Aspects of Stability of the Toroidal AdS Schwarzschild Black Hole

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Chapter 2 is based on the paper: *The Klein-Gordon equation on the toric AdS-Schwarzschild black hole, Classical and Quantum Gravity 2016.*

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Abstract

In this thesis various aspects about the dynamical stability of the toroidally symmetric Schwarzschild AdS black hole are discussed and proven.

The first chapter of the thesis is a literature review. This covers the key relevant results within the area and provides context for the results of the later chapters.

The second chapter concerns the Klein-Gordon equation with Dirichlet, Neumann and Robin boundary conditions on the exterior of the toroidally symmetric Schwarzschild AdS black hole. Through the vector field method energy estimates, and degenerating Morawetz estimates are proven. From these it is seen that the energy of the solutions on these spacetimes are bounded and decay polynomially in time. Furthermore, it is shown that there exist null geodesics on this spacetime remain exterior to the event horizon boundary for arbitrary coordinate time. Through a Gaussian beam argument, it follows that the degeneration in the Morawetz estimates is necessary.

The third chapter proves the non-linear stability of the toroidally symmetric Schwarzschild AdS black hole as a solution to the AdS-Einstein–Klein-Gordon system within the class of square toroidal symmetries where the field satisfies Dirichlet or Neumann boundary conditions. This is done through establishing wellposedness of the system in a region near null infinity. Then for initial data 'sufficiently small' it is shown through bootstrap arguments that the energy remains bounded by the initial data on the regular region exterior to the black hole. This is then used to establish the orbital stability of the spacetime. Then through the vector field method, exponential decay of the field on the regular region exterior to the black hole is established. From this the asymptotic stability follows. Finally, a vacuum stability result is established in the toroidal symmetry class where the periods of the torus are allowed to vary.

For Claire, Steve, and Lewis

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This thesis is a corollary of the Banach-Tarski paradox. It proves that the fun, excitement, and challenges of one PhD can be split into two disjoint subsets and reassembled into one with twice as much! As such many people have supported and inspired me throughout.

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Introduction

1.1 Introduction

In 1915 Albert Einstein published his revolutionary theory of general relativity. It is of a geometric nature. The role of space and time is played by a four dimensional Lorentzian manifold $(\mathcal{M}, g)$, where the metric g satisfies the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{1.1}$$

The left hand side of these equations consists of geometric quantities. $R_{\mu\nu}$ is the Ricci curvature of the metric, R is the scalar curvature given by the trace of $R_{\mu\nu}$ with respect to g, and Λ is a scalar known as the cosmological constant. The right hand side consists of physical quantities. $T_{\mu\nu}$ is the stress-energy tensor which describes the distribution of stress and energy of the universe. G is the gravitational constant, and c is the speed of light. Typically one works in geometrised units and takes $G = c = 1$. The unification of these two, seemingly unrelated, concepts was elegantly summarised by John Wheeler as: *'Spacetime tells matter how to move; matter tells spacetime how to curve'*.
Specialising now to the case $T_{\mu\nu} = 0$, (1.1) reduces to the Einstein vacuum equations

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{1.2}$$

Imposing the further restriction $\Lambda = 0$, one can see that the manifold $\mathcal{M} = \mathbb{R}^4$ equipped with the metric

$$\eta = -dt^2 + dx^2 + dy^2 + dz^2, \tag{1.3}$$

solves (1.2). This is not hugely surprising, as if we are to interpret gravity through the curvature of a spacetime and reconcile this with the Newtonian notion that gravity is related to mass, then one should expect the flat spacetime as a solution. This isn't the only solution to (1.2), in 1916 Karl Schwarzschild [Sch16] published his solution, $\mathcal{M} = \mathbb{R} \times (2M, \infty) \times \mathbb{S}^2$ with metric

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\mathbb{S}^2, \tag{1.4}$$

which is a one parameter family (in M) of solutions to (1.2). It is thought to model spherically symmetric static stars. Originally it was expected that the radius of the star would exceed $r_{star} > 2M$ [Buc59] and a vacuum model was no longer appropriate in the region $r \leq r_{star}$. At the time this would have appeared fortuitous as there appears to be a singularity at $r = 2M$. However it turns out that this is only a coordinate singularity, and the solution can be extended beyond $r = 2M$. Within this new region it can be seen that all future directed causal curves remained trapped to $r < 2M$, forming what we now call a black hole. The surface $\mathcal{H} = \{r = 2M\}$ is known as the event horizon. This solution however does have a genuine singularity at $r = 0$, which can be seen by the blow up of the Kretschmann scalar $K = R_{abcd}R^{abcd}$. (For Schwarzschild this takes the value $K_{Sch} = \frac{48M^2}{r^6}$). Forty seven years later Kerr discovered a larger family of solutions to (1.2) [Ker63]. These are rotating black holes with the additional parameter $g_{M,\alpha}$ where α represents the angular momentum of the black hole.

While these black hole solutions are mathematically interesting their physical relevance could be contested. A popular idea in physics is that of gravitational collapse. It is believed the end state of large stars can be modelled by a Kerr solution. However when solving for Schwarzschild or Kerr solutions one imposes symmetry restrictions on the spacetime that *a priori* the universe may not exhibit (and is almost certainly not present in gravitational collapse). Furthermore, there are currently no known explicit collapsing matter solutions with the Kerr exterior as their final state [BC05]. Nevertheless objects resembling black holes have been observed in experiments such as LIGO's detection of gravitational waves [Aea17]. It is therefore apparent that in order to explore the significance of these solutions within the physical theory one needs to study (1.2) as a dynamical problem.

1.2 The Cauchy problem

When studying equations of motion of hyperbolic type as in (1.2) one typically considers a Cauchy problem. An example would be given by the wave equation on $\mathbb{R}^{1,3}$ with coordinates $(t, \mathbf{x})$,

$$\begin{aligned} -\psi_{tt}(t, \mathbf{x}) + \Delta\psi(t, \mathbf{x}) &= 0, \\ \psi(0, \mathbf{x}) &= f_1(\mathbf{x}), \\ \psi_t(0, \mathbf{x}) &= f_2(\mathbf{x}). \end{aligned} \tag{1.5}$$

In order to have a well posed problem the function ψ and its first time derivative at $t = 0$ must be specified.

Due to the geometrical nature of (1.2), it proved difficult to find a similar appropriate formulation of the problem. This was resolved for the $\Lambda = 0$ case in the work of Choquet-Bruhat–Geroch

[CBG69]. One defines an initial data set as a spacelike three dimensional manifold Σ, with a Riemannian metric h, and a symmetric 2 tensor K, satisfying the Einstein constraint equations

$$\begin{aligned} R_h + (K^a_a)^2 - K^b_a K^a_b &= 0, \\ \nabla_b K^b_a - \nabla_a K^b_b &= 0. \end{aligned} \tag{1.6}$$

Formulating the equations in the wave gauge

$$\Gamma^{\alpha}{}_{\mu}{}^{\mu} = 0, \tag{1.7}$$

(Γ are the connection coefficients) the equations form a system of quasilinear wave equations

$$-\frac{1}{2} g^{\mu\alpha} \frac{\partial^2 g_{\sigma\nu}}{\partial x^\mu \partial x^\alpha} + \mathcal{N}_{\sigma\nu}(g, \partial g) = 0. \tag{1.8}$$

The system is then solvable by standard methods, while checking the gauge condition (1.7) is propagated. This leads to the result

Theorem 1.2.1 (Choquet-Bruhat-Geroch). *For a smooth initial data set* (Σ, h, K) *satisfying* (1.6)*, there exists a unique smooth maximal four dimensional Lorentzian manifold* $(\mathcal{M}, g)$ *satisfying* (1.2) *(*$\Lambda = 0$*) with a smooth embedding* $\iota : \Sigma \to \mathcal{M}$ *such that* $(\mathcal{M}, g)$ *is globally hyperbolic with Cauchy surface* $\iota(\Sigma)$ *and* h*,* K *are the induced first and second fundamental forms on* $\iota(\Sigma)$ *respectively.*

In comparison to (1.5), one can think of $h \sim \psi|_{t=0}$ and $K \sim \psi_t|_{t=0}$. The theorem proven is a local wellposedness result, because while it implies the existence of spacetimes, it doesn't tell us about global properties, such as completeness or structure. These need to be inferred from other methods. It is worth remarking that in establishing existence of the maximal development, the original authors invoked Zorn's lemma. This has since been shown to be unnecessary as seen in [Sbi16]. The regularity of the solution is also not optimal in this theorem; there has been further work culminating in the proof of the bounded L^2 curvature conjecture [KRS15] which shows wellposedness of (1.2) in a very low level of regularity. Scaling arguments suggest this level of regularity is however still not optimal.

1.3 Stability problems

With (1.2) now understood as a locally wellposed initial value problem for an initial data set (Σ, h, K), one can start asking questions about global properties of the solution. One really wants to know whether the black hole solutions can actually form from a large class of initial data [Chr09], or if they are just artefacts of symmetry. It seems that this problem is beyond the reach of the current methodology. A simpler question to understand their physical significance is about their stability as solutions. Indeed if they aren't stable as solutions to (1.2) then their natural presence is cast into doubt. However before the stability of black holes is tackled, there is the simpler, yet still highly non-trivial problem of the non-linear stability of Minkowski space.

This was proven in the work of Christodoulou-Klainerman [CK93]

Theorem 1.3.1. *Consider a strongly asymptotically flat initial data set* (Σ, h, K)*, assume it is sufficiently close to Minkowski space in some weighted* H^k *sense. The maximal development is geodesically complete and approaches Minkowski space in all directions. Furthermore a complete null infinity can be attached to the spacetime.*

The proof was later simplified in the work of [LR05] who worked in the wave gauge rather than the more invariant formulation of [CK93]. This result has since been extended including the addition of various matter fields [LM17], [Tay17], [LT17], [Bie09], and to higher dimensional theories of relativity [Wya17]. It had already been established in the de-Sitter ($\Lambda > 0$) setting [Fri86]. The extension of this result to black hole spacetimes without symmetry assumptions remains largely still open. However there has been success for the slowly rotating ($|\alpha| \ll M$) Kerr de-Sitter black hole ($\Lambda > 0$) by Hintz and Vasy [HV16], who show that the solutions decay exponentially fast to another Kerr de Sitter metric.

1.4 Linear waves on black hole spacetimes

While the full non-linear stability of black holes has not yet been proven, a typical approach would be to work on linear problems, and to build up methodology from there. Perhaps the simplest linearisation that can be attempted is to study the wave equation on the fixed target black hole solution. The Cauchy problem is given by

$$\begin{aligned} \Box_g \psi &= 0, \\ \psi|_\Sigma &= \Psi_1, \\ n|_\Sigma \psi|_\Sigma &= \Psi_2, \end{aligned} \tag{1.9}$$

where $n|_\Sigma$ is the future directed unit normal to the spacelike surface Σ, and $\Box_g$ is the wave operator given by

$$\Box_g \psi = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \psi \right). \tag{1.10}$$

In view of (1.8) we may loosely think that ψ is standing in for the metric tensor, and when g is a black hole spacetime, many of the difficulties seen in the non-linear stability problem manifest themselves in proving boundedness and decay of ψ. Indeed the decay of waves and boundedness by initial data is akin to asymptotic and orbital stability in the non-linear problem respectively.

1.4.1 Vector field method

A robust method to prove boundedness and decay of waves on black hole spacetimes is known as the vector field method. The core idea is to define a quantity known as the energy momentum tensor

$$\mathbb{T}_{\mu\nu}[\psi] = \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \psi \nabla_\sigma \psi, \tag{1.11}$$

which satisfies $\nabla^\mu \mathbb{T}_{\mu\nu}[\psi] = 0$ when $\Box_g \psi = 0$. For a smooth vector field X one then defines the vector field multiplier current as

$$J^X_\mu[\psi] = \mathbb{T}_{\mu\nu}[\psi] X^\nu, \tag{1.12}$$

and its associated bulk term

$$K^X[\psi] = {}^X\pi^{\mu\nu}\mathbb{T}_{\mu\nu}[\psi], \tag{1.13}$$

where ${}^X\pi = \frac{1}{2}\mathcal{L}_X g$. Crucially the following identity holds

$$\nabla^\mu J^X_\mu[\psi] = K^X[\psi], \tag{1.14}$$

which through the divergence theorem has many applications when integrating over spacetime regions of the manifold.

Bounded Energy

Considering the case where the spacetime is Minkowski. Let Σ_τ to be the surface $\{t = \tau\}$ where τ is constant, and let $X = \partial_t$ which is Killing. The current is computed as

$$2J^T_\mu n^\mu|_{\Sigma_t} = (\nabla_t\psi)^2 + (\nabla_x\psi)^2 + (\nabla_y\psi)^2 + (\nabla_z\psi)^2 . \tag{1.15}$$

So $J^T_\mu n^\mu|_{\Sigma_t}$ forms the energy density of ψ. Integrating (1.14) for the field decaying sufficiently fast to spacelike infinity we derive the result

$$\int_{\Sigma_t} J^T_\mu n^\mu|_{\Sigma_t} dS_{\Sigma_t} = \int_{\Sigma_0} J^T_\mu n^\mu|_{\Sigma_0} dS_{\Sigma_0}, \tag{1.16}$$

which is nothing more than the classical conservation of energy. One can now extract pointwise bounds on the field ψ in terms of its initial data through commuting the equation with all the Killing fields, elliptic estimates, and Klainerman-Sobolev inequalities [Kla87]. The same methodology works in black hole spacetimes however there are additional issues. Many of these can be seen on the Schwarzschild geometry. In order to have a regular horizon one uses the Gullstrand-Painlevé coordinate system

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + 2\sqrt{\frac{2M}{r}} dtdr + dr^2 + r^2 d\mathbb{S}^2. \tag{1.17}$$

In this setting $T = \partial_t$ is still a timelike Killing field. Let Σ be a Cauchy hypersurface (that is every inextendible causal curve of the spacetime intersects Σ precisely once), furthermore let it be asymptotically flat. Then define Σ_τ to be $\phi_\tau(\Sigma \cap \{r \geq 2M\})$ where ϕ_τ is the one parameter family of diffeomorphisms generated by T. Noting that the flux along $\mathcal{H}$ is positive and integrating (1.14) yields a result about a bounded energy

$$\int_{\Sigma_t} \left(1 - \frac{2M}{r}\right) \left((\nabla_t\psi)^2 + (\nabla_r\psi)^2 + \left|\nabla\!\!\!/\psi\right|^2\right) dS_{\Sigma_t} \leq C \int_{\Sigma_0} J^T_\mu n^\mu|_{\Sigma_0} dS_{\Sigma_0}, \tag{1.18}$$

where $C > 0$ is a uniform constant. While the same methodology as in Minkowski space can still be applied, this estimate degenerates as $r \to 2M$. Boundedness can only be obtained away from the horizon. Fortunately this issue can be rectified by exploiting an interesting phenomena of the black hole, known as the red shift effect. The idea being that if two observers A and B are transmitting photons to each other, while falling into the black hole with B entering at a later (coordinate) time, then the frequency of the light that B sees from A is exponentially decaying (being shifted to the red). This suggests there is an additional decay mechanism in a neighbourhood of the surface $\mathcal{H}$. This is exploited by [DR08], [DR09b], and manifests itself in the existence of a vector field N, which is timelike, acts like T away from the horizon and importantly doesn't have a degenerate current at the horizon. Combining this with the T estimate as before, one can remove the degeneration in the estimates, [DR08] and apply the same theory to prove boundedness as the Minkowski case.

Energy Decay

To prove that the fields are decaying in time one turns to techniques pioneered by Morawetz [Mor61], [Mor68], [Mor66]. The idea is to use the vector field method, but to also consider vector fields that aren't Killing. In the context of wave equation on the exterior of an obstacle, the conformal Killing field $K = t\partial_t + r\partial_r$ was used in [Mor61]. This choice of K leads to a t weight appearing in (1.12) along with the standard energy density. From this time decay of the energy can be inferred. In [Mor68] an alternative idea was seen for the non-linear Klein-Gordon equation. The idea is to use vector fields of the form $X = h(r)\partial_r$, for a radial function $h(r)$. This results in what are now known as Morawetz or integrated energy decay estimates (IED). To illustrate the idea, consider a spherically symmetric, static, asymptotically flat spacetime with a smooth time function t, area radius r, and polar angles θ, ϕ. Let $\mathcal{E}_\psi(t, r, \theta, \phi)$ be a density for $E_\psi(t)$ the energy of ψ (associated to T). A Morawetz estimate for ψ would typically take the form

$$\int_0^T \int_{\Sigma_t} f(r)\mathcal{E}_\psi(t, r, \theta, \phi)dSdt \leq CE_\psi[0], \tag{1.19}$$

where $C > 0$ is uniform and $f : (0, \infty) \to [0, \infty)$. Typically f degenerates on certain surfaces of the spacetime, or as $r \to \infty$. When this estimate is coupled with a bounded energy estimate, providing the weight of the function f is sufficiently strong, one can then prove decay of the energy on hyperboloidal slices [DR09a]. These decay rates are typically polynomial in nature however in the instance where no degeneration of f occurs, exponential decay can be established. The standard commutation methods and Sobolev estimates are then invoked to imply pointwise estimates. In order to prove Morawetz type estimates, one returns to the vector field method but instead of using T as a multiplier other more specialised choices are used. The idea is to chose X in such a way the the bulk term $K^X[\psi] \geq 0$ and $J^X[\psi]$ is controllable by the energy (which has been proven bounded typically through the vector fields T and N). In the case of the Schwarzschild geometry, the work of [LS00], [BS06] employed these ideas in the context of the non-linear Schrödinger equation. This was later extended to the wave equation in [DR09b], [DR07a], where the use of growing t weights in X was seen to not be necessary. Uniform decay

$|\psi| \leq C\tau^{-1}$ was proven by using the more robust vector fields of the form $f(r)\partial_r$. Locally this was improved by [Luk10] and then later [AAG18] to $|\psi| \leq C\tau^{-3}$ for $r < R$ with $R > 0$. In the case of the Kerr spacetime for the range $|\alpha| < M$, uniform decay of rate $\tau^{-\frac{1}{2}}$ and for bounded r, the rate $\tau^{-\frac{3}{2}+\delta}$ was established in [DR10], and [DRSR14]. Alternatively [AB15] proved Morawetz estimates, and uniform boundedness for slowly rotating Kerr, ($|\alpha| \ll M$) by using a more generalised version of the vector field method, exploiting Killing tensors. In the case of Schwarzschild de-Sitter, polynomial decay was established in [DR07b] and extended to exponential decay in [Dya10], for the Kerr de-Sitter setting. With the strong cosmic censorship conjecture in mind, there has also been a lot of work on linear waves for the Reissner-Nördstrom spacetime; here the focus shifts towards studying the interior of the black hole in order to understand the behaviour of the Cauchy horizon [Are11], [Fra16], [AAG18].

Trapping and the Photon sphere

In the case of proving energy decay for black hole spacetimes, it is often the case that one cannot prove a Morawetz estimate where f does not degenerate either on compact surfaces or as $r \to \infty$. The key obstacle to decay that appears for black holes is the photon sphere. In the case of Schwarzschild the surface $r = 3M$ forms a surface of trapped null geodesics. Its effect on the estimates can be seen by taking Reggie-Wheeler coordinates (r^*, t), and choosing a radial vector field of the form $X = f(r^*)\partial_{r^*}$ for a C^1 function f. From [DR08] we get a bulk term

$$K^X[\psi] = \frac{f'}{1-\frac{2M}{r}}\left(\partial_{r^*}\psi\right)^2 + \frac{f}{r}\left(1 - \frac{3M}{r}\right)\left|\nabla\!\!\!/\psi\right|^2 - \frac{1}{4}\left(2f' + 4\frac{r-2M}{r^2}f\right)\nabla^\sigma\psi\nabla_\sigma\psi. \quad (1.20)$$

With modification to $J^X_\mu[\psi]$ (adding terms of the form $w\nabla_\mu(\psi^2)$ and $\psi^2\nabla_\mu(w)$ for a smooth function w) it is possible to choose f so that this is positive semi-definite. There is the problem of degeneration at $r = 3M$. This can resolved by commuting the equation with the angular momentum operators, proving estimates of the form

$$\int_0^T \int_{\Sigma_t} \mathcal{E}_\psi(t, r, \mathbf{x})dSdt \leq C\tilde{E}_\psi[0]. \quad (1.21)$$

where $\tilde{E}_\psi$ is a higher order energy. This is often referred to as 'having to lose a derivative'. The geometric optics approximation shows that high frequency disturbances propagate along null geodesics hence the photon sphere is forming an obstacle for decay. In the work of [Sbi15] this idea was made rigorous and it was proven that if there are trapped null geodesics in a region of bounded radius on a spacetime then one cannot prove estimates of the form

$$\int_0^T \int_{\Sigma_t \cap \{r \leq R\}} \mathcal{E}_\psi(t, r, \mathbf{x})dSdt \leq CE_\psi[0]. \quad (1.22)$$

To contradict an estimate of the form (1.22) one constructs Gaussian beams localised to null geodesics, whose energy remains arbitrarily close to the geodesic's energy. As Gaussian beams approximately solve the wave equation, one can construct actual solutions that remain arbitrarily close the Gaussian beam (in the energy norm) for a given time. These solutions then naturally decay very slowly in time. They have to overcome this trapping before the decaying effects of the black hole event horizon or dispersion to infinity can occur.

1.5 Spacetimes with symmetry

When moving from the linear regime of the wave equation to non-linear problems, it is helpful to first restrict to the spherical symmetry class. This reduces the dimensionality of the problem to $1+1$ dimensions, where more methodology for the study of PDE has been developed. Unfortunately if one studies (1.2) within the class of spherical symmetry, through Birkhoff's theorem, they see the only solution is (an isometric subset of) the Schwarzschild solution. As this solution is static there are no dynamics in the problem. To study a time dependent problem, one has to revisit (1.1) where the system is coupled to a matter model.

1.5.1 Scalar field system

The simplest matter model one can consider for $\Lambda = 0$ is to couple to a scalar field ψ. This is done with the stress energy tensor

$$\mathbb{T}_{\mu\nu}[\psi] = \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \psi \nabla_\sigma \psi, \tag{1.23}$$

as the LHS of (1.1) is divergence free, the stress energy tensor must also be. This implies the field satisfies

$$\Box_g \psi = 0. \tag{1.24}$$

This system was studied comprehensively by Christodoulou. Through a series of papers a full picture of the dynamics was established. In the paper [Chr86b], Christodoulou shows that ψ governs the dynamics of (1.1) within this symmetry class, and that one can express the system as a non-linear evolution equation for ψ. Applying contraction map arguments, local wellposedness for the system is established. The result is then extended to global existence for small initial data, and it is shown that the spacetime converges to Minkowski space with polynomial decay rates. Finally, it is shown that the Bondi mass converges to zero. (This result could be thought of as a specialisation of stability of Minkowski space, for the scalar field system under spherical symmetry). For larger data the existence of a generalised solution was shown [Chr86a], uniqueness and the fact that generalised solutions extend classical solutions was shown in [Chr87b]. Finally, in the paper [Chr87a], it was shown that if the Bondi mass does not converge to zero, a black hole forms, surrounded by a vacuum. A similar result has been proven for the de-Sitter setting [CAN13], where for small initial data the solution decays back to de-Sitter exponentially.

An alternative to studying Schwarzschild as a 1+1 dimensional PDE but as a solution of

(1.2), is to move to five dimensional relativity and impose biaxial Bianchi IX symmetry. In this setting Birkhoff's theorem fails to hold. Orbital stability was shown in [DH06] and the asymptotic in [Hol10b].

Cosmic Censorship and other matter models

As well as being a simplified non-linear problem to study the stability of black holes, spherically symmetric spacetimes have been studied in detail to understand the cosmic censorship conjectures.

Conjecture 1.5.1 (Weak Cosmic Censorship). *For admissible initial data to* (1.1) *there exists a generic subclass in which future null infinity is complete.*

The physical intuition behind this conjecture is that any singularities must form behind an event horizon, and observers at future null infinity can exist for all time.

Conjecture 1.5.2 (Strong Cosmic Censorship). *For admissible initial data to* (1.1) *there exists a generic subclass for which the the maximal future development of the solution is future inextendible as a suitably regular Lorentzian metric.*

This conjecture is a statement about the determinism of general relativity as a physical theory. In spite of their names the strong and weak cosmic censorship conjectures are logically independent of each other. This can be seen when viewed from a PDE perspective. The weak cosmic censorship is a statement about global existence, and the strong is about global uniqueness, which are *a priori* independent concepts. In the case of the scalar field system under spherical symmetry both of these conjectures have been proven to be true [Chr98].
There has been further work on these conjectures in other spherically symmetric spacetimes notably in the work of Dafermos [Daf05]. The author shows that for spherically symmetric spacetimes obeying certain conditions (notably, an energy condition and an extension principle on the regular and marginally trapped regions) then they have a complete null infinity. In the case of the Einstein–Maxwell-Klein-Gordon system, Kommemi [Kom13], was able to prove a stronger extension principle which also holds in the trapped region. Price's law (polynomial decay rate along the horizon) has also been shown [DR05].

Spherical symmetry is not the only symmetry studied in (1.1), the case of toroidal symmetries leading to Gowdy spacetimes [Gow74], (which form toy models for big bang cosmology) has also been studied in [LS15].

1.6 Anti de-Sitter space

Historically, most of the theory in relation to (1.1) has been considering the case $\Lambda \geq 0$. There are two main reasons for this. Firstly from a physical perspective these have been traditionally thought of as the interesting cases, $\Lambda = 0$ models isolated gravitational systems. $\Lambda > 0$ models an expanding universe, useful for inflation theories and the universe on a large scale. $\Lambda < 0$ would model a collapsing universe which seems unphysical. There is however now a great deal of interest from the high energy physics community about these spacetimes, coming from the AdS/CFT correspondence [Mal99]. Loosely speaking it is believed that conformal field theories in $n-1$ dimensions are in duality with n dimensional solutions to (1.1) with $\Lambda < 0$. The other reason they were less studied is that they are qualitatively different to the $\Lambda \geq 0$ case. If we look for a solution to (1.1) with $\Lambda < 0$ and $T_{\mu\nu} = 0$, with the maximal amount of symmetry, one finds the anti de-Sitter spacetime. It is the manifold $\mathcal{M} = \mathbb{R}^4$ with metric

$$g = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\mathbb{S}^2, \tag{1.25}$$

where $l^2 = \frac{-3}{\Lambda}$. Inspecting the Penrose diagram of this solution [HE73],

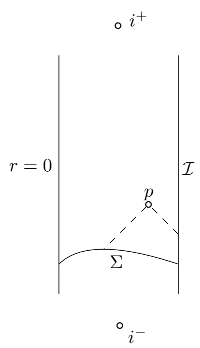


Figure 1.1: Penrose diagram of AdS

one can see that for any attempt to define a Cauchy hypersurface, one will run into trouble. Past directed causal curves passing through points like p can intersect with null infinity ($\mathcal{I}$) (which is now timelike). This means that AdS is not globally hyperbolic, and nor are spacetimes with similar asymptotics known as asymptotically AdS spacetimes (aAdS). When looking at the Cauchy problem in these spacetimes and applying the result from theorem 1.2.1, we cannot expect to get AdS as the maximal development of any initial data set. In order to resolve this issue one needs to discuss initial boundary value problems (IBVPs) [Fri95].

1.6.1 The Kottler solutions

The setting $\Lambda < 0$ also has its own analogue of the Schwarzschild solution, however in this setting a curious new phenomenon occurs. It is no longer necessary that static black holes with compact horizons must be spherical. The solutions are known as the Kottler metrics [Kot18], [Lem95], [Bir99] or Schwarzschild AdS. Defining $r_{+,k}$ to be the unique real root of $f(r) = k - \frac{2M}{r} + \frac{r^2}{l^2}$ where $k \in \{-1, 0, 1\}$, the solutions are given by $\mathcal{M} = \mathbb{R} \times (r_{+,k}, \infty) \times \mathcal{G}_k$, equipped with

$$g = -\left(k - \frac{2M}{r} + \frac{r^2}{l^2}\right) dt^2 + \left(k - \frac{2M}{r} + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\gamma_k^2, \tag{1.26}$$

and

$$\mathcal{G}_k = \begin{cases} \mathbb{S}^2 \text{ for } k = 1, \\ \mathfrak{T}^2 \text{ for } k = 0, \\ \Sigma_g \text{ for } k = -1, \end{cases} \tag{1.27}$$

where $\mathbb{S}^2$ is a sphere, $\mathfrak{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a torus, Σ_g is a surface of genus $g \geq 2$ with a metric of constant curvature -1, and $d\gamma_k^2$ is the unit metric on these spaces.

The symmetries of the geometry are not tied to the staticity. There are static solutions to (1.1) with $\Lambda < 0$, in which the geometry at $\mathcal{I}$ of the above solutions is perturbed providing a non-degeneracy condition holds [ACD02]. A rigidity result (see [ACD02] for a precise formulation) exists in the first two cases. If the topology at $\mathcal{I}$ is given by $\mathbb{R} \times \mathbb{S}^2$, then the AdS metric (1.25) or the regular region of the spherical Schwarzschild AdS ($M > 0$) are the unique static globally hyperbolic (in a sense of manifold with boundary) solutions to (1.2), with boundary metric at null infinity $\gamma = -dt^2 + d\gamma_1^2$. In the case of toroidal solutions, there is a similar result where the regular region of the toroidal AdS Schwarzschild black hole and the AdS soliton [HM98], are the unique static, globally hyperbolic (in a sense of manifold with boundary) solutions to (1.2), with boundary metric at null infinity $\gamma = -dt^2 + d\gamma_0^2$. Due to the symmetries of $(\Sigma_g, d\gamma_{-1}^2)$ only being local, a rigidity result in the hyperbolic case is still open.

Extensions and coordinate transformations

As is typical in Schwarzschildean coordinates, it can be seen that the metric in (1.26) becomes singular on the future and past event horizons, $\mathcal{H}^+ = \{r = r_{+,k}, t > 0\}$ and
$\mathcal{H}^- = \{r = r_{+,k}, t < 0\}$ respectively. The Penrose diagram for these solutions is given by

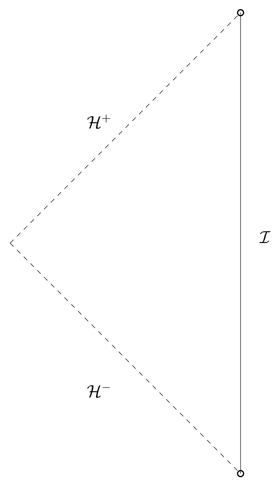


Figure 1.2: Penrose diagram of the Kottler solutions in Schwarzschildean coordinates.

These coordinates are largely problematic for analysis. We will often want to study regions of the spacetime containing the event horizon. In order to remedy this we will illustrate a maximal extension of this solution which will include the event horizons and the black hole region. We will then exhibit coordinate systems on restrictions of this extension. These restrictions will cover the regions of interest considered in this thesis.

Kruskal–Szekeres extension

We follow a similar construction to that in [Hem04], generalising slightly for $k \in \{-1, 0, 1\}$. We define

$$F(X) := \left(k - \frac{2M}{X} + \frac{X^2}{l^2}\right), \tag{1.28}$$

and the tortoise coordinate r_* by

$$r_*(r) := \int_{2r_{+,k}}^{r} F^{-1}(X)dX. \tag{1.29}$$

A change of variable $w = F(X)$, shows us that $\lim_{r\to r_{+,k}} r_*(r) = -\infty$. Now through a Taylor expansion, we see that r_* has the following behaviour near $r_{+,k}$

$$r_* = \frac{1}{F'(r_{+,k})}\left(\ln\left(r - r_{+,k}\right) + \mathcal{O}\left(r - r_{+,k}\right)\right). \tag{1.30}$$

Now the metric (1.26), can be written as

$$g = -F(r)\left(dt^2 - dr_*^2\right) + r^2 d\gamma_k^2. \tag{1.31}$$

We change to null coordinates, defined by

$$u = t - r_*, \quad v = t + r_*, \tag{1.32}$$

where $u, v \in (-\infty, \infty)$. The metric now takes the Eddington-Finkelstein form

$$g = -F(r)dudv + r^2 d\gamma_k^2. \tag{1.33}$$

Through the chain rule we see the following relationships

$$-2r_u = 2r_v = F. \tag{1.34}$$

The metric still degenerates at $r_{+,k}$, to remove this degeneration consider the change of coordinates given by

$$U = -\exp\left(-\frac{u}{2}\cdot F'(r_{+,k})\right), \quad V = \exp\left(\frac{v}{2}\cdot F'(r_{+,k})\right). \tag{1.35}$$

where $U \in (-\infty, 0)$, $V \in (0, \infty)$. The metric now takes the form

$$g = \frac{-4}{(F'(r_{+,k}))^2} F(r) e^{-F'(r_{+,k}) r_*(U,V)} dU dV + r^2 d\gamma_k^2. \tag{1.36}$$

We can now extend the spacetime so that $U, V \in (-\infty, \infty)$. Computing the limit

$$\lim_{r \to r_{+,k}} F(r) e^{-F'(r_{+,k}) r_*} = \lim_{r \to r_{+,k}} \frac{F(r)}{r - r_{+,k}} = \lim_{r \to r_{+,k}} \left(\frac{r}{l^2} + \frac{r_{+,k}}{l^2} + \frac{r_{+,k}^2}{l^2 r} + \frac{k}{r_{+,k}} \right) = \frac{3r_{+,k}^2 + kl^2}{l^2 r_{+,k}} > 0. \tag{1.37}$$

We see the metric no longer degenerates at the horizon. The function $r(U, V)$ is now defined implicitly through the relation

$$UV = -e^{F'(r_{+,k}) r_*(r(U,V))}. \tag{1.38}$$

We note that the sets $\{U = 0\}$ and $\{V = 0\}$ correspond to the set $\{r = r_{+,k}\}$ the event horizons. Now through the symmetry $(U, V) \mapsto (-U, -V)$ we see now that $r = const$ curves have two solutions. So we deduce that the spacetime has two regions containing a singularity and two causally disconnected exterior regions. This results in the Penrose diagram

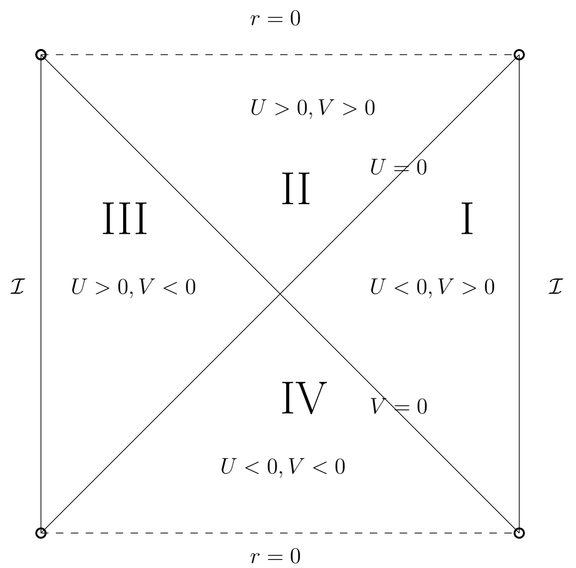


Figure 1.3: Kruskal extension of the Kottler solution.

Region I is the exterior of the black hole. Region II is the black hole, in the sense that all causal curves remained trapped in this region. Region III is also an exterior region and is isometric to I but causally disconnected. In region IV we find that all causal curves must leave this region in finite affine time, we refer to it as a white hole. This extension is maximal in the sense that geodesics cannot be continued into other regions.

Extended Eddington-Finkelstein chart

As curious as regions III and IV of the maximal extension are, they would seem to be unphysical. Furthermore, the metric given by (1.36), is poorly suited for identifying that the metric is indeed asymptotically AdS. For later applications in chapter three we remedy this by going back to the form in (1.33). Recall these coordinates cover region I but degenerate at $r = r_{+,k}$. To fix this we choose a surface $\{V = V_0 > 0\}$ and denote its past intersection with $\mathcal{I}$ by (u_0, V_0). We now make the coordinate transformation in u along this surface given by the solution to the ODE

$$\frac{d\hat{u}}{du} = \frac{\left(k - \frac{2M}{r} + \frac{r^2}{l^2}\right)}{k + \frac{r^2}{l^2}}, \tag{1.39}$$

with $\hat{u}(u_0) = u_0$. This metric is now regular at $r = r_{+,k}$ and the coordinates cover the region shown in the Penrose diagram

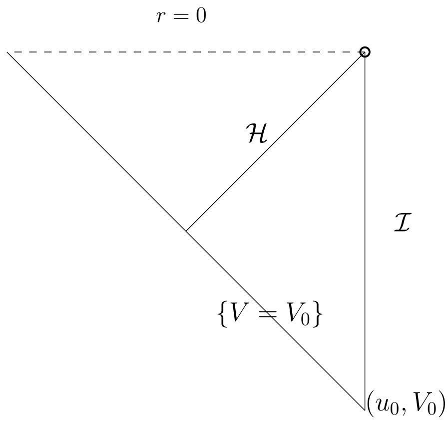


Figure 1.4: Penrose diagram for regularised Eddington Finkelstein chart.

We now have that the radial function satisfies

$$-2r_{\hat{u}} = \left(k + \frac{r^2}{l^2}\right), \tag{1.40}$$

along this surface.

We can see through the chain rule that specifying r to satisfy these relations on this ray is equivalent to fixing the u coordinate along it.

Gullstrand-Painlevé chart

An alternative to using the tortoise coordinate to remove the degeneration is to change the time coordinate. This is done in the Gullstrand-Painlevé coordinate system.

Defining

$$t_* = t - f(r), \tag{1.41}$$

with

$$f'(r) = -\frac{2M}{r\left(k + \frac{r^2}{l^2}\right)F^2}, \tag{1.42}$$

then the metric (1.26) takes the non-diagonal form

$$g = -Fdt_*^2 + \frac{4M}{r}\frac{1}{k + \frac{r^2}{l^2}}drdt_* + \frac{\left(k + \frac{2M}{r} + \frac{r^2}{l^2}\right)}{\left(k + \frac{r^2}{l^2}\right)^2}dr^2 + r^2 d\gamma_k^2. \tag{1.43}$$

We see that at $r = r_{+,k}$ that there is no degeneration of the metric. The coordinates cover the region given in the following Penrose diagram

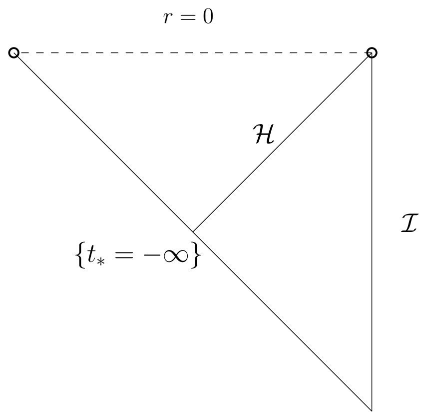


Figure 1.5: Penrose diagram for Gullstrand-Painlevé chart.

This chart will be of use in chapter two.

1.6.2 Linear waves on AdS

Due to the lack of global hyperbolicity of aAdS spacetimes the wellposedness of the wave equation is now non-trivial. On AdS if one considers the Klein-Gordon equation

$$\Box_g \psi - \frac{2a}{l^2}\psi = 0, \tag{1.44}$$

as in [BF82], and exploits the $SO(2,3)$ symmetries of the spacetime, the equation separates. The radial component of the field has a regular singular point at $\mathcal{I}$. It locally admits the expansion

$$\psi(t,r,\theta,\phi) = \frac{1}{r^{\beta^+}}\left(\psi^+(t,\theta,\phi) + \mathcal{O}\left(r^{-2}\right)\right) + \frac{1}{r^{\beta^-}}\left(\psi^-(t,\theta,\phi) + \mathcal{O}\left(r^{-2}\right)\right), \tag{1.45}$$

where $\beta^\pm = -\frac{3}{2} \pm \underbrace{\sqrt{\frac{9}{4} + 2a}}_{=:\kappa}$. Restricting to the range $-\frac{9}{8} < a < 0$, both of the branches decay towards $\mathcal{I}$. For a wellposed problem, conditions on the functions $\psi^\pm$ at $\mathcal{I}$ must be imposed. The case $\psi^- = 0$ would be the analogue of homogeneous Dirichlet boundary conditions. $\psi^+ = 0$ would correspond to homogeneous Neumann boundary conditions. Combinations such as $\psi^+ + \beta\psi^- = 0$, where β is a function along $\mathcal{I}$, correspond to Robin boundary conditions. The latter two boundary conditions are only wellposed in the range $-\frac{9}{8} < a < -\frac{5}{4}$. The conditions on a have become known as the Breitenlohner-Freedman bounds. Using the vector field method one can see that that the energy (or renormalised energy in the Neumann case), is conserved for these boundary conditions. As these are reflective boundary conditions there is no dissipation at $\mathcal{I}$. There are no obvious decay mechanisms in the spacetime and in general for these boundary conditions one cannot expect decay. This is confirmed in [BF82] where time periodic solutions are constructed. It is for this reason it is suspected that pure AdS is not stable in the full non-linear problem. Wellposedness of (1.44) at H^2 regularity was extended to aAdS spacetimes with Dirichlet boundary conditions in [Hol12]. This was then extended in [War13] to a H^1 level of regularity, but also to a much wider class of boundary conditions, such as inhomogeneous Dirichlet/Neumann as well as Robin. The core difficulty of moving from Dirichlet homogeneous boundary conditions to the others can be seen in (1.45). If $\psi^- \neq 0$ then the slowly decaying branch of the solution is still present. Attempting to use $X = \partial_t$ in the vector field method one finds the energy flux is infinite on the surface $\mathcal{I}$. For all discussed boundary conditions, with the exception of homogeneous Dirichlet, a renormalisation of the energy is required. In [BF82] this was done by considering the tensor

$$\overline{\mathbb{T}_{\mu\nu}} = \mathbb{T}_{\mu\nu} + \kappa \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu + R_{\mu\nu} \right) \psi^2, \tag{1.46}$$

which satisfies

$$\nabla_\mu \overline{\mathbb{T}^\mu{}_\nu} = \frac{\kappa}{2} \left(\partial_\nu R \right) \psi^2, \tag{1.47}$$

for pure AdS this will give a conserved renormalised energy. This new energy is related to the classical Killing energy, differing only by a surface term (which is diverging for the classical energy). This idea was exploited further in [War13] where the problem is rephrased in terms of 'twisted derivatives'. Formally, one takes a function that captures the radial decay of the field f, and then defines a new derivative operator by

$$\tilde{\nabla}_\mu \psi := f \nabla_\mu \left(f^{-1} \psi \right). \tag{1.48}$$

Expressing the problems in terms of twisted derivatives one finds that the new energy arising from the T Killing field is finite at $\mathcal{I}$. From here using standard techniques adapted to twisted Sobolev spaces, one can show wellposedness in the class $\underline{H}^1$ (the twisted equivalent of H^1).

The problem of boundedness and decay of linear waves on asymptotically AdS spacetimes is more complex than in the asymptotically flat counter part. This is because the natural boundary conditions (homogeneous Dirichlet or Neumann) don't allow for the wave to disperse at $\mathcal{I}$. In the case of pure AdS the existence of time periodic solutions means that we can't expect decay in general. In order to add a decay mechanism to the problem one can consider spacetimes with a black hole. One would expect that energy would fall through the horizon providing a decay mechanism. In the cases of Robin boundary conditions, one might expect that the energy may not even be bounded, as the Robin function may be permitting energy to enter the system through $\mathcal{I}$. Energy boundedness for slowly rotating Kerr-AdS (and thus including Spherical Schwarzschild-AdS) with Dirichlet boundary conditions was first proven in [Hol10a], which used the vector field method, coupled with Hardy estimates to compensate for the fact that $\mathbb{T}_{\mu\nu}$ no longer satisfies the dominant energy condition. The problem of the other boundary conditions was addressed in [HW14]. It was shown that in the case of AdS Schwarzschild for Dirichlet, Neumann, and Robin boundary conditions (either positive time independent Robin function, or a negative Robin constant greater than a critical value) that the energy arising from twisting is finite, non-increasing, and positive definite. This result can be extended to general aAdS black holes (with positive surface gravity), providing the smallest eigenvalue of an associated bilinear form is strictly positive. It was shown that if the smallest eigenvalue was the only one with negative sign, then there are solutions with energy growing faster than any power of t. Finally in the case of Kerr-AdS, there exists boundary data depending on the black holes parameters which gives rise to a linear hair solution (i.e. a non-trivial stationary solution).

As for decay of linear waves, in the paper [HS13a] Holzegel and Smulevici, show that under: the Hawking-Reall bound $r_+^2 > |\alpha|\, l^2$ [HR99], slowly rotating Kerr AdS, or restrictions on the Klein-Gordon mass, then the H^1 energy of a field arising from H^2 data decays logarithmically in time. Compared to the results for asymptotically flat Schwarzschild and Kerr, this is notably slower. The cause of this slower decay can be understood from the spherical Schwarzschild case. Much like the asymptotically flat case there is a photon sphere at $r = 3M$. However the asymptotically AdS end coupled with the boundary conditions provides a type of reflective barrier for null geodesics. In this setting, a stable trapping like phenomenon occurs and the only way the waves are able to decay is due to tunnelling through a potential barrier. Another barrier for decay in the Kerr AdS setting is due to superradiance. If the Hawking-Reall bound is violated this effect is present in Kerr AdS and corresponds to waves being amplified in an ergoregion. Dold [Dol17b] showed in this setting it is possible to construct mode solutions to (1.44) on Kerr-AdS with homogeneous Dirichlet or Neumann boundary conditions which grow exponentially. This provides evidence that Kerr-AdS is also likely unstable, in the superradiant regime.

An alternative problem for studying waves on AdS was undertaken in [HLSW15]. In this setting one considers dissipative boundary conditions. This adds an alternative decay mechanism where some of the energy of the wave leaves the spacetime through $\mathcal{I}$. In the case of the conformal Klein-Gordon equation ($a = -1$), Maxwell's equations, and the Bianchi equations boundedness of the energy was established, along with a Morawetz estimate degenerating at $\mathcal{I}$. This was then extended to a non-degenerate energy estimate with a derivative loss. It was further shown that some derivative loss was necessary using the Gaussian beam method of [Sbi15]. These results have led to the belief that under an appropriate formulation of the non-linear problem, AdS is stable if dissipative boundary conditions are imposed.

In chapter two of this thesis, we will consider the problem of linear waves on the toroidal AdS Schwarzschild black hole. The key contrast to the spherical setting is best expressed in terms of the null geodesics of the space time. Considering the geodesic equations one can see the radial coordinate obeys the equation

$$E^2 = \dot{r}^2 - \underbrace{\frac{d^2}{r^2}\left(k - \frac{2M}{r} + \frac{r^2}{l^2}\right)}_{V(r)}, \tag{1.49}$$

where E and d are some quantities relating to the integrals of motion. Dot denotes differentiation with respect to an affine parameter. Studying the null geodesics leads to the equation

$$0 = V'(r) = \frac{2d^2}{r^4}(3M - kr)\,. \tag{1.50}$$

We sketch the potential in Figure 1.6 and we see that there are only trapped null geodesics in the spherical setting ($r = 3M$).

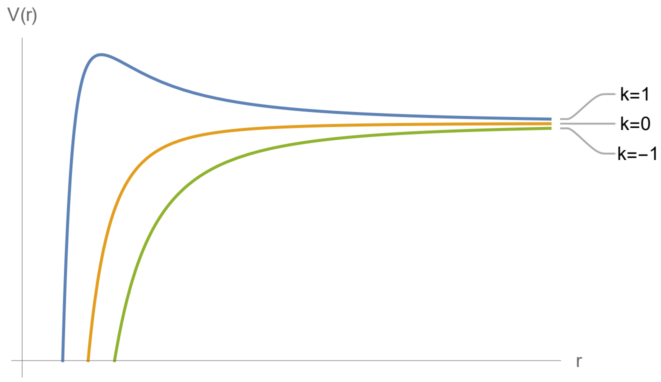


Figure 1.6: Plots of $V(r)$ for the values of k.

Loosely one can think of the sign of k representing either an attractive (negative) or repulsive (positive) force for the null geodesics. Approximating solutions of wave equations with Gaussian beams [Sbi15], and recalling that for AdS spacetimes with Dirichlet/Neumann boundary conditions there isn't naturally a dissipative effect at $\mathcal{I}$, the plots indicate that for large r it could take a photon a long time to fall into the black hole. This would be an obstacle for decay. In the case of $k = 1$ with the lack of dissipation at $\mathcal{I}$ we can interpret this confining effect in the spirit of stable trapping. This provides a heuristic interpretation of the results in [HS13a]. In the case of $k = 0$ there is no photon sphere and the stable trapping-like effect appears to be absent. One would expect a faster decay rate for the waves, and the quantification of such a rate would be the first step in proving a non-linear stability result for an aAdS black hole.

In section 2.2 the IBVP is formulated, and wellposedness is established at a $\underline{H}^1$ level for Dirichlet, Neumann, and Robin boundary conditions using the results from [War13]. In section 2.3, using the vector field method but on an adapted version of the twisted energy momentum tensor the problem of bounded energy from [Dun14] is revisited. Then in section 2.4, we use radial vector fields as multipliers and modified energy currents to construct Morawetz estimates at a $\underline{H}^1$ level. The degeneration lies only on the tangential and time derivatives of the field as it approaches $\mathcal{I}$. Through commuting the equation with ∂_t, a non-degenerate integrated decay estimate is established, and t^{-1} decay is proven for $\underline{H}^2$ regular initial data. Faster polynomial decay is shown for more regular data. Finally in section 2.6 we establish through a Gaussian beam method that some loss of derivative is necessary. This is done by constructing null geodesics which remain outside the event horizon for arbitrarily long coordinate time. These fast decay rates show that more investigation into the non-linear stability is warranted.

1.6.3 Symmetric spacetimes

In much the same way as in the asymptotically flat case, if one wants to study a 1+1 system of Einstein's equations by imposing spherical symmetry, a Birkhoff theorem [SW10] forces one to consider a non-trivial matter model for there to be any dynamics. Analogous to the asymptotically flat case the simplest such matter field one can consider is given by the Klein-Gordon equation. The first problem of studying this system is to prove it is locally wellposed near $\mathcal{I}$. In a double null coordinate system the metric takes the form

$$g = -\Omega^2(u,v)dudv + r^2(u,v)d\mathbb{S}^2, \tag{1.51}$$

where $r(u,v)$ is the area radius of the sphere at point (u,v). The system then reduces to the Einstein–Klein-Gordon system

$$\partial_u \left(\frac{r_u}{\Omega^2}\right) = -4\pi r \frac{\psi_u^2}{\Omega^2}, \tag{1.52}$$

$$\partial_v \left(\frac{r_v}{\Omega^2}\right) = -4\pi r \frac{\psi_v^2}{\Omega^2}, \tag{1.53}$$

$$r_{uv} = -\frac{r_u r_v}{r} + \frac{2a\pi}{l^2} r\psi^2\Omega^2 - \frac{3}{4l^2} r\Omega^2 - \frac{\Omega^2}{4r}, \tag{1.54}$$

$$(\log \Omega)_{uv} = -4\pi\psi_u\psi_v + \frac{r_u r_v}{r^2} + \frac{\Omega^2}{4r^2}, \tag{1.55}$$

$$\psi_{uv} = -\frac{r_v}{r}\psi_u - \frac{r_u}{r}\psi_v - \frac{a}{2l^2}\Omega^2\psi. \tag{1.56}$$

This was first studied in [HS12] where local wellposedness was proven for Dirichlet boundary conditions at a H^2 level of regularity. As the spacetime is expected to be aAdS, the variables are expected to become singular on the boundary $\mathcal{I}$. At the expense of clear geometric interpretations an equivalent renormalised system is studied. Furthermore it is fruitful to consider a new dynamical variable; the Hawking mass

$$\varpi = \frac{r}{2}\left(1 + \frac{4r_u r_v}{\Omega^2}\right) + \frac{r^3}{2l^2}. \tag{1.57}$$

It satisfies the cleaner boundary conditions $\varpi|_{\mathcal{I}} = M$, and is invariant under a change of null coordinates, unlike Ω^2. From (1.52), (1.53) (the Raychaudhuri equations) and (1.54) it can be seen that this variable satisfies

$$\partial_u \varpi = -8\pi r^2 \frac{r_v}{\Omega^2}\psi_u^2 + \frac{4\pi r^2 a}{l^2} r_u \psi^2, \tag{1.58}$$

$$\partial_v \varpi = -8\pi r^2 \frac{r_u}{\Omega^2}\psi_v^2 + \frac{4\pi r^2 a}{l^2} r_v \psi^2. \tag{1.59}$$

These equations can then be used to replace (1.52), and (1.53) for Ω^2. Furthermore, the transport equations imply (after a Hardy inequality) that ϖ forms a potential for the H^1 energy of ψ. The authors show that to construct initial data one needs to provide r and ψ on constant v null ray. As initial data sets for (1.1) must obey constraint equations, the other variables are determined by integrating transport equations. Furthermore, fixing r on a $v = const$ ray is equivalent to specifying the u coordinate along the ray and thus the only free data is the field ψ. This shows the advantage of working in the double null gauge. A contraction map argument is then used to prove the existence of a solution. Higher regularity is then shown *a posteriori*. The authors also provide further results for global analysis of the solution. A unique maximal development of the solution is shown to exist. Crucially, the geometric invariance of the boundary conditions is required to prove uniqueness. Extension principles are also proven: firstly in the regions near $\mathcal{I}$ where one needs future control over various quantities of the solution, and secondly in the regions of bounded, non-zero r, and finite spacetime volume. This extension principle is very much in the spirit of [Kom13] allowing one to fill in the spacetime away from $\mathcal{I}$, and describe first singularity formation. Due to the work of [War13] it is possible to pose the problem at a lower level of regularity, and with other boundary conditions. In the paper [HW13], Holzegel and Warnick show that the system is locally wellposed for a wider class of Dirichlet, Neumann, and Robin boundary conditions. The same energy flux issues appear in this problem, and the renormalisation scheme needs to be phrased in terms of twisted derivatives. Another renormalisation of the Hawking mass is also required. Integrating along characteristics and using energy estimates a contraction map argument can be formulated. The result is then extended to consider non-linear potentials for the field (i.e. self interacting fields), which is of interest in exploring the dynamics of hairy black holes. For smoother initial data, H^2 solutions are also constructed.

The stability of spherical Schwarzschild AdS, under spherical symmetry, with Dirichlet boundary conditions was proven in [HS13b]. Establishing orbital stability in this setting is non-trivial as the energy conditions of [Daf05] no longer hold. The key idea of using the Hawking mass as a potential is still retained, however the quantity is no longer monotonic. It is however coercive in the regular region of the spacetime (in an integrated sense after establishing Hardy inequalities). Proving this requires a complicated bootstrap argument on the location of the horizon. The coercivity is then used to show that the Hawking mass is indeed bounded (for small initial data). Coupled with a redshift argument, the H^1 norm along with pointwise norms of the field, and metric functions, are shown to be bounded by initial data. Then using a vector field argument, a non-degenerate Morawetz estimate is constructed for the range $a \geq -1$ showing that the fields are decaying exponentially to zero. Commuting the equations with $\mathcal{T}$, the Kodama vector field [Kod80], higher regularity estimates are shown, and the completeness of null infinity can be shown from the extension principle.

Motivated by these results, in chapter three we will consider the stability of toroidal AdS Schwarzschild black holes, for homogeneous Dirichlet, and Neumann boundary conditions at a $\underline{H}^1$ level of regularity. The main interest being in if these results can be established in spite of the slower decay from the Neumann conditions. The toroidal setting has an advantage over spherical. This is that the 'right' twisting function is simpler. As previously stated, proving orbital stability in the non-linear problem is similar to boundedness in the linear problem. The twisting function for the spherical problem from [HW14], is comparatively complex to the one used in chapter two. The latter having a nicer relation to the standard derivative operator. This simpler form is then exploited to see monotonicity properties.

In section 3.3 we state the problem and discuss the symmetry reduction. Wishing to maintain a similar system to the spherical setting, a metric ansatz of the form

$$g = -\Omega^2(u,v)dudv + r^2(u,v)\left(dx^2+dy^2\right), \tag{1.60}$$

is posed. The topology is the product manifold $\mathcal{Q}^+ \times \mathfrak{T}^2$. Where $\mathcal{Q}^+$ is a two dimensional Lorentzian manifold with boundary, and $\mathfrak{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Geometrically we are imposing a flat square toroidal symmetry on our solution (see definition 3.3.2). In this symmetry reduction we see that the toroidal Einstein–Klein-Gordon system takes the form

$$\partial_u\left(\frac{r_u}{\Omega^2}\right) = -4\pi r\frac{(\partial_u\psi)^2}{\Omega^2}, \tag{1.61}$$

$$\partial_v\left(\frac{r_v}{\Omega^2}\right) = -4\pi r\frac{(\partial_v\psi)^2}{\Omega^2}, \tag{1.62}$$

$$r_{uv} = -\frac{r_u r_v}{r} + \frac{2\pi a r}{l^2}\Omega^2\psi^2 - \frac{3}{4}\frac{r}{l^2}\Omega^2, \tag{1.63}$$

$$(\log\Omega)_{uv} = -4\pi\partial_u\psi\partial_v\psi + \frac{r_u r_v}{r^2}, \tag{1.64}$$

$$\partial_u\partial_v\psi = -\frac{r_u}{r}\psi_v - \frac{r_v}{r}\psi_u - \frac{\Omega^2 a}{2l^2}\psi. \tag{1.65}$$

We can now see that these equations are similar to (1.52) - (1.56). The differing terms being in (1.63) and (1.64). The terms are sub-leading in powers of r. In section 3.4 we discuss that after a suitable modification to the renormalised Hawking mass of [HW13], the same proof of wellposedness carries through. In section 3.4.3, geometric uniqueness of the solution is shown.

In section 3.5, we prove the analogues of the extension results from [HS12]. As the regularity of the system is now posed at a lower level, the conditions for extension near $\mathcal{I}$ are now simpler (only requiring $\underline{H}^1$ regularity to propagate). The local extension result is also similar, however as ψ_v is now only in L^2, appropriate modifications need to be made for local wellposedness and the extension principle in the interior. This is included in appendix A and generalises for all Λ, k, a.

Smallness of the initial data is vital for the stability results of chapter three. In section 3.6 we quantify this smallness in terms of norms. We then use this to show the maximal development contains a black hole region.

In section 3.7 we prove orbital stability for solutions arising from small initial data. This is done by exploiting monotonicity of a new variable that we will call the final renormalised Hawking mass. To illustrate why we do this, first consider the natural analogue of the Hawking mass as in [HS12]

$$\varpi_1 = \frac{2r r_u r_v}{\Omega^2} + \frac{r^3}{2l^2}. \tag{1.66}$$

For Neumann boundary conditions this leads to the expected divergent energy fluxes. As in [HW13] this variable needs to be renormalised. If we consider the renormalisation used for local existence in [HW13]

$$\varpi_2 = \varpi_1 - 2\pi \left(-\frac{3}{2} + \kappa\right) \frac{r^3}{l^2}\psi^2, \tag{1.67}$$

then the energy fluxes remain finite. However the equations now satisfy

$$\begin{aligned} \partial_u \varpi_2 = &-8\pi r^2 \frac{r_v}{\Omega^2}(\tilde{\nabla}_u \psi)^2 - 8\pi \left(-\frac{3}{2} + \kappa\right) \left(\varpi_2 + 2\pi \left(-\frac{3}{2} + \kappa\right) \frac{r^3}{l^2}\psi^2\right) \psi \tilde{\nabla}_u \psi \\ &- 4\pi \psi^2 r_u \left(-\frac{3}{2} + \kappa\right)^2 \frac{\left(\varpi_2 + 2\pi \left(-\frac{3}{2} + \kappa\right) \frac{r^3}{l^2}\psi^2\right)}{r}, \end{aligned} \tag{1.68}$$

$$\begin{aligned} \partial_v \varpi_2 = &-8\pi r^2 \frac{r_u}{\Omega^2}(\tilde{\nabla}_v \psi)^2 - 8\pi \left(-\frac{3}{2} + \kappa\right) \left(\varpi_2 + 2\pi \left(-\frac{3}{2} + \kappa\right) \frac{r^3}{l^2}\psi^2\right) \psi \tilde{\nabla}_v \psi \\ &- 4\pi \psi^2 r_v \left(-\frac{3}{2} + \kappa\right)^2 \frac{\left(\varpi_2 + 2\pi \left(-\frac{3}{2} + \kappa\right) \frac{r^3}{l^2}\psi^2\right)}{r}, \end{aligned} \tag{1.69}$$

where the twisting is done with respect to $r^{-\frac{3}{2}+\kappa}$. There is no obvious monotonicity to these equations or even integrated coercivity. In order to fix this, we consider another renormalisation of the Hawking mass. The idea is to iteratively use the product rule on the cross terms involving $\psi \tilde{\nabla}_\mu \psi$ and $\varpi \psi \tilde{\nabla}_\mu \psi$. At each iteration we move the total derivative term to the LHS and define a new Hawking mass. This generates a series which can be summed to

$$\varpi = \frac{2 r_u r_v r}{\Omega^2} e^{4\pi\left(-\frac{3}{2}+\kappa\right)\psi^2} + \frac{r^3}{2l^2}, \tag{1.70}$$

which satisfies

$$\begin{aligned} \partial_u \varpi &= -\frac{8\pi r^2 r_v}{\Omega^2}(\tilde{\nabla}_u \psi)^2 e^{4\pi\left(-\frac{3}{2}+\kappa\right)\psi^2} + \frac{4\pi \left(-\frac{3}{2} + \kappa\right)^2 r_u}{r} \varpi \psi^2 + \frac{r_u r^2}{l^2} f(\psi^2), \\ \partial_v \varpi &= -\frac{8\pi r^2 r_u}{\Omega^2}(\tilde{\nabla}_v \psi)^2 e^{4\pi\left(-\frac{3}{2}+\kappa\right)\psi^2} + \frac{4\pi \left(-\frac{3}{2} + \kappa\right)^2 r_v}{r} \varpi \psi^2 + \frac{r_v r^2}{l^2} f(\psi^2). \end{aligned} \tag{1.71}$$

where $f(\psi^2)$ are higher order terms of ψ. From here establishing that $\varpi > 0$ shows monotonicity, providing $f \geq 0$ (which we will show follows for $\psi^2 \ll 1$). In establishing this, we use a bootstrap argument for the magnitude of the field. This is arguably simpler that the bootstrap argument of [HS12] as bootstrapping on the location of the horizon is no longer needed. Coupled with a redshift argument one finds that the $\underline{H}^1$ energy of the field is controlled by the initial data. This allows the bootstrap argument to be closed from the smallness of the initial data. We then establish further estimates for the field and metric functions, and the orbital stability follows for $\kappa \in (0, \frac{1}{2}]$. Then in section 3.8, we use vector field arguments to show exponential decay of the field, and a Penrose inequality for $\kappa \in (0, \frac{1}{2})$. This establishes the stability of the toroidal Schwarzschild black hole within this symmetry class. It is worth contrasting this result to [HS13b] as under suitable modifications to that argument one expects that the toroidal symmetry result would hold at a H^2 level for Dirichlet boundary conditions.

This result is stronger in two senses. Firstly much rougher initial data can now be considered. Secondly the scale of the perturbation can be thought of as larger (in a Sobolev sense). The norms of [HS13b], would diverge for a subset of data admissible by these methods.

Finally in section 3.10 we show a vacuum result. Moving from constant square toroidal symmetry to varying rectangular, an extra field $B(u,v)$ can be introduced. Posing a metric of the form

$$g = -\Omega^2(u,v)dudv + r^2(u,v)\left(e^{-\sqrt{8\pi}B(u,v)}dx^2 + e^{\sqrt{8\pi}B(u,v)}dy^2\right), \tag{1.72}$$

and considering the vacuum system (1.2), which reduces to (1.61) - (1.65), where $B = \psi$, and $a = 0$. A massless scalar field system. However from the work of [War13] this has currently not been shown to be wellposed with Neumann boundary conditions, and the main result of chapter three cannot be directly applied here. However we will have shown that many of the key results from [HS12] and [HS13b] can be applied to the Dirichlet case, and this system is stable within that regime. This shows a curious, intimate link between dynamics of the scalar field and the vacuum system, within this symmetry setting.

2

The Klein-Gordon Equation on the Toroidal AdS Black Hole

2.1 The Results

In this chapter we discuss the Klein-Gordon equation on the toroidal AdS Schwarzschild black hole (TAdSS), first seen in [DW16]. The goal of this chapter will be to prove a precise version of four results. For brevity in the statements we make reference to some quantities that will be later defined. Firstly let ψ be a solution to the Klein-Gordon equation given by (2.23), satisfying suitable Dirichlet, Neumann, or Robin boundary conditions. $\mathcal{E}[\psi]$ is a non-degenerate energy density of ψ defined in (2.57).

The first result is a Morawetz estimate that holds at a purely $\underline{H}^1$ level of regularity. It forms the basis of energy decay estimates, and gives indication to a trapping like issue occurring in the tangential directions.

Theorem 2.1.1 (Morawetz estimate)**.** *For $T_1 < T_2$, there exists an open set of Klein-Gordon masses, such that the following estimate holds*

$$\int_{\{T_1<t<T_2\}} \frac{1}{r^3}\mathcal{E}[\psi]drdxdydt \leq C_{M,l,\kappa}\int_{\{t=T_1\}}\mathcal{E}[\psi]drdxdy, \tag{2.1}$$

where $C_{M,l,\kappa} > 0$.

Exploiting the staticity of the spacetime and commuting with the Killing field $T = \partial_t$ we extend the Morawetz estimate to a full integrated decay estimate.

Theorem 2.1.2 (Integrated decay estimate with derivative loss)**.** *For $T_1 < T_2$, there exists an* *open set of Klein-Gordon masses, such that the following estimate holds*

$$\int_{\{T_1<t<T_2\}} \mathcal{E}[\psi] drdxdydt \leq C_{M,l,\kappa} \int_{\{t=T_1\}} \left(\mathcal{E}[\psi] + \mathcal{E}[\psi_t]\right) drdxdy, \tag{2.2}$$

where $C_{M,l,\kappa} > 0$.

A corollary of theorem 2.1.2 is polynomial decay of the field.

Corollary 2.1.1 (Polynomial decay of energy)**.** *For $T_1 < T$, there exists an open set of Klein-Gordon masses, such that the following estimate holds*

$$\int_{\{t=T\}} \mathcal{E}[\psi] drdxdy \leq \frac{C}{(1+T)^n} \sum_{k=0}^{n} \int_{\{t=T_1\}} \mathcal{E}[\partial_t^k \psi] drdxdy, \tag{2.3}$$

where $C_{n,M,l,\kappa} > 0$.

Finally we show that some degeneration in estimate (2.1) is necessary.

Theorem 2.1.3 (Necessity of derivative loss)**.** *There exists no constant $C > 0$, independent of T, such that the estimate*

$$\int_{\{0<t<T\}} \mathcal{E}[\psi] drdxdydt \leq C \int_{\{t=0\}} \mathcal{E}[\psi] drdxdy, \tag{2.4}$$

holds for all smooth solutions of (2.23)*.*

2.2 The spacetime and the Klein-Gordon equation

2.2.1 The toroidal AdS Schwarzschild black hole

Fix $M, l > 0$ and define the value $r_+ := (2Ml^2)^{\frac{1}{3}}$, let $\mathfrak{T}^2$ denote the two dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. The exterior of the toroidal AdS Schwarzschild black hole is then defined to be the manifold with boundary given by

$$\mathcal{M} = \mathbb{R}_{t\geq 0} \times \mathbb{R}_{r\geq r_+} \times \mathfrak{T}^2, \tag{2.5}$$

with Lorentzian metric

$$g = -\left(\frac{-2M}{r} + \frac{r^2}{l^2}\right) dt^2 + \frac{4Ml^2}{r^3} dtdr + \left(\frac{2Ml^4}{r^5} + \frac{l^2}{r^2}\right) dr^2 + r^2\left(dx^2 + dy^2\right). \tag{2.6}$$

Here we use Gullstrand–Painlevé coordinates as discussed in section 1.6.1. The parameter M represents the mass of the black hole, l the AdS radius. The set

$$\mathcal{H} = \{(t, r, x, y) \in \mathcal{M} : r = r_+\}, \tag{2.7}$$

forms the event horizon of this spacetime. The cometric takes the form

$$g^{-1} = -\left(\frac{2Ml^4}{r^5} + \frac{l^2}{r^2}\right)\partial_t^2 + \frac{4Ml^2}{r^3}\partial_t\partial_r + \left(\frac{-2M}{r} + \frac{r^2}{l^2}\right)\partial_r^2 + \frac{1}{r^2}\left(\partial_x^2 + \partial_y^2\right). \tag{2.8}$$

Define the function

$$\tilde{r} = \frac{1}{r}, \tag{2.9}$$

which is a boundary defining function for the null infinity $\mathcal{I}$ of the spacetime. We attach the set $\{\tilde{r} = 0\}$ as a boundary. Formally

$$\mathcal{I} = \{(t, r, x, y) \in \overline{\mathcal{M}} : \tilde{r} = 0\}. \tag{2.10}$$

2.2.2 Hypersurfaces

Throughout this chapter we will make extensive use of the divergence theorem. As this relates 4-volume integrals to flux integrals across hypersurfaces it will be helpful to define various geometric quantities related to this specific spacetime. This set up is similar to the spherical problem in [HW14] but has been adapted to the toroidal setting.

We introduce the following slab of spacetime

$$\mathcal{M}_{[T_1,T_2]} = \mathcal{M} \cap \{t \in [T_1, T_2]\}, \tag{2.11}$$

which has volume form

$$dV = r^2 dt dr dx dy =: r^2 d\eta. \tag{2.12}$$

We will denote surfaces of constant t by Σ_t. These surfaces have future directed unit normal given by

$$n = \sqrt{\frac{2l^4M}{r^5} + \frac{l^2}{r^2}}\partial_t - \frac{2lM}{\sqrt{2l^4M + r^4}}\partial_r, \tag{2.13}$$

and surface measure

$$dS_{\Sigma_t} = \sqrt{l^2\left(\frac{2l^2M}{r} + r^2\right)} dr dx dy. \tag{2.14}$$

We see that Σ_t is a regular spacelike surface up to and including the horizon.

We define a subset of this surface by

$$\Sigma_t^{[R_1,R_2]} = \Sigma_t \cap \{r \in [R_1, R_2]\}. \tag{2.15}$$

We also denote the surfaces of constant r by $\hat{\Sigma}_r$. These surfaces have unit normal given by

$$m = \frac{2l^2M}{r^3\sqrt{\frac{-2M}{r} + \frac{r^2}{l^2}}}\partial_t + \sqrt{\frac{-2M}{r} + \frac{r^2}{l^2}}\partial_r, \tag{2.16}$$

and surface measure

$$dS_{\hat{\Sigma}_r} = r^2\sqrt{-\frac{2M}{r}+\frac{r^2}{l^2}}drdxdy, \tag{2.17}$$

while m becomes singular, and $dS_{\hat{\Sigma}_r}$ degenerates as we approach the null surface $\mathcal{H}$, we note that the combination $m^\mu dS_{\hat{\Sigma}_r}$ is well defined on this surface, and gives the appropriate surface measure.

We again define a subset of this surface by

$$\hat{\Sigma}_t^{[T_1,T_2]} = \hat{\Sigma}_r \cap \{r \in [T_1, T_2]\}. \tag{2.18}$$

We will also define for fixed t and r the associated torus

$$\mathfrak{T}^2_{t,r} = \Sigma_t \cap \hat{\Sigma}_r. \tag{2.19}$$

This surface has induced measure

$$dS_{\mathfrak{T}^2_{t,r}} = r^2 dxdy. \tag{2.20}$$

The Divergence Theorem

We now state the divergence theorem for this spacetime.

Let

$$D = \{(t,r,x,y) \in \mathcal{M} \quad | \quad t \in [T_1,T_2], r \in [R_1,R_2]\}, \tag{2.21}$$

be a region of the spacetime, then for J^μ a C^1 vector field on $\mathcal{M}$, the following form of the divergence theorem holds

$$\begin{aligned}\int_D -\nabla_\mu J^\mu dV = &\int_{\Sigma_{T_2}^{[R_1,R_2]}} J_\mu n^\mu dS_{\Sigma_{T_2}} - \int_{\Sigma_{T_1}^{[R_1,R_2]}} J_\mu n^\mu dS_{\Sigma_{T_1}} \\ &+ \int_{\hat{\Sigma}_{R_1}^{[T_1,T_2]}} J_\mu m^\mu dS_{\Sigma_{R_1}} - \int_{\hat{\Sigma}_{R_2}^{[T_1,T_2]}} J_\mu m^\mu dS_{\Sigma_{R_2}}.\end{aligned} \tag{2.22}$$

We may extend D to the set $\mathcal{M}_{[T_1,T_2]}$ by taking the limits $R_1 \to r_+$ and $R_2 \to \infty$, provided they exist.

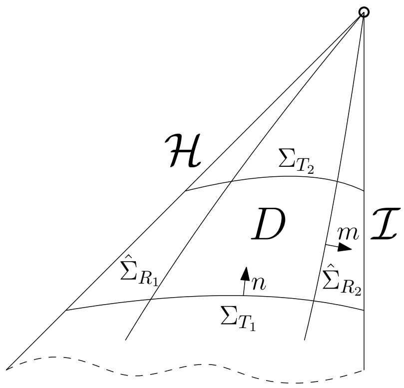


Figure 2.1: Penrose diagram of the spacetime

2.2.3 The Klein-Gordon equation

We now turn to study the Klein-Gordon equation on this spacetime. It is given by

$$\Box_g \psi - \frac{2a}{l^2}\psi = 0, \tag{2.23}$$

where the constant

$$-a < \frac{9}{8}, \tag{2.24}$$

obeys the Breitenlohner-Freedman bound [BF82]. It will be useful at this point to define the parameter

$$\kappa = \sqrt{\frac{9}{4} + 2a}, \tag{2.25}$$

the main reason for doing this will become apparent in the wellposedness section.
Expanding this equation in Gullstrand–Painlevé coordinates we see that the equation has the following form

$$\begin{aligned} &-\left(\frac{2Ml^4}{r^5} + \frac{l^2}{r^2}\right)\psi_{tt} + \frac{1}{r^2}\partial_r\left(r^2\left(\frac{-2M}{r} + \frac{r^2}{l^2}\right)\psi_r\right) + \frac{4Ml^2}{r^3}\psi_{rt} \\ &- \frac{2Ml^2}{r^4}\psi_t + \frac{1}{r^2}\Delta_{(x,y)}\psi - \frac{2a}{l^2}\psi = 0. \end{aligned} \tag{2.26}$$

we see that as $r \to r_+$ the second order radial derivatives are degenerating. Expressed in the operator form

$$-\psi_{tt} + B\psi_t + L\psi = 0, \tag{2.27}$$

where B is a first order spatial operator and L a second order spatial operator. It is clear that L is not strongly elliptic on the set $\mathcal{M}$. For this reason standard energy methods will prove insufficient to prove boundedness of the full H^1 norm. This issue can be resolved by exploiting the redshift effect for black holes [DR08], [DR09b]. Furthermore as the spacetime is not globally hyperbolic we will need to provide boundary conditions to solve this equation uniquely. With the exception of the class of Dirichlet boundary conditions the standard energy fluxes associated to the Killing field ∂_t will not be finite. In order to resolve this issue we use a renormalisation process first developed in [War13].

The Twisted Derivative

The basis of the renormalisation scheme is the twisted derivative. The core idea being once the Klein-Gordon equation is expressed with twisted derivatives, that when restricted to finite spatial domains the associated 'Killing' energy differs to the standard one by only a surface flux. It is this flux that that is diverging for the standard energy as the domain is expanded to infinity, but the twisted Killing energy remains finite, and serves as a good quantity for energy methods.
We define a twisting function f by the property

$$fr^{\frac{3}{2}-\kappa} = 1 + \mathcal{O}\left(r^{-2}\right). \tag{2.28}$$

The twisted derivative with respect to f is given by

$$\tilde{\nabla}_\mu \psi = f \nabla_\mu \left(f^{-1}\psi\right), \tag{2.29}$$

and the formal L^2 adjoint of $\tilde{\nabla}$ by

$$\tilde{\nabla}^\dagger_\mu \psi = -f^{-1}\nabla_\mu \left(f\psi\right). \tag{2.30}$$

We can bring the Klein-Gordon equation to the form

$$-\tilde{\nabla}^\dagger_\mu \tilde{\nabla}^\mu \psi - V\psi = 0, \tag{2.31}$$

where the potential function V is given by

$$V = -\left(f^{-1}\nabla_\mu \nabla^\mu f - \frac{2a}{l^2}\right). \tag{2.32}$$

2.2.4 Boundary conditions

For this section we will study three classes of boundary conditions. For $\psi \in C^1(\mathcal{M}, \mathbb{R})$ we say it obeys

- Dirichlet boundary conditions, if $\kappa > 0$ and

$$r^{\frac{3}{2}-\kappa}\psi \to 0, \text{ as } r \to \infty, \tag{2.33}$$

- Neumann boundary conditions, if $\kappa \in (0, 1)$ and

$$r^{\frac{5}{2}+\kappa}\tilde{\nabla}_r \psi \to 0 \text{ as } r \to \infty, \tag{2.34}$$

- Robin boundary conditions, if $\kappa \in (0,1)$ and

$$r^{\frac{5}{2}+\kappa}\tilde{\nabla}_r\psi + \beta r^{\frac{3}{2}-\kappa}\psi \to 0, \text{ as } r \to \infty. \tag{2.35}$$

  where $\beta \in C^1(\mathcal{I})$. We further will require that $\partial_t\beta = 0$ and $\beta \geq 0$. We remark that this constraint is not necessary for the wellposedness results.

2.2.5 Twisted Sobolev spaces, wellposedness, and asymptotics

Twisted Sobolev Spaces

Let Σ be a spacelike hypersurface extending to $\mathcal{I}$, which it intersects orthogonally. We define the norms

$$||\psi||^2_{\underline{L}^2(\Sigma)} = \int_\Sigma \frac{\psi^2}{r} dS_\Sigma, \tag{2.36}$$

$$||\psi||^2_{\underline{H}^1(\Sigma,\kappa)} = \int_\Sigma \left(\left|\tilde{\nabla}\psi\right|^2 + \frac{\psi^2}{r^2}\right) r dS_\Sigma, \tag{2.37}$$

and now define the space $\underline{H}^1_0(\Sigma,\kappa)$ as the completion of smooth functions supported away from $\mathcal{I}$. Different choices of twisting functions give rise to equivalent norms provided the function satisfies (2.28).

We now also define the renormalised energy on a constant t-slice associated to this problem by

$$\begin{aligned} E_t[\psi] =& \frac{1}{2}\int_{\Sigma_t}\left(-g^{tt}\left(\nabla_t\psi\right)^2 + g^{rr}\left(\tilde{\nabla}_r\psi\right)^2 + \left|\nabla\!\!\!/\psi\right|^2 + V(r)\psi^2\right) r^2 dr dx dy \\ &+ \frac{1}{2l^2}\int_{\mathfrak{T}^2_{t,\infty}}\left(r^{\frac{3}{2}-\kappa}\psi\right)^2 \beta dx dy. \end{aligned} \tag{2.38}$$

where $\nabla\!\!\!/$ denotes the connection of the induced metric on tori of constant t and r, (in these coordinates we see that $\left|\nabla\!\!\!/\psi\right|^2 = \frac{1}{r^2}\left(\psi_x^2 + \psi_y^2\right))$. We will often refer to $\nabla\!\!\!/\psi$ as 'tangential terms'. The later integral is understood in the limiting sense as $r \to \infty$.

Wellposedness

For completeness we summarise and state the wellposedness theorem, and asymptotic expansion as found in [HW14] (theorem 1.1 and theorem 1.2 respectively), and proven in [War13] for (2.23). We will denote by $\mathcal{D}^+(\Sigma)$ the future Cauchy development of Σ together with the subset of $\mathcal{I}$ lying in the future of Σ. Let n_Σ be the future directed unit normal of Σ and define

$$\hat{n}_\Sigma = r n_\Sigma. \tag{2.39}$$

Theorem 2.2.1. • *Let $\kappa > 0$, and $\psi_0 \in \underline{H}^1_0(\Sigma,\kappa)$, $\psi_1 \in \underline{L}^1(\Sigma)$. Then there exists a unique* *weak solution ψ to the equation*

$$\Box_g \psi + \frac{1}{l^2}\left(\frac{9}{4} - \kappa^2\right)\psi = 0, \tag{2.40}$$

in $\mathcal{D}^+(\Sigma)$ with Dirichlet boundary conditions on $\mathcal{I}$, such that $\psi|_\Sigma = \psi_0$ and $\hat{n}|_\Sigma \psi = \psi_1$. Furthermore for a spacelike surface $\mathcal{S} \subset \mathcal{D}^+(\Sigma)$ intersecting $\mathcal{I}$ orthogonally we have that $u|_\mathcal{S} \in \underline{H}^1_0(\mathcal{S}, \kappa), \hat{n}_\Sigma \psi|_\mathcal{S} \in \underline{L}^2(\mathcal{S})$.

- *Let $\kappa \in (0,1)$, $\psi_0 \in \underline{H}^1(\Sigma, \kappa)$, $\psi_1 \in \underline{L}^1(\Sigma)$. Then there exists a unique weak solution ψ to the equation*

$$\Box_g \psi + \frac{1}{l^2}\left(\frac{9}{4} - \kappa^2\right)\psi = 0, \tag{2.41}$$

in $\mathcal{D}^+(\Sigma)$ with Neumann or Robin (for given β) boundary conditions on $\mathcal{I}$, such that $\psi|_\Sigma = \psi_0$ and $\hat{n}|_\Sigma \psi = \psi_1$. Furthermore for a spacelike surface $\mathcal{S} \subset \mathcal{D}^+(\Sigma)$ intersecting $\mathcal{I}$ orthogonally we have that $\psi|_\mathcal{S} \in \underline{H}^1(\mathcal{S}, \kappa), \hat{n}_\Sigma \psi|_\mathcal{S} \in \underline{L}^2(\mathcal{S})$.

If it is the case that the renormalised energy $E_t[\psi]$ is coercive and if

- *ψ satisfies Neumann or Robin boundary conditions with $\kappa \in (0,1)$, and the initial data satisfies*

$$\sum_{i=0}^{1} \left|\left| (\hat{n}_{\Sigma_0})^i \psi \right|\right|_{\underline{H}^1(\Sigma_0,\kappa)} + \left|\left| (\hat{n}_{\Sigma_0})^2 \psi \right|\right|_{\underline{L}^2(\Sigma_0)} < \infty, \tag{2.42}$$

then ψ is locally C^0 and,

$$\sup_{\Sigma_t} \left| r^{\frac{3}{2}-\kappa} \psi \right| \leq C \left(\sum_{i=0}^{1} \left|\left| (\hat{n}_{\Sigma_0})^i \psi \right|\right|_{\underline{H}^1(\Sigma_0,\kappa)} + \left|\left| (\hat{n}_{\Sigma_0})^2 \psi \right|\right|_{\underline{L}^2(\Sigma_0)} \right), \tag{2.43}$$

where $C > 0$ is a t independent constant.

- *ψ satisfies Dirichlet boundary conditions, $\kappa > 0$, and the initial data satisfies*

$$\sum_{i=0}^{1} \left|\left| (\hat{n}_{\Sigma_0})^i \psi \right|\right|_{\underline{H}^1(\Sigma_0,\kappa)} + \left|\left| (\hat{n}_{\Sigma_0})^2 \psi \right|\right|_{\underline{L}^2(\Sigma_0)} < \infty, \tag{2.44}$$

then ψ is locally C^0 and for all $\epsilon > 0$ there exists a t independent $C_\epsilon > 0$ such that

$$\sup_{\Sigma_t} \left| r^{\frac{3}{2}-\epsilon} \psi \right| \leq C_\epsilon \left(\sum_{i=0}^{1} \left|\left| (\hat{n}_{\Sigma_0})^i \psi \right|\right|_{\underline{H}^1(\Sigma_0,\kappa)} + \left|\left| (\hat{n}_{\Sigma_0})^2 \psi \right|\right|_{\underline{L}^2(\Sigma_0)} \right). \tag{2.45}$$

By using higher order energies we can also gain uniform control over derivatives of ψ.
For sufficiently smooth initial data we have that $\psi|_\mathcal{S} \in H^k_{loc.}(\mathcal{S}), \hat{n}_\Sigma \psi|_\mathcal{S} \in H^{k-1}_{loc.}(\mathcal{S})$, and the field ψ admits the asymptotic expansion

$$\psi = r^{-\frac{3}{2}+\kappa}\left[\psi_0^- + \mathcal{O}\left(r^{-1-\kappa}\right)\right] + r^{-\frac{3}{2}-\kappa}\left[\psi_1^+ + \mathcal{O}\left(r^{\kappa-1}\right)\right], \tag{2.46}$$

where the functions $\psi_i^{\pm} \in H^{k-1-i}(\mathcal{I})$ *satisfy*

- $\psi_0^- = 0$ *for Dirichlet boundary conditions,*
- $\psi_1^+ = 0$ *for Neumann boundary conditions,*
- $2\kappa\psi_1^+ - \beta\psi_0 = 0$ *for Robin boundary conditions.*

Remark 2.2.1. *From* (2.43) *we can see that the twisting function* f *is capturing the decay of the field.*

Remark 2.2.2. *For convenience we will now assume that our initial data is chosen such that solutions to* (2.23) *are smooth. A density argument shows that this assumption can be removed later. As such we will have asymptotic expansions to all orders which will allow us to quantify the decay of the field and derivatives.*

Corollary 2.2.1. *For sufficiently smooth solutions to* (2.23) *for* $\kappa \in (0,1)$, *we have the following asymptotics*
Dirichlet:

$$\begin{aligned} &\psi = \mathcal{O}\left(r^{-\frac{3}{2}-\kappa}\right), \quad \nabla_t\psi = \mathcal{O}\left(r^{-\frac{3}{2}-\kappa}\right), \\ &\left|\nabla\!\!\!/\psi\right| = \mathcal{O}\left(r^{-\frac{3}{2}-\kappa}\right), \quad \tilde{\nabla}_r\psi = \mathcal{O}\left(r^{-\frac{5}{2}-\kappa}\right), \end{aligned} \tag{2.47}$$

and for Neumann and Robin:

$$\begin{aligned} &\psi = \mathcal{O}\left(r^{-\frac{3}{2}+\kappa}\right), \quad \nabla_t\psi = \mathcal{O}\left(r^{-\frac{3}{2}+\kappa}\right), \\ &\left|\nabla\!\!\!/\psi\right| = \mathcal{O}\left(r^{-\frac{3}{2}+\kappa}\right), \quad \tilde{\nabla}_r\psi = \mathcal{O}\left(r^{-\frac{5}{2}+\kappa}\right). \end{aligned} \tag{2.48}$$

Twisted energy momentum tensor

Rather than working with the classical energy momentum tensor (1.11), we work with a tensor adapted to the twisting. The twisted energy momentum tensor.

Definition 2.2.1. *For* $\psi \in C^1(\mathcal{M})$ *we define the twisted energy momentum tensor as*

$$\tilde{\mathbb{T}}_{\mu\nu}[\psi] = \tilde{\nabla}_\mu\psi\tilde{\nabla}_\nu\psi - \frac{1}{2}g_{\mu\nu}\left(\tilde{\nabla}_\sigma\psi\tilde{\nabla}^\sigma\psi + V\psi^2\right), \tag{2.49}$$

recalling that

$$V = -\left(f^{-1}\nabla_\mu\nabla^\mu f - \frac{2a}{l^2}\right). \tag{2.50}$$

In contrast to the classical twisted energy momentum tensor, this tensor does not satisfy the property of vanishing divergence for when ψ is a solution (2.23). It does however satisfy some useful algebraic relations.

Corollary 2.2.2.
Taken from [HW14]

- *For* $\phi \in C^2(\mathcal{M})$
$$\nabla_\mu \tilde{\mathbb{T}}^\mu{}_\nu[\phi] = \left(-\tilde{\nabla}^\dagger_\mu \tilde{\nabla}^\mu \phi - V\phi\right) \tilde{\nabla}_\nu \phi + \tilde{S}_\nu[\phi], \tag{2.51}$$
with
$$\tilde{S}_\nu[\phi] = \frac{\tilde{\nabla}^\dagger_\nu (fV)}{2f}\phi^2 + \frac{\nabla^\dagger_\nu f}{2f}\tilde{\nabla}_\sigma \phi \tilde{\nabla}^\sigma \phi. \tag{2.52}$$
For ψ *a solution to* (2.23) *and* X *a smooth vector field.*
We define the current
$$\tilde{J}^X_\mu[\psi] = \tilde{\mathbb{T}}_{\mu\nu}[\psi]X^\nu, \tag{2.53}$$
the bulk term
$$\tilde{K}^X[\psi] = {}^X\pi_{\mu\nu}\tilde{\mathbb{T}}^{\mu\nu}[\psi] + X^\nu \tilde{S}_\nu[\psi], \tag{2.54}$$
where
$${}^X\pi_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu X_\nu + \nabla_\nu X_\mu\right) = \frac{1}{2}\left(\mathcal{L}_X g\right)_{\mu\nu} \tag{2.55}$$
is the deformation tensor. Then
$$\nabla^\mu \tilde{J}^X_\mu[\psi] = \tilde{K}^X[\psi]. \tag{2.56}$$

- *If the twisting function* f *is chosen such that* $V \geq 0$ *then* $\tilde{\mathbb{T}}_{\mu\nu}$ *satisfies the dominant energy condition.*

- *If* Z *is a Killing field of the spacetime that satisfies* $\mathcal{L}_Z(f) = 0$ *then we have that* $\tilde{J}^Z_\mu[\psi]$ *is a conserved current.*

As $\tilde{S}_\mu$ and thus $\tilde{K}^X$ only depend on the 1−jet of ψ, we observe that $\tilde{J}^X_\mu[\psi]$ is a compatible current in the sense of Christodoulou [Chr16].

2.3 Bounded energy

In this section we recall the bounded energy result first shown in [Dun14].

Theorem 2.3.1. *For* ψ *a solution to* (2.23), *with* $\kappa \in (0,1)$, *satisfying Dirichlet, Neumann or Robin boundary conditions, for* $T_1 < T_2$ *there exists a uniform constant* $C_{M,l,\kappa} > 0$ *such that the renormalised energy density given by*
$$\mathcal{E}[\psi] := \frac{1}{r}\psi^2 + r^4\left(\tilde{\nabla}_r \psi\right)^2 + (\nabla_t \psi)^2 + r^2 \left|\not{\nabla}\psi\right|^2, \tag{2.57}$$
satisfies the following energy inequality
$$\int_{\Sigma_{T_2}} \mathcal{E}[\psi] dr dx dy \leq C_{M,l,\kappa}\int_{\Sigma_{T_1}} \mathcal{E}[\psi] dr dx dy. \tag{2.58}$$

Proof. First making the choice of twisting function

$$f(r) = r^{-\frac{3}{2}+\kappa}, \tag{2.59}$$

yields that

$$V(r) = \frac{(3-2\kappa)^2 M}{2r^3} > 0. \tag{2.60}$$

We now consider the Killing field

$$T = \partial_t, \tag{2.61}$$

it is trivial to see

$$\mathcal{L}_T(f) = 0, \tag{2.62}$$

so we have that $\tilde{J}^T_\mu[\psi]$ is a conserved quantity, that is

$$\nabla^\mu \tilde{J}^T_\mu[\psi] = 0. \tag{2.63}$$

Integrating over the spacetime slab D gives

$$\int_{\Sigma_{T_2}^{[R_1,R_2]}} \tilde{J}^T_\mu[\psi] n^\mu dS_{\Sigma_{T_2}} - \int_{\Sigma_{T_1}^{[R_1,R_2]}} \tilde{J}^T_\mu[\psi] n^\mu dS_{\Sigma_{T_1}} = \int_{\hat{\Sigma}_{R_2}^{[T_1,T_2]}} \tilde{J}^T_\mu[\psi] m^\mu dS_{\hat{\Sigma}_{R_2}} - \int_{\hat{\Sigma}_{R_1}^{[T_1,T_2]}} \tilde{J}^T_\mu[\psi] m^\mu dS_{\hat{\Sigma}_{R_1}}. \tag{2.64}$$

expanding the constant t slices in coordinates we see that

$$\int_{\Sigma_t^{[R_1,R_2]}} \tilde{J}^T_\mu[\psi] n^\mu dS_{\Sigma_t} = \frac{1}{2} \int_{\Sigma_t^{[R_1,R_2]}} \left(-g^{tt} (\nabla_t \psi)^2 + g^{rr} \left(\tilde{\nabla}_r \psi \right)^2 + \left| \not\nabla \psi \right|^2 + V(r)\psi^2 \right) r^2 dr dx dy, \tag{2.65}$$

which we see due to the choice of f is a coercive quantity. We may also expand the contribution on constant r surfaces to see

$$\int_{\hat{\Sigma}_r^{[T_1,T_2]}} \tilde{J}^T_\mu[\psi] m^\mu dS_{\hat{\Sigma}_r} = \int_{\hat{\Sigma}_r^{[T_1,T_2]}} \left(g^{rt} (\nabla_t \psi)^2 + g^{rr} (\nabla_t \psi) \left(\tilde{\nabla}_r \psi \right) \right) r^2 dt dx dy. \tag{2.66}$$

Now taking the limit $R_1 \to r_+$ we see that

$$\int_{\hat{\Sigma}_{r_+}^{[T_1,T_2]}} \tilde{J}^T_\mu[\psi] m^\mu dS_{\hat{\Sigma}_r} = \int_{\mathcal{H}_{[T_1,T_2]}} g^{rt} (\nabla_t \psi)^2 r_+^2 dt dx dy =: F(\psi; [T_1, T_2]) \geq 0, \tag{2.67}$$

and taking $R_2 \to \infty$ we see that for Dirichlet and Neumann boundary conditions

$$\lim_{r\to\infty} \int_{\hat{\Sigma}_r^{[T_1,T_2]}} \tilde{J}^T_\mu[\psi] m^\mu dS_{\hat{\Sigma}_r} = 0, \tag{2.68}$$

and for Robin

$$\lim_{r\to\infty} \int_{\hat{\Sigma}_r^{[T_1,T_2]}} \tilde{J}^T_\mu[\psi] m^\mu dS_{\hat{\Sigma}_r} = \frac{1}{2l^2} \int_{\mathfrak{T}^2_{T_1,\infty}} \left(r^{\frac{3}{2}-\kappa} \psi \right)^2 \beta dx dy - \frac{1}{2l^2} \int_{\mathfrak{T}^2_{T_2,\infty}} \left(r^{\frac{3}{2}-\kappa} \psi \right)^2 \beta dx dy, \tag{2.69}$$

where $\mathfrak{T}^2_{T_1,\infty}$ is the torus at infinity, and we understand the r terms in the integral to mean

$\lim_{r\to\infty} r^{\frac{3}{2}-\kappa}\psi$. We have shown the energy identity

$$E_{T_2}[\psi] = E_{T_1}[\psi] - F(\psi;[T_1,T_2]). \tag{2.70}$$

An application of the redshift effect [DR08] removes the degeneration in the g^{tt} term, and we conclude

$$\int_{\Sigma_{T_2}} \mathcal{E}[\psi]drdxdy \leq C_{M,l,\kappa} \int_{\Sigma_{T_1}} \mathcal{E}[\psi]drdxdy. \tag{2.71}$$

□

Corollary 2.3.1. *Pointwise bounds for the field can now be proven. For a sufficient smooth solution we apply T, and the redshift vector field as commutators. Then coupled with elliptic estimates, the result follows from a Sobolev embedding.*

2.4 Energy decay rates

Now that we have established bounds in time on the energy, we seek to show that it is decaying to zero. In view of Dirichlet and Neumann boundary conditions this would imply that all the energy of the field is falling into the black hole region. Proving decay of the field in this setting will serve as a blueprint for the nonlinear problems of chapter three.

2.4.1 Morawetz estimate

Theorem 2.4.1. *There exists $\kappa^* \in (\frac{3}{4}, 1]$, such that for $\kappa \in (0,\kappa^*)$, ψ satisfying (2.23), with Dirichlet, Neumann or Robin boundary conditions, and for $T_1 < T_2$, the following estimate holds*

$$\int_{\mathcal{M}_{[T_1,T_2]}} \left(\frac{1}{r}\psi^2 + r^4\left(\tilde{\nabla}_r\psi\right)^2 + \frac{1}{r}\left(\nabla_t\psi\right)^2 + \frac{1}{r}\left|\nabla\!\!\!/\psi\right|^2\right) dtdrxdy \leq C_{M,l,\kappa}\int_{\Sigma_{T_1}} \mathcal{E}[\psi]drdxdy \tag{2.72}$$

for some $C_{M,l,\kappa} > 0$.

To prove this we first prove three lemmas

Lemma 2.4.1 (Hardy Inequality)**.** *Let $\phi \in C^\infty([r_+,\infty),\mathbb{R})$ be such that $\lim_{r\to\infty} r^{\frac{1}{2}}\phi = 0$. Then the following inequality holds*

$$\int_{r_+}^\infty \phi^2 \leq C\left(\int_{r_+}^\infty \frac{\phi^2}{r}dr + \int_{r_+}^\infty (\tilde{\nabla}_r\phi)^2 r^2 dr\right), \tag{2.73}$$

where $C = C(r_+)$.

Proof. Define a cut off function by

$$\chi(r) = \begin{cases} 0 & \text{if } r \leq 2r_+, \\ 1 & \text{if } r \geq 4r_+, \\ \text{Smooth} & \text{if } r \in [2r_+, 4r_+], \end{cases} \tag{2.74}$$

with the property that $\chi'(r) \leq \frac{C}{r}$ for some $C > 0$, and monotone. Now express

$$||\psi||_{L^2} = ||(1-\chi)\psi + \chi\psi||_{L^2} \tag{2.75}$$

after an application of the triangle inequality we can estimate the terms separately.

$$||(1-\chi)\psi||^2_{L^2} = \int_{r_+}^{\infty} (1-\chi)^2\psi^2 dr \leq \int_{r_+}^{4r_+} r \cdot \frac{\psi^2}{r} dr \leq 4r_+ \int_{r_+}^{\infty} \frac{\psi^2}{r} dr, \tag{2.76}$$

and for the second term

$$\begin{aligned} ||\chi\psi||^2_{L^2(r_+,\infty)} &= \int_{r_+}^{\infty} (\chi\psi)^2 dr = \int_{2r_+}^{\infty} (\chi\psi)^2 dr = \int_{2r_+}^{\infty} \left(\chi\psi r^{\frac{3}{2}-\kappa}\right)^2 \partial_r \left(\frac{r^{-2+2\kappa}}{2\kappa - 2}\right) dr \\ &= \left[\frac{1}{2\kappa-2}(\chi\psi)^2 r\right]_{2r_+}^{\infty} + \frac{2}{2-2\kappa} \int_{2r_+}^{\infty} \chi\psi \left(\tilde{\nabla}_r \chi\psi\right) r dr \\ &\leq \frac{1}{1-\kappa} ||\chi\psi||_{L^2(r_+,\infty)} \left|\left|r\tilde{\nabla}_r\chi\psi\right|\right|_{L^2(2r_+,\infty)}. \end{aligned} \tag{2.77}$$

We then estimate the term

$$\begin{aligned} \left|\left|r\tilde{\nabla}_r\chi\psi\right|\right|^2_{L^2(2r_+,\infty)} &= \int_{2r_+}^{\infty} r^2(\tilde{\nabla}_r\chi\psi)^2 dr \\ &\leq C\int_{2r_+}^{\infty} r^2(\chi\tilde{\nabla}_r\psi)^2 + r^2(u\partial_r\chi)^2 dr \\ &\leq C\left(\int_{r_+}^{\infty} (\tilde{\nabla}_r\psi)^2 r^2 dr + \int_{r_+}^{\infty} \frac{\psi^2}{r} dr\right), \end{aligned} \tag{2.78}$$

the result then follows from these estimates. □

Lemma 2.4.2. *Let ψ satisfy (2.23) with Dirichlet, Neumann or Robin boundary conditions. Define the modified energy current as*

$$\tilde{J}^\mu[\psi] = \tilde{\mathbb{T}}_\nu{}^\mu[\psi]X^\nu + w_1(r)\psi\tilde{\nabla}^\mu\psi + w_2(r)\psi^2 X^\mu, \tag{2.79}$$

with

$$X = r\partial_r, \tag{2.80}$$

then following equality holds

$$\begin{aligned}\nabla_\mu \tilde{J}^\mu[\psi] = &\left(2rw_2(r) + \left(\frac{r^2}{l^2} - \frac{2M}{r}\right) w_1'(r)\right)\psi\tilde{\nabla}_r\psi + \frac{2l^2M}{r^3}w_1'(r)\psi\nabla_t\psi \\ &+ \left(rw_2'(r) + 2\kappa w_2(r) + \frac{(3-2\kappa)^3 M}{4r^3} + \frac{(3-2\kappa)^2 M}{2r^3}w_1(r)\right)\psi^2 \\ &+ 4\left(\frac{(2-\kappa)l^2M}{r^3} + \frac{l^2M}{r^3}w_1(r)\right)\tilde{\nabla}_r\psi\nabla_t\psi + ((1-\kappa)+w_1(r))\left|\nabla\!\!\!/\psi\right|^2 \\ &+ \left(-\frac{\kappa r^2}{l^2} - \frac{(3-2\kappa)}{r} + \left(\frac{r^2}{l^2} - \frac{2M}{r}\right)w_1(r)\right)\left(\tilde{\nabla}_r\psi\right)^2 \\ &+ \left(-\frac{(5-2\kappa)l^4M}{r^5} - \frac{2l^4M}{r^5}w_1(r) - ((1-\kappa)+w_1(r))\frac{l^2}{r^2}\right)(\nabla_t\psi)^2.\end{aligned} \tag{2.81}$$

Proof. We proceed term by term. Recall the identity

$$\nabla^\mu\left(\tilde{\mathbb{T}}_{\mu\nu}X^\nu\right) = {}^X\pi_{\mu\nu}\tilde{\mathbb{T}}^{\mu\nu} + X^\nu\tilde{S}_\nu. \tag{2.82}$$

We compute

$$\tilde{S}_r = \frac{1}{r}\left((3-\kappa)V(r)\psi^2 + \frac{(3-2\kappa)}{2}\tilde{\nabla}_\mu\psi\tilde{\nabla}^\mu\psi\right), \tag{2.83}$$

and deduce

$$X^\nu\tilde{S}_\nu = (3-\kappa)V(r)\psi^2 + \frac{(3-2\kappa)}{2}\tilde{\nabla}_\mu\psi\tilde{\nabla}^\mu\psi. \tag{2.84}$$

Turning to the first term of (2.82), we first compute the deformation tensor as

$${}^X\pi = g - \frac{3M}{r}dt^2 - \frac{8l^2M}{r^3}dtdr - \frac{l^2(5l^2M+r^3)}{r^5}dr^2, \tag{2.85}$$

so we see that X is asymptotically conformally Killing. Contracting the metric into the twisted energy momentum tensor we get

$$g_{\mu\nu}\tilde{\mathbb{T}}^{\mu\nu} = -\tilde{\nabla}_\mu\psi\tilde{\nabla}^\mu\psi - 2V(r)\psi^2, \tag{2.86}$$

we then combine this with the $\tilde{S}$ terms

$$g_{\mu\nu}\tilde{\mathbb{T}}^{\mu\nu} + X^\nu\tilde{S}_\nu = (1-\kappa)V(r)\psi^2 + \frac{(1-2\kappa)}{2}g^{\mu\nu}\tilde{\nabla}_\mu\psi\tilde{\nabla}_\nu\psi. \tag{2.87}$$

Contracting with the remaining terms produces

$$\frac{6l^2M}{r^3}\tilde{\nabla}_r\psi\nabla_t\psi - \frac{l^2(8l^2M+r^3)M}{2r^5}(\nabla_t\psi)^2 - \frac{4l^2M+r^3}{2l^2r}(\tilde{\nabla}_r\psi)^2 + \frac{M(3-2\kappa)^2}{4r^3}\psi^2 + \frac{1}{2}\left|\nabla\!\!\!/\psi\right|^2. \tag{2.88}$$

We now compute the divergence of the w_1 term

$$\nabla_\mu\left(w_1(r)\psi\tilde{\nabla}^\mu\psi\right) = \nabla_\mu(w_1\psi)\tilde{\nabla}^\mu\psi + w_1\psi\nabla_\mu\tilde{\nabla}^\mu\psi, \tag{2.89}$$

noting the relation for $\phi \in C^1$

$$\nabla_\mu \phi = \phi \tilde{\nabla}^\dagger_\mu 1 - \tilde{\nabla}^\dagger_\mu \phi, \tag{2.90}$$

and that $\tilde{\nabla}^\dagger_r 1 = \tilde{\nabla}_r 1$, we now compute

$$\nabla_\mu \tilde{\nabla}^\mu \psi = V\psi + \tilde{\nabla}_r 1 \tilde{\nabla}^r \psi. \tag{2.91}$$

(2.89) becomes

$$\nabla_\mu \left(w_1(r)\psi \tilde{\nabla}^\mu \psi \right) = w_1' \psi \tilde{\nabla}^r \psi + w_1 \tilde{\nabla}_\mu \psi \tilde{\nabla}^\mu \psi + w_1 V \psi^2, \tag{2.92}$$

which expands to

$$\begin{aligned} \nabla_\mu \left(w_1(r)\psi \tilde{\nabla}^\mu \psi \right) = & \frac{2l^2 M}{r^3} w_1' \psi \nabla_t \psi + w_1' \left(\frac{r^2}{l^2} - \frac{2M}{r} \right) \psi \tilde{\nabla}_r \psi + \frac{4l^2 M}{r^2} \tilde{\nabla}_r \psi \nabla_t \psi \\ & + w_1 \left(\frac{r^2}{l^2} - \frac{2M}{r} \right) \left(\tilde{\nabla}_r \psi \right)^2 + w_1 \left(-\frac{2l^4 M}{r^5} - \frac{l^2}{r^2} \right) (\nabla_t \psi)^2 \\ & + w_1 \frac{M(3-2\kappa)^2}{4r^3} \psi^2 + w_1 \left| \nabla\!\!\!/ \psi \right|^2 . \end{aligned} \tag{2.93}$$

For the final term

$$\begin{aligned} \nabla_\mu \mu \left(w_2 \psi^2 X^\mu \right) &= \left(r w_2' + 3 w_2 \right) \psi^2 + 2 r w_2 \psi \nabla_r \psi \\ &= \left(r w_2' + 2\kappa w_2 \right) \psi^2 + 2 r w_2 \psi \tilde{\nabla}_r \psi, \end{aligned} \tag{2.94}$$

the result then follows. □

Lemma 2.4.3. *For solutions of* (2.23) *with Dirichlet, Neumann or Robin boundary conditions, consider a current of the form*

$$\tilde{J}^\mu[\psi] = \tilde{\mathbb{T}}_\nu{}^\mu[\psi] X^\nu + w_1(r)\psi \tilde{\nabla}^\mu \psi + w_2(r) \psi^2 X^\mu, \tag{2.95}$$

that satisfies

- $X = r\partial_r$
- $w_1 = -k_1 + f(r)$ *with* $f(r) \in \mathcal{O}(r^{-3})$ *and* $k_1 > 0$,
- $w_2 = \frac{k_2}{r^3}$, *with* $0 \le k_2 < \frac{(3-2\kappa)^2}{4} M$.

It follows that

$$\int_{\mathcal{M}_{[T_1,T_2]}} -\nabla_\mu \tilde{J}^\mu[\psi] dVol \le C \int_{\Sigma_{T_1}} \mathcal{E}[\psi] dr dx dy, \tag{2.96}$$

for some constant $C > 0$ *independent of* T_1 *and* T_2.

Proof. We start by studying surfaces of constant r, then take limits to evaluate the boundary contributions. We compute

$$\begin{aligned}
&\int_{\hat{\Sigma}_r^{[T_1,T_2]}} \tilde{J}_\mu m^\mu dS_{\hat{\Sigma}_r} = \\
&\int_{\hat{\Sigma}_r^{[T_1,T_2]}} \left(r^3 w_2(r) - \frac{(3-2\kappa)^2}{4} M \right) \psi^2 + \frac{r^2}{2l^2}(r^3 - 2Ml^2)(\tilde{\nabla}_r \psi)^2 + \left(\frac{l^4 M}{r^2} + \frac{l^2 r}{2} \right) (\nabla_t \psi)^2 \\
&- \frac{1}{2} r^2 \left| \not{\nabla} \psi \right|^2 + w_1(r) \frac{r}{l^2} \left(r^3 - 2Ml^2 \right) \psi \tilde{\nabla}_r \psi + \frac{2l^2 M w_1(r)}{r} \psi \nabla_t \psi dt dx dy.
\end{aligned} \tag{2.97}$$

We study the terms at $\mathcal{I}$ first. One can easily see from the decay in r for the field that most of terms are converging to 0. For the non-obvious ones we compute

$$\begin{aligned}
&\lim_{r\to\infty} \frac{r^2}{2l^2} \left(r^3 - 2Ml^2 \right) (\tilde{\nabla}_\psi)^2 + w_1(r) \frac{r}{l^2} \left(r^3 - 2Ml^2 \right) \psi \tilde{\nabla}_r \psi \\
&= \lim_{r\to\infty} \frac{1}{2l^2} r^{-2\kappa} \left(r^{\frac{5}{2}+\kappa} \tilde{\nabla}_r \psi \right)^2 - \frac{k_1}{l^2} \left(r^{\frac{3}{2}-\kappa} \psi \right) r^{\frac{5}{2}+\kappa} \tilde{\nabla}_r \psi.
\end{aligned} \tag{2.98}$$

The first term is also clearly converging to 0, in the case of Dirichlet or Neumann conditions we see that the latter term also converges to 0. In the case of Robin we find that it converges to

$$\lim_{r\to\infty} \frac{k_1}{l^2} \beta \left(r^{\frac{3}{2}-\kappa} \psi \right)^2. \tag{2.99}$$

We summarise this as

$$\lim_{r\to\infty} \int_{\hat{\Sigma}_r^{[T_1,T_2]}} \tilde{J}_\mu m^\mu dS_{\hat{\Sigma}_r} = \frac{k_1}{l^2} \int_{T_1}^{T_2} \int_{T^2_{t,\infty}} \beta \left(r^{\frac{3}{2}-\kappa} \psi \right)^2 dx dy dt, \tag{2.100}$$

where $\beta = 0$ for Dirichlet and Neumann boundary conditions.
We now turn to the terms at the horizon, applying Young's inequality with ϵ, we see that

$$\begin{aligned}
&\lim_{r\to r_+} \int_{\hat{\Sigma}_r^{[T_1,T_2]}} \tilde{J}_\mu m^\mu dS_{\hat{\Sigma}_r} \leq \\
&\lim_{r\to r_+} \int_{\hat{\Sigma}_r^{[T_1,T_2]}} \left(r^3 w_2(r) - \frac{(3-2\kappa)^2}{4} M + \epsilon \right) \psi^2 + \frac{r^2}{2l^2} \left(r^3 - 2Ml^2 \right) (\tilde{\nabla}_r \psi)^2 \\
&\quad + \left(\frac{l^4 M}{r^2} + \frac{l^2 r}{2} + \frac{l^4 M w_1^2(r)}{\epsilon r^2} \right) (\nabla_t \psi)^2 - \frac{1}{2} r^3 \left| \not{\nabla} \psi \right|^2 + w_1(r) \frac{r}{l^2} (r^3 - 2Ml^2) \psi \tilde{\nabla}_r \psi dt dx dy \\
&\leq \int_{\hat{\Sigma}_{r_+}^{[T_1,T_2]}} \left(k_2 - \frac{(3-2\kappa)^2}{4} M + \epsilon \right) \psi^2 + CF(\psi; [T_1, T_2]).
\end{aligned} \tag{2.101}$$

For $k_2 < \frac{(3-2\kappa)^2}{4} M$ we can always find an $\epsilon > 0$, such that the integrand is negative. We thus have

$$\lim_{r\to r_+} \int_{\hat{\Sigma}_r^{[T_1,T_2]}} \tilde{J}_\mu m^\mu dS_{\hat{\Sigma}_r} \leq CF(\psi; [T_1, T_2]) \leq CE_{T_1}[\psi]. \tag{2.102}$$

We now study the surfaces of constant t, we compute

$$\begin{aligned}
&\int_{\Sigma_t} \tilde{J}_\mu n^\mu dS_{\Sigma_t} \\
=&\int_{\Sigma_t} -2l^2 M(\tilde{\nabla}_r\psi)^2 + \left(\frac{2l^4M}{r^2} + l^2 r\right)\tilde{\nabla}_r\psi\nabla_t\psi + w_1(r)\left(\frac{2l^4M}{r^3}+l^2\right)\psi\nabla_t\psi \\
&-\frac{2l^2Mw_1(r)}{r}\psi\tilde{\nabla}_r\psi drdxdy \\
=&\int_{\Sigma_t} -2l^2 M(\tilde{\nabla}_r\psi)^2 + \left(\frac{2l^4M}{r^2} + l^2 r\right)\tilde{\nabla}_r\psi\nabla_t\psi - k_1 l^2\psi\nabla_t\psi \\
&+\left(\frac{2l^4M\left(f(r)-k_1\right)}{r^3} + l^2 f(r)\right)\psi\nabla_t\psi - \frac{2l^2Mw_1(r)}{r}\psi\tilde{\nabla}_r\psi drdxdy \\
=&\int_{\Sigma_t} -2l^2 M(\tilde{\nabla}_r\psi)^2 + h_1(r)\tilde{\nabla}_r\psi\nabla_t\psi - k_1 l^2\psi\nabla_t\psi + h_2(r)\psi\nabla_t\psi - h_3(r)\psi\tilde{\nabla}_r\psi drdxdy,
\end{aligned} \tag{2.103}$$

where we have $h_1 \in \mathcal{O}(r)$, $h_2 \in \mathcal{O}(r^{-1})$, and $h_3 \in \mathcal{O}(r^{-1})$. We now apply Young's inequality

$$\begin{aligned}
&\int_{\Sigma_t} \tilde{J}_\mu n^\mu dS_{\Sigma_t} \\
\leq &\int_{\Sigma_t} \left(\frac{1}{2\epsilon_1} + \frac{k_1 l^4}{2\epsilon_3} + \frac{1}{2}\left|h_2(r)\right|\right)(\nabla_t\psi)^2 + \left(\frac{1}{2}\left|h_2(r)\right| + \frac{1}{2\epsilon_2}\left|h_3(r)\right|\right)\psi^2 drdxdy \\
+&\int_{\Sigma_t} \left(\frac{\epsilon_2}{2}\left|h_3(r)\right| + \frac{\epsilon_1}{2}(h_1(r))^2 - 2l^2M\right)(\tilde{\nabla}_r\psi)^2 + \frac{k_1^2 l^4\epsilon_3}{2}\psi^2 drdxdy.
\end{aligned} \tag{2.104}$$

The first integral can clearly be controlled by a constant multiple of the energy at time T_1. For the latter one we invoke the Hardy estimate

$$\begin{aligned}
&\int_{\Sigma_t} \tilde{J}_\mu n^\mu dS_{\Sigma_t} \leq CE_{T_1}[\psi] \\
&+\int_{\Sigma_t} \left(C_2\frac{k_1^2 l^4\epsilon_3}{2}r^2 + \frac{\epsilon_2}{2}\left|h_3(r)\right| + \frac{\epsilon_1}{2}(h_1(r))^2 - 2l^2M\right)(\tilde{\nabla}_r\psi)^2 + \frac{k_1^2 l^4\epsilon_3}{2}\psi^2 + C_2\frac{k_1^2 l^4\epsilon_3}{2r}\psi^2 drdxdy.
\end{aligned} \tag{2.105}$$

We now choose ϵ_i's small enough such that

$$C_2\frac{k_1^2 l^4\epsilon_3}{2}r_+^2 + \frac{\epsilon_2}{2}\left|h_3(r_+)\right| + \frac{\epsilon_1}{2}(h_1(r_+))^2 < 2l^2M, \tag{2.106}$$

we may then find a $C > 0$ independent of T_1 and T_2 such that

$$\int_{\Sigma_t} \tilde{J}_\mu n^\mu dS_{\Sigma_t} \leq CE_{T_1}[\psi]. \tag{2.107}$$

From here we simply apply the divergence theorem

$$\int_{\mathcal{M}_{[T_1,T_2]}} -\nabla_\mu \tilde{J}^\mu[\psi]dV \leq CE_{T_1}[\psi] - \tilde{C}\int_{T_1}^{T_2}\int_{\mathfrak{T}^2_{t,\infty}} \beta(r^{\frac{3}{2}-\kappa}\psi)^2 dxdydt, \tag{2.108}$$

from here the result follows. □

We are now able to prove theorem 2.4.1. From inspection of the of the bulk terms in lemma 2.4.2, we can see that tangential terms appear with the opposite sign to the higher order coefficients of the time derivative terms. In order to get a signed bulk term we cancel these off. This gives us our first estimate. We can then reintroduce the tangential terms controlling the negative terms by the first estimate.

Proof of theorem 2.4.1. Define the first current as

$$\tilde{J}_1^\mu = \tilde{\mathbb{T}}^\mu_\nu X^\mu + (\kappa - 1)\psi\tilde{\nabla}^\mu\psi + \frac{(3-2\kappa)M}{2r^3}\left(\frac{1}{2} + \frac{\epsilon}{2}\right)\psi^2 X^\mu, \tag{2.109}$$

its easy to see that the current satisfies lemma 2.4.3, provided $\epsilon < 2(1-\kappa)$. Then from lemma 2.4.2 we compute

$$\begin{aligned} -\nabla_\mu \tilde{J}_1^\mu[\psi]\cdot r^2 = &\frac{3l^4 M}{r^3}(\nabla_t\psi)^2 - \frac{4Ml^2}{r}\tilde{\nabla}_r\psi\nabla_t\psi + \frac{r^4}{l^2}(\tilde{\nabla}_r\psi)^2 \\ &+ \frac{(3-2\kappa)^2 M\epsilon}{4r}\psi^2 - \frac{1}{2}(3-2\kappa)(1+\epsilon)M\psi\tilde{\nabla}_r\psi + Mr(\tilde{\nabla}_r\psi)^2. \end{aligned} \tag{2.110}$$

Two applications of Young's inequality give

$$\begin{aligned} -\nabla_\mu \tilde{J}_1^\mu[\psi]\cdot r^2 \geq &\frac{l^4 M}{r^3}\left(3 - \frac{4}{\delta_1}\right)(\nabla_t\psi)^2 + \left(\frac{r^4}{l^2} - Mr\left(\delta_1 + \frac{(1+\epsilon)^2}{2\delta_2} - 1\right)\right)(\tilde{\nabla}_r\psi)^2 \\ &+ \frac{M}{r}\left(\epsilon - \frac{\delta_1}{2}\right)\left(\frac{3-2\kappa}{2}\right)^2\psi^2, \end{aligned} \tag{2.111}$$

if we restrict to

$$\delta_2 < 2\epsilon, \quad \delta_1 > \frac{4}{3}, \quad \delta_1 + \frac{(1+\epsilon)^2}{2\delta_2} - 1 < 2, \tag{2.112}$$

we have a positive bulk term. These conditions can be met provided

$$3\epsilon^2 - 14\epsilon + 3 < 0, \tag{2.113}$$

that is

$$\epsilon \in \left(\frac{1}{3}(7 - 2\sqrt{10}), \frac{1}{3}\left(7 + 2\sqrt{10}\right)\right), \tag{2.114}$$

with the constraint

$$\kappa < 1 - \frac{\epsilon}{2}. \tag{2.115}$$

This provides the upper bound $\kappa < \kappa^* = \frac{1}{6}\left(2\sqrt{10} - 1\right) \approx 0.887$, for which our result holds. Choosing $\epsilon = 1 - \kappa$, we see that we have positivity and boundedness for the range

$$\kappa \in \left(0, \frac{1}{6}\left(2\sqrt{10} - 1\right)\right). \tag{2.116}$$

Within this range we may now find a constant $c = c(M, l, \kappa)$, such that

$$-\nabla_\mu \tilde{J}_1^\mu[\psi] \cdot r^2 \geq c\left(\frac{1}{r^3}\left(\nabla_t \psi\right)^2 + r^4\left(\tilde{\nabla}_r \psi\right)^2 + \frac{1}{r}\psi^2\right). \tag{2.117}$$

We apply lemma 2.4.3 to deduce

$$\int_{\mathcal{M}_{[T_1,T_2]}} \left(\frac{1}{r^3}(\nabla_t\psi)^2 + r^4(\tilde{\nabla}_r\psi)^2 + \frac{1}{r}\psi^2\right) dtdrdxdy \leq C\int_{\Sigma_{T_1}} \mathcal{E}[\psi]drdxdy. \tag{2.118}$$

To control the tangential terms, we consider the current

$$\tilde{J}_2^\mu = \tilde{\mathbb{T}}_\nu^\mu X^\mu - \left(\frac{1}{r^3} + (1-\kappa)\right)\psi\tilde{\nabla}^\mu\psi. \tag{2.119}$$

Computing the divergence of this current we see

$$\begin{aligned}
-\nabla_\mu \tilde{J}_2^\mu[\psi] \cdot r^2 = &\left(\frac{6M}{r^3} - \frac{3}{l^2}\right)\psi\tilde{\nabla}_r\psi - \frac{6l^2M}{r^5}\psi\nabla_t\psi - \frac{4l^2M(r^3-1)}{r^4}\tilde{\nabla}_r\psi\nabla_t\psi \\
&+ \left(\frac{r^4+r}{l^2} + M\left(r - \frac{2}{r^2}\right)\right)(\tilde{\nabla}_r\psi)^2 + \frac{(l^4M\left(3r^3-2\right) - l^2r^3)}{r^6}(\nabla_t\psi)^2 \\
&- \frac{\left(3-2\kappa\right)^2 M\left(r^3-2\right)}{4r^4}\psi^2 + \frac{1}{r}\left|\nabla\!\!\!/\psi\right|^2.
\end{aligned} \tag{2.120}$$

From (2.117) we control all the non-tangential terms, and may deduce the estimate

$$\frac{1}{r}\left|\nabla\!\!\!/\psi\right|^2 \leq -\nabla_\mu\tilde{J}_2^\mu[\psi]\cdot r^2 + C\left(\frac{1}{r^3}(\nabla_t\psi)^2 + r^4(\tilde{\nabla}_r\psi)^2 + \frac{1}{r}\psi^2\right). \tag{2.121}$$

Integrating this estimate yields the result. □

2.4.2 Integrated decay estimate with derivative loss

We now seek to improve the weights of the Morawetz estimate. Unfortunately this will require 'losing' a derivative that is

Theorem 2.4.2. *For $\kappa \in (0, \kappa^*)$ and ψ a solution to (2.23) with Dirichlet, Neumann, or Robin boundary conditions the following integral inequality holds*

$$\int_{\mathcal{M}_{[T_1,T_2]}} \mathcal{E}[\psi]dtdrdxdy \leq C\int_{\Sigma_{T_1}} \left(\mathcal{E}[\psi] + \mathcal{E}[\psi_t]\right)drdxdy, \tag{2.122}$$

for a constant $C = C(M, l, \kappa) > 0$.

Proof. An application of lemma 2.4.1 to the estimate in theorem 2.4.1 yields

$$\int_{\mathcal{M}_{[T_1,T_2]}} \psi^2 dtdrdxdy \leq C\int_{\Sigma_{T_1}} \mathcal{E}[\psi]drdxdy. \tag{2.123}$$

We now exploit some symmetry of the spacetime. Noting that ∂_t is a Killing field we have the commutator relation

$$\left[\partial_t, \Box_g - \frac{2a}{l^2}\right] = 0, \tag{2.124}$$

which in turn implies, for smooth enough initial data, that

$$\int_{\mathcal{M}_{[T_1,T_2]}} (\nabla_t\psi)^2\, dtdrdxdy \leq C \int_{\Sigma_{T_1}} \mathcal{E}[\psi_t] drdxdy, \tag{2.125}$$

combining this estimate with theorem 2.4.1 yields

$$\int_{\mathcal{M}_{[T_1,T_2]}} (\nabla_t\psi)^2 + r^4\left(\tilde{\nabla}_r\psi\right)^2 + \psi^2 dtdrdxdy \leq C\left(\int_{\Sigma_{T_1}} \mathcal{E}[\psi] + \mathcal{E}[\psi_t] drdxdy\right). \tag{2.126}$$

All that remains is to recover the tangential derivatives. While we could at this point exploit further symmetry of the spacetime by using the ∂_x and ∂_y Killing fields, we will favour a more robust method. This allows these results to be applied to the perturbed solutions from [ACD02], and will be more useful with non-linear applications in mind. Consider the current defined by

$$\tilde{J}^\mu_3[\psi] = \tilde{\mathbb{T}}^\mu_\nu[\psi]X^\nu - (2-\kappa)\psi\tilde{\nabla}^\mu\psi, \tag{2.127}$$

which has divergence

$$-\nabla_\mu \tilde{J}^\mu_3 \cdot r^2 = \left(\frac{l^4 M}{r^3} - l^2\right)(\nabla_t\psi)^2 + \left(\frac{2r^4}{l^2} - Mr\right)\left(\tilde{\nabla}_r\psi\right)^2 + \frac{M(3-2\kappa)^2}{4r}\psi^2 + r^2\left|\nabla\!\!\!/\psi\right|^2. \tag{2.128}$$

This current satisfies the conditions of lemma 2.4.3 and as we control all the non tangential terms we have

$$\int_{\mathcal{M}_{[T_1,T_2]}} r^2\left|\nabla\!\!\!/\psi\right|^2 dtdrdxdy \leq C\left(\int_{\Sigma_{T_1}} \mathcal{E}[\psi] + \mathcal{E}[\psi_t] drdxdy\right), \tag{2.129}$$

and thus

$$\int_{\mathcal{M}_{[T_1,T_2]}} (\nabla_t\psi)^2 + r^4\left(\tilde{\nabla}_r\psi\right)^2 + \psi^2 + r^2\left|\nabla\!\!\!/\psi\right|^2 dtdrdxdy \leq C\left(\int_{\Sigma_{T_1}} \mathcal{E}[\psi] + \mathcal{E}[\psi_t] drdxdy\right). \tag{2.130}$$

□

2.5 Polynomial energy decay

We now extract a statement about energy decay from the integrated decay estimate. We prove the following theorem

Corollary 2.5.1. *For $\kappa \in (0, \kappa^*)$, and ψ a solution to* (3.5) *with Dirichlet, Neumann or Robin* *conditions. Then*

$$\int_{\Sigma_t} \mathcal{E}[\psi] dr dx dy \leq \frac{C}{(1+t)^n} \sum_{k=0}^{n} \int_{\Sigma_0} \mathcal{E}[\partial_t^k \psi] dr dx dy, \tag{2.131}$$

for some $C = C(n, M, l, \kappa) > 0$.

We proceed to prove this with two technical results. Firstly a Gronwall type estimate,

Lemma 2.5.1 (Gronwall type estimate)**.** *Let* $k > 0$*, and* $f \in C^1([T_1, \infty))$ *satisfy*

$$f'(t) \leq -\varkappa f(t) + \frac{A}{(1+t-T_1)^k}, \tag{2.132}$$

for some $A > 0$*. There exists* $C = C(\varkappa, k) > 0$ *such that*

$$f(t) \leq f(T_1) e^{-\varkappa(t-T_1)} + \frac{CA}{(1+t-T_1)^k}. \tag{2.133}$$

Proof. From (2.132) we derive

$$\frac{d}{dt}\left(f(t)e^{\varkappa t}\right) \leq \frac{Ae^{\varkappa t}}{(1+t-T_1)^k}, \tag{2.134}$$

integrating this quantity gives

$$f(t)e^{\varkappa t} - f(T_1)e^{\varkappa T_1} \leq A \int_{T_1}^{t} \frac{e^{\varkappa s}}{(1+s-T_1)^k} ds. \tag{2.135}$$

A simple change of variable yields

$$f(t)e^{\varkappa t} - f(T_1)e^{\varkappa T_1} \leq Ae^{\varkappa T_1} \int_0^{t-T_1} \frac{e^{\varkappa s'}}{(1+s')^k} ds', \tag{2.136}$$

and finally

$$f(t)e^{\varkappa t} - f(T_1)e^{\varkappa T_1} \leq Ae^{\varkappa T_1} \frac{e^{\varkappa(t-T_1)}}{(1+t-T_1)^k}. \tag{2.137}$$

For the last estimate we have used

$$\begin{aligned} \int_0^t \frac{e^{\varkappa s'}}{(1+s')^k} ds' &= \left[\frac{e^{\varkappa s'}}{\varkappa(1+s')^k}\right]_0^t + k \int_0^t \frac{e^{\varkappa s'}}{\varkappa(1+s')^{k+1}} ds' \\ &\leq \frac{e^{\varkappa t}}{\varkappa(1+t)^k} + kt \max_{s \in (0,t)} \left| \frac{e^{\varkappa s}}{\varkappa(1+s)^{k+1}} \right| \\ &\leq C \frac{e^{\varkappa t}}{(1+t)^k}. \end{aligned} \tag{2.138}$$

From here the result follows. □

Secondly, the following quantitative form of the red shift from [War15], theorem 3.8.

Lemma 2.5.2. *There exists a modified energy $\mathbb{E}_t[\psi]$, the redshift energy such that we have a uniform $C > 0$ such that*

$$C^{-1} \int_{\Sigma_t} \mathcal{E}[\psi] dr dx dy \leq \mathbb{E}_t[\psi] \leq C \int_{\Sigma_t} \mathcal{E}[\psi] dr dx dy, \tag{2.139}$$

and for ψ a solution to (2.23) with Dirichlet, Neumann or Robin boundary conditions then,

$$\frac{d}{dt}\mathbb{E}_t[\psi] \leq -\varkappa \mathbb{E}_t[\psi] + C E_t[\psi], \tag{2.140}$$

for some $\varkappa > 0$.

Proof of theorem 2.5.1. It is trivial to see from 2.38 that

$$E_t[\psi] \leq C \int_{\Sigma_t} \mathcal{E}[\psi] dr dx dy, \tag{2.141}$$

we then integrate in time and apply theorem 2.4.2 to see

$$\int_{T_1}^{T_2} E_s[\psi] \leq C \left(\int_{\Sigma_{T_1}} \mathcal{E}[\psi] + \mathcal{E}[\psi_t] dr dx dy \right). \tag{2.142}$$

We now write

$$\begin{aligned} (1 + t - T_1) E_t[\psi] &= E_t[\psi] + \int_{T_1}^{t} \frac{d}{ds} \left((s - T_1) E_s[\psi] \right) ds \\ &= E_t[\psi] + \int_{T_1}^{t} E_s[\psi] + (s - T_1) \dot{E}_t[\psi] ds \\ &\leq E_t[\psi] + \int_{T_1}^{t} E_s[\psi] ds \\ &\leq C \left(\int_{\Sigma_t} \mathcal{E}[\psi] + \mathcal{E}[\psi_t] dr dx dy \right). \end{aligned} \tag{2.143}$$

Where we have used to monotonicity property of $E_t[\psi]$. We thus deduce

$$E_t[\psi] \leq \frac{C}{(1 + t - T_1)} \left(\int_{\Sigma_{T_1}} \mathcal{E}[\psi] + \mathcal{E}[\psi_t] dr dx dy \right). \tag{2.144}$$

Now an application of lemma 2.5.1 to (2.140) gives

$$
\begin{aligned}
\mathbb{E}_t[\psi] &\leq \mathbb{E}_{T_1}[\psi]e^{-\varkappa(t-T_1)} + \frac{C}{1+t-T_1}\left(\int_{\Sigma_{T_1}} \mathcal{E}[\psi] + \mathcal{E}[\psi_t]drdxdy\right) \\
&\leq \mathbb{E}_{T_1}[\psi]e^{-\varkappa(t-T_1)} + \frac{C}{1+t-T_1}\left(\int_{\Sigma_{T_1}} \mathcal{E}[\psi] + \mathcal{E}[\psi_t]drdxdy\right) \\
&\leq Ce^{-\varkappa(t-T_1)}\int_{\Sigma_{T_1}} \mathcal{E}[\psi]drdxdy + \frac{C}{1+t-T_1}\left(\int_{\Sigma_{T_1}} \mathcal{E}[\psi] + \mathcal{E}[\psi_t]drdxdy\right) \\
&\leq \frac{C}{1+t-T_1}\left(\int_{\Sigma_{T_1}} \mathcal{E}[\psi] + \mathcal{E}[\psi_t]drdxdy\right),
\end{aligned}
\tag{2.145}
$$

from which we obtain

$$
\int_{\Sigma_t} \mathcal{E}[\psi]drdxdy \leq \frac{C}{1+t-T_1}\left(\int_{\Sigma_{T_1}} \mathcal{E}[\psi] + \mathcal{E}[\psi_t]drdxdy\right). \tag{2.146}
$$

From here we proceed inductively, noting that taking $T_1 = 0$ proves the case $n = 1$. The rest follows as in [HLSW15], lemma 5.8. □

2.6 Gaussian beams and derivative loss

The result of this section establishes that some derivative loss or degeneration of weights is indeed necessary.

Theorem 2.6.1. *There exists no constant $C > 0$, independent of T, such that the estimate*

$$
\int_{\mathcal{M}_{[0,T]}} \mathcal{E}[\psi]dtdrdxdy \leq C\int_{\Sigma_0} \mathcal{E}[\psi]drdxdy, \tag{2.147}
$$

holds for all smooth solutions of (2.23).

The proof is this is an adaptation of the Gaussian beam method of Sbierski [Sbi15]. The idea is to show the existence of null geodesics that remain outside the event horizon for arbitrary amounts of coordinate time. One then constructs approximate solutions (Gaussian beams) to (2.23), supported in a tubular neighbourhood $\mathcal{N}$ of these geodesics. The energy of the Gaussian beam remains arbitrarily close to the Killing energy of the geodesic in a neighbourhood of $\mathcal{N}$. Furthermore it also remains close to a true solution of (2.23) in the energy norm. This true solution is then used to contradict inequalities of the form (2.147).

Lemma 2.6.1. *Fix $T > 0$ then there exists a null geodesic γ such that $Im(\gamma) \subset \mathcal{M}_{[0,T]} \cap \{\frac{3}{2}r_+ < r < R(T)\}$ for some large $R(T)$.*

Proof. We proceed by using a Hamiltonian method. We construct integrals of motion from the Killing fields $\partial_t, \partial_x, \partial_y$, and from the fact that $\dot{\gamma}$ is null. That is there are three constants a, b, c

such that

$$
\begin{aligned}
a &= g(\dot{\gamma}, \partial_t), \\
b &= g(\dot{\gamma}, \partial_x), \\
c &= g(\dot{\gamma}, \partial_y), \\
0 &= g(\dot{\gamma}, \dot{\gamma}).
\end{aligned} \tag{2.148}
$$

Working in coordinates and taking τ as an affine parameter, we get the following geodesic equations

$$
\begin{aligned}
\dot{t} &= \frac{-1}{\frac{-2M}{r} + \frac{r^2}{l^2}} \left(a - \dot{r}\frac{2Ml^2}{r^3} \right), \\
\dot{r} &= -\left(d^2 \left(\frac{2M}{r} - \frac{1}{l^2} \right) + a^2 \right)^{\frac{1}{2}}, \\
\dot{x} &= \frac{b}{r^2}, \\
\dot{y} &= \frac{c}{r^2},
\end{aligned} \tag{2.149}
$$

where $d^2 = b^2 + c^2$. In order that we are heading towards the event horizon and not null infinity we choose the negative root of $\dot{r}^2$. As we are interested in a photon moving in the tangential directions at a distance R, we give the following initial conditions

$$
\begin{aligned}
r(0) &= R, \\
\dot{r}(0) &= 0.
\end{aligned} \tag{2.150}
$$

We solve for a

$$
a = -d \left(\frac{1}{l^2} - \frac{2M}{R} \right)^{\frac{1}{2}}, \tag{2.151}
$$

where we again choose the negative root. This is the geodesic is future directed i.e. $\dot{t} \geq 0$. The geodesic equation becomes

$$
\begin{aligned}
\dot{t} &= \frac{drl^2}{r^3 - 2Ml^2} \left(\left(\frac{1}{l^2} - \frac{2M}{R^3} \right)^{\frac{1}{2}} - \frac{2Ml^2}{r^3}\sqrt{2M} \left(\frac{1}{r^3} - \frac{1}{R^3} \right)^{\frac{1}{2}} \right), \\
\dot{r} &= -d\sqrt{2M} \left(\frac{1}{r^3} - \frac{1}{R^3} \right)^{\frac{1}{2}}, \\
\dot{x} &= \frac{b}{r^2}, \\
\dot{y} &= \frac{c}{r^2}.
\end{aligned} \tag{2.152}
$$

So as long as $R > r_+$ we see all of these are signed (away from R), and t, r, x, y are monotonic. As r is monotonic and decreasing it never reaches $\mathcal{I}$. Furthermore $\dot{t}, \dot{r}, \dot{x}, \dot{y}$ are also monotonic. We deduce that γ is a smooth embedding.

We now turn to computing the coordinate time taken for the photon to fall a distance $\frac{R}{2}$, remaining on the exterior of the black hole. We proceed by computing the affine time $\tau_{\frac{R}{2}}$, this is done using (2.152)

$$\tau_{\frac{R}{2}} = \frac{1}{d\sqrt{2M}} \int_{\frac{R}{2}}^{R} \left(\frac{1}{r^3} - \frac{1}{R^3} \right)^{-\frac{1}{2}} dr. \tag{2.153}$$

Rescaling the integral we get

$$\tau_{\frac{R}{2}} = \frac{R^{\frac{5}{2}}}{d\sqrt{2M}} \int_{\frac{1}{2}}^{1} \sqrt{\frac{y^3}{1-y^3}} dy. \tag{2.154}$$

Now

$$0 < \int_{\frac{1}{2}}^{1} \sqrt{\frac{y^3}{1-y^3}} dy \leq \int_{0}^{1} \sqrt{\frac{y^3}{1-y^3}} dy = \sqrt{\pi} \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})} < \infty, \tag{2.155}$$

so this integral is simply some positive constant. We denote it as K_F and write

$$\tau_{\frac{R}{2}} = \frac{K_F R^{\frac{5}{2}}}{d\sqrt{2M}}. \tag{2.156}$$

We use this to prove an estimate for the coordinate time. In order to do this we consider bounding $\dot{t}$ below on the interval $r \in \left[\frac{R}{2}, R\right]$

$$\dot{t} \geq \frac{dl^2}{r^2} \left(\frac{1}{l^2} - \frac{2M}{R^3} \right)^{\frac{1}{2}} - \frac{4dl^2}{(R^3 - 16Ml^2)\, R^{\frac{7}{2}}} \underbrace{\left(16Ml^2\sqrt{14M}\right)}_{=:\tilde{K}}, \tag{2.157}$$

now fix $\epsilon > 1$, and set $C = l^{-1}\left(1 - \frac{1}{\epsilon}\right)$, then provided $R^3 \geq 2\epsilon Ml^2$, we have

$$\begin{aligned} \dot{t} &\geq \frac{dl^2 C}{R^2} - \frac{4dl^2\tilde{K}}{R^{\frac{7}{2}}(R^3 - 16Ml^2)} \\ &= \frac{dl^2}{R^2} \left(C - \frac{4\tilde{K}}{R^{\frac{3}{2}}(R^3 - 16Ml^2)} \right). \end{aligned} \tag{2.158}$$

If we then choose $\epsilon = 9$, we have

$$\dot{t} \geq \frac{dl^2}{R^2} \left(\frac{8}{9l} - \frac{4\tilde{K}}{R^{\frac{3}{2}} 2Ml^2} \right). \tag{2.159}$$

Insisting $R^3 \geq \max\left\{18Ml^2, \left(\frac{18\tilde{K}}{7M}\right)^2\right\}$ we find

$$\dot{t} \geq \frac{dl}{9} \frac{1}{R^2}. \tag{2.160}$$

We thus have an inequality for the fall time (for large R)

$$\Delta_{\frac{R}{2}} t = \int_0^{\tau_{\frac{R}{2}}} \dot{t} d\tau \geq K \frac{dl^2}{R^2} \cdot \tau_{\frac{R}{2}} = K_F R^{\frac{1}{2}}. \tag{2.161}$$

Now fix T and let $R = \max\left\{\frac{9T^2}{4K_F^2}, (18Ml^2)^{\frac{1}{3}}, \left(\frac{18\tilde{K}}{7M}\right)^{\frac{2}{3}}\right\}$, we then have

$$\Delta_{\frac{R}{2}} t \geq \frac{3T}{2}, \tag{2.162}$$

or in other words a geodesic with $r > \frac{3}{2}r_+$, for $0 \leq t \leq T$.

□

Remark 2.6.1. *To visualise this geodesic it is helpful to return to equations* (2.151) *and rewrite them as*

$$d^2 = \frac{a^2}{R - 2Ml^2}. \tag{2.163}$$

From here we can see that the value of d (a tangential momentum) has to be non zero for the construction of the geodesic. We can think of these null geodesics as curves spiralling in towards the horizon, typically starting at a distance far away.

We now need a result from Sbierski [Sbi15], it is slightly modified to requiring the geodesic being a smooth embedding rather than working on a globally hyperbolic manifold (see comment following definition 2.35 of [Sbi15]).

Theorem 2.6.2. *Let $(\mathcal{M}, g)$ be a time orientated Lorentzian manifold with time function t, and foliated by the level sets $\Sigma_\tau = \{t = \tau\}$. Furthermore let γ be a smooth geodesic embedding that intersects Σ_0, and let N be a timelike, future directed vector field. Then for any neighbourhood $\mathcal{N}$ of γ and $T > 0$ with $\Sigma_T \cap Im(\gamma) \neq \emptyset$ there exists a Gaussian beam ψ_λ of the form*

$$\psi_\lambda(x) = a_{\mathcal{N}} e^{i\lambda\phi(x)}, \tag{2.164}$$

such that the following hold

$$||\Box_g \psi_\lambda||_{L^2(\mathcal{M}_{[0,T]})} \leq C(T), \tag{2.165}$$

where $C(T)$ depends on $a_{\mathcal{N}}, \phi$ and T but not on λ. Furthermore

$$E_t[\psi_\lambda] \to \infty, \quad \text{as } \lambda \to \infty, \tag{2.166}$$

and

$$\psi_\lambda \text{ is supported in } \mathcal{N}, \tag{2.167}$$

provided that on $\mathcal{M}_{[0,T]} \cap J^+(\mathcal{N} \cap \Sigma_0)$,

$$|n_{\Sigma_\tau}|^{-1} \leq C, \quad g(N, N) \leq -c < 0, \quad -g(N, n_{\Sigma_\tau}) \leq C, \tag{2.168}$$

and

$$\left|g(\nabla_{n_{\Sigma_\tau}} N, n_{\Sigma_\tau})\right|, \left|g(\nabla_{n_{\Sigma_\tau}} N, e_i)\right|, |g(\nabla_{e_i} N, e_j)| \leq C, \tag{2.169}$$

for $1 \leq i, j \leq 3$, c and where C are positive constants and $\{n_{\Sigma_\tau}, e_1, e_2, e_3\}$ are an orthonormal frame.

For our problem we use $N = \partial_t$, and use the time coordinate t as the time function. We also use the result from [Sbi15]

Lemma 2.6.2.

$$\mathfrak{I}(\phi|_\gamma) = \mathfrak{I}(\nabla\phi|_\gamma) = 0 \tag{2.170}$$

and

$$\mathfrak{I}(\nabla\nabla\phi|_\gamma) \text{ is positive definite on a 3D subspace transversal to } \dot{\gamma} \tag{2.171}$$

where $\mathfrak{I}$ denotes the imaginary part.

Combining this lemma with the fact that $a_{\mathcal{N}}$ is independent of λ we see that the L^2 norm of ψ_λ is also independent of λ. That is

Lemma 2.6.3. *Let ψ_λ be the Gaussian beam constructed in theorem 2.6.2 then there exists a constant $C(T) > 0$ independent of λ such that the following estimate holds*

$$||\psi_\lambda||_{L^2(\mathcal{M}_{[0,T]})} \leq C(T). \tag{2.172}$$

We now show that we can approximate solutions of (2.23) in the energy norm with Gaussian beams.

Lemma 2.6.4. *For all $\epsilon > 0$ there exists a solution ψ of (2.23) satisfying Dirichlet, Neumann or Robin boundary conditions, and initial data supported away from the horizon, with*

$$E_0[\psi] = 1, \tag{2.173}$$

and a Gaussian beam $\tilde{\psi_\lambda}$ such that

$$\left|E_t[\psi] - E_t[\tilde{\psi_\lambda}]\right| < \epsilon, \quad \forall 0 \leq t \leq T. \tag{2.174}$$

Proof. Construct ψ_λ from theorem 2.6.2, and use the geodesic from lemma 2.6.1. Then ensuring that $\mathcal{N}$ is bounded away from $\mathcal{I}$, and $\mathcal{H}$ we define

$$\tilde{\psi_\lambda} := \frac{\psi_\lambda}{\sqrt{E_0[\psi_\lambda]}}. \tag{2.175}$$

It follows from the triangle inequality that

$$\left|\left|\Box_g\tilde{\psi_\lambda} + \frac{-2a}{l^2}\tilde{\psi_\lambda}\right|\right|_{L^2(\mathcal{M}_{[0,T]})} \to 0, \tag{2.176}$$

as $\lambda \to \infty$.

Now take ψ to be the solution to

$$\begin{aligned} \Box_g \psi - \frac{2a}{l^2}\psi = 0, \\ \psi|_{\Sigma_0} = \tilde{\psi}_\lambda|_{\Sigma_0}, \\ n_{\Sigma_0}\psi|_{\Sigma_0} = n|_{\Sigma_0}\tilde{\psi}_\lambda, \end{aligned} \tag{2.177}$$

with Dirichlet, Neumann or Robin boundary conditions.
Applying the standard energy estimates to $\tilde{\psi}_\lambda$ we get

$$E_t[\tilde{\psi}_\lambda] \le E_0[\tilde{\psi}_\lambda] + \left|\left|\left(\Box_g \tilde{\psi}_\lambda - \frac{2a}{l^2}\tilde{\psi}_\lambda\right) \nabla_t \tilde{\psi}_\lambda\right|\right|^2_{L^2(\mathcal{M}_{[0,T]})}, \tag{2.178}$$

using that $\mathcal{N}$ is bounded from $\mathcal{I}$, applying Cauchy Schwartz, taking supremums, and absorbing with Young's inequality we get

$$E_t[\tilde{\psi}_\lambda] \le C(T)\left(E_0[\tilde{\psi}_\lambda] + \left|\left|\Box_g \tilde{\psi}_\lambda - \frac{2a}{l^2}\tilde{\psi}_\lambda\right|\right|_{L^2(\mathcal{M}_{[0,T]})}\right). \tag{2.179}$$

Using this inequality on the difference $\psi - \tilde{\psi}_\lambda$ yields

$$\left|E_t[\psi - \tilde{\psi}_\lambda]\right| \le C(T)\left|\left|\Box_g \tilde{\psi}_\lambda - \frac{2a}{l^2}\tilde{\psi}_\lambda\right|\right|_{L^2(\mathcal{M}_{[0,T]})}, \tag{2.180}$$

then for $\epsilon > 0$ we simply choose a large enough λ and the result follows. □

We now quote theorem 2.36 from [Sbi15]. This tells us that the Gaussian beam energy is localised around the geodesic Killing energy.

Lemma 2.6.5. *For all $\epsilon > 0$ there exists a neighbourhood $\mathcal{N}_0$ of $\mathcal{N}$ such that*

$$\left|E_t[\tilde{\psi}_\lambda|_{\mathcal{N}_0}] - (-g(T,\dot{\gamma})|_{Im(\gamma)\cap\Sigma_t})\right| < \epsilon, \tag{2.181}$$

for all $0 \le t \le T$.

We now show that the constructed solutions to (2.23) are losing energy very slowly.

Lemma 2.6.6. *Let $T > 0$ then for all $\epsilon > 0$ there exists a solution ψ of* (2.23) *with initial data supported away from $\mathcal{H}$, satisfying Dirichlet, Neumann or Robin boundary conditions with*

$$E_0[\psi] = 1, \tag{2.182}$$

such that

$$E_t[\psi] \le 1 - \epsilon, \tag{2.183}$$

for all $0 \le t \le T$.

Proof. It follows from ∂_t being Killing that $-g(T,\dot{\gamma})|_{Im(\gamma)\cap\Sigma_t}$ is a constant. We may renormalise this to 1 when solving for γ. The result then follows from the triangle inequality. □

With these results we can now prove theorem 2.6.1

Proof of theorem 2.6.1. Assume there exists a constant C independent of T, and ψ, such that follow estimate holds for all solutions (2.23)

$$\int_0^T E_t[\psi]dt \leq CE_0[\psi]. \tag{2.184}$$

As the energy is a decreasing function of t we have

$$TE_T[\psi] = \int_0^T E_T[\psi]dt \leq \int_0^T E_t[\psi]dt \leq CE_0[\psi]. \tag{2.185}$$

Choosing $T = 2C$, and constructing ψ from lemma 2.6.6, with the choice $\epsilon = \frac{1}{4}$ we deduce

$$\frac{3}{2}C \leq C. \tag{2.186}$$

A clear contradiction. We now extend this result to the non-degenerate energy by observing that for ψ supported away from $\mathcal{H}$ we have the estimate

$$\int_{\Sigma_0} \mathcal{E}[\psi] \leq CE_0[\psi], \tag{2.187}$$

and the estimate

$$\int_0^T E_t[\psi]dt \leq C\int_{\mathcal{M}_{[0,T]}} \mathcal{E}[\psi], \tag{2.188}$$

from which we construct the same contradiction. □

3

The Einstein–Klein-Gordon System

3.1 Introduction

3.1.1 The results

In this chapter the Einstein–Klein-Gordon system, within the class of square flat toroidal symmetry is presented (1.61)-(1.65), and five results are proven about its analysis.

Theorem 3.1.1 (Wellposedness)**.** *Given suitable initial data on a characteristic surface intersecting null infinity, and Dirichlet or Neumann boundary conditions for the field ψ. The Einstein–Klein-Gordon system* (1.61)-(1.65) *(with $\kappa \in (0, \frac{2}{3})$) has a unique toroidally symmetric weak solution in a future neighbourhood of this surface.*

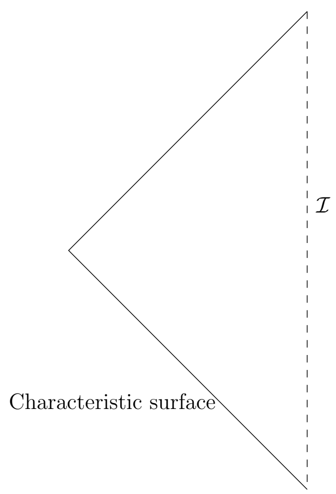


Figure 3.1: Region of solution

Furthermore the maximal development of this solution is unique.

Theorem 3.1.2 (Orbital Stability). *Assume we are given initial data on a characteristic surface intersecting null infinity that is 'close' to the TAdSS data, and Dirichlet or Neumann boundary conditions for the field ψ. The maximal development for the Einstein–Klein-Gordon system (with $\kappa \in (0, \frac{1}{2}]$) contains a black hole, and exterior region with complete null infinity. The solution is qualitatively similar to TAdSS. Furthermore there exists a constant $D > 0$, depending only on the initial data, Klein-Gordon 'mass', and AdS radius such that*

$$\left| r^{\frac{3}{2}-\kappa}\psi(u,v) \right| \leq D, \tag{3.1}$$

holds on the intersection of the regular region of the spacetime and the exterior of the black hole. Where r is a radial function, and u, v are Eddington-Finkelstein coordinates on the spacetime constructed as part of the proof.

Theorem 3.1.3 (Asymptotic Stability). *Assume we are given initial data on a characteristic surface intersecting null infinity that is 'close' to the TAdSS data, and Dirichlet or Neumann boundary conditions for the field ψ. On the maximal development for the Einstein–Klein-Gordon system (with $\kappa \in (0, \frac{1}{2})$), we have constants $C, D > 0$ depending only on the initial data, Klein-Gordon 'mass', and AdS radius such that*

$$\left| r^{\frac{3}{2}-\kappa}\psi(u,v) \right| \leq D\exp(-Cv), \tag{3.2}$$

holds on the intersection of the regular region of the spacetime and the exterior of the black hole. Where v is an Eddington-Finkelstein coordinate.

Theorem 3.1.4. *We have that for $\kappa \in (0, \frac{1}{2})$, M the initial final renormalised Hawking mass at null infinity, $\mathcal{H}$ the event horizon of the spacetime, and $r_+ := (2Ml^2)^{\frac{1}{3}}$ that the Lorentzian Penrose inequality*

$$\sup_{\mathcal{H}} r \leq r_+, \tag{3.3}$$

holds. Furthermore we have along $\mathcal{H}$ that r converges to r_+ exponentially in v an Eddington-Finkelstein coordinate.

These can all be summarised in the following theorem

Theorem 3.1.5. *Assume we are given initial data on a characteristic surface intersecting null infinity that is 'close' to the TAdSS data, and Dirichlet or Neumann boundary conditions for the field ψ. The associated maximal development (with $\kappa \in (0, \frac{1}{2})$), is a black hole spacetime with a regular future horizon, and a complete null infinity. Furthermore the estimate*

$$\left| r^{\frac{3}{2}-\kappa}\psi(u,v) \right| \leq D\exp(-Cv), \tag{3.4}$$

holds on the intersection of the regular region of the spacetime and the exterior of the black hole. From which we may deduce that the metric is converging exponentially in v, uniformly in u, to a toroidal AdS Schwarzschild solution with mass M, in the Eddington-Finkelstein gauge. It is in this sense we say that the toroidal AdS Schwarzschild is stable within the class of square flat toroidal symmetries.

3.2 Structure of the argument

The proofs of this chapter are quite long and complex; the following flowchart has been included to help the reader understand the structure and dependences.

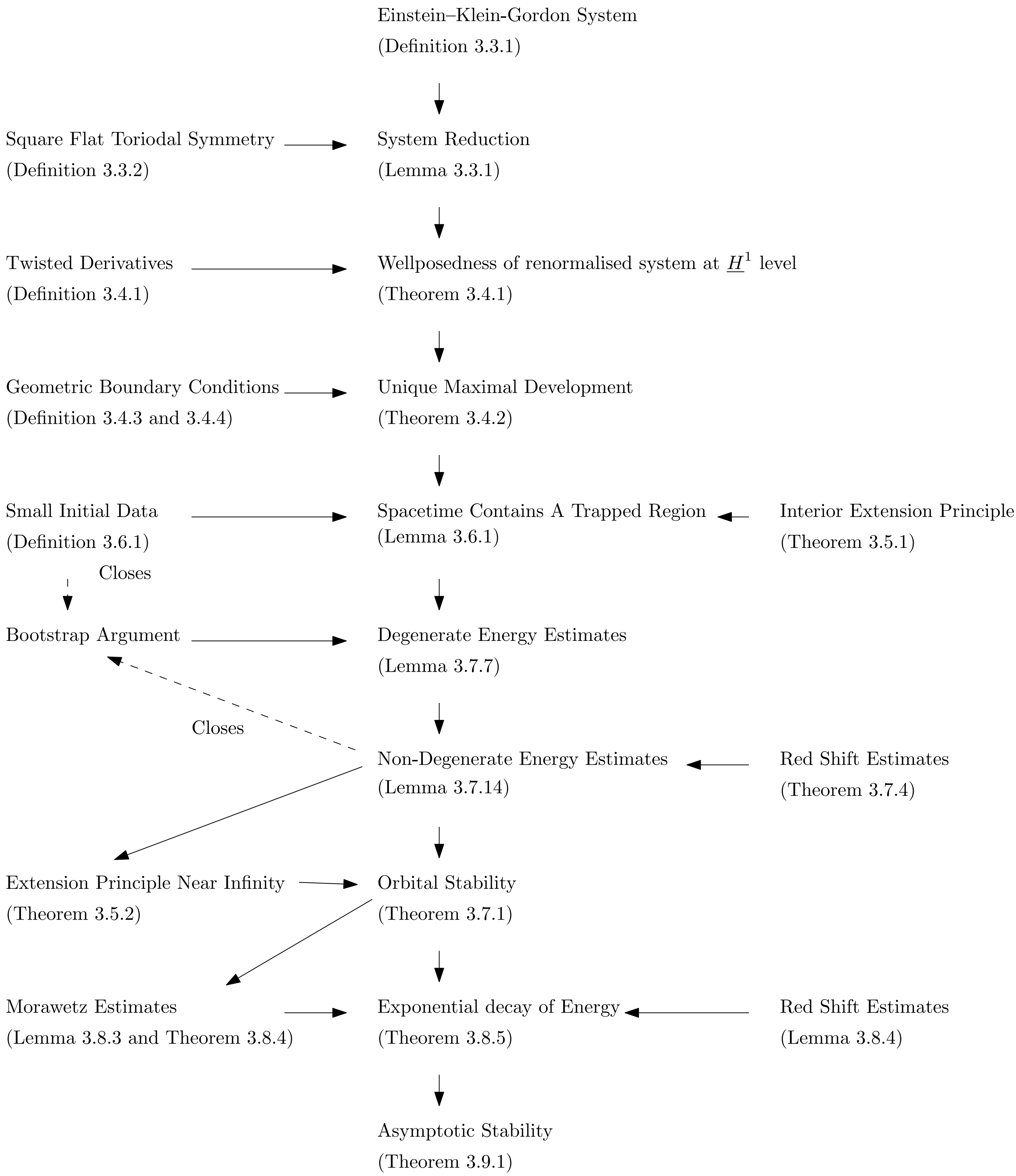


Figure 3.2: Structure of the arguments

Definition 3.3.1. *The Einstein–Klein-Gordon system (EKG) in an asymptotically anti de-Sitter space time is given by*

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{3}{l^2} g_{\mu\nu} &= 8\pi T_{\mu\nu}, \\ \Box_g \psi - \frac{2a}{l^2}\psi &= 0, \\ \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \psi \nabla^\sigma \psi - g_{\mu\nu} \frac{a}{l^2}\psi^2 &= T_{\mu\nu}. \end{aligned} \tag{3.5}$$

Here g, ψ are the Lorentzian metric and Klein-Gordon field which we are solving for respectively. $R_{\mu\nu}$ is the Ricci curvature, R the scalar curvature, l the AdS radius related the cosmological constant of the system Λ through the relationship $\Lambda = \frac{-3}{l^2}$, a is a negative constant which can be thought of as the mass of the Klein-Gordon equation. Keeping with the notation of [HW13] we define a parameter $\kappa = \sqrt{\frac{9}{8} + 2a}$ with $\kappa \in (0, 1)$.

System reduction

Label the spacetime coordinates (u, v, x, y). As seen in [Gow74], imposing a global $\mathfrak{T}^2$ symmetry on the spacetime enforces the product form $\mathcal{M} = \mathcal{Q}^+ \times \mathfrak{T}^2$. Where $\mathcal{Q}^+$ is a two dimensional Lorentzian manifold, and $\mathfrak{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Furthermore, the metric may be put into the form

$$g = -\,\Omega^2(u,v)dudv + r^2(u,v)\left(A(u,v)dx + B(u,v)dy\right)^2 + (B(u,v)dx + C(u,v)dy)^2, \tag{3.6}$$

where $AC - B^2 = 1$, and $A + C > 0$.
Choosing $A = C = 1$ and $B = 0$ retains similarity to the spherical problem. This gives the torus the properties of being square and flat.

Definition 3.3.2. *If a Lorentzian manifold has topology $\mathcal{M} = \mathcal{Q}^+ \times \mathfrak{T}^2$, and a metric of the form*

$$g = -\Omega^2(u,v)dudv + r^2(u,v)\left(dx^2 + dy^2\right). \tag{3.7}$$

We say the spacetime has a square flat toroidal symmetry.

Lemma 3.3.1. *For a metric of the form* (3.7)*, the system* (3.5) *reduces to*

$$\partial_u\left(\frac{r_u}{\Omega^2}\right) = -4\pi r \frac{(\partial_u\psi)^2}{\Omega^2}, \tag{3.8}$$

$$\partial_v\left(\frac{r_v}{\Omega^2}\right) = -4\pi r \frac{(\partial_v\psi)^2}{\Omega^2}, \tag{3.9}$$

$$r_{uv} = -\frac{r_u r_v}{r} + \frac{2\pi a r}{l^2}\Omega^2\psi^2 - \frac{3}{4}\frac{r}{l^2}\Omega^2, \tag{3.10}$$

$$(\log\Omega)_{uv} = -4\pi\partial_u\psi\partial_v\psi + \frac{r_u r_v}{r^2}, \tag{3.11}$$

$$\partial_u \partial_v \psi = -\frac{r_u}{r}\psi_v - \frac{r_v}{r}\psi_u - \frac{\Omega^2 a}{2l^2}\psi. \tag{3.12}$$

Proof. To see this we need to compute the Ricci curvature. (3.8) and (3.9) are the uu and vv components respectively. (3.10) comes from the uv component and (3.11), (3.12) follow from the other components. Conversely for the metric and field given by a solution to (3.8)-(3.12), the system of equations (3.5) holds. □

Renormalised Hawking mass:

We consider a suitable modification to [HW13], [HS12], and [HS13b]. We define the first renormalised Hawking mass as

$$\varpi_1 = \frac{2r_u r_v r}{\Omega^2} + \frac{r^3}{2l^2}. \tag{3.13}$$

It can be seen that provided equations (3.8)-(3.12) hold, this quantity satisfies the transport equations:

$$\partial_u \varpi_1 = -8\pi r^2 \frac{r_v}{\Omega^2}(\partial_u \psi)^2 + \frac{4\pi r^2 a}{l^2} r_u \psi^2, \tag{3.14}$$

$$\partial_v \varpi_1 = -8\pi r^2 \frac{r_u}{\Omega^2}(\partial_v \psi)^2 + \frac{4\pi r^2 a}{l^2} r_v \psi^2. \tag{3.15}$$

We may replace some of the equations in lemma 3.3.1 with these transport equations. This follows from the following lemma (where we assume derivatives to be taken in a weak sense).

Lemma 3.3.2. *Suppose* (3.10), (3.12), (3.14) *and* (3.15), *hold (where* Ω *is defined through* (3.13)*). Then we have that* (3.8) *and* (3.9) *holds. Furthermore if* (3.10) *can be differentiated in* u*, then* (3.11) *holds.*

We may also express the wave equation for r in terms of ϖ_1 as

$$r_{uv} = -\frac{\Omega^2}{2}\left(\frac{\varpi_1}{r^2} + \frac{r}{l^2}\right) + \frac{2\pi a}{l^2}\Omega^2\psi^2. \tag{3.16}$$

Similar to the system [HS13b], (3.14), and (3.15) imply that we can think of the Hawking mass as potential for a weighted H^1 energy.

3.4 Wellposedness of the initial-boundary-value problem

We will now discuss the wellposedness of the Einstein–Klein-Gordon system (3.8)-(3.12). This section follows in a similar fashion to [HW13].

The variables in (3.8)-(3.12), while having a clear geometrical meaning, are inconvenient to analyse the system. We anticipate the behaviour $\Omega^2 \sim r^2, \psi \sim r^{-\frac{3}{2}+\kappa}$ at the conformal boundary (where we expect $r \to \infty$). Furthermore when we introduce Neumann boundary conditions to the problem, the variable ϖ_1 will no longer form a potential for a useful H^1 energy. The quantity will diverge as $r \to \infty$. To rectify these issues we proceed by solving an equivalent system that has undergone a renormalisation scheme, as in [HW13]. It is worth remarking that the modifications to the toroidal case are sub-leading, and the system behaves almost identically in the analysis of wellposedness. We will assume for now that the solutions have enough regularity for the (3.8) to (3.12) to be understood in a weak sense but will postpone the discussion until section 3.4.3.

3.4.1 Renormalised system

Motivated by [HW13] we introduce the twisted derivative.

Definition 3.4.1. *Let $f \in C^1$, then the twisted derivative is defined as the differential operator given by*

$$\tilde{\nabla}_\mu \psi = f \nabla_\mu \left(\frac{\psi}{f} \right). \tag{3.17}$$

Following [HW13], and considering the linear problem in [DW16], the canonical choice of twisting function is

$$r^g, \text{ where } g = -\frac{3}{2} + \kappa. \tag{3.18}$$

We define the second renormalised Hawking mass to be

$$\varpi_2 = \varpi_1 - 2\pi g \frac{r^3}{l^2} \psi^2. \tag{3.19}$$

The latter term has been introduced to cancel the diverging term as $\mathcal{I}$ is approached.

Lemma 3.4.1. *Define the variables*

$$r = \frac{1}{\tilde{r}}, \quad \varpi_1 = \varpi_2 + 2\pi g \frac{r^3}{l^2} \psi^2, \tag{3.20}$$

then the EKG system is equivalent to

$$\partial_u \varpi_2 = -8\pi r^2 \frac{r_v}{\Omega^2} (\tilde{\nabla}_u \psi)^2 - 8\pi g \left(\varpi_2 + 2\pi g \frac{r^3}{l^2} \psi^2 \right) \psi \tilde{\nabla}_u \psi - 4\pi \psi^2 r_u g^2 \frac{\left(\varpi_2 + 2\pi g \frac{r^3}{l^2} \psi^2 \right)}{r}, \tag{3.21}$$

$$\tilde{r}_{uv} = \Omega^2 \tilde{r}^2 \left(\frac{3}{2} \varpi_2 \tilde{r}^2 - \frac{2\pi g^2}{l^2 \tilde{r}} \psi^2 \right), \tag{3.22}$$

$$\partial_v \left(r \tilde{\nabla}_u \psi \right) = \left(\kappa - \frac{1}{2} \right) r_u \tilde{\nabla}_v \psi - \frac{\Omega^2}{4} r V \psi, \tag{3.23}$$

with auxiliary variables

$$\Omega^2 = -\frac{4 r^4 \tilde{r}_u \tilde{r}_v}{\mu_1}, \quad \mu_1 = -\frac{2\varpi_1}{r} + \frac{r^2}{l^2}, \quad V = \frac{2g^2}{r^3} \varpi_1 + \frac{8\pi a g}{l^2} \psi^2, \tag{3.24}$$

and constraint equation

$$\partial_v \varpi_2 = -8\pi r^2 \frac{r_u}{\Omega^2}(\tilde{\nabla}_v\psi)^2 - 8\pi g\left(\varpi_2 + 2\pi g\frac{r^3}{l^2}\psi^2\right)\psi\tilde{\nabla}_v\psi - 4\pi\psi^2 r_v g^2\frac{\left(\varpi_2 + 2\pi g\frac{r^3}{l^2}\psi^2\right)}{r}. \quad (3.25)$$

We remark that equation (3.23) *can also be expressed as*

$$\partial_u\left(r\tilde{\nabla}_v\psi\right) = \left(\kappa - \frac{1}{2}\right) r_v\tilde{\nabla}_u\psi - \frac{\Omega^2}{4} rV\psi. \quad (3.26)$$

Proof. This is just calculation. We simply study the derivatives of the variables $\tilde{r}$, ϖ_2, and use the equations (3.8) - (3.12) to simplify. For the Klein-Gordon equation we substitute in the definition of the twisted derivative and simplify. □

The domain

We define the triangular domain:

$$\Delta_{\delta,u_0} := \{(u,v) \in \mathbb{R}^2 : u_0 \leq v \leq u_0 + \delta, v < u \leq u_0 + \delta\}, \quad (3.27)$$

and the boundary piece

$$\mathcal{I} = \overline{\Delta}_{\delta,u_0} \backslash \Delta_{\delta,u_0} = \{(u,v) \in \overline{\Delta}_{\delta,u_0} : u = v\}. \quad (3.28)$$

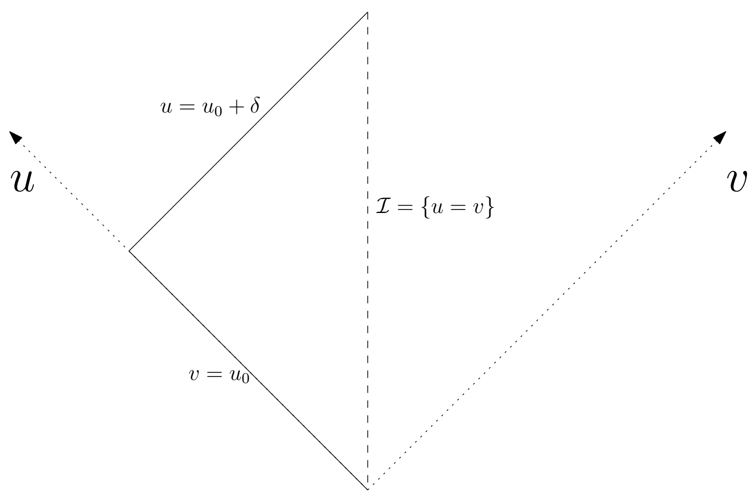


Figure 3.3: Diagram of Δ_{δ,u_0}

A solution to (3.21)-(3.23) obeying constraint (3.25) can be constructed on Δ_{δ,u_0}. We will need to specify initial data, and boundary conditions for this system. We follow the generality of [HW13], but keep in mind that we are mainly interested in perturbations of the TAdSS solution.

The Klein-Gordon mass

Throughout this chapter three quantities related to the Klein-Gordon mass are used fairly interchangeably. This is largely to clean up the algebra. In order improve clarity we collect them here

- a, denotes the Klein-Gordon mass,
- g, denotes the radial decay of the field, $g = -\frac{3}{2} + \sqrt{\frac{9}{4} + 2a}$,
- κ, denotes the radical part of g, $\kappa = \sqrt{\frac{9}{4} + 2a}$.

We now collect some of the key values of the quantities and how they relate in a table below

	BF Lower Bound	Conformal	BF Upper Bound
a	$-\frac{9}{8}$	-1	$-\frac{5}{8}$
κ	0	$\frac{1}{2}$	1
g	$-\frac{3}{2}$	-1	$-\frac{1}{2}$

3.4.2 Initial data and boundary conditions

Initial data

Definition 3.4.2. *Let $\mathcal{N} = (u_0, u_1]$ be a real interval. Then a free data set is a pair of functions $(\bar{\tilde{r}}, \overline{\psi}) \in C^2(\mathcal{N}) \times C^1(\mathcal{N})$ such that:*

- $\bar{\tilde{r}} > 0$ *and* $\bar{\tilde{r}}_u > 0$ *in* $\mathcal{N}$, *as well as* $\lim_{u\to u_0} \bar{\tilde{r}}_u = \frac{1}{2l^2}$ *and* $\lim_{u\to u_0} \bar{\tilde{r}}_{uu} = 0$.
- *The following bounds hold on the initial data*
$$\int_{u_0}^{u_1} \left[\left(\bar{\tilde{\nabla}}_u \overline{\psi}\right)^2 + \overline{\psi}^2 \right] (u - u_0)^{-2} du < \infty, \tag{3.29}$$
- $$\sup_{\mathcal{N}} \left| \overline{\psi} \cdot \bar{\tilde{r}}^{-\frac{3}{2}+\kappa} \right| + \sup_{\mathcal{N}} \left| r^{\frac{1}{2}} \bar{\tilde{\nabla}}_u \overline{\psi} \right| < \infty. \tag{3.30}$$

Where $\bar{\tilde{\nabla}}_u$ is a twisted derivative with twisting function: $\overline{f} = \left(\frac{1}{2}(u - u_0)\right)^{\frac{3}{2}-\kappa}$.

With a free data set we are now able to construct a *complete initial data* set $(\bar{\tilde{r}}, \overline{\psi}, \overline{\varpi_2}, \overline{\tilde{r}_v})$ Let $M_N > 0$, we define $\overline{\varpi_2}$ as the unique $C^1(\mathcal{N})$ solution to:

$$\begin{aligned}\partial_u \overline{\varpi_2} =& 2\pi \frac{\overline{r}^2}{\overline{r}_u}\left(-\frac{2\overline{\varpi_2}}{\overline{r}} + \frac{\overline{r}^2}{l^2} - 4\pi g \frac{\overline{r}^2}{l^2}\overline{\psi}^2\right)(\overline{\tilde{\nabla}}_u \overline{\psi})^2 - 8\pi g\left(\overline{\varpi_2} + 2\pi g \frac{\overline{r}^3}{l^2}\overline{\psi}^2\right)\overline{\psi}\,\overline{\tilde{\nabla}}_u\overline{\psi} \\ & - 4\pi \overline{\psi}^2 \overline{r}_u g^2 \frac{\left(\overline{\varpi_2} + 2\pi g \frac{\overline{r}^3}{l^2}\overline{\psi}^2\right)}{\overline{r}},\end{aligned} \tag{3.31}$$

with boundary condition

$$\lim_{u \to u_0} \overline{\varpi_2} = M_N. \tag{3.32}$$

We define $\overline{\tilde{r}_v}$ is a similar way, as the unique $C^1(\mathcal{N})$ solution of the ODE

$$\partial_u \overline{\tilde{r}_v} = \frac{\bar{\tilde{r}}^2 \overline{\tilde{r}}_u \overline{\tilde{r}_v}}{-\frac{2\overline{\varpi_2}}{\overline{r}} + \frac{\overline{r}^2}{l^2} - 4\pi g \frac{\overline{r}^2}{l^2}\overline{\psi}^2}\left(\frac{3}{2}\overline{\varpi_2}\bar{\tilde{r}}^2 - \frac{2\pi g^2}{l^2 \bar{\tilde{r}}}\overline{\psi}^2\right), \tag{3.33}$$

with boundary condition

$$\lim_{u \to u_0} \overline{\tilde{r}_v} = -\frac{1}{2l^2}. \tag{3.34}$$

Remark 3.4.1. *The choice of* $\bar{\tilde{r}}$ *is equivalent to choosing the scale of the* u *coordinate along* $\mathcal{N}$*. It represents the gauge choice of this problem. Typically one will choose* $\bar{\tilde{r}} = \frac{u-u_0}{2l^2}$*.*
The choice of $\overline{\psi}$ *is free, providing the conditions* (3.29)*, and* (3.30) *hold. The value* M_N *is free provided it is strictly positive.*
The choice of boundary condition for $\overline{\tilde{r}_v}$ *is to ensure that initially* $\overline{\tilde{r}}_u + \overline{\tilde{r}_v} = 0$*, this will be propagated along the boundary* $\mathcal{I}$ *boundary conditions on* $\mathcal{I}$ *as defined in the next section.*

Boundary conditions and weak formulations

Notation

We now define the function

$$\rho = \frac{u - v}{2}, \tag{3.35}$$

and the boundary coordinate

$$t = \frac{u + v}{2}, \tag{3.36}$$

which we can see parametrises $\mathcal{I}$.
We denote twisting with the function ρ by

$$\hat{\nabla}_\mu \psi := \rho^{\frac{3}{2}-\kappa}\nabla_\mu(\psi \rho^{-\frac{3}{2}+\kappa}). \tag{3.37}$$

The $\underline{H}^1$ norm over a set $\mathcal{U} \subset \Delta_{\delta,u_0}$ is given by

$$||\psi||^2_{\underline{H}^1(\mathcal{U})} := \int_{\mathcal{U}} \left(\rho^{-2}\left(\left|\hat{\nabla}\psi\right|^2\right) + \rho^{-2}\psi^2\right) dudv. \tag{3.38}$$

The space $\underline{H}^1_0(\Delta_{\delta,u_0})$ is given by the completion of $C^\infty_c(\Delta_{\delta,u_0})$ in the $\underline{H}^1$ norm.

Boundary conditions for $\tilde{r}$

In order that we produce a spacetime that is asymptotically AdS we will insist

$$\tilde{r}|_{\mathcal{I}} = 0. \tag{3.39}$$

As a consequence of $T := \partial_u + \partial_v$ being tangent to $\mathcal{I}$, we see that

$$T(\tilde{r}) = 0, \tag{3.40}$$

along $\mathcal{I}$.

Weak formulations for ψ

We want to impose homogeneous Dirichlet and Neumann boundary conditions on the field. As we will be solving at a $C^0\underline{H}^1$ level of regularity, following the weak formulation for hyperbolic equations in [Lad85], we see this requires the following weak forms for (3.23).

Definition 3.4.3. *We say ψ weakly solves* (3.23) *with Dirichlet boundary conditions if*

$$\int_{\Delta_{\delta,u_0}} \left(\tilde{\nabla}^\mu \phi \tilde{\nabla}_\mu \psi - V\psi\phi\right) dVol = 0, \tag{3.41}$$

holds for all $\phi \in \underline{H}^1_0(\Delta_{\delta,u_0})$, and $\rho^{-\frac{3}{2}+\kappa}\psi = 0$ on $\mathcal{I}$ in a trace sense.

Definition 3.4.4. *We say ψ weakly solves* (3.23) *with Neumann boundary conditions if*

$$\int_{\Delta_{\delta,u_0}} \left(\tilde{\nabla}^\mu \phi \tilde{\nabla}_\mu \psi - V\psi\phi\right) dVol = 0, \tag{3.42}$$

holds for all $\phi \in \underline{H}^1(\Delta_{\delta,u_0})$ with $\phi|_{\{v=u_0\}} = \phi|_{\{u=u_0+\delta\}} = 0$ in a trace sense.

Assuming higher regularity on the solution, we can define the following classical notions of boundary conditions:

We will say a solution classically satisfies Neumann boundary conditions on $\mathcal{I}$ if

$$\rho^{-\frac{1}{2}-\kappa}\left(\tilde{\nabla}_\rho \psi\right) = 0. \tag{3.43}$$

We will say a solution classically satisfies Dirichlet boundary conditions on $\mathcal{I}$ if

$$\rho^{-\frac{3}{2}+\kappa}\psi = 0. \tag{3.44}$$

3.4.3 Wellposedness

Regularity

In this section we define the regularity we require for a weak solution to exist. We define the

$C^0\underline{H}^1$ norm by

$$\begin{aligned}||\psi||^2_{C^0\underline{H}^1(\Delta_{\delta,u_0})} := &\sup_{(u,v)\in\Delta_{\delta,u_0}} \int_v^u \left(\rho^{-2}\left(\hat{\nabla}_u\psi\right)^2 + \rho^{-2}\psi^2\right) du' \\ &+ \sup_{(u,v)\in\Delta_{\delta,u_0}} \int_{v_0}^v \left(\rho^{-2}\left(\hat{\nabla}_v\psi\right)^2 + \rho^{-2}\psi^2\right) dv'.\end{aligned} \tag{3.45}$$

Definition 3.4.5. *A weak solution to the renormalised EKG system is an element of the function space:*

$$\mathfrak{W} = \{(\tilde{r}, \varpi_2, \psi) : \tilde{r} \in C^1_{loc.}, \psi \in C^0\underline{H}^1, \varpi_2 \in W^{1,1}_{loc}, \quad \tilde{r}_{uv}, \tilde{r}_{uu}, \psi_u, (\varpi_2)_u \in C^0_{loc.}\}, \tag{3.46}$$

that satisfies (3.21)-(3.23) *in a weak sense. That is equations* (3.21) *and* (3.22) *hold classically,* (3.25) *holds almost everywhere and* (3.23) *holds weakly in the sense of 3.41 or 3.42.*

Remark 3.4.2. *We note the mixed regularity of the solution. Notably $\psi_u \in C^0_{loc.}$ is allowing some equations to hold pointwise. From this it follows that if we have a weak solution to the EKG system we necessarily have that $\tilde{r}_{uuv}, \Omega, \Omega_u \in C^0_{loc.}$.*

Lemma 3.4.2. *If we have a weak solution to the renormalised EKG equations, then the EKG equations* (3.8)-(3.12) *hold weakly. We say the metric* (3.7) *solves* (3.5) *weakly, and is C^0.*

Theorem 3.4.1. *Fix $0 < \kappa < \frac{2}{3}$, let $(\overline{\tilde{r}}, \overline{\psi})$ be a free data set on $\mathcal{N} = (u_0, u_1]$, and fix Neumann or Dirichlet boundary conditions. Then there exists a $\delta > 0$, such that the following holds. There exists a weak solution $(\tilde{r}, \varpi_2, \psi) \in \mathfrak{W}$, of the renormalised Einstein Klein-Gordon equations in the triangle Δ_{δ,u_0}, such that*

- *$\tilde{r} \to 0$ as $\mathcal{I}$ is approached,*
- *ψ satisfies either Dirichlet or Neumann boundary conditions weakly,*
- *The functions ψ and $\tilde{r}$ agree as C^1 functions with $\overline{\psi}$ and $\overline{\tilde{r}}$, respectively when restricted to $\{v = u_0\}$.*

Proof. The proof of this theorem is essentially identical to that found in [HW13]. There is a minor difference in that there is a slight change in sub-leading terms compared to the spherical case. Largely this manifests as certain terms not being present so the argument is slightly cleaner. It follows a Banach fixed point theorem argument of establishing a map whose fixed point is a solution to (3.21)-(3.23) and then establishing that it is a contraction over a ball of radius b in the space $\mathfrak{W}$.

□

Remark 3.4.3. *If more regularity is assumed on the initial data then just as in [HW13] it may be shown that we have a classical solution. The boundary conditions hold classically, and $\mathcal{T}\psi := -\frac{r_u}{\Omega^2}\partial_v\psi + \frac{r_v}{\Omega^2}\partial_u\psi$ decays like $\rho^{\frac{3}{2}-\kappa}$, as the boundary is approached.*

Remark 3.4.4. *As the field $\psi \in C^0\underline{H}^1$, it obeys energy estimates. These are shown by working at this higher level of regularity, and recovered by a density argument (similar to proposition 8.1 in [HW13]). In later sections when deriving these energy estimates, we will see boundary terms that won't make sense at the current level of regularity. We can however see they vanish at a higher level of regularity, and thus may be dropped from the estimates.*

Further properties of the local solution

Definition 3.4.6. *We say a spacetime is asymptotically AdS if the metric has the following form*

$$g = -\frac{1}{\tilde{r}^2}\left(\left(\frac{1}{l^2} + \mathcal{O}\left(\tilde{r}^{2+\epsilon}\right)\right) d\hat{u}d\hat{v} + d\mathfrak{T}^2\right), \tag{3.47}$$

and weakly asymptotically AdS if it has the form

$$g = -\frac{1}{\tilde{r}^2}\left(\left(\frac{1}{l^2} + \mathcal{O}\left(\tilde{r}^{\epsilon}\right)\right) d\hat{u}d\hat{v} + d\mathfrak{T}^2\right). \tag{3.48}$$

for some $\epsilon > 0$.

Remark 3.4.5. *We note that the $\mathfrak{W}$ solution constructs an asymptotically AdS set. If one makes the definitions:*

$$f(u) := \tilde{r}_u(u,u), \quad g(v) := \tilde{r}_v(v,v), \tag{3.49}$$

(these are bounded functions on Δ_{δ,u_0} from the wellposedness proof), and then the coordinate transformation

$$\frac{d\hat{u}}{du} = 2l^2 f(u), \quad \frac{d\hat{v}}{dv} = -2l^2 g(v), \tag{3.50}$$

then on $\mathcal{I}$ we have $\tilde{r}_u = \frac{1}{2l^2}$, and $\tilde{r}_v = -\frac{1}{2l^2}$. Furthermore from the analysis of [HW13], or similarly in section 3.5.2 one can see that in this coordinate system

$$\tilde{r}_{\hat{u}\hat{v}} = \mathcal{O}\left(\tilde{r}^{2-2\kappa}\right). \tag{3.51}$$

We deduce that

$$\tilde{r}_u = \frac{1}{2l^2} + \mathcal{O}\left(\tilde{r}^{3-2\kappa}\right), \quad \tilde{r}_v = -\frac{1}{2l^2} + \mathcal{O}\left(\tilde{r}^{3-2\kappa}\right), \tag{3.52}$$

and the conformal factor satisfies

$$\Omega^2 = \frac{-4r^4\tilde{r}_u\tilde{r}_v}{\mu_1} = \frac{1}{\tilde{r}^2 l^2}\left(1 + \mathcal{O}\left(\tilde{r}^{3-2\kappa}\right)\right). \tag{3.53}$$

This implies the metric has the form

$$g = -\frac{1}{\tilde{r}^2}\left(\left(\frac{1}{l^2} + \mathcal{O}\left(\tilde{r}^{3-2\kappa}\right)\right) d\hat{u}d\hat{v} + d\mathfrak{T}^2\right). \tag{3.54}$$

We remark that for $\kappa \geq \frac{1}{2}$ the spacetime is only weakly asymptotically AdS.

Remark 3.4.6. *We note that the local* $\mathfrak{W}$ *solution will obey all the estimates of lemma 5.1 of [HW13]. We list them here*

$$\begin{aligned} &e^{-b} \leq \frac{\tilde{r}}{\rho} \leq e^{b}, \quad e^{-b} \leq 2\tilde{r}_u \leq e^{b}, \quad e^{-b} \leq -2\tilde{r}_v \leq e^{b}, \quad |\tilde{r}_{uv}| \leq b, \\ &|T(\tilde{r})| \leq b \cdot \rho, \quad \varpi_1 \leq C_b \cdot \tilde{r}^{-2\kappa}, \quad \left|\frac{\Omega^2}{r^2}\right| \leq C_b, \quad |\psi| \leq B_b \cdot \tilde{r}^{\frac{3}{2}-\kappa}, \quad \left|\hat{\nabla}_u \psi\right| \leq C_b \cdot \tilde{r}^{\frac{1}{2}-\frac{s}{4}}. \end{aligned} \tag{3.55}$$

We first note the estimate $e^{-b} \leq \frac{\tilde{r}}{\rho} \leq e^{b}$ *implies*

$$\lim_{u \to v} \tilde{r} = 0. \tag{3.56}$$

We remark further that as $\Omega^2 := \frac{4r^4 \tilde{r}_u \tilde{r}_v}{\frac{-2\varpi_1}{r} + \frac{r^2}{l^2}}$ *the estimate*

$$\left|\frac{r^2}{\Omega^2}\right| \leq C_b, \tag{3.57}$$

immediately follows.

Remark 3.4.7. *It follows that corollary 5.3 of [HW13] holds, that is*

$$\left|\frac{\tilde{r}_u}{\tilde{r}} - \frac{1}{2\rho}\right| \leq 3b \cdot e^{b}. \tag{3.58}$$

The proof is identical. We remark this result is important for establishing decay of the twisted derivative with respect to r^g

$$\tilde{\nabla}_u \psi = \hat{\nabla}_u \psi + \psi \left(\frac{3}{2} - \kappa\right) \left(\frac{\tilde{r}_u}{\tilde{r}} - \frac{1}{2\rho}\right). \tag{3.59}$$

This allows us to see that the decay of $\tilde{\nabla}_u \psi$ *is as strong as* $\hat{\nabla}_u \psi$*, and that twisting with* ρ *and* $\tilde{r}$ *are equivalent.*

Remark 3.4.8. *For* $\mathcal{U} \subset \mathbb{R}^{1+1}$ *we can define the following norm*

$$||\psi||^2_{\underline{H}^1_g(\mathcal{U})} := \int_{\mathcal{U}} \left(\tilde{r}^{-2} \left(\left|\tilde{\nabla}\psi\right|^2\right) + \tilde{r}^{-2}\psi^2\right) dudv. \tag{3.60}$$

We see that in Δ_{δ,u_0} *this is equivalent to the* $\underline{H}^1$ *norm. We can equivalently define* $\underline{H}^1_{0,g}(\Delta_{\delta,u_0})$ *as the completion of* $C^\infty_c(\Delta_{\delta,u_0})$ *with the* $\underline{H}^1_g$ *norm.*
This norm can be promoted to subsets of $\mathbb{R}^{1+1} \times \mathfrak{T}^2$ *by also integrating over the toroidal variables. We will slightly abuse notation and use it to mean both as all our functions are toroidally symmetric and pose no change to its value.*

We now define some important geometric quantities.

Definition 3.4.7. *We define the Kodama vector field*

$$\mathcal{T} = -\frac{r_v}{\Omega^2}\partial_u + \frac{r_u}{\Omega^2}\partial_v, \tag{3.61}$$

its orthogonal complement

$$\mathcal{R} = -\frac{r_v}{\Omega^2}\partial_u - \frac{r_u}{\Omega^2}\partial_v = (dr)^\sharp, \tag{3.62}$$

and the operators

$$\tilde{\mathcal{T}}\psi = \mathcal{T}^\mu \tilde{\nabla}_\mu \psi, \quad \tilde{\mathcal{R}}\psi = \mathcal{R}^\mu \tilde{\nabla}_\mu \psi. \tag{3.63}$$

We also note

$$\tilde{\mathcal{T}}\psi = \mathcal{T}\psi. \tag{3.64}$$

Definition 3.4.8. *It is now convenient to define the final renormalised Hawking mass as*

$$\varpi = \frac{2r_u r_v r}{\Omega^2} e^{4\pi g\psi^2} + \frac{r^3}{2l^2}, \tag{3.65}$$

it is clear that it is invariant under change in null coordinates.

Lemma 3.4.3. ϖ *satisfies the following differential equations*

$$\begin{aligned}
\partial_u \varpi &= -\frac{8\pi r^2 r_v}{\Omega^2}(\tilde{\nabla}_u \psi)^2 e^{4\pi g\psi^2} + \frac{4\pi g^2 r_u}{r}\varpi\psi^2 + \frac{r_u r^2}{l^2} f(\psi^2), \\
\partial_v \varpi &= -\frac{8\pi r^2 r_u}{\Omega^2}(\tilde{\nabla}_v \psi)^2 e^{4\pi g\psi^2} + \frac{4\pi g^2 r_v}{r}\varpi\psi^2 + \frac{r_v r^2}{l^2} f(\psi^2),
\end{aligned} \tag{3.66}$$

where

$$f(\psi^2) = e^{4\pi g\psi^2}\left(4\pi a\psi^2 - \frac{3}{2}\right) - 2\pi g^2\psi^2 + \frac{3}{2}. \tag{3.67}$$

Proof. We begin by directly studying the u derivative of ϖ

$$\partial_u \varpi = \partial_u\left(\frac{2r_u r_v r}{\Omega^2}\right) e^{4\pi g\psi^2} + \frac{16 g\pi r_u r_v r}{\Omega^2}\psi\psi_u e^{4\pi g\psi^2} + \frac{3}{2l^2} r^2 r_u. \tag{3.68}$$

Equation (3.8) and (3.10) imply that

$$\partial_u\left(\frac{2r_u r_v r}{\Omega^2}\right) = -8\pi\frac{r^2 r_v}{\Omega^2}\psi_u^2 + \frac{4\pi r^2 a}{l^2} r_u\psi^2 - \frac{3}{2l^2} r^2 r_u, \tag{3.69}$$

so

$$\partial_u \varpi = -8\pi\frac{r^2 r_v}{\Omega^2}\psi_u^2 e^{4\pi g\psi^2} + \frac{4\pi r^2 a}{l^2} r_u \psi^2 e^{4\pi g\psi^2} - \frac{3}{2l^2} r^2 r_u e^{4\pi g\psi^2} + \frac{16 g\pi r_u r_v r}{\Omega^2}\psi\psi_u e^{4\pi g\psi^2} + \frac{3}{2l^2} r^2 r_u. \tag{3.70}$$

Recalling the twisting function

$$f = r^g, \tag{3.71}$$

and the identity

$$\psi_u^2 = (\tilde{\nabla}_u \psi)^2 + \frac{2g r_u}{r}\psi\psi_u + \frac{g^2 r_u^2}{r^2}\psi^2. \tag{3.72}$$

We see that

$$\partial_u \varpi = -\frac{8\pi r^2 r_v}{\Omega^2}(\tilde{\nabla}_u \psi)^2 e^{4\pi g\psi^2} + \frac{8\pi g^2 r_u^2 r_v}{\Omega^2}\psi^2 e^{4\pi g\psi^2} + \frac{4\pi r^2 a}{l^2} r_u \psi^2 e^{4\pi g\psi^2} + \frac{3}{2l^2} r^2 r_u\left(1 - e^{4\pi g\psi^2}\right). \tag{3.73}$$

We turn to studying

$$F := \frac{8\pi g^2 r_u^2 r_v}{\Omega^2}\psi^2 e^{4\pi g\psi^2} + \frac{4\pi r^2 a}{l^2} r_u \psi^2 e^{4\pi g\psi^2} + \frac{3}{2l^2} r^2 r_u \left(1 - e^{4\pi g\psi^2}\right). \tag{3.74}$$

Recalling

$$\varpi - \frac{r^3}{2l^2} = \frac{2r_u r_v r}{\Omega^2} e^{4\pi g\psi^2}, \tag{3.75}$$

giving

$$F = \frac{4\pi g^2 r_u}{r}\varpi\psi^2 - 2\pi\frac{r_u r^2 g^2}{l^2}\psi^2 + \frac{4\pi r^2 a}{l^2} r_u \psi^2 e^{4\pi g\psi^2} + \frac{3}{2l^2} r^2 r_u \left(1 - e^{4\pi g\psi^2}\right). \tag{3.76}$$

Expanding (3.76) reveals hows the twisting function removes the divergent terms

$$\begin{aligned} F =& \frac{4\pi g^2 r_u}{r}\varpi\psi^2 - 2\pi\frac{r_u r^2 g^2}{l^2}\psi^2 + \frac{4\pi r_u r^2 a}{l^2}\psi^2 + \frac{4\pi r_u r^2 a}{l^2}\left(e^{4\pi g\psi^2} - 1\right) + \frac{-3\cdot 4\pi g\psi^2}{2l^2} r^2 r_u \\ &+ \frac{3}{2l^2} r^2 r_u \left(1 + 4\pi g\psi^2 - e^{4\pi g\psi^2}\right), \end{aligned} \tag{3.77}$$

factoring

$$\begin{aligned} F =& \frac{4\pi g^2 r_u}{r}\varpi\psi^2 + \frac{2\pi}{l^2} r_u r^2 \left(-g^2 + 2a - 3g\right)\psi^2 + \frac{4\pi r_u r^2 a}{l^2}\underbrace{\left(e^{4\pi g\psi^2} - 1\right)}_{\sim\psi^2}\psi^2 \\ &+ \frac{3}{2l^2} r^2 r_u \underbrace{\left(1 + 4\pi g\psi^2 - e^{4\pi g\psi^2}\right)}_{\sim\psi^4}, \end{aligned} \tag{3.78}$$

recalling the relation

$$-g^2 + 2a - 3g = 0, \tag{3.79}$$

we see the divergent terms are no longer present, and

$$F = \frac{4\pi g^2 r_u}{r}\varpi\psi^2 + \frac{4\pi r_u r^2 a}{l^2}\left(e^{4\pi g\psi^2} - 1\right)\psi^2 + \frac{3}{2l^2} r^2 r_u \left(1 + 4\pi g\psi^2 - e^{4\pi g\psi^2}\right), \tag{3.80}$$

which factorises to

$$F = \frac{4\pi g^2 r_u}{r}\varpi\psi^2 + \frac{r_u r^2}{l^2}\left(e^{4\pi g\psi^2}\left(4\pi a\psi^2 - \frac{3}{2}\right) - 2\pi g^2\psi^2 + \frac{3}{2}\right). \tag{3.81}$$

From here the result follows. By the symmetry of the equations the $\partial_v\varpi$ result is analogous. □

Lemma 3.4.4. *ϖ is constant along $\mathcal{I}$.*

Remark 3.4.9. *The proof of this result is largely technical, and comes down to understanding a regularity issue. Formally as in [HW13] we could compute*

$$T\varpi|_{\mathcal{I}} = \lim_{\rho\to 0} 4\pi r^2 e^{4\pi g\psi^2}\tilde{\nabla}_\rho\psi\cdot\mathcal{T}\psi, \tag{3.82}$$

which for Dirichlet or Neumann conditions can be seen to be 0, for $\psi \in \underline{H}^2$. However at a $\underline{H}^1$ level we do not have enough decay to infer this result. To get around this problem we use the fact that we expect ϖ to be an energy potential for ψ. Performing energy methods with a rescaling of the Kodama vector field gives rise to this expected energy equality. However in this latter setting we can exploit the $C^0\underline{H}^1$ regularity of ψ to see this boundary term is zero. We then integrate the curl of the divergence of ϖ over a smaller triangle, bounded away from $\mathcal{I}$ and take limits to recover the result.

Proof of lemma 3.4.4. From the wave equations (3.23) and (3.26) we derive the following formula for multipliers

$$\begin{aligned}\partial_u\left(h(\tilde{\nabla}_v\psi)^2\right)+\partial_v\left(w(\tilde{\nabla}_u\psi)^2\right)&=2\left(\kappa-\frac{1}{2}\right)\tilde{\nabla}_u\psi\tilde{\nabla}_v\psi\left(\frac{r_v}{r}h+\frac{r_u}{r}w\right)\\&+2\left(\frac{1}{2}h_u-\frac{r_u}{r}h\right)(\tilde{\nabla}_v\psi)^2+2\left(\frac{1}{2}w_v-\frac{r_v}{r}w\right)(\tilde{\nabla}_u\psi)^2\\&-\frac{\Omega^2}{2}V\psi\left(h\tilde{\nabla}_v\psi+w\tilde{\nabla}_u\psi\right).\end{aligned} \tag{3.83}$$

Where h, w are C^1 functions. We expect that Kodama vector field $\mathcal{T}$ should give rise to a conservation law. However from the modifications made to the Hawking mass we will choose a rescaling of this vector field by $X = 8\pi r^2 e^{4\pi g\psi^2}\mathcal{T}$. This amounts to choosing the following functions as multipliers

$$w = 8\pi\frac{r_v}{\Omega^2}r^2e^{4\pi g\psi^2}, \quad h = 8\pi\frac{-r_u}{\Omega^2}r^2e^{4\pi g\psi^2}. \tag{3.84}$$

From (3.83) we see the cross terms involving $\tilde{\nabla}_u\psi\tilde{\nabla}_v\psi$ cancel. We now study the term

$$\frac{\Omega^2}{2}V\psi\left(h\tilde{\nabla}_v\psi+w\tilde{\nabla}_u\psi\right)=4\pi r^2e^{4\pi g\psi^2}V\psi\left(-r_u\nabla_v\psi+r_v\nabla_u\psi\right), \tag{3.85}$$

the twisted derivative terms cancel here, (as we are effectively taking a T derivative but only twist with r). Expanding the potential V in terms of ϖ we have

$$V=\frac{2g^2}{r^3}\varpi e^{-4\pi g\psi^2}-\frac{g^2}{l^2}e^{-4\pi g\psi^2}+\frac{g^2}{l^2}+\frac{8\pi a g}{l^2}\psi^2, \tag{3.86}$$

and thus we derive the equation

$$\begin{aligned}4\pi r^2e^{4\pi g\psi^2}V\psi\left(-r_u\nabla_v\psi+r_v\nabla_u\psi\right)&=-8\pi\frac{g^2}{r}r_u\varpi\psi\nabla_v\psi+8\pi\frac{g^2}{r}r_v\varpi\psi\nabla_u\psi\\&-\frac{4\pi g^2}{l^2}r^2\psi\left(r_v\nabla_u\psi-r_u\nabla_v\psi\right)\\&+\frac{4\pi g^2}{l^2}r^2\psi e^{4\pi g\psi^2}\left(r_v\nabla_u\psi-r_u\nabla_v\psi\right)\\&+\frac{32\pi^2ag}{l^2}r^2\psi^3e^{4\pi g\psi^2}\left(r_v\psi_u-r_u\psi_v\right).\end{aligned} \tag{3.87}$$

Now let us define

$$f(\psi^2) = e^{4\pi g\psi^2}\left(4\pi a\psi^2 - \frac{3}{2}\right) - 2\pi g^2\psi^2 + \frac{3}{2}, \tag{3.88}$$

and consider its T derivative

$$\partial_u\left(\frac{r_v r^2}{l^2} f(\psi^2)\right) + \partial_v\left(\frac{-r_u r^2}{l^2} f(\psi^2)\right), \tag{3.89}$$

it can be quickly seen this will cancel down to

$$\frac{r_v r^2}{l^2}\partial_u\left(f(\psi^2)\right) - \frac{r_u r^2}{l^2}\partial_v\left(f(\psi^2)\right). \tag{3.90}$$

Computing

$$\partial_u\left(f(\psi^2)\right) = 4\pi g^2 e^{4\pi g\psi^2}\psi\psi_u - 4\pi g^2\psi\psi_u + 32\pi^2 a g e^{4\pi g\psi^2}\psi^3\psi_u, \tag{3.91}$$

we see that

$$\begin{aligned}&\partial_u\left(\frac{r_v r^2}{l^2} f(\psi^2)\right) + \partial_v\left(\frac{-r_u r^2}{l^2} f(\psi^2)\right) = \\ &-\frac{4\pi g^2}{l^2} r^2\psi\left(r_v\nabla_u\psi - r_u\nabla_v\psi\right) + \frac{4\pi g^2}{l^2} r^2\psi e^{4\pi g\psi^2}\left(r_v\nabla_u\psi - r_u\nabla_v\psi\right) \\ &+\frac{32\pi^2 a g}{l^2} r^2\psi^3 e^{4\pi g\psi^2}\left(r_v\psi_u - r_u\psi_v\right).\end{aligned} \tag{3.92}$$

Returning to (3.87) we can now write this as a combination of flux terms and a bulk term as follows

$$\begin{aligned}-\frac{\Omega^2}{2}V\psi\left(h\tilde{\nabla}_v\psi + w\tilde{\nabla}_u\psi\right) &= -4\pi r^2 e^{4\pi g\psi^2}V\psi\left(-r_u\nabla_v\psi + r_v\nabla_u\psi\right) \\ &= \partial_v\left(4\pi\frac{g^2}{r}r_u\varpi\psi^2\right) + \partial_u\left(-4\pi\frac{g^2}{r}r_v\varpi\psi^2\right) \\ &+\partial_u\left(-\frac{r_v r^2}{l^2}f(\psi^2)\right) + \partial_v\left(\frac{r_u r^2}{l^2}f(\psi^2)\right) \\ &-\psi^2\partial_v\left(4\pi\frac{g^2}{r}r_u\varpi\right) + \psi^2\partial_u\left(4\pi\frac{g^2}{r}r_v\varpi\right).\end{aligned} \tag{3.93}$$

We then compute using the Hawking mass equations (and symmetry) that

$$-\psi^2\partial_v\left(4\pi\frac{g^2}{r}r_u\varpi\right) + \psi^2\partial_u\left(4\pi\frac{g^2}{r}r_v\varpi\right) = \frac{32\pi^2 g^2 r r_u^2}{\Omega^2}\psi^2\left(\tilde{\nabla}_v\psi\right)^2 e^{4\pi g\psi^2} - \frac{32\pi^2 g^2 r r_v^2}{\Omega^2}\psi^2\left(\tilde{\nabla}_u\psi\right)^2 e^{4\pi g\psi^2}. \tag{3.94}$$

We now compute the bulk terms

$$2\left(\frac{1}{2}h_u - \frac{r_u}{r}h\right) = \frac{32\pi^2 r^3}{\Omega^2}e^{4\pi g\psi^2}\left(\tilde{\nabla}_u\psi\right)^2 - \frac{32\pi^2 g^2 r r_u^2}{\Omega^2}e^{4\pi g^2\psi^2}\psi^2, \tag{3.95}$$

and

$$2\left(\frac{1}{2}w_v - \frac{r_v}{r}w\right) = -\frac{32\pi^2 r^3}{\Omega^2}e^{4\pi g\psi^2}\left(\tilde{\nabla}_v\psi\right)^2 + \frac{32\pi^2 g^2 r r_v^2}{\Omega^2}e^{4\pi g^2\psi^2}\psi^2. \tag{3.96}$$

It follows from the choice of multipliers and (3.83) that

$$\begin{aligned}&\partial_u\left(-8\pi\frac{r_u r^2}{\Omega^2}e^{4\pi g\psi^2}\left(\tilde{\nabla}_v\psi\right)^2 + 4\pi\frac{g^2}{r}r_v\varpi\psi^2 + \frac{r_v r^2}{l^2}f(\psi^2)\right)\\ &+\partial_v\left(-\left(-8\pi\frac{r_v r^2}{\Omega^2}e^{4\pi g\psi^2}\left(\tilde{\nabla}_u\psi\right)^2 + 4\pi\frac{g^2}{r}r_u\varpi\psi^2 + \frac{r_u r^2}{l^2}f(\psi^2)\right)\right)\\ &= 0.\end{aligned} \tag{3.97}$$

We now perform the energy estimate by integrating over the domain Δ_{δ,u_0}

$$\begin{aligned}0 =& \int_{t(u_0,u_0)}^{t(u_0+\delta,u_0+\delta)} 4\pi r^2 e^{4\pi g\psi^2}\tilde{\nabla}_\rho\psi\cdot\mathcal{T}\psi dt\\ &+\int_{u_0}^{u_0+\delta}\left(-8\pi\frac{r_u r^2}{\Omega^2}e^{4\pi g\psi^2}\left(\tilde{\nabla}_v\psi\right)^2 + 4\pi\frac{g^2}{r}r_v\varpi\psi^2 + \frac{r_v r^2}{l^2}f(\psi^2)\right)(u,u_0)du\\ &-\int_{u_0}^{u_0+\delta}\left(8\pi\frac{r_v r^2}{\Omega^2}e^{4\pi g\psi^2}\left(\tilde{\nabla}_u\psi\right)^2 - 4\pi\frac{g^2}{r}r_u\varpi\psi^2 - \frac{r_u r^2}{l^2}f(\psi^2)\right)(u_0+\delta,v)dv,\end{aligned} \tag{3.98}$$

here we have used that $T(\tilde{r}) = 0$ on the boundary.
From the boundary conditions we get the energy estimate

$$\begin{aligned}&\int_{u_0}^{u_0+\delta}\left(-8\pi\frac{r_u r^2}{\Omega^2}e^{4\pi g\psi^2}\left(\tilde{\nabla}_v\psi\right)^2 + 4\pi\frac{g^2}{r}r_v\varpi\psi^2 + \frac{r_v r^2}{l^2}f(\psi^2)\right)(u,u_0)du\\ &=\int_{u_0}^{u_0+\delta}\left(8\pi\frac{r_v r^2}{\Omega^2}e^{4\pi g\psi^2}\left(\tilde{\nabla}_u\psi\right)^2 - 4\pi\frac{g^2}{r}r_u\varpi\psi^2 - \frac{r_u r^2}{l^2}f(\psi^2)\right)(u_0+\delta,v)dv.\end{aligned} \tag{3.99}$$

Now to show the Hawking mass is constant along this surface we consider a smaller triangle Δ_{δ,u_*} where $u_* > u_0$ so we are bounded away from null infinity. We thus see that

$$\varpi(u_0+\delta,u_0+\delta) - \varpi(u_0,u_0) = \lim_{u_*\to u_0}\int_{t(u_*)}^{t(u_*+\delta)} T\varpi dt. \tag{3.100}$$

Now define a vector field

$$X = -\varpi_v\partial_u + \varpi_u\partial_v, \tag{3.101}$$

we then see from the divergence theorem and lemma 3.4.3 that

$$\begin{aligned}0 =& \int_{t(u_*)}^{t(u_*+\delta)} T\varpi dt + \int_{u_*}^{u_*+\delta}\left(-8\pi\frac{r_u r^2}{\Omega^2}e^{4\pi g\psi^2}\left(\tilde{\nabla}_v\psi\right)^2 + 4\pi\frac{g^2}{r}r_v\varpi\psi^2 + \frac{r_v r^2}{l^2}f(\psi^2)\right)(u,u_*)du\\ &-\int_{u_*}^{u_*+\delta}\left(8\pi\frac{r_v r^2}{\Omega^2}e^{4\pi g\psi^2}\left(\tilde{\nabla}_u\psi\right)^2 - 4\pi\frac{g^2}{r}r_u\varpi\psi^2 - \frac{r_u r^2}{l^2}f(\psi^2)\right)(u_*+\delta,v)dv.\end{aligned} \tag{3.102}$$

Sending $u_* \to u_0$ we see the second two integrals cancel from the energy estimate leaving us to conclude that

$$\varpi(u_0 + \delta, u_0 + \delta) = \varpi(u_0, u_0). \tag{3.103}$$

As δ is arbitrary we see that the Hawking mass is constant along the boundary, and equal to its initial value. One can now also make the interpretation that ϖ forms a potential for the energy of the system. □

Geometric uniqueness

When solving system (3.21) - (3.26) it is important to note that we have made a choice of gauge to define the boundary of the space-time. Thus a priori we might expect our solution to be dependant on this gauge. This is essentially the problem of geometric uniqueness as discussed in [Fri09]. In order to address this issue we note that the weak formulations of the Klein-Gordon equation are invariant under a change of null coordinates.

Lemma 3.4.5. *Let $\mathcal{D} = (\bar{\tilde{r}}, \overline{\psi})$ be a data set satisfying the initial and boundary conditions of section 3.4.2. Let $(\mathcal{M}_i, g_i, \psi_i)$ be two developments of $\mathcal{D}$. Then both $(\mathcal{M}_i, g_i, \psi_i)$ are extensions of a common development.*

Theorem 3.4.2. *A data set $\mathcal{D} = (\tilde{r}, \psi)$ satisfying the initial and boundary conditions of section 3.4.2 admits a maximal development. This development is unique up to isometry.*

We remark at this point that the following proof follows similarly to [HS13b]. In this vein we make the following definition

Definition 3.4.9. *Let $\mathcal{N}$ be an interval of the form $\mathcal{N} = (u_0, u_1]$. Given an initial data set $(\bar{\tilde{r}}, \overline{\psi})$ satisfying the initial conditions in section 3.4.2 we say a development $\mathfrak{D}$ is a triple $(\mathcal{M}, g, \psi)$ such that $(\mathcal{M}, g)$ is a smooth Lorentzian manifold with C^0 metric, ψ is $C^0\underline{H}^1$ function on $\mathcal{M}$, and the following hold*

- *$(\mathcal{M}, g, \psi)$ is a square flat toroidally symmetric weak solution to the EKG system, with area radius r being a C^1 function with $r > 0$.*
- *The quotient manifold $\mathcal{Q} = \mathcal{M}/\mathfrak{T}^2$ with its induced Lorentzian metric is a manifold with boundary $\mathcal{N}_{\mathcal{Q}}$ which is a null ray, diffeomorphic to a subset $\mathcal{N}$ of the form $(u_0, u_0+\epsilon)$, for some $\epsilon > 0$. If φ is such a diffeomorphism: $\varphi : \mathcal{N}_{\mathcal{Q}} \to (u_0, u_0+\epsilon)$, then $\psi \circ \varphi = \overline{\psi}|_{(u_0,u_0+\epsilon)}$ and $\tilde{r} \circ \varphi = \bar{\tilde{r}}|_{(u_0+u_0+\epsilon)}$.*
- *$\mathcal{Q}$ admits a a system of global bounded null coordinates, and may be embedded conformally into a subset of $\mathbb{R}^{1+1}$. The boundary of $\mathcal{Q}$ with respect to the topology of $\mathbb{R}^{1+1}$ is composed of a future boundary $\mathcal{N}_{\mathcal{F}}$, a past boundary which coincides with $\mathcal{N}_{\mathcal{Q}}$ and a C^1 time-like boundary $\mathcal{I}$ given by the level set $\mathcal{I} = \{(u,v) \in \mathbb{R}^{1+1} : \tilde{r}(u,v) = 0\}$. The metric g is asymptotically AdS in the sense of* (3.47).
- *The field ψ is in $C^0\underline{H}^1$, and satisfies the following weak formulations of* (3.23)

– *Dirichlet*

$$\int_{\mathcal{M}} \left(\tilde{\nabla}^\mu \phi \tilde{\nabla}_\mu \psi - V\psi\phi \right) dVol = 0, \tag{3.104}$$

holds for all $\phi \in \underline{H}^1_{0,g}(\mathcal{M})$, *and* $\tilde{r}^{-\frac{3}{2}+\kappa}\psi = 0$ *on* $\mathcal{I}$ *in a trace sense.*

– *Neumann*

$$\int_{\mathcal{M}} \left(\tilde{\nabla}^\mu \phi \tilde{\nabla}_\mu \psi - V\psi\phi \right) dVol = 0, \tag{3.105}$$

holds for all $\phi \in \underline{H}^1_{g}(\mathcal{M})$ *with* $\phi|_{\mathcal{N}_\mathcal{Q}} = \phi|_{\mathcal{N}_\mathcal{F}} = 0$ *in a trace sense.*

- *'Global hyperbolicity' holds in the sense that all past directed inextendible causal curves in* $\mathcal{Q}$ *either intersect* $\mathcal{N}_\mathcal{Q}$, *or have a limit point on* $\mathcal{I}$.
- *The field* ψ *satisfies the following integrability conditions:*
  For each constant v *ray,* $R_v \subset \mathcal{Q}$ *we have:*

$$\int_{R_v} \frac{r^4}{|r_u|}(\tilde{\nabla}_u \psi)^2 + \frac{|r_u|}{r}\psi^2 d\bar{u} < \infty, \tag{3.106}$$

and each constant u *ray,* $R_u \subset \mathcal{Q}$ *we have:*

$$\int_{R_u} \left| \frac{r^2 r_u}{\Omega^2} \right| (\tilde{\nabla}_v \psi)^2 + \frac{|r_v|}{r}\psi^2 d\bar{v} < \infty. \tag{3.107}$$

It follows from the definition that the Penrose diagram of $\mathfrak{D}$ *has the form:*

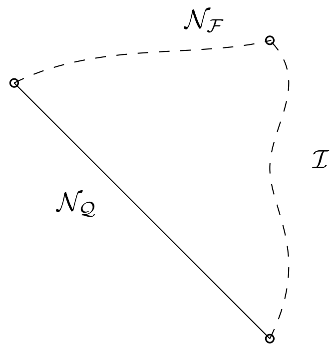


Figure 3.4: Penrose diagram of a development.

We now define what we mean by an extension of a development. Let $(\mathcal{M}_i, g_i, \psi_i)$, *be developments with associated null ray diffeomorphisms* φ_i *mapping* $\mathcal{N}_{\mathcal{Q}_i} \mapsto \mathcal{N}$ *for* $i = 1, 2$. *We say that* $(\mathcal{M}_1, g_1, \psi_1)$ *is an extension of* $(\mathcal{M}_2, g_2, \psi_2)$ *if there exists an isometric embedding* φ *of* $(\mathcal{M}_2, g_2)$ *into* $(\mathcal{M}_1, g_1)$ *which maps* ψ_2 *to* ψ_1, *and* $\varphi_1^{-1} \circ \varphi \circ \varphi_2$ *is the identity map on* $\mathcal{N}$.
This definition makes the set of developments into a partially ordered set. The maximal development is the maximal element. We remark at the stage that while we are invoking Zorn's lemma the work of [Sbi16] means this is likely unnecessary.

We now prove lemma 3.4.5:

Proof. Let $(\mathcal{M}, g, \psi)$ be a development with $\mathcal{N} = (u_0, u_1]$. Let $\mathcal{N}_{\mathcal{Q}}$ be the initial null ray as in definition 3.4.9. We thus may apply a change of u coordinate so that $\mathcal{N}_{\mathcal{Q}}$ may be identified with $(u_0, u_0 + \epsilon)$ of $\mathcal{N}$. In this coordinate system we have $\tilde{r}_u = \overline{\tilde{r}}_u$, $\tilde{\nabla}_u \psi = \overline{\tilde{\nabla}_u \psi}$ etc on $\mathcal{N}_{\mathcal{Q}}$. As $\mathcal{I}$ is a C^1 boundary we can find a function f with $f' > 0$ such that $u = f(v)$, on $\mathcal{I}$, defining $V = f(v)$ we see that in the (u, V) coordinate system $\mathcal{I} = u = V$, and the boundary has been straightened out. As this is a C^1 coordinate transform the invariant norms of ψ remain finite. We thus have the metric is at least C^0. Hence the variables $(\tilde{r}, \varpi_2, \psi)$ of the development satisfy (3.21) - (3.23) in the coordinates used in the proof of theorem 3.4.1. Furthermore they have the same regularity and satisfy the same weak formulations of the theorem as these are invariant under a change of double null coordinates. By the uniqueness property of solutions to theorem 3.4.1 the solutions must agree in the intersection of their domain of definitions, notably in a neighbourhood of $\mathcal{N} \cap \mathcal{I}$. □

3.5 Extension principles

We now wish to control aspects of the maximal development's geometry, in particular how singularities may form. For this we prove two extension principles for the space time.

3.5.1 Interior extension principle

Theorem 3.5.1. *Let $(\mathcal{Q}^+ \times \mathfrak{T}^2, g, \psi)$ denote the maximal $\mathfrak{W}$ extension of an asymptotically AdS initial data set for the system* (3.8)-(3.12)*. Suppose $p = (U, V) \in \overline{\mathcal{Q}^+}$. If the set*

$$\mathcal{D} = [U', U] \times [V', V] \backslash \{p\} \subset Q^+, \tag{3.108}$$

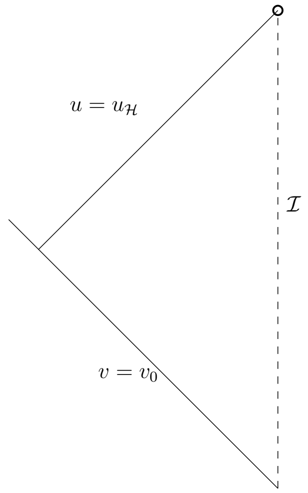
is non-empty, has finite spacetime volume, and there exist constants

$$0 < r_0 \leq r \leq R < \infty, \quad \textit{for all } (u, v) \in \mathcal{D}, \tag{3.109}$$

then $p \in \mathcal{Q}^+$.

Proof. The proof of this is similar to [Kom13] and [HS12]. The key difference is that we are working with a slightly lower level of regularity. In particular the function ψ_v is in L^2 but may not also be in C^0. This means that standard contraction map argument cannot be done in just C^k spaces. The extension principle then follows in the same manner but exploiting the absolute continuity of ψ. The proofs for this can be found as an appendix to this thesis.

Corollary 3.5.1. *For a free data set that contains a (marginally) trapped surface (that is a point on $\mathcal{N}$ such that $\overline{r_v} \leq 0$), the quotient of the maximal development of the initial data set contains a subset as shown in the Penrose diagram:*



Here $u_{\mathcal{H}}$ is the boundary of u-constant null rays on which $\tilde{r} \to 0$ on. Furthermore this set is belongs to $\mathcal{Q}^+$.

□

Proof. The local wellposedness gives us a solution in a small triangle where $\tilde{r} \to 0$ along a constant u ray. Now the initial data contains points where $r_v \leq 0$. From the Raychaudhuri equation (3.9) we see that this inequality propagates in v, hence there are points in the spacetime that cannot reach $\mathcal{I}$. As we have a solution in a small triangle in which $\tilde{r} \to 0$ along any $u = const$ ray, there exists some $u_{\mathcal{H}}$ such that for $u < u_{\mathcal{H}}$, $\tilde{r} \not\to 0$ along these rays. Finally we can see that that the ray $u = u_{\mathcal{H}}$ is regular, as r is monotonic, the extension principle (for finite v) forbids singularities along it. □

3.5.2 Extension principle near infinity

Theorem 3.5.2. *For a $\mathfrak{W}$ solution to* (3.5)-(3.12) *in a triangular region Δ_{d,u_0}, assume that we have:*

- *The corner condition*

$$\lim_{v \to u_0 + d} \tilde{r}(u_0 + d, v) = 0. \tag{3.110}$$

- *For any constant v-ray ($\mathcal{N}(v)$) contained Δ_{d,u_0} and intersecting $\mathcal{I}$, there exists a constant $K > 0$*

$$\int_{u_{\mathcal{I}}}^{u} \frac{r^4}{-r_u}(\tilde{\nabla}_u \psi)^2(\bar{u}, v) - r_u \psi^2(\bar{u}, v) d\bar{u} + \sup_{\mathcal{N}(v)} \left| \frac{r^{\frac{5}{2}}}{-r_u} \tilde{\nabla}_u \psi \right| + \sup_{\mathcal{N}(v)} \left| r^{\frac{3}{2}-\kappa} \psi \right| + \sup_{\mathcal{N}(v)} |\varpi - M|^{\frac{1}{2}} < K. \tag{3.111}$$

These are all geometric quantities which will be used to form an initial data set on the ray $(\mathcal{N}(v))$.

- *A non-degeneracy condition*

$$\min\left(\inf_{\Delta_{d,u_0}}\left|\frac{1}{l^2}-\frac{2\varpi}{r^3}\right|,\inf_{\Delta_{d,u_0}}\left|\frac{1}{l^2}-\frac{2\varpi_1}{r^3}\right|\right)>c. \tag{3.112}$$

- *There exists a constant $r_{min}>0$ such that in Δ_{d,u_0}*

$$r\geq r_{min}. \tag{3.113}$$

Then there exists a $\delta^*>0$ such that the solution can be extended to the set $\Delta_{d+\delta^*,u_0}$.

Proof. We proceed with the following three lemmas.

Lemma 3.5.1. *In Δ_{d,u_0} we have the following estimates*

$$\tilde{r}(u,v)>0\ ,\tilde{r}_u(u,v)>0,\ \lim_{u\to v}\left|\tilde{r}_u-\frac{1}{2l^2}\right|=0,\ \textit{and}\ \lim_{u\to v}\tilde{r}(u,v)=0. \tag{3.114}$$

Proof. Recall from the boundary conditions

$$\lim_{u\to v}\tilde{r}(u,v)=0, \tag{3.115}$$

We now prove estimates for the quantity $\tilde{r}_u$.
Using the wave equation for the radial component (3.10) (rewritten in terms of Ω^2)

$$\tilde{r}_{uv}=\frac{-3\tilde{r}_u r_v}{r}+\frac{3}{4l^2}\frac{\Omega^2}{r}-\frac{2\pi a}{l^2}\frac{\psi^2\Omega^2}{r}, \tag{3.116}$$

we may express this as the first order equation

$$\partial_v\tilde{r}_u=\tilde{r}_u\cdot f(u,v)r_v, \tag{3.117}$$

where

$$f(u,v)=\left(\frac{6\varpi}{r^2}+\frac{3r}{l^2}\left(1-e^{-4\pi g\psi^2}\right)+\frac{6\varpi}{r^2}\left(e^{-4\pi g\psi^2}-1\right)-\frac{8\pi a}{l^2}r\psi^2\right)\frac{e^{4\pi g\psi^2}}{\frac{r^2}{l^2}-\frac{2\varpi}{r}}. \tag{3.118}$$

For fixed u, integrating (3.116) in v from $v=v_0$, we see

$$\tilde{r}_u=\frac{1}{2l^2}\exp\left(\int_{r(v_0)}^{r(v)}f(u,v)dr\right). \tag{3.119}$$

We now wish to show that f is integrable. Elementary estimation and noting that

$$\left|1-e^{-4\pi g\psi^2}\right|\leq C_{l,g}r^{-3+2\kappa}, \tag{3.120}$$

shows

$$|f| \leq \frac{1}{\left|\frac{r^2}{l^2} - \frac{2\varpi}{r}\right|} \left(\left|6Mr^{-2}\right| + \left|C_{l,g} r^{-2+2\kappa}\right| + \left|C_{M,l,g} r^{-3+2\kappa}\right| + \left|\frac{2\pi(-a)}{l^2} K r^{-2+2\kappa}\right| \right). \quad (3.121)$$

Due to the given inequality

$$\frac{r^2}{l^2} - \frac{2\varpi}{r} \geq cr^2, \quad (3.122)$$

we see that

$$|f| \leq C_{M,l,g} r^{-4+2\kappa}. \quad (3.123)$$

We thus have a constant $C_{M,l,g}$ such that

$$\frac{1}{2l^2} \exp\left(-C_{M,l,g} \int_{r(v_0)}^{r(v)} r^{-4+2\kappa} dr \right) \leq \tilde{r}_u \leq \frac{1}{2l^2} \exp\left(C_{M,l,g} \int_{r(v_0)}^{r(v)} r^{-4+2\kappa} dr \right), \quad (3.124)$$

which integrates to

$$\frac{1}{2l^2} \exp\left(-C_{M,l,g} \left(r^{-3+2\kappa} - \overline{r}^{-3+2\kappa} \right) \right) \leq \tilde{r}_u \leq \frac{1}{2l^2} \exp\left(C_{M,l,g} \left(\overline{r}^{-3+2\kappa} - r^{-3+2\kappa} \right) \right). \quad (3.125)$$

From here we can clearly see (from continuity) that as $u \to v$, we have

$$\lim_{u \to v} \left| \tilde{r}_u - \frac{1}{2l^2} \right| = 0. \quad (3.126)$$

So we can find a constant such that

$$0 < \frac{1}{2l^2} \tilde{C'} \leq \tilde{r}_u \leq \frac{1}{2l^2} C'. \quad (3.127)$$

We may also integrate the above inequality in u to see that

$$\tilde{r}(u,v) \geq \overline{\tilde{r}}(u) > 0. \quad (3.128)$$

□

Lemma 3.5.2. *In Δ_{d,u_0} we have the following estimates*

$$\lim_{u \to v} \tilde{r}_{uu}(u,v) = 0. \quad (3.129)$$

The idea is to integrate the quantity $\tilde{r}_{uuv}$ in v, this however requires control over more asymptotics of the solution, and its derivatives in the region Δ_{d,u_0}. The majority of this lemma is showing the required decay.

Proof. Estimates for r_v:
It follows from

$$\tilde{r}_u|_{\mathcal{I}} + \tilde{r}_v|_{\mathcal{I}} = 0, \quad (3.130)$$

that

$$-\tilde{r}_v|_{\mathcal{I}} = \frac{1}{2l^2}. \tag{3.131}$$

Returning to the evolution equation for $\tilde{r}$, (3.10) (rewritten in terms of Ω^2)

$$\tilde{r}_{uv} = \frac{-3r_u\tilde{r}_v}{r} + \frac{3}{4l^2}\frac{\Omega^2}{r} - \frac{2\pi a}{l^2}\frac{\psi^2\Omega^2}{r}, \tag{3.132}$$

we are in the similar situation of the previous lemma

$$\partial_u\tilde{r}_v = \tilde{r}_v \cdot f(u,v)r_u, \tag{3.133}$$

where

$$f(u,v) = \left(\frac{6\varpi}{r^2} + \frac{3r}{l^2}\left(1 - e^{-4\pi g\psi^2}\right) + \frac{6\varpi}{r^2}\left(e^{-4\pi g\psi^2} - 1\right) - \frac{8\pi a}{l^2}r\psi^2\right)\frac{e^{4\pi g\psi^2}}{\frac{r^2}{l^2} - \frac{2\varpi}{r}}. \tag{3.134}$$

Integrating in along u for fixed v, and using the estimates on the value at $\mathcal{I}$

$$-\frac{1}{2l^2}\exp\left(\int_{r(u)}^{\infty} f(u,v)dr\right) \leq \tilde{r}_v \leq -\frac{1}{2l^2}\exp\left(\int_{r(u)}^{\infty} f(u,v)dr\right). \tag{3.135}$$

Recalling that

$$|f| \leq C_{M,l,g}r^{-4+2\kappa}, \tag{3.136}$$

we deduce the bound

$$-C_{M,l,g} \leq \tilde{r}_v \leq -C_{M,l,g}. \tag{3.137}$$

Estimates for Ω^2:
Recall that

$$\Omega^2 = \frac{-4r_ur_v}{\frac{r^2}{l^2} - \frac{2\varpi}{r}}e^{4\pi g\psi^2}, \tag{3.138}$$

so using our previous estimates we can quickly see for some constants $\tilde{C}, C > 0$

$$Cr^2 \leq \Omega^2 \leq \tilde{C}r^2. \tag{3.139}$$

Estimates for ϖ_2:
First note

$$\varpi_2 = \varpi e^{-4\pi g\psi^2} + \left(1 - 4\pi g\psi^2 - e^{-4\pi g\psi^2}\right)\frac{r^3}{2l^2}, \tag{3.140}$$

and that

$$1 - 4\pi g\psi^2 - e^{-4\pi g\psi^2} \leq -4\pi^2g^2\psi^4 + \mathcal{O}\left(r^{-9+6\kappa}\right) \leq C_gK^2r^{-6+4\kappa}. \tag{3.141}$$

So we easily see

$$|\varpi_2| \leq C_{g,M}. \tag{3.142}$$

Estimates for $\partial_u\varpi_2$:

Recalling the equation (3.25)

$$\partial_u \varpi_2 = -8\pi r^2 \frac{r_v}{\Omega^2}(\tilde{\nabla}_u \psi)^2 - 8\pi g \left(\varpi_2 + 2\pi g \frac{r^3}{l^2}\psi^2 \right) \psi \tilde{\nabla}_u \psi - 4\pi \psi^2 r_u g^2 \frac{\left(\varpi_2 + 2\pi g \frac{r^3}{l^2}\psi^2\right)}{r}, \tag{3.143}$$

we can now estimate all these quantities pointwise to deduce

$$|\partial_u \varpi_2| \leq C_{M,g,l} r. \tag{3.144}$$

Estimates for $\tilde{r}_{uv}$:
Rewriting $\tilde{r}_{uv}$ equation (3.22) as

$$\tilde{r}_{uv} = \varpi \Omega^2 e^{-4\pi g \psi^2} \frac{3}{4r^4} - \frac{1}{rl^2}\Omega^2 \left(\frac{3}{4} e^{-4\pi g \psi^2} - \frac{3}{4} + 2\pi a \psi^2 \right), \tag{3.145}$$

estimating the bracketed term

$$\left| \frac{3}{4} e^{-4\pi g \psi^2} - \frac{3}{4} + 2\pi a \psi^2 \right| = \left| \pi g^2 \psi^2 + \mathcal{O}\left(r^{-6+4\kappa}\right) \right| \leq C_g r^{-3+2\kappa}, \tag{3.146}$$

so from here we can conclude that

$$|\tilde{r}_{uv}| \leq C_{l,g,M} r^{-2+2\kappa}. \tag{3.147}$$

Estimates for ϖ_1:
We now prove that

$$\left| \frac{\varpi_1 - M}{r^{2\kappa}} \right| \leq C_{M,g}. \tag{3.148}$$

(The motivation for the $r^{-2\kappa}$ term is to ensure a finiteness of this quantity as we approach $\mathcal{I}$.)
We recall the estimates

$$M - K \leq \varpi \leq M + K, \tag{3.149}$$

and note

$$-C_g r^{-3+2\kappa} \leq 1 - e^{-4\pi g \psi^2} \leq 0. \tag{3.150}$$

Then we expand

$$\frac{\varpi_1 - M}{r^{2\kappa}} = \frac{1}{r^{2\kappa}} \left(\varpi e^{-4\pi g \psi^2} - M \right) + \left(1 - e^{-4\pi g \psi^2} \right) \frac{r^{3-2\kappa}}{2l^2}, \tag{3.151}$$

estimating we see

$$\frac{\varpi_1 - M}{r^{2\kappa}} \leq \frac{M}{r^{2\kappa}} \left(e^{-4\pi g \psi^2} - 1 \right) + \frac{K}{r^{2\kappa}} e^{-4\pi g \psi^2} \leq C_{M,g}. \tag{3.152}$$

In the other direction

$$\begin{aligned}\frac{\varpi_1 - M}{r^{2\kappa}} &\geq \frac{1}{r^{2\kappa}}\left((M-K)e^{-4\pi g\psi^2} - M\right) - C_{l,g} \\ &\geq \frac{M}{r^{2\kappa}}\left(e^{-4\pi g\psi^2} - 1\right) - \frac{K}{r^{2\kappa}}e^{-4\pi g\psi^2} - C_{l,g} \\ &\geq \frac{M}{r^{2\kappa}} \cdot C_g r^{-3+2\kappa} - \frac{K}{r^{2\kappa}}C_K - C_{l,g} \\ &\geq -C_{M,g,l},\end{aligned} \tag{3.153}$$

thus concluding the proof.

Estimates for $\tilde{r}_{uu}$:

Recalling

$$\mu_1 = \frac{r^2}{l^2} - \frac{2\varpi_1}{r}, \tag{3.154}$$

We easily see from (3.112) and estimates for ϖ_1 it follows that

$$cr^2 \leq \mu_1 \leq C_{l,M,g}\left(r^2 + r^{2\kappa-1}\right) \leq C_{l,M,g}r^2. \tag{3.155}$$

We will prefer the form

$$\tilde{C}_{l,M,g}r^{-2} \leq \frac{1}{\mu_1} \leq cr^{-2}. \tag{3.156}$$

We will also need to bound $|\partial_u \mu_1|$. We start with

$$|\partial_u \varpi_1| = \left|\partial_u\left(\varpi_2 - 2\pi g\frac{r^3}{l^2}\psi^2\right)\right|, \tag{3.157}$$

preserving only the powers of r terms we get

$$\partial_u\left(\varpi_2 - 2\pi g\frac{r^3}{l^2}\psi^2\right) \sim \partial_u\varpi_2 + r^2 r_u\psi^2 + r^3\psi\tilde{\nabla}_u\psi + r^2 r_u\psi^2 \sim r + r^{1+2\kappa} + r^{2+\kappa} + r^{1+\kappa}, \tag{3.158}$$

so the dominant behaviour is $r^{2+\kappa}$ and thus

$$|\partial_u\varpi_1| \leq C_{M,l,g}r^{2+\kappa}. \tag{3.159}$$

We then deduce the following bound for $|\partial_u\mu_1|$

$$|\partial_u\mu_1| = \left|\frac{2rr_u}{l^2} - \frac{2\varpi_{1,u}}{r} + \frac{2\varpi_1}{r^2}\right| \leq C_{M,l,g}\left(r^3 + r^{1+\kappa} + r^{-2+2\kappa}\right) \leq C_{M,l,g}r^3. \tag{3.160}$$

We now need to study $\tilde{r}_{uvu}$, it will be worthwhile recalling the relation

$$\frac{\Omega^2}{r^2} = \frac{-4\tilde{r}_u\tilde{r}_v}{\tilde{r}^2\mu_1}. \tag{3.161}$$

Then we compute

$$\begin{aligned}\tilde{r}_{uvu} =&\partial_u\left(\frac{\Omega^2}{r^2}\left(3\varpi_2\tilde{r}^2-\frac{g^2\pi}{\tilde{r}l^2}\psi^2\right)\right)\\ =&\partial_u\left(\frac{-4\tilde{r}_u\tilde{r}_v}{\mu_1}\left(3\varpi_2-\frac{g^2\pi}{\tilde{r}^3l^2}\psi^2\right)\right)\\ =&-\tilde{r}_{uu}\frac{4\tilde{r}_v}{\mu_1}\left(3\varpi_2-\frac{g^2\pi}{\tilde{r}^3l^2}\psi^2\right)+\left(\frac{-4\tilde{r}_u\tilde{r}_{uv}}{\mu_1}+\frac{4\tilde{r}_u\tilde{r}_v\mu_{1,u}}{\mu_1^2}\right)\left(3\varpi_2-\frac{g^2\pi}{\tilde{r}^3l^2}\psi^2\right)\\ &+\frac{-4\tilde{r}_u\tilde{r}_v}{\mu_1}\left(3\varpi_{2,u}+2\psi\psi_u\frac{1}{\tilde{r}^3}-\frac{3\tilde{r}_u}{\tilde{r}^4}\psi^2\right).\end{aligned} \tag{3.162}$$

We thus have a first order equation of the form

$$\partial_v(\tilde{r}_{uu})+F\tilde{r}_{uu}=G, \tag{3.163}$$

where

$$F=\frac{4\tilde{r}_v}{\mu_1}\left(3\varpi_2-\frac{g^2\pi}{\tilde{r}^3l^2}\psi^2\right), \tag{3.164}$$

and

$$G=\left(\frac{-4\tilde{r}_u\tilde{r}_{uv}}{\mu_1}+\frac{4\tilde{r}_u\tilde{r}_v\mu_{1,u}}{\mu_1^2}\right)\left(3\varpi_2-\frac{g^2\pi}{\tilde{r}^3l^2}\psi^2\right)+\frac{-4\tilde{r}_u\tilde{r}_v}{\mu_1}\left(3\varpi_{2,u}+2\psi\psi_u\frac{1}{\tilde{r}^3}-\frac{3\tilde{r}_u}{\tilde{r}^4}\psi^2\right), \tag{3.165}$$

with initial data

$$\bar{\tilde{r}}_{uu}=0. \tag{3.166}$$

We solve for

$$\tilde{r}_{uu}(u,v)=\exp\left(-\int_{v_0}^v Fdv\right)\int_{v_0}^v\exp\left(\int_{v_0}^v Fdv\right)Gdv, \tag{3.167}$$

and estimate by

$$|\tilde{r}_{uu}|\le\exp\left(\int_{v_0}^v|F|\,dv\right)\int_{v_0}^v\exp\left(\int_{v_0}^v|F|\,dv\right)|G|\,dv. \tag{3.168}$$

Term by term we estimate

$$|F|=\left|\frac{4\tilde{r}_v}{\mu_1}\left(3\varpi_2-\frac{g^2\pi}{\tilde{r}^3l^2}\psi^2\right)\right|\le C_{M,l,g}r^{-2+2\kappa}, \tag{3.169}$$

so

$$\int_{v_0}^v|F|\,dv\le C_{M,l,g}\int_{v_0}^v r^{-2+2\kappa}r^{-2}r_vdv=\int_{r_0}^r r^{-4+2\kappa}dr\le C_{M,l,g}. \tag{3.170}$$

So the asymptotics of G will determine those of $\tilde{r}_{uu}$. We examine the terms of G separately

$$\left(\frac{-4\tilde{r}_u\tilde{r}_{uv}}{\mu_1}+\frac{4\tilde{r}_u\tilde{r}_v\mu_{1,u}}{\mu^2}\right)\le C_{M,l,g}\left(r^{-2}\cdot r^{-2+2\kappa}+\frac{1\cdot 1\cdot r^3}{r^4}\right)\le C_{M,l,g}r^{-1}, \tag{3.171}$$

then we have

$$\left(3\varpi_2-\frac{g^2\pi}{\tilde{r}^3l^2}\psi^2\right)\le C_{M,l,g}\left(1+r^{2\kappa}\right)\le C_{M,l,g}r^{2\kappa}, \tag{3.172}$$

so we have for the first term

$$\left(\frac{-4\tilde{r}_u\tilde{r}_{uv}}{\mu_1}+\frac{4\tilde{r}_u\tilde{r}_v\mu_{1,u}}{\mu^2}\right)\left(3\varpi_2-\frac{g^2\pi}{\tilde{r}^2l^2}\psi^2\right)\leq C_{M,l,g}r^{-1+2\kappa}. \tag{3.173}$$

Then for

$$\begin{aligned}\frac{-4\tilde{r}_u\tilde{r}_v}{\mu_1}\left(2\psi\psi_u\frac{1}{\tilde{r}^3}-\frac{2\tilde{r}_u}{\tilde{r}^4}\psi^2\right)&=\frac{-4\tilde{r}_u\tilde{r}_v}{\mu_1}\left(2\psi\tilde{\nabla}_u\psi\frac{1}{\tilde{r}^3}+2\psi^2\frac{gr_u}{\tilde{r}^2}-\frac{2\tilde{r}_u}{\tilde{r}^3}\psi^2\right)\\&=\frac{C_{M,l,g}}{r^2}\left(r^{-2+\kappa}r^3+r^{-3+2\kappa}r^2r^2+r^3r^{-3+2\kappa}\right)\\&\leq C_{M,l,g}r^{-1+2\kappa},\end{aligned} \tag{3.174}$$

thus

$$|G|\leq C_{M,l,g}r^{-1+2\kappa}. \tag{3.175}$$

Finally we see

$$\begin{aligned}\int_{v_0}^{v}|G|\,dv\leq\int_{v_0}^{v}C_{M,l,g}r^{-1+2\kappa}dv\leq C_{M,l,g}\int_{v_0}^{v}r^{-3+2\kappa}r_vdv\\\leq C_{M,l,g}\left(r^{-2+2\kappa}-\overline{r}^{-2+2\kappa}\right)\leq C_{M,l,g}r^{-2+2\kappa}.\end{aligned} \tag{3.176}$$

Proving

$$\lim_{u\to v}|\tilde{r}_{uu}|=0. \tag{3.177}$$

Furthermore, we can find a constant such that

$$|\tilde{r}_{uu}|\leq C_{M,l,g}. \tag{3.178}$$

□

Lemma 3.5.3. *There exists a constant $C_{g,l,M}>0$ such that*

$$\left|\frac{\tilde{r}_u}{\tilde{r}}-\frac{1}{2\rho}\right|<C_{g,M,l}. \tag{3.179}$$

Proof. We split the proof into two parts. The first part is to provide estimates on the radial function and its T derivative. These are then used to estimate a solution to a differential equation.

Estimates for $\tilde{r}$:

Recalling that $\tilde{r}|_{\mathcal{I}}=\tilde{r}(u,u)=\tilde{r}(v,v)=0$,

$$\tilde{r}(u,v)=\int_v^u\tilde{r}_udu\leq C_l(u-v)=C_l\rho, \tag{3.180}$$

and similarly

$$\tilde{r}(u,v)=\int_v^u\tilde{r}_udu\geq C_l(u-v)=C_l\rho. \tag{3.181}$$

Thus

$$\tilde{C}_l \rho \leq \tilde{r} \leq C_l \rho. \tag{3.182}$$

Estimates for $T(\tilde{r})$:
Noting that

$$T(\tilde{r})(u,v) = 0 + \int_v^u \tilde{r}_{vu} + \tilde{r}_{uu} du, \tag{3.183}$$

we deduce

$$|T(\tilde{r})| \leq \int_v^u |\tilde{r}_{uv}| + |\tilde{r}_{uu}| \, du \leq C_{M,l,g}(u-v) = C_{M,l,g}\rho \leq C_{M,l,g}\tilde{r}. \tag{3.184}$$

Now consider the following equation

$$\partial_v \left(\rho \tilde{r}_u - \frac{1}{2}\tilde{r} \right) = -\frac{1}{2}T(\tilde{r}) + \rho \cdot \tilde{r}_{uv}, \tag{3.185}$$

not that the bracketed term on the LHS is 0 on the boundary $\mathcal{I}$. Integrating

$$\int_v^u \partial_v \left(\rho \tilde{r}_u - \frac{1}{2}\tilde{r} \right) dv \leq \int_v^u C \cdot \rho dv \leq C\rho^2, \tag{3.186}$$

we thus conclude that

$$\frac{\tilde{r}_u}{\tilde{r}} - \frac{1}{2\rho} \leq C\frac{\rho}{\tilde{r}}. \tag{3.187}$$

Similarly

$$-\frac{\tilde{r}_u}{\tilde{r}} + \frac{1}{2\rho} \leq C\frac{\rho}{\tilde{r}}. \tag{3.188}$$

Finally from (3.182)

$$\left| \frac{\tilde{r}_u}{\tilde{r}} - \frac{1}{2\rho} \right| < C. \tag{3.189}$$

□

Corollary 3.5.2. *We have that there exists a constant $C_{g,M,l} > 0$ such that*

$$\frac{1}{C_{g,M,l}} \left(\left(\hat{\nabla}_u \psi \right)^2 + \psi^2 \right) \rho^{-2} \leq \left(\left(\tilde{\nabla}_u \psi \right)^2 + \psi^2 \right) r^2 \leq C_{g,M,l} \left(\left(\hat{\nabla}_u \psi \right)^2 + \psi^2 \right) \rho^{-2}. \tag{3.190}$$

That is twisting with r and ρ are equivalent in H^1 type norms.

We now have the relevant estimates to prove the theorem:

Proof of the Theorem 3.5.2. From the interior wellposedness results we can extend to the set $\Delta_{d+\tilde{\delta},u_0} \cap \{v \leq u_0 + d + \tilde{\delta} - \epsilon\}$, for some $\tilde{\delta} > 0$, which depends on ϵ from the continuity. We now extend to a triangle $\Delta_{d+\delta^*,u_0}$. We note that from the previous lemmas we have on each $v = const$ ray in Δ_{d,u_0} that the function $\tilde{r}$ restricted to this space is admissible as part of an initial data set. Lemma 3.5.1, and corollary 3.5.2 show us that ψ restricted to the ray is also admissible as part of an initial data set. Now let δ be the time of existence of a solution using this data set, but with K replaced by $2K$ and c by $\frac{c}{2}$. Now by choosing the ray

$v_c = u_0 + d - \frac{\delta}{2}$. By the above argument (and using continuity), we can extend our solution to the ray $(u_0 + d - \frac{\delta}{2} + \delta^*] \times \{u_0 + d - \frac{\delta}{2}\}$ for some $\delta^* < \frac{\delta}{2}$ such that the conditions (3.111), and (3.112) hold on $(u_0 + d - \frac{\delta}{2} + \delta^*] \times \{u_0 + d - \frac{\delta}{2}\}$ with K replaced by $2K$ and c by $\frac{c}{2}$. We then apply the local existence result to extend the solution to $\Delta_{d+\delta^*,u_0}$. □

3.6 Perturbed toroidal AdS Schwarzschild data and maximal development

In view of Birkhoff's theorem we know that if we choose $\psi = 0$, then we will solve for an isometric subset of the TAdSS solution. We thus choose initial data for ψ that is quantifiably close to 0 and considered to be small. Under this smallness assumption we will then prove various estimates about the derived quantities on the initial data ray that we will need in the evolution.

3.6.1 Initial data

The free data

Let $b > 0$, we will later take this to be a sufficiently small quantity.

Definition 3.6.1. *Let $\mathcal{N} = (u_0, u_1] \times \{v_0\}$. We define our initial radial function*

$$\bar{\bar{r}}(u) = \frac{u - u_0}{2l^2}. \tag{3.191}$$

The free data consists of a $C^1(\mathcal{N})$ function ψ such that

$$\left(\int_{\mathcal{N}} \left(\left(\bar{\tilde{\nabla}}_u \overline{\psi}\right)^2 + \overline{\psi}^2\right) \overline{r}^2 du\right)^{\frac{1}{2}} + \sup_{\mathcal{N}} \left|\overline{\psi}\overline{r}^{\frac{3}{2}-\kappa}\right| + \sup_{\mathcal{N}} \left|r^{\frac{1}{2}+\frac{s}{2}} \left(\bar{\tilde{\nabla}}_u \overline{\psi}\right)\right| =: b^2, \tag{3.192}$$

where $0 < s < 1$. (The choice of s is technical and we only expect to see $r^{-\frac{1}{2}}$ decay of the u derivative propagating in the system, we do however need this initial smallness in the problem in order to prove various results about the spacetime).

Deduced quantities

From $\overline{\psi}$ and $\overline{\bar{r}}$, we can define the following derived quantities for our system:
Initial Hawking mass:

We define $\overline{\varpi}$ as the unique $C^1(\mathcal{N})$ solution to

$$\begin{aligned}\partial_u\overline{\varpi} =&\frac{2\pi\overline{r}^2}{\overline{r}_u}\left(\frac{\overline{r}^2}{l^2}-\frac{2\overline{\varpi}}{\overline{r}}\right)(\tilde{\overline{\nabla}}_u\overline{\psi})^2+\frac{4\pi g^2\overline{r}_u}{\overline{r}}\overline{\varpi}\overline{\psi}^2\\ &+\frac{\overline{r}_u\overline{r}^2}{l^2}\left(e^{4\pi g\overline{\psi}^2}\left(4\pi a\overline{\psi}^2-\frac{3}{2}\right)+2\pi(3g-2a)\overline{\psi}^2+\frac{3}{2}\right),\end{aligned} \tag{3.193}$$

with boundary condition

$$\lim_{u\to u_0}\overline{\varpi}=M. \tag{3.194}$$

From this we then define

$$\overline{\varpi_2}=\overline{\varpi}e^{-4\pi g\overline{\psi}^2}-\frac{\overline{r}^3}{2l^2}\left(e^{-4\pi g\overline{\psi}^2}+4\pi g\overline{\psi}^2-1\right), \tag{3.195}$$

(recall we need to define $\overline{\varpi_2}$ to solve the system but we'd like $\overline{\varpi}$ to have certain properties).

The quantity $\overline{r_v}$:

Recall equation (3.8) holds classically. Defining the variable

$$\chi:=-\frac{\Omega^2}{4r_u}, \tag{3.196}$$

we can rewrite (3.8) as

$$\partial_u\log\chi=\frac{4\pi r}{r_u}\left(\nabla_u\psi\right)^2. \tag{3.197}$$

Solving this ODE and using the definition of the Hawking mass, one gets the following expression for r_v

$$r_v=\chi|_{\mathcal{I}}\cdot\left(\frac{-2\varpi}{r}+\frac{r^2}{l^2}\right)e^{4\pi g\psi^2}\exp\left(\int_{u_0}^{u}\frac{4\pi r}{r_u}(\nabla_u\psi)^2du\right), \tag{3.198}$$

we will later on make a gauge choice where $\chi|_{\mathcal{I}}=\frac{1}{2}$. We choose

$$\overline{r_v}=\frac{1}{2}\left(\frac{-2\overline{\varpi}}{\overline{r}}+\frac{\overline{r}^2}{l^2}\right)e^{4\pi g\overline{\psi}^2}\exp\left(\int_{u_0}^{u}\frac{4\pi\overline{r}}{\overline{r}_u}(\nabla_u\overline{\psi})^2du\right), \tag{3.199}$$

we remark that while we have not used a twisted derivative in the definition of $\overline{r_v}$, the initial data choices allow us to see that is indeed an integrable quantity. It is also easy to see that $\overline{r_v}$ is independent of choice of u-coordinate on the data.

The quantity Ω^2:

We finally define the $C^1(\mathcal{N})$ quantity

$$\overline{\Omega}^2 = -\frac{4\overline{r}_u\overline{r_v}}{\left(\frac{-2\overline{\varpi}}{\overline{r}} + \frac{\overline{r}^2}{l^2}\right)} e^{4\pi g \overline{\psi}^2}. \tag{3.200}$$

3.6.2 Consequences of the smallness

Defining the regular and marginally trapped region of the initial data ray to be

$$\mathscr{R}_{v_0} \cup \mathcal{A}_{v_0} = \mathcal{N} \cap \{u \in \mathcal{N} : \overline{r_v}(u) > 0\} \cup \{u \in \mathcal{N} : \overline{r_v}(u) = 0\}. \tag{3.201}$$

Lemma 3.6.1. *We have that for $b > 0$ sufficiently small, on $\mathscr{R}_{v_0} \cup \mathcal{A}_{v_0}$,*

$$\sup_{u \in \mathscr{R}_{v_0} \cup \mathcal{A}_{v_0}} |\overline{\varpi} - M| \leq C_{l,g} b^2, \tag{3.202}$$

so for small inital data $\overline{\varpi}$ is positive.
For the toroidal AdS Schwarzschild value

$$\overline{r_v}^s(u) := \frac{1}{2}\left(-\frac{2M}{\overline{r}(u)} + \frac{\overline{r}(u)^2}{l^2}\right), \tag{3.203}$$

we have

$$\sup_{u \in \mathscr{R}_{v_0} \cup \mathcal{A}_{v_0}} |\overline{r_v} - \overline{r_v}^s| \leq C_{l,a,M} b^2. \tag{3.204}$$

For small enough initial data there exists points on $\mathcal{N}$ such that

$$\overline{r_v} \leq 0. \tag{3.205}$$

Furthermore there is a unique $u^* \in \mathcal{N}$ such that

$$\overline{r_v}(u^*) = 0. \tag{3.206}$$

Defining $r_{min} := \overline{r}(u^*)$ we have

$$|r_{min} - r_+| \leq C(b), \tag{3.207}$$

where $C(b) \to 0$ as $b \to 0$. Where $r_+ := (2Ml^2)^{\frac{1}{3}}$.

Proof. We will perform a bootstrap argument along $\mathcal{N}$. Define a bootstrap region to be

$$\mathcal{B}_{v_0} := \mathscr{R}_{v_0} \cup \mathcal{A}_{v_0} \cap \left\{r \geq \frac{r_+}{2}\right\}, \tag{3.208}$$

Clearly this set is closed, non empty, and connected. We need to show it's open to complete the bootstrap argument.

We define

$$f\left(\overline{\psi}^2\right) = \left(e^{4\pi g\overline{\psi}^2}\left(4\pi a\overline{\psi}^2 - \frac{3}{2}\right) + 2\pi(3g-2a)\overline{\psi}^2 + \frac{3}{2}\right). \tag{3.209}$$

The equation for $\overline{\varpi}$ (3.193) is thus

$$\begin{aligned}
\partial_u\overline{\varpi} &= \frac{2\pi\overline{r}^2}{\overline{r}_u}\left(\frac{\overline{r}^2}{l^2} - \frac{2\overline{\varpi}}{\overline{r}}\right)(\overline{\tilde{\nabla}}_u\overline{\psi})^2 + \frac{4\pi g^2\overline{r}_u}{\overline{r}}\overline{\varpi}\overline{\psi}^2 + \frac{\overline{r}_u\overline{r}^2}{l^2}f(\overline{\psi}^2) \\
&= \varpi\underbrace{\left(-\frac{4\pi\overline{r}}{\overline{r}_u}\left(\overline{\tilde{\nabla}}_u\overline{\psi}\right)^2 + \frac{4\pi g\overline{r}_u}{\overline{r}}\overline{\psi}^2\right)}_{=:h} + \frac{2\pi\overline{r}^4}{l^2\overline{r}_u}\left(\overline{\tilde{\nabla}}_u\overline{\psi}\right)^2 + \frac{\overline{r}_u\overline{r}^2}{l^2}f(\overline{\psi}^2) \\
&= \overline{\varpi}h - 4\pi\overline{r}\left(\overline{r}\overline{\tilde{\nabla}}_u\overline{\psi}\right)^2 + \frac{\overline{r}_u\overline{r}^2}{l^2}f(\overline{\psi}^2).
\end{aligned} \tag{3.210}$$

Solving this equation

$$\begin{aligned}
&\overline{\varpi} - M\exp\left(\int_{u_0}^u h du\right) \\
&= \exp\left(\int_{u_0}^u h du\right)\int_{u_0}^u -4\pi\overline{r}\left(\overline{r}^{\frac{1}{2}}\overline{\tilde{\nabla}}_u\overline{\psi}\right)^2\exp\left(-\int_{u_0}^u h du\right) + \frac{\overline{r}_u\overline{r}^2}{l^2}f(\overline{\psi}^2)\exp\left(-\int_{u_0}^u h du\right)du.
\end{aligned} \tag{3.211}$$

We estimate h by

$$\begin{aligned}
|h| &= \left|\left(-\frac{4\pi\overline{r}}{\overline{r}_u}\left(\overline{\tilde{\nabla}}_u\overline{\psi}\right)^2 + \frac{4\pi g\overline{r}_u}{\overline{r}}\overline{\psi}^2\right)\right| \\
&= -\overline{r}_u\left|-\frac{4\pi\overline{r}}{\overline{r}_u^2}\left(\overline{\tilde{\nabla}}_u\overline{\psi}\right)^2 + \frac{4\pi g}{\overline{r}}\overline{\psi}^2\right| \\
&= -\overline{r}_u\left|-\frac{4\pi\overline{r}}{\overline{r}_u^2}\left(\overline{\tilde{\nabla}}_u\overline{\psi}\right)^2 + \frac{4\pi g}{\overline{r}}\overline{\psi}^2\right| \\
&= -\overline{r}_u\left|-\frac{16\pi l^2}{\overline{r}^3}\left(\overline{\tilde{\nabla}}_u\overline{\psi}\right)^2 + \frac{4\pi g}{\overline{r}}\overline{\psi}^2\right| \\
&= -\overline{r}_u\left|-\frac{16\pi l^2}{\overline{r}^{4+s}}\left(\overline{r}^{\frac{1}{2}+\frac{s}{2}}\overline{\tilde{\nabla}}_u\overline{\psi}\right)^2 + 4\pi g\overline{r}^{-4+2\kappa}\left(\overline{r}^{\frac{3}{2}-\kappa}\overline{\psi}\right)^2\right| \\
&\leq C_{l,g}b^2\left(\overline{r}^{-4+s} + \overline{r}^{-4+2\kappa}\right)(-\overline{r}_u).
\end{aligned} \tag{3.212}$$

We see that in $\mathcal{B}_{v_0}$

$$\int_{u_0}^u |h|\, du \leq C_{l,g}\frac{b^2}{r_+}. \tag{3.213}$$

As $|h| = h$, we have

$$\begin{aligned}\overline{\varpi} - M \geq& \overline{\varpi} - M \exp\left(\int_{u_0}^{u} h du\right) \\ =& \exp\left(\int_{u_0}^{u} h du\right) \int_{u_0}^{u} \partial_u \left(\overline{\varpi} \exp\left(\int_{u_0}^{u} h du\right)\right) du \\ =& \exp\left(\int_{u_0}^{u} h du\right) \left(\int_{u_0}^{u} -4\pi \overline{r}^{1-s} \left(\overline{r}^{\frac{1}{2}+\frac{s}{2}} \overline{\tilde{\nabla}}_u \overline{\psi}\right)^2 \exp\left(-\int_{u_0}^{u} h du\right)\right. \\ &\left. + \frac{\overline{r}_u \overline{r}^2}{l^2} f(\overline{\psi}^2) \exp\left(-\int_{u_0}^{u} h du\right) du\right) \\ \geq& \exp\left(\int_{u_0}^{u} h du\right) \int_{u_0}^{u} -4\pi \overline{r}^{1-s} b^2 \exp\left(C_{l,g} \frac{b^2}{r_+}\right) + \frac{\overline{r}_u \overline{r}^2}{l^2} f(\overline{\psi}^2) \exp\left(-\int_{u_0}^{u} h du\right) du.\end{aligned} \tag{3.214}$$

(Note $\overline{r}^{1-s} du \sim \overline{r}^{-1-s} d\overline{r}$ hence the need for our initial data to have the additional s smallness). Continuing the estimation

$$\begin{aligned}\overline{\varpi} - M \geq& \exp\left(\int_{u_0}^{u} h du\right) \int_{u_0}^{u} -4\pi \overline{r}^{1-s} b^2 \exp\left(C_{l,g} \frac{b^2}{r_+}\right) + \frac{\overline{r}_u \overline{r}^2}{l^2} f(\overline{\psi}^2) \exp\left(-\int_{u_0}^{u} h du\right) du \\ \geq& -4\pi \exp\left(\int_{u_0}^{u} h du\right) b^2 \exp\left(C_{l,g} \frac{b^2}{r_+}\right) \int_{u_0}^{u} \overline{r}^{1-s} du \\ &+ \exp\left(\int_{u_0}^{u} h du\right) \int_{u_0}^{u} \frac{\overline{r}_u \overline{r}^2}{l^2} f(\overline{\psi}^2) \exp\left(-\int_{u_0}^{u} h du\right) du \\ =& -8\pi l^2 \exp\left(\int_{u_0}^{u} h du\right) b^2 \exp\left(C_{l,g} \frac{b^2}{r_+}\right) \int_{\infty}^{\overline{r}} \overline{r}^{-1-s} d\overline{r} \\ &+ \exp\left(\int_{u_0}^{u} h du\right) \int_{u_0}^{u} \frac{\overline{r}_u \overline{r}^2}{l^2} f(\overline{\psi}^2) \exp\left(-\int_{u_0}^{u} h du\right) du \\ \geq& -C_{l,g} \exp\left(C_{l,g} \frac{b^2}{r_+}\right) \left(\frac{r_+}{2}\right)^{-s} b^2 + \exp\left(\int_{u_0}^{u} h du\right) \int_{u_0}^{u} \frac{\overline{r}_u \overline{r}^2}{l^2} f(\overline{\psi}^2) \exp\left(-\int_{u_0}^{u} h du\right) du.\end{aligned} \tag{3.215}$$

For b^2 small

$$f(\overline{\psi}^2) \leq 8\pi a g \overline{\psi}^4, \tag{3.216}$$

so

$$\begin{aligned}\exp\left(\int_{u_0}^{u} h du\right) \int_{u_0}^{u} \frac{\overline{r}_u \overline{r}^2}{l^2} f(\overline{\psi}^2) \exp\left(-\int_{u_0}^{u} h du\right) du \geq& -C_{g,l} \exp\left(C_{l,g} \frac{b^2}{r_+}\right) b^4 \int_{\infty}^{\overline{r}} r^{-4+4\kappa} d\overline{r} \\ \geq& -C_{g,l} \exp\left(C_{l,g} \frac{b^2}{r_+}\right) b^4 r_+^{-3+4\kappa}.\end{aligned} \tag{3.217}$$

And thus

$$\overline{\varpi} - M \geq -C_{l,g} \exp\left(C_{l,g} \frac{b^2}{r_+}\right) r_+^{-s} b^2 - C_{g,l} \exp\left(C_{l,g} \frac{b^2}{r_+}\right) b^4 r_+^{-3+4\kappa} \geq -b^2 f(b), \tag{3.218}$$

where $f(b)$ goes to a positive constant as $b \to 0$.
Estimating the other direction

$$\partial_u \overline{\varpi} \leq h\overline{\varpi}, \tag{3.219}$$

we quickly see

$$\begin{aligned}\overline{\varpi} - M &\leq M\left(\exp\left(\int_{u_0}^{u} hdu\right) - 1\right)\\ &\leq M\left(\exp\left(C_{l,g}\frac{b^2}{r_+}\right) - 1\right)\\ &\leq \frac{C_{M,l,g}}{r_+}b^2.\end{aligned} \tag{3.220}$$

We conclude that

$$|\overline{\varpi} - M| \leq f(b)b^2, \tag{3.221}$$

where $f(b) \to C > 0$ as $b \to 0$.
We now show the second statement (3.204). Recall that

$$\overline{r_v} = \frac{1}{2}\left(\frac{-2\overline{\varpi}}{r} + \frac{\overline{r}^2}{l^2}\right)\exp\left(4\pi g\overline{\psi}^2 + \int_{u_0}^{u}\frac{-8\pi l^2}{\overline{r}}(\nabla_u\overline{\psi})^2du\right), \tag{3.222}$$

so

$$\begin{aligned}|\overline{r_v} - \overline{r_v}^s| &= \left|\frac{1}{2}\left(\frac{-2\overline{\varpi}}{r} + \frac{\overline{r}^2}{l^2}\right)\exp\left(4\pi g\overline{\psi}^2 + \int_{u_0}^{u}\frac{-8\pi l^2}{\overline{r}}(\nabla_u\overline{\psi})^2du\right) - \frac{1}{2}\left(\frac{-2M}{r} + \frac{\overline{r}^2}{l^2}\right)\right|\\ &= \frac{1}{2}\left|\frac{2}{\overline{r}}(M - \overline{\varpi}) + \left(\frac{-2\overline{\varpi}}{\overline{r}} + \frac{\overline{r}^2}{l^2}\right)\left(\exp\left(4\pi g\overline{\psi}^2 + \int_{u_0}^{u}\frac{-8\pi l^2}{\overline{r}}(\nabla_u\overline{\psi})^2du\right) - 1\right)\right|\\ &\leq \frac{1}{\overline{r}}|\overline{\varpi} - M| + \left|\frac{-2\overline{\varpi}}{\overline{r}} + \frac{\overline{r}^2}{l^2}\right|\left|\left(\exp\left(4\pi g\overline{\psi}^2 + \int_{u_0}^{u}\frac{-8\pi l^2}{\overline{r}}(\nabla_u\overline{\psi})^2du\right) - 1\right)\right|\\ &\leq \frac{C_{l,M,a}}{r}b^2 + \left|\frac{-2\overline{\varpi}}{r} + \frac{\overline{r}^2}{l^2}\right|\left|\left(\exp\left(4\pi g\overline{\psi}^2 + \int_{u_0}^{u}\frac{-8\pi l^2}{\overline{r}}(\nabla_u\overline{\psi})^2du\right) - 1\right)\right|.\end{aligned} \tag{3.223}$$

Noting the estimate

$$\left|\nabla_u\overline{\psi}\right|^2 \leq C_{l,g}b^2\overline{r}^{-3+2\kappa}, \tag{3.224}$$

we quickly can see that for $b < 1$

$$\left|\left(\exp\left(4\pi g\overline{\psi}^2 + \int_{u_0}^{u}\frac{-8\pi l^2}{\overline{r}}(\nabla_u\psi)^2du\right) - 1\right)\right| \leq C_{g,l}b^2\overline{r}^{-3+2\kappa}. \tag{3.225}$$

From here it follows that

$$|\overline{r_v} - \overline{r_v}^s| \leq b^2\left(\frac{C_{l,M,a}}{r} + C_{g,l}\overline{r}^{-1+2\kappa}\right). \tag{3.226}$$

Restricting to $\kappa \in (0, \frac{1}{2}]$ we can find a constant $C_{M,l,a} > 0$, such that

$$|\overline{r_v} - \overline{r_v}^s| \leq C_{M,l,g} b^2, \tag{3.227}$$

proving (3.204).

For statements (3.205) and (3.206), define u^+ by $\overline{r}(u^+) = r_+$. Now in our coordinate system we have

$$(\overline{r_v}^s)_u = -\frac{M}{l^2} - \frac{\overline{r}^3}{l^4} < 0, \tag{3.228}$$

showing that $\overline{r_v}^s$ is monotone on $\mathcal{N}$. For $\epsilon > 0$ consider $\tilde{u} = u^+ + \epsilon$ so

$$\overline{r_v}^s(\tilde{u}) < 0, \tag{3.229}$$

coupled with the estimate

$$\overline{r_v}^s(\tilde{u}) - Cb^2 \leq \overline{r_v}(\tilde{u}) \leq Cb^2 + \overline{r_v}^s(\tilde{u}), \tag{3.230}$$

implies for small initial data there is a point on $\mathcal{N}$ where $\overline{r_v} \leq 0$. Clearly we can repeat this argument and find a b^2 and u value say u_m where $\overline{r_v}(u_m) \geq 0$.

From continuity we know that there exists at least one value of $u \in [u_m, \tilde{u}]$ such that $\overline{r_v}(u) = 0$. Viewing the radial equation in terms of ϖ

$$(\overline{r_v})_u = -\frac{\overline{\Omega^2}}{2}\left(\frac{\overline{r}}{2l^2}\left(3 - e^{4\pi g \overline{\psi}^2}\right) + \frac{\overline{\varpi}}{\overline{r}^2} e^{4\pi g \overline{\psi^2}}\right) + \frac{2\pi a}{l^2}\overline{\Omega^2 \psi}^2 < 0, \tag{3.231}$$

for small b. We have $\overline{r_v}$ is monotonic, and this zero is unique. Let u^* denote this value, and denote it's $\overline{r}$ value by r_{min}. Recalling the definition of $\overline{r_v}$ we see that for this value of u the relationship

$$r_{min}^3 = 2l^2 \overline{\varpi}(u^*). \tag{3.232}$$

So

$$\left|r_{min}^3 - r_+^3\right| = 2l^2 \left|\overline{\varpi}(u^*) - M\right| \leq Cb^2. \tag{3.233}$$

This implies the inequality

$$r_{min} \geq \left(r_+^3 - Cb^2\right)^{\frac{1}{3}}, \tag{3.234}$$

and we deduce that

$$r_{min} \geq r_+ - C(b), \tag{3.235}$$

where $C(b) \to 0$ as $b \to 0$. From here we see that for b chosen small enough, the inequality in $\mathcal{B}_{v_0}$

$$r \geq r_{min} \geq r_+ - C(b) > \frac{r_+}{2}, \tag{3.236}$$

holds. Hence $\mathcal{B}_{v_0}$ is open and

$$\mathscr{R}_{v_0} \cup \mathcal{A}_{v_0} \cap \left\{r \geq \frac{r_+}{2}\right\} = \mathscr{R}_{v_0} \cup \mathcal{A}_{v_0}. \tag{3.237}$$

□

Maximal development and set up

We let $\mathcal{Q} \subset \mathbb{R}^2$ denote the quotient by $\mathfrak{T}^2$, of the maximal development from perturbed TAdSS data. From the geometric uniqueness statement we know that this is unique up to diffeomorphism.
We then denote the regular region $\mathscr{R} = \{(u,v) \in \mathcal{Q} : r_v(u,v) > 0\}$, and let $N(u) \subset \mathcal{Q}$ denote the outgoing characteristic null-line $u = const$ emanating from the initial data.

Lemma 3.6.2. *In the maximal development we have the following properties*

- *The set*
$$\{u > u_0 : N(u) \subset \mathscr{R} \text{ and } \tilde{r} \to 0 \text{ along } N(u)\} \neq \emptyset. \tag{3.238}$$
- *For*
$$u_{\mathcal{H}} := \sup_{u>u_0} \{u : N(u) \subset \mathscr{R} \text{ and } \tilde{r} \to 0 \text{ along } N(u)\}, \tag{3.239}$$
and
$$\mathscr{R}_{\mathcal{H}} := \mathscr{R} \cap \{u_0 < u < u_{\mathcal{H}}\}, \quad \text{and} \quad \overline{\mathscr{R}_{\mathcal{H}}} := \mathscr{R}_{\mathcal{H}} \cup N(u_{\mathcal{H}}), \tag{3.240}$$
we have that
$$\sup_{\mathscr{R}_{\mathcal{H}}} v = \sup_{N(u_{\mathcal{H}})} v. \tag{3.241}$$
- *Define 'null infinity' $\mathcal{I} = \{(u, v_\infty(u)) | u_0 < u < u_{\mathcal{H}}\}$, where $v_\infty(u)$ is the value of v such that: $\lim_{v \to v_{\infty(u)}} \tilde{r}(u,v) = 0$. Which we can reparametrise by $\{u_{\mathcal{I}}(v), v | v \in \mathcal{Q}\}$, where $u_{\mathcal{I}}(v)$ is the u coordinate of the past limit point where the $v = const$ ray intersects $\mathcal{I}$. Then there exists a double null system (u,v) covering $\mathscr{R}_{\mathcal{H}}$, such that*
$$\chi|_{\mathcal{I}} = \frac{1}{2}, \quad \bar{\tilde{r}}_u = \frac{1}{2l^2}. \tag{3.242}$$

Proof. From continuity the data set contains a point where $r_v < 0$, then we simply apply corollary 3.5.1. As $N(u_{\mathcal{H}})$ is regular (3.241) follows from the fact that a first singularity cannot form along it. Letting
$$\hat{u} = h(u), \quad \hat{v} = g(v), \tag{3.243}$$
we see that under these transforms we have that
$$\hat{\chi} := \frac{\hat{\Omega}^2}{-4r_{\hat{u}}} = \frac{\chi}{g'}, \tag{3.244}$$
and
$$-r_{\hat{u}} = \frac{-r_u}{h'}. \tag{3.245}$$
So choosing
$$g'(v) = \frac{\Omega^2}{-2r_u}(v,v) = 2\chi(v,v), \tag{3.246}$$

and

$$f'(u) = \frac{-2l^2 r_u}{r^2}(u, v_0), \tag{3.247}$$

we then have that at $\mathcal{I}$

$$\hat{\chi} = \frac{1}{2}, \tag{3.248}$$

and on the initial data ray $v = v_0$

$$-r_{\hat{u}} = \frac{r^2}{2l^2}. \tag{3.249}$$

We then switch to these coordinates and drop the hats.

In these coordinates null infinity is no longer a straight line and the future limit point of the ray $u = u_{\mathcal{H}}$ is not included as a part of null infinity, as it is a priori possible for $\tilde{r} \to 0$ along this ray. We will, as part of the proof of orbital stability, show this not to be the case. $\square$

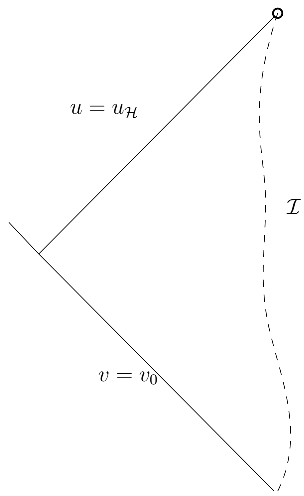


Figure 3.5: Depiction of (a subset of) the Penrose diagram

Lemma 3.6.3. *In $\mathscr{R}_{\mathcal{H}}$ we have that*

$$r \geq r_{min}, \tag{3.250}$$

Proof. We can write

$$r(u, v) = \overline{r}(u) + \int_{v_0}^{v} r_v(u, \hat{v}) d\hat{v}, \tag{3.251}$$

as we are in the regular region we know the integral is positive ($r_v > 0$), and lower bounds on the initial data ($\overline{r} \geq r_{min} > 0$) prove the result. $\square$

Lemma 3.6.4. *We have in $\mathcal{Q}^+$ that*

$$r_u < 0. \tag{3.252}$$

Proof. Integrating equation (3.8) from $\mathcal{I}$ yields the inequality

$$\frac{r_u}{\Omega^2} \leq -\frac{1}{2}, \tag{3.253}$$

the result follows. $\square$

Curves of constant r_X and r_Y

We define r_X to be the solution to

$$-\frac{2M}{r_X}+\frac{r_X^2}{l^2}=d, \tag{3.254}$$

where $d>0$ has been chosen small enough so that

$$\log\frac{r_X}{r_{min}}<\frac{1}{2\left|a\right|}. \tag{3.255}$$

This may be chosen from the continuity of r.
Now we define r_Y to be the solution to

$$-\frac{2M}{r_Y}+\frac{r_Y^2}{l^2}=c, \tag{3.256}$$

with $c<d$. So $r_Y<r_X$. Our Penrose diagram looks like:

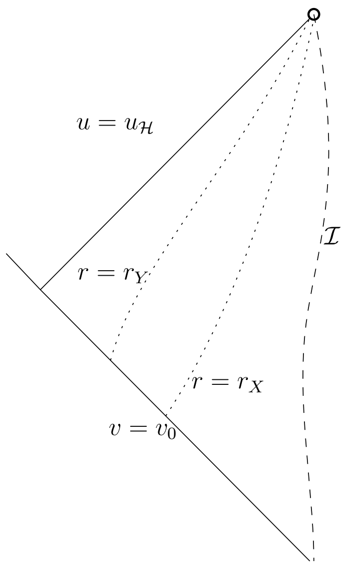


Figure 3.6: Penrose diagram of spacetime depicting $r=const$ curves.

Geometric norms

- The H_d^1 norm

$$||\psi||^2_{H^1_d}(u,v)=\int_{u_{\mathcal{I}}}^{u}\frac{r^2 r_v}{\Omega^2}(\nabla_u\psi)^2+\frac{(-r_u)}{r}\psi^2 d\bar{u}+\int_{v_0}^{v}-\frac{r^2 r_u}{\Omega^2}(\nabla_v\psi)^2+\frac{r_v}{r}\psi^2 d\bar{v}. \tag{3.257}$$

This is the standard degenerate energy norm one expects to see from exploiting the Kodama vector field $\mathcal{T}$ of the system (in context of the energy momentum tensor of the field). It is however not finite for our boundary conditions but will be useful when considering regions of bounded r.

- The $\underline{H}^1_d$ norm

$$||\psi||^2_{\underline{H}^1_d}(u,v) = \int_{u_\mathcal{I}}^{u} \frac{r^2 r_v}{\Omega^2}(\tilde{\nabla}_u\psi)^2 + \frac{(-r_u)}{r}\psi^2 d\bar{u} + \int_{v_0}^{v} -\frac{r^2 r_u}{\Omega^2}(\tilde{\nabla}_v\psi)^2 + \frac{r_v}{r}\psi^2 d\bar{v}. \quad (3.258)$$

This norm naturally arises from considering the renormalised Hawking mass as an energy potential. It suffers from degeneration on the first order terms at the apparent horizon (where $r_v = 0$), and a sub optimal weight on the zeroth order terms. This norm will be the basis of our estimates.

- The $\underline{H}^1_1$ norm

$$||\psi||^2_{\underline{H}^1_1}(u,v) = \int_{u_\mathcal{I}}^{u} \frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + \frac{(-r_u)}{r}\psi^2 d\bar{u} + \int_{v_0}^{v} -\frac{r^2 r_u}{\Omega^2}(\tilde{\nabla}_v\psi)^2 + \frac{r_v}{r}\psi^2 d\bar{v}. \quad (3.259)$$

After the redshift estimates we will be able to show control of this norm on the spacetime. It does not suffer degeneration at the apparent horizon. In fact it actually offers more control on for the weights of the zeroth order term but this is only clear after exploiting a Hardy type inequality.

- The $\underline{H}^1$ norm

$$||\psi||^2_{\underline{H}^1}(u,v) = \int_{u_\mathcal{I}}^{u} \frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 - r_u\psi^2 d\bar{u} + \int_{v_0}^{v} -\frac{r^2 r_u}{\Omega^2}(\tilde{\nabla}_v\psi)^2 + \frac{r_v}{r}\psi^2 d\bar{v}. \quad (3.260)$$

After a Hardy inequality we will be able to see this is equivalent to the $\underline{H}^1_1$ norm. We will however find this form more useful.

- The $\underline{L}^2$ norm

$$||\psi||^2_{\underline{L}^2}(u,v) = \int_{u_\mathcal{I}}^{u} -r_u\psi^2 d\bar{u} + \int_{v_0}^{v} \frac{r_v}{r}\psi^2 d\bar{v}. \quad (3.261)$$

- For convenience we also define the flux quantity

$$\mathbb{F}(u,v) = ||\psi||^2_{\underline{H}^1}(u,v) + ||\psi||^2_{\underline{H}^1}(u,v_0). \quad (3.262)$$

It is worth noting that these norms are all independent of change of double null coordinates and are thus geometric in their nature.

Boundary conditions

In the maximal development with enough regularity the Dirichlet boundary conditions are

$$\tilde{r}^{-\frac{3}{2}+\kappa}\psi = 0, \quad \text{on } \mathcal{I}, \quad (3.263)$$

and the Neumann boundary conditions are

$$\tilde{r}^{-\frac{1}{2}-\kappa}\tilde{\mathcal{R}}\psi = 0, \quad \text{on } \mathcal{I}. \quad (3.264)$$

Furthermore we may also deduce

Lemma 3.6.5.

$$\varpi|_{\mathcal{I}} = M. \tag{3.265}$$

Proof. This is the same argument as in lemma 3.4.4. We see that $\mathcal{T}$ is transverse to $\tilde{r} = const > 0$ surfaces and integrate the curl of ϖ over a triangular domain with boundary $\{\tilde{r} = const\} \cup \{u = const\} \cup \{v = const\}$. The boundary contribution on $\{\tilde{r} = const\}$ is given by

$$\mathcal{T}\varpi = 8\pi e^{4\pi g\psi^2} r^2 \tilde{\mathcal{R}}\psi\mathcal{T}\psi. \tag{3.266}$$

which is the term that vanishes in the energy estimates for ψ. So we take the limit $\tilde{r} \to 0$ to deduce the result. □

3.7 Orbital stability and completeness of null infinity

The goal of this section is to prove the two key theorems:

Theorem 3.7.1. *(Orbital Stability: Basic Estimates)*
In $\mathscr{R}_{\mathcal{H}}$, for $\kappa \leq \frac{1}{2}$, for $b > 0$ sufficiently small, such that we have the following estimates

$$|2\chi(u,v) - 1|^{\frac{1}{2}} + |\varpi(u,v) - M|^{\frac{1}{2}} + \left|\frac{r^{\frac{5}{2}}}{-r_u}\tilde{\nabla}_u\psi(u,v)\right| + \left|\frac{r^{\frac{5}{2}-\kappa}}{-r_u}\psi_u(u,v)\right| + \left|r^{\frac{3}{2}-\kappa}\psi(u,v)\right|$$
$$+ ||\psi||_{\underline{H}^1}(u,v) \leq C_{l,M,\kappa}\left(||\psi||_{\underline{H}^1}(u_{\mathcal{H}},v_0) + \sup_{I(v_0)}\left(\left|\frac{r^{\frac{5}{2}}}{-r_u}\tilde{\nabla}_u\psi\right| + \left|r^{\frac{3}{2}-\kappa}\psi\right|\right)\right). \tag{3.267}$$

Where b was defined in (3.6.1)*.*

Theorem 3.7.2. *(Completeness of Null Infinity)*
Let

$$v_m = \sup_{v \geq v_0}\{v|(u_{\mathcal{H}},v) \in \mathcal{Q}\}. \tag{3.268}$$

Then it is the case that $v_m = \infty$.

We will use a bootstrap argument to establish these results. The core idea is to bootstrap on the size of the field ψ.

3.7.1 Basic estimates

The bootstrap

Let $\tilde{u} \in [u_0, u_{\mathcal{H}}]$ we then define the region

$$\hat{\mathcal{B}}(\tilde{u}) = \mathcal{R}_{\mathcal{H}} \cap \{u_0 < u < \tilde{u}\}. \tag{3.269}$$

We will bootstrap on the condition

$$\left| r^{\frac{3}{2}-\kappa} \psi \right| < b. \tag{3.270}$$

So we let

$$u_{max} := \sup_u \left\{ u \in \mathscr{R} : \left| r^{\frac{3}{2}-\kappa} \psi \right| < b \right\}, \tag{3.271}$$

the bootstrap region is then defined as

$$\mathcal{B} = \hat{\mathcal{B}}(u_{max}) \subset \mathscr{R}_{\mathcal{H}}. \tag{3.272}$$

We aim to prove that $\mathcal{B} = \mathscr{R}_{\mathcal{H}} = \hat{\mathcal{B}}(u_{\mathcal{H}})$. It is clear that $\mathcal{B}$ is open, connected, and non empty. Hence we aim to show that $\mathcal{B}$ is a closed subset of $\mathscr{R}_{\mathcal{H}}$. For this we assume that $u_{max} < u_{\mathcal{H}}$ is fixed, and that in $\overline{\mathcal{B}}$ we can improve the bound (3.270) (it trivially holds at $u = u_0$ as this is an initial data point).

The renormalised Hawking mass

Definition 3.7.1. *Recall that we defined the final renormalised Hawking mass as*

$$\varpi = \frac{2 r_u r_v r}{\Omega^2} e^{4\pi g \psi^2} + \frac{r^3}{2l^2}. \tag{3.273}$$

The final renormalised Hawking mass provides a potential for the $\underline{H}^1$ geometric energy, and for small enough ψ it satisfies monotonicity properties. Coupled with a redshift estimate this leads to an energy estimate for ψ. From here Sobolev embeddings can be used to recover the bootstrap assumption.

Monotonicity and boundedness for small ψ

In order to get a sign for the derivatives of ϖ we need to see that the function $f(\psi^2)$ is indeed positive in the region $\mathcal{B}$. Plotting the curve shows the following global behaviour.

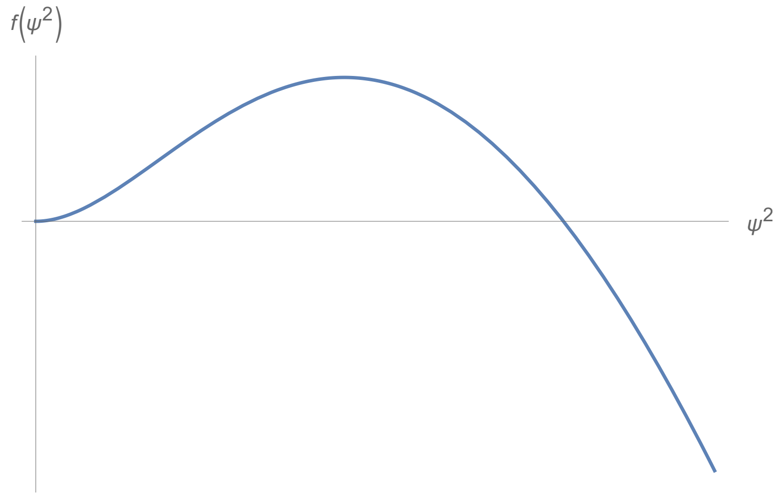


Figure 3.7: Plot of $f(\psi^2)$.

We now seek to quantify the local positive behaviour in $\mathcal{B}$ with the following lemma.

Lemma 3.7.1 (Bounds for $f(\psi^2)$).
For

$$f(\psi^2) = e^{4\pi g\psi^2} 4\pi a\psi^2 - \frac{3}{2}e^{4\pi g\psi^2} - 2\pi g^2\psi^2 + \frac{3}{2}, \tag{3.274}$$

we have that in $\mathcal{B}$

$$f(\psi^2) \geq 0, \tag{3.275}$$

and

$$f(\psi^2) \leq 8\pi^2 ag\psi^4. \tag{3.276}$$

Proof. Define

$$f(x) = e^{4\pi gx} 4\pi ax - \frac{3}{2}e^{4\pi gx} - 2\pi g^2 x + \frac{3}{2}, \tag{3.277}$$

and consider its values for $x \geq 0$. As $f(0) = 0$, we proceed to study the functions behaviour near this point.
We compute

$$f'(x) = 16\pi^2 gaxe^{4\pi gx} + 2\pi g^2 e^{4\pi gx} - 2\pi g^2, \tag{3.278}$$

so $f'(0) = 0$. Further computation shows

$$f''(x) = 16\pi^2 g^2 \left(4\pi ax + \kappa\right) e^{4\pi gx}, \tag{3.279}$$

which remains positive for $x \leq \frac{-\kappa}{4\pi a} = \frac{1}{2\pi}\frac{\kappa}{\frac{9}{4}-\kappa^2}$.
We thus have the following differential inequality that for $x \in \left[0, \frac{1}{2\pi}\frac{\kappa}{\frac{9}{4}-\kappa^2}\right]$

$$f''(x) \geq 0, \quad f(0) = f'(0) = 0. \tag{3.280}$$

Solving this differential inequality yields the first result.
For the second result consider

$$g(x) := f(x) - 8\pi^2 agx^2, \tag{3.281}$$

again we see that $g(0) = 0$. Compute

$$g'(x) = 16\pi^2 gax\left(e^{4\pi gx} - 1\right) + 2\pi g^2\left(e^{4\pi gx} - 1\right) \leq 0, \tag{3.282}$$

solving this differential inequality yields that

$$f(x) \leq 8\pi^2 agx^2, \tag{3.283}$$

for non-negative x. □

Corollary 3.7.1. *In* $\mathcal{B}$ *we have*

$$\varpi(u,v) \leq \varpi(u_{\mathcal{I}}, v) = M. \tag{3.284}$$

Proof. Estimating the $\partial_u \varpi$ from (3.7.1) we have

$$\partial_u \varpi \leq \frac{4\pi g^2 r_u}{r} \varpi \psi^2. \tag{3.285}$$

A Gronwall estimate then implies

$$\varpi(u,v) \leq \varpi(u_{\mathcal{I}}, v) \exp\left(\int_{u_{\mathcal{I}}}^{u} 4\pi g^2 \frac{r_u}{r} \psi^2 du \right) \leq \varpi(u_{\mathcal{I}}, v) = M. \tag{3.286}$$

□

Remark 3.7.1. *We have in $\mathcal{B}$*

$$\varpi(u, v_0) := \varpi_0 = \overline{\varpi} \geq M - C_{l,g} b^2 \geq \frac{M}{2} > 0. \tag{3.287}$$

for b^2 small enough.

Proof. We use the smallness of b, and our initial data estimate (3.202). □

Corollary 3.7.2. *In $\mathcal{B}$ we have*

$$0 < \frac{M}{2} \leq \varpi. \tag{3.288}$$

Proof. Identical to the proof of corollary 3.7.1, we estimate $\partial_v \varpi$ and use the lower bound for ϖ_0. □

Corollary 3.7.3. *In $\mathcal{B}$ we have*

$$\partial_u \varpi \geq 8\pi^2 g \frac{a r_u r^2}{l^2} \psi^4 - \frac{8\pi r^2 r_v}{\Omega^2} (\tilde{\nabla}_u \psi)^2 + \frac{4\pi g^2 r_u}{r} M \psi^2, \tag{3.289}$$

$$\partial_v \varpi \leq 8\pi^2 g \frac{a r_v r^2}{l^2} \psi^4 - \frac{8\pi r^2 r_u}{\Omega^2} (\tilde{\nabla}_v \psi)^2 + \frac{4\pi g^2 r_v}{r} M \psi^2. \tag{3.290}$$

Proof. This follows immediately from the lemmas 3.4.3, 3.7.1 and corollary 3.7.1. □

Corollary 3.7.4. *In $\mathcal{B}$ we have that*

$$\partial_u \varpi \leq -\frac{8\pi r^2 r_v}{\Omega^2} (\tilde{\nabla}_u \psi)^2 e + \frac{4\pi g^2 r_u}{r} M \psi^2 \leq 0, \tag{3.291}$$

$$\partial_v \varpi \geq -\frac{8\pi r^2 r_u}{\Omega^2} (\tilde{\nabla}_v \psi)^2 + \frac{4\pi g^2 r_v}{r} M \psi^2 \geq 0. \tag{3.292}$$

Proof. This follows immediately from the lemmas 3.4.3, 3.7.1 and corollary 3.7.2. □

Corollary 3.7.5. *We have in $\mathcal{B}$*

$$|\varpi - M| \leq C_{a,M,l} \left(||\psi||^2_{\underline{H}^1_d}(u,v) + b^4 \right). \tag{3.293}$$

Proof. The bound from above follows trivially from corollary 3.7.4, from below we integrate $\partial_u \varpi$ and use estimate (3.289)

$$\begin{aligned}\varpi - M = \varpi - \varpi(u_{\mathcal{I}}, v) &= \int_{u_{\mathcal{I}}}^{u} -\frac{8\pi r^2 r_v}{\Omega^2}(\tilde{\nabla}_u\psi)^2 e^{4\pi g\psi^2} + \frac{4\pi g^2 r_u}{r}\varpi\psi^2 + \frac{r_u r^2}{l^2} f(\psi^2) du \\ &\geq \int_{u_{\mathcal{I}}}^{u} -\frac{8\pi r^2 r_v}{\Omega^2}(\tilde{\nabla}_u\psi)^2 + \frac{2\pi g^2 r_u}{r} M\psi^2 du + \int_{u_{\mathcal{I}}}^{u} \frac{8\pi^2 ag}{l^2}\psi^4 r^2 r_u du \\ &- C_{a,M} \left|\left|\psi\right|\right|^2_{\underline{H}^1_d}(u,v) - C_{a,l} b^4 \int_{u_{\mathcal{I}}}^{u} r_u r^{-4+2\kappa} du \\ &\geq -C_{a,M,l}\left(\left|\left|\psi\right|\right|^2_{\underline{H}^1_d}(u,v) + b^4\right).\end{aligned} \tag{3.294}$$

□

ϖ_1 **estimates:**

Lemma 3.7.2. *We have, in $\mathcal{B}$,*

$$\left|\frac{\varpi_1 - M}{r^{2\kappa}}\right| \leq C_{M,g,l} b^2. \tag{3.295}$$

Proof. Expanding

$$\frac{\varpi_1 - M}{r^{2\kappa}} = \frac{1}{r^{2\kappa}}\left(\varpi e^{-4\pi g\psi^2} - M\right) + \left(1 - e^{-4\pi g\psi^2}\right)\frac{r^{3-2\kappa}}{2l^2}, \tag{3.296}$$

and applying the estimates

$$M - C_{M,g,l} b^2 \leq \varpi \leq M, \tag{3.297}$$

and (for small b)

$$8\pi g\psi^2 \leq 1 - e^{-4\pi g\psi^2} \leq 4\pi g\psi^2 < 0. \tag{3.298}$$

We get

$$\begin{aligned}\frac{\varpi_1 - M}{r^{2\kappa}} &\leq \frac{M}{r^{2\kappa}}\left(e^{-4\pi g\psi^2} - 1\right) \\ &\leq \frac{M}{r^{2\kappa}} \cdot 8\pi(-g) r^{-3+2\kappa} b^2 \\ &\leq 8\pi M(-g) r_{min}^{-3} b^2 \leq C_{M,g} b^2.\end{aligned} \tag{3.299}$$

In the other direction

$$
\begin{aligned}
\frac{\varpi_1 - M}{r^{2\kappa}} &\geq \frac{1}{r^{2\kappa}}\Big((M - C_{M,g,l}b^2)e^{-4\pi g\psi^2} - M\Big) + \frac{8\pi g r^{3-2\kappa}\psi^2}{2l^2} \\
&\geq \frac{M}{r^{2\kappa}}\Big(e^{-4\pi g\psi^2} - 1\Big) - \frac{C_{M,g,l}b^2}{r^{2\kappa}}e^{-4\pi g\psi^2} + \frac{8\pi g r^{3-2\kappa}\psi^2}{2l^2} \\
&\geq \frac{M}{r^{2\kappa}}\cdot(-4\pi g\psi^2) - \frac{C_{M,g,l}b^2}{r^{2\kappa}}e + \frac{4\pi g}{l^2}b^2 \\
&\geq -M4\pi g b^2 r^{-3} - \frac{C_{M,g,l}b^2}{r^{2\kappa}}e + \frac{4\pi g}{l^2}b^2 \\
&\geq -C_{M,g,l}b^2,
\end{aligned}
\tag{3.300}
$$

thus concluding the proof. □

Lemma 3.7.3. *Let $\alpha + \beta > 2\kappa$ be non negative numbers, and $D > 0$, a positive constant. The following estimate holds in $\mathcal{B}$ (for b small enough)*

$$
\frac{\varpi_1}{r^\alpha} + Dr^\beta \geq D(1+\epsilon)r^\beta \geq Dr^\beta > 0,
\tag{3.301}
$$

where $0 < \epsilon \leq \frac{M}{D}r_{min}^{-(\beta+\alpha)} - c_{M,g,l}b^2$.

Proof.

$$
\begin{aligned}
\frac{\varpi_1}{r^\alpha} + Dr^\beta &\geq Dr^\beta + \frac{M}{r^\alpha} - C_{M,g,l}b^2 r^{2\kappa-\alpha} \\
&\geq Dr^\beta\left(1 + \frac{M}{D}r_{min}^{-(\beta+\alpha)} - \frac{C_{M,g,l}b^2}{D}r^{2\kappa-(\alpha+\beta)}\right) \\
&\geq Dr^\beta\left(1 + \frac{M}{D}r_{min}^{-(\beta+\alpha)} - \frac{C_{M,g,l}b^2}{D}r_{min}^{2\kappa-(\alpha+\beta)}\right) \\
&\geq D(1+\epsilon)r^\beta \geq Dr^\beta > 0.
\end{aligned}
\tag{3.302}
$$

□

Remark 3.7.2. *This result is purely technical, while we know that $\varpi > 0$ we didn't a priori have positivity for ϖ_1 in $\mathscr{R}_{\mathcal{H}}$. Estimates for ϖ_1 are useful due to how the quantity algebraically interacts with the system. It is often coupled with a term like Dr^β which we will want to keep in our estimates.*

Estimates for r_u

We now seek to control the growth of function r_u.

Theorem 3.7.3. *In the bootstrap region $\mathcal{B}$, there exists a uniform constant $C = C(l, g) > 0$ such that*

$$
\frac{1}{C}r^2 \leq -r_u.
\tag{3.303}
$$

We split the proof into three lemmas.

Lemma 3.7.4. *In the region $\mathcal{B} \cap \{r_{min} \leq r \leq r_X\}$, we have that there exists $C = C(l, X) > 0$ such that*

$$r_X^{-2} C r^2 \leq -r_u. \tag{3.304}$$

Proof. We restrict ourselves to the region $r_{min} \leq r \leq r_X$, and consider (3.10),

$$r_{uv} = -\frac{r_u r_v}{r} + \frac{2\pi a r}{l^2}\Omega^2\psi^2 - \frac{3}{4}\frac{r}{l^2}\Omega^2. \tag{3.305}$$

It follows that

$$r_{uv} \leq -\frac{r_u r_v}{r}, \tag{3.306}$$

which we integrate and estimate

$$-r_u \geq -\overline{r_u} \cdot \overline{r} \cdot \frac{1}{r} \geq \frac{r_{min}^3}{2l^2 r_X} =: C > 0. \tag{3.307}$$

We extend to

$$r_X^{-2} C r^2 \leq -r_u. \tag{3.308}$$

□

We now study the set where r is unbounded.

Lemma 3.7.5 (Region Splitting)**.** *We have on the region $\mathcal{B} \cap \{r \geq r_Y\}$, the following inequality*

$$\frac{r^2}{l^2} - \frac{2M}{r} \geq C_{Y,l,M} r^2. \tag{3.309}$$

Proof. Let $C_{Y,l} = \min(\frac{1}{2l^2}, \frac{d}{r_Y^2})$ and define

$$f(r) = \frac{r^2}{l^2} - \frac{2M}{r} - C_{Y,l,M} r^2. \tag{3.310}$$

Then

$$f(r_Y) = d - C_{Y,l,M} r_Y^2 \geq 0, \tag{3.311}$$

and

$$f'(r) = \frac{1}{r}\left(\frac{r^2}{l^2} + \frac{2M}{r}\right) + 2r\left(\frac{1}{2l^2} - C_{Y,l,M}\right) \geq 0, \tag{3.312}$$

whence the result follows. □

Lemma 3.7.6. *We have in the region $\mathcal{B} \cap \{r \geq r_Y\}$ that*

$$\frac{1}{C_{Y,M,l}} r^2 \leq -r_u \leq C_{Y,M,l} r^2. \tag{3.313}$$

Proof. Using the wave equation for the radial function

$$\tilde{r}_{uv} = \frac{-3\tilde{r}_u r_v}{r} + \frac{3}{4l^2}\frac{\Omega^2}{r} - \frac{2\pi a}{l^2}\frac{\psi^2\Omega^2}{r}, \tag{3.314}$$

we may write this as

$$\tilde{r}_{uv} = \tilde{r}_u \cdot f(u,v) r_v, \tag{3.315}$$

where

$$f(u,v) = \left(\frac{6\varpi}{r^2} + \frac{3r}{l^2}\left(1 - e^{-4\pi g \psi^2}\right) + \frac{6\varpi}{r^2}\left(e^{-4\pi g \psi^2} - 1\right) - \frac{8\pi a}{l^2} r \psi^2 \right) \frac{e^{4\pi g \psi^2}}{\frac{r^2}{l^2} - \frac{2\varpi}{r}}. \tag{3.316}$$

We have that

$$\tilde{r}_u = \frac{1}{2l^2} \exp\left(\int_{r(v_0)}^{r(v)} f(u,v) dr \right). \tag{3.317}$$

We now wish to show that f is integrable. Elementary estimation, and noting that

$$\left| 1 - e^{-4\pi g \psi^2} \right| \leq -8\pi g \psi^2, \tag{3.318}$$

for b^2 small, we see

$$|f| \leq \frac{1}{\left| \frac{r^2}{l^2} - \frac{2M}{r} \right|} \left(\left| 6Mr^{-2} \right| + \left| \frac{24\pi(-g)b^2}{l^2} r^{-2+2\kappa} \right| + \left| 48\pi(-g) b^2 r^{-3+2\kappa} \right| + \left| \frac{2\pi(-a)}{l^2} b^2 r^{-2+2\kappa} \right| \right). \tag{3.319}$$

We now restrict to $r \geq r_Y$, here we have

$$|f| \leq C_{Y,M} r^{-4+2\kappa}. \tag{3.320}$$

So there exists a constant C_Y such that

$$\frac{1}{2l^2} \exp\left(-C_{Y,M} \int_{r(v_0)}^{r(v)} r^{-4+2\kappa} dr \right) \leq \tilde{r}_u \leq \frac{1}{2l^2} \exp\left(C_{Y,M} \int_{r(v_0)}^{r(v)} r^{-4+2\kappa} dr \right). \tag{3.321}$$

Integrating gives

$$\frac{1}{2l^2} \exp\left(C_{Y,M} \left(r^{-3+2\kappa} - \overline{r}^{-3+2\kappa} \right) \right) \leq \tilde{r}_u \leq \frac{1}{2l^2} \exp\left(C_{Y,M} \left(\overline{r}^{-3+2\kappa} - r^{-3+2\kappa} \right) \right), \tag{3.322}$$

and we deduce the bound

$$\frac{1}{2l^2} \exp\left(-C_{Y,M} r_{min}^{-3+2\kappa} \right) \leq \tilde{r}_u \leq \frac{1}{2l^2} \exp\left(C_{Y,M} r_{min}^{-3+2\kappa} \right). \tag{3.323}$$

From here the result follows. □

From these three lemmas the proof of theorem 3.7.3 follows.

Degenerate energy estimates from the Hawking mass:

Lemma 3.7.7. *In $\mathcal{B}$ we have there exists a $C_{g,M,\kappa,Y} > 0$, such that*

$$||\psi||_{\underline{H}_d^1}(u,v) \leq C_{g,M,\kappa,Y}\left(||\psi||_{\underline{H}_d^1}(u_{\mathcal{H}},v_0) + b^4\right). \tag{3.324}$$

Proof. We start by noting

$$\begin{aligned}\int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} -\partial_{\bar{u}}\varpi(\bar{u},v_0)d\bar{u} \geq \int_{u_{\mathcal{I}}}^{u} -\partial_{\bar{u}}\varpi(u,v_0)d\bar{u}& \\ =&\varpi|_{\mathcal{I}} - \varpi(u,v_0) + \varpi(u,v) - \varpi(u,v) \\ =&\int_{u_{\mathcal{I}}}^{u} -\partial_{\bar{u}}\varpi(\bar{u},v)d\bar{u} + \int_{v_0}^{v} \partial_{\bar{v}}\varpi(u,\bar{v})d\bar{v} \\ \geq&\int_{u_{\mathcal{I}}}^{u} \frac{8\pi r^2 r_v}{\Omega^2}(\tilde{\nabla}_u\psi)^2 e^{-1} + \frac{2\pi g^2(-r_u)}{r}M\psi^2 d\bar{u} \\ &+ \int_{v_0}^{v} -\frac{8\pi r^2 r_u}{\Omega^2}(\tilde{\nabla}_v\psi)^2 + \frac{2\pi g^2 r_v}{r}M\psi^2 d\bar{v} \\ \geq& C_{g,M}\left(\int_{u_{\mathcal{I}}}^{u} \frac{r^2 r_v}{\Omega^2}(\tilde{\nabla}_u\psi)^2 + \frac{(-r_u)}{r}\psi^2 d\bar{u} + \int_{v_0}^{v} -\frac{r^2 r_u}{\Omega^2}(\tilde{\nabla}_v\psi)^2 + \frac{r_v}{r}\psi^2 d\bar{v}\right).\end{aligned} \tag{3.325}$$

This gives us control of the $\underline{H}_d^1$ norm from initial data, providing we can prove the LHS integral is controlled by the $\underline{H}_d^1$ norm. Recall

$$-\partial_u\varpi \leq \frac{8\pi r^2 r_v}{\Omega^2}(\tilde{\nabla}_u\psi)^2 e^{4\pi g\psi^2} - \frac{4\pi g^2 r_u}{r}\varpi\psi^2 - \frac{r_u r^2}{l^2}8\pi^2 a g\psi^4, \tag{3.326}$$

we see the first two terms form the $\underline{H}_d^1$ norm. We need to control the final term.
From the bootstrap assumption we form the bound

$$\psi^4 r^2 \leq b^4 r^{-4+4\kappa}, \tag{3.327}$$

using this in the integral

$$\begin{aligned}\int_{u_{\mathcal{I}}}^{u} -8\pi^2 g\frac{a r_u r^2}{l^2}\psi^4 du &\leq b^4 C_g \int_{u_{\mathcal{I}}}^{u} \frac{r_u}{3-4\kappa}\partial_u\left(r^{-3+4\kappa}\right)du \\ &= \frac{b^4 C_g}{3-4\kappa}\left[r^{-3+4\kappa}\right]_{u_{\mathcal{I}}}^{u} = \frac{b^4 C_g}{3-4\kappa}r^{-3+4\kappa} \\ &\leq C_{g,r_{min}}\frac{1}{3-4\kappa}b^4 \leq C_{g,r_{min}}b^4.\end{aligned} \tag{3.328}$$

We conclude

$$\int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} -\partial_{\bar{u}}\varpi(\bar{u},v_0)d\bar{u} \leq C_{M,g}\,||\psi||^2_{\underline{H}_d^1}(u_{\mathcal{H}},v_0) + C_g b^4. \tag{3.329}$$

The result then follows. □

Energy estimates in $\{r \geq r_Y\}$

We now wish to improve our estimates to the $\underline{H}^1$ norm and recover pointwise estimates. We begin in the region away from the degenerative issues of r_v. We will see that the standard theory of Hardy and Sobolev estimates can be recovered with twisted derivatives.

Lemma 3.7.8 (Norm equivalence away from degeneration)**.**
For $\mathcal{B} \cap \{r \geq r_Y\}$ *we have that*

$$C_l \,||\psi||^2_{\underline{H}^1_d}(u,v) \leq ||\psi||^2_{\underline{H}^1_1}(u,v) \leq C_{l,Y}\,||\psi||^2_{\underline{H}^1_d}(u,v). \tag{3.330}$$

Proof. Note that on the domain $\{r \geq r_Y\} \cap \mathcal{B}$

$$\frac{r^2 r_v}{\Omega^2} = \left(\frac{r^2}{l^2} - \frac{2\varpi}{r}\right)\frac{e^{-4\pi g\psi^2}}{-4r_u} r^2, \tag{3.331}$$

estimating from either side

$$C_l \frac{r^4}{-r_u} \geq \left(\frac{r^2}{l^2} - \frac{2\varpi}{r}\right)\frac{e^{-4\pi g\psi^2}}{-4r_u} r^2 \geq \left(\frac{r^2}{l^2} - \frac{2M}{r}\right)\frac{r^2}{-4r_u} \geq C_{l,Y}\frac{r^4}{-r_u}. \tag{3.332}$$

□

Lemma 3.7.9 (Hardy Inequality)**.** *We have in* $\mathcal{B}$

$$\frac{1}{C_g}\,||\psi||^2_{\underline{H}^1_1}(u,v) \leq ||\psi||^2_{\underline{H}^1}(u,v) \leq C_g\,||\psi||^2_{\underline{H}^1_1}(u,v). \tag{3.333}$$

Proof. Fix $u_1 < u_2 < u_{\mathcal{I}} \in \mathcal{B}$, and let χ be a bump function with the following properties

$$\chi(r(u)) = \begin{cases} 0 & u \leq u_1, \\ 1 & u \geq u_2, \\ \text{Smooth and bounded by 1} & \text{otherwise.} \end{cases} \tag{3.334}$$

Then

$$||(1-\chi)\psi||^2_{\underline{L}^2} = \int_{u_2}^{u} -r_u(1-\chi)^2\psi^2 du \leq \sup_{(u_2,u)} r \cdot \int_{u_2}^{u} \frac{-r_u}{r}\psi^2 du \leq C_{r(u_2)}\int_{u_{\mathcal{I}}}^{u}\frac{-r_u}{r}\psi^2 du. \tag{3.335}$$

Looking at

$$\begin{aligned} ||\chi\psi||^2_{\underline{L}^2} &= \int_{u_{\mathcal{I}}}^{u_1}\left(\chi\psi r^{\frac{3}{2}-\kappa}\right)^2 \partial_u\left(\frac{r^{-2+2\kappa}}{2-2\kappa}\right)du \\ &= \left[\frac{r(\chi\psi)^2}{2-2\kappa}\right]^{u_1}_{u_{\mathcal{I}}} + \int_{u_{\mathcal{I}}}^{u_1}\frac{1}{1-\kappa}\chi r\psi\tilde{\nabla}_u(\chi\psi)\,r du \\ &\leq \frac{1}{1-\kappa}\,||\chi\psi||_{\underline{L}^2}\cdot\left(\int_{u_{\mathcal{I}}}^{u_1}\left(\tilde{\nabla}_u(\chi\psi)\right)^2\frac{r^2}{-r_u}du\right)^{\frac{1}{2}}, \end{aligned} \tag{3.336}$$

we have

$$
\begin{aligned}
||\chi\psi||^2_{\underline{L}^2} &\leq \frac{1}{(1-\kappa)^2}\int_{u_{\mathcal{I}}}^{u_1}\left(\tilde{\nabla}_u(\chi\psi)\right)^2\frac{r^2}{-r_u}du \\
&\leq C_\kappa\int_{u_{\mathcal{I}}}^{u_1}\chi^2(\tilde{\nabla}_u\psi)^2\frac{r^2}{-r_u}+\frac{r^2}{-r_u}\left(\psi\partial_u\chi\right)^2du.
\end{aligned}
\tag{3.337}
$$

Studying the latter term

$$
\frac{r^2}{-r_u}\left(\psi\partial_u\chi\right)^2=\frac{r^2}{-r_u}\left(r_u\psi\chi'\right)^2=-r_u\psi^2r^2(\chi')^2,
\tag{3.338}
$$

note that due to regularity of χ and compactness, there exists a $C>0$, such that

$$
\int_{u_{\mathcal{I}}}^{u_1}-r_u\psi^2r^2(\partial_r\chi)^2du=\int_{u_2}^{u_1}-r_u\psi^2r^2(\partial_r\chi)^2du\leq C\int_{u_{\mathcal{I}}}^{u}\frac{-r_u}{r}\psi^2du.
\tag{3.339}
$$

This then implies

$$
\frac{1}{C_\kappa}||\chi\psi||^2_{\underline{L}^2}\leq\int_{u_{\mathcal{I}}}^{u_1}(\tilde{\nabla}_u\psi)^2\frac{r^2}{-r_u}du+\int_{u_{\mathcal{I}}}^{u}\frac{-r_u}{r}\psi^2du,
\tag{3.340}
$$

combining all the results

$$
||\psi||^2_{\underline{L}^2}=\int_{u_{\mathcal{I}}}^{u}-r_u\psi^2du\leq C_\kappa\left(\int_{u_{\mathcal{I}}}^{u_1}(\tilde{\nabla}_u\psi)^2\frac{r^2}{-r_u}du+\int_{u_{\mathcal{I}}}^{u}\frac{-r_u}{r}\psi^2du\right).
\tag{3.341}
$$

The result follows. □

Lemma 3.7.10 (Sobolev Inequality)**.** *We have in for $\psi\in\underline{H}^1(\mathcal{B})$ there exists $C_{g,M,l}>0$. such that*

$$
\left|r^{-g}\psi\right|(u,v)\leq C_{g,M,l}\left(||\psi||_{\underline{H}^1}(u,v)+||\psi||_{\underline{L}^2}(u_{\mathcal{H}},v)\right).
\tag{3.342}
$$

Proof. We begin by writing

$$
\begin{aligned}
r^{-g}\psi(u,v)-r^{-g}\psi(z,v) &= \int_z^u\partial_{u'}(r^{-g}\psi)(u',v)du'=\int_z^u(\tilde{\nabla}_{u'}\psi)r^{-g}du' \\
&= \int_z^u\left(\frac{\left(\tilde{\nabla}_{u'}\psi\right)r^2}{\sqrt{-r_{u'}}}\cdot r^{-2-g}\sqrt{-r_{u'}}\right)(u',v)du' \\
&\leq \left(\int_z^u\left(\frac{(\tilde{\nabla}_{u'}\psi)^2r^4}{-r_{u'}}\right)(u',v)du'\right)^{\frac{1}{2}}\left(\int_z^u\left(-r^{-4-2g}r_{u'}\right)(u',v)du'\right)^{\frac{1}{2}} \\
&\leq \left(\int_{u_{\mathcal{I}}}^u\left(\frac{(\tilde{\nabla}_{u'}\psi)^2r^4}{-r_{u'}}\right)(u',v)du'\right)^{\frac{1}{2}}\left(\int_{u_{\mathcal{I}}}^u\left(-r^{-4-2g}r_{u'}\right)(u',v)du'\right)^{\frac{1}{2}} \\
&\leq ||\psi||_{\underline{H}^1}(u,v)\left(\int_{u_{\mathcal{I}}}^u\left(-r^{-4-2g}r_{u'}\right)(u',v)du'\right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.343}
$$

Now as

$$\int_{u_{\mathcal{I}}}^{u} \left(-r^{-4-2g} r_{u'}\right)(u',v)du' = \left[\frac{r^{-3-2g}}{-3-2g}\right]_{\infty}^{r(u,v)} = \frac{1}{2\kappa} r^{-2\kappa}(u,v), \tag{3.344}$$

we have

$$\left|r^{-g}\psi(u,v)\right| \leq \left|r^{-g}\psi(z,v)\right| + \frac{1}{2\kappa} r^{-\kappa}(u,v) \left|\left|\psi\right|\right|_{\underline{H}^1}(u,v). \tag{3.345}$$

Integrating z over the whole u ray

$$r^{-g}\psi(u,v)\int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} dz \leq \int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} r^{-g}\left|\psi(z,v)\right| dz + Cr^{-\kappa}(u,v)\left|\left|\psi\right|\right|_{\underline{H}^1}(u,v)\int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} dz. \tag{3.346}$$

As $\int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} dz = C_{dom.}$ a domain dependant constant. We need only worry about

$$\begin{aligned}
\int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} r^{-g}\left|\psi(z,v)\right| dz &= \int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} \sqrt{-r_z}\left|\psi(z,v)\right| \cdot \frac{r^{-g}}{\sqrt{-r_z}}(z,v)dz \\
&\leq \left(\int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} -r_z\psi^2(z,v)dz\right)^{\frac{1}{2}} \left(\int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} \frac{r^{-2g}}{-r_z}(z,v)dz\right)^{\frac{1}{2}} \\
&\leq \left|\left|\psi\right|\right|_{\underline{L}^2}(u_{\mathcal{H}},v)\left(\int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} \frac{r^{-2g}}{-r_z}(z,v)dz\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.347}$$

Estimating the latter term

$$\int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} \frac{r^{-2g}}{-r_z}(z,v)dz = \int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} \frac{r^{-2g}}{r_z^2}\cdot -r_z(z,v)dz = \int_{r_{\mathcal{H}}}^{\infty} \frac{r^{-2g}}{r_z^2}dr \leq C\left[\frac{r^{-2g-3}}{-2g-3}\right]_{r_{\mathcal{H}}}^{\infty} < C_g r_{min}^{-2\kappa}. \tag{3.348}$$

(recalling the results of theorem 3.7.3). We thus conclude

$$\left|r^{-g}\psi(u,v)\right| \leq C_{g,M,l}\left(\left|\left|\psi\right|\right|_{\underline{H}^1}(u,v) + \left|\left|\psi\right|\right|_{\underline{L}^2}(u_{\mathcal{H}},v)\right). \tag{3.349}$$

where $C_{g,M,l}$ is a positive constant depending on the domain. The result then follows. □

Corollary 3.7.6. *For $\psi \in \underline{H}^1_d(\{r \geq r_Y\} \cap \mathcal{B})$ we have that*

$$\left|r^{\frac{3}{2}-\kappa}\psi\right|(u,v) \leq C_{g,M,l}\left(\left|\left|\psi\right|\right|_{\underline{H}^1_d}(u,v) + \left|\left|\psi\right|\right|_{\underline{L}^2}(u_{\mathcal{H}},v)\right). \tag{3.350}$$

Proof. This is just an application of lemma 3.7.8, 3.7.9 and 3.7.10. □

Red shift estimates in $\{r \leq r_X\}$

In this region we are bounded away from $\mathcal{I}$. As such we are not worried about divergent fluxes there, and we do not need to work within the twisted framework. Primarily we are concerned by the degenerative properties of r_v. To combat this we use a redshift argument from [HS13b] adapted to this setting.

Lemma 3.7.11. *In the region* $\{r \le r_X\} \cap \mathcal{B}$ *we have a constant* $C_{X,g} > 0$*, such that*

$$\frac{1}{C_{X,g}} ||\psi||_{\underline{H}^1_d}(u,v) \le ||\psi||_{H^1_d}(u,v) \le C_{X,g} ||\psi||_{\underline{H}^1_d}(u,v). \tag{3.351}$$

Proof. The first key estimate is that

$$(\tilde{\nabla}_u\psi)^2 \le 2(\nabla_u\psi)^2 + 2\frac{g^2r_u^2}{r^2}\psi^2 \tag{3.352}$$

Estimating

$$\begin{aligned}
\frac{r^2r_v}{\Omega^2}(\tilde{\nabla}_u\psi)^2 &\le 2\frac{r^2r_v}{\Omega^2}(\nabla_u\psi)^2 + 2g^2\frac{r_u^2r_v}{\Omega^2}\psi^2 \\
&\le 2\frac{r^2r_v}{\Omega^2}(\nabla_u\psi)^2 - 2g^2r_u\left(\frac{r^2}{l^2} - \frac{2\varpi}{r}\right)e^{-4\pi g\psi^2}\psi^2 \\
&\le 2\frac{r^2r_v}{\Omega^2}(\nabla_u\psi)^2 - 2eg^2r_u\frac{r^2}{l^2}\psi^2 \\
&\le 2\frac{r^2r_v}{\Omega^2}(\nabla_u\psi)^2 - 2e\frac{g^2}{l^2}r_X^3\frac{-r_u}{r}\psi^2 \\
&\le C_{X,g}\left(\frac{r^2r_v}{\Omega^2}(\nabla_u\psi)^2 + \frac{-r_u}{r}\psi^2\right).
\end{aligned} \tag{3.353}$$

Similarly

$$(\nabla_u\psi)^2 \le 2(\tilde{\nabla}_u\psi)^2 + 2\frac{g^2r_u^2}{r^2}\psi^2. \tag{3.354}$$

We simply repeat the previous argument. □

Lemma 3.7.12. *We have in the region* $\{r \le r_X\} \cap \mathcal{B}$

$$\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right| \le C_{X,g}\left(\left|\frac{r^{\frac{5}{2}}}{-r_u}\tilde{\nabla}_u\psi\right| + \left|r^{\frac{3}{2}-\kappa}\psi\right|\right). \tag{3.355}$$

Proof.

$$\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right| = \left|\frac{r^{\frac{5}{2}}}{r_u}\left(\tilde{\nabla}_u\psi + \frac{gr_u}{r}\psi\right)\right| \le C_{X,g}\left(\left|\frac{r^{\frac{3}{2}}}{-r_u}\tilde{\nabla}_u\psi\right| + \left|r^{\frac{5}{2}-\kappa}\psi\right|\right). \tag{3.356}$$

□

Lemma 3.7.13 (Basic Red Shift Estimate)**.** *We have that in the region* $\{r \le r_X\} \cap \mathcal{B}$

$$\left|\frac{r\psi_u}{r_u}(u,v)\right| \le C_{a,l,X}\left[\sup_{D(u,v)} ||\psi||_{\underline{H}^1_d} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right| + \sup_{I(v_0)}\left|r^{\frac{3}{2}-\kappa}\psi\right|\right] + |a|\,|\psi|\,(u,v). \tag{3.357}$$

Proof. We adapt the proof of [HS13b] and [DR05] to this setting. We express the Klein-Gordon equation in the following form

$$\partial_v \left(\frac{r\psi_u}{r_u}(u,v) \right) = -\psi_v + \frac{2ar\chi\psi}{l^2} - \frac{r\psi_u}{r_u}\rho, \tag{3.358}$$

where

$$\rho := 2\chi \left[\frac{\varpi_1}{r^2} + \frac{r}{l^2} - 8\pi r \frac{a}{l^2}\psi \right]. \tag{3.359}$$

Now using lemma 3.7.3 we see that

$$\frac{\varpi_1}{r^2} + \frac{r}{l^2} \geq \frac{3}{4l^2} r, \tag{3.360}$$

we see that for b sufficiently small enough that

$$\frac{\rho}{\chi} > \frac{3r_{\min}}{2l^2} > 0. \tag{3.361}$$

Integrating (3.358) using the Duhamel formula gives

$$\begin{aligned} \frac{r\psi_u}{r_u}(u,v) = &\left(\frac{r\psi_u}{r_u}(u,v_0) \right) \cdot \exp\left(\int_{v_0}^{v} -\rho(u,\bar{v})d\bar{v} \right) \\ &+ \int_{v_0}^{v} \left[\exp\left(-\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v} \right) \left(-\psi_v + \frac{2r\chi a\psi}{l^2}(u,\bar{v}) \right) \right] d\bar{v}. \end{aligned} \tag{3.362}$$

The first term is bounded by initial data, so we concern ourselves with the latter inhomogeneous term. We start with the ψ_v term

$$\begin{aligned} &\int_{v_0}^{v} \exp\left(-\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v} \right) \psi_v \\ \leq &\left(\int_{v_0}^{v} \frac{\chi}{r^2} \cdot \exp\left(-2\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v} \right) \right)^{\frac{1}{2}} \left(\int_{v_0}^{v} \frac{\psi_v^2}{\chi} r^2(u,\bar{v})d\bar{v} \right)^{\frac{1}{2}}. \end{aligned} \tag{3.363}$$

The latter integral can be clearly controlled by the H^1_d energy. Turning to the other integral we may rewrite the integrand as

$$\int_{v_0}^{v} \frac{\chi}{r^2} \cdot -\exp\left(-2\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v} \right) = \int_{v_0}^{v} \frac{\chi}{2\rho r^2} \cdot \partial_{\bar{v}} \exp\left(-2\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v} \right) d\bar{v}. \tag{3.364}$$

We note the bound

$$\left| \frac{\chi}{2\rho r^2} \right| < \frac{l^2}{4r_{min}r_X^2}, \tag{3.365}$$

and estimate (3.364) by

$$\begin{aligned}\int_{v_0}^{v} \frac{\chi}{r^2} \cdot \exp\left(-2\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v}\right) dv &< \frac{l^2}{4r_{min}r_X^2} \int_{v_0}^{v} \left|\partial_{\bar{v}} \exp\left(-2\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v}\right)\right| d\bar{v} \\ &= \frac{l^2}{4r_{min}r_X^2} \int_{v_0}^{v} \partial_{\bar{v}} \exp\left(-2\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v}\right) d\bar{v} \\ &= \frac{l^2}{4r_{min}r_X^2} \left(1-\exp\left(-2\int_{v_0}^{v} \rho(u,\hat{v})d\hat{v}\right)\right) \\ &\leq \frac{l^2}{4r_{min}r_X^2}.\end{aligned} \tag{3.366}$$

We note that evaluating the derivative $\partial_{\bar{v}} \exp\left(-2\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v}\right)$, shows that the quantity has a positive sign. We now study the term

$$\int_{v_0}^{v} \exp\left(-\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v}\right) \frac{2r\chi a\psi}{l^2} dv, \tag{3.367}$$

which we rewrite as

$$\int_{v_0}^{v} \partial_{\bar{v}} \exp\left(-\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v}\right) \cdot \frac{2r\chi a\psi}{\rho l^2} d\bar{v}. \tag{3.368}$$

Integrating this by parts, and studying the surface terms we see that

$$\begin{aligned}\left[\exp\left(-\int_{\bar{v}}^{v} \rho(u,\hat{v})d\hat{v}\right) \cdot \frac{2r\chi a\psi}{\rho l^2}\right]_{v_0}^{v} &= \frac{2r\chi a\psi}{\rho l^2}|_v - \exp\left(-\int_{v_0}^{v} \rho(u,\hat{v})d\hat{v}\right) \cdot \frac{2r\chi a\psi}{\rho l^2}|_{v_0} \\ &\leq \psi\frac{ar}{l^2} \cdot \frac{1}{\frac{\varpi_1}{r^2}+\frac{r}{l^2}-4\pi r\frac{a}{l^2}\psi^2} + \left|\psi\frac{ar}{l^2} \cdot \frac{1}{\frac{\varpi_1}{r^2}+\frac{r}{l^2}-4\pi r\frac{a}{l^2}\psi^2}|_{v_0}\right| \\ &\leq |a\psi| + |a\psi|\,|_{v_0} \\ &\leq |a\psi| + \sup_{N(v_0)} C_{a,X}\left|r^{\frac{3}{2}-\kappa}\psi\right|.\end{aligned} \tag{3.369}$$

Where we note we have have invoked lemma (3.7.3) to control the $\frac{1}{\frac{\varpi_1}{r^2}+\frac{r}{l^2}-4\pi r\frac{a}{l^2}\psi^2}$ term.
Turning to the bulk terms, first we compute

$$\partial_v\left(\psi\frac{2r\chi}{\rho}\right) = \frac{2r\chi}{\rho}\left(1+\frac{2r\chi}{\rho}\frac{8\pi a}{l^2}\psi^2\right)\psi_v + \left(\frac{2r\chi}{\rho}\right)^2\left(\frac{3\varpi_1}{r^4}-\frac{4\pi a}{rl^2}\psi^2\right)\psi r_v - \left(\frac{2r\chi}{\rho}\right)^2 2\pi\frac{\psi}{r}\frac{\psi_v^2}{\chi}. \tag{3.370}$$

Noting that

$$\frac{2r\chi}{\rho}, \psi, \varpi_1 \tag{3.371}$$

are bounded above and below on this domain. We clean up (3.370) to an expression of the form

$$\partial_v\left(\psi\frac{2r\chi}{\rho}\right) \leq C_1|\psi_v| + C_2|\psi|\,r_v + C_3\left|\frac{\psi}{r}\frac{\psi_v^2}{\chi}\right|. \tag{3.372}$$

The C_1 term can be dealt with as in (3.363). We look at the C_2 term, firstly recalling

$$\mu_1 = \frac{-2\varpi_1}{r} + \frac{r^2}{l^2} = \frac{-4r_u r_v}{\Omega^2}, \tag{3.373}$$

so

$$\mu_1 \chi = r_v. \tag{3.374}$$

From lemma 3.7.3 we have that μ_1 is bounded on this domain. We estimate term by term

$$\begin{aligned} &\int_{v_0}^{v} d\bar{v} \left[\exp\left(-\int_{\bar{v}}^{v} \rho(u,\hat{v}) d\hat{v} \right) |\psi| \, r_v \right] \\ &\leq \left(\int_{v_0}^{v} \frac{\psi^2}{r} r_v \right)^{\frac{1}{2}} \cdot \left(\int_{v_0}^{v} r\chi\mu_1 \cdot \exp\left(-2\int_{v_0}^{v} \rho(u,\hat{v}) d\hat{v} \right) \right)^{\frac{1}{2}}. \end{aligned} \tag{3.375}$$

As r, and μ_1, are bounded, the latter term can be estimated by

$$\sup \left(\mu_1 \frac{r\chi}{\rho} \right) \int_{v_0}^{v} \partial_{\bar{v}} \exp\left(-2\int_{v_0}^{v} \rho(u,\hat{v}) d\hat{v} \right) d\bar{v}, \tag{3.376}$$

thus this term can be controlled by $||\psi||_{\underline{H}^1_d}$.
For the C_3 term we need only apply the pointwise bound on ψ to get $||\psi||_{\underline{H}^1_d}$ control (after bounding the r weights).

□

Theorem 3.7.4 (Red Shift Estimate)**.** *In the region $\{r \leq r_X\} \cap \mathcal{B}$ we have a constant $C_{g,l,X} > 0$ such that*

$$\left| r^{\frac{3}{2}} \frac{r\psi_u}{r_u}(u,v) \right| \leq C_{g,l,X} \left[\sup_{D(u,v)} ||\psi||_{\underline{H}^1_d} + \sup_{I(v_0)} \left| r^{\frac{3}{2}} \frac{r\psi_u}{r_u} \right| + \sup_{I(v_0)} \left| r^{\frac{3}{2}-\kappa}\psi \right| \right], \tag{3.377}$$

furthermore we also have

$$\left| r^{\frac{3}{2}-\kappa}\psi \right|(u,v) \leq C_{g,l,X} \left[\sup_{D(u,v)} ||\psi||_{\underline{H}^1_d} + \sup_{I(v_0)} \left| r^{\frac{3}{2}} \frac{r\psi_u}{r_u} \right| + \sup_{I(v_0)} \left| r^{\frac{3}{2}-\kappa}\psi \right| \right]. \tag{3.378}$$

Proof. We want to drop the ψ term on the right hand side of lemma 3.7.13, this is done by integrating from the r_X curve toward the horizon in u.

$$|\psi|(u,v) \leq |\psi(u_{r_X}, v)| + \int_{u_{r_X}}^{u} \left| \frac{r\psi_u}{r_u} \right| \frac{-r_u}{r} du. \tag{3.379}$$

For clarity we quickly remark that from (3.255) we can construct the estimate

$$\int_{u_{r_X}}^{u} \frac{-r_u}{r} du = \ln\left(\frac{r_X}{r(u,v)} \right) \leq \ln\left(\frac{r_X}{r_{min}} \right) < \frac{1}{2|a|}. \tag{3.380}$$

Inserting (3.380) into lemma 3.7.13, we see

$$|\psi|(u,v) \leq |\psi(u_{r_X},v)| + C_{a,l,X}\left[\sup_{D(u,v)}||\psi||_{\underline{H}^1_d} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right| + \sup_{I(v_0)}\left|r^{\frac{3}{2}-\kappa}\psi\right|\right] + \frac{1}{2}|\psi|(u,v). \tag{3.381}$$

Absorbing the ψ term on the LHS gives

$$|\psi|(u,v) \leq |\psi(u_{r_X},v)| + C_{a,l,X}\left[\sup_{D(u,v)}||\psi||_{\underline{H}^1_d} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right| + \sup_{I(v_0)}\left|r^{\frac{3}{2}-\kappa}\psi\right|\right]. \tag{3.382}$$

Which in this region implies

$$\left|r^{\frac{3}{2}-\kappa}\psi\right|(u,v) \leq \left|r^{\frac{3}{2}-\kappa}\psi(u_{r_X},v)\right| + C_{a,l,X}\left[\sup_{D(u,v)}||\psi||_{\underline{H}^1_d} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right| + \sup_{I(v_0)}\left|r^{\frac{3}{2}-\kappa}\psi\right|\right]. \tag{3.383}$$

Recalling that $\{r \leq r_X\} \subset \{r \leq r_Y\}$, and invoking corollary 3.7.6

$$\left|r^{\frac{3}{2}-\kappa}\psi\right|(u,v) \leq C_{a,l,X}\left[\sup_{D(u,v)}||\psi||_{\underline{H}^1_d} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right| + \sup_{I(v_0)}\left|r^{\frac{3}{2}-\kappa}\psi\right|\right], \tag{3.384}$$

showing (3.378). (3.377) then follows. □

Corollary 3.7.7. *In the region $\{r \leq r_X\} \cap \mathcal{B}$ we have a constant $C_{a,l,X} > 0$, such that*

$$\left|\frac{r^{\frac{5}{2}}}{r_u}\tilde{\nabla}_u\psi\right| \leq C_{a,l,X}\left[\sup_{D(u,v)}||\psi||_{\underline{H}^1_d} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right| + \sup_{I(v_0)}\left|r^{\frac{3}{2}-\kappa}\psi\right|\right]. \tag{3.385}$$

Proof. We trivially estimate by

$$\begin{aligned}\frac{r^{\frac{3}{2}}}{-r_u}\tilde{\nabla}_u\psi &= \frac{r^{\frac{3}{2}}}{-r_u}\nabla_u\psi + gr^{\frac{1}{2}}\psi\\ \left|\frac{r^{\frac{3}{2}}}{-r_u}\tilde{\nabla}_u\psi\right| &\leq C_X\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right| + \left|gr^{\frac{3}{2}}\psi\right|.\end{aligned} \tag{3.386}$$

Whence the result follows. □

Estimates in the whole bootstrap region

Lemma 3.7.14 (Energy estimate)**.** *We have in $\mathcal{B}$*

$$||\psi||_{\underline{H}_1^1}(u,v) \leq C\left[\sup_{D(u,v)} ||\psi||_{\underline{H}_d^1} + \sup_{I(v_0)}\left(\left|\frac{r^{\frac{5}{2}}}{-r_u}\tilde{\nabla}_u\psi\right| + \left|r^{\frac{3}{2}-\kappa}\psi\right|\right)\right]. \tag{3.387}$$

Proof. Firstly let us note

$$\begin{aligned}\int_{u_{\mathcal{I}}}^{u} \frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + \frac{(-r_u)}{r}\psi^2 du = &\int_{u_{\mathcal{I}}}^{u} \mathbb{1}_{\{r\geq r_X\}}\left(\frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + \frac{(-r_u)}{r}\psi^2\right)du \\ &+ \int_{u_{\mathcal{I}}}^{u} \mathbb{1}_{\{r\leq r_X\}}\left(\frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + \frac{(-r_u)}{r}\psi^2\right)du.\end{aligned} \tag{3.388}$$

Now from lemma 3.7.8 we have

$$\int_{u_{\mathcal{I}}}^{u} \mathbb{1}_{\{r\geq r_X\}}\left(\frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + \frac{(-r_u)}{r}\psi^2\right)du \leq C_{l,Y}\,||\psi||_{\underline{H}_d^1}(u,v), \tag{3.389}$$

and from lemma 3.7.11

$$\int_{u_{\mathcal{I}}}^{u} \mathbb{1}_{\{r\leq r_X\}}\left(\frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + \frac{(-r_u)}{r}\psi^2\right)du \leq C_{X,g}\int_{u_{\mathcal{I}}}^{u} \mathbb{1}_{\{r\leq r_X\}}\left(\frac{r^4}{-r_u}(\nabla_u\psi)^2 + \frac{(-r_u)}{r}\psi^2\right)du. \tag{3.390}$$

Using the results of theorem 3.7.4 we see

$$\begin{aligned}\int_{u_{\mathcal{I}}}^{u} \mathbb{1}_{\{r\leq r_X\}}\frac{r^4}{-r_u}(\nabla_u\psi)^2 du &= \int_{u_{\mathcal{I}}}^{u} \mathbb{1}_{\{r\leq r_X\}}\left(r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right)^2\frac{-r_u}{r}du \\ &\leq \sup_{r\leq r_X}\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right|^2 \ln\left(\frac{r_X}{r_{min}}\right) \\ &\leq C_{g,l,X}\left[\sup_{D(u,v)}||\psi||_{\underline{H}_d^1} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right| + \sup_{I(v_0)}\left|r^{\frac{3}{2}-\kappa}\psi\right|\right].\end{aligned} \tag{3.391}$$

An application of corollary 3.7.7 gives

$$\left|r^{\frac{3}{2}}\frac{r\psi_u}{r_u}\right| = \left|\frac{r^{\frac{5}{2}}}{r_u}\left(\tilde{\nabla}_u\psi + \frac{gr_u}{r}\psi\right)\right| \leq C_g\left(\left|\frac{r^{\frac{5}{2}}}{-r_u}\tilde{\nabla}_u\psi\right| + \left|r^{\frac{3}{2}-\kappa}\psi\right|\right), \tag{3.392}$$

and the result follows. □

Corollary 3.7.8. *We have in the bootstrap region*

$$||\psi||_{\underline{H}^1}(u,v) \leq C_{g,M,\kappa}\left(||\psi||_{\underline{H}^1}(u_{\mathcal{H}},v_0) + \sup_{I(v_0)}\left(\left|r^{\frac{1}{2}}\tilde{\nabla}_u\psi\right| + \left|r^{\frac{3}{2}-\kappa}\psi\right|\right) + b^4\right). \tag{3.393}$$

Corollary 3.7.9. *In the region $\mathcal{B}$ we have that for b sufficiently small*

$$\left|r^{\frac{3}{2}-\kappa}\psi\right| < b^{\frac{3}{2}}. \tag{3.394}$$

Proof. From lemma 3.7.10 we have that

$$\begin{aligned}\left|r^{\frac{3}{2}-\kappa}\psi\right|(u,v) &\leq C_{g,M,l}\left(||\psi||_{\underline{H}^1}(u,v)+||\psi||_{\underline{L}^2}(u_{\mathcal{H}},v)\right)\\ &\leq C_{g,M,l}\left(||\psi||_{\underline{H}^1}(u_{\mathcal{H}},v_0)+\sup_{I(v_0)}\left(\left|\frac{r^{\frac{5}{2}}}{-r_u}\tilde{\nabla}_u\psi\right|+\left|r^{\frac{3}{2}-\kappa}\psi\right|\right)+b^4\right),\end{aligned} \tag{3.395}$$

so from the smallness of the initial data we conclude that

$$\left|r^{\frac{3}{2}-\kappa}\psi\right| < C_{g,M,\kappa}b^2 < b^{\frac{3}{2}}. \tag{3.396}$$

□

Theorem 3.7.5. *We have that $\mathcal{B}=\mathscr{R}_{\mathcal{H}}$.*

Proof. We know that $\mathcal{B}$ is an open non empty subset of $\mathscr{R}_{\mathcal{H}}$. Now fix a point $(u^*,v^*)\in\mathcal{B}$, and take a sequence $(u_n,v_n)\to(u^*,v^*)$, as $n\to\infty$, from the continuity of r and ψ we must have that

$$\left|r^{\frac{3}{2}-\kappa}\psi\right|(u^*,v^*)\leq b^{\frac{3}{2}}<b. \tag{3.397}$$

So we conclude that $(u^*,v^*)\in\mathcal{B}$. $\mathcal{B}$ is then closed, and hence $\mathcal{B}=\mathscr{R}_{\mathcal{H}}$. □

Remark 3.7.3. *It follows from theorem 3.7.5 that*

$$\begin{aligned}&|\varpi(u,v)-M|^{\frac{1}{2}}+\left|r^{\frac{3}{2}-\kappa}\psi(u,v)\right|+||\psi||_{\underline{H}^1}(u,v)\\ &\qquad\leq C_{l,M,g}\left(||\psi||_{\underline{H}^1}(u_{\mathcal{H}},v_0)+\sup_{I(v_0)}\left(\left|\frac{r^{\frac{5}{2}}}{-r_u}\tilde{\nabla}_u\psi\right|+\left|r^{\frac{3}{2}-\kappa}\psi\right|\right)\right).\end{aligned} \tag{3.398}$$

holds in $\mathscr{R}_{\mathcal{H}}$.

Consequence of the bootstrap estimates

Metric function estimates

Lemma 3.7.15. *We have in the regular region the estimate*

$$|2\chi-1|^{\frac{1}{2}}\leq C_{g,M,\kappa}\left(||\psi||_{\underline{H}^1}(u_{\mathcal{H}},v_0)+\sup_{I(v_0)}\left(\left|\frac{r^{\frac{5}{2}}}{-r_u}\tilde{\nabla}_u\psi\right|+\left|r^{\frac{3}{2}-\kappa}\psi\right|\right)\right). \tag{3.399}$$

Proof. Recall that $\chi=\frac{\Omega^2}{-4r_u}$ satisfies the equation

$$\partial_u(\ln\chi)=\frac{4\pi r}{r_u}(\partial_u\psi)^2. \tag{3.400}$$

Integrating this equation gives

$$\chi=\chi|_{\mathcal{I}}\exp\left(\int_{u_{\mathcal{I}}}^{u}-\frac{4\pi r}{-r_u}(\partial_u\psi)^2\,du\right). \tag{3.401}$$

Using a Young estimate in one direction

$$\frac{r}{-r_u}\psi_u^2 \leq 2\frac{r}{-r_u}\left(\tilde{\nabla}_u\psi\right)^2 + \frac{2g^2(-r_u)}{r}\psi^2 \leq C_g\left(\frac{r^4}{-r_u}\left(\tilde{\nabla}_u\psi\right)^2 + \frac{-r_u}{r}\psi^2\right), \tag{3.402}$$

and the negativity of the integrand in the other, we have

$$\frac{1}{2}\exp\left(-C_g\,||\psi||^2_{\underline{H}^1}(u,v))\right) \leq \chi \leq \frac{1}{2}. \tag{3.403}$$

The result now follows from the energy estimate (3.398). □

Corollary 3.7.10. *There exists a constant $C_g > 0$ such that in the regular region*

$$-2\left(1 + C_g b^2\right) r_u \leq \Omega^2 \leq -2r_u. \tag{3.404}$$

Corollary 3.7.11. *In the set $\{r \geq r_Y\} \cap \mathscr{R}_{\mathcal{H}}$ we have from lemma 3.7.6 that*

$$\Omega^2 \leq C_Y r^2. \tag{3.405}$$

Lemma 3.7.16. *We have that*

$$\tilde{r}_v|_{\mathcal{I}} = -\frac{1}{2l^2}. \tag{3.406}$$

Proof. Recall that from the definition of the Hawking mass

$$r_v = \frac{\Omega^2}{-4r_u}\left(\frac{r^2}{l^2} - \frac{2\varpi}{r}\right)e^{-4\pi g\psi^2}. \tag{3.407}$$

Which implies

$$\tilde{r}_v = -\chi\left(\frac{1}{l^2} - \frac{2\varpi}{r^3}\right)e^{-4\pi g\psi^2}. \tag{3.408}$$

Taking the limit $r \to \infty$, we have that

$$\tilde{r}_v|_{\mathcal{I}} = -\frac{1}{2l^2}. \tag{3.409}$$

□

Lemma 3.7.17. *In $\mathscr{R}_{\mathcal{H}}$*

$$r_v \leq \frac{1}{2l^2}\left(1 + C_{g,l}b^2\right)r^2. \tag{3.410}$$

Proof. Firstly it is possible to rewrite the $\tilde{r}_{uv}$ equation (3.22) as

$$\tilde{r}_{uv} = \varpi\Omega^2 e^{-4\pi g\psi^2}\frac{3}{4r^4} - \frac{1}{rl^2}\Omega^2\left(\frac{3}{4}e^{-4\pi g\psi^2} - \frac{3}{4} + 2\pi a\psi^2\right). \tag{3.411}$$

For ψ^2 small

$$0 \leq \frac{3}{4}e^{-4\pi g\psi^2} - \frac{3}{4} + 2\pi a\psi^2 \leq \pi g^2\psi^2. \tag{3.412}$$

So in (3.411) we drop the positive terms, and bound by

$$\begin{aligned}\tilde{r}_{uv} &\geq -\frac{1}{rl^2}\Omega^2\left(\frac{3}{4}e^{-4\pi g\psi^2}-\frac{3}{4}+2\pi a\psi^2\right)\\ &\geq -C_{g,l}b^2r^{-4+2\kappa}(-r_u).\end{aligned} \tag{3.413}$$

We then integrate this inequality, and use lemma 3.7.16 to see

$$\tilde{r}_v \geq -\frac{1}{2l^2}\left(1+C_{g,l}b^2\right). \tag{3.414}$$

The result then follows. □

Lemma 3.7.18. *In $\mathscr{R}_{\mathcal{H}}$*

$$r_v \leq \frac{1}{2}\left(1+Cb^2\right)\Omega^2. \tag{3.415}$$

Proof. Integrating (3.9) shows

$$\begin{aligned}\frac{r_v}{\Omega^2}(u,v) \leq \frac{r_v}{\Omega^2}(u,v_0) &= \frac{1}{4}\left(\frac{r^2}{l^2}-\frac{2\varpi}{r}\right)\frac{e^{-4\pi g\psi^2}}{-r_u}(u,v_0)\\ &= \frac{1}{2}\left(1-\frac{l^2\varpi}{r^3}\right)e^{-4\pi g\psi^2}(u,v_0) \leq \frac{1}{2}\left(1+Cb^2\right).\end{aligned} \tag{3.416}$$

From here the result follows. □

We now collect these estimates in the convenient corollary:

Corollary 3.7.12 (Global Estimates)**.** *There exists constants $C_i > 0$ depending on g, M, l such that in $\mathscr{R}_{\mathcal{H}}$*

$$r_v \leq C_1 r^2 \leq -C_2 r_u \leq C_3\Omega^2 \leq -C_4 r_u. \tag{3.417}$$

Lemma 3.7.19. *In the region $\{r \geq r_Y\} \cap \mathscr{R}_{\mathcal{H}}$ we have*

$$\Omega^2 \leq C_{Y,l} r_v. \tag{3.418}$$

Proof.

$$\Omega^2 = \frac{-4r_u r_v}{-\frac{2\varpi}{r}+\frac{r^2}{l^2}}e^{4\pi g\psi^2} \leq C_{Y,l} r_v. \tag{3.419}$$

□

Corollary 3.7.13. *In the region $\{r \geq r_Y\} \cap \mathscr{R}_{\mathcal{H}}$ we have*

$$r^2 \leq C_{Y,l} r_v. \tag{3.420}$$

Corollary 3.7.14 (Stronger estimates away from the degeneration)**.** *In the region $\mathscr{R}_{\mathcal{H}} \cap \{r \geq r_Y\}$, there exists constants $C_i > 0$, depending on Y, g, M, l such that*

$$r^2 \leq C_1 r_v \leq -C_2 r_u \leq C_3\Omega^2 \leq C_4 r^2. \tag{3.421}$$

Lemma 3.7.20. *In $\mathscr{R}_{\mathcal{H}}$ we have*

$$\mu_1 = e^{-4\pi g\psi^2}\mu. \tag{3.422}$$

Proof. This follows from the observation

$$\varpi_1 = \varpi e^{-4\pi g\psi^2} - \frac{r^3}{2l^2}\left(e^{-4\pi g\psi^2} - 1\right). \tag{3.423}$$

□

Corollary 3.7.15. *In $\mathscr{R}_{\mathcal{H}}$ we have*

$$e\mu \geq \mu_1 \geq \mu. \tag{3.424}$$

Pointwise $u-$derivative decay

To complete our estimates we need to control $\tilde{\nabla}_u\psi$ in the region containing $\mathcal{I}$. We proceed by integrating the Klein-Gordon equation in this region.

Lemma 3.7.21. *We have in $\{r \geq r_Y\} \cap \mathscr{R}$*

$$\left|r^{\frac{1}{2}}\tilde{\nabla}_u\psi\right| \leq C_{g,M,\kappa,Y}\left(||\psi||_{\underline{H}^1}(u,v) + \sup_{I(v_0)}\left(\left|r^{\frac{1}{2}}\tilde{\nabla}_u\psi\right|\right)\right). \tag{3.425}$$

Proof. Recall equation (3.23)

$$\partial_v\left(r\tilde{\nabla}_u\psi\right) = -r_u\left(-\frac{1}{2} + \kappa\right)\tilde{\nabla}_v\psi - \frac{\Omega^2}{4}rV\psi, \tag{3.426}$$

where

$$V = \frac{2g^2}{r^3}\varpi_1 + \frac{8\pi ag}{l^2}\psi^2. \tag{3.427}$$

Using the results of the bootstrap arguments, and the metric function estimates we can easily see

Corollary 3.7.16. *In $\{r \geq r_Y\} \cap \mathscr{R}$*

$$\left|V - \frac{2g^2M}{r^3}\right| \leq C_{M,g,l}b^2r^{-3+2\kappa}. \tag{3.428}$$

From which the following estimate follows

$$|V| \leq C_{M,l,g}r^{-3+2\kappa}. \tag{3.429}$$

Integrate equation (3.23) to get

$$\left|r\tilde{\nabla}_u\psi(u,v)\right| \leq \left|r\tilde{\nabla}_u\psi(u,v_0)\right| + \left|\int_{v_0}^{v} -r_u\left(-\frac{1}{2}+\kappa\right)\tilde{\nabla}_v\psi dv\right| + \left|\int_{v_0}^{v} -\frac{\Omega^2}{4}rV\psi dv\right|. \tag{3.430}$$

Estimating term by term, working from left to right

$$\left|r\tilde{\nabla}_u\psi(u,v_0)\right| = \left|r^{\frac{1}{2}}\cdot r^{\frac{1}{2}}\tilde{\nabla}_u\psi(u,v_0)\right| \leq \sup_{I(v_0)}\left|r^{\frac{1}{2}}\tilde{\nabla}_u\psi(u,v)\right|\cdot C\cdot r^{\frac{1}{2}}(u,v_0). \tag{3.431}$$

As we are in the regular region we have that $r(u,v)\geq r(u,v_0)$. Thus

$$\left|r\tilde{\nabla}_u\psi(u,v_0)\right| \leq C\sup_{I(v_0)}\left|r^{\frac{1}{2}}\tilde{\nabla}_u\psi(u,v)\right|\cdot r^{\frac{1}{2}}(u,v). \tag{3.432}$$

For the next term

$$\begin{aligned}\int_{v_0}^{v} -r_u\left(-\frac{1}{2}+\kappa\right)\tilde{\nabla}_v\psi &= \int_{v_0}^{v} -r_u\cdot\left(\frac{r}{\Omega}\tilde{\nabla}_v\psi\right)\left(\left(-\frac{1}{2}+\kappa\right)\frac{\Omega}{r}\right)dv\\ &\leq C_\kappa\left(\int_{v_0}^{v}\frac{-r_ur^2}{\Omega^2}\left(\tilde{\nabla}_v\psi\right)^2dv\right)^{\frac{1}{2}}\left(\int_{v_0}^{v}-r_u\frac{\Omega^2}{r^2}dv\right)^{\frac{1}{2}}\\ &\leq C_\kappa\left|\left|\psi\right|\right|_{\underline{H}^1_d}(u,v)\left(\int_{v_0}^{v}-r_u\frac{\Omega^2}{r^2}dv\right)^{\frac{1}{2}}.\end{aligned} \tag{3.433}$$

From theorem 3.7.3

$$\begin{aligned}\int_{v_0}^{v}-r_u\frac{\Omega^2}{r^2}dv &\leq C_{M,l,g}\int_{v_0}^{v}r_vdv\\ &\leq C_{M,l,g}r.\end{aligned} \tag{3.434}$$

So we have

$$\left|\int_{v_0}^{v}-r_u\left(-\frac{1}{2}+\kappa\right)\tilde{\nabla}_v\psi dv\right| \leq C_{M,l,g}r^{\frac{1}{2}}\left|\left|\psi\right|\right|_{\underline{H}^1_d}(u,v). \tag{3.435}$$

For the final term we compute

$$\begin{aligned}\int_{v_0}^{v}-\frac{\Omega^2}{4}rV\psi dv &\leq C_{M,l,g}\int_{v_0}^{v}r_vr^{-2+2\kappa}\psi dv\\ &\leq C_{M,l,g}\left(\int_{v_0}^{v}\frac{r_v}{r}\psi^2dv\right)^{\frac{1}{2}}\left(\int_{v_0}^{v}r_vr^{-3+4\kappa}dv\right)^{\frac{1}{2}}\\ &\leq C_{M,l,g}\left|\left|\psi\right|\right|_{\underline{H}^1_d}(u,v)\left(\int_{v_0}^{v}r^{-3+4\kappa}r_vdv\right)^{\frac{1}{2}}\\ &\leq C_{M,l,g}r^{-1+2\kappa}\left|\left|\psi\right|\right|_{\underline{H}^1_d}(u,v)\\ &\leq C_{M,l,g}r^{\frac{1}{2}}\left|\left|\psi\right|\right|_{\underline{H}^1_d}(u,v).\end{aligned} \tag{3.436}$$

Combining all these estimates we have

$$\left|r\tilde{\nabla}_u\psi(u,v)\right| \leq C_{M,l,g}r^{\frac{1}{2}}\left(\left|\left|\psi\right|\right|_{\underline{H}^1_d}(u,v)+\sup_{I(v_0)}\left|\tilde{\nabla}_u\psi(u,v)\right|\right), \tag{3.437}$$

from which we deduce the result. □

Corollary 3.7.17. *Using theorem 3.7.3 we see this estimate is equivalent to*

$$\left|\frac{r^{\frac{5}{2}}}{-r_u}\tilde{\nabla}_u\psi\right| \leq C_{g,M,l,Y}\left(||\psi||_{\underline{H}^1}(u,v) + \sup_{I(v_0)}\left(\left|\frac{r^{\frac{5}{2}}}{-r_u}\tilde{\nabla}_u\psi\right| + \left|r^{\frac{3}{2}-\kappa}\psi\right|\right)\right), \tag{3.438}$$

or alternatively

$$\left|\frac{r^{\frac{5}{2}-\kappa}}{-r_u}\psi_u\right| \leq C_{g,M,l,Y}\left(||\psi||_{\underline{H}^1}(u,v) + \sup_{I(v_0)}\left(\left|\frac{r^{\frac{5}{2}}}{-r_u}\tilde{\nabla}_u\psi\right| + \left|r^{\frac{3}{2}-\kappa}\psi\right|\right)\right). \tag{3.439}$$

Concluding the proof of theorem 3.7.1.

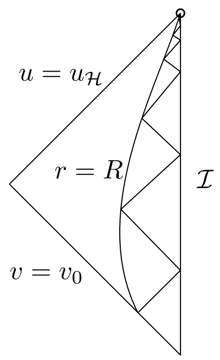
3.7.2 Completeness of null infinity

Proposition 3.7.1. *Let* $v_m = \sup_{v\geq v_0}\{v|(u_{\mathcal{H}},v)\in\mathcal{Q}\}$. *Then it is the case that* $v_m = \infty$.

Proof. This result is an adaptation of the proof in [HS13b] to this setting. Consider curves of constant r. In $\mathscr{R}_{\mathcal{H}}$ we have that these are timelike and foliate $\mathscr{R}_{\mathcal{H}}$. We now have two cases:

- None of the constant r curves have a future limit $(u_{\mathcal{H}}, v_m)$, (i.e. they all intersect the horizon).
- There is an R, such that $r = R$ has a future limit point $(u_{\mathcal{H}}, v_m)$. (And hence also true for all $r = R'$ with $R' > R$).

We deal with the latter case first.
Consider the infinite 'zig-zag' curve as depicted below:



Now we see that the v-length of each constant u-piece $\mathcal{U}_i$ is uniformly bounded below. This is done by checking the bounds on χ, and $\frac{\mu_1}{r^2} \leq \frac{2}{l^2}$ for large enough R. In this case we have

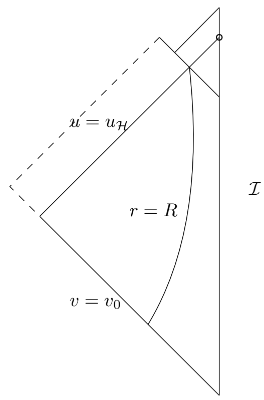
$$\frac{l^2}{2}\chi\frac{\mu_1}{r^2} \leq \frac{1}{2} < 1. \tag{3.440}$$

Recalling that $\chi\mu_1 = r_v$, we derive

$$\int_{\mathcal{U}_i} dv \geq \frac{l^2}{2}\int \chi\frac{\mu_1}{r^2}dv = \frac{l^2}{2}\int \frac{r_v}{r^2}dv \geq \frac{l^2}{2R}. \tag{3.441}$$

There are infinity many $\mathcal{U}_i$ in the zig-zag curve. (If there were a finite number then there must be some $N \in \mathbb{N}$ such that $\mathcal{U}_N$ is the ray $\gamma : (u_{\mathcal{H}}, v), v \in (v_N, v_m)$. This ray is bounded to right of $r = R$, so we must have that $r = R$ has become null, a contradiction). It follows that $v_m = \infty$.

We now deal with the first case, here we must have that $\lim_{v\to v_m} r(u_{\mathcal{H}}, v) = \infty$. We will assume that $v_m = V < \infty$, and contradict that $u = u_{\mathcal{H}}$ is the last u-ray in which $r = \infty$ can be reached. First pick $r = R$ very large, in view of the bounds on ϖ, ϖ_1, χ we have that $\mu_1 > c > 0$, and $\mu \geq e^{-1}c$ hold in $\overline{\mathscr{R}_{\mathcal{H}}}$. This is trivial in $\overline{\mathscr{R}_{\mathcal{H}}} \cap \{r \geq R\}$ by computation. For $\overline{\mathscr{R}_{\mathcal{H}}} \cap \{r \leq R\}$ we have its true by compactness, (since $r_v = 0$ cannot hold, as this would contradict that $r \to \infty$ along any $u = const$ ray in $\overline{\mathscr{R}_{\mathcal{H}}}$). Note that $\mu \geq e^{-1}\mu_1 = \frac{r_v}{\eta}e^{-1} > ce^{-1} > 0$. Thus showing (3.112) holds. By a change of u coordinate we can locally straighten out the boundary and achieve $u = v$ there. We now satisfy the conditions of the extension principle near infinity. We extend our spacetime to the depicted triangle



This contradicts the assumption that $u_{\mathcal{H}}$ is the last ray along which $r \to \infty$ can be reached.

□

3.8 Asymptotic stability

We have now established that we have a complete black hole spacetime, which is asymptotically AdS. We seek now to prove that the $\underline{H}^1$ norm of the field ψ is decaying exponentially in the v-coordinate to 0. From here we can see that across any $u = const$ slice, that metric is decaying uniformly to a toroidal AdS Schwarzschild solution of mass M. This is in contrast to what was seen the linear problem of the previous chapter. In that setting it was proven that polynomial decay of the field held, but exponential decay did not. The barrier to exponential decay was shown to be linked to null geodesics being far away from the horizon, and possessing non zero angular momentum. For solutions to our toroidally symmetric problem we now show that this is no longer the case. The symmetry restrictions no longer allow us to construct null geodesics with this property, and we will see the field decays exponentially. We establish asymptotic stability through Morawetz estimates. The core result we aim to prove in this section is

$$\int_D \frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + (-r_u)\psi^2 + \frac{-r_u r^2}{\Omega^2}(\tilde{\nabla}_v\psi)^2 + \frac{r_v}{r}\psi^2 dudv \leq C_{l,g,M}\mathbb{F}(u,v). \tag{3.442}$$

From this we can extract exponential decay of a key flux quantity, implying decay of the field. We follow the vector field method of the previous chapter. However due to the complexities from the non-linearities and the ungeometrical nature of twisting, we prefer to work with the standard energy-momentum tensor. We use this to prove a global but low weighted integrated decay estimate. The low weights ensure that the technicalities of infinite fluxes are not present, and twisting is not needed. We then optimise the weights by directly multiplying the Klein-Gordon equation in twisted form, as seen in the classical methods of Morawetz [Mor61].

It will be fruitful in this section to make and remind ourselves of the following definitions

$$\chi := \frac{\Omega^2}{-4r_u}, \quad \gamma := \frac{\Omega^2}{4r_v}. \tag{3.443}$$

In order to make the proof of this result more manageable we split it into three theorems

Theorem 3.8.1 (Low Weighted Degenerate Global Estimate). *In $\mathscr{R}_{\mathcal{H}}$, for $\kappa < \frac{1}{2}$, we have the following estimate*

$$\mathbb{I}_{deg}[\psi] := \int_D r^{-6}\left(\frac{1}{\gamma^2}\psi_u^2 + \frac{1}{\chi^2}\psi_v^2\right)\frac{\Omega^2 r^2}{2}dudv + \int_D \left(\frac{1-|a|}{r}\right)\psi^2\frac{\Omega^2}{2}r^2 dudv \leq C_{l,g,M}\mathbb{F}(u,v). \tag{3.444}$$

Remark 3.8.1. *While this estimate is low weighted it has the advantage of hold globally on the spacetime. It is insufficient to prove exponential decay of ψ, due the degeneration at the boundary of the regular region appearing in the γ factor, and that the powers of r are too low to control an integrated $\underline{H}^1$ norm. However it allow us to localise estimates to either a region near $\mathcal{I}$, or to a region $\{r \leq r_X\}$, where more specialised vector fields can be used. The proof of this theorem is inspired from §5.3 of [HS13b] but has been expanded into more detail (in particular towards the boundary terms and generalised to cover more choices of multiplier).*

Theorem 3.8.2 (Red Shift estimate). *In $\mathscr{R}_{\mathcal{H}}$ for $\kappa < \frac{1}{2}$, we have the following estimate*

$$\begin{aligned} &\int_D \frac{1}{r^7}\left(\frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + \frac{-r_u r^2}{\Omega^2}(\tilde{\nabla}_v\psi)^2\right)d\bar{u}d\bar{v} \\ &+ \int_D ((1-|a|)\,r)\,(-r_u)\psi^2 + ((1-|a|)\,r)\,r_v\psi^2 d\bar{u}d\bar{v} \leq C_{l,g,M}\mathbb{F}(u,v). \end{aligned} \tag{3.445}$$

Remark 3.8.2. *With the global estimate proven we use smoothed cut-off functions in order to remove the degeneration coming from the γ term. This is done using a redshift vector field localised to the region $\{r \leq r_X\}$. We may also convert back to using twisted derivatives. This estimate implies local energy decay, that is in any compact region we have the field is decaying exponentially but it is insufficient for a global decay statement.*

Theorem 3.8.3 (Morawetz Estimate)**.** *In $\mathscr{R}_{\mathcal{H}}$, for $\kappa < \frac{1}{2}$, we have the following estimate*

$$\int_D \frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + (-r_u)\psi^2 + \frac{-r_u r^2}{\Omega^2}(\tilde{\nabla}_v\psi)^2 + \frac{r_v}{r}\psi^2 dudv \leq C_{l,g,M}\mathbb{F}(u,v). \tag{3.446}$$

Remark 3.8.3. *To show this, we localise a vector field to a region near $\mathcal{I}$ where the spacetime is behaving like AdS. In this region the estimates of corollary 3.7.14 hold. With these estimates we can sharpen the r weights of theorem 3.8.2 to a global integrated decay estimate. Exponential decay of the fields follows from this estimate.*

3.8.1 Useful estimates and identities

Lemma 3.8.1 (Hardy estimate in v)**.** *In $\{r \geq r_Y\} \cap \mathscr{R}_{\mathcal{H}}$ the following estimate holds*

$$\int_{v_0}^{v} \psi^2 r_v dv \leq C_{Y,l,g}\mathbb{F}(u,v). \tag{3.447}$$

Proof.

$$\begin{aligned}
\int_{v_0}^{v} \psi^2 r_v dv &= \int_{v_0}^{v} \left(\psi r^{\frac{3}{2}-\kappa}\right)^2 \partial_v \left(\frac{r^{-2+2\kappa}}{-2+2\kappa}\right) dv \\
&= \left[r\psi^2 \frac{1}{-2+2\kappa}\right]_{v_0}^{v} + \frac{1}{2-2\kappa}\int_{v_0}^{v} r\psi\tilde{\nabla}_v\psi dv \\
&\leq C_g \left|\left|\psi\right|\right|_{\underline{H}^1}(u,v_0) + \int_{v_0}^{v} 2\left(\sqrt{r_v}\psi\right)\left(\frac{r}{\sqrt{r_v}\left(2-2\kappa\right)}\tilde{\nabla}_v\psi\right) dv \\
&\leq C_g \left|\left|\psi\right|\right|_{\underline{H}^1}(u,v_0) + \epsilon\int_{v_0}^{v} \psi^2 r_v dv + \frac{1}{\epsilon\left(4-4\kappa\right)^2}\int_{v_0}^{v} \frac{r^2}{r_v}\left(\tilde{\nabla}_v\psi\right)^2 dv,
\end{aligned} \tag{3.448}$$

choosing $\epsilon < 1$ and using corollary 3.7.14 completes the proof. □

Vector field identities

Let X be a vector field of the following form, $X^u(u,v)\partial_u + X^v(u,v)\partial_v$.
We define the deformation tensor as

$$2^X\pi^{\alpha\beta} = \nabla^\alpha X^\beta + \nabla^\beta X^\alpha = g^{\alpha\gamma}\partial_\gamma X^\beta + g^{\beta\delta}\partial_\delta X^\alpha + g^{\alpha\gamma}g^{\beta\delta}g_{\gamma\delta,\mu}X^\mu. \tag{3.449}$$

We'll usually suppress the X in the notation for convenience.
We compute the non zero components

$$\pi^{uu} = -\frac{2}{\Omega^2}\partial_v X^u, \quad \pi^{vv} = -\frac{2}{\Omega^2}\partial_u X^v, \tag{3.450}$$

$$\pi^{uv} = -\frac{1}{\Omega^2}\left(\partial_v X^v + \partial_u X^u\right) - \frac{2}{\Omega^2}\left(\frac{\Omega_u}{\Omega}X^u + \frac{\Omega_v}{\Omega}X^v\right), \tag{3.451}$$

$$\pi^{xx} = \frac{1}{r^3}\left(r_u X^u + r_v X^v\right), \quad \pi^{yy} = \frac{1}{r^3}\left(r_u X^u + r_v X^v\right). \tag{3.452}$$

Defining the energy momentum tensor

$$\mathbb{T}_{\mu\nu}[\psi] = \nabla_u \psi \nabla_\nu - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \psi \nabla^\sigma \psi - \frac{a}{l^2}\psi^2 g_{\mu\nu}, \tag{3.453}$$

The non zero components are

$$\mathbb{T}_{uu} = (\nabla_u \psi)^2, \quad \mathbb{T}_{vv} = (\nabla_v \psi)^2, \quad \mathbb{T}_{uv} = \frac{a\Omega^2}{2l^2}\psi^2, \tag{3.454}$$

$$\mathbb{T}_{xx} = \mathbb{T}_{yy} = \frac{2}{\Omega^2} r^2 \nabla_u \psi \nabla_v \psi - \frac{a}{l^2} r^2 \psi^2. \tag{3.455}$$

Computing the divergence of $\mathbb{T}$, we get the usual relation

$$\nabla^\mu \mathbb{T}_{\mu\nu}[\psi] = (\nabla_\nu \psi)\left(\Box_g \psi - \frac{2a}{l^2}\psi\right) = 0. \tag{3.456}$$

Now, defining the energy current

$$J^X_\mu[\psi] = \mathbb{T}^X_{\mu\nu} X^\nu, \tag{3.457}$$

and the associated bulk term

$$K^X[\psi] = \nabla^\mu J^X_\mu = \mathbb{T}_{\mu\nu}\pi^{\mu\nu}. \tag{3.458}$$

We can expand K^X as

$$\begin{aligned} K^X[\psi] =& -\frac{2}{\Omega^2}(\partial_v X^u)(\partial_u \psi)^2 - \frac{2}{\Omega^2}(\partial_u X^v)(\partial_v \psi)^2 \\ &+ (\partial_u \psi)(\partial_v \psi)\left(\frac{4r_u}{\Omega^2 r}X^u + \frac{4r_v}{\Omega^2 r}X^v\right) \\ &- \frac{a}{l^2}\psi^2\left(\partial_u X^u + \left(2\frac{r_u}{r} + 2\frac{\Omega_u}{\Omega}\right)X^u + \partial_v X^v + \left(2\frac{r_v}{r} + 2\frac{\Omega_v}{\Omega}\right)X^v\right). \end{aligned} \tag{3.459}$$

Motivated heavily by the linear theory in chapter 2 ($X = F(r)\mathcal{R}$), we consider a vector field of the form

$$X = -\frac{r_v}{\Omega^2}\cdot F(r)\partial_u + -\frac{r_u}{\Omega^2}\cdot F(r)\partial_v. \tag{3.460}$$

Where F is bounded and sufficiently smooth. We thus compute

$$\partial_v X^u = 4\pi r \frac{(\partial_v \psi)^2}{\Omega^2} F - \frac{r_v^2}{\Omega^2} F', \tag{3.461}$$

$$\partial_u X^v = 4\pi r \frac{(\partial_u \psi)^2}{\Omega^2} F - \frac{r_u^2}{\Omega^2} F', \tag{3.462}$$

$$\partial_u X^u + 2\frac{\Omega_u}{\Omega}X^u + \partial_v X^v + 2\frac{\Omega_v}{\Omega}X^v = -2\frac{r_u r_v}{\Omega^2}F' - 2\frac{r_{uv}}{\Omega^2}F. \tag{3.463}$$

Then we express K^X as

$$K^X = K^X_{main} + K^X_{error}, \tag{3.464}$$

where the terms are defined by

$$\begin{aligned} K^X_{main} &= 2F'\left(\frac{r_v}{\Omega^2}\psi_u + \frac{r_u}{\Omega^2}\psi_v\right)^2 \\ &+ \psi_u\psi_v\left(-\frac{4r_ur_v}{\Omega^4}\left(F' + \frac{2}{r}F\right)\right) \\ &- \frac{a}{l^2}\psi^2\left(-2f - 2\frac{r_vr_u}{\Omega^2}\left(F' + \frac{2}{r}F\right) - \frac{2r_{uv}}{\Omega^2}F\right), \\ K^X_{error} &= \frac{-16}{\Omega^4}\pi r\psi_u^2\psi_v^2 F. \end{aligned} \tag{3.465}$$

3.8.2 Low weighted global energy estimate

We now prove the low weighted degenerate global Morawetz estimate. To do this we will study multipliers of the form

$$F(r) = -r^{-n}. \tag{3.466}$$

We need several lemmas to construct the estimate.

Lemma 3.8.2. *In $\mathscr{R}_{\mathcal{H}}$ for $\kappa < \frac{1}{2}$*

$$\int_{D(u,v)\times\mathbb{T}^2} \nabla^\mu J^X_\mu \leq C_{l,g,M}\mathbb{F}(u,v). \tag{3.467}$$

Proof. Studying the surface terms

$$\begin{aligned} \int_{D(u,v)\times\mathfrak{T}^2} \nabla^\mu J^X_\mu = &\int_{v_0}^{v} \left(\mathbb{T}_{vv}V^v + \mathbb{T}_{uv}X^u\right) r^2(u,\overline{v})d\overline{v} + \int_{u_{\mathcal{I}}}^{u} \left(\mathbb{T}_{uu}X^u + \mathbb{T}_{uv}X^v\right) r^2(\overline{u},v)d\overline{u} \\ &- \int_{u_0}^{u} \left(\mathbb{T}_{uu}X^u + \mathbb{T}_{uv}X^v\right) r^2(\overline{u},v_0)d\overline{u} - \int_{\mathcal{I}} \mathbb{T}_{\mu\nu}X^\mu\hat{n}^\nu d\sigma_{\mathcal{I}}. \end{aligned} \tag{3.468}$$

$\mathcal{I}$ surface

The metric restricted to constant $\tilde{r}$ surfaces is given by

$$h = -\Omega^2\frac{r_v}{-r_u}dv^2 + r^2\left(dx^2 + dy^2\right). \tag{3.469}$$

The surface form is given by

$$d\sigma_{\tilde{r}=const} = \sqrt{\left|\Omega^2\frac{r_v}{-r_u}r^4\right|}dvdxdy, \tag{3.470}$$

and the unit normal

$$\hat{n} = \sqrt{\frac{-r_v}{\Omega^2 r_u}}\partial_u - \sqrt{\frac{-r_u}{\Omega^2 r_v}}\partial_v. \tag{3.471}$$

As this surface is timelike we consider the inward pointing unit vector. We compute

$$\hat{n}\sqrt{|h|} = r^2 \frac{r_v}{-r_u}\partial_u - r^2\partial_v. \tag{3.472}$$

Exploring the flux terms we see

$$\begin{aligned}
\mathbb{T}_{\mu\nu}X^\mu \hat{n}^\nu d\sigma_{\mathcal{I}} =& \left(\mathbb{T}_{uu}X^u + \mathbb{T}_{uv}X^v\right)\frac{r_v}{-r_u}r^2 - \left(\mathbb{T}_{vv}X^v + \mathbb{T}_{uv}X^u\right)r^2 \\
=& \left((\nabla_u\psi)^2\frac{r_v}{\Omega^2}r^{-n} + \frac{a\Omega^2}{2l^2}\psi^2\frac{r_u}{\Omega^2}r^{-n}\right)\frac{r_v}{-r_u}r^2 - \left((\nabla_v\psi)^2\frac{r_u}{\Omega^2}r^{-n} + \frac{a\Omega^2}{2l^2}\psi^2\frac{r_v}{\Omega^2}r^{-n}\right)r^2 \\
=& r^{2-n}\frac{\Omega^2}{-r_u}\left((\mathcal{R}\psi)^2 - \left(\frac{r_u+r_v}{\Omega^2}\right)\nabla_u\psi\nabla_v\psi + \frac{a}{l^2}\frac{r_u r_v}{\Omega^2}\psi^2\right) \\
=& r^{2-n}\frac{\Omega^2}{-r_u}\left(\mathcal{R}\psi\right)^2,
\end{aligned} \tag{3.473}$$

(we may neglect the latter terms on the boundary through the density argument). Now using the relationship

$$\mathcal{R}\psi = \tilde{\mathcal{R}}\psi - \frac{2gr_u r_v}{r\Omega^2}\psi, \tag{3.474}$$

we see that all these terms vanish on the boundary.

Fixed u, v surfaces

We study the flux on the surface of fixed u. Splitting the region into a section where $r \le r_Y$. We see

$$\begin{aligned}
\int_{v_0}^{v}\left(\mathbb{T}_{vv}X^v + \mathbb{T}_{uv}X^u\right)r^2(u,\bar{v})d\bar{v} =& \int_{v_0}^{v}\frac{-r_u}{\Omega^2 r^{n-2}}\psi_v^2 - \frac{a}{2l^2 r^{n-2}}r_v\psi^2 dv \\
\le& \int_{v_0}^{v}\frac{-2r_u}{\Omega^2 r^{n-2}}(\tilde{\nabla}_v\psi)^2 + \frac{-2r_u}{\Omega^2}\cdot\frac{g^2r_v^2}{r^n}\psi^2 - \frac{a}{2l^2r^{n-2}}r_v\psi^2 dv.
\end{aligned} \tag{3.475}$$

The middle term of the integrand can be estimated by

$$\frac{-2r_u}{\Omega^2}\cdot\frac{g^2r_v^2}{r^n}\psi^2 \le C_{M,l,g}r^{2-n}r_v\psi^2 \le C_{Y,M,l,g}\frac{r_v}{r}\psi^2. \tag{3.476}$$

The final term of the integrand, for $n \ge 3$ we can bound by

$$-\frac{a}{2l^2r^{n-2}}r_v\psi^2 \le -\frac{a}{2l^2}\frac{r_v}{r}\psi^2. \tag{3.477}$$

For $n < 3$ we can estimate by

$$-\frac{a}{2l^2r^{n-2}}r_v\psi^2 \le -\frac{ar_Y^{-n+3}}{2l^2}\frac{r_v}{r}\psi^2. \tag{3.478}$$

In the region $r \geq r_Y$, assuming $n \leq 2$

$$\frac{-2r_u}{\Omega^2} \cdot \frac{g^2 r_v^2}{r^n}\psi^2 \leq C_{Y,l,g} r^{2-n} r_v \psi^2 \leq C_{Y,l,g} r_v \psi^2. \tag{3.479}$$

We can control this by the initial energy using lemma (3.8.1). We thus see that

$$\int_{v_0}^{v} \left(\mathbb{T}_{vv} X^v + \mathbb{T}_{uv} X^u\right) r^2(u, \bar{v}) d\bar{v} \leq C_{l,g,M} \mathbb{F}(u, v). \tag{3.480}$$

As the integrand for the other surface is of the form

$$\left(\mathbb{T}_{uu} X^u + \mathbb{T}_{uv} X^v\right) r^2 = \frac{-r_v}{\Omega^2 r^{n-2}}\psi_u^2 + \frac{a}{2l^2 r^{n-2}} r_u \psi^2, \tag{3.481}$$

it may be treated in the same way.

□

Lemma 3.8.3. *In $\mathscr{R}_{\mathcal{H}}$, for $\kappa < \frac{1}{2}$, we have the following global integrated decay estimate*

$$\mathbb{I}_{deg}[\psi] := \int_D r^{-6}\left(\frac{1}{\gamma^2}\psi_u^2 + \frac{1}{\chi^2}\psi_v^2\right)\frac{\Omega^2 r^2}{2} du dv + \int_D \left(\frac{1-|a|}{r}\right)\psi^2 \frac{\Omega^2}{2} r^2 du dv \leq C_{l,g,M}\mathbb{F}(u, v). \tag{3.482}$$

Proof.

Bulk terms

We now look at the bulk term

$$\begin{aligned}
K^{X,0}_{main} =& (2+n) r^{-n-1}\left(\frac{r_v}{\Omega^2}\psi_u + \frac{r_u}{\Omega^2}\psi_v\right)^2 \\
&- 4(2-n)\frac{r_u r_v}{\Omega^4} r^{-n-1}\psi_u\psi_v + \frac{a}{l^2}\psi^2 r^{-n}\left(2(n-2)\frac{r_u r_v}{r\Omega^2} - 2\frac{r_{uv}}{\Omega^2}\right) \\
=& (2+n) r^{-n-1}\left(\frac{r_v}{\Omega^2}\psi_u + \frac{r_u}{\Omega^2}\psi_v\right)^2 + (n-2) r^{-n-1}\left(\frac{r_v}{\Omega^2}\psi_u - \frac{r_u}{\Omega^2}\psi_v\right)^2 \\
&+ \frac{a}{l^2}\psi^2 r^{-n}\left(2(n-2)\frac{r_u r_v}{r\Omega^2} - 2\frac{r_{uv}}{\Omega^2}\right) \\
=& (2+n) r^{-n-1}\left(\mathcal{R}\psi\right)^2 + (n-2) r^{-n-1}\left(\mathcal{T}\psi\right)^2 + \frac{a}{l^2}\psi^2 r^{-n}\left(2(n-2)\frac{r_u r_v}{r\Omega^2} - 2\frac{r_{uv}}{\Omega^2}\right),
\end{aligned} \tag{3.483}$$

so the first order terms have a good sign for $n \geq 2$.

As for the zeroth order terms

$$\underbrace{\frac{a}{l^2}\psi^2 r^{-n-1}\frac{2(n-2) r_u r_v}{\Omega^2}}_{\geq 0} - \frac{2a}{l^2}\psi^2 r^{-n}\frac{r_{uv}}{\Omega^2}. \tag{3.484}$$

The first term has a good sign for ($n \geq 2$). For the second term we analyse through the r_{uv} equation (3.10). We recall this may be expressed as

$$r_{uv} = -\frac{\Omega^2}{2}\left(\frac{\varpi_1}{r^2} + \frac{r}{l^2} - \frac{4\pi a r}{l^2}\psi^2\right). \tag{3.485}$$

The term thus has the form

$$-\frac{a}{l^2}r^{-n}\psi^2\left(2\frac{r_{uv}}{\Omega^2}\right) = \frac{a}{l^2r^n}\underbrace{\left(\frac{\varpi_1}{r^2}+\frac{r}{l^2}-\frac{4\pi ar}{l^2}\psi^2\right)}_{\geq 0}\psi^2, \tag{3.486}$$

so it has a negative sign. We proceed by proving a Hardy type inequality to absorb it. First note that from the (ϖ_1) Hawking mass equations we can show that

$$\frac{1}{2r_u}\partial_u\mu_1 - \frac{8\pi r r_v}{r_u\Omega^2}(\partial_u\psi)^2 = \frac{r}{l^2}+\frac{\varpi_1}{r^2}-\frac{4\pi ar}{l^2}\psi^2, \tag{3.487}$$

and

$$\frac{1}{2r_v}\partial_v\mu_1 - \frac{8\pi r r_u}{r_v\Omega^2}(\partial_v\psi)^2 = \frac{r}{l^2}+\frac{\varpi_1}{r^2}-\frac{4\pi ar}{l^2}\psi^2. \tag{3.488}$$

We now integrate (3.486) (recall $\sqrt{g}=\frac{\Omega^2}{2}r^2$)

$$\begin{aligned}&\int_{D(u,v)}\frac{a}{l^2r^n}\left(\frac{\varpi_1}{r^2}+\frac{r}{l^2}-\frac{4\pi ar}{l^2}\psi^2\right)\psi^2\frac{\Omega^2}{2}r^2dudv\\ =&\frac{1}{2}\int_D\frac{a}{l^2r^{n-2}}\psi^2\left(-\chi\cdot\mu_{1,u}-\frac{4\pi r r_v}{r_u}\left(\partial_u\psi\right)^2\right)dudv\\ &+\frac{1}{2}\int_D\frac{a}{l^2r^{n-2}}\psi^2\left(\gamma\cdot\mu_{1,v}-\frac{4\pi r r_u}{r_v}\left(\partial_v\psi\right)^2\right)dudv.\end{aligned} \tag{3.489}$$

Integrating the first term by parts and recalling the following relation

$$\mu_1 = \frac{-4r_ur_v}{\Omega^2}, \tag{3.490}$$

and from the Raychaudhuri equations

$$\partial_u\chi = -\frac{\Omega^2}{r_u^2}r\pi\psi_u^2, \tag{3.491}$$

and

$$\partial_v\gamma = \frac{\Omega^2}{r_v^2}r\pi\psi_v^2. \tag{3.492}$$

We can then compute

$$\mu_1\partial_u\left(\chi r^{-n+2}\psi^2\right) = r^{-n+2}\left(\frac{(2-n)r_ur_v}{r}\psi^2+2r_v\psi\psi_u+\frac{4\pi r r_v}{r_u}\psi_u^2\right), \tag{3.493}$$

and

$$-\mu_1\partial_v\left(\gamma r^{-3}\psi^2\right) = r^{-n+2}\left(\frac{(2-n)r_ur_v}{r}\psi^2+2r_u\psi\psi_v+\frac{4\pi r r_u}{r_v}\psi_v^2\right). \tag{3.494}$$

We see that

$$\int_{D(u,v)} \frac{a}{l^2 r^n}\left(\frac{\varpi_1}{r^2}+\frac{r}{l^2}-\frac{4\pi a r}{l^2}\psi^2\right)\psi^2\frac{\Omega^2}{2}r^2 dudv = \\ \int_D \frac{4a}{l^2 r^{n-2}}\gamma\chi\mu_1\psi\left(\frac{1}{4\gamma}\psi_u-\frac{1}{4\chi}\psi_v\right)+\frac{(2-n)a}{l^2 r^{n-1}}r_u r_v\psi^2 dudv-\int_{u_0}^{u}\frac{a}{l^2 r^{n-2}}\psi^2(-r_u)(\bar{u},v_0)d\bar{u} \\ +\int_{u_{\mathcal{I}}}^{u}\frac{a}{l^2 r^{n-2}}\psi^2(-r_u)(\bar{u},v)d\bar{u}-\int_{v_0}^{v}\frac{a}{l^2 r^{n-2}}\psi^2 r_v(u,\bar{v})d\bar{v}. \tag{3.495}$$

We rewrite as

$$\int_{D(u,v)} \frac{-a}{l^2 r^n}\left(\frac{\varpi_1}{r^2}+\frac{r}{l^2}-\frac{4\pi a r}{l^2}\psi^2\right)\psi^2\frac{\Omega^2}{2}r^2 dudv = \\ \int_D \frac{-4a}{l^2 r^{n-2}}\gamma\chi\mu_1\psi\left(\frac{1}{4\gamma}\psi_u-\frac{1}{4\chi}\psi_v\right)+\frac{(n-2)a}{l^2 r^{n-1}}r_u r_v\psi^2 dudv \\ -\int_{u_0}^{u}\frac{-a}{l^2 r^{n-2}}\psi^2(-r_u)(\bar{u},v_0)d\bar{u}+\int_{u_{\mathcal{I}}}^{u}\frac{-a}{l^2 r^{n-2}}\psi^2(-r_u)(\bar{u},v)d\bar{u}-\int_{v_0}^{v}\frac{-a}{l^2 r^{n-2}}\psi^2 r_v(u,\bar{v})d\bar{v}. \tag{3.496}$$

Now using Young's inequality we see that

$$\int_D \frac{4|a|}{l^2 r^{n-2}}\gamma\chi\mu_1\psi\left(\frac{1}{4\gamma}\psi_u-\frac{1}{4\chi}\psi_v\right)dudv \\ \le \int_D \frac{|a|}{2l^2 r^n}\psi^2\left(\frac{r}{l^2}+\frac{\varpi_1}{r^2}-\frac{4\pi r a\psi^2}{l^2}\right)\frac{r^2\Omega^2}{2}dudv \\ +\int_D \frac{32|a|}{l^2 r^{n-2}}\frac{\gamma^2\chi^2\mu_1^2}{\Omega^4}\left(\frac{r}{l^2}+\frac{\varpi_1}{r^2}-\frac{4\pi r a\psi^2}{l^2}\right)^{-1}\left(\frac{\psi_u}{4\gamma}-\frac{\psi_v}{4\chi}\right)^2\frac{\Omega^2}{2}dudv. \tag{3.497}$$

Noting the relation

$$\frac{\gamma^2\chi^2\mu_1^2}{\Omega^4}=\frac{1}{16}, \tag{3.498}$$

we see that

$$\int_D \frac{4|a|}{l^2 r^{n-2}}\gamma\chi\mu\psi\left(\frac{1}{4\gamma}\psi_u-\frac{1}{4\chi}\psi_v\right)dudv \\ \le \int_D \frac{|a|}{2l^2 r^n}\psi^2\left(\frac{r}{l^2}+\frac{\varpi_1}{r^2}-\frac{4\pi r a\psi^2}{l^2}\right)\frac{r^2\Omega^2}{2}dudv \\ +\int_D \frac{2|a|}{l^2 r^{n-1}}\left(\frac{1}{l^2}+\frac{\varpi_1}{r}-\frac{4\pi a\psi^2}{l^2}\right)^{-1}\left(\frac{\psi_u}{4\gamma}-\frac{\psi_v}{4\chi}\right)^2\frac{\Omega^2}{2}dudv. \tag{3.499}$$

For ease we now define

$$\alpha := \left(1+\frac{\varpi_1 l^2}{r^3}-4\pi a\psi^2\right)^{-1}, \tag{3.500}$$

substituting (3.499) into (3.496) gives

$$-\int_D \frac{2(n-2)a}{l^2r^{n-1}}r_ur_v\psi^2 dudv + \int_D \frac{-a}{l^2r^n}\psi^2\left(\frac{r}{l^2}+\frac{\varpi_1}{r^2}-\frac{4\pi ra\psi^2}{l^2}\right)\frac{\Omega^2}{2}r^2dudv \leq C_{X,Y,l,g,M}\mathbb{F}(u,v)$$
$$+\int_D \frac{4|a|\alpha}{r^{n-1}}\left(\frac{1}{4\gamma}\psi_u - \frac{1}{4\chi}\psi_v\right)^2\frac{\Omega^2}{2}dudv. \tag{3.501}$$

Now

$$\int_D -\frac{2(n-2)a}{l^2r^{n-1}}r_ur_v\psi^2dudv = \frac{(n-2)a}{l^2}\int_D \frac{1}{r^n}\left(\frac{-2\varpi_1}{r^2}+\frac{r}{l^2}\right)\psi^2\frac{\Omega^2r^2}{2}dudv. \tag{3.502}$$

We then rewrite (3.499) as the following Hardy estimate

$$\begin{aligned}
&\int_D \frac{2(n-2)a}{l^2r^{n-1}}r_ur_v\psi^2dudv - \int_D \frac{-a}{l^2r^n}\psi^2\left(\frac{r}{l^2}+\frac{\varpi_1}{r^2}-\frac{4\pi ra\psi^2}{l^2}\right)\frac{\Omega^2}{2}r^2dudv\\
&= \int_D \frac{-a}{l^2r^n}\psi^2\left(\frac{r(n-1)}{l^2}+\frac{\varpi_1(5-2n)}{r^2}-\frac{4\pi ra\psi^2}{l^2}\right)\frac{\Omega^2}{2}r^2dudv \leq C_{l,g,M}\mathbb{F}(u,v)\\
&+\int_D \frac{4|a|\alpha}{r^{n+1}}(\mathcal{R}\psi)^2\frac{\Omega^2r^2}{2}dudv.
\end{aligned} \tag{3.503}$$

The LHS is clearly non negative for $1 \leq n \leq \frac{7}{2}$. Examining the integral of K^X we have

$$\begin{aligned}
\int_D K^{X,0}dudv\frac{\Omega^2r^2}{2}dudv &= \int_D (2+n)r^{-n+1}(\mathcal{R}\psi)^2\frac{\Omega^2}{2}dudv + \int_D (n-2)r^{-n-1}(\mathcal{T}\psi)^2\frac{\Omega^2r^2}{2}dudv\\
&+\int_D \frac{a}{l^2}\psi^2r^{1-n}(n-2)r_ur_vdudv\\
&+\int_{D(u,v)} \frac{a}{l^2r^n}\left(\frac{\varpi_1}{r^2}+\frac{r}{l^2}-\frac{2\pi ar}{l^2}\psi^2\right)\psi^2\frac{\Omega^2}{2}r^2dudv\\
&\geq \int_D (n+2-4|a|\alpha)r^{-n+1}(\mathcal{R}\psi)^2\frac{\Omega^2}{2}dudv\\
&+\int_D (n-2)r^{-n-1}(\mathcal{T}\psi)^2\frac{\Omega^2r^2}{2}dudv\\
&-\int_D \frac{a}{l^2}\psi^2r^{1-n}(n-2)r_ur_vdudv - C_{l,g,M}\mathbb{F}(u,v)\\
&= \int_D (n+2-4|a|\alpha)r^{-n+1}(\mathcal{R}\psi)^2\frac{\Omega^2}{2}dudv\\
&+\int_D (n-2)r^{-n-1}(\mathcal{T}\psi)^2\frac{\Omega^2r^2}{2}dudv\\
&+\frac{(n-2)a}{2l^2}\int_D \frac{1}{r^n}\left(\frac{-2\varpi_1}{r^2}+\frac{r}{l^2}\right)\psi^2\frac{\Omega^2r^2}{2}dudv - C_{l,g,M}\mathbb{F}(u,v).
\end{aligned} \tag{3.504}$$

That is

$$C_{l,g,M}\mathbb{F}(u,v)+\int_D K^{X,0}dudv\frac{\Omega^2r^2}{2}dudv \geq \int_D (n+2-4|a|\alpha)\, r^{-n-1}(\mathcal{R}\psi)^2\frac{\Omega^2r^2}{2}dudv \\ +\int_D (n-2)r^{-n-1}(\mathcal{T}\psi)^2\frac{\Omega^2r^2}{2}dudv+\frac{(n-2)a}{2l^2}\int_D\frac{1}{r^n}\left(\frac{-2\varpi_1}{r^2}+\frac{r}{l^2}\right)\psi^2\frac{\Omega^2r^2}{2}dudv. \tag{3.505}$$

$1-\alpha$ **estimates**

$$\begin{aligned} 1-\alpha = 1-|a|\left(1+\frac{\varpi_1 l^2}{r^3}-4\pi a\phi^2\right)^{-1} &= \frac{1-|a|}{1+\frac{\varpi_1 l^2}{r^3}-4\pi a\phi^2}+\frac{\frac{\varpi l^2}{r^3}-4\pi a\phi^2}{1+\frac{\varpi l^2}{r^3}-4\pi a\phi^2} \\ &\geq \frac{1}{1+\frac{Ml^2+cb^2}{r^3}}\left(1-|a|+\frac{\varpi_1 l^2}{r^3}\right) \\ &\geq \frac{1}{1+\frac{Ml^2}{r^3_{min}}+\frac{cb^2}{r^3_{min}}}\left(1-|a|+\frac{\varpi_1 l^2}{r^3}\right). \end{aligned} \tag{3.506}$$

Then recalling $r_+ = 2Ml^2$ using the estimate

$$\left|r_+^3 - r_{min}^3\right| \leq Cb^2, \tag{3.507}$$

we see

$$\begin{aligned} 1-|a|\left(1+\frac{\varpi l^2}{r^3}-4\pi a\phi^2\right)^{-1} &\geq \frac{1}{1+\frac{Ml^2}{r^3_{min}}+\frac{cb^2}{r^3_{min}}}\left(1-|a|+\frac{\varpi l^2}{r^3}\right) \\ &\geq \frac{1}{1+\frac{1}{2}+\frac{cb^2}{r^3_{min}}}\left(1-|a|+\frac{\varpi l^2}{r^3}\right) \\ &\geq \frac{1}{2}\left(1-|a|+\frac{Ml^2}{4r^3}\right). \end{aligned} \tag{3.508}$$

$n=2$ **estimate**

If we choose $n=2$ and restrict to $|a|<1$, then we have

$$\int_D (1-|a|)\, r^{-3}(\mathcal{R}\psi)^2\frac{\Omega^2r^2}{2}dudv \leq C_{l,g,M}\mathbb{F}(u,v), \tag{3.509}$$

and the L^2 estimate

$$\int_D (1-|a|)\, r^{-1}\psi^2\frac{\Omega^2}{2}r^2dudv \leq C_{l,g,M}\mathbb{F}(u,v). \tag{3.510}$$

Low weighted estimates for $|a| < 1$

Choosing $n = 5$ in (3.505) gives

$$\begin{aligned} &C_{l,g,M}\mathbb{F}(u,v) + \int_D K^X dudv \frac{\Omega^2 r^2}{2} dudv \geq \int_D (7 - 4|a|)\, r^{-6} (\mathcal{R}\psi)^2 \frac{\Omega^2 r^2}{2} dudv \\ &+ \int_D 3r^{-6} (\mathcal{T}\psi)^2 \frac{\Omega^2 r^2}{2} dudv + \frac{3a}{2l^2} \int_D \frac{1}{r^5} \left(\frac{-2\varpi_1}{r^2} + \frac{r}{l^2} \right) \psi^2 \frac{\Omega^2 r^2}{2} dudv. \end{aligned} \tag{3.511}$$

Then using estimate (3.510)

$$\begin{aligned} &C_{l,g,M}\mathbb{F}(u,v) + \int_D K^X dudv \frac{\Omega^2 r^2}{2} dudv \\ &\geq \int_D r^{-6} (\mathcal{R}\psi)^2 \frac{\Omega^2 r^2}{2} dudv + \int_D r^{-6} (\mathcal{T}\psi)^2 \frac{\Omega^2 r^2}{2} dudv + \int_D \frac{(1 - |a|)}{r} \psi^2 \frac{\Omega^2}{2} r^2 dudv, \end{aligned} \tag{3.512}$$

allows us to recover the $\mathcal{T}\psi$ term. Expressing in terms of u and v derivatives

$$\int_D r^{-6} \left(\frac{1}{\gamma^2} \psi_u^2 + \frac{1}{\chi^2} \psi_v^2 \right) \frac{\Omega^2 r^2}{2} dudv + \int_D \left(\frac{1 - |a|}{r} \right) \psi^2 \frac{\Omega^2}{2} r^2 dudv \leq C_{l,g,M}\mathbb{F}(u,v). \tag{3.513}$$

□

Remark 3.8.4. *This lemma is the origin of the restriction $\kappa < \frac{1}{2}$. This is due to the choice of $n = 2$, it produces terms of the form $r^n \psi^2$ which will only decay for $\kappa < \frac{1}{2}$ for the boundary conditions we are considering. If one were to choose $n < 2$ to remedy this problem, the issue of positivity from* (3.505) *arises. The bulk term can't be seen to be positive. However as in [HS13b] we expect faster decay for Dirichlet boundary conditions. If one were to follow that scheme in the toroidal setting at the H^2, level they would expect to extend the result to $\kappa = \frac{1}{2}$. Beyond this value seems to out of reach technically. It is expected to require more sophisticated multipliers.*

3.8.3 The redshift vector field

We now seeking to remove the degeneration in the estimates due to the $\left(\frac{1}{\gamma^2}\right)$ factor. To do this we localise a vector field to a region near the horizon, and exploit a red shift effect. The result of lemma 3.8.4 is an adapted version of [HS13b] to this setting.

Lemma 3.8.4. *In $\mathscr{R}_{\mathcal{H}}$ we have the following estimate for $\kappa \in (0, \frac{1}{2})$*

$$\begin{aligned} &\int_D \frac{1}{r^7} \left(\frac{r^4}{-r_u} (\tilde{\nabla}_u \psi)^2 + \frac{-r_u r^2}{\Omega^2} (\tilde{\nabla}_v \psi)^2 \right) d\bar{u} d\bar{v} \\ &+ \int_D ((1 - |a|)\, r)\, (-r_u) \psi^2 + ((1 - |a|)\, r)\, r_v \psi^2 d\bar{u} d\bar{v} \leq C_{l,g,M}\mathbb{F}(u,v). \end{aligned} \tag{3.514}$$

Proof. Firstly fix $r_W > r_X$ and define

$$Y = (-r_u)^{-1}\eta(r)\partial_u, \tag{3.515}$$

where

$$\eta(r) = \begin{cases} 0 & \text{for } r_W \leq r, \\ \text{smooth, monotonic with bounded derivative} & \text{for } r_X \leq r \leq r_W, \\ 1 & \text{for } r \leq r_X. \end{cases} \tag{3.516}$$

We see that

$$K^Y = \left(-\frac{2}{\Omega^2}\partial_v\left(\frac{-1}{r_u}\right)(\partial_u\psi)^2 + (\partial_u\psi)(\partial_v\psi)\frac{-4}{\Omega^2 r} + \frac{2a}{l^2}\psi^2 + \frac{4a\pi}{r_u^2}r\psi^2\psi_u^2\right)\eta(r) + \left(\frac{2}{\Omega^2}\psi_u^2 + \frac{a}{l^2}\psi^2\right)\eta'(r). \tag{3.517}$$

The first term is equal to

$$\left(\frac{\psi_u^2}{2r_u^2}\left(\frac{\varpi_1}{r^2} + \frac{2r}{l^2}\right) + \frac{\psi_u}{r_u}\frac{1}{r\chi}\psi_v + \frac{a}{l^2}\psi^2\frac{2}{r}\right)\eta(r). \tag{3.518}$$

We also can quickly compute that

$$J^Y[\psi](Y,\partial_u) = \frac{\psi_u^2}{-r_u}\eta, \quad J^Y[\psi](Y,\partial_v) = \frac{a\chi}{l^2}\eta\psi^2, \tag{3.519}$$

asymptotically it is clear that these terms are 0 at $\mathcal{I}$. Estimating (3.518), we see

$$\begin{aligned} &\left(\frac{\psi_u^2}{2r_u^2}\left(\frac{\varpi_1}{r^2} + \frac{2r}{l^2}\right) + \frac{\psi_u}{r_u}\frac{1}{r\chi}\psi_v + \frac{a}{l^2}\psi^2\frac{2}{r}\right)\eta(r) \\ \geq &\left(\frac{3r\psi_u^2}{4l^2r_u^2} + \frac{r^{\frac{1}{2}}\psi_u}{lr_u}\frac{l}{r^{\frac{3}{2}}\chi}\psi_v + \frac{a}{l^2}\psi^2\frac{2}{r}\right)\eta(r) \\ \geq &\left(\frac{r\psi_u^2}{4l^2r_u^2} - \frac{l^2}{2}\frac{1}{r^3\chi^2}\psi_v^2 - \frac{2\left|a\right|}{r}\psi^2\right)\eta(r). \end{aligned} \tag{3.520}$$

Integrating K^Y this gives the estimate

$$\begin{aligned} &\int_{u_{\mathcal{I}}}^{u}\frac{\psi_u^2}{r_u^2}r^2(-r_u)(\bar{u},v)\eta d\bar{u} + \int_D\frac{r\psi_u^2}{r_u^2}\Omega^2r^2\eta d\bar{u}d\bar{v} \\ \leq &C_{M,l,a}\int_D\left(\frac{\psi_v^2}{r^3\chi^2} + \frac{\psi^2}{r}\right)\eta(r)\Omega^2r^2d\bar{u}d\bar{v} + \int_D\left(\frac{2}{\Omega^2}\psi_u^2 + \frac{a}{l^2}\psi^2\right)\eta'(r)\Omega^2r^2d\bar{u}d\bar{v} \\ &+ C_{M,l,a}\int_{v_0}^{v}\frac{1}{l^2}\chi\psi^2r^2(u,\bar{v})\eta d\bar{v} + \int_{u_{\mathcal{I}}}^{u}\frac{\psi_u^2}{r_u^2}r^2(-r_u)\eta(\bar{u},v_0)d\bar{u}. \end{aligned} \tag{3.521}$$

It is clear from corollary 3.7.14 that

$$\int_D\left(\frac{2}{\Omega^2}\psi_u^2 + \frac{a}{l^2}\psi^2\right)\eta'(r)\Omega^2r^2d\bar{u}d\bar{v} \leq C_{M,l,g,Y}\mathbb{I}_{deg}[\psi](u,v). \tag{3.522}$$

Now the first term on the right hand side of (3.521), we already control from lemma 3.8.3, the last term is the an initial data norm that we control after a few trivial estimates. We are left to deal with the third term

$$\int_{v_0}^{v} \frac{1}{l^2}\chi\psi^2 r^2(u,\bar{v})\eta d\bar{v} = \int_{v_0}^{u} d\bar{u}\partial_u \int_{v_0}^{v^*(u)} \frac{1}{l^2}\chi\psi^2 r^2 \eta d\bar{v}. \tag{3.523}$$

Where $v^*(u)$ is the v-value where the ray of constant u intersects either $\mathcal{I}$, or the constant v ray. We pass the derivative through

$$\int_{v_0}^{v} \frac{1}{l^2}\chi\psi^2 r^2(u,\bar{v})\eta d\bar{v} = \int_D \left(-r\pi\frac{\psi_u^2}{r_u^2}\psi^2 - \frac{1}{2}\psi\frac{\psi_u}{r_u} - \frac{1}{2r}\psi^2\right)\eta\Omega^2 r^2 d\bar{u}d\bar{v} + \int_D \frac{1}{l^2}\chi\psi^2 r^2 \eta' r_u d\bar{u}d\bar{v}, \tag{3.524}$$

The second integrand is estimated using corollary 3.7.12. The η' allows us to disregard the r weights. The first integrand may be estimated by dropping the negative terms, and a Young inequality

$$\int_{v_0}^{v} \frac{1}{l^2}\chi\psi^2 r^2(u,\bar{v})\eta d\bar{v} \leq \int_D \left(\epsilon\frac{r\psi_u^2}{r_u^2} + \frac{1}{16\epsilon}\frac{\psi^2}{r}\right)\eta\Omega^2 r^2 d\bar{u}d\bar{v} + C_{M,l,g,Y}\mathbb{I}_{deg}[\psi](u,v). \tag{3.525}$$

We can an absorb an ϵ amount of the the derivative terms with the left hand side. Control of the zeroth order terms has already been established.
This leaves us with

$$\begin{aligned}&\int_{u_{\mathcal{I}}}^{u} \frac{\psi_u^2}{r_u^2} r^2(-r_u)(\bar{u},v)\eta d\bar{u} + \int_D \frac{r\psi_u^2}{r_u^2}\eta\Omega^2 r^2 dudv \\ \leq &C_{M,l,a}\mathbb{I}_{deg}[\psi](u,v) + \int_{u_{\mathcal{I}}}^{u} \frac{\psi_u^2}{r_u^2} r^2(-r_u)\eta d\bar{u}.\end{aligned} \tag{3.526}$$

Combining this with the global estimate we have that

$$\int_D r^{-6}\left(r^4\frac{\psi_u^2}{r_u^2} + \frac{1}{\chi^2}\psi_v^2\right)\frac{\Omega^2 r^2}{2} dudv + \int_D \frac{(1-|a|)}{r}\psi^2\frac{\Omega^2}{2} r^2 d\bar{u}d\bar{v} \leq C_{X,Y,l,g,M}\mathbb{F}(u,v). \tag{3.527}$$

We now show this holds in a twisted setting. We estimate

$$r^4\frac{\psi_u^2}{r_u^2} \geq C\left(r\frac{\psi_u^2}{r_u^2}\right) = C\left(\frac{r}{r_u^2}(\tilde{\nabla}_u\psi)^2 + \frac{2g}{r_u}\psi\tilde{\nabla}_u\psi + \frac{1}{r}g^2\psi^2\right). \tag{3.528}$$

We apply Young's inequality to get

$$r^4\frac{\psi_u^2}{r_u^2} \geq C\left(\frac{r}{2r_u^2}(\tilde{\nabla}_u\psi)^2 - \frac{1}{r}g^2\psi^2\right). \tag{3.529}$$

We thus have

$$\int_D \frac{1}{r^6}\left(\frac{r}{2r_u^2}(\tilde{\nabla}_u\psi)^2 - \frac{1}{r}g^2\psi^2\right)\Omega^2 r^2(\bar{u},\bar{v})d\bar{u}d\bar{v} \leq C_{l,g,M}\mathbb{F}(u,v). \tag{3.530}$$

Adding a multiple of the zeroth order terms of (3.527) to get

$$\int_D \frac{1}{r^6}\left(\frac{r}{r_u^2}(\tilde{\nabla}_u\psi)^2 + \frac{\psi_v^2}{\chi^2}\right)\Omega^2 r^2(\bar{u},\bar{v})d\bar{u}d\bar{v} + \int_D \left(\frac{1-|a|}{r}\right)\psi^2\frac{\Omega^2}{2}r^2 d\bar{u}d\bar{v} \leq C_{X,Y,l,g,M}\mathbb{F}(u,v). \tag{3.531}$$

With the use of corollary 3.7.12 we estimate the ψ_v terms by

$$\begin{aligned}
\frac{\psi_v^2}{\chi^2} &\geq \frac{C}{r^3}\frac{16r_u^2}{\Omega^4}\left((\tilde{\nabla}_v\psi)^2 + \frac{2gr_v}{r}\psi\tilde{\nabla}_v\psi + \frac{r_v^2}{r^2}\psi^2\right)\\
&\geq C_l\frac{-r_u}{\Omega^2}\left(\frac{1}{r^3}(\tilde{\nabla}_v\psi)^2 + \frac{2gr_v}{r^4}\psi\tilde{\nabla}_v\psi + \frac{r_v^2}{r^5}\psi^2\right)\\
&\geq C_l\frac{-r_u}{\Omega^2}\left(\frac{1}{2r^3}(\tilde{\nabla}_v\psi)^2 - \frac{r_v^2}{r^5}\psi^2\right)\\
&\geq C_l\frac{-r_u}{\Omega^2}\left(\frac{1}{2r^3}(\tilde{\nabla}_v\psi)^2 - \frac{1}{r}\psi^2\right).
\end{aligned} \tag{3.532}$$

This allows us to conclude that

$$\begin{aligned}
&\int_D \frac{1}{r^{11}}\left(\frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + \frac{-r_u r^2}{\Omega^2}(\tilde{\nabla}_v\psi)^2\right)\Omega^2 r^2(\bar{u},\bar{v})d\bar{u}d\bar{v}\\
&+\int_D \left(\frac{1-|a|}{r}\right)\psi^2\frac{\Omega^2}{2}r^2 d\bar{u}d\bar{v} \leq C_{l,g,M}\mathbb{F}(u,v),
\end{aligned} \tag{3.533}$$

or as

$$\begin{aligned}
&\int_D \frac{1}{r^7}\left(\frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + \frac{-r_u r^2}{\Omega^2}(\tilde{\nabla}_v\psi)^2\right)d\bar{u}d\bar{v}\\
&+\int_D (1-|a|)\,r(-r_u)\psi^2 + (1-|a|)\,rr_v\psi^2 d\bar{u}d\bar{v} \leq C_{l,g,M}\mathbb{F}(u,v).
\end{aligned} \tag{3.534}$$

□

3.8.4 Morawetz estimate

Theorem 3.8.4. *In $\mathscr{R}_{\mathcal{H}}$, for $\kappa < \frac{1}{2}$, the following Morawetz estimate holds*

$$\int_D \frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2 + (-r_u)\psi^2 + \frac{-r_u r^2}{\Omega^2}(\tilde{\nabla}_v\psi)^2 + \frac{r_v}{r}\psi^2 dudv \leq C_{l,g,M}\mathbb{F}(u,v). \tag{3.535}$$

Proof. Fix $\infty > r_M > r_Z > r_Y$, (we will specify the conditions they need to satisfy later) and let $\eta(r)$ be the cut off function defined by

$$\eta(r) = \begin{cases} 0 & \text{for } r_Z \geq r,\\ \text{smooth with bounded derivative} & \text{for } r_M \geq r \geq r_Z,\\ 1 & \text{for } r \geq r_M. \end{cases} \tag{3.536}$$

From equations (3.23) and (3.26) we derive

$$\begin{aligned}\frac{1}{2}\partial_u\left(h(\tilde{\nabla}_v\psi)^2\right)+\frac{1}{2}\partial_v\left(f(\tilde{\nabla}_u\psi)^2\right)=&\left(\kappa-\frac{1}{2}\right)\tilde{\nabla}_u\psi\tilde{\nabla}_v\psi\left(\frac{r_v}{r}h+\frac{r_u}{r}f\right)\\&+\left(\frac{1}{2}h_u-\frac{r_u}{r}h\right)(\tilde{\nabla}_v\psi)^2+\left(\frac{1}{2}f_v-\frac{r_v}{r}f\right)(\tilde{\nabla}_u\psi)^2\\&-\frac{\Omega^2}{4}V\psi\left(h\tilde{\nabla}_v\psi+f\tilde{\nabla}_u\psi\right).\end{aligned}\tag{3.537}$$

We now make the choice

$$f=-\frac{r_v}{\Omega^2}r\eta,\quad h=-\frac{r_u}{\Omega^2}r\eta.\tag{3.538}$$

Flux estimates
Applying the divergence theorem and examining the flux terms

$$\begin{aligned}&\int_D\frac{1}{2}\partial_u\left(h(\tilde{\nabla}_v\psi)^2\right)+\frac{1}{2}\partial_v\left(f(\tilde{\nabla}_u\psi)^2\right)du'dv'\\=&\frac{1}{2}\int_{v_0}^{v}-\frac{r_u}{\Omega^2}\eta r(\tilde{\nabla}_v\psi)^2(u,v')dv'+\frac{1}{2}\int_{u_{\mathcal{I}}}^{u}\frac{r_v}{\Omega^2}\eta r(\tilde{\nabla}_u\psi)^2(u',v)du'\\&+\frac{1}{2}\int_{u_0}^{u}\frac{r_v}{\Omega^2}\eta r(\tilde{\nabla}_u\psi)^2(u',v_0)du'+\int_{\mathcal{I}}\frac{1}{2}\frac{\Omega^2}{-r_u}r\eta\left(\frac{r_v^2}{\Omega^4}\left(\tilde{\nabla}_u\psi\right)^2+\frac{r_u^2}{\Omega^2}\left(\tilde{\nabla}_v\psi\right)^2\right)\\\leq&C_{M,l,g}\mathbb{F}(u,v)+\frac{1}{12}\int_{\mathcal{I}}\frac{\Omega^2}{-r_u}r\left(\tilde{\mathcal{R}}\psi\right)^2.\end{aligned}\tag{3.539}$$

The latter term then vanishes due to the boundary conditions (it decays like $r^{-2\kappa}$). The cross terms are written in terms of $\mathcal{T}$ and $\tilde{\mathcal{R}}$ derivatives which can be seen to be 0 on the boundary, from the density argument and boundary conditions.
Bulk terms
Defining $\hat{f}$ and $\hat{h}$ through

$$f=\hat{f}\eta,\quad h=\hat{h}\eta,\tag{3.540}$$

and noting the identity

$$\begin{aligned}\frac{1}{2}\partial_u\left(\eta\hat{h}(\tilde{\nabla}_v\psi)^2\right)+\frac{1}{2}\partial_v\left(\eta\hat{f}(\tilde{\nabla}_u\psi)^2\right)=&\eta\left(\frac{1}{2}\partial_u\left(\hat{h}(\tilde{\nabla}_v\psi)^2\right)+\frac{1}{2}\partial_v\left(\hat{f}(\tilde{\nabla}_u\psi)^2\right)\right)\\&+\eta'\left(r_u\hat{h}(\tilde{\nabla}_v\psi)^2+r_v\hat{f}(\tilde{\nabla}_u\psi)^2\right).\end{aligned}\tag{3.541}$$

We see there are two regions of interest $r\geq r_M$, and $r_M\geq r\geq r_Z$.

We deal with the former first

$$\begin{aligned}
&\eta\left(\frac{1}{2}\partial_u\left(\hat{h}(\tilde{\nabla}_v\psi)^2\right)+\frac{1}{2}\partial_v\left(\hat{f}(\tilde{\nabla}_u\psi)^2\right)\right)\\
=&-2\left(\kappa-\frac{1}{2}\right)\tilde{\nabla}_u\psi\tilde{\nabla}_v\psi\left(\frac{r_vr_u}{\Omega^2}\right)\\
&+\left(\frac{1}{2}\frac{r_u^2}{\Omega^2}+\frac{2\pi r^2}{\Omega^2}\psi_u^2\right)(\tilde{\nabla}_v\psi)^2+\left(\frac{1}{2}\frac{r_v^2}{\Omega^2}+\frac{2\pi r^2}{\Omega^2}\psi_v^2\right)(\tilde{\nabla}_u\psi)^2\\
&+\frac{1}{4}rV\psi\left(r_u\tilde{\nabla}_v\psi+r_v\tilde{\nabla}_u\psi\right)\\
\geq&-2\left(\kappa-\frac{1}{2}\right)\tilde{\nabla}_u\psi\tilde{\nabla}_v\psi\left(\frac{r_vr_u}{\Omega^2}\right)+\left(\frac{1}{2}\frac{r_u^2}{\Omega^2}\right)(\tilde{\nabla}_v\psi)^2+\left(\frac{1}{2}\frac{r_v^2}{\Omega^2}\right)(\tilde{\nabla}_u\psi)^2\\
&+\frac{1}{4}rV\psi\left(r_u\tilde{\nabla}_v\psi+r_v\tilde{\nabla}_u\psi\right).
\end{aligned}\tag{3.542}$$

We apply Young's inequality to see

$$\begin{aligned}
\frac{1}{2}\partial_u\left(\hat{h}(\tilde{\nabla}_v\psi)^2\right)+\frac{1}{2}\partial_v\left(\hat{f}(\tilde{\nabla}_u\psi)^2\right)\geq&\left(\left(\frac{1}{2}-\left|\kappa-\frac{1}{2}\right|\right)\frac{r_u^2}{\Omega^2}\right)(\tilde{\nabla}_v\psi)^2+\left(\left(\frac{1}{2}-\left|\kappa-\frac{1}{2}\right|\right)\frac{r_v^2}{\Omega^2}\right)(\tilde{\nabla}_u\psi)^2\\
&+\frac{1}{4}rV\psi\left(r_u\tilde{\nabla}_v\psi+r_v\tilde{\nabla}_u\psi\right).
\end{aligned}\tag{3.543}$$

So the first row terms are manifestly positive. We then estimate

$$\begin{aligned}
\frac{1}{4}rV\psi\left(r_u\tilde{\nabla}_v\psi+r_v\tilde{\nabla}_u\psi\right)\leq&\frac{1}{8}r^2\psi^2+\frac{1}{8}V^2\left(r_u^2\left(\tilde{\nabla}_v\psi\right)^2+r_u^2\left(\tilde{\nabla}_u\psi\right)^2\right)\\
\leq&C_{Y,l,M,g}\left(r^2\psi^2+r^{-2+4\kappa}\left(\left(\tilde{\nabla}_u\psi\right)^2+\left(\tilde{\nabla}_v\psi\right)^2\right)\right).
\end{aligned}\tag{3.544}$$

Using the estimates in corollary (3.7.14), we have in this region for r_M chosen large enough

$$\int_D\frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2+(-r_u)\psi^2+\frac{-r_ur^2}{\Omega^2}(\tilde{\nabla}_v\psi)^2+\frac{r_v}{r}\psi^2dudv\leq C_{l,g,M}\mathbb{F}(u,v).\tag{3.545}$$

Then in the latter region

$$\int_D\eta\left(\frac{1}{2}\partial_u\left(\hat{h}(\tilde{\nabla}_v\psi)^2\right)+\frac{1}{2}\partial_v\left(\hat{f}(\tilde{\nabla}_u\psi)^2\right)\right)+\eta'\left(r_u\hat{h}(\tilde{\nabla}_v\psi)^2+r_v\hat{f}(\tilde{\nabla}_u\psi)^2\right)dv.\tag{3.546}$$

We see that as the derivative of η is bounded, and r is bounded above and below in the region where the stronger estimates (3.7.14) hold. We can then trivially bound these terms above by the global estimate (3.8.3). We then combine this higher weighted estimate with the global one to see

$$\int_D\frac{r^4}{-r_u}(\tilde{\nabla}_u\psi)^2+(-r_u)\psi^2+\frac{-r_ur^2}{\Omega^2}(\tilde{\nabla}_v\psi)^2+\frac{r_v}{r}\psi^2dudv\leq C_{l,g,M}\mathbb{F}(u,v).\tag{3.547}$$

□

3.8.5 Exponential decay

Theorem 3.8.5. *Defining the v-flux,*

$$\mathcal{F}(v) = \int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} \left(\frac{r^4}{-r_u} (\tilde{\nabla}_u \psi)^2 + (-r_u)\psi^2 \right) (\bar{u}, v) d\bar{u}, \tag{3.548}$$

and the region $\tilde{D}(v_1, v_2) = D(u_{\mathcal{H}}, v_2) \cap \{v \geq v_1\}$. *Then for* $\kappa < \frac{1}{2}$,

$$\mathcal{F}(v) \leq \tilde{C}_{M,l,g} \mathcal{F}(v_0) e^{-\alpha v}. \tag{3.549}$$

for some uniform $\alpha > 0$.

Proof. Applying the estimate (3.535) on the region $\tilde{D}(v_n, v_{n+1})$ yields the estimate

$$\mathcal{F}(v_{n+1}) + c_{M,l} \int_{v_n}^{v_{n+1}} \mathcal{F}(\bar{v}) d\bar{v} \leq C_{M,l,g} \mathcal{F}(v_n). \tag{3.550}$$

Now take $v \leq v_n \leq v_{n+1}$ and thus

$$\int_{v_n}^{v_{n+1}} \mathcal{F}(\bar{v}) d\bar{v} \leq \int_{v}^{v_{n+1}} \mathcal{F}(\bar{v}) d\bar{v} \leq \frac{C_{M,l,g}}{c_{M,l}} \mathcal{F}(v), \tag{3.551}$$

implying

$$c_{M,l}(v_{n+1} - v_n) \cdot \inf_{\nu \in [v_n, v_{n+1}]} F(\nu) \leq c_{M,l} \int_{v_n}^{v_{n+1}} \mathcal{F}(v) dv \leq C_{M,l,g} \mathcal{F}(v). \tag{3.552}$$

Choosing $v_n = 2^n + v$ yields

$$\inf_{\nu \in [2^n + v, 2^{n+1} + v]} \mathcal{F}(\nu) \leq \frac{C_{M,l,g}}{c_{M,l} 2^n} \mathcal{F}(v). \tag{3.553}$$

Now take $V \in (v, \infty)$, find $n \in \mathbb{Z}$ such that $2^{n+1} < v - V \leq 2^{n+2}$, then

$$\mathcal{F}(V) \leq C_{M,l,g} \inf_{\nu \in [2^n + v, 2^{n+1} + v]} \mathcal{F}(\nu) \leq \frac{C_{M,l,g}^2}{c_{M,l} 2^n} \mathcal{F}(v). \tag{3.554}$$

As we have $V - v \leq 2^{n+2}$, we deduce

$$\mathcal{F}(V) \leq \frac{4C_{M,l,g}^2}{c_{M,l}(V - v)} \mathcal{F}(v), \tag{3.555}$$

which holds for $0 \leq v < V$.
Now define $K = \frac{4C_{M,l,g}^2}{c_{M,l}}$, and choose $V = eK$. We then have

$$\mathcal{F}(v + V) \leq \frac{1}{e} \mathcal{F}(v). \tag{3.556}$$

Now assume that the following holds for $n \in \mathbb{N}$

$$\mathcal{F}(v+nV) \leq \frac{1}{e^n}\mathcal{F}(v), \tag{3.557}$$

so we see that

$$\mathcal{F}(v+(n+1)V) = \mathcal{F}((v+V)+nV) \leq \frac{1}{e^n}\mathcal{F}(v+V) \leq \frac{1}{e^{n+1}}\mathcal{F}(v). \tag{3.558}$$

Hence we have that

$$\mathcal{F}(v+nV) \leq \frac{1}{e^n}\mathcal{F}(v), \tag{3.559}$$

holds by induction.
Now take $\nu \geq 0$ write it as $\nu = nV + v$, where $0 \leq v \leq V$. We then have that

$$\mathcal{F}(\nu) \leq \frac{1}{e^n}f(v) \leq \frac{C_{M,l,g}}{e^n}\mathcal{F}(v_0) = \frac{C_{M,l,g}}{e^{\frac{\nu}{V}-\frac{v}{V}}}\mathcal{F}(v_0) \leq C_{M,l,g}\mathcal{F}(v_0)e^{-\frac{\nu}{V}+1} \leq \tilde{C}_{M,l,g}\mathcal{F}(v_0)e^{-\alpha\nu} \tag{3.560}$$

for some $\alpha > 0$. □

Corollary 3.8.1. *We have in $\mathscr{R}_{\mathcal{H}}$ for $\kappa < \frac{1}{2}$, there exists a constant $\alpha > 0$ such that*

$$\sup_u |2\chi(u,v) - 1| + \sup_u |\varpi(u,v) - M| \leq \tilde{C}_{M,l,g}\exp(-\alpha \cdot v), \tag{3.561}$$

and

$$|\psi(u,v)| \leq \tilde{C}_{M,l,g} r^{-\frac{3}{2}+\kappa}\exp(-\alpha \cdot v). \tag{3.562}$$

Proof. This follows from (3.267) and (3.549). □

Remark 3.8.5. *It is in this sense that we say the metric is converging to a Toroidal Schwarzschild-AdS metric of mass M, exponentially in v, in the Eddington Finkelstein gauge.*

Corollary 3.8.2. *In $\mathscr{R}_{\mathcal{H}}$, for $\kappa < \frac{1}{2}$, there exists a constant $\alpha > 0$ such that*

$$\left|\frac{\varpi_1(u,v) - M}{r^{2\kappa}}\right| \leq \tilde{C}_{M,l,g}\exp(-\alpha \cdot v). \tag{3.563}$$

Proof. Write

$$\begin{aligned}\frac{\varpi_1(u,v) - M}{r^{2\kappa}} =& r^{-2\kappa}(\varpi(u,v) - M) \\ &+ r^{-2\kappa}\left(e^{-4\pi g\psi^2} - 1\right)\varpi - \frac{r^{3-2\kappa}}{2l^2}\left(e^{-4\pi g\psi^2} - 1\right).\end{aligned} \tag{3.564}$$

Taking absolute values the result then follows from theorem 3.7.1 and corollary 3.8.1. □

Corollary 3.8.3. *We have that for $\kappa < \frac{1}{2}$, the Lorentzian Penrose inequality*

$$\sup_{\mathcal{H}} r \leq r_+. \tag{3.565}$$

Furthermore we have along $\mathcal{H}$ that r converges to r_+ exponentially in v.

Proof. This is an adaptation of the proof of proposition 3.7 in [HS13b] to this setting.
Assume for contradiction that $r \geq r_+ + \delta$ for some $\delta > 0$, held along $\mathcal{H}$. Then by corollary 3.8.2 we have the existence of a $v_i > v_0$ such that

$$\frac{\mu_1}{r^2} \geq \left(\frac{1}{l^2} - \frac{2M}{r^3}\right) - \frac{2\,|M - \varpi_1|}{r^3} \geq c_{M,l}\delta, \tag{3.566}$$

holds on $\mathcal{H} \cap \{v \geq v_i\}$. Now integrating $r_v r^{-2}$ we see

$$\int_{v_i}^{v} \frac{r_v}{r^2} d\bar{v} = -\frac{1}{r(v)} + \frac{1}{r(v_i)} \leq C, \tag{3.567}$$

holds for some uniform $C > 0$. However from (3.566) we have

$$\int_{v_i}^{r_v} \frac{r_v}{r^2} d\bar{v} = \int_{v_i}^{v} \frac{\mu}{r^2} \chi d\bar{v} \geq C_{M,l}\delta \cdot (v - v_i)\,, \tag{3.568}$$

which is clearly a contradiction.
Now that we have seen r is bounded along $\mathcal{H}$, we can prove the exponential decay through an integrated decay estimate.

$$\begin{aligned}\int_{v}^{\infty} (r_+ - r(u_{\mathcal{H}}, \bar{v}))\, d\bar{v} &\leq C_{M,l} \int_{v}^{\infty} \mu_1 d\bar{v} + \tilde{C}_{M,l,g} e^{(-B_{M,l,g}v)} \\ &\leq C_{M,l} \int_{v}^{\infty} r_v d\bar{v} + \tilde{C}_{M,l,g} e^{-B_{M,l,g}v} \\ &\leq C_{M,l}\,(r_+ - r(u_{\mathcal{H}}, v)) + \tilde{C}_{M,l,g} e^{-B_{M,l,g}v}.\end{aligned} \tag{3.569}$$

From the positivity of $r_+ - r(u_{\mathcal{H}}, v)$ we derive the integrated decay statement

$$\int_{v_1}^{v_2} (r_+ - r(u_{\mathcal{H}}, \bar{v}))\, d\bar{v} \leq C_{M,l}\,(r_+ - r(u_{\mathcal{H}}, v_1)) + \tilde{C}_{M,l,g} e^{-B_{M,l,g}v_1}. \tag{3.570}$$

Exponential decay follows in a similar manner to theorem 3.8.5. □

3.9 The main theorem

Theorem 3.9.1. *Consider a weak solution of the Einstein–Klein-Gordon system arising from small initial data, within the class of square flat toroidal symmetries with ψ satisfying Dirichlet or Neumann boundary conditions and a Klein-Gordon mass bound $\kappa \leq \frac{1}{2}$. The associated maximal development of the solution is a black hole spacetime, with a regular future horizon and a complete null infinity. Furthermore for $\kappa < \frac{1}{2}$ the estimates of* (3.267) *and* (3.442) *hold for any (u, v) in the regular region exterior to the apparent horizon. This implies that ψ decays exponentially in v on this region.*

We may remark that we can use these techniques to study toroidally symmetric solutions of the Klein-Gordon equation on a fixed toroidal AdS Schwarzschild background. In this decoupled setting we have $\chi = \gamma = \frac{1}{2}$ and $\mathcal{T} = \partial_t$ and the following corollary.

Corollary 3.9.1. *Let $(\mathcal{M}, g)$ be a fixed toroidal Schwarzschild AdS spacetime with Eddington Finkelstein coordinate system (u, v). Let $\kappa < \frac{1}{2}$ then the toroidally symmetric solutions of the Klein-Gordon equation decay exponentially in the v coordinate on the black hole exterior.*

It is worth contrasting this with the non symmetric results of chapter where only polynomial decay can be established.

3.10 Vacuum result

So far the restriction to a square flat toroidal symmetry was to emulate the spherical symmetry situation of a Birkhoff theorem. As seen in [Gow74] there are more degrees of freedom to a flat metric on a torus than a round metric on a sphere. We can evade trivial vacuum dynamics within a rectangular flat toroidal symmetry class by making the following metric ansatz

$$g = -\Omega^2(u,v)dudv + r^2(u,v)\left(e^{-\sqrt{8\pi}B(u,v)}dx^2 + e^{\sqrt{8\pi}B(u,v)}dy^2\right). \tag{3.571}$$

Here the periods of tori are allow to vary under a scalar field B. At different points (u, v), we get rectangular tori which, unlike the case $B =$ *constant* we cannot scale back to a unit torus through coordinate transformations of x and y. If we study the now dynamical vacuum equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{3}{l^2}g_{\mu\nu} = 0, \tag{3.572}$$

with this ansatz, they reduce to

$$\partial_u\left(\frac{r_u}{\Omega^2}\right) = -4\pi r\frac{(B_u)^2}{\Omega^2}, \tag{3.573}$$

$$\partial_v\left(\frac{r_v}{\Omega^2}\right) = -4\pi r\frac{(B_v)^2}{\Omega^2}, \tag{3.574}$$

$$r_{uv} = -\frac{r_u r_v}{r} - \frac{3}{4}\frac{r}{l^2}\Omega^2, \tag{3.575}$$

$$(\log\Omega)_{uv} = \frac{r_u r_v}{r^2} - 4\pi B_u B_v, \tag{3.576}$$

$$B_{uv} = -\frac{r_u}{r}B_v - \frac{r_v}{r}B_u. \tag{3.577}$$

We notice this system is equivalent to (3.8) - (3.12) where the Klein-Gordon field is massless ($a = 0$). In contrast to the Bianchi IX system as studied in [Dol17a], the scalar curvature of the group orbits is 0. Consequently (3.577) is a linear wave equation, making the analysis much simpler. However as $a = 0$ corresponds to $\kappa = \frac{3}{2}$, we cannot currently hope to pose any other boundary conditions other than Dirichlet. Intuitively this makes sense, imposing Dirichlet boundary conditions would mean fixing the periods of the torus at null infinity. Unfortunately the main results of this thesis cannot be directly used. However as discussed there are only very minor differences between the spherical and toroidal systems (at the reduced equation level). One could study this system in the same way as in [HS12], and [HS13b], (many of the similarities have already been exhibited) to deduce stability.

Theorem 3.10.1. *Consider an initial free data set* $(\bar{r}, \bar{B})$*, obeying the* ϵ*-perturbed Schwarzschild-AdS data set conditions (as defined in [HS13b]) with* $a = 0$ *on a null ray* $N(v_0)$*, and Dirichlet boundary conditions. The associated maximal development is a black hole spacetime with a regular future horizon, and a complete null infinity. Furthermore the estimate*

$$\left| r^{\frac{3}{2}-\kappa} B(u, v) \right| \leq D \exp(-Cv), \tag{3.578}$$

holds on the intersection of the regular region of the spacetime and the exterior of the black hole. From which we may deduce that the metric is converging exponentially in v*, uniformly in* u*, to a toroidal AdS Schwarzschild solution with mass* M*, in the Eddington-Finkelstein gauge.*

Thus the toroidal AdS black hole is indeed a stable solution to the vacuum equations within the symmetry class imposed by the metric ansatz.

References

[AAG18] Y. Angelopoulos, S. Aretakis, and D. Gajic. Late-time asymptotics for the wave equation on spherically symmetric, stationary spacetimes. *Advances in Mathematics*, 323:529–621, January 2018.

[AB15] L. Andersson and P. Blue. Hidden symmetries and decay for the wave equation on the Kerr spacetime. *Annals of Mathematics*, 182(3):787–853, November 2015.

[ACD02] M. Anderson, P. T. Chrusciel, and E. Delay. Non-trivial, static, geodesically complete, vacuum space-times with a negative cosmological constant. *Journal of High Energy Physics*, 2002(10):063–063, October 2002.

[Aea17] B. P. Abbott et al. GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence. *Physical Review Letters*, 119(14), October 2017.

[Are11] S. Aretakis. Stability and instability of extreme Reissner-Nordström black hole spacetimes for linear scalar perturbations I. *Communications in mathematical physics*, 307(1):17, 2011.

[BC05] R. Beig and P. T. Chruściel. Stationary Black Holes. *arXiv:gr-qc/0502041*, February 2005. arXiv: gr-qc/0502041.

[BF82] P. Breitenlohner and D. Freedman. Stability in gauged extended supergravity. *Annals of Physics*, 144(2):249–281, December 1982.

[Bie09] L. Bieri. An extension of the stability theorem of the Minkowski space in general relativity. *J. Differential Geom. 86 (2010), no. 1, 17–70.*, April 2009.

[Bir99] D. Birmingham. Topological black holes in anti-de Sitter space. *Classical and Quantum Gravity*, 16(4):1197–1205, April 1999.

[BS06] P. Blue and A. Soffer. Errata for "Global existence and scattering for the nonlinear Schrodinger equation on Schwarzschild manifolds", "Semilinear wave equations on the Schwarzschild manifold I: local decay estimates", and "The wave equation on the Schwarzschild metric II: Local decay for the spin 2 Regge Wheeler equation". *arXiv:0608073 [gr-qc]*, August 2006.

[Buc59] H. A. Buchdahl. General relativistic fluid spheres. *Phys. Rev.*, 116, November 1959.

[CAN13] L. Costa, A. Alho, and J. Natário. The problem of a self-gravitating scalar field with positive cosmological constant. *Annales Henri Poincaré*, 14(5):1077–1107, July 2013.

[CBG69] Y. Choquet-Bruhat and R. Geroch. Global aspects of the Cauchy problem in general relativity. *Communications in Mathematical Physics*, 14(4):329–335, 1969.

[Chr86a] D. Christodoulou. Global existence of generalized solutions of the spherically symmetric Einstein-scalar equations in the large. *Communications in Mathematical Physics*, 106(4):587–621, 1986.

[Chr86b] D. Christodoulou. The problem of a self-gravitating scalar field. *Communications in Mathematical Physics*, 105(3):337–361, 1986.

[Chr87a] D. Christodoulou. A mathematical theory of gravitational collapse. *Communications in Mathematical Physics*, 109(4):613–647, 1987.

[Chr87b] D. Christodoulou. The structure and uniqueness of generalized solutions of the spherically symmetric Einstein-scalar equations. *Communications in Mathematical Physics*, 109(4):591–611, 1987.

[Chr98] D. Christodoulou. The instability of naked singularities in the gravitational collapse of a scalar field. *Annals of Mathematics*, Second Series, Vol. 149, No. 1 (Jan., 1999):183–217, December 1998.

[Chr09] D. Christodoulou. *The formation of black holes in general relativity.* European Mathematical Society, 2009.

[Chr16] D. Christodoulou. *The action principle and partial differential equations. (AM-146).* Princeton University Press, March 2016.

[CK93] D. Christodoulou and S. Klainerman. *The global nonlinear stability of the Minkowski space.* Princeton : Princeton University Press, 1993.

[Daf05] M. Dafermos. Spherically symmetric spacetimes with a trapped surface. *Classical and Quantum Gravity*, 22(11):2221–2232, June 2005.

[DH06] M. Dafermos and G. Holzegel. On the nonlinear stability of higher-dimensional triaxial Bianchi IX black holes. *Adv. Theor. Math. Phys. 10 (2006)*, pages 503–523, 2006.

[Dol17a] D. Dold. Global dynamics of asymptotically locally AdS spacetimes with negative mass. *arXiv:1711.06700 [gr-qc, physics:hep-th]*, November 2017.

[Dol17b] D. Dold. Unstable mode solutions to the Klein-Gordon equation in Kerr-anti-de Sitter spacetimes. *Communications in Mathematical Physics*, 350(2):639–697, March 2017.

[DR05] M. Dafermos and I. Rodnianski. A proof of Price's law for the collapse of a self-gravitating scalar field. *Inventiones mathematicae*, 162(2):381–457, 2005.

[DR07a] M. Dafermos and I. Rodnianski. A note on energy currents and decay for the wave equation on a Schwarzschild background. *arXiv:0710.0171 [gr-qc]*, September 2007.

[DR07b] M. Dafermos and I. Rodnianski. The wave equation on Schwarzschild-de Sitter spacetimes. *arXiv:0709.2766 [gr-qc]*, September 2007.

[DR08] M. Dafermos and I. Rodnianski. Lectures on black holes and linear waves. *arXiv:0811.0354 [gr-qc, physics:math-ph]*, November 2008.

[DR09a] M. Dafermos and I. Rodnianski. A new physical-space approach to decay for the wave equation with applications to black hole spacetimes. *XVIth International Congress on Mathematical Physics*, pages 421–433, October 2009.

[DR09b] M. Dafermos and I. Rodnianski. The red-shift effect and radiation decay on black hole spacetimes. *Communications on Pure and Applied Mathematics*, 62(7):859–919, July 2009.

[DR10] M. Dafermos and I. Rodnianski. Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases |a| << M or axisymmetry. *arXiv:1010.5132 [gr-qc]*, October 2010.

[DRSR14] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman. Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case |a|< M. *Annals of Mathematics*, 183(3):787–913, 2014.

[Dun14] J. Dunn. Stability problems in AdS spacetimes. Master's thesis, The University of Warwick, August 2014.

[DW16] J. Dunn and C. Warnick. The Klein–Gordon equation on the toric AdS-Schwarzschild black hole. *Classical and Quantum Gravity*, 33(12):125010, 2016.

[Dya10] S. Dyatlov. Exponential energy decay for Kerr-de Sitter black holes beyond event horizons. *Mathematical Research Letters*, 18(5), October 2010.

[Fra16] A. Franzen. Boundedness of massless scalar waves on Reissner-Nordström interior backgrounds. *Communications in Mathematical Physics*, 343(2):601–650, April 2016.

[Fri86] H. Friedrich. On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure. *Communications in Mathematical Physics*, 107(4):587–609, 1986.

[Fri95] H. Friedrich. Einstein equations and conformal structure: existence of anti-de Sitter-type space-times. *Journal of Geometry and Physics*, 17(2):125–184, 1995.

[Fri09] H. Friedrich. Initial boundary value problems for Einstein's field equations and geometric uniqueness. *General Relativity and Gravitation*, 41(9):1947–1966, 2009.

[Gow74] R. Gowdy. Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: Topologies and boundary conditions. *Annals of Physics*, 83(1):203–241, March 1974.

[HE73] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*, volume Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York, 1973.

[Hem04] S. Hemming. *Aspects of quantum fields and strings on AdS black holes.* PhD Thesis, University of Helsinki, 2004.

[HLSW15] G. Holzegel, J. Luk, J. Smulevici, and C. Warnick. Asymptotic properties of linear field equations in anti-de Sitter space. *arXiv preprint arXiv:1502.04965*, 2015.

[HM98] Gary T. Horowitz and Robert C. Myers. AdS-CFT correspondence and a new positive energy conjecture for general relativity. *Physical Review D*, 59(2):026005, December 1998.

[Hol10a] G. Holzegel. On the massive wave equation on slowly rotating Kerr-AdS spacetimes. *Communications in Mathematical Physics*, 294(1):169, February 2010.

[Hol10b] G. Holzegel. Stability and decay-rates for the five-dimensional Schwarzschild metric under biaxial perturbations. *Advances in Theoretical and Mathematical Physics*, 14(5):1245–1372, 2010.

[Hol12] G. Holzegel. Well-posedness for the massive wave equation on asymptotically anti-de Sitter spacetimes. *Journal of Hyperbolic Differential Equations*, 09, June 2012.

[HR99] S. W. Hawking and H. S. Reall. Charged and rotating AdS black holes and their CFT duals. *Physical Review D*, 61(2), December 1999.

[HS12] G. Holzegel and J. Smulevici. Self-gravitating Klein–Gordon fields in asymptotically anti-de-Sitter spacetimes. *Annales Henri Poincaré*, 13(4):991–1038, May 2012.

[HS13a] G. Holzegel and J. Smulevici. Decay properties of Klein-Gordon Fields on Kerr-AdS spacetimes. *Communications on Pure and Applied Mathematics*, 66(11):1751–1802, November 2013.

[HS13b] G. Holzegel and J. Smulevici. Stability of Schwarzschild-AdS for the spherically symmetric Einstein-Klein-Gordon system. *Communications in Mathematical Physics*, 317(1):205–251, January 2013.

[HV16] P. Hintz and A. Vasy. The global non-linear stability of the Kerr-de Sitter family of black holes. *arXiv:1606.04014 [gr-qc, physics:math-ph]*, June 2016.

[HW13] G. Holzegel and C. Warnick. The Einstein-Klein-Gordon-AdS system for general boundary conditions. *Journal of Hyperbolic Differential Equations*, 12, December 2013.

[HW14] G. Holzegel and C. Warnick. Boundedness and growth for the massive wave equation on asymptotically anti-de Sitter black holes. *Journal of Functional Analysis*, 266(4):2436–2485, February 2014.

[Ker63] R. Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Physical Review Letters*, 11(5):237–238, September 1963.

[Kla87] S. Klainerman. Remarks on the global sobolev inequalities in the minkowski space Rn+1. *Communications on Pure and Applied Mathematics*, 40(1):111–117, January 1987.

[Kod80] H. Kodama. Conserved energy flux for the spherically symmetric system and the backreaction problem in the black hole evaporation. *Progress of Theoretical Physics*, 63(4):1217–1228, April 1980.

[Kom13] J. Kommemi. The global structure of spherically symmetric charged scalar field spacetimes. *Communications in Mathematical Physics*, 323(1):35–106, August 2013.

[Kot18] F. Kottler. Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie. *Annalen der Physik*, 361(14):401–462, January 1918.

[KRS15] S. Klainerman, I. Rodnianski, and J. Szeftel. The bounded L2 curvature conjecture. *Inventiones mathematicae*, 202(1), August 2015.

[Lad85] O. Ladyzhenskaya. *The boundary value problems of mathematical physics*, volume 49 of *Applied mathematical sciences*. Springer-Verlag, 1985.

[Lem95] J. Lemos. Two-dimensional black holes and planar general relativity. *Classical and Quantum Gravity*, 12(4):1081–1086, April 1995.

[LM17] P. LeFloch and Y. Ma. The global nonlinear stability of Minkowski space. Einstein equations, f(R)-modified gravity, and Klein-Gordon fields. *arXiv:1712.10045 [gr-qc]*, December 2017.

[LR05] H. Lindblad and I. Rodnianski. Global existence for the Einstein vacuum equations in wave coordinates. *Communications in Mathematical Physics*, 256(1):43–110, May 2005.

[LS00] I. Laba and A. Soffer. Global existence and scattering for the nonlinear Schrodinger equation on Schwarzschild manifolds. *arXiv:math-ph/0002030*, February 2000.

[LS15] P. LeFloch and J. Smulevici. Weakly regular T2 symmetric spacetimes. The global geometry of future developments. *Journal of the European Mathematical Society*, 17(5):1883–1292, 2015.

[LT17] H. Lindblad and M. Taylor. Global stability of Minkowski space for the Einstein–Vlasov system in the harmonic gauge. *arXiv:1707.06079 [gr-qc]*, July 2017.

[Luk10] J. Luk. Improved decay for solutions to the linear wave equation on a Schwarzschild black hole. *Annales Henri Poincaré*, 11(5):805–880, October 2010.

[Mal99] J. Maldacena. The large N limit of superconformal field theories and supergravity. *International Journal of Theoretical Physics*, 38(4):1113–1133, 1999.

[Mor61] C. Morawetz. The decay of solutions of the exterior initial-boundary value problem for the wave equation. *Communications on Pure and Applied Mathematics*, 14(3):561–568, August 1961.

[Mor66] C. Morawetz. Exponential decay of solutions of the wave equation. *Communications on Pure and Applied Mathematics*, 19(4):439–444, November 1966.

[Mor68] C. Morawetz. Time decay for the nonlinear Klein-Gordon equation. *Proc. R. Soc. Lond. A*, 306(1486):291–296, September 1968.

[Sbi15] J. Sbierski. Characterisation of the energy of Gaussian beams on Lorentzian manifolds: with applications to black hole spacetimes. *Anal. PDE*, 8(6):1379–1420, 2015.

[Sbi16] J. Sbierski. On the existence of a maximal Cauchy development for the Einstein equations - a Dezornification. *Annales Henri Poincaré*, 17(2):301–329, February 2016.

[Sch16] K. Schwarzschild. Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie. *Sitzungsber. d. Preuss. Akad. d. Wissenschaften*, 1916.

[SW10] K. Schleich and D. Witt. A simple proof of Birkhoff's theorem for cosmological constant. *Journal of Mathematical Physics*, 51(11):112502, November 2010.

[Tay17] M. Taylor. The global nonlinear stability of Minkowski space for the massless Einstein–Vlasov system. *Annals of PDE*, 3(9), June 2017.

[War13] C. Warnick. The massive wave equation in asymptotically AdS spacetimes. *Communications in Mathematical Physics*, 321(1):85–111, July 2013.

[War15] C. Warnick. On quasinormal modes of asymptotically anti-de Sitter black holes. *Communications in Mathematical Physics*, 333(2):959–1035, January 2015.

[Wya17] Z. Wyatt. The weak null condition and Kaluza-Klein spacetimes. *arXiv:1706.00026 [gr-qc]*, May 2017.

A

Appendix

A.1 Introduction

In this appendix we consider a slightly lower regularity local wellposedness result, and extension principle for the Einstein–Klein-Gordon system. This extension principle is motivated heavily by the work of Kommemi [Kom13], however for the problem of interest we need to relax the regularity of ψ_v to being in L^2 as opposed to in C^0.

A.2 Local Wellposedness in the Interior

A.2.1 The system

We will consider the system

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} &= 8\pi T_{\mu\nu}, \\ T_{\mu\nu} &= \nabla_\mu \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \psi \nabla^\sigma \psi - \frac{m}{2} g_{\mu\nu}\psi^2, \\ \Box_g \psi - m\psi &= 0, \end{aligned} \tag{A.1}$$

where m and Λ are real parameters. We then further restrict to a metric of the form

$$g = -\Omega^2(u,v)dudv + r^2(u,v)d\sigma_k^2, \tag{A.2}$$

where

$$d\sigma_k^2 = \begin{cases} d\theta^2 + \sin^2(\theta)\, d\varphi^2 & \text{for } k = 1 \\ dx^2 + dy^2 & \text{for } k = 0 \\ d\rho^2 + \sinh^2(\rho)\, d\varphi^2 & \text{for } k = -1 \end{cases} \tag{A.3}$$

and u, v are null coordinates.
The system then reduces to

$$\partial_u \left(\frac{r_u}{\Omega^2} \right) = -4\pi r \frac{\psi_u^2}{\Omega^2} \tag{A.4}$$

$$\partial_v \left(\frac{r_v}{\Omega^2} \right) = -4\pi r \frac{\psi_v^2}{\Omega^2}, \tag{A.5}$$

$$r_{uv} = -\frac{r_u r_v}{r} + \pi m r \psi^2 \Omega^2 + \frac{\Lambda}{4} r \Omega^2 - k \frac{\Omega^2}{4r}, \tag{A.6}$$

$$(\log \Omega)_{uv} = -4\pi \psi_u \psi_v + \frac{r_u r_v}{r^2} + k \frac{\Omega^2}{4r^2}, \tag{A.7}$$

$$\psi_{uv} = -\frac{r_v}{r}\psi_u - \frac{r_u}{r}\psi_v - \frac{m}{4}\Omega^2 \psi. \tag{A.8}$$

We remark that a choice of

$$\Lambda = \frac{-3}{l^2}, \quad m = \frac{2a}{l^2}, \quad k = 0, \tag{A.9}$$

reduces us to the AdS system we are interested in.

A.2.2 The domain

Define $X = [0, d] \times \{0\} \cup \{0\} \times [0, d]$, the two null rays emanating from the origin. We then let $0 < \delta \leq d$, and define

$$\square_\delta = [0, \delta] \times [0, \delta], \tag{A.10}$$

to be our domain.

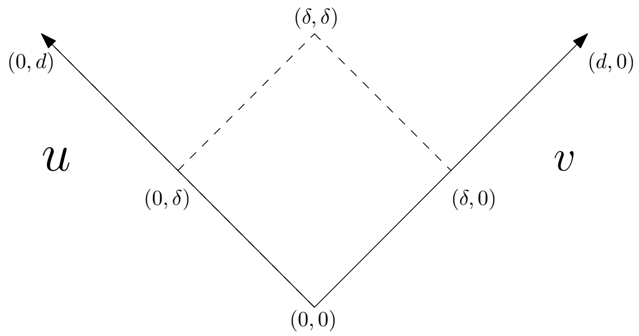


Figure A.1: Diagram of the sets.

Then we denote the restricted initial data rays as

$$X' = [0, \delta] \times \{0\} \cup \{0\} \times [0, \delta]. \tag{A.11}$$

Proposition A.2.1 (Local Wellposedness)**.** *Consider the set $X = [0, d] \times \{0\} \cup \{0\} \times [0, d]$. Let $r \in C^1(X)$ be a positive function with the additional property $r_{uv}, r_{uu} \in C^0(X)$. Let $\Omega \in C^0(X)$ also be a positive function with $\Omega_u \in C^0(X)$. Let $\psi \in C^0 H^1(X)$ with $\psi_u \in C^0(X)$. Suppose that the equations* (3.8) *and* (3.9) *hold on $[0, d] \times \{0\}$, and $\{0\} \times [0, d]$ respectively. And let*

C_u^n denote the $C^n(u)$ norm on $[0,d] \times \{0\}$ and C_v^n denote the $C^n(v)$ norm on $\{0\} \times [0,d]$. We adopt a similar notation for the Sobolev norms. For a domain $\mathcal{D}$ define the following norm

$$N(\mathcal{D}) = \sup \Big\{ ||\psi||_{C_v^0 H_u^1}, ||\psi||_{C_u^0 H_v^1}, ||\psi_u||_{C_u^0}, ||\psi_u||_{C_v^0}, ||r||_{C_u^1}, ||r||_{C_v^1}, ||r_{uu}||_{C_u^0}, ||r_{uu}||_{C_v^0}, \\ ||r_{uv}||_{C_u^0}, ||r_{uv}||_{C_v^0}, \left|\left|r^{-1}\right|\right|_{C_u^0}, \left|\left|r^{-1}\right|\right|_{C_v^0}, ||\Omega||_{C_u^0}, ||\Omega||_{C_v^0}, ||\Omega_u||_{C_u^0}, ||\Omega_u||_{C_v^0}, \left|\left|\Omega^{-1}\right|\right|_{C_u^0}, \left|\left|\Omega^{-1}\right|\right|_{C_v^0} \Big\}. \tag{A.12}$$

Where all the norms are taken to be over $\mathcal{D}$.
Further more assume that

$$\frac{1}{N} \leq \Omega \leq N, \tag{A.13}$$

and

$$0 < r_0 \leq r \leq R < \infty. \tag{A.14}$$

Then there exists a $\delta > 0$ depending only on N, such that the Einstein–Klein-Gordon system has a unique $\mathfrak{W}$ solution (as defined in definition 3.4.5) on the set $[0,\delta] \times [0,\delta]$.

A.2.3 Function spaces

We define the metrics

$$d_\Omega(\Omega_1, \Omega_2) = \left|\left|\log\left(\frac{\Omega_1}{\Omega_2}\right)\right|\right|_{C^0} + \left|\left|\log\left(\frac{\Omega_1}{\Omega_2}\right)_u\right|\right|_{C^0}, \tag{A.15}$$

$$d_r(r_1, r_2) = ||r_1 - r_2||_{C_0} + ||r_{1,u} - r_{2,u}||_{C_0} + ||r_{1,v} - r_{2,v}||_{C_0} + ||r_{1,uv} - r_{2,uv}||_{C_0} + ||r_{1,uu} - r_{2,uu}||_{C_0}, \tag{A.16}$$

and finally

$$d_\psi(\psi_1, \psi_2) = ||\psi_1 - \psi_2||_{C^0 H^1} + ||\psi_1 - \psi_2||_{C^0} + ||\psi_{1,u} - \psi_{2,u}||_{C^0}, \tag{A.17}$$

where we recall that

$$||\psi||^2_{C^0 H^1} = \sup_{(u,v)\in\square_\delta} \int_0^u \psi_u^2 + \psi^2 du + \sup_{(u,v)\in\square_\delta} \int_0^v \psi_v^2 + \psi^2 dv. \tag{A.18}$$

For notational ease we will define $\Sigma_i = (r_i, \Omega_i^2, \psi_i)$ to be a solution triple, and denote the metric on this space by

$$d(\Sigma_1, \Sigma_2) = d_r(r_1, r_2) + d_\Omega(\Omega_1, \Omega_2) + d_\psi(\psi_1, \psi_2). \tag{A.19}$$

We then define

$$\overline{\Omega} = \frac{\Omega(u,0)\Omega(0,v)}{\Omega(0,0)}, \tag{A.20}$$

we will insist on $\bar{\Omega} > \frac{1}{N} > 0$. Where N is a constant.
Furthermore define

$$\overline{r} = r(0,v) + r(u,0) - r(0,0), \tag{A.21}$$

$$\overline{\psi} = \frac{1}{2}\left(\psi(0,v) + \psi(u,0)\right). \tag{A.22}$$

We now define $C^{1+}_r(\Box_\delta)$ to be the space of positive functions $r \in C^1(\Box_\delta)$, such that r_{uv}, $r_{uu} \in C^0(\Box_\delta)$, and agree with $\overline{r}$ on X'. We define $C^{0+}_\psi H^1(\Box_\delta)$ as the space of continuously differentiable in u functions, that agree with $\bar{\psi}$ on X', and are both continuous in u with values in $H^1_v(\Box_\delta)$ and continuous in v with values in $H^1_u(\Box_\delta)$. We also define $C^{0+}_\Omega(\Box_\delta)$as the space of real valued functions Ω, such that Ω are C^0 in $\Box_\delta$ agree with $\bar{\Omega}$ on X', and are such that the u derivative exists and is continuous. We then define $\mathcal{C} = C^{1+}_r(\Box_\delta) \times C^{0+}_\psi H^1(\Box_\delta) \times C^{0+}_\Omega(\Box_\delta)$ and equip it with the the metric $d(\Sigma_1, \Sigma_2)$. This is then a complete metric space.
We then define the ball of radius b by these norms and centre it at $(\overline{r}, \overline{\Omega}, \bar{\psi})$. We denote it by $\mathcal{B}_b$.

Domain restriction

We from continuity we can restrict δ, such that on the restricted initial data ray we have

$$||\psi(0,v)||_{C^0H^1([0,\delta])} < \frac{b}{8}, \tag{A.23}$$

$$||\psi(u,0)||_{C^0H^1([0,\delta])} < \frac{b}{8}, \tag{A.24}$$

$$||\psi_u(u,0)||_{C^0([0,\delta])} < \frac{b}{8}, \tag{A.25}$$

$$||\overline{\Omega}||_{C^0(X')} + ||\overline{\Omega}_u||_{C^0(X')} < N. \tag{A.26}$$

A.2.4 The map

We define a map from $C^{1+}_r(\Box_\delta) \times C^{0+}_\psi H^1(\Box_\delta) \times C^{0+}_\Omega(\Box_\delta)$ by

$$\log \hat{\Omega}(u,v) = \log \bar{\Omega} + \int_0^u \int_0^v -4\pi \psi_u \psi_v + \frac{r_u r_v}{r^2} + k\frac{\Omega^2}{4r^2} dv du, \tag{A.27}$$

$$\hat{r}(u,v) = \bar{r} + \int_0^u \int_0^v -\frac{r_u r_v}{r} + \pi m r \psi^2 \Omega^2 + \frac{\Lambda}{4} r\Omega^2 - k\frac{\Omega^2}{4r} du dv, \tag{A.28}$$

$$\begin{aligned} \hat{\psi} = \text{unique } H^1 \text{ solution of: } & \partial_u \partial_v \hat{\psi} = -\frac{r_v}{r}\psi_u - \frac{r_u}{r}\psi_v - \frac{m}{4}\Omega^2 \psi \\ & \text{with initial data } \bar{\psi}|_{\{v=0\}} =: \psi_2 \text{ and boundary data } \bar{\psi}|_{\{u=0\}} =: \psi_1. \end{aligned} \tag{A.29}$$

A.2.5 Useful lemmas

Lemma A.2.1. *let $\psi \in C^{0+}_\psi H^1(\Box_\delta)$ then*

$$\int_0^u \int_0^v \psi_u \psi_v dv du \leq \delta \, ||\psi||^2_{C^0H^1}. \tag{A.30}$$

Proof. By the Cauchy Schwartz inequality we have

$$\begin{aligned}\int_0^u \int_0^v \psi_u \psi_v dvdu &\leq \left(\int_0^u \int_0^v \psi_u^2 dvdu\right)^{\frac{1}{2}} \left(\int_0^u \int_0^v \psi_v^2 dvdu\right)^{\frac{1}{2}} \\ &= \left(\int_0^v \int_0^u \psi_u^2 dudv\right)^{\frac{1}{2}} \left(\int_0^u \int_0^v \psi_v^2 dvdu\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^v ||\psi||_{C^0H^1}^2 \, dv\right)^{\frac{1}{2}} \left(\int_0^u ||\psi||_{C^0H^1}^2 \, du\right)^{\frac{1}{2}} \\ &= \delta \, ||\psi||_{C^0H^1}^2 \, .\end{aligned} \tag{A.31}$$

□

A.2.6 Mapping back to the ball

$\hat{\Omega}$ Estimates

$\log(\hat{\Omega})$ estimates

As $\Omega \in \mathcal{B}_b$, we get the following estimate

$$\left|\log\left(\frac{\Omega}{\overline{\Omega}}\right)\right| < b. \tag{A.32}$$

This implies that

$$Ne^{-b} \leq \overline{\Omega}e^{-b} \leq \Omega \leq \overline{\Omega}e^{b} < Ne^{b}. \tag{A.33}$$

From directly studying the map, applying lemma A.2.1, and ML inequalities we see that

$$\left|\log \frac{\hat{\Omega}}{\overline{\Omega}}\right| \leq \delta \left(4\pi \, ||\psi||_{C^0H^1}^2 + \frac{\delta}{r_0^2} \, ||r_u||_{C^0} \, ||r_v||_{C^0} + \frac{|k|}{4r_0^2}\delta N^2 e^{2b}\right) = \delta C_b. \tag{A.34}$$

For δ sufficiently small we map back to the ball.

The following inequality then follows

$$\frac{1}{N} \cdot e^{-\delta C_b} \leq \hat{\Omega} \leq Ne^{\delta C_b} =: \tilde{C}_b. \tag{A.35}$$

$\log(\hat{\Omega})_u$ estimates

Differentiating (A.27)

$$\left(\log \hat{\Omega}\right)_u = \left(\log \overline{\Omega}\right)_u + \int_0^v -4\pi \psi_u \psi_v + \frac{r_u r_v}{r^2} + k\frac{\Omega^2}{4r^2} dv, \tag{A.36}$$

we deduce

$$\left|\left(\log \frac{\hat{\Omega}}{\overline{\Omega}}\right)_u\right| \leq \delta^{\frac{1}{2}} \left(4\pi \, ||\psi||_{C^0H^1} \, ||\psi_u||_{C^0} + \frac{\delta^{\frac{1}{2}}}{r_0^2} \, ||r_u||_{C^0} \, ||r_v||_{C^0} + \frac{|k|}{4r_0^2}\delta^{\frac{1}{2}} N^2 e^{2b}\right), \tag{A.37}$$

providing the δ required smallness.

$\hat{r}$ ESTIMATES

$\hat{r}$ estimate

It is fairly easy to see from the map, and continuity estimates that

$$|\hat{r}-\overline{r}| \leq \delta^2 \left(\frac{1}{r_0} ||r_u||_{C^0} ||r_v||_{C^0} + \pi |m| \, ||r||_{C^0} ||\psi||^2_{C^0} \frac{b^2}{64} e^{\delta^2 C_b^2} + \frac{|\Lambda|}{4} ||r||_{C^0} N^2 e^{\delta^2 C_b^2} \right. \\ \left. + |k| \frac{1}{4r_0} N^2 e^{\delta^2 C_b^2} ||\psi||_{C^0} \right). \tag{A.38}$$

$\hat{r}_u$ and $\hat{r}_v$ estimates

We can compute these quantities by differentiating (A.28). As the integrand is continuous and the set compact we get that it is in L^1, and we may use Fubini's theorem. Applying the fundamental theorem of calculus, we see

$$\hat{r}(u,v)_u = \overline{r}_u + \int_0^v -\frac{r_u r_v}{r} + \pi m r \psi^2 \Omega^2 + \frac{\Lambda}{4} r \Omega^2 - k \frac{\Omega^2}{4r} dv, \tag{A.39}$$

$$\hat{r}(u,v)_v = \overline{r}_v + \int_0^u -\frac{r_u r_v}{r} + \pi m r \psi^2 \Omega^2 + \frac{\Lambda}{4} r \Omega^2 - k \frac{\Omega^2}{4r} du, \tag{A.40}$$

from which

$$|\hat{r}_u - \overline{r}_u| \leq \delta \left(\frac{1}{r_0} ||r_u||_{C^0} ||r_v||_{C^0} + \pi |m| \, ||r||_{C^0} ||\psi||^2_{C^0} \frac{b^2}{64} e^{\delta^2 C_b^2} \right. \\ \left. + \frac{|\Lambda|}{4} ||r||_{C^0} N^2 e^{\delta^2 C_b^2} + |k| \frac{1}{4r_0} N^2 e^{\delta^2 C_b^2} ||\psi||_{C^0} \right), \tag{A.41}$$

and

$$|\hat{r}_v - \overline{r}_v| \leq \delta \left(\frac{1}{r_0} ||r_u||_{C^0} ||r_v||_{C^0} + \pi |m| \, ||r||_{C^0} ||\psi||^2_{C^0} \frac{b^2}{64} e^{\delta^2 C_b^2} \right. \\ \left. + \frac{|\Lambda|}{4} ||r||_{C^0} \frac{b^2}{64} e^{\delta^2 C_b^2} + |k| \frac{1}{4r_0} \frac{b^2}{64} e^{\delta^2 C_b^2} ||\psi||_{C^0} \right), \tag{A.42}$$

follow.

$\hat{r}_{uu}$ and $\hat{r}_{uv}$ estimates

As $\Omega \in \mathcal{B}_b$ we have

$$\left| \left(\log \frac{\Omega}{\bar{\Omega}} \right)_u \right| < b, \tag{A.43}$$

this implies

$$\Omega_u \leq (b + \Omega) \frac{\bar{\Omega}_u}{\bar{\Omega}} \leq \left(b + N e^b \right). \tag{A.44}$$

Writing $\hat{r}_{uv}$ and $\hat{r}_{uu}$ as

$$\hat{r}_{uv} = \int_0^u \partial_u \left(\hat{r}_{uv} \right) du, \tag{A.45}$$

and

$$\hat{r}_{uu} = \int_0^v \partial_u \left(\hat{r}_{uv}\right) dv. \tag{A.46}$$

We then compute

$$\begin{aligned}\hat{r}_{uvu} = & - r_{uu} r_v r^{-1} - r_u r_{uv} r^{-1} + r_u^2 r_v r^{-2} + \pi m r_u \psi^2 \Omega^2 + 2\pi m r \psi \psi_u \Omega^2 + 2\pi m r \psi^2 \Omega \Omega_u \\ & + \frac{\Lambda}{4} r_u \Omega^2 + \frac{\Lambda}{2} r \Omega \Omega_u - \frac{k}{2} \Omega \Omega_u r^{-1} + \frac{k}{4} \Omega^2 r^{-2} r_u,\end{aligned} \tag{A.47}$$

Recall the Ω and Ω_u terms are bounded. All the other functions are continuous, and we can put these in the supremum norms. Integrating over the domain we get the δ smallness.

$\hat{\psi}$ ESTIMATES

$C^0 H^1$ and C^0 estimates

Lemma A.2.2. *For $F \in L^2(\square_\delta)$ there exists a sequence of functions $F_\epsilon \in C^\infty(\square_\delta^\circ)$ such that $F_\epsilon \to F$ in L^2.*

Now define

$$\left(\psi_v\right)_\epsilon (0, v) = \psi_v * \eta_\epsilon(0, v), \tag{A.48}$$

where η_ϵ is the standard smoothing kernel, then define

$$\psi_\epsilon(0, v) = \psi(0, 0) + \int_0^v \left(\psi_v\right)_\epsilon (0, v') dv'. \tag{A.49}$$

Similarly

$$\left(\psi_u\right)_\epsilon (u, 0) = \psi_u * \eta_\epsilon(u, 0), \tag{A.50}$$

and

$$\psi_\epsilon(u, 0) = \psi(0, 0) + \int_0^u \left(\psi_u\right)_\epsilon (u', 0) du', \tag{A.51}$$

from this construction it is standard theory to see that

$$\psi_\epsilon \to \psi, \quad (\psi_\epsilon)_u \to \psi_u, \quad (\psi_\epsilon)_v \to \psi_v, \tag{A.52}$$

in L^2 as $\epsilon \to 0$ with $\psi_\epsilon(u, 0) \in C^\infty\left([0, \delta)\right)$, $\psi_\epsilon(0, v) \in C^\infty\left([0, \delta)\right)$.
We remark that these functions have been constructed to agree at $(0, 0)$ are are therefore admissible
Now we solve the equation

$$\partial_v \partial_u \psi_\epsilon = F_\epsilon, \tag{A.53}$$

with initial data $\psi_\epsilon(u, 0)$ and boundary data $\psi_\epsilon(0, v)$. As we have a smooth solution we derive the following estimates (see section A.4.1) after passing $\epsilon \to 0$

$$\left|\left|\hat{\psi}\right|\right|_{C^0 H^1} \leq ||\psi_1||_{L^2(\mathcal{N}')} + ||\psi_2||_{L^2(\mathcal{N})} + \delta \, ||\psi_{2,v}||_{L^2(\mathcal{N})} + \delta \, ||\psi_{1,u}||_{L^2(\mathcal{N}')} + 2\delta^{\frac{3}{2}} \, ||F||_{L^2(\square_\delta)}. \tag{A.54}$$

Showing

$$\left|\left|\hat{\psi} - \overline{\psi}\right|\right|_{C^0 H^1} \leq 4 \cdot \frac{b}{8} + \delta \left|\left|\psi_{2,v}\right|\right|_{L^2(\mathcal{N})} + \delta \left|\left|\psi_{1,u}\right|\right|_{L^2(\mathcal{N}')} + 2\delta^{\frac{3}{2}} \left|\left|F\right|\right|_{L^2(\Box_\delta)} . \tag{A.55}$$

As F can quite clearly be put into L^2, We have for small δ, that we map back to the ball. From the Sobolev inequality

$$\left|\left|\psi - \overline{\psi}\right|\right|_{C^0} \leq \left|\left|\hat{\psi}\right|\right|_{C^0 H^1} , \tag{A.56}$$

we immediately see that we map back to the ball.
Finally for the ψ_u we use the following lemma:

Lemma A.2.3. *For $F \in C^0_u L^1_v(\Box_\delta)$ we can find a sequence of functions $F_\epsilon \in C^\infty(\Box^\circ_\delta)$ such that $F_\epsilon \to F$ in $C^0_u L^1_v$.*

(See section A.4.1 for proof).
Studying equation

$$\partial_v \left(\psi_\epsilon\right) = F_\epsilon, \tag{A.57}$$

by integrating and taking the limit as $\epsilon \to 0$,

$$\left|\left|\psi_u(u,v) - \overline{\psi}_u(u,v)\right|\right|_{C^0_u} \leq \frac{1}{2} \left|\left|\psi_{1,u}\right|\right|_{C^0(\mathcal{N}')} + \left|\left|F\right|\right|_{C^0_u L^1_v(\Box_\delta)} . \tag{A.58}$$

Evaluating the norm of F, we see

$$\left|\left|F\right|\right|_{C^0_u L^1_v(\Box_\delta)} \leq \delta \frac{1}{r_0} \left|\left|r_v\right|\right|_{C^0} \left|\left|\psi_u\right|\right|_{C^0} + \delta \frac{m}{4} \left|\left|\Omega\right|\right|^2_{C^0} \left|\left|\psi\right|\right|_{C^0} + \delta^{\frac{1}{2}} \frac{1}{r_0} \left|\left|r_u\right|\right|_{C^0} \left|\left|\psi_v\right|\right|_{C^0 H^1} \tag{A.59}$$

so for δ small enough we map back to the ball.

A.2.7 Contraction map

Throughout the contraction map argument we will often make use of the following estimates.

Lemma A.2.4. *There exists a constant $C_b > 0$ such that*

$$\left|\Omega_1 - \Omega_2\right| \leq C_b \left|\log\left(\frac{\Omega_1}{\Omega_2}\right)\right| . \tag{A.60}$$

Proof.

$$\begin{aligned} \left|\Omega_1 - \Omega_2\right| &= \left|e^{\log \Omega_1} - e^{\log \Omega_2}\right| \\ &\leq \left|e^{\log \Omega_2}\left(e^{\log\left(\frac{\Omega_1}{\Omega_2}\right)} - 1\right)\right| \\ &\leq e^{Ne^b + C_b} \left|\log\left(\frac{\Omega_1}{\Omega_2}\right)\right| . \end{aligned} \tag{A.61}$$

□

Lemma A.2.5. *Let $A_1, A_2 > 0$, and B_1, B_2 be functions. The following inequalities hold*

$$\begin{aligned} |A_1B_1 - A_2B_2| &\le |A_1|\,|B_1 - B_2| + |B_2|\,|A_1 - A_2|\,, \\ \left|A_1^2 - A_2^2\right| &\le (|A_1| + |A_2|)\,|A_1 - A_2|\,, \\ \left|\frac{1}{A_1} - \frac{1}{A_2}\right| &\le \frac{1}{|A_1A_2|}\,|A_1 - A_2|\,. \end{aligned} \tag{A.62}$$

Contraction for $\hat{\Omega}$

$\log\hat{\Omega}$ estimates
We can see the contraction for this directly

$$\left|\log\frac{\hat{\Omega}_1}{\hat{\Omega}_2}\right| = \left|\int_0^u\int_0^v (-4\pi\psi_{1,u}\psi_{1,v} + 4\pi\psi_{2,u}\psi_{2,v}) + \left(\frac{r_{1,u}r_{1,v}}{r_1^2} - \frac{r_{2,u}r_{2,v}}{r_2^2}\right) + \left(k\frac{\Omega_1^2}{4r_1^2} - k\frac{\Omega_1^2}{4r_1^2}\right)dvdu\right|, \tag{A.63}$$

we have three terms to deal with. The second two are fairly simple, estimating with (A.61), (A.62), and that we map back to the ball. From the integrals we get a δ^2 smallness here. The first term requires more care, due to the regularity.
Firstly factor

$$-4\pi\psi_{1,u}\psi_{1,v} + 4\pi\psi_{2,u}\psi_{2,v} = 4\pi\left(\psi_{1,u}\left(\psi_{1,v} - \psi_{2,v}\right) + \psi_{2,v}\left(\psi_{1,u} - \psi_{2,u}\right)\right). \tag{A.64}$$

Then performing ML, and Cauchy Schwartz inequalities we estimate

$$\begin{aligned} \int_0^u\int_0^v -4\pi\psi_{1,u}\psi_{1,v} + 4\pi\psi_{2,u}\psi_{2,v}dvdu &\le ||\psi_{1,u}||_{C^0}\int_0^u\int_0^v|\psi_{1,v} - \psi_{2,v}|\,dudv \\ &+ \int_0^u\int_0^v|\psi_{2,v}|\,dudv\,||\psi_{1,u} - \psi_{2,u}||_{C^0} \\ &\le \delta^{\frac{3}{2}}4\pi\,||\psi_{1,u}||_{C^0}\,||\psi_1 - \psi_2||_{C^0H^1} \\ &+ \delta^{\frac{3}{2}}4\pi\,||\psi_2||_{C^0H^1}\,||\psi_{1,u} - \psi_{2,u}||_{C^0} \\ &\le \delta^{\frac{3}{2}}C_b\cdot d(\Sigma_1,\Sigma_2). \end{aligned} \tag{A.65}$$

We thus have

$$\left|\log\frac{\hat{\Omega}_1}{\hat{\Omega}_2}\right| \le \delta^{\frac{3}{2}}C_b\cdot d(\Sigma_1,\Sigma_2). \tag{A.66}$$

$\log\hat{\Omega}_u$ estimates
Recall the map for this variable is

$$\left(\log\hat{\Omega}\right)_u = \left(\log\overline{\Omega}\right)_u + \int_0^v -4\pi\psi_u\psi_v + \frac{r_ur_v}{r^2} + k\frac{\Omega^2}{4r^2}dv. \tag{A.67}$$

This is a similar argument as the map for $\log \hat{\Omega}$, only the integral in u is not present. (This corresponded to a higher power of δ). We immediately see that

$$\left| \log \left(\frac{\hat{\Omega}_1}{\hat{\Omega}_2} \right)_u \right| \leq \delta^{\frac{1}{2}} C_b \cdot d(\Sigma_1, \Sigma_2) \tag{A.68}$$

Contraction for $\hat{r}$

The maps for $\hat{r}$ and its derivatives only contain continuous functions. This means the estimates (A.61), and (A.62), will suffice to estimate the integrand in the desired form. Integrating then gets us the required δ or δ^2 smallness factor

$$d(\hat{r}_1, \hat{r}_2) \leq \min\left(\delta, \delta^2\right) C_b d(\Sigma_1, \Sigma_2). \tag{A.69}$$

For clarity we include that we may write

$$\begin{aligned} \hat{r}_{uvu} = & - r_{uu} r_v r^{-1} - r_u r_{uv} r^{-1} + r_u^2 r_v r^{-2} + \pi m r_u \psi^2 \Omega^2 + 2\pi m r \psi \psi_u \Omega^2 + 2\pi m r \psi^2 \Omega \Omega_u \\ & + \frac{\Lambda}{4} r_u \Omega^2 + \frac{\Lambda}{2} r \Omega \Omega_u - \frac{k}{2} \Omega \Omega_u r^{-1} + \frac{k}{4} \Omega^2 r^{-2} r_u, \end{aligned} \tag{A.70}$$

as

$$\begin{aligned} \hat{r}_{uvu} = & - r_{uu} r_v r^{-1} - r_u r_{uv} r^{-1} + r_u^2 r_v r^{-2} + \pi m r_u \psi^2 \Omega^2 + 2\pi m r \psi \psi_u \Omega^2 + 2\pi m r \psi^2 \Omega^2 \log\left(\Omega\right)_u \\ & + \frac{\Lambda}{4} r_u \Omega^2 + \frac{\Lambda}{2} r \Omega^2 \log\left(\Omega\right)_u - \frac{k}{2} \Omega^2 \log\left(\Omega\right)_u r^{-1} + \frac{k}{4} \Omega^2 r^{-2} r_u. \end{aligned} \tag{A.71}$$

Contraction for $\hat{\psi}$

Contraction in the $C^0 H^1$ norm

It follows from the energy estimates that

$$\left|\left| \hat{\psi_1} - \hat{\psi_2} \right|\right|^2_{C^0 H^1} \leq \int_0^\delta \int_0^\delta (F_1 - F_2)^2 du dv. \tag{A.72}$$

Expanding the RHS integrand we see

$$F_1 - F_2 = -\frac{r_{1,v}}{r_1} \psi_{1,u} - \frac{r_{1,u}}{r_1} \psi_{1,v} - \frac{m}{4} \Omega_1^2 \psi_1 - \left(-\frac{r_{2,v}}{r_2} \psi_{2,u} - \frac{r_{2,u}}{r_2} \psi_{2,v} - \frac{m}{4} \Omega_1^2 \psi_2 \right), \tag{A.73}$$

which we regroup into the low regularity terms L and the continuous terms C.

$$F_1 - F_2 = \underbrace{\left(\frac{r_{2,u}}{r_2} \psi_{2,v} - \frac{r_{1,u}}{r_1} \psi_{1,v} \right)}_{L} + \underbrace{\left(\frac{r_{2,v}}{r_2} \psi_{2,u} - \frac{r_{1,v}}{r_1} \psi_{1,u} + \frac{m}{4} \Omega_1^2 \psi_2. - \frac{m}{4} \Omega_1^2 \psi_1 \right)}_{C}. \tag{A.74}$$

Estimating with Cauchy's inequality we see we can estimate by

$$(F_1 - F_2)^2 \leq 2L^2 + 2C^2, \tag{A.75}$$

the C terms are all continuous, and thus trivial to deal with. We look at L terms which require more care. First write the L terms by

$$L = \frac{r_{2,u}}{r_2}\left(\psi_{2,v} - \psi_{1,v}\right) + \psi_{1,v}\frac{1}{r_2}\left(r_{2,u} - r_{1,u}\right) + \psi_{1,v}\frac{r_{1,u}}{r_1 r_2}\left(r_1 - r_2\right). \tag{A.76}$$

Using Cauchy's inequality iteratively we have

$$\frac{L^2}{4} \leq \left(\frac{r_{2,u}}{r_2}\right)^2 \left(\psi_{2,v} - \psi_{1,v}\right)^2 + \psi_{1,v}^2 \left(\frac{1}{r_2}\right)^2 \left(r_{2,u} - r_{1,u}\right)^2 + \psi_{1,v}^2 \left(\frac{r_{1,u}}{r_1 r_2}\right)^2 \left(r_1 - r_2\right)^2. \tag{A.77}$$

Estimating term by term

$$\int_0^\delta \int_0^\delta \left(\frac{r_{2,u}}{r_2}\right)^2 \left(\psi_{2,v} - \psi_{1,v}\right)^2 dudv \leq \left|\left|\frac{r_{2,u}}{r_2}\right|\right|_{C^0} \delta \sup_u \int_0^\delta \left|\psi_{2,v} - \psi_{1,v}\right|^2 dv \leq \delta \left|\left|\frac{r_{2,u}}{r_2}\right|\right|_{C^0} ||\psi_1 - \psi_2||^2_{C^0 H^1}, \tag{A.78}$$

$$\int_0^\delta \int_0^\delta \psi_{1,v}^2 \left(\frac{1}{r_2}^2\right)^2 \left(r_{2,u} - r_{1,u}\right)^2 dudv \leq \delta \left|\left|\frac{1}{r_2}\right|\right|_{C^0} \sup_u \left(\int_0^\delta \psi_{1,v}^2 dv\right) ||r_{1,u} - r_{2,u}||^2_{C^0}, \tag{A.79}$$

$$\int_0^\delta \int_0^\delta \psi_{1,v}^2 \left(\frac{r_{1,u}}{r_1 r_2}\right)^2 \left(r_1 - r_2\right)^2 dudv \leq \delta \left|\left|\left(\frac{r_{1,u}}{r_1 r_2}\right)^2\right|\right|_{C^0} \sup_u \left(\int_0^v \psi_{1,v}^2 dv\right) ||r_1 - r_2||^2_{C^0}. \tag{A.80}$$

Thus we are able to conclude that

$$\left|\left|\hat{\psi}_1 - \hat{\psi}_1\right|\right|^2_{C^0 H^1} \leq C_{l,b,a}\delta \cdot d(\Sigma_1, \Sigma_2)^2. \tag{A.81}$$

Contraction in C^0 for ψ

By the fundamental theorem of calculus

$$\begin{aligned}
\left|\hat{\psi}_1(u,v) - \hat{\psi}_2(u,v)\right| &= \left|\int_0^u \hat{\psi}_{1,u} - \hat{\psi}_{2,u} du\right| \\
&\leq \left(\int_0^u 1 du\right)^{\frac{1}{2}} \left(\int_0^u \left(\hat{\psi}_{1,u} - \hat{\psi}_{2,u}\right)^2 du\right)^{\frac{1}{2}} \\
&\leq \delta^{\frac{1}{2}} \left|\left|\hat{\psi}_1 - \hat{\psi}_2\right|\right|_{C^0 H^1} \\
&\leq C_{b,l,a}\delta^{\frac{3}{2}} \cdot d(\Sigma_1, \Sigma_2).
\end{aligned} \tag{A.82}$$

Contraction in C^0 for for ψ_u

The wave equation implies the following estimate

$$\left|\hat{\psi}_{1,u}(u,v) - \hat{\psi}_{2,u}(u,v)\right| \leq \int_0^\delta \left|\frac{r_{2,u}}{r_2}\psi_{2,v} - \frac{r_{1,u}}{r_1}\psi_{1,v} + \frac{r_{2,v}}{r_2}\psi_{2,u} - \frac{r_{1,v}}{r_1}\psi_{1,u} + \frac{m}{4}\Omega_2^2\psi_2 - \frac{m}{4}\Omega_1^2\psi_1\right| dv. \tag{A.83}$$

From the continuity of the functions, only the first two terms in the integrand are non trivial. Explicitly we are dealing with

$$\begin{aligned}\int_0^\delta \left|\frac{r_{2,u}}{r_2}\psi_{2,v} - \frac{r_{1,u}}{r_1}\psi_{1,v}\right| dv &\leq \int_0^\delta \left|\frac{r_{2,u}}{r_2}\left(\psi_{2,v} - \psi_{1,v}\right)\right| dv + \int_0^\delta \left|(r_{2,u} - r_{1,u})\frac{\psi_{1,v}}{r_2} + (r_1 - r_2)\frac{r_{1,u}}{r_1 r_2}\psi_{1,v}\right| dv \\ &\leq \left|\left|\frac{r_{2,u}}{r_2}\right|\right|_{C^0} \delta^{\frac{1}{2}} \left|\left|\psi_1 - \psi_2\right|\right|_{C^0 H^1} + \delta^{\frac{1}{2}} \left|\left|r_{1,u} - r_{2,u}\right|\right|_{C^0} \left|\left|\frac{1}{r_2}\right|\right|_{C^0} \sup_u \left(\int_0^\delta \psi_{1,v}^2 dv\right)^{\frac{1}{2}} \\ &+ \delta^{\frac{1}{2}} \left|\left|r_1 - r_2\right|\right|_{C^0} \left|\left|\frac{r_{1,u}}{r_1 r_2}\right|\right|_{C^0} \left(\int_0^v \psi_{1,v}^2 dv\right)^{\frac{1}{2}}.\end{aligned} \tag{A.84}$$

It thus follows that

$$\left|\left|\hat{\psi}_{1,u}(u,v) - \hat{\psi}_{2,u}(u,v)\right|\right|_{C^0} \leq C_{b,l,a}\delta^{\frac{1}{2}} \cdot d(\Sigma_1, \Sigma_2). \tag{A.85}$$

Hence for δ sufficiently small we have a contraction. A unique solution of the desired regularity follows from Banach's fixed point theorem.

A.2.8 Propagation of constraints

By calculation it follows from equations (A.6)-(A.8) that the following equations hold

$$\partial_v\left(\Omega^2\partial_u\left(\frac{r_u}{\Omega^2}\right) + 4\pi r\psi_u^2\right) = -\frac{r_v}{r}\left(r_{uu} - 2r_u\frac{\Omega_u}{\Omega} + 4\pi r\psi_u^2\right) = -\frac{r_v}{r}\left(\Omega^2\partial_u\left(\frac{r_u}{\Omega^2}\right) + 4\pi r\psi_u^2\right), \tag{A.86}$$

and

$$\partial_u\left(\Omega^2\partial_v\left(\frac{r_u}{\Omega^2}\right) + 4\pi r\psi_v^2\right) = -\frac{r_u}{r}\left(\Omega^2\partial_v\left(\frac{r_v}{\Omega^2}\right) + 4\pi r\psi_v^2\right). \tag{A.87}$$

These are homogeneous equations, as the constraint holds on the initial data rays by assumption, they must propagate. That is

$$\partial_v\left(\frac{r_v}{\Omega^2}\right) = -4\pi r\frac{\psi_v^2}{\Omega^2}, \tag{A.88}$$

holds almost everywhere and that

$$\partial_u\left(\frac{r_u}{\Omega^2}\right) = -4\pi r\frac{\psi_u^2}{\Omega^2}, \tag{A.89}$$

holds classically.

A.3 Extendibility criterion

Definition A.3.1. *Define the timelike future/past of a point p*

$$I^{\pm}(p) := p \cup \{q \in \mathcal{M} | \exists \gamma : [0,1] \to \mathcal{M}, \gamma(0) = p, \gamma(1) = q | \dot{\gamma} \text{ future(+) or past(-) directed, timelike}\} \tag{A.90}$$

and the causal future/past of p

$$J^{\pm}(p) := p \cup \{q \in \mathcal{M} | \exists \gamma : [0,1] \to \mathcal{M}, \gamma(0) = p, \gamma(1) = q | \dot{\gamma} \text{ future(+) or past(-) directed, causal}\}. \tag{A.91}$$

Proposition A.3.1 (Extendibility criterion)**.** *Let $p = (U, V) \in \overline{\mathcal{Q}^+} \backslash \mathcal{Q}^+$, and $q = (U', V') \in (I^-(p) \cap \mathcal{Q}^+) \backslash \{p\}$ be such that the set*

$$\mathcal{D} = \left(J^+(q) \cap J^-(p)\right) \backslash \{p\} \subset \mathcal{Q}^+, \tag{A.92}$$

then we have that

$$N(\mathcal{D}) = \infty. \tag{A.93}$$

Proof. Assume the contrapositive. Suppose that $N = 2N(\mathcal{D}) < \infty$. Now corresponding to this N there is a $\delta > 0$, from the previous proposition that we can solve on a domain $\square_\delta$. Consider the point $(U - \frac{1}{2}\delta, V - \frac{1}{2}\delta)$. Take δ small enough so this point is in $\mathcal{Q}^+$. Translate so this point is at the origin $(0, 0)$. As $\mathcal{Q}^+$ is open we have from continuity the existence of $\delta^* \in (\frac{1}{2}\delta, \delta)$ such that

$$X^* = \{0\} \times [0, \delta^*] \cup [0, \delta^*] \times \{0\} \subset \mathcal{Q}^+, \tag{A.94}$$

and that the assumptions of the previous proposition hold. Thus there exists a unique solution in

$$\mathcal{E} = [0, \delta^*] \times [0, \delta^*]. \tag{A.95}$$

By uniqueness this coincides with previous solution on the subset $\mathcal{D} \cap \mathcal{E}$. As $\mathcal{E} \cup \mathcal{Q}^+$ is the quotient of a maximal development of initial data, we get from the maximality of $\mathcal{Q}^+$ that $\mathcal{E} \cup \mathcal{Q}^+ \subset \mathcal{Q}^+$. So we have that $p \in \mathcal{Q}^+$. □

A.4 Interior extension principle

Let $\mathcal{G}$ be either $\mathbb{S}^2$, $\mathfrak{T}^2$ or Σ_g.

Proposition A.4.1. *Let $(\mathcal{Q}^+ \times \mathcal{G}, g, \psi)$ denote the maximal $\mathfrak{W}$ extension of an asymptotically AdS initial data set as defined in 3.4.2. Suppose $p = (U, V) \in \overline{\mathcal{Q}^+}$. If*

- $$\mathcal{D} = [U', U] \times [V', V] \backslash \{p\} \subset Q^+, \tag{A.96}$$

  has finite spacetime volume,

- *and there exist constants*

$$0 < r_0 \leq r \leq R < \infty, \quad \textit{for all } (u,v) \in \mathcal{D}, \tag{A.97}$$

then $p \in \mathcal{Q}^+$.

Proof. Firstly defining $\chi = \frac{\Omega^2}{4r_u}$. Write the Raychaudhuri equation (3.8) as

$$\partial_u \log \chi = \frac{4\pi r}{r_u}(\partial_u \psi)^2 \sim r_u r^{-4+2\kappa}, \tag{A.98}$$

we know that at $\mathcal{I}$, we have $r_u < 0$ (spacetime is aAdS) so we have that $\chi|_{\mathcal{I}} < 0$. This implies

$$\chi = \chi|_{u_{\mathcal{I}}} \exp\left(\int_{\mathcal{I}}^{u} \frac{4\pi r}{r_u}(\partial_u \psi)^2 du\right), \tag{A.99}$$

and thus $r_u < 0$ in the maximal development.
Now from our assumptions on $\mathcal{D}$ we have

$$\int_{V'}^{V} \int_{U'}^{U} \Omega^2 dU dV < C, \tag{A.100}$$

and

$$\frac{1}{C} < r_0 \leq r(u,v) \leq R < C, \tag{A.101}$$

for some constant $C > 0$. Continuity and compactness tells us that on $[U', U] \times \{V'\}$ and $\{U'\} \times [V', V]$ we have the estimates

$$\begin{aligned}&\frac{1}{N} < -r r_u < N, \quad |r r_v| < N, \quad |r\psi| < N, \quad |\psi_u| < N, \quad |r_{uu}| < N, \quad |r_{uv}| < N, \\ &|\Omega_u| < N, \quad |\log \Omega| < N.\end{aligned} \tag{A.102}$$

for some constant N. We now write (A.6) in the form

$$\partial_u (r r_v) = \pi m r^2 \psi^2 \Omega^2 + \frac{\Lambda}{4} r^2 \Omega^2 - k\frac{\Omega^2}{4}. \tag{A.103}$$

Integrating the above twice, and using our bounds on the spacetime volume and r, we get

$$\int_{V'}^{V} \int_{U'}^{U} \psi^2 \Omega^2 du dv \leq \tilde{C}. \tag{A.104}$$

Where $\tilde{C}$ depends only on C, m, Λ and k. We can also form the pointwise estimate

$$\sup_{[U',U]} |r r_v| \leq N + \frac{1}{4}\int_{U'}^{U} \left(|k| + |\Lambda| r^2\right) \Omega^2 du + \int_{U'}^{U} \pi |m| r^2 \psi^2 \Omega^2 du. \tag{A.105}$$

We integrate this to

$$\int_{V'}^{V} \sup_{[U',U]} |rr_v|\, dv \leq N(V - V') + \tilde{C}, \tag{A.106}$$

and similarly

$$\int_{U'}^{U} \sup_{[V',V]} |rr_u|\, du \leq N(U - U') + \tilde{C}. \tag{A.107}$$

We now partition $\mathcal{D}$ into sub diamonds given by

$$\mathcal{D}_{jk} = [u_j, u_{j+1}] \times [v_k, v_{k+1}] \quad j, k = 0, \ldots, N, \tag{A.108}$$

with $u_0 = U'$, $u_N = U$, $v_0 = V'$, $v_N = V$, and such that for a given $\epsilon > 0$ we have

$$\int_{v_k}^{v_{k+1}} \int_{u_j}^{u_{j+1}} \Omega^2 \psi^2 du dv + \int_{v_k}^{v_{k+1}} \int_{u_j}^{u_{j+1}} \Omega^2 du dv + \int_{v_k}^{v_{k+1}} \sup_{[U',U]} |rr_v|\, dv + \int_{u_j}^{u_{j+1}} \sup_{[V',V]} |rr_u|\, du < \epsilon. \tag{A.109}$$

This is possible in view of the uniform bounds we proved. □

Define

$$P_{jk} = \sup_{\mathcal{D}_{jk}} |r\psi(u, v)|\,, \tag{A.110}$$

and pick an arbitrary point $(u^*, v^*) \in \mathcal{D}_{jk}$, and consider (A.8) written as

$$\partial_v \partial_u (r\psi) = \psi \partial_u (r_v) - \frac{m}{4} r \Omega^2 \psi, \tag{A.111}$$

as the right hand side is continuous we note that this holds almost everywhere. We now wish to integrate this in both variables, we first study

$$\int_{v_k}^{v^*} \int_{u_j}^{u^*} \frac{m}{4} r \Omega^2 \psi du dv \leq C_m P_{jk} \cdot \epsilon, \tag{A.112}$$

as for the term

$$\begin{aligned} &\int_{v_k}^{v^*} \int_{u_j}^{u^*} \psi \partial_u (r_v)\, du dv = \\ &\int_{v_k}^{v^*} \int_{u_j}^{u^*} \psi \left(-\frac{r_u r_v}{r} + \pi m r \psi^2 \Omega^2 + \frac{\Lambda}{4} r \Omega^2 - k \frac{\Omega^2}{4r} \right) du dv \leq P_{jk} \cdot C_{l,r_0} \epsilon, \end{aligned} \tag{A.113}$$

this estimate is obviously true for all the terms on the RHS with the exception of the first. We can estimate this by:

$$\begin{aligned} &\int_{v_k}^{v^*} \int_{u_k}^{u^*} -\frac{r_u r_v}{r} \psi du dv \leq P_{jk} \int_{v_k}^{v^*} \int_{u_k}^{u^*} \frac{-r_u}{r^3} |rr_v|\, du dv \\ &\leq P_{jk} \int_{v_k}^{v^*} \sup_{[u_j, u_{j+1}]} |rr_v|\, dv \int_{u_j}^{u^*} \frac{-r_u}{r^3} du \leq P_{jk} \cdot \frac{C_l}{r_0^2} \cdot \epsilon. \end{aligned} \tag{A.114}$$

So integrating (A.111), in u and v for sufficiently small ϵ, yields the the uniform bound

$$P_{jk} < 2\left(\sup_{[u_j,u_{j+1}]\times\{v_k\}} |r\psi| + \sup_{\{u_j\}\times[v_k,v_{k+1}]} |r\psi|\right) < 2\left(P_{j,k-1} + P_{j-1,k}\right). \tag{A.115}$$

Inductively step back to $P_{0,k}$ and $P_{j,0}$, yielding a uniform bound for P_{jk} in terms of the initial data. Taking a maximum over all the sub diamonds thus yields

$$\sup_{\mathcal{D}} |r\psi| < \tilde{C}. \tag{A.116}$$

The next step is to prove a pointwise bound for $\log(\Omega)$. Recalling the equation (A.7) the only term we don't yet know how to deal with is

$$\begin{aligned}\int_{U'}^{U}\int_{V'}^{V} \psi_u\psi_v du dv &= \int_{U'}^{U}\int_{V'}^{V} \frac{1}{2}\partial_v\partial_u\left(\psi^2\right) - \psi\left(-\frac{r_v}{r}\psi_u - \frac{r_u}{r}\psi_v - \frac{m}{4}\Omega^2\psi\right) du dv \\ &= \int_{U'}^{U}\int_{V'}^{V} \frac{1}{2}\partial_v\partial_u\left(\psi^2\right) + \frac{m}{4}\Omega^2\psi^2 + \frac{r_u}{2r}\partial_v\left(\psi^2\right) + \frac{r_v}{2r}\partial_u\left(\psi^2\right) du dv.\end{aligned} \tag{A.117}$$

We control the first two terms of the integral by previous estimates. We turn to look at the third (the fourth is analogous),

$$\int_{V'}^{V}\int_{U'}^{U} \frac{r_u}{2r}\partial_v\left(\psi^2\right) du dv = \int_{V'}^{V}\left[\frac{r_u}{2r}\psi^2\right]_{U'}^{U} dv + \int_{V'}^{V}\int_{U'}^{U}\left(\frac{r_{uv}}{r} - \frac{r_u r_v}{r}\right)\psi^2 du dv. \tag{A.118}$$

We can control the surface terms in view of (A.106) and the bounds on r. As for the other terms

$$\int_{V'}^{V}\int_{U'}^{U}\left(\frac{r_{uv}}{r} - \frac{r_u r_v}{r}\right)\psi^2 du dv = \int_{V'}^{V}\int_{U'}^{U} -2\frac{r_u r_v}{r}\psi^2 + \pi m r\psi^4\Omega^2 + \frac{\Lambda}{4}r\Omega^2\psi^2 - k\frac{\Omega^2}{4r}\psi^2 du dv, \tag{A.119}$$

term by term we can see that we control this from previous estimates. We conclude that

$$|\log(\Omega)| < \tilde{C}. \tag{A.120}$$

So we have uniform constants c_0, c_1 such that

$$c_0 < \Omega^2 < c_1. \tag{A.121}$$

With this estimate reviewing (A.105) we thus can control

$$\sup_{\mathcal{D}} |r r_u| + \sup_{\mathcal{D}} |r r_v| < \tilde{C}. \tag{A.122}$$

Where the $\tilde{C}$ depends on the domain values. A uniform estimate for r_{uv} now immediately follows from the definition. Integrating (A.111) in v we get that ψ_u is uniformly bounded. With these bounds we can thus integrate (A.4), and (A.5), to see that we have $\dot{H}^1$ control of ψ. Using this and the uniform bound on ψ we can upgrade this to full C^0H^1 control. Now integrating (3.11) in v, estimating the $\psi_u\psi_v$ term by bounding ψ_u uniformly, and applying the Cauchy-Schwartz estimate on the ψ_v term, we see that we control Ω_u uniformly. Finally the r_{uu} estimate follows from expanding the derivative of (3.8). We thus have that in $\mathcal{D}$ the existence of a $0 < M < \infty$, where

$$\begin{aligned} M = \sup \Big\{ & ||\psi||_{H^1_u}, ||\psi||_{H^1_v}, ||\psi_u||_{C^0_u}, ||\psi_u||_{C^0_v}, ||r||_{C^1_u}, ||r||_{C^1_v}, ||r_{uu}||_{C^0_u}, ||r_{uu}||_{C^0_v}, \\ & ||r_{uv}||_{C^0_u}, ||r_{uv}||_{C^0_v}, \left|\left|r^{-1}\right|\right|_{C^0_u}, \left|\left|r^{-1}\right|\right|_{C^0_v}, ||\Omega||_{C^0_u}, ||\Omega||_{C^0_v}, ||\Omega_u||_{C^0_u}, ||\Omega_u||_{C^0_v}, \left|\left|\Omega^{-1}\right|\right|_{C^0_u}, \left|\left|\Omega^{-1}\right|\right|_{C^0_v} \Big\}. \end{aligned} \tag{A.123}$$

An application of proposition A.3.1 shows us that we are done.

A.4.1 Wave equation theory, and technical results

Lemma A.4.1. *For any $\delta > 0$ and $F \in C^\infty(\Box^\circ_\delta)$ the equation*

$$\partial_v\partial_u\psi = F. \tag{A.124}$$

With initial data $\psi(u,0) = \psi_0$, and boundary data $\psi(0,v) = \psi_1$, which satisfy the compatibility condition $\psi_0(0) = \psi_1(0)$ and are C^∞. We have the following energy estimates for ψ

$$||\psi||_{C^0H^1} \leq ||\psi_1||_{L^2(\mathcal{N}')} + ||\psi_2||_{L^2(\mathcal{N})} + \delta\,||\psi_{2,v}||_{L^2(\mathcal{N})} + \delta\,||\psi_{1,u}||_{L^2(\mathcal{N}')} + 2\delta^{\frac{3}{2}}\,||F||_{L^2(\Box_\delta)}. \tag{A.125}$$

Proof. Firstly we define the quantity

$$E_v(u) = \int_0^v \psi_v^2(u,v')dv'. \tag{A.126}$$

Now by multiply (A.124) by ψ_v to get the equality

$$\frac{1}{2}\partial_u\left(\psi_v^2\right) = \psi_v F. \tag{A.127}$$

Integrating twice gives

$$\int_0^v \psi_v^2(u,v') - \psi_v^2(0,v')dv' = 2\int_0^v\int_0^u F(u',v')\psi_v(u',v')dudv. \tag{A.128}$$

We apply Fubini's theorem, and write the expression as

$$\int_0^v \psi_v^2(u,v')dv' - \int_0^v \psi(0,v')dv' = 2\int_0^u\int_0^v F(u',v')\psi_v(u',v')dvdu. \tag{A.129}$$

Taking derivatives in u we see

$$\begin{aligned} \frac{d}{du}E_v(u) &= 2\int_0^v F(u,v')\psi_v(u,v')dv \\ &\leq 2E_v^{\frac{1}{2}}(u)\left(\int_0^v F(u,v')^2 dv\right)^{\frac{1}{2}}. \end{aligned} \tag{A.130}$$

So we have

$$\frac{d}{du}E_v(u)^{\frac{1}{2}} \leq \left(\int_0^v F(u,v')^2 dv\right)^{\frac{1}{2}}, \tag{A.131}$$

integrating back in u we have

$$\begin{aligned} \|\psi_v\|_{L_v^2}(u,v) &\leq \|\psi_{2,v}\|_{L_v^2}(0,v) + \int_0^u \left(\int_0^v F(u,v')^2 dv\right)^{\frac{1}{2}} dv \\ &\leq \|\psi_{2,v}\|_{L_v^2}(0,v) + \delta^{\frac{1}{2}} \|F\|_{L_v^2 L_u^2}(u,v). \end{aligned} \tag{A.132}$$

We repeat this in the u variable to deduce the inequality

$$\|\psi_v\|_{L_v^2}(u,v) + \|\psi_u\|_{L_u^2}(u,v) \leq \|\psi_{2,v}\|_{L_v^2}(0,v) + \|\psi_{1,u}\|_{L_u^2}(u,0) + 2\delta^{\frac{1}{2}} \|F\|_{L_v^2 L_u^2}(u,v). \tag{A.133}$$

Which extends to

$$\|\psi_v\|_{L_v^2}(u,\delta) + \|\psi_u\|_{L_u^2}(\delta,v) \leq \|\psi_{2,v}\|_{L_v^2}(0,\delta) + \|\psi_{1,u}\|_{L_u^2}(\delta,0) + 2\delta^{\frac{1}{2}} \|F\|_{L_v^2 L_u^2}(\delta,\delta). \tag{A.134}$$

Taking C^0 norms gives

$$\|\psi_v\|_{C_u^0 L_v^2(\Box_\delta)} + \|\psi_u\|_{C_v^0 L_u^2(\Box_\delta)} \leq \|\psi_{2,v}\|_{L^2(\mathcal{N})} + \|\psi_{1,u}\|_{L^2(\mathcal{N}')} + 2\delta^{\frac{1}{2}} \|F\|_{L^2(\Box_\delta)}. \tag{A.135}$$

We now need the first order terms. From the fundamental theorem of calculus

$$\begin{aligned} |\psi(u,v)| &= \left|\psi(0,v) + \int_0^u \psi_u(u',v)du'\right| \\ &\leq |\psi(0,v)| + \delta^{\frac{1}{2}} \|\psi_u\|_{L_u^2}(u,v), \end{aligned} \tag{A.136}$$

hence

$$\|\psi(u,v)\|_{C_v^0} \leq \|\psi(0,v)\|_{C_v^0} + \delta^{\frac{1}{2}} \|\psi_u\|_{C_v^0 L_u^2}(u,v). \tag{A.137}$$

Thus from estimating

$$\begin{aligned} \|\psi\|_{L_v^2}^2(u,v) &= \int_0^v |\psi(u,v')|^2 dv' \\ &\leq \delta \|\psi(0,v)\|_{L_v^2}^2 + \delta^2 \|\psi_u\|_{C_v^0 L_u^2}^2(u,v), \end{aligned} \tag{A.138}$$

we conclude

$$\|\psi\|_{L_v^2}(u,v) \leq \delta^{\frac{1}{2}} \|\psi(0,v)\|_{L_v^2} + \delta \|\psi_u\|_{C_v^0 L_u^2}(u,v). \tag{A.139}$$

Similarly

$$||\psi||_{L^2_u}(u,v) \le \delta^{\frac{1}{2}} ||\psi(u,0)||_{L^2_u} + \delta ||\psi_u||_{C^0_u L^2_v}(u,v). \tag{A.140}$$

From here we can then see

$$||\psi||_{C^0_u H^1_v(\Box_\delta)} + ||\psi||_{C^0_v H^1_u(\Box_\delta)} \le \delta^{\frac{1}{2}} ||\psi_1||_{L^2(\mathcal{N}')} + \delta^{\frac{1}{2}} ||\psi_2||_{L^2(\mathcal{N})} + \delta ||\psi_{2,v}||_{L^2(\mathcal{N})} + \delta ||\psi_{1,u}||_{L^2(\mathcal{N}')} + 2\delta^{\frac{3}{2}} ||F||_{L^2(\Box_\delta)}. \tag{A.141}$$

Finally

$$||\psi||_{C^0 H^1(\Box_\delta)} \le \delta^{\frac{1}{2}} ||\psi_1||_{L^2(\mathcal{N}')} + \delta^{\frac{1}{2}} ||\psi_2||_{L^2(\mathcal{N})} + \delta ||\psi_{2,v}||_{L^2(\mathcal{N})} + \delta ||\psi_{1,u}||_{L^2(\mathcal{N}')} + 2\delta^{\frac{3}{2}} ||F||_{L^2(\Box_\delta)}. \tag{A.142}$$

□

Lemma A.4.2.

$$||\psi - \overline{\psi}||_{C^0} \le ||\psi||_{C^0 H^1}, \tag{A.143}$$

Proof. From the fundamental theorem of calculus we see

$$\begin{aligned} |\psi(u,v) - \psi(0,v)| &= \left| \int_0^u \psi_u(u',v) du' \right| \\ &\le \delta^{\frac{1}{2}} ||\psi_u||_{L^2_u}(u,v) \\ &\le ||\psi||_{C^0 H^1}, \end{aligned} \tag{A.144}$$

and

$$\begin{aligned} |\psi(u,v) - \psi(0,v)| &= \left| \int_0^v \psi_u(u,v') dv' \right| \\ &\le \delta^{\frac{1}{2}} ||\psi_v||_{L^2_v}(u,v) \\ &\le ||\psi||_{C^0 H^1}. \end{aligned} \tag{A.145}$$

Applying the triangle inequality we get

$$|\psi(u,v) - \overline{\psi}(u,v)| \le \frac{1}{2} |\psi(u,v) - \psi(0,v)| + \frac{1}{2} |\psi(u,v) - \psi(u,0)| \le ||\psi||_{C^0 H^1}. \tag{A.146}$$

We conclude

$$||\psi - \overline{\psi}||_{C^0} \le ||\psi||_{C^0 H^1}. \tag{A.147}$$

□

Lemma A.4.3. *We also have the following estimate for the u derivative*

$$||\psi_u(u,v) - \overline{\psi}_u(u,v)||_{C^0_u} \le \frac{1}{2} ||\psi_{1,u}||_{C^0(\mathcal{N}')} + ||F||_{C^0_u L^1_v(\Box_\delta)}, \tag{A.148}$$

Proof. First integrate (A.8) to get

$$\psi_u(u,v) - \frac{1}{2}\psi_u(u,0) = \frac{1}{2}\psi_u(u,0) + \int_0^v F dv. \tag{A.149}$$

From taking absolute values the result follows. $\square$

Lemma A.4.4. *Let $F \in C^0_u L^1_v(\square_\delta)$ then there exists an $F_\epsilon \in C^\infty(\square_\delta)$ such that $F_\epsilon \to F$ in $C^0_u L^1_v(\square_\epsilon := [-\delta, 2\delta] \times [-\delta, 2\delta])$.*

Proof. First fix $1 > \epsilon > 0$ we then extend F by for $(u, v) \in \square_\epsilon$

$$\begin{aligned} F_e(u+\delta, v) =& F(\delta - u, v) \\ F_e(u-\delta, v) =& F(u, v) \\ F_e(u, v \pm \delta) =& F(u, v), \end{aligned} \tag{A.150}$$

(i.e. periodic in v and reflections in u). So $F_e \in C^0_u L^2_v(\square_\epsilon)$, and agrees with F on $\square_\delta$. Now let φ_ϵ be the standard smoothing kernel, and define on $\square_\epsilon$

$$F_\epsilon(u, v) = \int_{\mathbb{R}} \varphi_\epsilon(v') F_e(u, v - v') \chi_{(\epsilon, \delta - 2\epsilon)}(v') dv'. \tag{A.151}$$

Thus $F_\epsilon \in C^\infty(\square_\epsilon)$. We now extend to all of $C^\infty_c(\mathbb{R}^2)$ by using a smooth cut-off function to 0 in the annular set $[-2\delta, 3\delta] \times [-2\delta, 3\delta] \backslash \square_\epsilon$, and 0 on the set $[3\delta, \infty) \times [3\delta, \infty)$. It is now our goal to show that when restricted to $\square_\delta$ we have that $F_\epsilon \to F$ in $C^0_u L^1_v(\square_\epsilon)$. We compute

$$\begin{aligned} ||F_\epsilon(u, \cdot) - F(u, \cdot)||_{L^1_v(\square_\delta)} &= \int_0^\delta \left| \int_{\mathbb{R}} \varphi_\epsilon(v') \left(F(u, v - v') \chi_{(\epsilon, \delta-2\epsilon)}(v') - F(u, v) \right) dv' \right| dv \\ &\leq \int_0^\delta \int_{\mathbb{R}} |\varphi_\epsilon(v')| \left| \left(F(u, v - v') \chi_{(\epsilon, \delta-2\epsilon)}(v') - F(u, v) \right) \right| dv' dv \\ &= \int_{\mathbb{R}} \int_0^\delta |\varphi_\epsilon(v')| \left| \left(F(u, v - v') \chi_{(\epsilon, \delta-2\epsilon)}(v') - F(u, v) \right) \right| dv dv' \\ &= \int_{\mathbb{R}} |\varphi_\epsilon(v')| \int_0^\delta \left| \left(F(u, v - v') \chi_{(\epsilon, \delta-2\epsilon)}(v') - F(u, v) \right) \right| dv dv' \\ &= \int_{\mathbb{R}} |\varphi_\epsilon(w)| \int_0^{\frac{\delta}{\epsilon}} \left| \left(F(u, v - \epsilon w) \chi_{(\epsilon, \delta-2\epsilon)}(\epsilon w) - F(u, v) \right) \right| \epsilon dv dw. \end{aligned} \tag{A.152}$$

As $F \in C^0_u L^1_v(\square_\epsilon)$ we have that

$$\left| \left| \left(F(u, v - \epsilon w) \chi_{(\epsilon, \delta-2\epsilon)}(\epsilon w) - F(u, v) \right) \right| \right|_{L^1_v} \in C^0_u([-\delta, 2\delta]). \tag{A.153}$$

From compactness there exists a $u^* \in [-\delta, 2\delta]$, such that

$$\left| \left| \left(F(u, v - \epsilon w) \chi_{(\epsilon, \delta-2\epsilon)}(\epsilon w) - F(u, v) \right) \right| \right|_{L^1_v} \leq \left| \left| \left(F(u^*, v - \epsilon w) \chi_{(\epsilon, \delta-2\epsilon)}(\epsilon w) - F(u^*, v) \right) \right| \right|_{L^1_v}. \tag{A.154}$$

By Fatou's lemma

$$\sup_{u\in[0,\delta]} ||F_\epsilon(u,\cdot) - F(u,\cdot)||_{L^1_v} \leq \int_{\mathbb{R}} |\psi(w)| \sup_{u\in[0,\delta]} ||F_\epsilon(u,\cdot) - F(u,\cdot)||_{L^1_v} dw$$
$$\leq \int_{\mathbb{R}} |\psi(w)| \sup_{u\in[0,\delta]} \left|\left|\tau_{\epsilon w}\left(F(u,\cdot)\chi_{(2\epsilon,\delta-2\epsilon)}(\cdot)\right) - F(u,\cdot)\right|\right|_{L^1_v} dw, \tag{A.155}$$

where τ is the translation operator. The goal now is to show that

$$\lim_{\epsilon\to 0} \sup_{u\in[0,\delta]} ||F_\epsilon(u,\cdot) - F(u,\cdot)||_{L^1_v} = 0. \tag{A.156}$$

Assume the contrapositive. Then we may find $\beta > 0$, $\epsilon_i \to 0$, and u_i such that

$$\left|\left|\tau_{\epsilon w}\left(F(u,\cdot)\chi_{(2\epsilon,\delta-2\epsilon)}(\cdot)\right) - F(u,\cdot)\right|\right|_{L^1_v} \geq \beta. \tag{A.157}$$

Since $u \in [0,\delta]$, we can assume (after extracting a subsequence) that $u_i \to u^*$ with $u^* \in [0,\delta]$. Now note the following inequality

$$\begin{aligned}
\left|\left|\tau_{\epsilon w}\left(F(u_i,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon)}(\cdot)\right) - F(u_i,\cdot)\right|\right|_{L^1_v} &\leq \left|\left|\tau_{\epsilon_i w}\left(F(u_i,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot)\right) - \tau_{\epsilon_i w}\left(F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot)\right)\right|\right|_{L^1_v} \\
&+ \left|\left|\tau_{\epsilon_i w}\left(F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot)\right) - F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot)\right|\right|_{L^1_v} \\
&+ \left|\left|F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot) - F(u^*,\cdot)\right|\right|_{L^1_v} \\
&+ ||F(u^*,\cdot) - F(u_i,\cdot)||_{L^1_v} \\
&= \left|\left|F(u_i,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot) - F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot)\right|\right|_{L^1_v} \\
&+ \left|\left|\tau_{\epsilon_i w}\left(F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot)\right) - F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot)\right|\right|_{L^1_v} \\
&+ \left|\left|F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot) - F(u^*,\cdot)\right|\right|_{L^1_v} \\
&+ ||F(u^*,\cdot) - F(u_i,\cdot)||_{L^1_v}.
\end{aligned} \tag{A.158}$$

Examining each norm one by one. For the first term

$$\left|\left|F(u_i,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot) - F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot)\right|\right|_{L^1_v} \to 0, \tag{A.159}$$

this follows from $F \in C^0_u L^1_v$ and that $u_i \to u^*$. For the second

$$\left|\left|\tau_{\epsilon_i w}\left(F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot)\right) - F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot)\right|\right|_{L^1_v} \to 0, \tag{A.160}$$

this follows from the continuity of the translation map from $L^1 \to L^1$. For the third

$$\left|\left|F(u^*,\cdot)\chi_{(2\epsilon_i,\delta-2\epsilon_i)}(\cdot) - F(u^*,\cdot)\right|\right|_{L^1_v} \to 0, \tag{A.161}$$

the follows as integrand is dominated by $2\,|F(u^*,\cdot)|$, and that the indicator function will converge pointwise to unity. Finally for the fourth

$$||F(u^*, \cdot) - F(u_i, \cdot)||_{L^1_v} \to 0, \tag{A.162}$$

this follows from $F \in C^0_u L^1_v$ and that $u_i \to u^*$. So we have shown that

$$\left|\left|\tau_{\epsilon w}\left(F(u_i, \cdot)\chi_{(2\epsilon_i, \delta - 2\epsilon)}(\cdot)\right) - F(u_i, \cdot)\right|\right|_{L^1_v} \to 0, \tag{A.163}$$

which is a contradiction.

□