Perturbation solutions to the lnes equation for multiple lens planes

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Abstract

Continuing work initiated in an earlier publication (Asada, 2009, MNRAS, **394**, 818), we make a systematic attempt to determine, as a function of lens and source parameters, the positions of images by multi-plane gravitational lenses. By extending the previous single-plane work, we present a method of Taylor-series expansion to solve the multi-plane lens equation in terms of mass ratios. The advantage of this method is that it allows a systematic iterative analysis and clarifies the dependence on lens and source parameters. In concordance with the multi-plane lensed-image counting theorem that the lower bound on the image number is 2^N for N planes with a single point mass on each plane, our iterative results show directly that 2^N images are always realized as the minimum number of lensed images.

1 Introduction

Gravitational lensing has become one of important subjects in modern astronomy and cosmology (e.g., Schneider 2006, Weinberg 2008). It has many applications as gravitational telescopes in various fields ranging from extra-solar planets to dark matter and dark energy at cosmological scales (e.g., Refregier 2003 for a review). For instance, it is successful in detecting extra-solar planetary systems (Schneider and Weiss 1986, Mao and Paczynski 1991, Gould and Loeb 1992, Bond et al. 2004, Beaulieu et al. 2006). Gaudi et al. (2008) have found an analogy of the Sun-Jupiter-Saturn system through lensing. In recent, gravitational lensing has been used to constrain modified gravity at cosmological scale (Reyes et al. 2010).

It has long been a challenging problem to express the image positions as functions of lens and source parameters (Asada 2002, Asada, Hamana and Kasai 2003 and references therein). For this purpose, we present a method of Taylor-series expansion to solve the multi-plane lens equation in terms of mass ratios by extending the previous single-plane work (Asada 2009).

For N point lenses, Witt (1990) succeeded in recasting the lens equation into a single-complex-variable polynomial. This is in an elegant form and thus has been often used in investigations of point-mass lenses. The single-variable polynomial due to N point lenses on a single plane has the degree of N^2+1, though the maximum number of images is known as $5(N-1)$ (Rhie 2001, 2003, Khavinson and Neumann 2006, 2008). This means that unphysical roots are included in the polynomial (for detailed discussions on the disappearance and appearance of images near fold and cusp caustics for general lens systems, see also Petters, Levine and Wambsganss (2001) and references therein). Following Asada (2009), we consider the lens equation in dual complex variables, so that we can avoid inclusions of unphysical roots.

2 Basic Formulation

2.1 Multi-plane lens equation

We consider lens effects by N point masses, each of which is located at different distance D_i $(i = 1, 2, \cdots N)$ from the observer. For this case, we prepare N lens planes and assume the thin-lens approximation for each lens plane (Blandford, Narayan 1986, Yoshida, Nakamura, Omote 2005).

First of all, angular variables are normalised in the unit of the angular radius of the Einstein ring as

$$\theta_E = \sqrt{\frac{4GM_{tot}D_{1S}}{c^2 D_1 D_S}}, \tag{1}$$

where we put the total mass on the first plane at D_1, G denotes the gravitational constant, c means the light speed, M_{tot} is defined as the total mass $\sum_{i=1}^{N} M_i$ and D_1, D_S and D_{1S} denote distances between the observer and the first mass, between the observer and the source, and between the first mass and the source, respectively.

Recursively one can write down the multi-plane lens equation (Blandford and Narayan 1986, Schneider et al. 1992). In the vectorial notation, the double-plane lens equation is written as

$$\begin{aligned}\boldsymbol{\beta} &= \boldsymbol{\theta} - \left(\nu_1 \frac{\boldsymbol{\theta} - \boldsymbol{\ell}_1}{|\boldsymbol{\theta} - \boldsymbol{\ell}_1|^2} \right. \\ & \left. + \nu_2 d_2 \frac{\boldsymbol{\theta} - \nu_1 \delta_2 \frac{\boldsymbol{\theta} - \boldsymbol{\ell}_1}{|\boldsymbol{\theta} - \boldsymbol{\ell}_1|^2} - \boldsymbol{\ell}_2}{|\boldsymbol{\theta} - \nu_1 \delta_2 \frac{\boldsymbol{\theta} - \boldsymbol{\ell}_1}{|\boldsymbol{\theta} - \boldsymbol{\ell}_1|^2} - \boldsymbol{\ell}_2|^2} \right), \end{aligned} \tag{2}$$

where $\boldsymbol{\beta}$, $\boldsymbol{\theta}$, $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ denote the positions of the source, image, first and second lens objects, respectively. Here, ν_i denotes the mass ratio of each lens object, and we define d_2 and δ_2 as

$$d_2 \equiv \frac{D_1 D_{2s}}{D_2 D_{1s}}, \tag{3}$$

$$\delta_2 \equiv \frac{D_S D_{12}}{D_2 D_{1S}}. \tag{4}$$

It is convenient to use complex variables when algebraic manipulations are done. In a formalism based on complex variables, two-dimensional vectors for the source, image and lens positions are denoted as $w = \beta_x + i\beta_y$, $z = \theta_x + i\theta_y$, and $\epsilon_i = \ell_{ix} + i\ell_{iy}$, respectively. Figure 1 shows our notation for the multi-plane lens system. Here, z is on the complex plane corresponding to the first lens object that finally deflects light rays and thus z means the direction of a lensed image.

By employing this formalism, the double-plane lens equation is rewritten as

$$w = z - \left(\frac{1-\nu}{z^*} + \frac{\nu d_2}{z^* - \epsilon^* - \frac{(1-\nu)\delta_2}{z^*}} \right), \tag{5}$$

where the asterisk $*$ means the complex conjugate and we use the identity as $\nu_1 + \nu_2 = 1$ to delete ν_1 and ν denotes ν_2. Note that we choose the center of the complex coordinate as the first mass. Then, we have $\epsilon_1 = 0$ and simply denote $\epsilon \equiv \epsilon_2$, which is the projected relative position of the second mass with respect to the first one. The lens equation is non-analytic because it contains not only z but also z^*.

2.2 Iterative solutions

The mass ratio does not exceed the unity by its definition. Therefore, we use a simple-minded method of making expansions in terms of the mass ratios. One can delete ν_1 by noting the identity as $\sum_i \nu_i = 1$. And the location of the first lens is chosen as the origin of the complex coordinates.

Formal solutions are expressed in Taylor series as

$$z = \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \cdots \sum_{p_N=0}^{\infty} \nu_2^{p_2} \nu_3^{p_3} \cdots \nu_N^{p_N} z_{(p_2)(p_3)\cdots(p_N)}, \tag{6}$$

where the coefficients $z_{(p_2)(p_3)\cdots(p_N)}$ are independent of any ν_i. What we have to do is to determine each coefficient $z_{(p_2)(p_3)\cdots(p_N)}$ iteratively.

At the zeroth order, we have always a single-plane lens equation as the limit of $\nu_1 \to 1$ ($\nu_2 = \cdots = \nu_N \to 0$). We have the two roots for it. In addition, we have more roots for a multi-plane lens equation as *seeds* for our iterative calculations. An algorithm for doing such things is explained in next section.

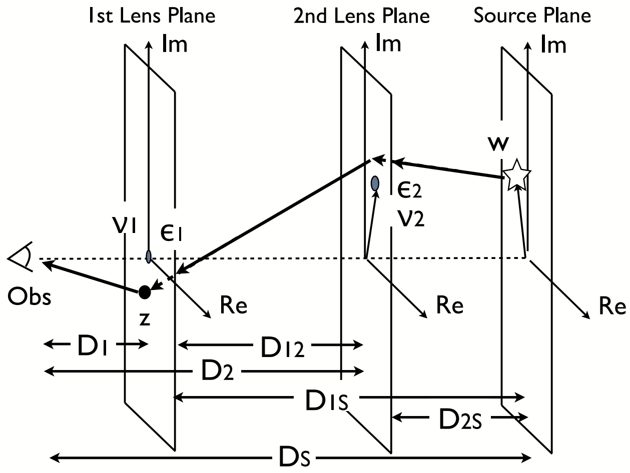

Figure 1: Notation: The source and image positions on complex planes are denoted by w and z, respectively. Locations of N masses are denoted by ϵ_i for $i = 1, \cdots, N$. Here, we assume the thin lens approximation for each deflector. The several distances among the observer, source and each lens object are also defined.

3 Image Positions

3.1 Double lens planes

At zeroth order ($\nu_2 \to 0$), the double-plane lens equation becomes simply

$$w = z_{(0)} - \frac{1}{z^*_{(0)}}, \tag{7}$$

which is rewritten as

$$z_{(0)} z^*_{(0)} - 1 = w z^*_{(0)}. \tag{8}$$

The L.H.S. of the last equation is purely real so that the R.H.S. must be real. Unless $w = 0$, therefore, one can put $z_{(0)} = Aw$ by introducing a certain real number A. By substituting $z_{(0)} = Aw$ into Eq. (8), one obtains a quadratic equation for A as

$$ww^* A^2 - ww^* A - 1 = 0. \tag{9}$$

This is solved as

$$\begin{aligned} A &= \frac{1}{2}\left(1 \pm \sqrt{1 + \frac{4}{ww^*}}\right) \\ &\equiv A_\pm, \end{aligned} \tag{10}$$

which gives $z_{(0)}$ as $A_\pm w$.

In the particular case of $w = 0$, Eq. (8) becomes $|z_{(0)}| = 1$, which is nothing but the Einstein ring. In the following, we assume a general case of $w \neq 0$.

We must consider $z_{(0)} = A_\pm w$, separately,

Table 1 shows a numerical example of image positions obtained iteratively and their convergence.

$z_{(1)}$ tells us an order-of-magnitude estimate of the effect by a separation between the two lens planes. Such a depth effect is characterised by δ_2, which enters the iterative expressions through z_+ and z_-.

Table 1: Example of image positions by the double-plane lens. We choose $\nu_1 = 9/10$, $\nu_2 = 1/10$, $\epsilon = 3/2$, $w = 2$, $D_1/D_S = 2/5$, $D_2/D_S = 3/5$. Iterative results (denoted as '0th'. '1st', '2nd' and '3rd') show a good convergence for the value (denoted as 'Num') that is obtained by numerically solving the lens equation.

Images	1	2	3	4
0th.	2.414213	-0.414213	1.780776	-0.280776
1st.	2.434312	-0.390217	1.731605	-0.276050
2nd.	2.430981	-0.388713	1.732327	-0.275043
3rd.	2.431474	-0.388781	1.732190	-0.274861
Num	2.431396	-0.388766	1.73220	-0.274833

4 Conclusion

We made a systematic attempt to determine, as a function of lens and source parameters, the positions of images by multi-plane gravitational lenses (Izumi, Asada 2010). We presented a method of Taylor-series expansion to solve the multi-plane lens equation in terms of mass ratios.

In concordance with the multi-plane lensed-image counting theorem that the lower bound on the image number is 2^N for N planes with a single point mass on each plane, our iterative results show directly that 2^N images are always realized as the minimum number of lensed images.

It is left as a future work to compare the present result with state-of-art numerical simulations.

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