Journal of Physics: Conference Series **68** (2007) 012001 doi:10.1088/1742-6596/68/1/012001

Gravitational field of relativistic gyratons

Valeri P. Frolov

Theoretical Physics Institute, Department of Physics,
University of Alberta, Edmonton, AB, Canada, T6G 2J1

E-mail: frolov@phys.ualberta.ca

Abstract. A gyraton is an object moving with the speed of light and having finite energy and internal angular momentum (spin). First we derive the gravitational field of a gyraton in the linear approximation. After this we study solutions of the vacuum Einstein equations for gyratons. We demonstrate that these solutions in 4 and higher dimensions reduce to two linear problems in a Euclidean space. A similar reduction is also valid for gyraton solutions of the Einstein-Maxwell gravity and in supergravity. Namely, we demonstrate that in the both cases the solutions in 4 and higher dimensions reduce to linear problems in a Euclidean space.

1. Introduction

A gyraton is an object moving with the speed of light and having finite energy and internal angular momentum (spin). A physically interesting example of a gyraton-like object is a spinning (circular polarized) beam-pulse of the high-frequency electromagnetic or gravitational radiation. Studies of the gravitational fields of beams and pulses of light have a long history. Tolman [1] found a solution in the linear approximation. Peres [2, 3] and Bonnor [4] obtained exact solutions of the Einstein equations for a pencil of light. Polarization effects were studied in [5]. In the present paper we discuss the generalization of these solutions to the case where the beam of radiation carries angular momentum and the number of spacetime dimensions is arbitrary [6, 7]. Such solutions are important for study mini black hole formation in a high-energy collision of two particles with spin. We also discuss a generalisation of these solutions to the case when a gyraton has an electric [8] or Kalb-Ramon [9] charge.

2. Gravitational Field of a Gyraton in a Linear Approximation

We consider first the gravitational field of a spinning massive point-like object in the linearized gravity. We write the linearized gravitational field in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat metric. We denote by D the total number of spacetime dimensions and by $(\bar{t}, \mathbf{X})$ the Cartesian coordinates in the Minkowski spacetime. The linearized Einstein equations has the form

$$\Box h_{\mu\nu} = -16\pi G\,(T_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}\eta^{\alpha\beta}T_{\alpha\beta})\,, \tag{1}$$

where G is the $D-$dimensional gravitational coupling constant. If a compact (point-like) rotating object is at rest its stress-energy components are

$$T_{\bar{t}\bar{t}} = M\delta^{D-1}(\mathbf{X}) , \quad T_{\bar{t}a} = J_{ab}\partial_b\delta^{D-1}(\mathbf{X}) , \tag{2}$$

where M is the mass and J_{ab} is the angular momentum of the body. In what follows we shall consider a motion of the point-like spinning object with a constant velocity. We choose one of the spatial coordinates, $\bar{\xi}$, to be in the direction of motion, and denote the other spatial coordinates in the direction transverse to the motion by $\mathbf{x} = (x^a)$. We use notations $\bar{t}$ and $\bar{\xi}$ for the coordinates in the reference frame where the source is at rest. For simplicity we also assume that $T_{\bar{t}\bar{\xi}} = 0$.

By solving equations (1), one obtains the linearized metric in the form

$$ds^2 = -d\bar{t}^2 + d\bar{\xi}^2 + d\mathbf{x}^2 + 2\bar{A}_{ab}x^b dx^a d\bar{t} + \bar{\Phi}[d\bar{t}^2 + \frac{1}{D-3}(d\bar{\xi}^2 + d\mathbf{x}^2)] , \tag{3}$$

$$\bar{\Phi} \sim \frac{M}{\bar{r}^{D-3}} , \quad \bar{A}_{ab} \sim \frac{J_{ab}}{\bar{r}^{D-1}} , \quad \bar{r}^2 = \bar{\xi}^2 + r^2 , \quad r^2 = \mathbf{x}^2 . \tag{4}$$

Because of the linearity, a solution for an extended one-dimensional (line-like) object oriented in $\bar{\xi}-$direction can be written in the same form (3) with the following functions $\bar{\Phi}$ and $\bar{A}_{ab}$

$$\bar{\Phi} \sim \int \frac{d\bar{\xi}' \ \bar{\varepsilon}(\bar{\xi}')}{[(\bar{\xi} - \bar{\xi}')^2 + \mathbf{x^2}]^{(D-3)/2}} , \tag{5}$$

$$\bar{A}_{ab} \sim \int \frac{d\bar{\xi}' \ \bar{j}_{ab}(\bar{\xi}')}{[(\bar{\xi} - \bar{\xi}')^2 + \mathbf{x^2}]^{(D-1)/2}} , \tag{6}$$

Here $\bar{\varepsilon}$ and $\bar{j}_{ab}$ are the mass and angular momentum densities, respectively.

To obtain a metric for a spinning source moving with the speed of light we boost the solution (3), (5), (6), that is consider this solution in a reference frame which is moving with a constant velocity. We denote by t and ξ the coordinates in the frame which is moving along $\bar{\xi}$ axis with the velocity β (in the negative direction) and $\gamma = (1-\beta^2)^{-1/2}$

$$\bar{\xi} = \gamma(\xi - \beta t) = \frac{\gamma}{\sqrt{2}}[(1-\beta)v + (1+\beta)u] ,$$

$$\bar{t} = \gamma(t - \beta\xi) = \frac{\gamma}{\sqrt{2}}[(1-\beta)v - (1+\beta)u] .$$

Here $u = (\xi - t)/\sqrt{2}$ and $v = (\xi + t)/\sqrt{2}$ are null coordinates in the flat background spacetime. If the length l of the source in the $\bar{\xi}-$direction remains the same, its length in the moving frame because of the Lorentz contraction becomes γ^{-1}. In order to obtain an on object moving with the speed of light and having *finite duration* (length) we assume that transition to the boosted frame is accompanied by the scaling with the factor γ of the initial length of the object. We also assume that the energy, $E = \gamma M$, and the angular momentum, J_{ab}, remain fixed.

In the so-called Penrose limit, that is when $\beta \to 1$, one has

$$\bar{t} \sim -\sqrt{2}\gamma u , \quad \bar{\xi} \sim \sqrt{2}\gamma u . \tag{7}$$

Under the above conditions one also has

$$E = \gamma\bar{M} = \int \varepsilon(u)du , \ \bar{\varepsilon}(\bar{\xi}) = \frac{1}{\sqrt{2}\gamma^2}\varepsilon(u) , \tag{8}$$

$$J_{ab} = \bar{J}_{ab} = \int j_{ab}(u)du\,,\;\; \bar{j}_{ab}(\bar{\xi}) = \frac{1}{\sqrt{2}\gamma} j_{ab}(u)\,. \tag{9}$$

To obtain the metric we use the following relation

$$\lim_{\gamma\to\infty} \frac{\gamma}{(\gamma^2 y^2 + r^2)^{m/2}} = \frac{\sqrt{\pi}\Gamma((m-1)/2)}{\Gamma(m/2)} \frac{\delta(y)}{r^{m-1}}\,. \tag{10}$$

Using this relation one obtains

$$ds^2 = -2du\,dv + d\mathbf{x}^2 + 2A_{ab}x^b dx^a du + \Phi du^2\,, \tag{11}$$

$$\Phi \sim \frac{\varepsilon(u)}{r^{(D-4)/2)}}\,, \qquad A_{ab} \sim \frac{j_{ab}(u)}{r^{(D-2)/2)}}\,. \tag{12}$$

For Aichelburg-Sexl metric $\varepsilon = E\delta(u)$ and $A_{ab} = 0$

3. Reduction of the Einstein equations

Now our purpose is to obtain an exact solution of the vacuum Einstein equations (valid outside the region occupied by a gyraton) which has the asymptotic form (11)-(12) at large r. We assume that the metric has the same form as an asymptotic solution and write it as follows

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -2\;du\;dv + d\mathbf{x}^2 + \Phi\;du^2 + 2\;(\mathbf{A}, d\mathbf{x})du\,, \tag{13}$$

$$\Phi = \Phi(u, \mathbf{x})\,, \qquad A_a = A_a(u, \mathbf{x})\,. \tag{14}$$

This metric is known as Brinkmann [10] metric. Evidently, $l^\mu\partial_\mu = \partial_v$ is the null Killing vector.

When $\Phi = \mathbf{A} = 0$, the coordinates $x^1 = v = (t+\xi)/\sqrt{2}$ and $x^2 = u = (t-\xi)/\sqrt{2}$ are null. The coordinate u remains null for the metric (13). The metric is generated by an object moving with the velocity of light in the $\xi-$direction. The coordinates $(x^3, \ldots, x^D)$ are coordinates of an $n-$dimensional space ($n = D-2$) transverse to the direction of motion. We use boldface symbols to denote vectors in this space. For example, $\mathbf{x}$ is a vector with components x^a ($a = 3, \ldots, D$). We denote by r the length of this vector, $r = |\mathbf{x}|$. We also denote

$$d\mathbf{x}^2 = \sum_{a=3}^{D}(dx_a)^2\,, \qquad (\mathbf{A}, d\mathbf{x}) = \sum_{a=3}^{D} A_a\;dx^a\,, \tag{15}$$

$$\triangle = \sum_{a=3}^{D} \partial_a^2\,, \qquad \mathrm{div}\mathbf{A} = \sum_{a=3}^{D} A^a_{,a}\,. \tag{16}$$

We assume that the sum is taken over the repeated indices and omit the summation symbol. Working in the Cartesian coordinates we shall not distinguish between upper and lower indices.

The form of the metric (13) is invariant under the following (gauge) transformation

$$v \to v + \lambda(u, \mathbf{x})\,,\;\; A_a \to A_a + \lambda_{,a}\,,\;\; \Phi \to \Phi + 2\lambda_{,u}\,. \tag{17}$$

It is also invariant under rescaling

$$u \to au\,,\;\; v \to a^{-1}v\,,\;\; \Phi \to a^2\Phi\,,\;\; \mathbf{A} \to a\mathbf{A}\,. \tag{18}$$

It is easy to check that

$$l_{\mu;\nu} = 0\,. \tag{19}$$

It means that the null Killing vector $\mathbf{l}$ is covariantly constant. In the 4-dimensional case, spacetimes admitting a (covariantly) constant null vector field are called plane-fronted gravitational waves with parallel rays, or briefly pp-waves (see e.g. [11, 12, 13]). Similar terminology is often used for higher dimensional metrics (see e.g. [14, 15]).

It is possible to show that all the local scalar invariants constructed from the Riemann tensor and its covariant derivatives for the metric (13) vanish. This statement is valid *off shell*, that is the metric need not be a solution of the vacuum Einstein equations. This property is well known for 4-dimensional case, since pp-wave solutions are of Petrov type N. Generalization of this result to higher-dimensional metrics (13) with $\mathbf{A} = 0$ was given in [16, 17]. (For a general discussion of spacetimes with vanishing curvature invariants see [18, 19, 20, 21]).

Calculations give the following non-vanishing components for the Ricci tensor

$$R_{uu} = \partial_u \mathrm{div}\mathbf{A} - \frac{1}{2} \triangle \Phi + \frac{1}{4}\mathbf{F}^2\,, \tag{20}$$

$$R_{au} = \frac{1}{2}\partial_b F_a{}^{b}\,, \tag{21}$$

where

$$\mathrm{div}\mathbf{A} = \partial_a A^a\,, \qquad \mathbf{F}^2 = F_{ab}F^{ab}\,, \qquad \triangle\Phi = \partial_a\partial^a\Phi\,. \tag{22}$$

Thus, the metric (13) is a solution of vacuum Einstein equations if and only if the following equations are satisfied

$$\partial_b F_a{}^{b} = 0\,, \tag{23}$$

$$\triangle\Phi - 2\partial_u \mathrm{div}\mathbf{A} = \frac{1}{2}\mathbf{F}^2\,. \tag{24}$$

4. Electrically charged gyratons

We discuss now the electrically charged gyratons. We use the anzats (13) for the metric. We choose the electromagnetic vector potential A_μ in the form $A_\mu = A_u\delta^u_\mu + A_a\delta^a_\mu$, where the functions A_u and A_a are independent of the null coordinate v. The electromagnetic gauge transformations $A_\mu \to A_\mu + \Lambda_{,\mu}$ with $\Lambda = \Lambda(u, \mathbf{x})$ preserve the form of the potential. The Maxwell strength tensor for this vector potential is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

The quantities A_μ and $F_{\mu\nu}$ obey the relations

$$l^\epsilon A_\epsilon = l^\epsilon F_{\epsilon\mu} = 0\,. \tag{25}$$

The Einstein-Maxwell action in higher dimensions reads

$$S = \frac{1}{16\pi G}\int d^D\mathbf{x}\sqrt{|g|}\left[R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right].$$

Here G is the gravitational coupling constant in $D-$dimensional spacetime. The stress-energy tensor for the electromagnetic field is

$$T_{\mu\nu} = \frac{1}{16\pi G}\left[F_\mu{}^{\epsilon}F_{\nu\epsilon} - \frac{1}{4}g_{\mu\nu}\mathbf{F}^2\right], \tag{26}$$

Journal of Physics: Conference Series **68** (2007) 012001 doi:10.1088/1742-6596/68/1/012001

where $\mathbf{F}^2 = F_{\epsilon\sigma}F^{\epsilon\sigma}$.

Our aim now is to find solutions of the system of Einstein-Maxwell equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,, \tag{27}$$

$$F_{\mu}{}^{\nu}{}_{;\nu} = J_{\mu}\,, \tag{28}$$

for the adopted field ansatz (13) and (25). Direct calculations [7] show that for the metric (13) the scalar curvature vanishes $R = 0$ and the only nonzero components of the Ricci tensor are

$$R_{ua} = \frac{1}{2}f_{ab}{}^{,b}, \tag{29}$$

$$R_{uu} = -(a_u)^{;a}_{,a} + \frac{1}{4}f_{ab}f^{ab} + \partial_u(a_a{}^{,a})\,. \tag{30}$$

Since $R = \delta^{ab}R_{ab} = 0$ one has $\mathbf{F}^2 = 0$, and hence $F_{ab} = 0$. Thus in a proper gauge the transverse components of the electromagnetic vector potential vanish, $A_a = 0$. Let us denote $\mathcal{A} = A_u$, then the only non-vanishing components of $F_{\mu\nu}$ and $T_{\mu\nu}$ are

$$F_{ua} = -F_{au} = -\mathcal{A}_{,a}\,, \qquad T_{uu} = \frac{1}{16\pi G}(\nabla\mathcal{A})^2\,. \tag{31}$$

where $(\nabla\mathcal{A})^2 = \delta^{ab}\mathcal{A}_{,a}\mathcal{A}_{,b}$.

Thus the requirement that the electromagnetic field is consistent with the Einstein equations for the gyraton metric ansatz (13) implies that the vector potential a_μ can be chosen $a_\mu \sim l_\mu$, i.e., to be aligned with the null Killing vector. In this case $F_{\mu\nu}$ and all its covariant derivatives are also aligned with l_μ. Together with the orthogonality conditions (25) these properties can be used to prove that all local scalar invariants constructed from the Riemann tensor, the Maxwell tensor (31) and their covariant derivatives vanish. This property generalizes the analogous property for above discussed property of uncharged gyratons. It can also be used to prove that the charged gyraton solutions of the Einstein-Maxwell equations are also exact solutions of any other nonlinear electrodynamics and the Einstein equations [8].

Eventually the Einstein equations reduce to the following two sets of equations in $n-$dimensional flat space R^n

$$(a_u)^{;a}_{,a} - \partial_u(a_a{}^{,a}) = \frac{1}{4}f_{ab}f^{ab} - \frac{1}{2}(\nabla\mathcal{A})^2\,, \tag{32}$$

$$f_{ab}{}^{,b} = 0\,, \qquad f_{ab} = a_{b,a} - a_{a,b}\,. \tag{33}$$

We are looking for the field outside the region occupied by the gyraton, where $J_\mu = 0$. The Maxwell equations then reduce to the relation

$$\triangle\mathcal{A} = 0\,. \tag{34}$$

Here $\triangle$ is a flat $n-$dimensional Laplace operator. The electric charge of the gyraton is determined by the total flux of the electric field across the surface $\partial\Sigma$ surrounding it

$$Q = \frac{1}{16\pi G}\int_{\partial\Sigma} F^{\mu\nu}d\sigma_{\mu\nu}. \tag{35}$$

For a charged gyraton one has

$$Q \equiv \int_{u_1}^{u_2} du\, \rho(u). \tag{36}$$

where $\rho(u)$ is a linear charge density. By comparing (35) and (36) one can conclude that in the asymptotic region $r \to \infty$ the following relations are valid

$$F_{ur} \approx \begin{cases} \dfrac{16\pi G(n-2) g_n \rho(u)}{r^{n-1}}, & \text{for } n > 2, \\ \dfrac{8G\rho(u)}{r}, & \text{for } n = 2. \end{cases}$$

Here $g_n = \Gamma(\frac{n-2}{2})/(4\pi^{n/2})$.

Now we return to the Einstein equations. The combination which enters the left hand side of (32) is invariant under the transformation (17). One can use this transformation to put $a_a{}^{,a} = 0$. We shall use this "gauge" choice and denote $a_u = \frac{1}{2}\Phi$ in this "gauge". The equations (32)-(33) take the form ($\Phi = \varphi + \psi$)

$$\triangle \varphi = 0\,, \tag{37}$$

$$\triangle \psi = \frac{1}{2} f_{ab} f^{ab} - (\nabla \mathcal{A})^2\,, \tag{38}$$

$$\triangle a_a = 0\,. \tag{39}$$

The set of equations (34) and (37)–(39) determines the metric

$$ds^2 = d\bar{s}^2 + \Phi du^2 + 2a_a dx^a\, du \tag{40}$$

and the electromagnetic field $\mathcal{A}$ of a gyraton. (Let us emphasize again that these equations are valid only outside the region occupied by the gyraton.) These equations are linear equations in an Euclidean n-dimensional space R^n. The equations (34) and (37) coincide with the equations for electric potential created by a point like source, while the equation (39) formally coincides with the equation for the magnetic field. The last equation (38) is a linear equation in the Euclidean space which can be solved after one finds solutions for f_{ab} and $\mathcal{A}$. Thus for a chosen ansatz for the metric and the electromagnetic fields, the solution of the Einstein-Maxwell equations in $D-$dimensional spacetime reduce to linear problems in an Euclidean $n-$dimensional space ($n = D - 2$). Special solutions of these equations can be found in [8].

5. Gyratons in supergravity

We discuss now the gyratonic solutions in the supergravity [9]. We consider the massless bosonic sector of supergravity. We restrict ourselves by discussing what is called the common sector. The fields in the common sector are the metric $g_{\mu\nu}$, the Kalb-Ramond antisymmetric field $B_{\mu\nu}$ and the dilaton field ϕ. The corresponding action, which is also the low-energy superstring effective action, is

$$S = \frac{1}{16\pi G} \int d^D x\, \sqrt{|g|} e^{-2\phi} [R - 4(\nabla\phi)^2$$
$$-\frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda}] + \frac{1}{2} \int d^D x\, \sqrt{|g|} B_{\mu\nu} J^{\mu\nu} + \mathcal{S}_m\,. \tag{41}$$

Journal of Physics: Conference Series **68** (2007) 012001 doi:10.1088/1742-6596/68/1/012001

Here G is the $D-$dimensional gravitational (Newtonian) coupling constant, and $\mathcal{S}_m$ is the action for the string matter source. The string coupling constant g_s is determined by the vacuum expectation value of the dilaton field ϕ_0, $g_s = \exp(\phi_0)$. The 3-form flux

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\nu} + \partial_\lambda B_{\mu\nu} \tag{42}$$

is the Kalb-Ramond (KR) field strength and $B_{\mu\nu}$ is its anti-symmetric 2-form potential. The field $H_{\mu\nu\lambda}$ is invariant under the gauge transformation

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu. \tag{43}$$

$J^{\mu\nu}$ is the antisymmetric tensor of the current which plays the role of a source for the KR field.

For example, for the interaction of the KR field with a fundamental string described by the action

$$S_{int} = -\frac{q}{2}\int d^2\zeta \epsilon^{ab} B_{\mu\nu} \frac{\partial X^\mu}{\partial \zeta^a}\frac{\partial X^\nu}{\partial \zeta^b} \tag{44}$$

this current is

$$J^{\mu\nu}(x) = \frac{q}{2}\int d^2\zeta \frac{\delta^D(x - X(\zeta))}{\sqrt{|g|}} \epsilon^{ab} \frac{\partial X^\mu}{\partial \zeta^a}\frac{\partial X^\nu}{\partial \zeta^b}\,. \tag{45}$$

Here ϵ^{ab} is the antisymmetric symbol, $\zeta^a = (\tau, \sigma)$ are parameters on the string surface and the functions $X^\mu = X^\mu(\zeta)$ determine the embedding of the string worldsheet in the bulk (target) spacetime. The parameter q is the "string charge". The current $J^{\mu\nu}$ is tangent to the worldsheet of the string, $J^{[\mu\nu} X^{\lambda]}_{,c} = 0$.

We shall study a special class of gyraton solutions for which the dilaton field is constant, i.e., $e^\phi = g_s$. In this case the field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} + g_s^2 \kappa \mathcal{T}_{\mu\nu}\,, \tag{46}$$

$$H_{\mu\nu}{}^{l}{}_{;l} = 8\kappa J_{\mu\nu}. \tag{47}$$

Here the stress-energy tensor for the 3-form flux is

$$T_{\mu\nu} = \frac{1}{12}(3 H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} - \frac{1}{2} g_{\mu\nu} H_{\rho\sigma\lambda} H^{\rho\sigma\lambda})\,, \tag{48}$$

and $\kappa = 8\pi g_s^2 G$. $\mathcal{T}_{\mu\nu}$ which enters the equation (46) is the stress-energy of the matter (string) which we shall specify later.

Let Σ be a $(D-2)-$dimensional spacelike surface, and $\partial\Sigma$ be its boundary. We define the charge of the fundamental string intersecting Σ by Gauss's law as

$$Q := \int_{\partial\Sigma} d\sigma_{D-3} \;\, *_D\, H_3 = \int d\sigma_{\mu\nu\lambda} H_{\mu\nu\lambda}\,. \tag{49}$$

By using the Stoke's theorem and (47) one has

$$Q = \int d\sigma_{\mu\nu} J^{\mu\nu} = 8\kappa q\,. \tag{50}$$

Here $d\sigma_{\mu_1\cdots\mu_n} := i_{\mu_1}\cdots i_{\mu_n}(*1)$ in which $*1$ is the volume form of D-dimensional spacetime and $i_{\mu u} : \Lambda^p T^* \to \Lambda^{p-1} T^*$, $i_\mu dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_p} = p\delta_\mu^{[\nu_1} dx^{\nu_2} \wedge \cdots \wedge dx^{\nu_p]}$.

In what follows, we consider the gravitational and KR fields outside the sources, that is in the region where $\mathcal{T}_{\mu\nu} = 0$ and $J_{\mu\nu} = 0$. The relation (50) will be used to relate the parameters which enter a solution to the charge of the string.

6. Ansatz for Supergravity Gyraton

We use the same ansatz (13) for the metric as in the case of an electrically charged gyraton. For the KR field potential we use the ansatz similar to the one adopted for the electromagnetic gyratons (25). Namely we postulate that $B_{\mu\nu} = B_{\mu\nu}(u, \boldsymbol{x})$ and

$$l^\mu B_{\mu\nu} = 0\,. \tag{51}$$

It is easy to check that

$$l^\mu H_{\mu\nu\lambda} = 0. \tag{52}$$

The imposed constraints imply that the only non-vanishing components of $B_{\mu\nu}$ are $B_{ua}(u, \boldsymbol{x})$ and $B_{ab}(u, \boldsymbol{x})$, and of $H_{\mu\nu l}$ are $H_{uab}(u, \boldsymbol{x})$ and $H_{abc}(u, \boldsymbol{x})$. Moreover, to preserve the constraint (51) under gauge transformation, we should impose

$$\partial_v \Lambda_\mu - \partial_\mu \Lambda_v = 0. \tag{53}$$

The equation (52) implies that

$$H^2 = \boldsymbol{H}^2 \equiv H_{abc} H^{abc}. \tag{54}$$

We discuss now the ansatz for $\mathcal{T}_{\mu\nu}$ which enters the equation (47). We require that this tensor obeys the conservation low

$$\mathcal{T}^{\mu\nu}_{\ \ ;\nu} = 0\,, \tag{55}$$

and is aligned to the null Killing vector l_μ

$$\mathcal{T}_{\mu\nu} = l_{(\mu} p_{\nu)}\,, \qquad l^\mu p_\mu = 0\,. \tag{56}$$

The last condition guarantees that the trace of $\mathcal{T}^{\mu\nu}$ vanishes, $\mathcal{T}^\mu_\mu = 0$. For the metric (13) these conditions are satisfied when

$$p_\mu = p_\mu(u, x^a)\,, \qquad p^a{}_{,a} = 0\,. \tag{57}$$

This can be checked by using the condition $l_{\mu;\nu} = 0$. Bonnor [5] called such matter in 4-dimensional spacetime spinning null fluid.

7. Reduction of supergravity equations

The only non-vanishing components of the Ricci tensor for the metric (13) are given by equations (29) and (30). These relations imply $R = 0$. Since $\mathcal{T}^\mu_\mu = 0$ the field equation (46) yields $T^\mu_\mu = 0$. This equation implies $\boldsymbol{H}^2 = 0$, and hence $H_{abc} = 0$. Therefore, the only non-vanishing components of $H_{\mu\nu\lambda}$ are $H_{uab}(u, \boldsymbol{x})$. This means that $H_{\mu\nu\lambda} = l_{[\mu} P_{\nu\lambda]}$, so that the field strength $H_{\mu\nu\lambda}$ is aligned to the null Killing vector l_μ.

The field equations (46)–(47) reduce to

$$\begin{aligned}
(a_u)_{;a}^{;a} - \partial_u (a_a{}^{;a}) &= \frac{1}{4}\left(f_{ab} f^{ab} - H_{uab} H_u{}^{ab}\right) + \kappa p_u\,, && (58)\\
f_{ab}{}^{,b} &= -\kappa p_a\,, && (59)\\
H_{ua}{}^{b}{}_{,b} &= 8\kappa J_{ua}\,. && (60)
\end{aligned}$$

The last two relations are linear differential equation in the $n-$dimensional Euclidean space ($n = D - 2$). They can be solved for f_{ab} and H_{uab} once the source J_{ua} and the distribution for the source for gravito-magnetic field f_{ab} is given. After this we can solve the first equation for a_u, which for a given right-hand-side is also linear.

On the other hand, we can also solve the constraint $H_{abc} = 0$ by the following ansatz for the 2-form potential

$$B_{\mu\nu} = A_\mu l_\nu - A_\nu l_\mu \,. \tag{61}$$

From $l^\mu B_{\mu\nu} = 0$, we have $l^\mu A_\mu = 0$. This is equivalent to choose a gauge so that the only non-vanishing component of $B_{\mu\nu}$ is $B_{ua} = A_a(u, \boldsymbol{x})$, and of $H_{\mu\nu\lambda}$ is

$$H_{uab} = \partial_b A_a - \partial_a A_b \equiv F_{ba}. \tag{62}$$

Let us denote

$$\Phi = 2a_u\,, \quad \boldsymbol{a} = a_a\,, \quad \boldsymbol{f} = f_{ab} \tag{63}$$

$$\boldsymbol{A} = A_a\,, \quad \boldsymbol{F} = F_{ab}\,, \quad \boldsymbol{J} = J_{ua}\,, \quad \boldsymbol{p} = p_a\,. \tag{64}$$

In these notations the gyraton metric is

$$ds^2 = d\bar{s}^2 + \Phi du^2 + 2(\boldsymbol{a}, d\boldsymbol{x})du\,, \tag{65}$$

and the field equations (58) and (60) reduce to

$$\triangle\Phi - 2\partial_u(\boldsymbol{\nabla}\cdot\boldsymbol{a}) = \frac{1}{2}\left(\boldsymbol{f}^2 - \boldsymbol{F}^2\right) + 2\kappa p_u\,, \tag{66}$$

$$\triangle\boldsymbol{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) = 8\kappa\boldsymbol{J}. \tag{67}$$

Here $\boldsymbol{\nabla} = \partial_a$, and $\triangle$ is the Laplacian operator in the $n-$dimensional Euclidean space.

Using the coordinate and electromagnetic gauge transformations one can put

$$\nabla\cdot\boldsymbol{A} = 0\,, \qquad \nabla\cdot\boldsymbol{a} = 0\,. \tag{68}$$

For these gauge fixing conditions the equations (66), (59) and (67) take the form ($\Phi = \varphi + \psi$)

$$\triangle\varphi = 2\kappa p_u\,, \tag{69}$$

$$\triangle\psi = \frac{1}{2}\left(\boldsymbol{f}^2 - \boldsymbol{F}^2\right)\,, \tag{70}$$

$$\triangle\boldsymbol{a} = \kappa\boldsymbol{p}\,, \tag{71}$$

$$\triangle\boldsymbol{A} = 8\kappa\boldsymbol{J}\,. \tag{72}$$

It is interesting to note that the magnetic and gravitomagnetic terms enter the right hand side of (70) with the opposite signs. A special type of solutions is the case when these terms cancel one another, so that the equation for ψ outside the matter source becomes homogeneous. We call such solutions *saturated*. The condition of saturation is $\boldsymbol{f}^2 = \boldsymbol{F}^2$. This condition can be achieved by letting $\boldsymbol{p} = 8\boldsymbol{J}$ as suggested by (71) and (72).Solutions of the obtained field equations for a ring-like string configuration of gyratons are obtained in [9].

8. Summary and discussions

We demonstrated that the vacuum Einstein equations for the gyraton metrics (13) in an arbitrary number of spacetime dimensions D can be reduced to linear problems in the Euclidean $(D-2)-$dimensional space. These problems are: (1) To find a static electric field created by a point-like source; (2) To find a magnetic field created by a point-like source. The retarded time u plays the role of an external parameter. One can include u-dependence by making the coefficients in the harmonic decomposition for φ and $\mathbf{A}$ to be arbitrary functions of u. After choosing the solutions of these two problems one can define ψ by means of equations (38) and (70). A similar reduction is valid for the electrically charged gyratons and gyratons in supergravity. It should be emphasized that the point- or line-like sources are certainly an idealization. In [6] it was shown that gyraton solutions can describe the gravitational field of beam-pulse spinning radiation. In such a description one uses the geometric optics approximation. For its validity the size of the cross-section of the beam must be much larger than the wave-length of the radiation. In the presence of spin J one can expect additional restrictions on the minimal size of both, the cross-section size and the duration of the pulse. The obtained solutions are valid only outside some region surrounding the immediate neighborhood of the singularity.

The gyraton solutions might be used, for example, for study the gravitational interaction of ultrarelativistic particles with spin and charge. The gyraton metrics might be also interesting as possible exact solutions in the string theory. The generalization of the gyraton-type solutions to the case when a spacetime is asymptotically AdS was obtained recently in [22].

Acknowledgments

This work was supported by NSERC and by the Killam Trust.

References

[1] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology*, p.272, Oxford, Clarendon Press (1934).
[2] A. Peres, Phys. Rev. Lett. **3**, 571 (1959).
[3] A. Peres, Phys. Rev. **118**, 1105 (1960).
[4] W. B. Bonnor, Commun. Math. Phys. 13, 163 (1969).
[5] W. B. Bonnor, Int. J. Theor. Phys. **3**, 257 (1970).
[6] V. P. Frolov and D. V. Fursaev, Phys.Rev. **D71**,104034 (2005).
[7] V. P. Frolov, W. Israel and A. Zelnikov, Phys.Rev. **D72** 084031 (2005).
[8] V. P. Frolov and A. Zelnikov , *Class.Quant. Gravity* **23** 2119 (2006).
[9] V. P. Frolov and F.-L. Lin, Phys. Rev. **D73**, 104028 (2006).
[10] H.W. Brinkmann, Math. Ann. **94**, 119 (1925).
[11] P. Jordan, J. Ehlers and W. Kundt, *Akad. Wiss. Lit.(Mainz) Abhandl. Math.-Nat. Kl.* **2** 21 (1960).
[12] J. Ehlers and W. Kundt, in: *Gravitation: an introduction to current research*, ed. L. Witten, p.49 (Wiley, New York and London) (1962).
[13] H. Stephani *et al*, *Exact solutions to Einstein's Field Equations*, second edition, Chapter 31, Cambridge University Press (2003).
[14] M. Blau, M. O'Loughlin, G. Papadopoulos, and A. A. Tseytlin, Nucl.Phys. **B673**, 57 (2003).
[15] T. Ortín, *Gravity and Strings*, Chapter 10, Cambridge monographs on mathematical physics, Cambridge University Press (2004).
[16] D. Amati, C. Klimcik, Phys.Lett. **B219** (1989) 443.
[17] G.T. Horowitz, A.R. Steif, Phys.Rev.Lett. **64** (1990) 260.
[18] A. Coley, Phys. Rev. Lett. **89**, 281601 (2002).
[19] V. Pravda, A. Pravdova, A. Coley, and R. Milson, Class. Quantum Grav. **19**, 6213 (2002).
[20] A. Coley, R. Milson, V. Pravda, and A. Pravdova, Class. Quantum Grav. **21**, 5519 (2004).
[21] A. Coley, A. Fuster, S. Hervik, N. Pelavas, Class.Quant.Grav. **23**, 7431 (2006).
[22] V.P. Frolov and A. Zelnikov, Phys.Rev. **D72** 104005 (2005).