L}=\mu\mathcal{L}'\ ,\ \partial_t\mathcal{P}=\mu\mathcal{P}'\,, \tag{4.174}$$

where the prime denotes a derivative with respect to φ.

To have a well defined variational principle, we need to add boundary terms to the action:

$$\begin{aligned}
I_{\Sigma_o}&=-\frac{k}{4\pi}\int_{\Sigma_o}\mu\left\langle A_\varphi^2\right\rangle \mathrm{d}^2x-\frac{\kappa}{4\pi}\int_{\Sigma_o}(\mu C_\varphi^2+2\nu C_\varphi)\,\mathrm{d}^2x && (4.175)\\
&=-\int_{\Sigma_o}(\mu\mathcal{L}+\nu\mathcal{P})\,\mathrm{d}^2x\,,\\
I_{\Sigma_i}&=-\frac{k}{4\pi}\int_{\Sigma_i}\tilde{\mu}\left\langle A_\varphi^2\right\rangle \mathrm{d}^2x-\frac{\kappa}{4\pi}\int_{\Sigma_i}(\tilde{\mu}C_\varphi^2+2\tilde{\nu}C_\varphi)\,\mathrm{d}^2x && (4.176)\\
&=-\int_{\Sigma_i}(\tilde{\mu}\tilde{\mathcal{L}}+\tilde{\nu}\tilde{\mathcal{P}})\,\mathrm{d}^2x\,.
\end{aligned}$$

Our final action is then

$$S[A,C]=S_{CS}+I_{\Sigma_i}+I_{\Sigma_o}\,. \tag{4.177}$$

Solving the constraints $F_{r\varphi}=0$ and $\tilde{F}_{r\varphi}=0$ and taking care of the holonomy, we get:

$$\begin{aligned}
&A_r=G^{-1}\partial_rG\,, && A_\varphi=G^{-1}(\partial_\varphi+K(t))G\,,\\
&C_r=\partial_r\lambda\,, && C_\varphi=\partial_\varphi\lambda+k_0(t)\,,
\end{aligned} \tag{4.178}$$

where G is a group element of $SL(2,\mathbb{R})$ and is periodic, such as λ. $K(t)$ is a function parametrizing the holonomy and taking value in the Lie algebra.

First, we will focus on the $U(1)$-part of the action (4.177). The $U(1)$-part in (4.172) can be expressed as

$$S[\Phi(t,\varphi),\Psi(t,\varphi),k_0(t)] = \frac{\kappa}{4\pi}\int \left(\partial_\varphi\Phi\partial_t\Phi - \partial_\varphi\Psi\partial_t\Psi + 2k_0(\partial_t\Phi - \partial_t\Psi)\right) \mathrm{d}t\, \mathrm{d}\varphi\,, \tag{4.179}$$

where $\lambda(r=r_o,t,\varphi) = \Phi(t,\varphi)$ and $\lambda(r=r_i,t,\varphi) = \Psi(t,\varphi)$. From the boundary terms (4.175) and (4.176), we have

$$I_B = -\frac{\kappa}{4\pi}\int \Big(\mu(\partial_\varphi\Phi)^2 + \tilde{\mu}(\partial_\varphi\Psi)^2 + (\mu+\tilde{\mu})k_0^2 + 2(\nu+\tilde{\nu})k_0\Big)\, \mathrm{d}t\, \mathrm{d}\varphi\,. \tag{4.180}$$

Using the boundary conditions (4.173), the functions Φ and Ψ are constrained by

$$\begin{aligned} \partial_\varphi\Phi + k_0 &= \frac{2\pi}{\kappa}\mathcal{P}\,,\\ \partial_\varphi\Psi + k_0 &= \frac{2\pi}{\kappa}\tilde{\mathcal{P}}\,. \end{aligned} \tag{4.181}$$

So to summarize, we have

$$S[\Phi(t,\varphi),\Psi(t,\varphi),k_0(t)] = S^{(2)} + S^{(1)} + S^{(0)}\,, \tag{4.182}$$

with

$$\begin{aligned} S^{(2)} &= \frac{\kappa}{4\pi}\int \mathrm{d}^2x\, \Big(\partial_\varphi\Phi\partial_t\Phi - \mu(\partial_\varphi\Phi)^2\Big)\,,\\ S^{(1)} &= \frac{\kappa}{4\pi}\int \mathrm{d}^2x\, \Big(-\partial_\varphi\Psi\partial_t\Psi - \tilde{\mu}(\partial_\varphi\Psi)^2\Big)\,,\\ S^{(0)} &= \frac{\kappa}{2\pi}\int \mathrm{d}^2x\, \Big(k_0(\partial_t\Phi - \partial_t\Psi) - \frac{\mu+\tilde{\mu}}{2}k_0^2 - (\nu+\tilde{\nu})k_0\Big)\,. \end{aligned} \tag{4.183}$$

This action is invariant under the $U(1)$ gauge transformation

$$\Phi \to \Phi + \epsilon(t)\,, \qquad \Psi \to \Psi + \epsilon(t)\,, \qquad k_0 \to k_0\,, \tag{4.184}$$

and under the $\frac{\widehat{LG}\times\widehat{LG}}{G}$ global symmetry with $G = U(1)$ [72, 144]

$$\Phi \to \Phi + \alpha(\varphi)\,, \qquad \Psi \to \Psi + \beta(\varphi)\,, \tag{4.185}$$

where $\widehat{LG}$ is the loop group of a topological group G [196]. The quotient by G arises from the sharing of the same zero mode, related to the holonomy k_0, for Φ and Ψ.

If we want to focus only on one boundary:

$$S^{\Sigma_o}_{\text{bdy}} = \frac{\kappa}{4\pi}\int d^2x\, \Big(\Phi'\partial_-\Phi + 2k_0\dot{\Phi} - \mu k_0^2 - 2\nu k_0\Big)\,, \tag{4.186}$$

where we have defined $\partial_- = \partial_t - \mu\partial_\varphi$ and dropped a total derivative at the time boundaries. The remaining action has lost its gauge invariance and is left with only a $\widehat{LG}$ global symmetry.

Now, let us talk about the $Sl(2,\mathbb{R})$ part of the action (4.177). The computation was already done in Section 4.2.6 for similar boundary conditions. The only difference is the presence of μ in the boundary conditions (4.173) (set to 1 in Section 4.2.6) and the replacement of $\mathcal{L}$ by $\mathfrak{L}$. In summary, we have at the outer boundary:

$$S^{\Sigma_o}_{\text{bdy}} = \frac{k}{8\pi}\int_{\Sigma_o} \mathrm{d}t\mathrm{d}\varphi \left[\frac{\partial_- f' f''}{f'^2} + \alpha \partial_- f f'\right] , \tag{4.187}$$

and

$$\mathfrak{L} = \frac{\alpha}{4} f'^2 - \frac{1}{2}\{f;\varphi\} , \tag{4.188}$$

with α defined in (4.102) A similar action holds for the inner boundary. The fields Φ and f living on the outer boundary are coupled through the boundary conditions (4.173):

$$\begin{aligned} \Phi' + k_0 &= \frac{2\pi}{\kappa}\mathcal{P} , \\ \frac{\alpha}{2} f'^2 - \{f;\varphi\} &= \frac{4\pi}{k}(\mathcal{L} - \frac{\pi}{\kappa}\mathcal{P}^2) , \end{aligned} \tag{4.189}$$

and follow the action

$$S^{\Sigma_o}_{\text{bdy}} = \frac{k}{8\pi}\int_{\Sigma_o} \mathrm{d}^2x \left(\frac{\partial_- f' f''}{f'^2} + \alpha\partial_- f f'\right) + \frac{\kappa}{4\pi}\int \mathrm{d}^2 x \left(\Phi'\partial_-\Phi + 2k_0\dot{\Phi} - \mu k_0^2 - 2\nu k_0\right) . \tag{4.190}$$

We can now compare (4.190) with (4.170) and observe that

$$\mathcal{P}_0 = \frac{\kappa}{2\pi}k_0 , \qquad c_{\text{eff}} = 6k , \qquad \mathcal{L}_0 - \frac{\pi}{\kappa}\mathcal{P}_0^2 = \frac{k}{8\pi}\alpha . \tag{4.191}$$

The residual symmetries of (4.190) are a $U(1)$ gauge symmetry for the field f, in order to maintain the conjugacy class of the holonomy α, as long as $\alpha \neq -n^2$ [72] and $SL(2,\mathbb{R})^{(n)}$ when $\alpha = -n^2$, while the field Φ has a $\widehat{LG}$ global symmetry [72]. All of this should be in concordance with the little group of the coadjoint orbits for constant representatives in the case $\kappa \neq 0$.

4.3.5 Dimensional reduction and link with warped Schwarzian theory

The idea for this section is to perform the dimensional reduction of our action (4.190) as for Liouville theory where it ends up with a Schwarzian theory [75].

So the first step is to make a Wick rotation $t = iy$. Then we take the time circle to be very small, $\Delta y \to 0$, with $k' = k\Delta y$ and $\kappa' = \kappa\Delta y$. With that, we end up with

$$S = \frac{\mu k'}{8\pi}\int \mathrm{d}\varphi \left(\frac{f''^2}{f'^2} + \alpha f'^2\right) + \frac{\mu\kappa'}{4\pi}\int \mathrm{d}\varphi\left(\Phi'^2 + k_0^2 + 2\frac{\nu}{\mu}k_0\right) . \tag{4.192}$$

Now, up to some boundary terms, we can rewrite it with the Schwarzian derivative (2.28):

$$
\begin{aligned}
S &= \frac{\mu k'}{8\pi}\int \mathrm{d}\varphi \left(-2\left\{f;\varphi\right\} + \alpha f'^2\right) + \frac{\mu\kappa'}{4\pi}\int \mathrm{d}\varphi \left(\Phi'^2 + k_0^2 + 2\frac{\nu}{\mu}k_0\right) \\
&= -\frac{\mu k'}{8\pi}\int \mathrm{d}\varphi \left\{\tan\left(\frac{\sqrt{\alpha}}{2}f\right);\varphi\right\} + \frac{\mu\kappa'}{4\pi}\int \mathrm{d}\varphi\ \Phi'^2 + \int \mathrm{d}\varphi \left(\mu\mathcal{L}_0' + \nu\mathcal{P}_0'\right),
\end{aligned}
\tag{4.193}
$$

where for the last step we used the equations (4.191). The last equation is the Warped Schwarzian action in [169] with additional source terms.

4.3.6 One-loop partition function

We are now interested in computing the partition function for our action (4.190) in a similar manner as Section 5 of [75]. The general idea behind this strategy was outlined in [170,171,197]. The canonical ensemble partition function can be thought of an Euclidean functional integral weighted by the classical action with typical coupling constant $1/K$ (for AdS_3 gravity, this would be related to the Brown-Henneaux central charge $c = 24K$). At large K the dominant contribution to the path integral is given by the saddle point approximation, i.e. as a sum over classical saddles satisfying the appropriate boundary conditions of an exponential whose argument consist in a series expansion starting with the classical action evaluated on the saddle, followed by subleading terms representing quantum corrections to the effective action at n^{th} order in perturbation theory (as it turns out, there are instances where the partition function happens to be one-loop exact). On general grounds, the partition function is expected to be expressible as a sum of characters of the corresponding symmetry algebra - Virasoro for AdS_3 gravity, and BMS_3 for flat space holography. This indeed turns out to be the case [73,75,171]. In the present situation the relevant characters are those of the warped conformal algebra taking the following form [43] (see Section 2.1.4 for a brief review and notations):

$$
\chi_{h,p} = q^{h-\frac{c}{24}} y^p \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} \left(1-\delta_{\text{vac}}\, q\right) \qquad \text{for } p\in\mathbb{R}\,, \tag{4.194}
$$

$$
\chi_{h,p} = q^{h-\frac{c}{24}} y^p \prod_{n=1}^{\infty} \frac{1}{(1-q^{n})^2} \left(1-\delta_{\text{vac}}\, q\right) \qquad \text{for } p\in i\mathbb{R}\,. \tag{4.195}
$$

We will therefore expand the partition function for our action (4.190) in orders of k and κ and limit ourselves to the 1-loop contribution. We refer the reader to [73] for a sketch of the proof of the one-loop exactness of the partition function for any geometric action defined from the Kirillov-Konstant symplectic form. We will investigate whether this allows us to recover the above characters.

From now on, we will assume that $\mu = 0$ and $\nu = 1$, as suggested in [66]. This implies that the mass and angular momentum of the solutions with boundary conditions (4.175) are given by

$$M=\pi\mathcal{P}\,,\qquad J=-2\pi\mathcal{L}\,,\tag{4.196}$$

and the thermal entropy is the one of a spacelike warped AdS_3 black hole and a WCFT at finite temperature. The action (4.190) then reduces to

$$S^{\Sigma_o}_{\rm bdy}=\frac{k}{8\pi}\int_{\Sigma_o}\mathrm{d}^2x\left(\frac{\dot f' f''}{f'^2}+\alpha\dot f f'\right)+\frac{\kappa}{4\pi}\int\mathrm{d}^2x\left(\Phi'\dot\Phi+2k_0(\dot\Phi-1)\right)\,.\tag{4.197}$$

To compute the one-loop partition function, we first need the classical saddle point of the action (4.197) and then evaluate the euclidean action on this saddle point. The equations of motion derived from (4.197) for constant orbit representative are

$$\begin{aligned}\frac{1}{f'}\partial_t\left(\frac{\alpha}{2}f'^2-\{f;\varphi\}\right)&=0\,,\\ \partial_t\Phi'&=0\,.\end{aligned}\tag{4.198}$$

Using (4.189) and supposing $f'\neq 0$, we recover the equations of motion of the original Chern-Simons theory (4.174) for our choice of chemical potentials. The solutions of interest to the saddle point approximation are those with constant $\mathfrak{L}$ and $\mathcal{P}$, where their values are given by the zero modes $\mathcal{L}_0$ and $\mathcal{P}_0$, implying from (4.189) the equations

$$\frac{\alpha}{2}\left(f'^2-1\right)=\{f;\varphi\}\,,\qquad \Phi'=0\,.\tag{4.199}$$

We get as saddle point the solution:

$$f(\varphi,t)=\varphi+c_1(t)\,,\qquad \Phi(\varphi,t)=c_2(t)\,.\tag{4.200}$$

We make a Wick rotation $t=-iy$ and compute the euclidean action $iS_E=S^{\Sigma_o}_{\rm bdy}$,

$$S_E=-i\frac{k}{8\pi}\int_{\Sigma_o}\mathrm{d}^2x\left(\frac{\dot f' f''}{f'^2}+\alpha\dot f f'\right)-\frac{\kappa}{4\pi}\int\mathrm{d}^2x\left(i\Phi'\dot\Phi+2ik_0\dot\Phi-2k_0\right)\,.\tag{4.201}$$

For this part, we will separate the discussion into two cases. The first one for a real orbit representative $\mathcal{P}_0$ and the second one for a purely imaginary $\mathcal{P}_0$. Let us begin with a real $U(1)$ holonomy, which corresponds to warped black hole solutions, and derive the Hamiltonian as the real part of the action

$$H=\kappa k_0=2\pi\mathcal{P}_0\,,\tag{4.202}$$

which could already be seen from (4.168) with $\mu=0$ and $\nu=1$. It corresponds to an orbit representative $\mathcal{P}_0$ that is also real. For a given value of the holonomy, the Hamiltonian is always bounded from below. It is interesting to note that in the case of $\mu = 0$ and $\nu = 1$, the path integral is well-defined for any value of α, as long as k_0 is real, contrary to the Virasoro case where only values of α greater or equal to -1 were allowed.

Then, we periodically identify the euclidean time y. We get as periodicity conditions on the fields:

$$\begin{aligned} f(\varphi+\Omega\beta, y+\beta) &= f(\varphi, y) \quad , \quad & \Phi(\varphi+\Omega\beta, y+\beta) &= \Phi(\varphi, y)\,, \\ f(\varphi+2\pi, y) &= f(\varphi, y)+2\pi \quad , \quad & \Phi(\varphi+2\pi, y) &= \Phi(\varphi, y)\,. \end{aligned} \tag{4.203}$$

in terms of the fields $c_1(y)$ and $c_2(y)$, it gives

$$c_1(y+\beta) = c_1(y) - \Omega\beta \quad , \quad c_2(y+\beta) = c_2(y)\,. \tag{4.204}$$

So the solution is

$$f_0 = \varphi - \Omega y\,, \quad \Phi_0 = 0\,, \tag{4.205}$$

and the euclidean action evaluated on this saddle point (4.205) is

$$S_E^{(0)} = \beta\left(\kappa k_0 + ik\Omega\mathfrak{L}_0\right)\,. \tag{4.206}$$

From there, we can already state that the 0-loop contribution will give the factors

$$e^{-S_E^{(0)}} = q^{-\frac{c\alpha}{24}}\, y^{i\kappa k_0} \tag{4.207}$$

where we defined

$$q = e^{2\pi i\tau}\,, \qquad y = e^{2\pi i z}\,, \tag{4.208}$$

with $\beta\Omega = 2\pi\tau$ and $\beta = 2\pi z$. Furthermore, we are imposing that β and Ω are determined in terms of the holonomies α and k_0 [66]

$$\exp\left[\frac{i\beta\Omega}{2\pi}\int A_\varphi \mathrm{d}\varphi\right] = -\mathbb{1}\,, \quad \exp\left[\frac{i\beta}{2\pi}\left(\int C_t \mathrm{d}\varphi + \Omega\int C_\varphi \mathrm{d}\varphi\right)\right] = e^{2\pi i\gamma}\,, \tag{4.209}$$

leading to the following relations between β, Ω and the zero-modes:

$$\beta = 2\pi\left(\gamma - \frac{\pi\mathcal{P}_0}{\kappa\sqrt{\mathfrak{L}_0}}\right) \quad , \quad \Omega = \frac{1}{2\gamma\sqrt{\mathfrak{L}_0} - \frac{2\pi\mathcal{P}_0}{\kappa}} \quad , \quad \beta\Omega = \frac{\pi}{\sqrt{\mathfrak{L}_0}}\,. \tag{4.210}$$

The γ is an arbitrary constant and represents a kind of deformation at the level of the holonomy [66]. If it is an integer, one recovers a similar result to BTZ with an holonomy living in the center of the gauge group. However, it is expected that it is not necessarily the case for warped spacetimes. For instance, for the warped black hole in the canonical ensemble (1.13) [66],

$$\gamma = \frac{2\nu}{\nu^2+3}\,. \tag{4.211}$$

Now we can expand the fields $f(\varphi, y)$ and $\Phi(\varphi, y)$ around their saddle points

$$\begin{aligned} f(\varphi,y) &= f_0 + \sum_{m,n} \frac{f_{m,n}}{(2\pi)^2} e^{\frac{2\pi i m y}{\beta}} e^{inf_0}\,, \\ \Phi(\varphi,y) &= \sum_{m,n} \frac{\Phi_{m,n}}{(2\pi)^2} e^{\frac{2\pi i m y}{\beta}} e^{inf_0}\,. \end{aligned} \tag{4.212}$$

One can check that these series expansions satisfy the periodicity conditions (4.203). Plugging this into the euclidean action, it becomes

$$S_E = S_E^{(0)} - \frac{i}{16\pi^3}\sum_{m=-\infty}^{\infty}\sum_{n\neq 0} n(m-\tau n)\left(\frac{k}{2}|f_{m,n}|^2(n^2+\alpha) + \kappa|\Phi_{m,n}|^2\right) + \ldots\,, \tag{4.213}$$

where $f^*_{m,n} = f_{-m,-n}$, $\Phi^*_{m,n} = \Phi_{-m,-n}$ and the dots contains terms of the third order and more on the fields. One does not need to take into account the vacuum in this case since it possesses an imaginary value for $\mathcal{P}_0$ [66].

We can now compute the one-loop partition function by integrating out the fields $f_{m,n}$ and $\Phi_{m,n}$.

$$Z_{\text{1-loop}} = \int \mathcal{D}f\mathcal{D}\Phi\, e^{-S_E} = \mathcal{N} e^{-S_E^{(0)}} \prod_{m,n} n^{-1}(m-\tau n)^{-1}(n^2+\alpha)^{-1/2}\,, \tag{4.214}$$

where $\mathcal{N}$ is a normalization constant that did not depend of β and Ω. To perform the product for the m's, we take the derivative of the logarithm of the partition function with respect to τ:

$$\begin{aligned} \partial_\tau \log Z_{\text{1-loop}} &= -2\pi i k \mathfrak{L}_0 + \sum_{n\neq 0}\sum_{m=-\infty}^{\infty} \frac{n}{m-\tau n} \\ &= -2\pi i k \mathfrak{L}_0 - \sum_{n\neq 0}\left(\frac{1}{\theta} + 2n\sum_{m=1}^{\infty}\frac{\tau n}{\tau^2 n^2 - m^2}\right) \\ &= -2\pi i k \mathfrak{L}_0 - \pi \sum_{n\neq 0} n\cot(n\pi\tau) \\ &= -2\pi i k \mathfrak{L}_0 - 2\pi \sum_{n=1}^{\infty} n\cot(n\pi\tau) \end{aligned}$$

for $n\tau \notin \mathbb{Z}$. This sum diverges but there are several ways to deal with it. One is to use the zeta function regularization to write

$$\sum_{n=1}^{\infty} n\cot(n\pi\tau) = \sum_{n=1}^{\infty} n(\cot(n\pi\tau)+i) - i\sum_{n=1}^{\infty} n\,. \tag{4.215}$$

Now, the first sum on the right hand side converges for $\mathrm{Im}(\tau) > 0$. This requirement is not a problem for negative value of α, such as the vacuum. However for positive value of the holonomy, one needs to perform an analytical continuation $\tau \to \tau +$

$i\varepsilon$ as done in [173]. After exponentiation, the partition function reads (up to a normalization constant)

$$Z_{\text{1-loop}} = e^{-S_E^{(0)}} q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} = q^{-k\mathfrak{L}_0 - \frac{1}{24}} y^{i\kappa k_0} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} \,. \tag{4.216}$$

The 1-loop partition function reproduces the full WCFT partition function (4.195), originally computed in [43] for the precise value $\kappa = -1$,

$$\chi_{h,p}(\tau, z) = q^{h-c/24} y^p \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} \,, \tag{4.217}$$

for $p \in i\mathbb{R}$ which is our case here (see (4.216)). From there we read that

$$h = -2\pi\left(\mathcal{L}_0 - \frac{\pi}{\kappa}\mathcal{P}_0^2\right) + \frac{c-1}{24} \qquad , \qquad p = 2\pi i \mathcal{P}_0 \,. \tag{4.218}$$

It thus seems that starting from a real $\mathcal{P}_0$, solutions containing black holes, we recover the partition function of a primary field with imaginary charge p, like the vacuum, and its descendants.

Such a result may seem unusual, but a similar phenomenon was discussed in [47], where the authors computed the one-loop partition function in AdS_3 with CSS boundary conditions [55] using a quasinormal mode method [198]. Their primary focus was to describe the BTZ black hole, a unitary representation, but they unexpectedly obtained the vacuum character, also a unitary representation in this context, which they associated with thermal AdS. They argued that, akin to CFTs, an appropriate modular transformation relates thermal AdS to BTZ [42, 50], rendering them indistinguishable because they both derive from Euclidean AdS [31, 199, 200]. The only distinguishing factor is their contractible or non-contractible thermal circle, which explains why they obtained the vacuum determinant while computing the BTZ determinant.

We are now interested in purely imaginary values of $\mathcal{P}_0$, like the vacuum. Indeed, in that case, from [66], $\alpha = -(1+2j)^2$ where j is an integer and $k_0 = i\gamma$. We choose the branch $j = 0$ as in [66] to make the link with the WCFT vacuum values in [42]. When the orbit representative $\mathcal{P}_0$ is purely imaginary, $\mathfrak{L}_0$ stays real and the euclidean action (4.201) is instead

$$S_E = -\frac{ik}{2\pi}\int_{\Sigma_o} \mathrm{d}^2x \left(\frac{\dot{f}'f''}{4f'^2} + \mathfrak{L}_0 \dot{f} f'\right) - \int \mathrm{d}^2x \left(\frac{i\kappa}{8\pi}\Phi'\dot{\Phi} + i\mathcal{P}_0\dot{\Phi} - \mathcal{P}_0\right) \,. \tag{4.219}$$

The real part of the action is then

$$\mathrm{Re}[S_E] = -i\int \mathrm{d}\varphi \mathcal{P}_0 \dot{\Phi} \,. \tag{4.220}$$

Since we are now dealing with a non-unitary representation, (4.220) can no longer be interpreted as the Hamiltonian (4.168). For the field periodicity (4.203), a saddle point of the action (4.219) is also (4.205). One can show that any perturbations around this saddle point keep the real part of the action (4.220) bounded, such that the path integral derivation can also be performed. Following the previous computations, the one-loop partition function is simply

$$Z_{\text{1-loop}} = q^{-k\mathfrak{L}_0-\frac{1}{24}} y^{-\kappa k_0} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2}\,. \tag{4.221}$$

which is (4.216) with $k_0 \to ik_0$. This does not reproduce either the character for a real charge (4.194) or for an imaginary one (4.195).

In order to address this apparent puzzle, we momentarily pause on some particularities of WCFT partition functions. We have so far implicitly assumed that the bulk and boundary theories are defined by the following partition function

$$Z^{\text{CE}}(\beta,\theta) = \text{Tr}\; e^{-\beta P_0 + i\theta L_0} \tag{4.222}$$

which is said to be in *canonical* ensemble, and where L_0 and P_0 are the zero modes of the algebra (2.31). It appears however more natural in some situations [42] to define the following generators and partition function:

$$L_n^Q = L_n - \frac{2}{k}P_0P_n + \frac{1}{k}P_0^2\delta_n\,, \quad P_n^Q = -\frac{2}{k}P_0P_n + \frac{1}{k}P_0^2\delta_n\,. \tag{4.223}$$

and

$$Z^{\text{QE}}(\beta_R,\beta_L) = \text{Tr}\; e^{-\beta_R P_0^Q - \beta_L L_0^Q}\,. \tag{4.224}$$

The latter defines the so-called *quadratic* ensemble partition function. Importantly, we have in particular that $P_0^Q = -P_0^2/k$, hence states with opposite values of P_0 are identified.

We will refrain from reconsidering our previous analysis in quadratic ensemble, which would entail reformulating the appropriate bulk boundary conditions [105] in Chern-Simons/Lower-Spin Gravity language, then going through the steps of the Hamiltonian reduction, and identifying the corresponding geometric action. We will leave this for further work. However, it is natural to expect that the periodic boundary conditions on the quadratic ensemble $U(1)$ field, when expressed in terms of the canonical $U(1)$ field above, will lead to either periodic, *or anti-periodic* boundary conditions. Let us inspect the consequences of this observation, by considering the following periodicity condition:

$$\Phi(\varphi + \Omega\beta, y+\beta) = -\Phi(\varphi, y)\,. \tag{4.225}$$

The new saddle point satisfying the new antiperiodic condition (4.225) can be written in the form

$$\Phi_0 = \sin\left(\frac{\pi y}{\beta}\right)\,, \tag{4.226}$$

and the expansion of the field around his saddle point is

$$\Phi(\varphi, y) = \Phi_0 + \sum_{m,n} \frac{\Phi_{m,n}}{(2\pi)^2} e^{\frac{2\pi i m y}{\beta} + \frac{i\pi y}{\beta}} e^{i n f_0}\,. \tag{4.227}$$

The real part of the action (4.220) is also bounded for the perturbations (4.227) and we can perform the path integral derivation. Focusing on the $U(1)$ part of the action, the euclidean action around the saddle point is

$$S_E^{U(1)} = \kappa\beta k_0 + \frac{i}{2}\sum_{m=-\infty}^{\infty}\sum_{n\neq 0} \kappa \frac{|\Phi_{m,n}|^2}{(2\pi)^3} n\left(m - \tau n - \frac{1}{2}\right) + \dots\,. \tag{4.228}$$

And the one-loop $U(1)$ partition function becomes

$$Z_{\text{1-loop}}^{U(1)} = \mathcal{N} y^{i\kappa k_0} \prod_{m,n} n^{-1/2}\left(m - \tau n - \frac{1}{2}\right)^{-1/2}\,, \tag{4.229}$$

for $n \neq 0$. Using the same trick in Section 4.3.6, we derive the logarithm of the partition function according to τ to perform the sum over m, then we integrate and exponentiate to get the one-loop partition function. First we perform the logarithm and rewrite the sum as

$$\begin{aligned}
\log Z_{\text{1-loop}}^{U(1)} &= -\frac{1}{2}\sum_{n\neq 0}\sum_{m=-\infty}^{\infty} \log\left(m - \tau n - \frac{1}{2}\right) \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \left[\log\left(m - \tau n - \frac{1}{2}\right) + \log\left(m + \tau n - \frac{1}{2}\right]\right) \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \log\left(\frac{1}{4} - m + m^2 - n^2\tau^2\right)\,.
\end{aligned}$$

Now, deriving according to τ:

$$\begin{aligned}
\partial_\tau \log Z_{\text{1-loop}}^{U(1)} &= \sum_{n\neq 0}\sum_{m=-\infty}^{\infty} \frac{n^2\tau}{\frac{1}{4} - m + m^2 - n^2\tau^2} \\
&= \sum_{n\neq 0}\left(\sum_{m=-\infty}^{-1} \frac{n^2\tau}{\frac{1}{4} - m + m^2 - n^2\tau^2} + \frac{n^2\tau}{\frac{1}{4} - n^2\tau^2} + \sum_{m=1}^{\infty} \frac{n^2\tau}{\frac{1}{4} - m + m^2 - n^2\tau^2}\right) \\
&= \sum_{n\neq 0}\left(\sum_{m=1}^{\infty} \frac{n^2\tau}{\frac{1}{4} + m + m^2 - n^2\tau^2} + \frac{n^2\tau}{\frac{1}{4} - n^2\tau^2} + \sum_{m=1}^{\infty} \frac{n^2\tau}{\frac{1}{4} - m + m^2 - n^2\tau^2}\right) \\
&= \sum_{n\neq 0}\left(\frac{\pi}{2}\tan(n\pi\tau) + \frac{n^2\tau}{n^2\tau^2 - \frac{1}{4}} + \frac{n^2\tau}{\frac{1}{4} - n^2\tau^2} + \frac{\pi}{2}\tan(n\pi\tau)\right) \\
&= \sum_{n\neq 0} \pi\tan(n\pi\tau)\,.
\end{aligned}$$

Again this sum diverges but can be regularized using the zeta function regularization. After integration and exponentiation (up to a normalization constant), the $U(1)$ part of the one-loop partition function is

$$Z^{U(1)}_{\text{1-loop}} = \mathcal{N}' y^{i\kappa k_0} \prod_{n=1}^{\infty} \frac{1}{1+q^n}\,, \tag{4.230}$$

which is related to the $U(1)$ part of the Virasoro-Kač-Moody character in (3.11) of [43]. Combining this with the $SL(2,\mathbb{R})$ part that remains the same, we end up with

$$Z_{\text{1-loop}} = q^{-k\mathfrak{L}_0 - \frac{1}{24}} y^{i\kappa k_0} \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})}\,, \tag{4.231}$$

which is completely equivalent to (4.194) with weight and real charge (since P_0 is purely imaginary) (4.218).

For the vacuum, the computation of the $SL(2,\mathbb{R})$ part of the partition function needs to be improved because $\alpha=-1$ in (4.214), making start the product at $n=2$. It implies a slightly modified zeta regularization (4.215)

$$\sum_{n=2}^{\infty} n \to \zeta(-1) - 1 = -\frac{13}{12}\,. \tag{4.232}$$

The one-loop partition function for the vacuum, $\alpha=-1$ and $k_0=i\gamma$, is thus

$$Z^{\text{vac}}_{\text{1-loop}} = q^{-\frac{c+13}{24}} y^{-\kappa\gamma} \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})}(1-\delta_{\text{vac}} q)\,. \tag{4.233}$$

The factor 13 raised from the zeta regularization because we wanted to "regularize first, integrate later". In [73, 173], a different approach, "integrate first, regularize later", was performed which prevents the apparition of the regularization factor.

We can conclude from this section that for the natural periodic conditions (4.203), we only recover one of the warped Virasoro character. In order to find the other one, we need to impose the antiperiodic condition (4.225) which, for the moment, does not possess a physical motivation. Furthermore, the warped character are exchanged. When we tried to compute the warped character of a unitary representation, like a black hole, we get the warped character for an imaginary charge p and when we computed the warped charter of the vacuum, a non-unitary representation, we ended up with the warped character for a real charge p-modulo the strange antiperiodic condition. To confirm our results, it could be interesting to use a different approach such as the Heat Kernel method [171] or the quasinormal modes approach [47].

4.4 Summary and perspectives

In this chapter we have used the geometric action on the coadjoint orbits of the Warped conformal algebra to refine and further develop one of the simplest bulk models for Holographic WCFTs, consisting in Lower Spin Gravity. We have performed the Hamiltonian reduction of the classical gravity action in Chern-Simons form and obtained exactly the geometric action on the coadjoint orbits of the Warped Virasoro group. The orbit representatives correspond to the zero modes of the gravitational charges as we have shown by explicitly by taking into account the bulk holonomy, as was already observed in other holographic set-ups such as AdS_3 and flat space in 2+1 dimensions [72, 73, 75]. This makes the relationship between the different lower spin gravity saddles obeying the boundary conditions of [66] and the coadjoint orbits of the Warped conformal group explicit and provides an action principle for warped boundary gravitons. These are the excited states generated by boundary condition preserving diffeomorphisms of a given gravitational saddle and correspond to descendants of primary operators in the boundary WCFT. The action can be used to compute the contribution of Virasoro and KM descendants to the one-loop partition function of a given classical saddle on the torus, and we show how the result compares to the Warped Virasoro characters of [43].

Several extensions and further investigations could be envisioned following up on this chapter, of which we list some below. A direct one concerns a thorough analysis of Lower Spin Gravity in quadratic ensemble. This could allow to clarify some intriguing features observed in the one-loop computation. The effective action for Warped conformal transformations around a given background could be exploited for several further purposes. For instance, it could be used to compute boundary correlators from Wilson lines ending on the boundary as well as entanglement entropy and its leading order quantum correction. Furthermore, Wilson lines could be used to compute the Warped Virosoro identity block and its subleading terms, both for light operators and in the heavy-light limit, in the spirit of [73], and even more general blocks generalizing the open Wilson line networks of [201].

An interesting avenue to explore in our opinion would be a supersymmetric generalization of Lower Spin Gravity. Susy WCFTs have been addressed in [61] from a field theory perspective, and it would be interesting to generalize this to more supersymmetries and investigate the bulk counterpart, in the spirit of [202] for AdS3 or [203–206] for flat space. Also, a susy generlization of Lower Spin Gravity would open the door to an exact evaluation of its partition function using localization techniques, along the lines of [207, 208].

Another remarkable connection was made recently between the geometric action on the coadjoint orbits of the Virasoro group and complexity growth in 2d CFTs [209]. In that chapter a suitable definition for Nielsen complexity for 2D CFTs was introduced and led to the Alekseev-Shatashvili action as complexity functional for the CFT. Since the Alekseev-Shatashvili action also arises from the Hamiltonian reduction of AdS_3 gravity with Brown-Henneaux boundary conditions, one could view this as an explicit realization of the "complexity equals bulk action" proposal of [210]. It would be interesting to see if these arguments can also be applied to WCFTs and Lower Spin Gravity.

Chapter 5

Quasi-normal modes in warped black hole background

Quasinormal modes (QNMs) are a well-studied subject in General Relativity. They are considered characteristic modes of vibration of black holes, similar to the vibration of a bell. These modes appear as electromagnetic or gravitational perturbations of black holes. In fact, the study of QNMs began in the late 1950s by Regge and Wheeler [77], well before the term *black hole* was coined by Wheeler in 1967 during a talk at the NASA Goddard Institute of Space Studies [211,212]. Regge and Wheeler aimed to study the stability of the Schwarzschild metric under small perturbations independently of astrophysical context. In 1970, Zerilli [213] extended their work to include all perturbations. At that time, the term *quasinormal modes* did not exist yet; they were merely regarded as perturbations of a black hole.

The following year, Vishveshwara observed these modes in the scattering of gravitational waves by a Schwarzschild black hole [78], and Press subsequently named them *quasinormal modes* [214]. Since then, QNMs have appeared in various contexts, such as particles falling into Schwarzschild [215] and Kerr black holes [216,217], or as the gravitational collapse of a star into a black hole [79–81]. In 1975, Chandrasekhar and Detweiler numerically computed the QNMs of Schwarzschild black holes in a weakly damped regime [218].

The first link between QNMs and the unstable photon orbit around a black hole, called the *photon ring*, was established by Mashhoon in 1975 [87], where he proved Goebel's statement [219] that QNMs are essentially gravitational waves orbiting in spiral orbits around the photon ring and slowly leaking out. Their connection to gravitational waves allows us to use them to deduce the mass and angular momentum of an observed black hole [220] and to test the no-hair theorem of general relativity [221,222] through the LIGO and LISA experiments. Ferrari and Mashhoon formalized the connection between the photon ring and the QNMs of a Schwarzschild black hole in 1984 [86]. They shown that the dispersion relation of the angular frequency ω of the modes in the eikonal limit demonstrates a connection to the angular velocity Ω and the ratio of the Lyapunov exponent to the orbital period of the photon ring γ_L outside the outer horizon

$$\omega_{lmn} \overset{l \gg 1}{\approx} \left(l + \frac{1}{2}\right)\Omega - i\left(n + \frac{1}{2}\right)\gamma_L\,. \tag{5.1}$$

It was later generalized to stationary, spherically symmetric, asymptotically flat spacetimes in [84] and to Kerr black holes in [96]. After the advent of the AdS/CFT correspondence, Birmingham, Sachs, and Solodukhin [99] showed in 2001 that the QNM frequencies of BTZ black holes correspond to the poles of the retarded correlation function in the dual CFT. The following reviews are recommended for the interested readers [223–226].

In the context of Schwarzschild and Kerr black holes, the QNM spectrum cannot be computed exactly and requires approximation methods such as the inverse-potential approaches [86], Wentzel-Kramers-Brillouin (WKM) methods [88, 96, 227], geometric optics approximations [83, 84, 91, 93–96, 98, 228], or numerical methods [229, 230]. However, warped AdS_3 black holes have the particularity to have an exact QNM spectrum [82]. This characteristic is also observed in BTZ black holes [99] and self-dual $WAdS_3$ black holes [98]. However, the former do not exhibit a photon ring, while the latter are a near-horizon limit of near-extremal $WAdS_3$ black holes. This renders the $WAdS_3$ black holes particularly interesting to examine in this context.

We aim to calculate the QNMS as scalar solutions of the massless wave equations in a warped AdS_3 black hole background and connect them to the photon ring in the high-frequency (eikonal) limit. Considering only spinless solutions is adequate since the spin's influence is minimal in this limit [83]. The QNM results for asymptotically flat spacetimes were achieved by applying different boundary conditions at the horizon and at infinity, all of which are equivalent. An alternative definition of QNMs can be formulated based on their eikonal limit rather than the boundary conditions of the modes: QNMs are solutions to the massless wave equation that behave in the high-frequency limit like (5.1). Defining QNMs in this manner becomes relevant within the context of Warped Black Holes because, as we will see, the boundary conditions are not equivalent in this background. Consequently, we cannot base our definition upon them. We will select the relevant conditions leading to (5.1).

In this chapter, we will start by solving the massless scalar wave equation in a $WAdS_3$ background and determining the exact spectrum of QNMs. To achieve this, we will examine various boundary conditions at infinity, including finite flux, outgoing waves, and Dirichlet conditions. We will demonstrate that these conditions differ across coordinate systems for the WAdS black hole. To identify the appropriate boundary condition, we will explore several limits, such as the Penrose limit, the near-ring region limit, and the geometric optics approximation, which provide a physical interpretation related to the outer photon ring. We will then conclude that the outgoing waves condition at infinity is the most suitable. Finally, we will consider the BTZ limit and the near-horizon near-extremal limit to compare our findings with established results.

5.1 Warped AdS$_3$ Black Holes

The black hole solution we will be discussing in this work first appeared in [16, 34–36, 231], and was initially studied holographically in [33]. The metric is given by (1.13)

$$\mathrm{d}s^2 = -N(r)^2\mathrm{d}t^2 + \frac{\ell^2}{4R(r)^2N(r)^2}\mathrm{d}r^2 + R(r)^2\left(\mathrm{d}\theta + N^\theta(r)\mathrm{d}t\right)^2 \,, \tag{5.2}$$

where we defined

$$\begin{aligned} R(r)^2 &= \frac{r}{4}\left(3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+r_-(\nu^2+3)}\right) \,, \\ N(r)^2 &= \frac{1}{4R(r)^2}(\nu^2+3)(r-r_+)(r-r_-) \,, \\ N^\theta(r) &= \frac{2\nu r - \sqrt{r_+r_-(\nu^2+3)}}{2R(r)^2} \,, \end{aligned} \tag{5.3}$$

where $\theta \in [0, 2\pi[$ is an angle, $r \in [0, +\infty[$ is the radial coordinate and ν, r_+, r_- are constants. We assume $\nu^2 > 1$ to avoid closed timelike curves and will in addition restrict ourselves to $\nu > 1$ without loss of generality. The constants $r_\pm$ determine the positions of the inner and outer horizons of the black hole. We have $r_+ > r_- > 0$. These solutions are obtained as quotients by a discrete subgroup of the isometry group of spacelike warped AdS$_3$ [33], much like BTZ black holes are obtained from global AdS$_3$. Actually for $\nu = 1$, the metrics are locally AdS$_3$, and represent BTZ black holes, albeit in an unusual coordinate system.

5.2 Photon ring

Photon rings are gravitationally bound light rays. They are described (when they exist) by affinely parameterised null geodesics, $x^\mu(s) = \big(t(s), r(s), \theta(s)\big)$, whose r coordinate remains bounded. The two Killing vectors ∂_τ and ∂_θ lead to two conserved quantities. The corresponding conserved quantities are the energy

$$\frac{E}{\ell^2} = -\dot{t} + \frac{1}{2}\Big(-2r\nu + \sqrt{r_+r_-(\nu^2+3)}\Big)\dot{\theta} \,, \tag{5.4}$$

and the angular momentum

$$\frac{L}{\ell^2} = \frac{1}{4}\bigg(3r^2(\nu^2-1)\dot{\theta} - 2\sqrt{r_+r_-(\nu^2+3)}\dot{t} + r\Big((r_+ + r_-)(\nu^2+3)\dot{\theta} + 4\nu\Big(\dot{t} - \dot{\theta}\sqrt{r_+r_-(\nu^2+3)}\Big)\Big)\bigg) \,. \tag{5.5}$$

For null geodesics, $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$ gives

$$\dot{r}^2 + \frac{V(r)}{\ell^4} = 0 \,, \tag{5.6}$$

where

$$V(r) = 4\Big(L^2 + 2LER(r)^2N^\theta(r) + E^2R(r)^2\Big)\,. \tag{5.7}$$

Assuming $E \neq 0$, simultaneously solving the conditions

$$V(r) = V'(r) = 0\,, \tag{5.8}$$

provides solutions of the geodesic equation with

$$r = \tilde{r} = \text{const.}\,, \tag{5.9}$$

the photon ring[1]. We find as solutions, the two photon rings

$$\tilde{r}_\pm = \frac{r_+ + r_-}{2} \pm \frac{(r_+ - r_-)\nu}{\sqrt{3(\nu^2-1)}}\,. \tag{5.10}$$

By defining a critical energy-rescaled angular momentum,

$$\lambda = \frac{L}{E}\,, \tag{5.11}$$

each photon ring has

$$\lambda(\tilde{r}_\pm) := \tilde{\lambda}_\pm = \frac{1}{2}\left(\sqrt{r_+r_-(\nu^2+3)} - (r_+ + r_-)\nu \mp \frac{1}{2}(r_+ - r_-)\sqrt{3(\nu^2-1)}\right)\,. \tag{5.12}$$

For both photon rings, the impact parameter $\tilde{\lambda}_\pm$ is negative. It will be helpful when we examine the direction of propagation of the QNMs.

One can check that we have the following inequalities

$$\tilde{r}_+ > r_+ > r_- > \tilde{r}_-\,. \tag{5.13}$$

It is interesting to notice that, since the radial coordinate r cannot be negative [33], we have two photon rings as long as

$$\frac{r_-}{r_+} \geq \frac{2\nu - \sqrt{3(\nu^2-1)}}{2\nu + \sqrt{3(\nu^2-1)}}\,, \tag{5.14}$$

meaning that for a given ν, the ratio of the radii of the horizons cannot be too small. In the limit of large ν, the lower bound tends to an nonzero value indicating that even in that limit, the location of the horizons cannot be arbitrary. On the other hand, when ν goes to one, the only situation where we have two photon rings is the extremal case, where the two horizons coincide.

This condition can also be equivalently expressed in terms of ν as

$$\nu \geq \sqrt{\frac{3(r_+ + r_-)^2}{14r_+r_- - r_+^2 - r_-^2}}\,. \tag{5.15}$$

[1] In the special case, $E = 0$, (5.9) is fulfilled at $L = 0$.

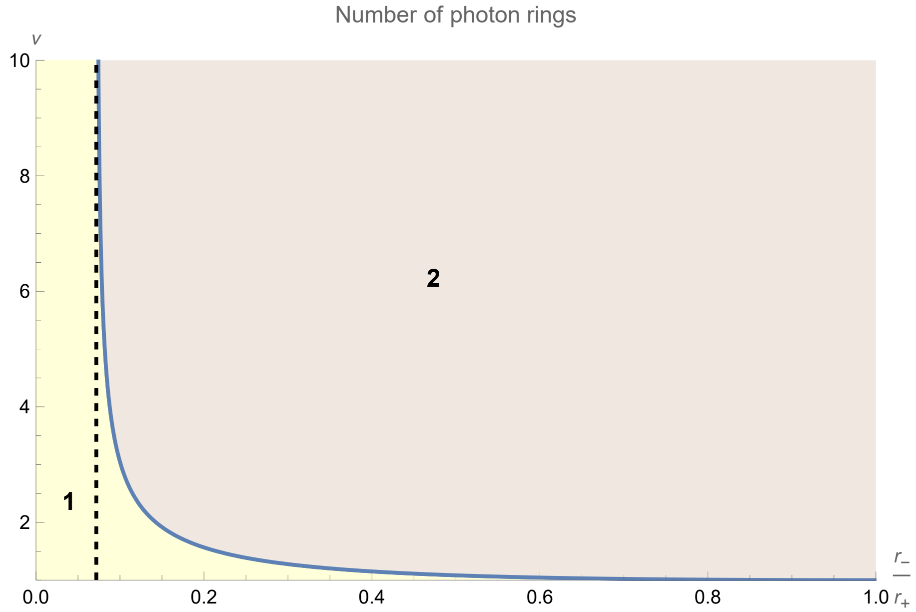

Figure 5.1: This represents the phase space of the different number of photon rings that you can have in your spacetime as a function of $\nu \in [1, +\infty[$ and the ration $r_-/r_+ \in]0, 1]$. In the red region, there are two photon rings $\tilde{r}_\pm$, while in the yellow region, there is only an outer photon ring $\tilde{r}$ outside the two horizons. On the blue line, we are at the boundary of the two zones where the inner photon ring is located at $\tilde{r}_- = 0$ and its angular velocity is infinite. The dashed vertical line at $r_-/r_+ = (2-\sqrt{3})/(2+\sqrt{3})$ shows the asymptotic behaviour of the boundary between the two zones.

This specific lower bound of ν is always greater than 1, as long as the radicand is positive. This implies that there will always be a certain value of ν where the inner photon ring disappears. If $\frac{r_-}{r_+} \leq \frac{2-\sqrt{3}}{2+\sqrt{3}}$, the radicand is negative, meaning that there is always only one photon ring for any value of ν, see Figure 5.1. When the inequalities are saturated, the inner photon ring is located at $\tilde{r}_- = 0$ and the impact parameter $\tilde{\lambda}_-$ vanishes, i.e.

$$\tilde{\lambda}_- \to 0 \qquad \text{when} \quad \nu = \sqrt{\frac{3(r_+ + r_-)^2}{14 r_+ r_- - r_+^2 - r_-^2}} \,. \tag{5.16}$$

The angular velocities at criticality $\tilde{\Omega}_\pm := \Omega(\tilde{r}_\pm)$ are given by

$$\tilde{\Omega}_\pm = \left.\frac{d\theta(s)}{dt(s)}\right|_{\tilde{r}_\pm, \tilde{\lambda}_\pm} = \frac{1}{\tilde{\lambda}_\pm}\,, \tag{5.17}$$

and the half-orbital period[2] is given by

$$\tau = -\frac{\pi}{\tilde{\Omega}_\pm} = -\pi \tilde{\lambda}_\pm \,. \tag{5.18}$$

[2] A light ray emitted close to the photon ring can perform a certain number of rotation around the black hole due to the presence of the photon ring before arriving to a distant observer. To quantize this number into integers, one may count in terms of half-orbits [232].

The bound geodesics are unstable because for the critical orbital radii,

$$V''(\tilde{r}_\pm) = -6E^2(\nu^2 - 1) < 0\,. \tag{5.19}$$

Any small perturbation will push them away from the photon ring. The photon ring image arises by these photons travelling on such "nearly-bound" geodesics. We consider two such nearby geodesics, one which is exactly bound at $\tilde{r}_\pm$, and the other one initially differing only by an infinitesimal radial separation δr_0. By solving the radial geodesic equation (5.6) for such geodesics, we get

$$\delta r(s) = e^{\sqrt{-\frac{1}{2}V''(\tilde{r}_\pm)}s}\,\delta r_0\,. \tag{5.20}$$

After $n = \frac{\Delta\theta}{\pi}$ half-orbits, the separation grows as

$$\delta r(n) = e^{\gamma\, n}\delta r_0\,, \tag{5.21}$$

where γ is the Lyapunov exponent and is defined as

$$\gamma \equiv \sqrt{-\frac{1}{2}V''(\tilde{r}_\pm)}\left|\frac{\pi}{\dot{\theta}}\right|_{\tilde{r}_\pm,\tilde{\lambda}_\pm}\,. \tag{5.22}$$

For our spacetime, it appears to be the same for both photon rings

$$\gamma = \frac{\pi}{4}(\nu^2 + 3)(r_+ - r_-)\,. \tag{5.23}$$

The period-averaged radial deviation as a function of time, t, becomes

$$\delta r(t) = e^{\gamma_L t}\delta r_0\,, \quad \gamma_L := \frac{\gamma}{\tau}\,, \tag{5.24}$$

where

$$\gamma_L \equiv \frac{\gamma}{\tau} = \sqrt{-\frac{1}{2}V''(\tilde{r}_\pm)}\left|\frac{1}{\dot{t}}\right|_{\tilde{r}_\pm,\tilde{\lambda}_\pm}\,. \tag{5.25}$$

In the following, we will often refer to γ_L as the Lyapunov exponent even if it is more precisely the ratio between the true Lyapunov exponent and the half-period. As the latter differs for each photon ring, we get different results

$$\gamma_{L\pm} = -\frac{\nu^2 + 3}{4}(r_+ - r_-)\tilde{\Omega}_\pm\,, \tag{5.26}$$

which are positive because $\tilde{\Omega}_\pm$ is negative. Together with $\phi \approx \tilde{\Omega}_\pm t$, we have the complete solution of the null geodesic in the near-ring region of the warped AdS_3 black hole.

5.2.1 Near-ring region

For our analysis, we consider nearly-bound geodesics in the near-ring region. To define this region, let us introduce the following parameters,

$$\eta := r_+ + r_- \,, \quad \sigma := r_+ - r_- \,, \tag{5.27}$$

and therefore, $\eta > \sigma$. We can rewrite the critical orbital radius (5.10) and the critical energy-rescaled angular momentum (5.12) as

$$\begin{aligned} \tilde{r}_\pm &= \frac{\eta}{2}\left(1 \pm \frac{2\sigma\nu}{\eta\sqrt{3(\nu^2-1)}}\right)\,, \\ \tilde{\lambda}_\pm &= \frac{\eta}{2}\left(\frac{1}{2}\sqrt{\left(1-\frac{\sigma^2}{\eta^2}\right)(\nu^2+3)} - \nu \mp \frac{\sigma}{2\eta}\sqrt{3(\nu^2-1)}\right)\,, \end{aligned} \tag{5.28}$$

where $0 < \frac{\sigma}{\eta} < 1$ by definition. Since η controls the scale of $\tilde{r}_\pm$ and $\tilde{\lambda}_\pm$, we can define the near-ring region in phase space as

$$\text{NEAR-RING REGION:} \quad \begin{cases} |\delta r| \ll \eta & (\text{near-peak})\,, \\ \left|\delta\lambda\right| \ll \eta & (\text{near-critical})\,, \end{cases} \tag{5.29}$$

where $\delta r = r - \tilde{r}_\pm$ and $\delta\lambda = \lambda - \tilde{\lambda}_\pm$. The first condition zooms into the spacetime of the bound photon orbit and the second condition zooms into the bound orbit in momentum space. The outer photon ring is defined by the following phase-space locus

$$\text{PHOTON RING:} \quad \delta r = 0 = \delta\lambda \tag{5.30}$$

5.3 Quasi-normal modes

5.3.1 Exact spectrum

We recall that scalar quasinormal modes are solutions to the wave equation and eigenfunctions of the operators ∂_t and ∂_θ (i.e. Killing vectors of the translational isometries along t and θ) with boundary conditions at the horizon and at infinity. Therefore, we consider massless scalar perturbations, which satisfy the wave equation

$$\nabla^2\Phi(t,\theta,r) = 0\,. \tag{5.31}$$

Although we are primarily interested in photons or gravitons, as the effects of spin are subleading in the geometric optics regime [83], it is sufficient for our purposes to solve the wave equation of a massless spin-zero field in a Warped AdS_3 background. Thus, we write:

$$\Phi(t,\theta,r) = e^{-i\omega t + ik\theta}\phi(r)\,, \tag{5.32}$$

where ω is the energy of the scalar mode and k is the quantized angular momentum. The radial part of the scalar wave equation takes the form

$$\Big[\varrho(r)\partial_r\left(\varrho(r)\partial_r\right)+V_{\mathrm{QNM}}(r)\Big]\phi(r)=0\,, \tag{5.33}$$

where we defined

$$\varrho(r)\equiv(\nu^2+3)(r-r_+)(r-r_-)=\frac{4}{\ell^2}R(r)^2N(r)^2\,, \tag{5.34}$$

and the wave potential is given by

$$V_{\mathrm{QNM}}(r)=4\Big(k^2+2k\omega R(r)^2N^\theta(r)+\omega^2R(r)^2\Big)\,. \tag{5.35}$$

In terms of the following change of coordinates

$$z=\frac{r-r_+}{r-r_-}\,, \tag{5.36}$$

we can rewrite the mode as:

$$\Phi(t,\theta,z)=e^{-i\omega t+ik\theta}\phi(z)\,. \tag{5.37}$$

Here, $z=0$ and $z=1$ correspond to the locations of the horizon and spatial infinity respectively, and $\phi(z)$ has to satisfy

$$z(1-z)\phi''(z)+(1-z)\phi'(z)-\left[\frac{A^2}{z}-B^2+\frac{C(C-1)}{1-z}\right]\phi(z)=0\,, \tag{5.38}$$

where we define

$$\begin{aligned}
A&=\frac{i}{(r_+-r_-)(\nu^2+3)}\Big(2k+\omega\Big(2\nu r_+-\sqrt{r_+r_-(\nu^2+3)}\Big)\Big)\,,\\
B&=\frac{i}{(r_+-r_-)(\nu^2+3)}\Big(2k+\omega\Big(2\nu r_--\sqrt{r_+r_-(\nu^2+3)}\Big)\Big)\,,\\
C&=\frac{1}{2}\left(1-\sqrt{1-\frac{12(\nu^2-1)\omega^2}{(\nu^2+3)^2}}\right)\,.
\end{aligned} \tag{5.39}$$

This linear second order differential equation can be brought to a standard hypergeometric differential equation by the function redefinition [82, 233]

$$\phi(z)=z^A\left(1-z\right)^Cf(z)\,. \tag{5.40}$$

Curious readers can refer to Appendix B for a review of the general properties of the hypergeometric function, which will be used throughout the chapter.

By defining,

$$\Psi(A,z)=z^A\,{}_2F_1(A+B+C,A-B+C;1+2A;z)\,, \tag{5.41}$$

the general solution can be expressed as

$$\Phi(t,\theta,z) = e^{-i\omega t + ik\theta}(1-z)^{C}\Big(C_+\ \Psi(A,z) + C_-\ \Psi(-A,z)\Big)\,, \tag{5.42}$$

with constant C_+ and C_-.

With the general solution to the massless wave equation in hand, our next objective is to impose the appropriate boundary conditions that yield the behavior (5.1) in the eikonal limit. To achieve this, we will analyze the various asymptotic behaviors of (5.42).

First, let us examine the region near the horizon. As $z \to 0$, we can approximate the mode $\Psi(A,z)$ as

$$\Psi(A,z) = z^{A} + (\text{subleading terms})\,. \tag{5.43}$$

For the sake of simplicity, let us express A as

$$A = ic_1 k + ic_2\omega\,, \tag{5.44}$$

where

$$c_1 = \frac{2}{(r_+ - r_-)(\nu^2+3)} > 0\,, \quad c_2 = \frac{\Big(2\nu r_+ - \sqrt{r_+ r_-(\nu^2+3)}\Big)}{(r_+ - r_-)(\nu^2+3)} > 0\,. \tag{5.45}$$

Since quasinormal mode frequencies are complex, we decompose them as

$$\omega = \omega_R - i\omega_I\,. \tag{5.46}$$

With this, we can express the complete wave solution as

$$\Phi_+(t,\theta,z) = \exp\left(-i\omega_R\left(t - (c_1\frac{k}{\omega_R} + c_2)\ln z\right) + ik\theta\right) e^{-\omega_I t - c_2\omega_I \ln z}\,. \tag{5.47}$$

Meanwhile, close to the horizon, the second solution $\Psi(-A,z)$ can be simplified to

$$\Psi(-A,z) = z^{-A} + (\text{subleading terms})\,. \tag{5.48}$$

which leads to the following asymptotic wave solution

$$\Phi_-(t,\theta,z) = \exp\left(-i\omega_R\left(t + (c_1\frac{k}{\omega_R} + c_2)\ln z\right) + ik\theta\right) e^{-\omega_I t - c_2\omega_I \ln z}\,. \tag{5.49}$$

To determine the direction of propagation for the two different solutions near the horizon, we examine the eikonal limit, where it is anticipated that

$$\omega_R \approx \tilde{\Omega}_\pm k = \frac{k}{\tilde{\lambda}_\pm}\,. \tag{5.50}$$

Then, one has to analyse the sign of the factor $c_1\tilde{\lambda}_\pm + c_2$. Indeed,

$$c_1\tilde{\lambda}_\pm + c_2 > 0\,, \tag{5.51}$$

which implies that while Φ_+ is an outgoing mode at the horizon, Φ_- is ingoing. To have only ingoing (outgoing) solutions, one must impose $C_+ = 0$ ($C_- = 0$).

We now examine the flux of the modes through the outer horizon. The flux of a complex scalar field is [82]

$$\mathcal{F} = \frac{\sqrt{-g}\,g^{rr}}{2i}\left(\Phi^*\partial_r\Phi - \Phi\partial_r\Phi^*\right) = \frac{\nu^2+3}{4i}(r_+ - r_-)z\left(\Phi^*\partial_z\Phi - \Phi\partial_z\Phi^*\right)\,. \tag{5.52}$$

Near the horizon, the flux is

$$\mathcal{F} \sim A_R\left(|C_+|^2 z^{2c_2\omega_I} + |C_-|^2 z^{-2c_2\omega_I}\right) - ic_2\omega_I\left(C_-^* C_+ z^{2iA_R} - C_+^* C_- z^{-2iA_R}\right)\,, \tag{5.53}$$

where A_R is the real part of (5.44). We can observe that an ingoing mode, $C_+ = 0$, will have a diverging flux at the horizon if ω_I is positive, which will be the case in the eikonal limit (see (5.71)). On the other hand, an outgoing mode, $C_- = 0$, will have a finite flux.

Let us look at the other boundary of our spacetime, spacial infinity, and compute the asymptotic behaviour of our solutions. Using a standard formula for the hypergeometric function , we can expand $\Phi(x)$ (5.42) near $z = 1$ as

$$\Phi(t,\theta,z) \approx e^{-i\omega t + ik\theta}\left(Q_+(1-z)^{\frac{1}{2}+\varpi} + Q_-(1-z)^{\frac{1}{2}-\varpi}\right)\,, \tag{5.54}$$

where we have defined

$$Q_+ = C_+\frac{\Gamma(1+2A)\Gamma(2C-1)}{\Gamma(A+B+C)\Gamma(A-B+C)} + C_-\frac{\Gamma(1-2A)\Gamma(2C-1)}{\Gamma(-A+B+C)\Gamma(-A-B+C)}\,, \tag{5.55}$$

$$Q_- = C_+\frac{\Gamma(1+2A)\Gamma(1-2C)}{\Gamma(1+A+B-C)\Gamma(1+A-B-C)} + C_-\frac{\Gamma(1-2A)\Gamma(1-2C)}{\Gamma(1-A+B-C)\Gamma(1-A-B-C)}\,.$$

Again, we have decomposed $C = \frac{1}{2} - \varpi$ for simplicity, where we define

$$\varpi = \sqrt{\frac{1}{4} - \frac{3(\nu^2-1)\omega^2}{(\nu^2+3)^2}} \equiv \sqrt{\frac{1}{4} - b\,\omega^2}\,, \tag{5.56}$$

where we have introduced

$$b = \frac{3(\nu^2-1)}{(\nu^2+3)^2}\,, \tag{5.57}$$

and because ϖ is complex, we further decompose $\varpi = \varpi_R + i\varpi_I$. We define the square root of a complex number such that the real part is positive, or, for a purely imaginary complex number, its imaginary part is positive. In the eikonal limit, we have

$$\begin{aligned}
\varpi &= \sqrt{\frac{1}{4} - b(\omega_R^2 - \omega_I^2) + 2ib\omega_R\omega_I} \\
&\approx \sqrt{b\,\omega_R}\,\sqrt{-\omega_R + 2i\omega_I} \\
&= \sqrt{\frac{b\,\omega_R}{2}}\left(\sqrt{\sqrt{\omega_R^2 + 4\omega_I^2} - \omega_R} + i\sqrt{\sqrt{\omega_R^2 + 4\omega_I^2} + \omega_R}\right) \\
&\approx \sqrt{b\,\omega_R}\left(\frac{\omega_I}{\sqrt{\omega_R}} + i\sqrt{\omega_R}\right) \\
&= i\sqrt{b}\,\omega
\end{aligned}$$

The imaginary part of ω is negative, so the real part of ϖ is indeed positive. Analogously to the case with the horizon, we consider the phase

$$-i\omega t + \left(\frac{1}{2} \pm \varpi\right)\log(1-z) \sim -i\omega_R\left(t \mp \sqrt{b}\log(1-z)\right)\,. \tag{5.58}$$

As the function $\log(1-z)$ is decreasing, we can conclude that a mode proportional to Q_+ is ingoing at infinity while a mode proportional to Q_- is outgoing at infinity.

For a flux condition at infinity, we have

$$\begin{aligned}
\mathcal{F} \propto i\varpi_I &\Big(|Q_-|^2(1-z)^{-2\varpi_R} - |Q_+|^2(1-z)^{2\varpi_R}\Big) \\
&- \varpi_R\Big(Q_-^* Q_+(1-z)^{2i\varpi_I} - Q_+^* Q_-(1-z)^{-2i\varpi_I}\Big)\,.
\end{aligned} \tag{5.59}$$

Because the real part of ϖ is positive, we note that an outgoing wave, $Q_+ = 0$, will lead to a diverging flux, while an ingoing wave, $Q_- = 0$, will have finite flux.

We could also look at a Dirichlet condition at infinity

$$\Phi(t,\theta,z) \xrightarrow{z\to 1} 0\,. \tag{5.60}$$

It does not impose a condition on Q_+ since the real part of $\frac{1}{2} + \varpi$ is always positive. Thus, the Dirichlet condition only implies $Q_- = 0$, similar to the finite flux condition.

One can notice that the natural proposal of ingoing waves at the horizon and outgoing waves at infinity, namely $C_+ = 0$ and $Q_+ = 0$, does not align with the boundary conditions proposed in [82], which specify ingoing waves at the horizon ($C_+ = 0$) and a finite flux at infinity ($Q_- = 0$). We will examine both cases independently and compare the implications.

Outgoing at infinity

For both cases mentioned above, we take ingoing modes at the horizon. To achieve vanishing value of the Q_+ coefficient, we have two possibilities

$$-A + B + C = -n \qquad \text{or} \qquad -A - B + C = -n\,, \tag{5.61}$$

for n a positive integer.

Case 1: $-A - B + C = -n$. This condition leads to

$$\frac{1}{2}\left(1 - \sqrt{1 - \frac{12(\nu^2-1)\omega^2}{(\nu^2+3)^2}}\right) - \frac{i(4k+\omega\delta)d}{(\nu^2+3)} = -n\,, \tag{5.62}$$

where we have introduced

$$\delta \equiv 2\Big(\nu(r_+ + r_-) - \sqrt{r_+ r_-(\nu^2+3)}\Big) \quad \text{and} \quad d \equiv \frac{1}{r_+ - r_-}\,. \tag{5.63}$$

To solve this equation, the easiest way is to isolate the square root and square the equation. It will provide us with two independent solutions, which we will substitute back into the original equation to determine the correct one. It give a quadratic equation in terms of ω of the form:

$$\mathcal{A}\omega^2 + \mathcal{B}\omega + \mathcal{C} = 0\,, \tag{5.64}$$

where

$$\begin{aligned}
\mathcal{A} &= a^2\delta^2 - b\,,\\
\mathcal{B} &= 2a\delta X_n\,,\\
\mathcal{C} &= X_n^2 + \frac{1}{4}\,,\\
X_n &= 4ak + i(n+\frac{1}{2})\,.
\end{aligned}$$

We defined two positive parameters to simplify the equations: $a \equiv \frac{1}{(r_+-r_-)(\nu^2+3)}$ and $b \equiv \frac{3(\nu^2-1)}{(\nu^2+3)^2}$. The general solution of (5.64) is

$$\omega_{1\pm} = \frac{-\mathcal{B} \pm \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}}}{2\mathcal{A}}\,. \tag{5.65}$$

Here and in the following cases, we label ω such that the first subscript "1" or "2" stands for the case and the second subscript stands for the two solutions "$\pm$". By taking the eikonal limit we obtain:

$$\omega_{1\pm} \sim \tilde{\Omega}_\pm k - i\left(n + \frac{1}{2}\right)\gamma_{L\pm} \tag{5.66}$$

with $\tilde{\Omega}_\pm$, $\gamma_{L\pm}$ given by (5.17) and (5.25). Now, to determine which solution satisfies the original equation (5.62), we need to substitute them into it and take the eikonal limit. Using the fact that in this limit, $\varpi \approx i\sqrt{b}\,\omega$ with b being defined in (5.57),

$$\omega_{1+} \overset{|k|\to\infty}{\sim} \tilde{\Omega}_+ k - i\left(n+\frac{1}{2}\right)\gamma_{L+}\,. \tag{5.67}$$

The full modes are then the solution (5.65) with the positive sign

$$\omega_{1+} = -\frac{\delta}{4}\chi + \frac{\mathrm{Sign}(k)}{4}\sqrt{3(\nu^2-1)(r_+-r_-)^2\chi^2 - 4\gamma_{L+}\gamma_{L-}}\,, \tag{5.68}$$

where

$$\chi = \tilde{\Omega}_+\tilde{\Omega}_-\left(k + \frac{i}{4}\left(n+\frac{1}{2}\right)(\nu^2+3)(r_+-r_-)\right)\,, \tag{5.69}$$

which can be rewritten as

$$\begin{aligned}\chi &= \tilde{\Omega}_+\left(\tilde{\Omega}_- k - i\left(n+\frac{1}{2}\right)\gamma_{L-}\right)\\ &= \tilde{\Omega}_-\left(\tilde{\Omega}_+ k - i\left(n+\frac{1}{2}\right)\gamma_{L+}\right)\,.\end{aligned} \tag{5.70}$$

The modes (5.68) exhibit a symmetry in terms of the parameters of both photon rings, but it is really the sign in front of the square root that will determine the behavior for large k and whether the modes depend on the inner or outer photon ring in that limit.

One could argue that these angular frequencies are not well defined when $\nu = \sqrt{\frac{3(r_+-r_-)^2}{14r_+r_- - r_+^2 - r_-^2}}$ because $\tilde{\Omega}_- = 1/\tilde{\lambda}_- \to +\infty$, cf. (5.16). However, by carefully taking the limit, we observe that the divergent parts cancel out, and the behavior of the modes near this singular point takes the same form as in the eikonal limit

$$\omega_{1+} \sim \tilde{\Omega}_+ k - i\left(n+\frac{1}{2}\right)\gamma_{L+}\,. \tag{5.71}$$

Case 2: $-A+B+C=-n$. This condition reads

$$\frac{1}{2}\left(1-\sqrt{1-\frac{12(\nu^2-1)\omega^2}{(\nu^2+3)^2}}\right) - \frac{2i\omega\nu}{(\nu^2+3)} = -n\,. \tag{5.72}$$

The 2 solutions for ω are

$$\omega_{2\pm} = -i\nu\Big(2n+1\Big) \pm i\sqrt{3(\nu^2-1)n(n+1)+\nu^2}\,. \tag{5.73}$$

By replacing these solutions in (5.72), we select the correct one

$$\omega_{2+} = -i\nu\Big(2n+1\Big) + i\sqrt{3(\nu^2-1)n(n+1)+\nu^2}\,. \tag{5.74}$$

These frequencies are purely imaginary and are independent of the angular momentum k. This implies that for these modes, there is no notion of ingoing or outgoing; they do not propagate. Furthermore, they only depend on the warp factor ν. One could argue that in the absence of an eikonal limit for these solutions, they cannot strictly be considered as QNMs.

Finite flux and Dirichlet condition at infinity

Another condition that one could naturally propose (and this was done in [82]) is a finite flux at infinity. This condition implies the same constraint as the Dirichlet condition which was mentioned before. We will see that the resulting modes are the ones that we reject in (5.65) and (5.73).

Case 1: $1 - A - B - C = -n$. This condition leads to

$$\frac{1}{2}\left(1+\sqrt{1-\frac{12(\nu^2-1)\omega^2}{(\nu^2+3)^2}}\right)-\frac{i(4k+\omega\delta)d}{(\nu^2+3)}=-n\,. \tag{5.75}$$

By isolating the square root and squaring the equation, the general solutions are also (5.65). However, only one of them is a solution to (5.75), such that

$$\omega_{1-}=-\frac{\delta}{4}\chi-\frac{\mathrm{Sign}(k)}{4}\sqrt{3(\nu^2-1)(r_+-r_-)^2\chi^2-4\gamma_{L+}\gamma_{L-}}\,, \tag{5.76}$$

with eikonal limit

$$\omega_{1-}\overset{|k|\to\infty}{\sim}\tilde{\Omega}_{-}k-i\left(n+\frac{1}{2}\right)\gamma_{L-}\,. \tag{5.77}$$

This time, the modes depend on the inner photon ring instead of the outer one. Although the modes diverge when $\nu=\sqrt{\frac{3(r_+-r_-)^2}{14r_+r_--r_+^2-r_-^2}}$, as was already pointed out in [82], the modes behave as in the eikonal limit in its vicinity.

Case 2: $1 - A + B - C = -n$. This condition reads

$$\frac{1}{2}\left(1+\sqrt{1-\frac{12(\nu^2-1)\omega^2}{(\nu^2+3)^2}}\right)-\frac{2i\omega\nu}{(\nu^2+3)}=-n\,. \tag{5.78}$$

This time, the solution is

$$\omega_{2-}=-i\nu\Big(2n+1\Big)-i\sqrt{3(\nu^2-1)n(n+1)+\nu^2}\,. \tag{5.79}$$

Once again, these modes are purely imaginary.

To conclude, if one requires outgoing waves at infinity, the resulting modes are, in the eikonal limit,

$$\begin{aligned}\omega_{1+}&=\tilde{\Omega}_{+}k-i\left(n+\frac{1}{2}\right)\gamma_{L+}\,,\\ \omega_{2+}&=-i\nu\Big(2n+1\Big)+i\sqrt{3(\nu^2-1)n(n+1)+\nu^2}\,.\end{aligned} \tag{5.80}$$

While, if one imposes a finite flux or a Dirichlet condition at infinity, one gets

$$\begin{aligned} \omega_{1-} &\sim \tilde{\Omega}\_k - i\left(n+\frac{1}{2}\right)\gamma_{L-}\,, \\ \omega_{2-} &\sim -i\nu\Big(2n+1\Big) - i\sqrt{3(\nu^2-1)n(n+1)+\nu^2}\,. \end{aligned} \tag{5.81}$$

For each condition at infinity, we found a purely imaginary solution that depends solely on the warp parameter ν and not on the black hole parameters. The next section will explore if this property persists in other coordinate systems and whether it is possible to interchange the modes $\omega_{1\pm}$, indicating that the boundary conditions are not independent.

5.3.2 Another set of coordinates?

The presence of purely imaginary modes in the QNM spectrum is troubling because no eikonal limit can be taken in this case. Since the dispersion relation depends on the coordinate system, one can wonder whether there exists one where the dispersion relation acquires a real part that could be relevant in the eikonal limit, and whether the modes found by imposing a finite flux at infinity can be obtained from the outgoing modes. We will limit ourselves to a linear and invertible transformation M for the t and θ coordinates:

$$\begin{pmatrix}\hat{t}\\ \hat{\theta}\end{pmatrix} = M\begin{pmatrix}t\\ \theta\end{pmatrix} \equiv \begin{pmatrix}A_t & B_t\\ A_\theta & B_\theta\end{pmatrix}\begin{pmatrix}t\\ \theta\end{pmatrix} \tag{5.82}$$

where A_t, A_θ and B_t, B_θ are real constants. A transformation of the radial coordinate is not relevant because it will have no impact on the modes, and we do not want cross terms mixing r with the other coordinates in the metric. Our starting metric is as before (5.2), and we will examine how the different parameters of the photon ring ($\tilde{r}_\pm$, $\tilde{\lambda}_\pm$, and $\gamma_{L\pm}$) will be modified after such a transformation. Our new metric is of the form

$$\mathrm{d}s^2 = g_{\hat{t}\hat{t}}(r)\mathrm{d}\hat{t}^2 + 2g_{\hat{t}\hat{\theta}}(r)\mathrm{d}\hat{t}\mathrm{d}\hat{\theta} + g_{\hat{\theta}\hat{\theta}}(r)\mathrm{d}\hat{\theta}^2 + g_{rr}(r)\mathrm{d}r^2\,. \tag{5.83}$$

A null geodesic satisfies the following equation

$$\dot{r}^2 + V(r) = 0\,, \tag{5.84}$$

where

$$V(r) = -\hat{E}^2\frac{g_{\hat{\theta}\hat{\theta}}(r) + 2\lambda g_{\hat{t}\hat{\theta}}(r) + \lambda^2 g_{\hat{t}\hat{t}}(r)}{g_{rr}(r)\left(g_{\hat{t}\hat{\theta}}(r)^2 - g_{\hat{\theta}\hat{\theta}}(r)g_{\hat{t}\hat{t}}(r)\right)}\,, \tag{5.85}$$

where $\hat{\lambda}$ is given by (5.11) and $\hat{E}$ and $\hat{L}$ are the constants of motion related to $\partial_{\hat{t}}$ and $\partial_{\hat{\theta}}$. We find the critical radius and the critical energy-rescaled angular momentum by examining the zeros of the potential and its first derivative. In terms of the old critical parameters (5.10, 5.12), the new parameters of the photon rings are

$$\begin{aligned} \hat{r}_\pm &= \tilde{r}_\pm\,, \\ \hat{\lambda}_\pm &= \frac{A_t\tilde{\lambda}_\pm + B_t}{A_\theta\tilde{\lambda}_\pm + B_\theta} = \frac{1}{\hat{\Omega}_\pm}\,. \end{aligned} \tag{5.86}$$

As the change of coordinates does not modify the radial coordinate, the critical radii remain identical, but the angular momenta $\Omega_\pm$ are modified.

For the Lyapunov exponent, the easiest method is to compute the second derivative of the geodesic potential (5.25)

$$\hat{\gamma}_{L\pm} = \sqrt{-\frac{1}{2}V''(r)}\frac{1}{\dot{t}}\Bigg|_{r=\hat{r}_\pm,\lambda=\hat{\lambda}_\pm} \tag{5.87}$$

$$= \mathrm{Sign}(\det M)\frac{\gamma_{L\pm}}{A_t + B_t\tilde{\Omega}_\pm}\,. \tag{5.88}$$

Now, we can examine how ω and k are modified in the dispersion relation. They are defined by the decomposition of the modes as

$$\Phi(x) = e^{-i\omega t + ik\theta}\phi(r)\,, \tag{5.89}$$

As we aim to describe the same quasinormal modes, the phase needs to remain identical

$$-i\omega t + ik\theta = -i\hat{\omega}\hat{t} + i\hat{k}\hat{\theta}\,. \tag{5.90}$$

It implies that under the change of coordinates (5.82)

$$\begin{aligned} \omega &= A_t\hat{\omega} - A_\theta\hat{k}\,, \\ k &= B_\theta\hat{k} - B_t\hat{\omega}\,. \end{aligned} \tag{5.91}$$

Now that we have observed how the different quantities are modified under the new coordinate system (5.82), we will explore how we can use that to modify the dispersion relation in the eikonal limit as desired. Originally, we have two sets of modes described by:

$$\begin{aligned} \omega_1 &= \tilde{\Omega}_\pm k - i(n+\frac{1}{2})\gamma_{L\pm}\,, \\ \omega_2 &= -if(n)\,. \end{aligned} \tag{5.92}$$

The second set consists entirely of an imaginary part. Ideally, we are looking for a coordinate system where the new modes satisfy, in the eikonal limit, the relations:

$$\begin{aligned} \hat{\omega}_1 &= \hat{\Omega}^{(1)}_\pm\hat{k} - i(n+\frac{1}{2})\hat{\gamma}^{(1)}_{L\pm}\,, \\ \hat{\omega}_2 &= \hat{\Omega}^{(2)}_\mp\hat{k} - if(n)\hat{\gamma}^{(2)}_{L\mp}\,, \end{aligned} \tag{5.93}$$

because the QNMs are expected to depend on the photon ring parameters. We chose to consider in $\hat{\omega}_2$ the parameter of the opposite photon ring of $\hat{\omega}_1$. It follows that

$$\begin{aligned} \hat{\Omega}^{(1)}_{\pm} &= \frac{A_\theta + B_\theta \tilde{\Omega}_{\pm}}{A_t + B_t \tilde{\Omega}_{\pm}}\,, \qquad & \hat{\Omega}^{(2)}_{\mp} &= \frac{A_\theta}{A_t}\,, \\ \hat{\gamma}^{(1)}_{L\pm} &= \frac{\gamma_{L\pm}}{A_t + B_t \tilde{\Omega}_{\pm}}\,, \qquad & \hat{\gamma}^{(2)}_{L\mp} &= \frac{1}{A_t}\,. \end{aligned} \tag{5.94}$$

The equations for $\hat{\Omega}^{(1)}_{\pm}$ and $\hat{\gamma}^{(1)}_{L\pm}$ are automatically satisfied according to (5.86) and (5.88). However, the equation for $\hat{\Omega}^{(2)}_{\mp}$ and (5.86) lead us to the constraint that the determinant of the transformation is zero, which is in contradiction with our initial supposition.

If we slightly loosen our requirements and only impose that the second set of modes acquire the Lyapunov exponent, without specifying a precise real part,

$$\hat{\omega}_2 = \hat{f}_2\,\hat{k} - i f(n) \hat{\gamma}^{(2)}_{L\mp}\,, \tag{5.95}$$

it implies that

$$\frac{B_t}{A_t} = \frac{\gamma_{L\mp} - 1}{\tilde{\Omega}_{\mp}}\,, \tag{5.96}$$

and the second modes take the final form

$$\hat{\omega}_2 = \frac{A_\theta \gamma_{L\mp}}{A_\theta + B_\theta \tilde{\Omega}_{\mp}} \hat{\Omega}^{(2)}_{\mp} \hat{k} - i f(n) \hat{\gamma}^{(2)}_{L\mp}\,. \tag{5.97}$$

The coefficient in front of $\hat{\Omega}^{(2)}_{\mp}$ cannot be set to 1 by a choice of A_θ and B_θ because it would imply the same condition (5.96) for them, leading to a non-reversible transformation. Therefore, the imaginary modes cannot be in the form (5.1) in any coordinate system. In this sense, it strongly suggests that those modes cannot be interpreted as QNMs.

It is really interesting to conclude that the form

$$\omega = \tilde{\Omega}_{\pm} k - i \left(n + \frac{1}{2}\right) \gamma_{L\pm}\,, \tag{5.98}$$

in the eikonal limit is conserved through linear transformation of the form (5.82). Although we have proven this starting from our metric (5.2), we expect that it should hold true for any metric.

The mode solutions for the two different boundary conditions, outgoing or finite flux at infinity, are fundamentally different in any coordinate system. In view of determining which boundary condition gives the most physically relevant modes outside the horizon, we need to analyze the spectrum near the outer photon ring. To do this, we will use three different methods and will examine each of them separately in the following sections.

5.4 Penrose limit

After exploring the potential modification of the modes (5.67) through a coordinate transformation, our attention will turn to analyzing how these modes behave in the vincinity of various photon rings. To define this vincinity, we will initially study the Penrose limit of the Warped AdS_3 black hole spacetime in this section, followed by an investigation into the region near the rings in the subsequent section.

In [97], an alternative method was proposed for computing the eikonal limit of the modes. This involves performing a Penrose limit of the geometry around the outer photon ring of (5.2), followed by solving the wave equation (see Figure 5.2).

$$\begin{array}{ccc} \text{General spacetime } (M, g) & \xrightarrow{\text{Penrose limit}_\gamma} & \text{Plane wave spacetime } (M_\gamma, g_\gamma) \\ \downarrow & & \downarrow \\ \text{Wave equation on } (M, g) & \xrightarrow[\text{Geometric optics}_\gamma]{} & \text{Wave equation on } (M_\gamma, g_\gamma) \end{array}$$

Figure 5.2: Geometrical optics approximation, including the leading amplitude evolution, to the wave equation based on the null geodesic γ on a spacetime M with metric g, is the exact wave equation on the Penrose limit spacetime M_γ with metric g_γ. This figure was taken from [97].

We will first compute the Penrose limit around any null geodesic of a general metric.

5.4.1 A Penrose limit guidebook

We will outline the steps provided in [234] to perform the Penrose limit for any metric of the relevant form. The approach is quite general but will be adapted for the specific class of metrics

$$\mathrm{d}s^2 = g_{tt}(r)\mathrm{d}t^2 + 2g_{t\theta}(r)\mathrm{d}t\mathrm{d}\theta + g_{\theta\theta}(r)\mathrm{d}\theta^2 + g_{rr}(r)\mathrm{d}r^2\,, \tag{5.99}$$

where we can interpret r as a radial coordinate.

The first step is to rewrite this in Penrose coordinates. The change of variables is

$$\begin{aligned} \mathrm{d}t &= -\mathrm{d}V + \lambda\,\mathrm{d}y + \dot{t}(U)\,\mathrm{d}U\,, \\ \mathrm{d}\theta &= \dot{\theta}(U)\,\mathrm{d}U + \mathrm{d}y\,, \\ \mathrm{d}r &= \dot{r}(U)\,\mathrm{d}U\,, \end{aligned} \tag{5.100}$$

where $\lambda = L/E$ is the impact parameter, L the conserved angular momentum, E the conserved energy and U is an affine parameter along a null geodesic such that

$\dot{t}$, $\dot{\theta}$ and $\dot{r}$ satisfy the following equations

$$\begin{aligned} g_{tt}\,\dot{t} + g_{t\theta}\,\dot{\theta} &= -1\,, \\ g_{\theta\theta}\,\dot{\theta} + g_{t\theta}\,\dot{t} &= \lambda\,, \\ \dot{r}^2 = \frac{g_{tt}\lambda^2 + 2g_{t\theta}\lambda + g_{\theta\theta}}{g_{rr}\,(g_{t\theta}^2 - g_{tt}g_{\theta\theta})} &\equiv V_{\text{eff}}(r)\,. \end{aligned} \tag{5.101}$$

Performing that change of coordinates gives the following metric:

$$\mathrm{d}s^2 = 2\,\mathrm{d}U\,\mathrm{d}V + g(U)\,\mathrm{d}y^2 + g_{tt}\,\mathrm{d}V^2 - 2\,(g_{tt}\lambda + g_{t\theta})\,\mathrm{d}y\,\mathrm{d}V\,. \tag{5.102}$$

where we defined $g(U) \equiv g_{tt}\lambda^2 + 2g_{t\theta}\lambda + g_{\theta\theta}$. We can now take the so-called Penrose limit. We perform a boost for the U and V coordinates combined with an overall rescaling of the coordinates:

$$(U, V, y) \overset{\text{boost}}{\to} (k^{-1}U, k\,V, y) \overset{\text{rescaling}}{\to} (U, k^2V, k\,y)\,. \tag{5.103}$$

Then, we rescale the metric as $\mathrm{d}s^2 \to k^{-2}\mathrm{d}s^2$ and consider the limit $k \to 0$. This process yields the Penrose limit of the original metric:

$$\mathrm{d}s_P^2 = 2\,\mathrm{d}U\,\mathrm{d}V + g(U)\,\mathrm{d}y^2\,. \tag{5.104}$$

The metric is a plane wave written in Rosen coordinates. We then need a last change of coordinates to rewrite it in terms of Brinkmann coordinates:

$$\begin{aligned} U &= u\,, \\ V &= v + \frac{1}{4}\frac{\dot{g}}{g}x^2\,, \\ y &= \frac{1}{\sqrt{g}}x\,. \end{aligned} \tag{5.105}$$

The final result is:

$$\mathrm{d}s_P^2 = 2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}x^2 + A(u)\,x^2\,\mathrm{d}u^2\,, \tag{5.106}$$

where

$$A(u) = \frac{1}{2}\frac{\ddot{g}}{g} - \frac{1}{4}\frac{\dot{g}^2}{g^2}\,. \tag{5.107}$$

Now the dots refer to derivatives according to the affine parameter u. If the wave profile A is constant, we can perform one last change of coordinates $u \to u/\sqrt{A}$ and $v \to \sqrt{A}\,v$ to get:

$$\mathrm{d}s_P^2 = 2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}x^2 + x^2\,\mathrm{d}u^2\,. \tag{5.108}$$

This approach involves two specific changes of coordinates.

We can apply this method for any null geodesics in the warped black hole spacetime (5.2) and obtain the wave profile

$$A = 3(\nu^2 - 1)\,. \tag{5.109}$$

Hence, as long as $\nu \neq 1$, the Penrose limit yields a non-trivial pp-wave and can be brought to the form (5.108). From its isometries, it is a Warped Flat space [235]. When $\nu = 1$, the Penrose limit of AdS_3 or BTZ is flat, as expected on general grounds.

However, if we continue in this manner, we will encounter a problem. When solving the wave equation in this spacetime, we eventually need to relate the coordinates of the Penrose limit back to the initial coordinates, particularly to impose appropriate boundary conditions for the pp-wave. These changes of coordinates become unclear when the null geodesic is precisely located on the photon ring. Therefore, we require an alternative method, which we will derive using Fermi coordinates in the next section.

5.4.2 Penrose limit with Fermi coordinates

In this section, we will derive the Penrose limit of both photon rings of warped AdS_3 black holes using Fermi coordinates and show that it takes the form

$$\frac{\mathrm{d}s^2}{l^2} = 2\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + x^2\mathrm{d}u^2\,. \tag{5.110}$$

For the convenience of the reader, we summarize the construction of the Penrose limit for both photon rings $\tilde{r}_\pm$ simultaneously. We first define a set of vectors that satisfy

$$u^\mu u_\mu = v^\mu v_\mu = u^\mu e_\mu = v^\mu e_\mu = 0\ ,\ u^\mu v_\mu = e^\mu e_\mu = 1\,, \tag{5.111}$$

on the photon ring with u^μ a tangent vector to the null geodesic, which implies that

$$u_\mu \mathrm{d}x^\mu = \mathrm{d}S\,, \tag{5.112}$$

where S satisfies the Hamilton-Jacobi equation

$$g^{\mu\nu}\,\partial_\mu S\,\partial_\nu S = 0\,, \tag{5.113}$$

with $g_{\mu\nu}$ given by (5.2). By separation of variables, a particular solution is expressed as

$$S = -t + \lambda\,\theta + \rho(r)\,, \tag{5.114}$$

where λ is determined by (5.11) and

$$\rho'(r) = D(r)\dot{r}\,, \tag{5.115}$$

with $\dot{r}$ given in (5.6) and the function $D(r)$ defined as

$$D(r) = \frac{1}{4R(r)^2N(r)^2}\,. \tag{5.116}$$

Then a set of vectors can be found on each photon ring as

$$\begin{aligned} u^\mu \partial_\mu &= b\left(\partial_t + \tilde{\Omega}_\pm \partial_\theta\right) , \\ v^\mu \partial_\mu &= -\left(1+\frac{b}{2}\right)\partial_t - \frac{b}{2}\tilde{\Omega}_\pm \partial_\theta , \\ e^\mu \partial_\mu &= \frac{1}{\sqrt{D(\tilde{r}_\pm)}}\partial_r , \end{aligned} \tag{5.117}$$

where we defined

$$b = \pm\frac{\sqrt{A}}{\gamma_{L\pm}} , \quad A = 3(\nu^2-1) , \quad \gamma_{L\pm} = -\frac{\nu^2+3}{4}(r_+ - r_-)\tilde{\Omega}_\pm , \tag{5.118}$$

with γ_L the Lyapunov exponent. This frame must be parallel transported along the geodesic. A parallel transported vector satisfies the equations:

$$\begin{aligned} \dot{V}^r(0) &= \mp\frac{\nu}{\sqrt{D(\tilde{r}_\pm)}}V^t(0) \pm \frac{\nu\,\tilde{\lambda}_\pm}{\sqrt{D(\tilde{r}_\pm)}}V^\theta(0) , \\ V^r &= \dot{V}^r(0)\tau + V^r(0) , \\ V^t &= 2\nu\sqrt{D(\tilde{r}_\pm)}\left(\frac{1}{2}\dot{V}^r(0)\tau^2 + V^r(0)\tau\right) + V^t(0) , \\ V^\theta &= 2\nu\lambda_\pm\sqrt{D(\tilde{r}_\pm)}\left(\frac{1}{2}\dot{V}^r(0)\tau^2 + V^r(0)\tau\right) + V^\theta(0) , \end{aligned} \tag{5.119}$$

where τ is an affine parameter and the dot represents the derivative with respect to this parameter. Solving these equations, we obtain a parallel transported null frame attached to the null geodesic:

$$\begin{aligned} u^\mu \partial_\mu &= b\left(\partial_t + \tilde{\Omega}_\pm \partial_\theta\right) , \\ v^\mu \partial_\mu &= -\partial_t + \left(\pm\tilde{\lambda}_\pm \nu^2 \sqrt{D(\tilde{r}_\pm)}\tau^2 - \frac{b}{2}\right)\left(\partial_t + \tilde{\Omega}_\pm\partial_\theta\right) \pm \frac{\nu}{\sqrt{D(\tilde{r}_\pm)}}\tau\partial_r , \\ e^\mu \partial_\mu &= 2\nu\sqrt{D(\tilde{r}_+)}\tau\left(\tilde{\lambda}_+\partial_t + \partial_\theta\right) + \frac{1}{\sqrt{D(\tilde{r}_\pm)}}\partial_r . \end{aligned} \tag{5.120}$$

Defining $\partial_u = u^\mu\partial_\mu$, $\partial_v = v^\mu\partial_\mu$ and $\partial_x = e^\mu\partial_\mu$, this frame allows us to perform the Penrose limit of the photon ring and obtain the metric:

$$\frac{\mathrm{d}s^2}{l^2} = 2\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + A\,x^2\mathrm{d}u^2 , \tag{5.121}$$

where $u = \tau$ is the affine parameter along the null geodesic, v is constant along the wave front, x is the transverse coordinate associated to the direction e^μ and

$$A = R_{\mu\alpha\nu\beta}u^\mu e^\alpha u^\nu e^\beta = 3(\nu^2-1) \tag{5.122}$$

is the wave profil. Now, we can solve (5.120) to find the linear change of coordinates:

$$\begin{aligned} t &= bu - \left(1 + \frac{b}{2}\right) v \,, \\ \theta &= \frac{b}{\tilde{\lambda}_\pm}\left(u - \frac{v}{2}\right) \,, \\ r &= \tilde{r}_\pm \mp \frac{b}{2\tilde{\lambda}_\pm} x \,. \end{aligned} \tag{5.123}$$

We can perform a last change of coordinates to absorb the constant wave profile A

$$u \to \frac{u}{\sqrt{A}} \,, \; v \to \sqrt{A} v \,. \tag{5.124}$$

5.4.3 Quasinormal modes from Penrose limit

Having demonstrated that the Penrose limit around the different photon rings can be formulated as (5.110), and having identified a coordinate transformation (5.123) linking (5.110) to our initial spacetime (5.2), we are now poised to investigate the quasinormal modes in this regime. As previously, we solve the wave equation for a massless scalar field Φ, albeit now in the background (5.110):

$$\nabla_\mu \nabla^\mu \Phi = 0 \,. \tag{5.125}$$

Using the ansatz

$$\Phi(u, v, x) = e^{ip_u u + ip_v v} \psi(x) \,, \tag{5.126}$$

where p_u and p_v are, a priori, complex numbers, the wave equation becomes

$$\psi''(x) = (2p_u p_v - p_v^2 x^2)\psi(x) \,. \tag{5.127}$$

The solution for this equation takes the form

$$\psi(x) = C_1 D_{a-\frac{1}{2}}((1+i)\sqrt{p_v}\, x) + C_2 D_{-a-\frac{1}{2}}((i-1)\sqrt{p_v}\, x) \,, \tag{5.128}$$

with $a = ip_u$ and where C_1, C_2 are integration constants and $D_\nu(x)$ are parabolic cylinder functions.

To determine the integration constants, we examine the behavior at both infinities, i.e $x \to \pm\infty$, and require for both photon rings an outgoing wave condition. Because of the exponential time dependence, every null geodesic will be arbitrarily far from the photon rings for a sufficiently large time. At both infinities, the function (5.128) behaves as:

$$\begin{aligned} \psi(x) &\overset{x\to\infty}{\sim} \left(C_1 + \frac{C_2}{\Gamma\left(\frac{1}{2} + a\right)}\right) e^{-ip_v x^2/2} x^{a-1/2} + C_2 e^{ip_v x^2/2} x^{-a-1/2} \,, \\ \psi(x) &\overset{x\to-\infty}{\sim} C_1 e^{-ip_v x^2/2} x^{a-1/2} + \left(\frac{C_1}{\Gamma\left(\frac{1}{2} - a\right)} + C_2\right) e^{ip_v x^2/2} x^{-a-1/2} \,. \end{aligned} \tag{5.129}$$

Relating back to the original metric, we can use the inverse transformation of (5.123) and (5.124):

$$\begin{aligned} u &= \frac{\sqrt{A}}{2}\left(-t + (1+\frac{2}{b})\tilde{\lambda}_{\pm}\theta\right) , \\ v &= \frac{1}{\sqrt{A}}(-t + \tilde{\lambda}_{\pm}\theta) , \\ x &= \sqrt{D(\tilde{r}_{\pm})}(r - \tilde{r}_{\pm}) , \end{aligned} \tag{5.130}$$

to rewrite our quasinormal modes. In this case, we have two waves of the form:

$$\Phi \sim e^{-\frac{i}{\sqrt{A}}\left(p_u \frac{A}{2} + p_v\right)t \pm i p_v D(\tilde{r}_{\pm})(r - \tilde{r}_{+})^2} . \tag{5.131}$$

To ensure outgoing waves near the photon rings, it is necessary for the factor in front of t and the one in front of r to have opposite signs. One can verify that choosing $\pm$ always imposes a condition on p_u and makes it purely imaginary. Since $D(\tilde{r}_+)$ and A are always positive, our focus should be on the terms involving p_v. Thus, we consistently need to choose the $+$ sign between the t and r terms. This requirement is equivalent to imposing

$$C_1 = 0 \text{ and } \frac{1}{2} + a = -n \;,\; n \in \mathbb{N} . \tag{5.132}$$

It implies then that $p_u = i(n + 1/2)$ and

$$\Phi_n \sim e^{-(n+1/2)u + i p_v (v + x^2/2)} H_n\left(-\sqrt{-i p_v}\, x\right) , \tag{5.133}$$

where $H_n(x)$ are the Hermite functions.

We can now compare to the quasinormal modes in the original spacetime:

$$\Phi(t, \theta, r) = e^{-i\omega t + ik\theta} f(r) . \tag{5.134}$$

Using the change of coordinates (5.123) and comparing the results to (5.133), we get:

$$-i\frac{b}{\sqrt{A}}\omega + i\frac{b}{\sqrt{A}}\tilde{\Omega}_{\pm}k = \mp(n + \frac{1}{2}) , \tag{5.135}$$

$$\left(1 + \frac{b}{2}\right)\omega - \frac{b}{2}\tilde{\Omega}_{\pm}k = p_v . \tag{5.136}$$

So we obtain the dispersion relation for each photon ring:

$$\omega = \tilde{\Omega}_{\pm}k - i\left(n + \frac{1}{2}\right)\gamma_{L\pm} . \tag{5.137}$$

We notice that around each photon ring, the modes depending on the corresponding parameters prevail.

5.5 Symmetries of the eikonal QNM spectrum

In this section, we will analyze the symmetries of the QNM spectrum within what we will denote as the near-ring region of phase space. Following this, the massless wave equation will be solved within this region. We expect the outcomes to mirror those derived from the Penrose limit, given that they share the same philosophy: exploring the QNM spectrum near the photon rings.

To define this region, we will once again study the QNM potential (5.35). In the eikonal limit, when ω_R and k are both large and of comparable magnitude, the real part of the QNM potential closely resembles the potential for null geodesics (5.7) if we identify

$$\omega_R \leftrightarrow E \qquad \text{and} \qquad k \leftrightarrow L \,. \tag{5.138}$$

It is therefore reasonable to define an analogous version of (5.29). We will describe both photon rings simultaneously using the subscript $\pm$ for quantities that differ between the two photon rings. We define the phase space of waves for warped AdS_3 black holes, known as the near-ring region, as [83, 98]:

$$\text{NEAR-RING REGION:} \quad \begin{cases} |\delta r| \ll \eta & \text{(near-peak)} \\ \left|\frac{k}{\omega_R} - \tilde{\lambda}_\pm\right| \ll \eta & \text{(near-critical)} \\ \frac{1}{\omega_R} \ll \eta & \text{(high-frequency)} \end{cases} \tag{5.139}$$

where η is defined by (5.27) and, in this section, $\delta r = r - \tilde{r}_\pm$. The final condition indicates that we are considering the eikonal limit of our modes. The families of modes that lack a real part are therefore not included in the description of this section.

In this near-ring region, the QNM potential (5.35) takes the form

$$V_{\text{QNM}}(\delta r) = 3(\nu^2-1)\omega_R \delta r^2 \mp 2i\frac{\nu^2+3}{\sqrt{3(\nu^2-1)}}(r_+ - r_-)\tilde{\lambda}_\pm\, \omega_R \omega_I \,. \tag{5.140}$$

The radial ODE (5.33) can be rewritten as

$$\mathcal{H}\,\phi(\delta r) = i\omega_I\,\phi(\delta r)\,, \tag{5.141}$$

where we defined the Hamiltonian

$$\mathcal{H} = -\frac{1}{2p_1\omega_R}\left(\partial_{\delta r}^2 + p_2^2\,\omega_R^2\,\delta r^2\right)\,, \tag{5.142}$$

with

$$p_1 = \frac{p_2}{\gamma_{L\pm}}\,, \qquad p_2 = \frac{12\sqrt{3}(\nu^2-1)^{3/2}}{(r_+ - r_-)^2(\nu^2+3)^2}\,. \tag{5.143}$$

This represents a time-independent Schrödinger equation where ϕ are the eigenstates of an inverted harmonic oscillator, and the eigenvalues $i\omega_I$ can be imaginary due to the presence of non-Hermitian boundary conditions. Based on [236, 237], we define the operators

$$a_\pm = \frac{e^{\pm p_2 t/p_1}}{\sqrt{2p_2\omega_R}}\Big(\mp i\partial_{\delta r} - p_2\omega_R\delta r\Big)\,, \quad L_0 = -\frac{i}{4}(a_+a_- + a_-a_+) = \frac{ip_1}{2p_2}\mathcal{H}\,, \quad L_\pm = \pm\frac{a_\pm^2}{2}\,, \tag{5.144}$$

where $a_\pm$ are the generators of the Heisenberg algebra $[a_+, a_-] = i\mathbb{1}$, and L_m satisfy the $SL(2,\mathbb{R})_{\text{QN}}$ algebra

$$[L_0, L_\pm] = \mp L_\pm\,, \quad [L_+, L_-] = 2L_0\,. \tag{5.145}$$

An important clarification is needed here regarding the notation. For $a_\pm$ and $L_\pm$, the subscript $\pm$ no longer refers to the different photon rings but follows the standard notation for ladder operators for the sake of clarity. In the definitions of these operators, the only dependence on the different photon rings is encapsulated in the coefficient p_1, and the definitions remain the same for both $\tilde{r}_\pm$. In principle, these operators are defined everywhere in our spacetime, but our focus lies in the near-ring region where L_0 is proportional to the Hamiltonian $\mathcal{H}$. We can also define the Casimir operator

$$L^2 = L_0^2 - L_0 - L_-L_+ = L_0^2 + L_0 + L_+L_-\,. \tag{5.146}$$

It commutes with every L_m, and consequently, by Schur's lemma, it is proportional to the identity. We can compute the proportionality coefficient

$$L^2 = -\frac{3}{16}\mathbb{1}\,. \tag{5.147}$$

The eigenstates of L_0 satisfy the relation

$$L_0\,\phi_h = h\,\phi_h\,, \tag{5.148}$$

with eigenvalues h. We thereby identify

$$\omega_I = -2\frac{p_2}{p_1}h = -2\gamma_{L\pm}h\,. \tag{5.149}$$

The mode ansatz (5.32) in the near-ring region (5.139) reduces to

$$\Phi(t,\delta r,\theta) = e^{-i\omega_R t + ik\theta}\Phi_h(t,\delta r)\,, \quad \Phi_h(t,\delta r) = e^{\omega_I t}\phi_h(\delta r) = e^{-2p_2ht/p_1}\phi_h(\delta r)\,. \tag{5.150}$$

Henceforth, we center our attention on the outer photon ring, but the computations and results are entirely identical for the inner photon ring. Since we desire outgoing boundary conditions for the fundamental modes, it is equivalent to impose the highest-weight condition [98]

$$L_+\Phi_h = 0 \quad \Longleftrightarrow \quad a_+^2\Phi_h = 0\,. \tag{5.151}$$

This requirement implies that

$$L^2\Phi_h = h(h-1)\Phi_h \quad \Rightarrow \quad h = \frac{1}{4} \text{ or } h = \frac{3}{4}\,. \tag{5.152}$$

As the highest-weight condition is a second-order differential equation, there exist two independent solutions $\Phi_{1,h}$ and $\Phi_{2,h}$ such that

$$a_+\Phi_{1,h} = 0 \qquad \text{and} \qquad \Phi_{2,h} = a_-\Phi_{1,h}\,. \tag{5.153}$$

One can verify that these equations imply (5.151) and that they are indeed independent. Using the commutation relation between $a_\pm$, we can also demonstrate that the solutions satisfy

$$\begin{aligned} a_+\Phi_{2,h} &= i\Phi_{1,h} \\ a_+a_-\Phi_{1,h} &= i\Phi_{1,h} \\ a_-a_+\Phi_{2,h} &= i\Phi_{2,h} \end{aligned} \tag{5.154}$$

and, as a consequence,

$$\begin{aligned} L_0\Phi_{1,h} &= \frac{1}{4}\Phi_{1,h}\,, \\ L_0\Phi_{2,h} &= \frac{3}{4}\Phi_{2,h}\,. \end{aligned} \tag{5.155}$$

Φ_1 is thus associated with the eigenvalue $h = \frac{1}{4}$ and Φ_2 with the eigenvalue $h = \frac{3}{4}$. We adjust our notation to clarify this association

$$\Phi_{\frac{1}{4}} \equiv \Phi_{1,h}\,, \qquad \Phi_{\frac{3}{4}} \equiv \Phi_{2,h}\,. \tag{5.156}$$

It remains to solve (5.153):

$$\Phi_{\frac{1}{4}} = e^{-\frac{1}{2}\gamma_{L_+}t+\frac{i}{2}p_2\omega_R\delta r^2}\,, \quad \Phi_{\frac{3}{4}} = \delta r\, e^{-\frac{3}{2}\gamma_{L_+}t+\frac{i}{2}p_2\omega_R\delta r^2}\,. \tag{5.157}$$

The higher overtones are obtained as the tower of the $SL(2,\mathbb{R})_{\text{QN}}$-descendants,

$$\begin{aligned} \Phi_{h,N}(t,\delta r) = L_-^N\Phi_h(t,\delta r) &= e^{-2\gamma_{L_+}(h+N)t}\phi_{h+N}(\delta r) \\ &\propto e^{-2\gamma_{L_+}(h+N)t}D_{2(h+N)-\frac{1}{2}}\left(\sqrt{-2ip_2\omega_R}\,\delta r\right)\,, \end{aligned} \tag{5.158}$$

where $D_n(\delta r)$ denotes the n^{th} parabolic cylinder function. With $n = 2(h+N) - \frac{1}{2}$, near the edges of the near-peak region, where $\delta r \to \pm\infty$, we have

$$\lim_{\delta r\to\pm\infty} D_n\left(\sqrt{-2ip_2\omega_R}\,\delta r\right) \approx \delta r^n e^{\frac{i}{2}p_2\omega_R\delta r^2}\,, \tag{5.159}$$

and the n^{th} overtone near the edges is therefore,

$$\Phi_n(t,\delta r,\theta) \approx e^{-\gamma_{L_+}(n+\frac{1}{2})t}\delta r^n e^{-i\omega_R\left(t-\frac{1}{2}p_2\delta r^2\right)+ik\theta}\,. \tag{5.160}$$

As in the near-ring region where $\omega_R = \tilde{\Omega}_+ k$, we observe the same behavior around the outer photon ring as we do from the Penrose limit

$$\omega = \tilde{\Omega}_+ k - i\left(n + \frac{1}{2}\right)\gamma_{L_+}\,. \tag{5.161}$$

As previously outlined, the discussion for the inner photon ring is exactly identical. Due to the exponential time dependence near the inner photon ring, we require outgoing boundary conditions which are equivalent to the highest-weight condition (5.151), and the overtones are derived from

$$\Phi_{h,N}(t,\delta r) = L_-^N \Phi_h(t,\delta r)\,. \tag{5.162}$$

Near the edges, the n^{th} overtone behaves like

$$\Phi_n(t,\delta r,\theta) \approx e^{-\gamma_{L_-}(n+\frac{1}{2})t}\delta r^n e^{-i\omega_R\left(t-\frac{1}{2}p_2\delta r^2\right)+ik\theta}\,, \tag{5.163}$$

and, as in the near-ring region of the inner photon ring $\omega_R = \tilde{\Omega}_- k$, the modes are

$$\omega = \tilde{\Omega}_- k - i\left(n + \frac{1}{2}\right)\gamma_{L_-}\,. \tag{5.164}$$

Once again, we observe an equivalent behavior as seen in the Penrose limit and the geometric optics approximation.

5.6 Quasinormal modes from geometric optics

The final method we will employ to derive the quasinormal mode expression is through the use of the geometric optics approximation. This method was primarily used for Schwarzschild and Kerr black holes where the exact spectrum is unknown. Despite being an approximation, it offers the advantage of having a clear interpretation and can be used to verify if the exact spectrum obtained from our boundary conditions behaves as expected. The geometric optics approximation connects solutions of the wave equation for a massless scalar field with null geodesic congruences in the eikonal limit. It has been demonstrated that when these null congruences occur near the photon ring, the approximation reproduces the eikonal limit of the quasinormal modes in many scenarios [83, 84, 91, 93–96, 98, 228].

When the wave frequencies are large compared to the local curvature, a solution of the massless wave equation (5.31) takes the approximate form

$$\Phi \approx A\, e^{iS}\,, \tag{5.165}$$

where $S(x^\mu)$ is a rapidly oscillating phase and $A(x^\mu)$ a slowly varying amplitude. The wave equation can then be expressed in terms of the gradient of the phase

$$p_\mu = \partial_\mu S\,, \tag{5.166}$$

as

$$-p_\mu p^\mu A + i\left(2p^\mu \nabla_\mu A + \nabla_\mu p^\mu A\right) + \nabla^2 A = 0\,. \tag{5.167}$$

The usual way to find a solution to this equation in the geometric optics approximation is to solve it order-by-order in inverse power of p. At leading order, p_μ is a null vector

$$p_\mu p^\mu = 0\,. \tag{5.168}$$

It also implies that the phase $S(x^\mu)$ is a solution to the Hamilton-Jacobi equation. As p_μ is a gradient, one can show that

$$p^\mu \nabla_\mu p_\nu = 0\,. \tag{5.169}$$

Indeed,

$$\begin{aligned} p^\mu \nabla_\mu p_\nu &= p^\mu \nabla_\mu \nabla_\nu S = p^\mu \nabla_\nu \nabla_\mu S \\ &= p^\mu \nabla_\nu p_\mu = \frac{1}{2}\nabla_\nu \left(p^\mu p_\mu\right) \\ &= 0\,. \end{aligned}$$

Thus, p_μ satisfies the geodesic equation and naturally defines an affine parameter s

$$\partial_s = p^\mu \partial_\mu\,. \tag{5.170}$$

In the subleading terms of (5.167), the expansion $\hat{\theta} = \nabla_\mu p^\mu$ is related to the parallel transport of the amplitude

$$p^\mu \nabla_\mu A = -\frac{1}{2}\hat{\theta}\, A\,. \tag{5.171}$$

If the expansion is constant, as it will be in our case, this equation can be solved in terms of the affine parameter s

$$\partial_s \log A = -\frac{1}{2}\hat{\theta} \quad\Longrightarrow\quad A = A_0\, e^{-\frac{1}{2}\hat{\theta} s}\,. \tag{5.172}$$

So for a positive expansion, the amplitude decays exponentially.

One may also show that if a function $u(x^\mu)$ does not vary along the null congruence

$$p^\mu \nabla_\mu u = 0\,, \tag{5.173}$$

one can use it to produce towers of approximate solutions. If A_0 is a solution up to the subleading equation (5.171), then $A_n = u^n A_0$ is also a solution.

We encountered the Hamilton-Jacobi equation while computing the Penrose limit (5.113) and found a particular solution through a separation of variables (5.114)

$$S = -\omega t + k\theta + \rho(r)\,, \tag{5.174}$$

where the function $\rho(r)$ satisfied the equation

$$\rho'(r)^2 = g_{rr}^2 V_{\mathrm{QNM}}(r)\,, \tag{5.175}$$

with the potential given in (5.35). We are seeking rays whose impact parameter is close to that of the outer photon ring, as they are asymptotically confined to it. It leads us to, $k/\omega = \tilde{\lambda}_+$,

$$\begin{aligned}\rho(r) &= \pm\omega\frac{\sqrt{3(\nu^2-1)}}{\nu^2+3}\int \frac{(r-\tilde{r}_+)}{(r-r_+)(r-r_-)}\mathrm{d}r \\ &= \pm\frac{\omega\nu}{\nu^2+3}\left(\log\left[\frac{r-r_-}{r-r_+}\right] + \frac{\sqrt{3(\nu^2-1)}}{2\nu}\log\left[(r-r_+)(r-r_-)\right]\right)\end{aligned} \tag{5.176}$$

In the near-ring region, this function approximates as

$$\rho(r) \approx C \pm \frac{6(\nu^2-1)\omega}{(\nu^2+3)(r_+-r_-)^2}\delta r^2\,, \tag{5.177}$$

where C is an irrelevant constant. This determines the behavior found in the Penrose limit and in the near-ring region approaches. Since we are interested in rays leaking out of the photon ring, we choose the positive sign in (5.177) and consequently in (5.176).

The expansion of these congruences is

$$\hat{\theta} = \sqrt{3(\nu^2-1)}\omega\,, \tag{5.178}$$

and is constant as mentioned earlier. We are now able to address the subleading order equation (5.171). Its solution is defined in the whole spacetime, yet we will restrict ourselves to the near-ring region, where it assumes a more elegant form and allows for comparison with the various previous approaches:

$$\left(2\delta r\partial_{\delta r} + \frac{2}{\gamma_{L+}}\partial_t + 1\right)A(t,\delta r) = 0\,. \tag{5.179}$$

A simple solution to this equation is

$$A_0(t,\delta r) = e^{-\frac{1}{2}\gamma_{L+}t}\,. \tag{5.180}$$

We then build the other approximate solutions

$$A_n(t,\delta r) = \delta r^n e^{-\left(n+\frac{1}{2}\right)\gamma_{L+}t}\,. \tag{5.181}$$

The general solutions in the geometric optics approximation in the near-ring region are

$$\Phi_{kn} \approx A_n(t,\delta r)e^{iS} = \delta r^n \exp\left(-i\omega t + k\theta + \frac{6(\nu^2-1)\tilde{\Omega}_+ k}{(\nu^2+3)(r_+-r_-)^2}\delta r^2\right)\,, \tag{5.182}$$

with

$$\omega = \tilde{\Omega}_+ k - i\left(n + \frac{1}{2}\right)\gamma_{L+}\,. \tag{5.183}$$

From the results of these different methods, we conclude the QNMs outside the horizon are primarily sourced by the outer photon ring. This indicates that the correct boundary conditions for QNMs in a warped black hole background are an ingoing condition at the horizon and an outgoing condition at infinity.

5.7 Observable conformal symmetry of the photon ring

In this section, we are interested in the consequences of the photon ring's presence for a distant observer looking at the black hole. Light rays traveling close to the outer photon ring will undergo several rotations around the black hole before escaping and reaching the distant observer. This implies that a single source can have multiple images, depending on their winding number. Similar to Schwarzschild, Kerr [83], and self-dual warped AdS3 black holes [98], we will construct the observable symmetry group SL(2,$\mathbb{R}$)$_{\text{PR}}$ of the photon ring. We will see that the dilation generator of this symmetry allows us to connect null geodesics with successive winding numbers. In $2+1$ dimensions, the observer screen is simply a line, and the black hole photon ring is a single critical point at infinity rather than a closed critical curve as in higher dimensions.

Let Γ be the phase space of null geodesics in (5.2) with coordinates $(r, \theta, p_r, p_\theta)$, and Ω the canonical symplectic form[3]

$$\mathrm{d}\Omega = \mathrm{d}p_r \wedge \mathrm{d}r + \mathrm{d}p_\theta \wedge \mathrm{d}\theta\,. \tag{5.184}$$

The Hamiltonian is obtained by solving the null condition $g^{\mu\nu}p_\mu p_\nu = 0$ for $p_t = -H$

$$H = -\frac{m(r)}{n(r)}p_\theta + \frac{1}{2n(r)D(r)}\sqrt{D(r)p_\theta^2 + n(r)p_r^2}\,, \tag{5.185}$$

which reduces at the bound orbit $(r, p_r) = (\tilde{r}_+, 0)$ to the critical energy $\tilde{H} = p_\theta/\tilde{\lambda}_+$. Since the system is integrable, it admits a canonical transformation to action-angle

[3]For this subsection, we are using different functions as the ones defined in (5.3). Their relations are

$$D(r) = \frac{1}{4R(r)^2N(r)^2}\ ,\ n(r) = R(r)^2\ ,\ m(r) = R(r)^2N^\theta(r)$$

Their use is only to produce more compact and elegant equations.

variables $(r, p_r, \theta, p_\theta) \to (T, H, \Theta, L)$ which preserves the symplectic form $\mathrm{d}\Omega$:

$$\begin{aligned} \mathrm{d}T &= \frac{2D(r)}{\sqrt{V(r)}} \left(n(r)H + m(r)L\right) \mathrm{d}r \,, \\ \mathrm{d}\Theta &= \mathrm{d}\theta + \frac{2D(r)}{\sqrt{V(r)}} \left(m(r)H + L\right) \mathrm{d}r \,, \\ L &= p_\theta \,, \end{aligned} \tag{5.186}$$

where $V(r)$ is related to the radial geodesic potential

$$V(r) \equiv \frac{p_r^2}{4D(r)^2} = \left(L^2 + 2m(r)HL + n(r)H^2\right) . \tag{5.187}$$

In these variables, the equations of motion are trivialized:

$$\begin{aligned} \dot{H} &= \{H, H\} = 0 \,, \qquad & \dot{L} &= \{L, H\} = 0 \,, \\ \dot{\Theta} &= \{\Theta, H\} = 0 \,, \qquad & \dot{T} &= \{T, H\} = 1 \,. \end{aligned} \tag{5.188}$$

The two equations on the first line indicate that the phase space Γ foliates into superselection sectors of fixed (H, L), which are conserved momenta. The first one on the second line implies that a photon with initial coordinates (r_i, θ_i, H, L) evolves to final coordinates (r_f, θ_f, H, L) according to:

$$\Delta\theta = \theta_f - \theta_i = \int_{\theta_i}^{\theta_f} \mathrm{d}\theta = -2 \int_{r_i}^{r_f} \frac{D(r)}{\sqrt{V(r)}} \left(m(r)H + L\right) \mathrm{d}r \,, \tag{5.189}$$

where the integral is evaluated along the photon trajectory. The last equation implies that the time elapsed along such trajectory is:

$$T = 2 \int_{r_i}^{r_f} \frac{D(r)}{\sqrt{V(r)}} \left(n(r)H + m(r)L\right) \mathrm{d}r \ . \tag{5.190}$$

Since we are focusing on optical images, we will only consider geodesics that begin and end at infinity, always remaining outside the ring orbit at $r = \tilde{r}_+$. An observer at infinity receives these null geodesics with impact parameter

$$\lambda = \frac{L}{H} > \tilde{\lambda}_+ \ , \tag{5.191}$$

and energy

$$\hat{H} \equiv H - \frac{L}{\tilde{\lambda}_+} \ . \tag{5.192}$$

The radius of closest approach $r_{\rm min}$ is reached when the radial momentum p_r vanishes. From (5.187), this is equivalent to requiring that $V(r_{\rm min}) = 0$:

$$r_{\rm min} = \tilde{r}_+ - \frac{2}{3(\nu^2 - 1)} \left(2\nu(\lambda - \tilde{\lambda}_+) - \sqrt{(\nu^2 + 3)(\lambda - \tilde{\lambda}_+)(\lambda - \tilde{\lambda}_-)}\right) \ . \tag{5.193}$$

Geodesics with $\hat{H} = 0$ are homoclinic, i.e they represent transitions between stable and unstable orbits in the phase space Γ, asymptoting to the photon orbit at $r = \tilde{r}+$ in the far past and/or future. Their impact parameter $\lambda = \tilde{\lambda}_+$ defines the critical point on the observed line at infinity. Since the variables (T, Θ, H, L) are canonical, it is possible to define an action of the conformal group $\mathrm{SL}(2,\mathbb{R})_{\mathrm{PR}}$ on the phase space Γ with the generators:

$$H_+ = \hat{H}\,, \quad H_0 = -T\hat{H}\,, \quad H_- = T^2\hat{H}\ . \tag{5.194}$$

This algebra commutes with the $U(1)$ algebra generated by L and acts within superselection sectors Γ_L of fixed angular momentum. However, it modifies the energy (5.192), as well as the impact parameter and the radius of closest approach for each geodesic.

The photon ring is an attractive fixed point for the flow generated by the dilation $e^{-\alpha H_0}$, under which [83, 98]

$$\hat{H}(0) \to \hat{H}(\alpha) = e^{-\alpha}\hat{H}(0)\ , \tag{5.195}$$

because $\hat{H}$ satisfies

$$\partial_\alpha \hat{H} = \{H_0, \hat{H}\} = -\hat{H}\,. \tag{5.196}$$

For large α, $\hat{H}$ becomes small and $T \to \infty$. Defining $\delta r = \frac{r-\tilde{r}_+}{\tilde{r}_+}$ such that it has no dimension, the point of closest approach (5.193) becomes to leading order as $\hat{H} \to 0$

$$\delta r^2_{\mathrm{min}} = \frac{4(\nu^2+3)}{9(\nu^2-1)^2}\frac{\tilde{\lambda}_+^2}{\tilde{r}_+^2}(\tilde{\lambda}_+ - \tilde{\lambda}_-)\frac{\hat{H}}{L}\,, \tag{5.197}$$

while

$$\mathrm{d}T \approx \frac{1}{\gamma_{L+}}\mathrm{d}\log\delta r \implies \delta r \approx \delta r_0\, e^{\gamma_{L+}T}\,, \tag{5.198}$$

as expected. It follows that under $\mathrm{SL}(2,\mathbb{R})_{\mathrm{PR}}$ dilation (5.195),

$$\partial_\alpha \log\delta r_{\mathrm{min}} = -\frac{1}{2}\ . \tag{5.199}$$

For a geodesic beginning and ending at $r = \infty$,

$$\Delta\theta = -4\int_{r_{\mathrm{min}}}^{\infty}\frac{D(r)}{\sqrt{V(r)}}\left(m(r)H + L\right)\mathrm{d}r \approx \frac{2\pi}{\gamma}\log\delta r_{\mathrm{min}}\,, \tag{5.200}$$

to leading order as $\delta r_{\mathrm{min}} \to 0$. Then, the winding number of the geodesics around the black hole, $w = |\Delta\theta|/2\pi$, diverges like

$$\partial_\alpha w = \frac{1}{2\gamma_{L+}}\,, \tag{5.201}$$

under dilations. If we consider a source at (r_s, θ_s) and an "observer" at (r_o, θ_o), there is an infinite number of null geodesics labeled by their winding number w between the two endpoints with the same angular shift $\Delta\theta$ modulo 2π. We conclude that for large w, or equivalently small $\hat{H}$ or small $\delta r_{\min}$, if $\alpha = 2\gamma_{L+}$, the dilation

$$D_0 = e^{-2\gamma_{L+}H_0}\,, \tag{5.202}$$

maps null geodesics with successive winding number $w \to w+1$. The products of D_0 form a discrete subgroup of SL(2,$\mathbb{R}$)$_{\mathrm{PR}}$ mapping the phase space of observed null geodesics to itself.

5.8 QNMs for Warped BTZ in the quadratic ensemble

First we will investigate the modes for Warped BTZ in the quadratic ensemble. Afterwards, we will take an appropriate limit to get the BTZ metric in the usual system of coordinates and compare them to the modes of BTZ [99]:

$$\begin{aligned}\hat{\omega}_L &= \hat{k} - 2i\,(\hat{r}_+ - \hat{r}_-)(n+1)\,,\\ \hat{\omega}_R &= -\hat{k} - 2i\,(\hat{r}_+ + \hat{r}_-)(n+1)\,.\end{aligned} \tag{5.203}$$

The starting point will be the metric (5.2) for which the change of coordinates to the quadratic ensemble is well known. The warped BTZ black hole in the quadratic ensemble has the metric in the ADM form (1.26)

$$\mathrm{d}s^2 = -N_{\mathrm{QE}}(\hat{r})^2\mathrm{d}\hat{t}^2 + \frac{(1-2H^2)}{R_{\mathrm{QE}}(\hat{r})^2 N_{\mathrm{QE}}(\hat{r})^2}\hat{r}^2\mathrm{d}\hat{r}^2 + R_{\mathrm{QE}}(\hat{r})^2(\mathrm{d}\hat{\theta} - N^{\hat{\theta}}(\hat{r})\mathrm{d}\hat{t})^2\,, \tag{5.204}$$

where

$$\begin{aligned}R_{\mathrm{QE}}(\hat{r})^2 &= \hat{r}^2 - 2H^2\left(\frac{\hat{r}^2 + \hat{r}_+\hat{r}_-}{\hat{r}_+ + \hat{r}_-}\right)^2,\\ N_{\mathrm{QE}}(\hat{r})^2 &= \frac{(1-2H^2)}{R_{\mathrm{QE}}(\hat{r})^2 L^2}(\hat{r}^2 - \hat{r}_+^2)(\hat{r}^2 - \hat{r}_-^2)\,,\\ N^{\hat{\theta}}(\hat{r}) &= \frac{1}{R_{\mathrm{QE}}(\hat{r})^2 L}\left((1-2H^2)\hat{r} + 2H^2\frac{(\hat{r}^2 - \hat{r}_+^2)(\hat{r}^2 - \hat{r}_-^2)}{(\hat{r}_+ + \hat{r}_-)}\right).\end{aligned} \tag{5.205}$$

This metric has the nice feature of leading to BTZ when $H^2 \to 0$. The change of coordinate from (5.204) to (5.2) is (1.41)

$$\begin{aligned}\hat{t} &= L\frac{t}{\mathbf{A}}\,,\\ \hat{\theta} &= -\theta - \frac{t}{\mathbf{A}}\,,\\ \hat{r}^2 &= \frac{\nu^2+3}{4\nu^2}\left(\mathbf{A}\nu r - \frac{3}{4}r_+ r_-(\nu^2 - 1)\right),\end{aligned} \tag{5.206}$$

with

$$\mathbf{A} = (r_+ + r_-)\nu - \sqrt{r_+ r_-(\nu^2+3)}\,. \tag{5.207}$$

The constants H^2 and L in the quadratic ensemble correspond to ν and l for the metric (5.2). They are related through

$$H^2 = -\frac{3(\nu^2-1)}{2(\nu^2+3)}\,, \qquad L = \frac{2l}{\sqrt{\nu^2+3}}\,. \tag{5.208}$$

We can now compute the parameters of the photon rings, their location $\tilde{\hat{r}}_\pm$, their angular velocity $\hat{\Omega}_\pm$ and their Lyapunov exponent $\hat{\gamma}_\pm$, and express them in terms of the parameters of the photon rings of the metric (5.2):

$$\begin{aligned}
\tilde{\hat{r}}^2_\pm &= \hat{r}(\tilde{r}_\pm) = \frac{\hat{r}_+^2+\hat{r}_-^2}{2} \pm \sqrt{1-\frac{1}{2H^2}}\frac{\hat{r}_+^2-\hat{r}_-^2}{2}\,,\\
\hat{\Omega}_\pm &= -\frac{\mathbf{A}\tilde{\Omega}_\pm + 1}{L} = \frac{\sqrt{1-\frac{1}{2H^2}}(\hat{r}_+ + \hat{r}_-) \mp (\hat{r}_+ - \hat{r}_-)}{\sqrt{1-\frac{1}{2H^2}}(\hat{r}_+ + \hat{r}_-) \pm (\hat{r}_+ - \hat{r}_-)}\,,\\
\hat{\gamma}_{L\pm} &= \frac{\mathbf{A}}{L}\gamma_{L\pm} = \frac{2\sqrt{1-\frac{1}{2H^2}}(\hat{r}_+^2 - \hat{r}_-^2)}{L^2\left(\sqrt{1-\frac{1}{2H^2}}(\hat{r}_+ + \hat{r}_-) \pm (\hat{r}_+ - \hat{r}_-)\right)}\,.
\end{aligned} \tag{5.209}$$

We can already look at the BTZ limit of those parameters

$$\begin{aligned}
\tilde{\hat{r}}_\pm &\to \pm\infty\,,\\
\hat{\Omega}_\pm &\to 1\,,\\
\hat{\gamma}_{L\pm} &\to 2(\hat{r}_+ - \hat{r}_-)\,.
\end{aligned} \tag{5.210}$$

The location of the inner photon ring at $-\infty$ must be taken carefully, because the domain of variation of r is $\mathbb{R}^+$. In this limit, one photon ring disappears while the other one is pushed at spacial infinity.

Starting with the modes in the eikonal limit

$$\begin{aligned}
\omega_1 &= \tilde{\Omega}_+ k - i(n+\frac{1}{2})\gamma_{L+}\,,\\
\omega_2 &= -i\nu(2n+1) + i\sqrt{3(\nu^2-1)n(n+1)+\nu^2}\,,
\end{aligned} \tag{5.211}$$

by defining the modes as

$$e^{-i\hat{\omega}\hat{t}+i\hat{k}\hat{\theta}} R\Big[\frac{\hat{r}^2-\hat{r}_+^2}{\hat{r}^2-\hat{r}_-^2}\Big] = e^{-i\omega t + ik\theta}R(z)\,, \tag{5.212}$$

we end up with

$$\begin{aligned}
&\hat{\omega}_1 = \hat{\Omega}_+\hat{k} - i(n+\frac{1}{2})\hat{\gamma}_{L+}\,,\\
&\hat{\omega}_2 = -\hat{k} - i\frac{(1-2H^2)(\hat{r}_2+\hat{r}_-)}{\sqrt{1+\frac{2}{3}H^2}\,L}\left(\sqrt{1-2H^2}(2n+1) - \sqrt{1-2H^2(1+4n(n+1))}\right)\,.
\end{aligned} \tag{5.213}$$

We can perform another limit, when $H^2 \to 0$, to see whether we recover the BTZ modes. For this, we need to start with the full version of the modes, not the eikonal limit. Taking this limit, we get

$$\hat{\omega}_1 = \hat{k} - i(\hat{r}_+ - \hat{r}_-)(2n + 1 - \text{sign}(\hat{k}))\,, \tag{5.214}$$

$$\hat{\omega}_2 = -\hat{k} - 2i(\hat{r}_+ + \hat{r}_-)n\,. \tag{5.215}$$

We can rewrite the modes $\hat{\omega}_1$ in terms of the parameters of the outer photon ring,

$$\hat{\omega}_1 = \hat{\Omega}_+\hat{k} - i(n + \frac{1 - \text{sign}(\hat{k})}{2})\hat{\gamma}_{L+}\,. \tag{5.216}$$

We observe that we do not exactly recover (5.203). However, the purely imaginary modes are necessary to obtain a form similar to $\hat{\omega}_R$. This difference arises because the boundary condition at infinity used in [99, 100], a vanishing Dirichlet condition, differs from the one we use. As previously mentioned, in non-asymptotically flat spacetimes, different boundary conditions are not equivalent. Since there is no photon ring in AdS_3, it was impossible to relate an eikonal limit of (5.203) to something like (5.1), as we did for the WAdS black hole.

5.9 Extremal limits

5.9.1 QNMs of extremal warped black holes

In this section, we want to adapt the computation to the case where the two horizons coincide, $r_+ = r_- = r_0$. In this scenario, the metric takes the form

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{d}t^2 &+ \frac{\ell^2\,\mathrm{d}r^2}{(\nu^2+3)^2(r-r_0)^2} + (2\nu r - r_0\sqrt{\nu^2+3})\,\mathrm{d}t\,\mathrm{d}\theta \\ &+ \frac{r}{4}\left(3(\nu^2-1)\,r - 2\,r_0\sqrt{\nu^2+3}(2\nu - \sqrt{\nu^2+3})\right)\mathrm{d}\theta^2\,. \end{aligned} \tag{5.217}$$

In this background, null geodesics follow

$$\dot{r}^2 + V(r) = 0\,, \tag{5.218}$$

where

$$V(r) = -4\left(L^2 + m(r)\,E\,L + n(r)\,E^2\right)\,. \tag{5.219}$$

In order to simplify the equations, we defined the functions $m(r)$ and $n(r)$ as

$$\begin{aligned} m(r) &= 2\nu r - r_0\sqrt{\nu^2+3}\,, \\ n(r) &= \frac{r}{4}\left(3(\nu^2-1)\,r - 2\,r_0\sqrt{\nu^2+3}(2\nu - \sqrt{\nu^2+3})\right)\,. \end{aligned} \tag{5.220}$$

As before, λ is the impact parameter and is defined as the ratio of the angular momentum L by the energy E. The photon ring satisfies

$$V(r) = 0 = V'(r)\,, \tag{5.221}$$

giving the unique solution

$$\tilde{r} = r_0\,, \qquad \tilde{\lambda} = -\frac{r_0}{2}(2\nu - \sqrt{\nu^2+3}) = \frac{1}{\tilde{\Omega}}\,. \tag{5.222}$$

As a result, the photon ring is located at the horizon, making it impossible to define a Lyapunov exponent in this context. In fact, for a slight perturbation $\delta r = r - r_0$ around the photon ring,

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta r = -\frac{(\nu^2+3)\sqrt{3(\nu^2-1)}}{4\nu\tilde{\lambda}}\,\delta r^2\,. \tag{5.223}$$

So the perturbation δr does not behave exponentially but as

$$\delta r(t) = \frac{4\nu\tilde{\lambda}}{(\nu^2+3)\sqrt{3(\nu^2-1)}}\frac{1}{t-t_0}\,, \tag{5.224}$$

where t_0 is an integration constant.

Quasi-normal modes in this background have the form

$$\Phi(t,\theta,r) = e^{-i\omega t + ik\theta}\phi(r)\,, \tag{5.225}$$

and satisfy the wave equation

$$\nabla^2\Phi(r,\theta,r) = 0\,. \tag{5.226}$$

The radial part of the wave equation is

$$(r-r_0)^2\partial_r\left((r-r_0)^2\partial_r\phi(r)\right) + V_{\mathrm{QNM}}(r)\;\phi(r) = 0\,, \tag{5.227}$$

where

$$V_{\mathrm{QNM}}(r) = \frac{4}{(\nu^2+3)^2}\left(k^2 + m(r)\,k\,\omega + n(r)\,\omega^2\right)\,. \tag{5.228}$$

We now perform the change of coordinate

$$z = \frac{1}{r-r_0}\,. \tag{5.229}$$

The horizon is located at $z \to \infty$, and the radial infinity is at $z \to 0$. Thus, the coordinate z is always positive in this range. We rewrite the equation (5.227) as

$$\partial_z^2\phi(z) = \left(A^2 + \frac{2AB}{z} + \frac{C(C-1)}{z^2}\right)\phi(z)\,, \tag{5.230}$$

where

$$A = \frac{i}{\nu^2+3}\left(2k + r_0(2\nu - \sqrt{\nu^2+3})\omega\right) ,$$
$$B = \frac{2i\nu\omega}{\nu^2+3} ,$$
$$C = \frac{1}{2}\left(1 - \sqrt{1 - \frac{12(\nu^2-1)}{(\nu^2+3)^2}\omega^2}\right) . \tag{5.231}$$

The definition of A, B and C allows for a choice of sign, but this choice is inconsequential because it merely transforms one of the two independent solutions into the other. The general solutions to (5.227) are

$$\begin{aligned}\phi(z) &= Q_-\, e^{-Az}\, z^C\, {}_1F_1(B+C; 2C; 2Az) \\ &\quad + Q_+\, e^{-Az}\, z^{1-C}\, {}_1F_1(1+B-C; 2-2C; 2Az) ,\end{aligned} \tag{5.232}$$

where ${}_1F_1(a,b,x)$ is the confluent hypergeometric function. We now want to impose the same boundary conditions as for the non-extremal case: ingoing at the horizon and outgoing at radial infinity ($z \to 0$). Let us first examine the second requirement. Near $z = 0$ the radial part of the modes behaves like

$$\phi(z) \overset{z\to 0}{\sim} Q_-\, z^{\frac{1}{2}-\varpi} + Q_+\, z^{\frac{1}{2}+\varpi} . \tag{5.233}$$

where we have defined $C = \frac{1}{2} - \varpi$, as before. We recall that we defined the square root of a complex number such that the real part is positive. Decomposing ϖ into its real and imaginary parts, the modes behave at infinity as

$$\Phi(x) \sim \exp\left(-i\omega_R(t \mp \sqrt{b}\log z)\right) , \tag{5.234}$$

where we focus only on the oscillating part in the last equation. In this geometry, an ingoing wave corresponds to a wave moving along decreasing z, so we need to impose $Q_+ = 0$.

Around the horizon, at $z \to \infty$, the modes become

$$\phi(z) \overset{z\to\infty}{\sim} C_+ e^{Az} z^B + C_- e^{-Az} z^{-B} , \tag{5.235}$$

with

$$C_+ = \frac{Q_+}{\Gamma(1+B-C)} + \frac{Q_-}{\Gamma(B+C)} , \qquad C_- = \frac{Q_+}{\Gamma(1-B-C)} + \frac{Q_-}{\Gamma(-B+C)} . \tag{5.236}$$

In the case of a QNM, where the real part of the energy in the eikonal limit is $k/\omega_R = \tilde{\lambda}$, $A = 0$ and one needs to look at the next order in z. Once again, looking at the propagation of a wave packet,

$$\Phi(x) \sim \exp\left(-i\omega_R\left(t \mp \frac{2\nu}{\nu^2+3}\log z\right)\right) . \tag{5.237}$$

The wave is ingoing at the horizon if it moves towards the positive values of z. It selects $C_- = 0$. Combined with the outgoing condition at infinity, one needs to impose that

$$\begin{aligned} & -B + C = -n \\ \iff & -\frac{2i\nu}{\nu^2+3}\omega + \frac{1}{2}\left(1 - \sqrt{1 - \frac{12(\nu^2-1)}{(\nu^2+3)^2}\omega^2}\right) = -n\,. \end{aligned} \tag{5.238}$$

It is exaclty the equation (5.72) where the solution was (5.74):

$$\omega_{2+} = -i\nu\Big(2n+1\Big) + i\sqrt{3(\nu^2-1)n(n+1)+\nu^2}\,, \tag{5.239}$$

a solution with no real part in the dispersion relation. It means that in order to compute QNMs with a non-zero real value for the energy, one needs to impose that A vanishes for any value of k, not only in the eikonal limit. In that case, the wave equation that we need to solve simplifies as

$$\partial_z^2\phi(z) = \frac{4\varpi^2-1}{4z^2}\phi(z)\,, \tag{5.240}$$

and the general solutions are

$$\phi(z) = Q_+\, z^{\frac{1}{2}+\varpi} + Q_-\, z^{\frac{1}{2}-\varpi}\,. \tag{5.241}$$

It exhibits the same behavior as (5.233), but in this case it is valid for any z. Since we are interested in QNMs leaking out of the photon ring, and since the photon ring is precisely located at the horizon, it suffices to require that the modes propagate towards spatial infinity, from $z = \infty$ to $z = 0$, i.e. $Q_+ = 0$. The dispersion relation for these solutions then becomes

$$\omega = \tilde{\Omega}k\,. \tag{5.242}$$

One can obtain this solution from sending $r_2 \to r_1 = r_0$ in (5.68).

5.9.2 QNMs in the near-horizon region of near-extremal warped black holes

Now, we want to take the near-horizon limit of the extremal warped black hole. The resulting metric is the self-dual Warped AdS_3 spacetime studied in [98]. To achieve this limit, we first need the Hawking temperature at the outer horizon (1.25)

$$T_H = \frac{\nu^2+3}{4\pi\ell}\frac{r_+ - r_-}{2\nu r_+ - \sqrt{r_+ r_-(\nu^2+3)}}\,, \tag{5.243}$$

and we define a rescaled dimensionless temperature

$$T_R = \frac{(2\nu - \sqrt{\nu^2+3})\ell}{\nu^2+3}\frac{T_H}{\varepsilon}\,. \tag{5.244}$$

We want to zoom close to the horizon in the extremal case while keeping the rescaled temperature T_R fixed. In other words, we rewrite the locations of the horizons

$$\begin{aligned} r_+ &= r_0(1+\varepsilon 2\pi T_R)\,, \\ r_- &= r_0(1-\varepsilon 2\pi T_R)\,, \end{aligned} \tag{5.245}$$

where r_0 is the location of the extremal horizon, and make the following change of coordinates:

$$\begin{aligned} \hat{t} &= -\frac{\nu^2+3}{2\nu-\sqrt{\nu^2+3}}\,\varepsilon\, t\,, \\ \hat{\theta} &= \frac{(2\nu-\sqrt{\nu^2+3})(\nu^2+3)}{4\nu}\, r_0\,\theta + \frac{\nu^2+3}{2\nu}\, t\,, \\ \hat{r} &= \frac{r-r_0}{r_0\,\varepsilon} - 2\pi T_R\,. \end{aligned} \tag{5.246}$$

We then perform the limit $\varepsilon \to 0$ and $T_H \to 0$ while keeping T_R fixed. The end result is the metric

$$\mathrm{d}s^2 = \frac{1}{\nu^2+3}\left(-\hat{r}(\hat{r}+4\pi T_R)\,\mathrm{d}\hat{t}^2 + \frac{\mathrm{d}\hat{r}^2}{\hat{r}(\hat{r}+4\pi T_R)} + \Lambda^2\left(\mathrm{d}\hat{\theta} + (\hat{r}+2\pi T_R)\,\mathrm{d}\hat{t}\right)^2\right)\,, \tag{5.247}$$

where

$$\Lambda = \frac{2\nu}{\sqrt{\nu^2+3}} \tag{5.248}$$

goes to 1 when $\nu \to 1$, as required in [98]. Using the change of coordinates (5.246), we can compute the locations of the new photon rings:

$$\hat{r}_\pm = \frac{\tilde{r}_\pm - r_0}{r_0\,\varepsilon} - 2\pi T_R = 2\pi T_R\left(\pm\frac{\Lambda}{\sqrt{\Lambda^2-1}} - 1\right)\,. \tag{5.249}$$

Also, as the change of coordinates is linear in (t,θ), we can compute the angular velocities and the Lyapunov exponents of the new photon rings using (5.86,5.88):

$$\hat{\Omega}_\pm = \mp 2\pi T_R\sqrt{1-\frac{1}{\Lambda^2}}\,, \qquad \hat{\gamma}_{L\pm} = 2\pi T_R\,. \tag{5.250}$$

Before applying the change of coordinates (5.246) and taking the near-extremal near-horizon limit of the modes (5.80), we will briefly review the self-dual results [98].

The ingoing modes solution of the scalar wave equation $\nabla^2\Phi = 0$ in the background (5.247) is

$$\Phi(\hat{t},\hat{r},\hat{\theta}) = e^{-i\hat{\omega}\hat{t}+i\hat{k}\hat{\theta}}\psi(\hat{r}) \tag{5.251}$$

with

$$\begin{aligned}\psi(r) = &\hat{r}^{-\frac{i}{2}\left(\frac{\hat{\omega}}{2\pi T_R}+\hat{k}\right)}\left(\frac{\hat{r}}{4\pi T_R}+1\right)^{\frac{i}{2}\left(\frac{\hat{\omega}}{2\pi T_R}-\hat{k}\right)} \\ &{}_2F_1\left[\frac{1}{2}+\beta-i\hat{k},\frac{1}{2}-\beta-i\hat{k};1-i\left(\frac{\hat{\omega}}{2\pi T_R}+\hat{k}\right);-\frac{\hat{r}}{2\pi T_R}\right],\end{aligned} \tag{5.252}$$

where

$$\beta = i\hat{k}\sqrt{\left(1-\frac{1}{\Lambda^2}\right)-\frac{1}{4\hat{k}^2}}\,. \tag{5.253}$$

This solution is symmetric under $\beta \leftrightarrow -\beta$. Close to infinity, the modes read

$$\psi(\hat{r}) \overset{\hat{r}\to\infty}{\approx} \hat{Q}_+\,\hat{r}^{-\frac{1}{2}+\beta} + \hat{Q}_-\,\hat{r}^{-\frac{1}{2}-\beta}\,. \tag{5.254}$$

where

$$\hat{Q}_\pm = \frac{\Gamma(\pm 2\beta)\,\Gamma\left(1-i\frac{\hat{\omega}}{2\pi T_R}-i\hat{k}\right)}{\Gamma(\frac{1}{2}\pm\beta-i\hat{k})\,\Gamma\left(\frac{1}{2}\pm\beta-i\frac{\hat{\omega}}{2\pi T_R}\right)}\,. \tag{5.255}$$

The oscillating part of the scalar wave is then

$$\Phi(\hat{t},\hat{r},\hat{\theta}) \sim \exp\left\{-i\hat{\omega}_R\left(t\mp\frac{\hat{k}}{\hat{\omega}_R}\sqrt{\left(1-\frac{1}{\Lambda^2}\right)-\frac{1}{4\hat{k}^2}}\log\hat{r}\right)\right\}\,. \tag{5.256}$$

When $\hat{k}$ becomes large, the real part of the frequencies is proportional to the angular velocity of one of the photon rings. We will look at each situation separately. First, when $\hat{\omega}_R \approx \hat{\Omega}_+\hat{k}$, it implies that

$$\Phi(\hat{t},\hat{r},\hat{\theta}) \sim \exp -i\hat{\omega}_R\left(t\pm\frac{\log\hat{r}}{2\pi T_R}\right)\,. \tag{5.257}$$

Thus, an outgoing mode needs $\hat{Q}_+ = 0$.

Secondly, when $\hat{\omega}_R \approx \hat{\Omega}_-\hat{k}$, one has

$$\Phi(x) \sim \exp -i\hat{\omega}_R\left(t\mp\frac{\log\hat{r}}{2\pi T_R}\right)\,, \tag{5.258}$$

and an outgoing mode satisfies $Q_- = 0$.

Solving both conditions $\hat{Q}_\pm = 0$ leads to

$$\frac{1}{2}\mp\beta-i\frac{\hat{\omega}}{2\pi T_R} = -n\,, \tag{5.259}$$

and thus

$$\hat{\omega}_{\pm} = \mp 2\pi T_R \hat{k}\sqrt{\left(1-\frac{1}{\Lambda^2}\right)-\frac{1}{4\hat{k}^2}} - i\left(n+\frac{1}{2}\right)2\pi T_R\,. \tag{5.260}$$

As for WAdS$_3$ black holes, the exact QNM spectrum can be computed in the self-dual WAdS$_3$ case. In the eikonal limit, the QNM modes are

$$\hat{\omega}_{\pm} \approx \hat{\Omega}_{\pm}\hat{k} - i\left(n+\frac{1}{2}\right)\hat{\gamma}_{L\pm}\,. \tag{5.261}$$

Contrary to the WAdS$_3$ case, both photon rings are necessary to recover the QNM spectrum.

Similar to our warped black holes, one can show that the various boundary conditions at infinity are not equivalent for self-dual WAdS$_3$. For instance, the flux at infinity is

$$\begin{aligned} F &= \frac{\sqrt{-g}\, g^{\hat{r}\hat{r}}}{2i}\left(\Phi^*\partial_{\hat{r}}\Phi - \Phi\partial_{\hat{r}}\Phi^*\right) \qquad (5.262)\\ &\overset{\hat{r}\to\infty}{\sim} e^{-2\hat{\omega}_I t}\left(\beta_I\left(|\hat{Q}_+|^2\,\hat{r}^{2\beta_R} - |\hat{Q}_-|^2\,\hat{r}^{-2\beta_R}\right) - i\beta_R\left(\hat{Q}_-^*\hat{Q}_+\,\hat{r}^{2i\beta_I} - \hat{Q}_+^*\hat{Q}_-\,\hat{r}^{-2i\beta_I}\right)\right)\,, \end{aligned}$$

where we have decomposed β into a real and an imaginary part $\beta = \beta_R + i\beta_I$. For the flux to remain finite at infinity, as $\beta_R > 0$ from our definition of the square root, we need to impose

$$\hat{Q}_+ = 0\,. \tag{5.263}$$

This condition is equivalent to

$$\frac{1}{2} + \beta - i\frac{\hat{\omega}}{2\pi T_R} = -n\,, \tag{5.264}$$

leading to the solution

$$\hat{\omega} = 2\pi T_R \hat{k}\sqrt{\left(1-\frac{1}{\Lambda^2}\right)-\frac{1}{4\hat{k}^2}} - i\left(n+\frac{1}{2}\right)2\pi T_R\,. \tag{5.265}$$

In the eikonal limit $|\hat{k}| \gg 1$, this solution becomes

$$\hat{\omega} = \hat{\Omega}_-\hat{k} - i\left(n+\frac{1}{2}\right)\hat{\gamma}_{L-}\,. \tag{5.266}$$

This mode solution, which is the only one obtained under the finite flux condition at infinity, differs from (5.267) and recovers partially (5.261).

We can now evaluate the near-extremal near-horizon limit of the modes (5.80). The first modes persist in the expected form in the eikonal limit

$$\hat{\omega}_1 = \hat{\Omega}_+\hat{k} - i\left(n+\frac{1}{2}\right)\hat{\gamma}_{L+}\,. \tag{5.267}$$

The purely imaginary mode diverges in this limit

$$\hat{\omega}_2 \sim O\left(\frac{1}{\varepsilon}\right) , \tag{5.268}$$

and therefore cannot be considered as physical. Only one of the modes (5.261) is obtained. Indeed, solutions that were previously excluded in the non-extremal scenario because of not satisfying the outgoing boundary condition, such as (5.81), now do in the near-extremal near-horizon limit (5.261). When we observed which modes of the WAdS$_3$ black hole were outgoing at infinity in (5.58), we studied the high-frequency regime of the function ϖ (5.56), which depends on ω. This allowed us to express the outgoing condition without specifying the real part of ω; in particular, it did not matter whether $\omega_R = \tilde{\Omega}_+ k$ or $\omega_R = \tilde{\Omega}_- k$. Subsequently, the outgoing condition implied $Q_- = 0$, which selected the modes $\omega_R = \tilde{\Omega}_+ k$. For the self-dual case, we needeed to examine the high-frequency regime of β, which only depends on $\hat{k}$. Thus, for the outgoing condition, we considered $\hat{\omega}_R = \hat{\Omega}_+ \hat{k}$ and $\hat{\omega}_R = \hat{\Omega}_- \hat{k}$ separately, and we observed that both modes could be outgoing: one with $\hat{Q}_+ = 0$ and the other one with $\hat{Q}_- = 0$. Therefore, it seems that there is a transition from a function depending on ω to one depending on $\hat{k}$, giving additional outgoing mode solutions. We also note that, after transforming ϖ under the change of coordinates (5.246) and taking the near-horizon limit, we obtain the function β:

$$\begin{aligned}
\varpi &= \sqrt{\frac{1}{4} - \frac{3(\nu^2-1)}{(\nu^2+3)^2}\omega^2} \\
&= \sqrt{\frac{1}{4} - 3(\nu^2-1)\left(\frac{\varepsilon\,\hat{\omega}}{2\nu - \sqrt{\nu^2+3}} + \frac{\hat{k}}{2\nu}\right)^2} \\
&\overset{\varepsilon\to 0}{=} \sqrt{\frac{1}{4} - \frac{3(\nu^2-1)}{4\nu^2}\hat{k}^2} \\
&= \sqrt{\frac{1}{4} - \frac{\Lambda^2-1}{\Lambda^2}\hat{k}^2} \\
&= \beta\,.
\end{aligned}$$

It would then be interesting to investigate more precisely how this transition affects the direction of propagation of the solutions.

Conclusions and Outlooks

Through this thesis, we worked with different holographic models for WCFTs: CSS boundary conditions for AdS_3, the $SL(2,\mathbb{R}) \times U(1)$ Chern-Simons theory and WAdS solutions in TMG, giving us an overview of various facets of the (W)AdS/WCFT correspondence.

In Chapter 3, we introduced a set of Quantum Energy Conditions (QECs) tailored for WCFTs. Using holographic computations of the entanglement entropy (EE) for vacuum solutions satisfying CSS boundary conditions, we demonstrated the saturation of these QECs. To establish the unsaturation of QECs for non-vacuum solutions, we sought solutions with non-trivial stress tensors on the boundary. For CFTs, a common method involves considering shockwave solutions sourced by bulk matter, where the minimal surface in the Ryu-Takayanagi (RT) prescription intersects it. Adapting this approach to WCFTs necessitates an improved RT prescription, even if we were still working in locally AdS spacetimes. Various proposals exist in the literature with distinct advantages and disadvantages, yet we have not determined the most suitable one for describing shockwave geometries dual to WCFTs. Nonetheless, we do not assert its impossibility and encourage further exploration. Another avenue could involve exploring alternative methods apart from shockwave solutions or use a Chern-Simons formulation to compute EE using Wilson lines. Furthermore, since we have not proven the non-saturation of QECs, it remains possible that the correct set of QECs is a linear combination of those we proposed, derived from currents associated with Virasoro-Kač-Moody charges

In Chapter 4, we constructed the geometric action on coadjoint orbits of the warped Virasoro group and successfully matched it with the action of Lower Spin Gravity on the boundary, employing Hamiltonian reduction under appropriate boundary conditions. This correspondence enabled us to relate the holonomies of our manifold to orbit representatives. Moreover, the residual symmetries arising from the reduced action are connected to the little group associated with each orbit. Such a correspondence was anticipated and has been previously established in AdS_3/CFT_2 [72] or flat space holography contexts [73].

Through dimensional reduction, we recovered the Warped Schwarzian action [169]. Subsequently, we computed the one-loop partition function to compare the result with the known warped Virasoro character. However, we only recovered one of the two characters. When attempting to compute the one-loop partition function for a warped black hole solution with a real orbit representative $\mathcal{P}_0$, we obtained the character of a non-unitary representation, akin to the vacuum state in the context of Lower Spin Gravity, and vice-versa. This mirrors a similar finding in [47], where the vacuum character was obtained while computing the BTZ character. However, their framework involved AdS_3 with CSS boundary conditions, where every representation, including the vacuum, is unitary. We additionally suggested imposing an antiperiodic condition around the thermal circle of the $U(1)$ field to derive the second warped character. While this ad-hoc proposition lacks strong physical motivation, it could be worthwhile to pursue further investigation in this direction.

To validate our results, one could compute the partition function using methods such as the quasinormal modes approach as in [47], or employ the Heat Kernel method [171].

For the last project, in Chapter 5, we aimed at revisiting the computation of the quasinormal modes in a warped black hole background, as previously done in [82], and relate their eikonal limit to the photon rings that the spacetime possesses, as has been done for Schwarzschild and Kerr black holes or self-dual WAdS. The advantage of warped black holes is that the QNM spectrum can be computed exactly, and they possess photon rings, unlike BTZ. However, contrary to asymptotically flat spacetimes where different boundary conditions at infinity are equivalent, things are different for asymptotically WAdS spacetimes. The finite flux condition, as originally studied in [82], leads to different solutions than the outgoing solution at infinity. Both solutions exhibit modes that are purely imaginary and others that are related to different photon rings in the high-frequency regime. To select the appropriate boundary condition, we performed several limits around the photon ring outside the event horizon, as well as inside, to capture the modes that physically exist outside the black hole. These limits include the Penrose limit, the near-ring region approximation, and the geometric optics approximation. Each limit indicates that the correct boundary condition at infinity is the outgoing condition, for which the modes depend on the outer photon ring in the eikonal limit. We also demonstrated that the purely imaginary modes cannot be brought to the form

$$\omega = \Omega\, k - i\left(n+\frac{1}{2}\right)\gamma_L\,,$$

in any coordinate system, indicating that they do not constitute proper quasinormal mode solutions. Additionally, we showed that this specific form of the modes is preserved under linear redefinitions of time t and angle θ starting from the warped black hole metric in the canonical ensemble. We expect that this preservation holds for any metric, and it would be interesting to provide a general proof of this statement in future work. Furthermore, we also demonstrated that, similar to Schwarzschild, Kerr, and self-dual warped black holes, a conformal symmetry emerges when observing the photon ring from a distant observer's perspective.

Finally, a natural next step would be to interpret these quasinormal modes in the context of the dual field theory. In AdS/CFT, quasinormal modes correspond to poles of thermal Green's functions, known as quantum Ruelle resonances [238]. In [239], a matching of the modes of a warped black hole obtained in [82] with CFT retarted Green's function was achieved. However, as we have discussed, we believe that these were not the correct modes to consider. Furthermore, this correspondence was established in the context of WAdS/CFT, predating the exploration of WCFTs. It would be interesting to adapt such analyses for WAdS/WCFT. In [46], the retarded Green's functions in WAdS/WCFT were computed. A deeper understanding of the computations in [238] and [46] could aid in constructing the appropriate dual description for our quasinormal modes and determining whether $WAdS_3$ is dual to a CFT or a WCFT.

It is interesting to point out that none of these projects followed a straight line to the expected results. The non-Lorentzian nature of WCFTs or the deformation of the $WAdS_3$ spacetime led us to ask profound questions about the holographic dictionary of the (W)AdS/WCFT correspondence, such as those regarding EE and the RT prescription, warped characters, and quasinormal modes and quantum Ruelle resonances. This underscores the significance of studying WCFTs and their various holographic duals. The summary of these questions and the answers we have been able to provide are presented in this thesis, and we hope it contributed to shed light on some aspects of the (W)AdS/WCFT correspondence.

Appendix A

The group $SL(2, \mathbb{R})$

In this section, we will provide a brief overview of the group $SL(2, \mathbb{R})$. This group has been referenced multiple times throughout this thesis, and since some of its properties are used, we have deemed it beneficial to consolidate them in a dedicated section.

The $SL(2, \mathbb{R})$ group is the special linear group of 2×2 real matrices with determinant equal to one:

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{R} \quad \text{and} \quad ad - bc = 1 \right\} . \tag{A.1}$$

It is a non-compact simple Lie group of dimension 3. Its Lie algebra, $sl(2, \mathbb{R})$, is the set of 2×2 real traceless matrices with:

$$sl(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{R} \quad \text{and} \quad ad = 0 \right\} , \tag{A.2}$$

such that any element G of the group can be expressed in terms of elements g of the algebra using the exponential map.

A common basis for the algebra is

$$L_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad L_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} , \tag{A.3}$$

where the generators L_n satisfies the commutation relations

$$[L_n, L_m] = (n - m) L_{n+m} . \tag{A.4}$$

Sometimes, for more clarity, we used the index $+, -$ instead of $1, -1$.

We can define a invariant bilinear form, which is the natural metric on $sl(2, \mathbb{R})$, using the trace:

$$\eta_{nm} \equiv \langle L_n L_m \rangle = \mathrm{Tr}(L_n L_m) = \left(\begin{array}{c|ccc} & L_1 & L_0 & L_{-1} \\ \hline L_1 & 0 & 0 & -1 \\ L_0 & 0 & \frac{1}{2} & 0 \\ L_{-1} & -1 & 0 & 0 \end{array} \right) . \tag{A.5}$$

The elements of the $SL(2,\mathbb{R})$ group can be classified in three conjugacy classes, up to conjugacy in $GL(2,\mathbb{R})$, depending on the trace of the elements. The eigenvalues λ of an element $G \in SL(2,\mathbb{R})$ satisfy the characteristic polynomial

$$\lambda^2 - \mathrm{Tr}(G)\,\lambda + 1 = 0\,. \tag{A.6}$$

The solutions of this equation are

$$\lambda = \frac{1}{2}\left(\mathrm{Tr}(G) \pm \sqrt{\mathrm{Tr}(G)^2 - 4}\right). \tag{A.7}$$

It implies the following classification:

- Elliptic: $|\mathrm{Tr}(G)| < 2$. The eigenvalues of such element are complex conjugates and take value on the unit circle. The corresponding element is conjugate to rotations on the Euclidean plane. For example, picking the generator $L_1 + L_{-1}$,

$$e^{x(L_1+L_{-1})} = \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}. \tag{A.8}$$

  Indeed, for $x \neq 0$, the absolute value of the trace of this matrix is always lower than 2.

- Parabolic: $|\mathrm{Tr}(G)| = 2$. The eigenvalues are degenerate and are both 1 or -1. The corresponding element is conjugate to shear mappings on the Euclidean plane. For example, picking the generator L_1 or L_{-1},

$$e^{xL_1} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \qquad e^{yL_{-1}} = \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix}. \tag{A.9}$$

  The trace of such element is always equal to 2.

- Hyperbolic: $|\mathrm{Tr}(G)| > 2$. The eigenvalues of such element are real and opposite to each other. The corresponding element is conjugate to hyperbolic rotations on the Euclidean plane. For exemple, picking the generator $L_1 - L_{-1}$ or L_0,

$$e^{x(L_1-L_{-1})} = \begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix}, \qquad e^{yL_0} = \begin{pmatrix} e^{y/2} & 0 \\ 0 & e^{-y/2} \end{pmatrix}. \tag{A.10}$$

The names of those different conjugacy classes, elliptic, parabolic and hyperbolic, come from the classification of conic sections by their eccentricity.

If G is an hyperbolic element, it can be decomposed in terms of simpler matrices:

$$G = e^{\alpha L_{-1}} e^{\beta L_0} e^{\gamma L_1}\,. \tag{A.11}$$

This decomposition is unique and is called *Gauss decomposition.* One could also write the Gauss decomposition like

$$G = e^{\alpha' L_1} e^{\beta' L_0} e^{\gamma' L_{-1}}\,, \tag{A.12}$$

where $(\alpha', \beta', \gamma')$ are related to (α, β, γ) through a field redefinition [72]. However this decomposition is not valid for any element of the group. If we seek a more general one, we can use the so-called *Iwasawa decomposition*. A group element $G \in SL(2,\mathbb{R})$ can be written as the product of three matrices

$$G = K\,A\,N\,, \tag{A.13}$$

where each of the matrices K, A, N belongs to different conjugacy classes (elliptic, hyperbolic and parabolic respectively).

Appendix B

Hypergeometric functions

In this appendix, we will review some properties of the hypergeometric functions that are used in Chapter 5 [240].

Hypergeometric functions are solutions of the differential equation

$$z(1-z)\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} + (c-(a+b+1)z)\,\frac{\mathrm{d}f(z)}{\mathrm{d}z} - ab\, f(z) = 0\,, \tag{B.1}$$

where a, b and c are some constant parameters. This equation has three regular singular points, in 0, 1 and ∞. One can solve the differential equation around any of those singular points and have two linearly independent solutions. For the situation in Chapter 5, z is a radial coordinate with domain between 0 and 1 such that we choose to solve the equation (B.1) around the singular point $z = 0$. Under the condition that c is not an integer, the two independent solutions are

$$\begin{aligned} f_1(z) &= {}_2F_1(a,b;c;z)\,,\\ f_2(z) &= z^{1-c}\,{}_2F_1(1+a-c,1+b-c;2-c;z)\,, \end{aligned} \tag{B.2}$$

where ${}_2F_1(a,b;c;z)$ is the so-called *hypergeometric function*. This function has the property to be symmetric under the exchange $a \leftrightarrow b$, which can already be seen from the differential equation (B.1).

For general knowledge, most usual functions in mathematics can be expressed in terms of hypergeometric function. For examples,

$$\begin{aligned} {}_2F_1(1,1;2;-z) &= \frac{\ln(1+z)}{z}\,,\\ {}_2F_1(a,b;b;z) &= \frac{1}{(1-z)^a}\,,\\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\right) &= \frac{\arcsin z}{z}\,. \end{aligned}$$

Any second order linear ordinary differential equation with three regular singular points can be turn into the differential equation (B.1). This is the reason why for the equation (5.38), we perform the function redefinition (5.40).

We are interested in Chapter 5 in series expansion at the boundaries, corresponding to the singular points $z = 0$ and $z = 1$. Around the special point $z = 0$, the hypergeometric function is simply

$$ {}_2F_1(a, b; c; z) \sim \mathcal{O}(1) \,, \tag{B.3}$$

while around the the point $z = 1$

$$ {}_2F_1(a, b; c; z) \sim \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} \,. \tag{B.4}$$

B.1 Confluent hypergeometric functions and parabolic cylinder functions

The confluent hypergeometric functions are solutions of the differential equation

$$ z\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} + (b-z)\frac{\mathrm{d}f(z)}{\mathrm{d}z} - af(z) = 0 \,, \tag{B.5}$$

with a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$. The two linearly independent solutions are

$$ \begin{aligned} f_1(z) &= {}_1F_1(a; b; z) \,, \\ f_2(z) &= z^{1-b}\, {}_1F_1(a+1-b; 2-b; z) \,. \end{aligned} \tag{B.6}$$

The confluent hypergeometric function can be view as a limit of a hypergeometric function where the regular singular point at $z = 1$ is send to the other regular singular point at $z = \infty$, making the latter irregular

$$ {}_1F_1(a; c; z) = \lim_{b\to\infty} {}_2F_1(a, b; c; z/b) \,. \tag{B.7}$$

As for the hypergeometric functions, some common mathematical functions can be written in terms of confluent hypergeometric function as

$$ \begin{aligned} {}_1F_1(0; b; z) &= 1 \,, \\ {}_1F_1(b; b; z) &= e^z \,, \\ {}_1F_1(1; 2; z) &= \frac{e^z - 1}{z} \,, \end{aligned}$$

and the parabolic cylinder functions $D_a(z)$ that we were also used in Chapter 5

$$ \begin{aligned} D_a(z) = \frac{2^{a/2}}{\sqrt{\pi}} e^{-z^2/4} \Bigg[&\cos\left(\frac{\pi a}{2}\right) \Gamma\left(\frac{a+1}{2}\right) {}_1F_1\left(-\frac{a}{2}; \frac{1}{2}; \frac{z^2}{2}\right) \\ &+ \sqrt{2}\, z \sin\left(\frac{\pi a}{2}\right) \Gamma\left(\frac{a}{2}+1\right) {}_1F_1\left(\frac{1-a}{2}; \frac{3}{2}; \frac{z^2}{2}\right) \Bigg] \,. \end{aligned} \tag{B.8}$$

The connection between the hypergeometric function ${}_2F_1(a,b;c;z)$ and the parabolic cylinder function $D_a(z)$ trough this limit is linked to the Penrose limit procedure, the near-ring region and the geometric optics approximation where limits are taken from the original phase space.

The expansion of the confluent hypergeometric function and the parabolic cylinder function around $z=0$ is

$$\begin{aligned} {}_1F_1(a;b;z) &\sim \mathcal{O}(1)\,, \\ D_a(z) &\sim \mathcal{O}(1)\,, \end{aligned} \tag{B.9}$$

while around the irregular singular point at $z=\infty$,

$$\begin{aligned} {}_1F_1(a;b;z) &\sim \Gamma(b)\left(\frac{e^z z^{a-b}}{\Gamma(a)}+\frac{(-z)^{-a}}{\Gamma(b-a)}\right)\,, \\ D_a(z) &\sim e^{-z^2/4} z^a\,. \end{aligned} \tag{B.10}$$

Furthermore, when a is a natural number, $a=n$, the parabolic cylinder function is expressible in terms of the Hermite function $H_n(z)$

$$D_n(z) = 2^{-n/2}\, e^{-z^2/4}\, H_n(z/\sqrt{2})\,. \tag{B.11}$$

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