Does the Entropy-Area Law hold for Schwarzschild-de Sitter spacetime ?

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Abstract

Multi-horizon means multi-temperature unless all of the Hawking temperatures of horizons coincide. Multi-temperature system is a nonequilibrium system, and generally the equation of state in nonequilibrium is different form that in equilibrium. This may imply that the horizon entropies in multi-horizon spacetime do not satisfy the entropy-area law which is an equation of state of a horizon in thermal equilibrium. This report examines whether the entropy-area law holds for Schwarzschild-de Sitter (SdS) spacetime, which is two-temperature system due to the difference of Hawking temperatures of black hole event horizon (BEH) and cosmological event horizon (CEH). We propose a reasonable evidence of breakdown of entropy-area law for CEH in SdS spacetime. The validity of the law for BEH in SdS spacetime can not be judged, but we point out the key issue for BEH's entropy.

1 Simple question from the Nonequilibrium viewpoint

Entropy-area law, which claims the entropy of horizon is equal to one quarter of its spatial area, is the equation of state for equilibrium systems which consist of horizons and matter fields. Now this law seems accepted as the universal law of thermodynamic aspects of any horizon in thermal equilibrium. However nothing is known about nonequilibrium situations of horizons.

On the other hand, generally in nonequilibrium physics, once the system under consideration comes in a nonequilibrium state, the equation of state for nonequilibrium case takes different form in comparison with that for equilibrium case. Especially the nonequilibrium entropy deviates from the equilibrium entropy (when a nonequilibrium entropy is well defined). Indeed, although a quite general formulation of nonequilibrium thermodynamics remains unknown at present [2], the differences of nonequilibrium entropy from equilibrium one are already revealed for some restricted class of nonequilibrium systems [1].

Then, for horizon systems, a simple question arises; *does the entropy-area law of horizons hold for horizon systems in nonequilibrium states?* We try to resolve this question and discuss *to what extent the entropy-area law is universal.* The representative nonequilibrium system of horizons may be multi-horizon spacetimes whose horizons have different Hawking temperatures. As a simple example of such system, we focus our discussion on Schwarzschild-de Sitter (SdS) spacetime, which is in two-temperature nonequilibrium state due to the difference of Hawking temperatures of black hole event horizon (BEH) and cosmological event horizon (CEH). We examine the entropy-area law for SdS spacetime.

However, applying some existing nonequilibrium thermodynamics to SdS spacetime is difficult at present. Then as one trial to search for SdS horizon entropy, we make a good strategy: We construct carefully two thermal *EQUILIBRIUM* systems separately for BEH and CEH which are designed so that the origin of nonequilibrium effect of CEH (BEH) on thermodynamic state of BEH (CEH) is retained and the Euclidean action method is applicable (see next section for a more concrete explanation). The subtraction term in Euclidean action is determined with referring to Schwarzschild thermodynamics for BEH and de Sitter thermodynamics for CEH. Although our systems are in thermal equilibrium states, some implication for nonequilibrium states of SdS horizons can be extracted, because the thermal states are "fine tuned" to include the origin of nonequilibrium nature of SdS spacetime. In this report, we will propose a reasonable evidence of breakdown of entropy-area law for CEH. The validity of the law for BEH can not be judged, but we will point out the key issue for BEH's entropy.

Recall that every existing verification of entropy-area law requires the thermal equilibrium of horizons. However, strictly speaking, it is not clear whether the thermal equilibrium is the necessary and sufficient

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[2] Or, a completely general formulation of nonequilibrium thermodynamics may not exist in our physical world.

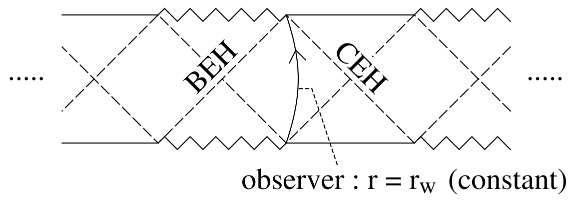
condition for the entropy-area law. Then, if the breakdown of the law will be confirmed for our thermal systems of BEH and CEH in SdS spacetime, it implies that the thermal equilibrium is not the necessary and sufficient condition but it is just the necessary condition for the entropy-area law. The sufficient condition for the law will be also suggested by this report.

2 Nonequilibrium nature of Schwarzschild-de Sitter spacetime

In general, the clear evidence of nonequilibrium is the existence of an energy flow inside the system under consideration, since no energy flow arises in thermal equilibrium systems. In SdS spacetime, because the Hawking temperature of BEH is always higher than that of CEH [2], a net energy flow arises inevitably from BEH to CEH. SdS spacetime is obviously a two-temperature nonequilibrium system. When one tries to analyze the SdS thermodynamics with the presence of the net energy flow, there arise difficult problems of nonequilibrium physics due to the net energy flow [1]; which quantity does the net energy flow raise as a new state variable to describe the degree of nonequilibrium nature?, how does the net energy flow cause the time evolution of the two-horizon system?, and so on. Therefore, at present, we need a good strategy to research the SdS thermodynamics with avoiding such difficult problems.

Here let us dare to ask: *Is the existence of net energy flow the principal origin of the nonequilibrium nature of SdS spacetime?* This can be rephrased as: *Does the net energy flow originate from some other physical factor?* The answer to the latter question seems Yes (No for the former) at least for SdS spacetime, because the energy flow between BEH and CEH is due to the difference of Hawking temperatures of horizons [3]. Furthermore, since the Hawking temperature is given by the surface gravity, the temperature difference is produced by the gravitational interaction between BEH and CEH. Therefore we recognize the gravitational interaction between BEH and CEH as the principal origin of nonequilibrium nature of SdS spacetime. Hence, if we can construct a system including a horizon (BEH or CEH) under the influence of gravitational interaction but excluding the net energy flow, then such system may reveal the nonequilibrium properties of BEH and CEH with avoiding the difficulties due to net energy flow. Then we introduce the following setup:

Setup (Heat Wall): Place a "heat wall" at $r = r_w$ in the region, $r_b < r < r_c$, as shown in the figure below, where r is the areal radius in SdS metric, and r_b and r_c are radii of BEH and CEH respectively. This heat wall reflects perfectly Hawking radiation of each horizon, and shields BEH (CEH) from the Hawking radiation emitted by CEH (BEH). The BEH (CEH) side of heat wall is regarded as a "heat bath" of Hawking temperature of BEH (CEH), and the net energy flow from BEH to CEH disappears. Then it is obvious that the region $\mathcal{M}_b$ enclosed by BEH and heat wall ($r_b < r < r_w$) forms a thermal *EQUILIBRIUM* system for BEH which is filled with Hawking radiation emitted by BEH and reflected by heat wall. Similarly the region $\mathcal{M}_c$ enclosed by CEH and heat wall ($r_w < r < r_c$) is also regarded as a thermal *EQUILIBRIUM* system for CEH. And we place the observer at the heat wall who measures all state variables of horizons. ($\mathcal{M}_b$ and $\mathcal{M}_c$ with the observer at r_w are already used to calculate Hawking temperatures in [2].)



As discussed hereafter, we can regard the thermal systems $\mathcal{M}_b$ and $\mathcal{M}_c$ as the desired systems which are under the influence of gravitational interaction between BEH and CEH without the net energy flow:

To explain it, we should remark that, while the heat wall shields the energy flow between the two horizons (which is mediated by matter fields of Hawking radiation), however the heat wall does not shield the gravitational interaction between the horizons (which is not mediated by matter field but

[3]For example, for ordinary gases in laboratory, an energy flow can arise by not only temperature difference but also viscosity, differences of pressure, number density and chemical potential, and so on. Energy flow is not the "cause" but the "effect" of nonequilibrium nature.

by gravitational field). This means the following; if we find that all state variables of thermal system $\mathcal{M}_b$ depend on a parameter of such gravitational interaction, then it is reasonable to regard the system $\mathcal{M}_b$ as a thermal equilibrium system for BEH under the influence of gravitational interaction of two horizons but excluding net energy flow, and similarly for $\mathcal{M}_c$ as a thermal system for CEH. Here it seems natural that the control parameter of the gravitational interaction between BEH and CEH are the mass parameter M and the cosmological constant Λ. Furthermore, concerning the state variables, we can assume very reasonably that every state variable of BEH depends on horizon radius r_b or its surface gravity κ_b, and similarly for CEH. Here the radii r_b, r_c and surface gravities κ_b, κ_c depend on M and Λ, because, for example, $r_b(M,\Lambda)$ and $r_c(M,\Lambda)$ are the positive roots of algebraic equation $f(r) = 0$, where $f(r) := -g_{tt} = 1-2M/r-\Lambda\, r^2/3$ and t is the time coordinate in the static chart of SdS metric $g_{\mu\nu}$. Then, obviously, every state variable of BEH (CEH) is under the influence of CEH's (BEH's) gravity through its dependence on horizon radii and surface gravities. Consequently, as mentioned above, we can regard the thermal systems $\mathcal{M}_b$ and $\mathcal{M}_c$ as the desired systems which are under the influence of gravitational interaction between the two horizons without the net energy flow.

Here note that the gravitational interaction on $\mathcal{M}_b$ is expressed as an external gravitational field produced by CEH which acts on BEH, and that on $\mathcal{M}_c$ is an external field by BEH acting on CEH. This situation is analogous to a magnetized gas under the influence of an external magnetic field. The magnetized gas consists of molecules possessing magnetic moment, and its thermodynamic state is characterized by three independent state variables; for example, temperature, volume and magnetization vector (response of the gas to external field), where the temperature and volume are variables required even for ordinary non-magnetized gas, and the magnetization vector is responsible for the magnetic property of the gas. The existence of three independent variables is mandatory for thermodynamic consistency of the magnetized gas. Then, as a strict thermodynamic requirement, our thermal systems $\mathcal{M}_b$ and $\mathcal{M}_c$ should also have three independent state variables to ensure thermodynamic consistency. This implies that every state variable of $\mathcal{M}_b$ and $\mathcal{M}_c$ is a function of three independent variables. Consequently, as a working hypothesis, we have to require that three parameters M, r_w and Λ are independent variables:

Working Hypothesis (three independent variables): To ensure thermodynamic consistency of our thermal systems $\mathcal{M}_b$ and $\mathcal{M}_c$, the radius of heat wall r_w, the mass parameter M and the cosmological constant Λ are regarded as three independent variables.

When one consider a non-variable Λ as a physical situation, it is obtained by setting the variation of Λ zero ($\delta\Lambda = 0$) in thermodynamics of $\mathcal{M}_b$ and $\mathcal{M}_c$ after constructing them with regarding Λ as an independent variable. In such case, the variable Λ is interpreted as a "working variable" to obtain SdS thermodynamics.

3 Entropies of horizons in Schwarzschild-de Sitter spacetime

As page space is limited, this section gives only a brief sketch of discussion of entropy-area law.

As a technique to obtain the state variables of $\mathcal{M}_b$ and $\mathcal{M}_c$, we make use of the Euclidean action method [3] which is applicable for any thermal equilibrium systems. The Euclidean action I_E is obtained by the imaginary unit i times the Wick rotation $t \to -i\tau$ of Lorentzian Einstein-Hilbert action,

$$I_E = \frac{1}{16\pi}\int_{\mathcal{M}} dx^4 \sqrt{g_E}\,(R_E - 2\,\Lambda) + \frac{1}{8\pi}\int_{\partial\mathcal{M}} dx^3 \sqrt{h_E}\, K_E - I_0 \,, \tag{1}$$

where R_E is the scalar curvature of Euclidean spacetime region $\mathcal{M}$, $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$, K_E is the trace of the extrinsic curvature of $\partial\mathcal{M}$, g_E is the determinant of Euclidean metric, h_E is the determinant of metric on $\partial\mathcal{M}$, and I_0 is the so-called *subtraction term*. I_0 is independent of the bulk metric $g_{E\mu\nu}$ of $\mathcal{M}$ and determines the integration constant of action integral with eliminating unexpected divergences of the other two integral terms. For our thermal systems for BEH and CEH in SdS spacetime, $\mathcal{M} = \mathcal{M}_b$ for BEH, $\mathcal{M} = \mathcal{M}_c$ for CEH, and $\partial\mathcal{M}$ is the heat wall for both horizons. (I_0 is determined later.)

The Euclidean spaces of $\mathcal{M}_b$ and $\mathcal{M}_c$ respectively have topology $D_2 \times S_2$, where D_2 is the time-radial part and S_2 reflects the spherical symmetry of Lorentzian SdS spacetime. The event horizon in Euclidean space is the center of D_2 and the boundary of D_2 has radius r_w. The regularity at the center of D_2 (excluding a conical singularity) determines the temperatures T_b of BEH and T_c of CEH, $T_b = \kappa_b/(2\pi\sqrt{f_w})$ and $T_c = \kappa_c/(2\pi\sqrt{f_w})$, where $f_w = f(r_w)$ and $f(r) = -g_{tt}$. The factor $\sqrt{f(r_w)}$ is equal to the so-called *Toleman factor* which expresses the gravitational redshift on the Hawking radiation propagating from horizon to observer. Therefore, these temperatures are consistent with our setting that the observer is at the heat wall.

We determine the subtraction term I_0 to match with Schwarzschild thermodynamics for BEH and de Sitter thermodynamics for CEH. Referring to [4] which established precisely the Schwarzschild thermodynamics, it is natural to set $I_0 = I_{\rm flat}$, where $I_{\rm flat}$ is the Euclidean action for Minkowski spacetime. On the other hand, the existing formulation of de Sitter thermodynamics does not introduce any boundary, since the spacetime is closed [5]. This corresponds to considering the micro-canonical ensemble, while the introduction of the boundary (heat wall) corresponds to the canonical ensemble. We can construct the canonical ensemble for de Sitter thermodynamics with introducing an appropriate boundary term in I_E to reproduce the same equations of state which the micro-canonical ensemble gives. Then, referring to the de Sitter's canonical ensemble, we find it is natural to set $I_0 = (1 - r_c/r_w)\sqrt{f_w}\, I_{\rm flat}$. With these subtraction terms, the Euclidean actions I_b for $\mathcal{M}_b$ and I_c for $\mathcal{M}_c$ are

$$I_b = \frac{\pi}{\kappa_b}\left[\, 3M - r_b + 2r_w\left(f_w - \sqrt{f_w}\right)\right] \quad , \quad I_c = \frac{\pi}{\kappa_b}\left[\, 3M - r_c + 2r_c f_w \,\right] . \tag{2}$$

Hence, following the argument of Euclidean action method [3], the free energies F_b for BEH and F_c for CEH are given by $F_b(T_b, A_w, X_b) := -T_b\, I_b$ and $F_b(T_c, A_w, X_c) := -T_c\, I_c$, where $A_w := 4\pi r_w^2$ is the extensive state variable of system size which mimics the volume of ordinary gases (see [4] for detail of this variable), and X_b and X_c are respectively the response of thermal systems $\mathcal{M}_b$ and $\mathcal{M}_c$ to the external gravitational field. These free energies are functions of three independent state variables as discussed at the working hypothesis in previous section.

Since X_b is the response of $\mathcal{M}_b$ to the external gravitational field by CEH, X_b should be a function of the quantity which characterizes the gravity of CEH and is measured by the observer at r_w. This implies,

$$X_b = r_w^2\, \Psi_b(\Lambda\, r_w^2) \quad \text{or} \quad X_b = r_w^2\, \Psi_b(\kappa_c\, r_w) \, , \tag{3}$$

where Ψ_b is an arbitrary function of single argument, and the factor r_w^2 is due to the detail of extensive nature of state variable [4] but not an essence of present discussion. Here we can not judge which of Λ and κ_c is appropriate as the characteristic quantity of CEH's gravity. Similarly, it is natural for X_c to require $X_c = r_w^2\, \Psi_c(M/r_w)$ or $X_c = r_w^2\, \Psi_c(\kappa_b\, r_w)$, where Ψ_c is an arbitrary function. Then, following the argument of thermodynamics, the entropy of BEH S_b and that of CEH S_c are define by the partial derivatives; $S_b := -\partial F_b(T_b, A_w, X_b)/\partial T_b$ and $S_b := -\partial F_c(T_c, A_w, X_c)/\partial T_c$, which are rearranged to be first order partial differential equations of Ψ_b and Ψ_c. We can find these differential equations imply:

Result for BEH: BEH's entropy $S_b = \pi\, r_b^2$ for the choice $X_b = r_w^2\, \Psi_b(\Lambda\, r_w^2)$, but $S_b \neq \pi\, r_b^2$ for the choice $X_b = r_w^2\, \Psi_b(\kappa_c\, r_w)$. The entropy-area law for BEH holds if $\partial_M X_b = 0$, but breaks down if $\partial_M X_b \neq 0$. This denotes that the sufficient condition of entropy-area law for BEH is $\partial_M X_b = 0$. Hence it is the dependence of X_b on M that determines the validity of entropy-area law for BEH.

Result for CEH: CEH's entropy $S_c \neq \pi\, r_c^2$ for either choice $X_c = r_w^2\, \Psi_c(M/r_w)$ and $X_c = r_w^2\, \Psi_c(\kappa_b\, r_w)$. The entropy-area law seems break down for CEH in SdS spacetime.

References

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