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Scalar particles mass spectrum and localization on FRW branes embedded in a 5D de Sitter bulk

Research Article

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Abstract: In this paper, we study the scalar fields evolving on a FRW brane embedded in a five-dimensional de Sitter bulk. The scale function and the warp factor, solutions of the Einstein equations, are employed in the five-dimensional Gordon equation describing the massive scalar field, whose wave function depends on the cosmic time and on the extra-dimension. We point out the existence of *bounded* states and find a minimum value of the effective four-dimensional mass. For the test (scalar) field envelope along the extra-dimension, we derive the corresponding Schrödinger-like equation which is formally that for the Pöschl-Teller potential. Accordingly, we have obtained the quantization law for the mass parameter of the tested scalar field.

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1. Introduction

From a cosmological view point, theories formulated on more than four dimensions, with ordinary matter trapped on the brane and gravitation only propagating through the entire bulk, that emerged from the pioneering work of Randall and Sundrum (RS) [1, 2], have offered an alternative scenario for explaining the late-time accelerated expansion of the universe [3–8].

In contrast to the standard RS setup with empty bulk (apart of the cosmological constant), and fields confined on the brane, it has been assumed that matter might evolve in the entire bulk and research has recently been dedicated to the cosmological consequences of this [9].

A particular form of bulk or brane matter, which is believed to have played an important role both in the early universe and in late-time acceleration, is the scalar field, with minimal or suitable choices of non-minimal couplings [10, 11]. Supported by data from high redshift Type Ia supernovae, these fields are seen as natural models of matter with negative pressure, called *quintessence* [12–16].

The revolutionary concept of inflation, as an early stage of the accelerated expansion of the universe, has offered a solution not only to the classic problems raised by the standard big bang cosmology, but also to the one related to the mechanism of structures formation in the universe. By assuming that the scalar field, with the corresponding energy density and pressure, is the only source of gravity, in some scenarios, the cosmic acceleration is driven by the

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four-dimensional field confined to the brane [17], while in others the massive scalar is filling the entire bulk [18–20]. In contrast to the localized zero scalar mode and a continuum of arbitrary light states, provided by the RS background, once one allows for a small (non-zero) mass, the massless bound state is replaced by a quasi-localized state with finite mass and width. The latter being related to the probability of state's decay into the bulk [21].
This paper follows previous investigations into a FRW brane embedded in a five-dimensional de Sitter bulk [22]. The derived scale function and warp factor are employed in the five-dimensional Klein–Gordon equation describing the massive scalar field whose wave function will depend on the cosmic time and on the extra-dimension.
Our results agree with those obtained by Langlois and Sasaki [23], who considered a test scalar field on a given background configuration and solved the Klein–Gordon equation in the bulk. They found an interesting relationship between the effective four-dimensional mass of the scalar states localized on the de Sitter brane and the five-dimensional mass.

2. The geometry

Let us start with the warped five-dimensional line element with an induced 3−brane with $k = 0$−FRW cosmological background

$$ds_5^2 = e^{2F(\tau,\zeta)}\eta_{ik}dx^i dx^k + (d\zeta)^2 \ , \quad i,k = \overline{1,4} \ , \qquad (1)$$

where the function F in the warp factor depends on the conformal time τ and on the extra-dimension coordinate, $x^5 = \zeta$, varying from $-\infty$ to ∞.

In terms of the orthonormal tetradic frames $e_a = \left\{e_i = e^{-F}\,\partial_i \, , \, e_5 = \partial_\zeta\right\}$ and $\omega^a = \left\{\omega^i = e^F\,dx^i \, , \, \omega^5 = d\zeta\right\}$, the Cartan formalism leads to the connection coefficients

$$\Gamma_{\alpha 4 \alpha} = F_{|4} \ , \quad \Gamma_{\alpha 5 \alpha} = F_{|5} \ , \quad \Gamma_{454} = -F_{|5} \ , \qquad (2)$$

where $\alpha = \overline{1,3}$, $F_{|a} = e_a F$, which furthermore leads to the following five-dimensional Einstein tensor components [22]

$$\begin{aligned} G_{\alpha\beta} &= \left\{-\left[2F_{|44} + 3\left(F_{|4}\right)^2\right] + 3\left[F_{|55} + 2\left(F_{|5}\right)^2\right]\right\}\delta_{\alpha\beta} \, , \\ G_{44} &= 3\left(F_{|4}\right)^2 - 3\left[F_{|55} + 2\left(F_{|5}\right)^2\right] , \\ G_{55} &= -3\left[F_{|44} + 2\left(F_{|4}\right)^2\right] + 6\left(F_{|5}\right)^2 \ , \\ G_{45} &= -3\,F_{|54} \, . \end{aligned} \qquad (3)$$

For a scalar field supporting this geometry, described by the energy-momentum tensor

$$T_{ab} = \phi_{|a}\phi_{|b} - \frac{1}{2}\eta_{ab}\left[\eta^{cd}\phi_{|c}\phi_{|d} + 2V(\phi)\right] ,$$

the Einstein equations,

$$G_{ab} = \kappa T_{ab} \, ,$$

where κ is Einstein's constant in five-dimensions, do explicitly read

$$\begin{aligned} -e^{-2F}\left[2F_{,44} + (F_{,4})^2\right] + 3\left[F_{,55} + 2(F_{,5})^2\right] &= \frac{\kappa}{2}\,e^{-2F}\,(\phi_{,4})^2 - \frac{\kappa}{2}\left[(\phi_{,5})^2 + 2V\right]; \\ 3e^{-2F}\,(F_{,4})^2 - 3\left[F_{,55} + 2(F_{,5})^2\right] &= \frac{\kappa}{2}\,e^{-2F}\,(\phi_{,4})^2 + \frac{\kappa}{2}\left[(\phi_{,5})^2 + 2V\right]; \\ -3e^{-2F}\left[F_{,44} + (F_{,4})^2\right] + 6\,(F_{,5})^2 &= \frac{\kappa}{2}\,e^{-2F}\,(\phi_{,4})^2 + \frac{\kappa}{2}\left[(\phi_{,5})^2 - 2V\right]; \\ -3F_{,54} &= \kappa\,\phi_{,4}\,\phi_{,5} \, . \end{aligned} \qquad (4)$$

For a conformally flat brane (as it actually stands for the case of $(k = 0)$-RW models), the most natural choice of the warp function F is

$$F(\tau,\zeta) = f(\tau) + h(\zeta) \, , \qquad (5)$$

so that the non-diagonal component of the Einstein's tensor, G_{45}, vanishes and the relationships in (2) become

$$\Gamma_{\alpha 4 \alpha} = e^{-f}e^{-h}\partial_4 f \ , \quad \Gamma_{\alpha 5 \alpha} = \partial_5 h \ , \quad \Gamma_{454} = -\partial_5 h \ , \qquad (6)$$

where $\partial_4 f = \partial_\tau f \equiv f'$ and $\partial_5 h = \partial_\zeta h \equiv h'$.
With respect to the scalar source, in view of the last equation in (4), this can be taken either as a function of the extra-dimension, ζ, or of time [22].
In the first case, i.e.

$$\phi(x^\alpha, \tau, \zeta) \;=\; \phi(\zeta)\,,$$

by switching to the proper time, t, by $f_{,4} = e^f \dot{f}$, where *dot* denotes the derivative with respect to t, the system (4) takes the explicit form

$$\begin{aligned}
&-e^{-2h}\left[2\ddot{f} + 3\left(\dot{f}\right)^2\right] + 3\left[h_{,55} + 2\,(h_{,5})^2\right] \\
&= -\frac{\kappa}{2}\left[(\phi_{,5})^2 + 2V\right]; \\
&3e^{-2h}\left(\dot{f}\right)^2 - 3\left[h_{,55} + 2\,(h_{,5})^2\right] = \frac{\kappa}{2}\left[(\phi_{,5})^2 + 2V\right]; \\
&-3e^{-2h}\left[\ddot{f} + 2\left(\dot{f}\right)^2\right] + 6\,(h_{,5})^2 = \frac{\kappa}{2}\left[(\phi_{,5})^2 - 2V\right].
\end{aligned} \tag{7}$$

By summing up the first two equations, one finds $\ddot{f} = 0$, i.e. $f = Ht$ and thus the cosmological dynamics of the ($k = 0$)-RW brane will be too simple, being turned into a permanently de Sitter one.
Secondly, let us consider the other important case which has been discussed in [22], i.e.

$$\phi\,(x^\alpha, \tau, \zeta) \;=\; \phi(\tau)\,.$$

The Einstein–Gordon equations (4) read

$$\begin{aligned}
&-e^{-2(f+h)}\left[2f_{,44} + (f_{,4})^2\right] + 3\left[h_{,55} + 2\,(h_{,5})^2\right] \\
&\quad= \frac{\kappa}{2}\,e^{-2(f+h)}\,(\phi_{,4})^2 - \kappa V; \\
&3e^{-2(f+h)}\,(f_{,4})^2 - 3\left[h_{,55} + 2\,(h_{,5})^2\right] \\
&\quad= \frac{\kappa}{2}\,e^{-2(f+h)}\,(\phi_{,4})^2 + \kappa V; \\
&-3e^{-2(f+h)}\left[f_{,44} + (f_{,4})^2\right] + 6\,(h_{,5})^2 \\
&\quad= \frac{\kappa}{2}\,e^{-2(f+h)}\,(\phi_{,4})^2 - \kappa V\,,
\end{aligned} \tag{8}$$

and they lead to the following relationship between the metric functions f and h [22]

$$e^{-2f}\left[f_{,44} + 2\,(f_{,4})^2\right] + 3\,e^{2h}\,h_{,55} \;=\; 0\,,$$

satisfied by [22, 24]

$$\begin{aligned}
&\text{(a)}\quad h(\zeta) \;=\; \ln\left[\frac{\omega}{Q_0}\cos\left(Q_0\zeta\right)\right]; \\
&\text{(b)}\quad f(t) \;=\; \frac{1}{3}\ln\left[\frac{b}{2\omega}\sinh\left(3\omega t\right)\right],
\end{aligned} \tag{9}$$

and $\kappa V = 6Q_0^2$, so that the scale factor of the brane is

$$a(t) = e^{f(t)} = \left[\frac{b}{2\omega}\sinh\left(3\omega t\right)\right]^{1/3}. \tag{10}$$

The constants Q_0 and ω are respectively proportional to the cosmological constant of the dS_5 bulk, Λ, and to the one on the visible brane, Λ_0, by [24]

$$Q_0 \;=\; \sqrt{\frac{\Lambda}{6}}\,,\quad \omega = \sqrt{\frac{\Lambda_0}{3}}\,,$$

while the parameter b can be related by $b = \sqrt{2}\omega a_*^3$ to the definite special value of the scale function

$$a_* = a(t_*) = \left[\frac{b}{\sqrt{2}\omega}\right]^{1/3}, \tag{11}$$

for which the acceleration parameter

$$q \;=\; \frac{\ddot{f} + (\dot{f})^2}{(\dot{f})^2} \;=\; 1 - \frac{3}{\cosh^2(3\omega t)} \tag{12}$$

vanishes and the expansion of the universe gets accelerated.
Thus, the scale factor (10) can be written as

$$a(t) = a_*\left[\frac{1}{\sqrt{2}}\sinh\left(\sqrt{3\Lambda_0}\,t\right)\right]^{1/3}. \tag{13}$$

3. Bosons in the bulk

In this section, we are going to construct the wave functions of the bosons, considered as test particles evolving in the five-dimensional bulk characterized by the line element (1), with the metric functions (9).
The real scalar field minimally coupled to bulk gravity is described by the following Lagrangian,

$$\mathcal{L}[\Phi] = \frac{1}{2}\eta^{ab}\Phi_{|a}\Phi_{|b} + U(\Phi)\,, \tag{14}$$

where the effective potential

$$U(\Phi) = -\frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4\,, \tag{15}$$

has two degenerate minima, $U_0 = -\mu^4/(4\lambda)$. For $\mu^2 > 0$, the zero KK Higgs mode has a non-vanishing vacuum expectation value and turns from a massless into a massive degree of freedom, as in the usual four-dimensional Higgs mechanism. After the Z_2-symmetry got spontaneously broken, near one of the degenerated vacua, $\Phi = \phi + \varphi$, we are going to keep only the mass term contribution, so that the potential is

$$U(\varphi) = \mu^2\varphi^2 + \dots \tag{16}$$

Thus, the Gordon-type equation

$$\eta^{ab}\varphi_{|ab} - \eta^{ab}\varphi_{|c}\Gamma^c{}_{ab} = \frac{\partial U}{\partial\varphi}\,, \tag{17}$$

in the pseudo-orthonormal frame with (6), where

$$\eta^{ab}\varphi_{|ab} = e^{-2f}e^{-2h}\left[\Delta\varphi - \frac{\partial^2\varphi}{\partial\tau^2} + \frac{\partial f}{\partial\tau}\frac{\partial\varphi}{\partial\tau}\right] + \frac{\partial^2\varphi}{d\zeta^2}$$

and

$$\eta^{ab}\varphi_{|c}\Gamma^c{}_{ab} = 3e^{-2f}e^{-2h}\frac{\partial f}{\partial\tau}\frac{\partial\varphi}{\partial\tau} - 4\frac{\partial h}{\partial\zeta}\frac{\partial\varphi}{\partial\zeta}\,,$$

reads

$$e^{-2f}e^{-2h}\left[\Delta\varphi - \frac{\partial^2\varphi}{\partial\tau^2} - 2\frac{\partial f}{\partial\tau}\frac{\partial\varphi}{\partial\tau}\right] + 4\frac{\partial h}{\partial\zeta}\frac{\partial\varphi}{\partial\zeta} + \frac{\partial^2\varphi}{d\zeta^2} = 2\mu^2\varphi\,. \tag{18}$$

The bulk equation (18) is separable for functions φ of the form

$$\varphi = g(\tau)Z(\zeta)\,, \tag{19}$$

where g is a function of the time alone. We switch from τ to t, by

$$\frac{\partial}{\partial\tau} = e^f\frac{\partial}{\partial t}\,,\quad \frac{\partial^2}{\partial\tau^2} = e^{2f}\left[\frac{\partial^2}{\partial t^2} + \frac{\partial f}{\partial t}\frac{\partial}{\partial t}\right],$$

and we come to the following system of decoupled equations:

$$\begin{aligned}
&(a)\quad \frac{d^2g}{dt^2} + 3\frac{df}{dt}\frac{dg}{dt} = Cg\,,\\
&(b)\quad \frac{d^2Z}{d\zeta^2} + 4\frac{dh}{d\zeta}\frac{dZ}{d\zeta} - 2\mu^2 Z = Ce^{-2h}Z\,,
\end{aligned} \tag{20}$$

where the sign of the constant C will be discussed below. With the metric function (9.b) and the new variable $\eta = 3\omega t$, the first equation in system (20) becomes

$$\frac{d^2g}{d\eta^2} + \coth\eta\frac{dg}{d\eta} - \frac{C}{9\omega^2}g = 0\,. \tag{21}$$

As in the case for spherical functions, considering complex duality reasons, $\theta \sim \pm i\eta$, it can be proven that the following quantized value of the separation constant C, namely

$$C_n \equiv 9\omega^2\left[n^2 - \frac{1}{4}\right],\quad n = 1, 2, 3, \dots \tag{22}$$

is the only one which leads to the mathematically well-behaved set of linearly independent functions, namely the thorus functions [25]

$$g_+(\eta) = \{\mathcal{P}_{n-1/2}(\cosh\eta)\,,\ \mathcal{Q}_{n-1/2}(\cosh\eta)\}\,. \tag{23}$$

These are "bounded", actual decaying states with no free particle interpretation, since, with the change of function

$$g_+(\eta) = \frac{1}{\sqrt{\sinh\eta}}\,u_+(\eta)\,,$$

the new functions u_+ satisfy the Schrödinger-like form

$$\frac{d^2u_+}{d\eta^2} + \left[-\left(n^2 + \frac{1}{4}\right) + \frac{1}{4}\coth^2\eta\right]u_+ = 0\,, \tag{24}$$

with negative energy parameter and negative potential

$$V_S = -\frac{1}{4}\coth^2\eta\,, \tag{25}$$

with u_+ being the Legendre functions

$$u_+(\eta) = \left\{P_n^{-1/2}(\coth\eta)\,,\ Q_n^{-1/2}(\coth\eta)\right\}.$$

In the opposite situation where C is negative, one can define a positive constant $k = -C > 0$ which acts in (20.a) as an effective four-dimensional mass. The function u_- is a solution of the equation

$$\frac{d^2u_-}{d\eta^2} + \left[\left(\frac{k}{9\omega^2} - \frac{1}{2}\right) + \frac{1}{4}\coth^2\eta\right]u_- = 0\,,$$

highlighting the inflection value k_0 of k, for $|\eta| \to \infty$ in the previous equation, i.e.

$$\frac{k_0}{9\omega^2} - \frac{1}{2} = -\frac{1}{4}\,.$$

The following condition for the parameter k,

$$k \geq k_0 = \frac{9\omega^2}{4} = \frac{3}{4}\Lambda_0 \tag{26}$$

leads to the lower limit of the effective four-dimensional mass.

4. Mass spectrum and localization

Let us now focus on the second equation of (20) and discuss the two cases corresponding to the opposite signs of C.

Using the relations (22) and (9.a), introducing the new variable $w = Q_0 \zeta$ and the "mass" parameter

$$\varepsilon^2 \equiv \frac{2\mu^2}{Q_0^2} \,, \tag{27}$$

the second equation in (20) becomes

$$\frac{d^2 Z_+}{dw^2} - 4 \tan w \frac{dZ_+}{dw} - \left[\varepsilon^2 + \frac{9\left(n^2 - 1/4\right)}{\cos^2 w} \right] Z_+ = 0 \,. \tag{28}$$

Let us consider a solution of the form $Z_+(w) = \cos^{-3/2}(w) \cdot v_+$, where the function v_+ satisfies the following differential equation:

$$\frac{d^2 v_+}{dw^2} - \tan w \frac{dv_+}{dw} + \left[\left(\frac{15}{4} - \varepsilon^2 \right) - \frac{9n^2}{\cos^2 w} \right] v_+ \; = \; 0 \,. \tag{29}$$

For

$$\frac{15}{4} - \varepsilon^2 = \nu(\nu + 1) \,,$$

leading to the spectrum

$$2\mu^2 = \left(\frac{15}{4} - \nu^2 - \nu \right) Q_0^2 \,, \tag{30}$$

the two linearly-independent solutions of (29) are the Legendre functions

$$v_+(w) \; = \; \{ P_{3n}^{\nu}(\sin w)) \,,\; Q_{3n}^{\nu}(\sin w)) \} \,, \tag{31}$$

which can be expressed in terms of hypergeometric functions [25] as

$$\begin{aligned} P_{3n}^{\nu}(\sin w) &= \frac{1}{\Gamma(1-\nu)} \left(\frac{1+\sin w}{1-\sin w} \right)^{\nu/2} {}_2F_1 \left(-3n,\, 3n+1;\, 1-\nu\,;\, \frac{1-\sin w}{2} \right) \\ Q_{3n}^{\nu}(\sin w) &= \frac{\pi}{2\sin(\nu\pi)} \left[P_{3n}^{\nu} \cos(\nu\pi) - \frac{\Gamma(3n+1+\nu)}{\Gamma(3n+1-\nu)} P_{3n}^{-\nu} \right] , \end{aligned} \tag{32}$$

with

$$\nu_{1,2} = -\frac{1}{2} \pm \sqrt{4 - \varepsilon^2} \,.$$

When the mass reaches the value corresponding to $\varepsilon^2 = 15/4$, the order ν_1 vanishes and the associated Legendre functions (31) simply become Legendre functions of first and second kind, respectively.

The condition $\varepsilon^2 = 4$ leads to a special value of the mass parameter μ, related to the dS_5 cosmological constant by

$$\mu_0^2 = \frac{\Lambda}{3} \,,$$

which, as in the RS model where

$$\frac{\Lambda}{6} = \left(\frac{M_5^3}{M_{Pl}^2} \right)^2 \,,$$

can be expressed in terms of the five-dimensional gravitational mass scale, M_5, and the effective Planck mass by

$$\mu_0 = \sqrt{2}\, \frac{M_5^3}{M_{Pl}^2} \,.$$

In this particular case, for which the functions (31) are

$$v_+^0(w) \;\; = \;\; \left\{ P_{3n}^{-1/2}(\sin w)) \,,\; \frac{\pi}{6n+1} P_{3n}^{1/2}(\sin w)) \right\} , \tag{33}$$

we obtain, using the first relation in (32), the periodic functions

$$\begin{aligned} v_+^0(w) \;\; = \;\; &\left\{ \sqrt{\frac{2}{\pi}} \frac{2}{6n+1} \frac{1}{\sqrt{\cos w}} \sin \left[\frac{6n+1}{2} \left(w - \frac{\pi}{2} \right) \right] , \right. \\ &\left. \sqrt{\frac{\pi}{2}} \frac{2}{6n+1} \frac{1}{\sqrt{\cos w}} \cos \left[\frac{6n+1}{2} \left(w - \frac{\pi}{2} \right) \right] \right\} \end{aligned} \tag{34}$$

and

$$Z_+^0(w) = \left\{ \sqrt{\frac{2}{\pi}} \frac{2}{6n+1} \frac{1}{\cos^2 w} \sin\left[\frac{6n+1}{2}\left(w - \frac{\pi}{2}\right)\right], \sqrt{\frac{\pi}{2}} \frac{2}{6n+1} \frac{1}{\cos^2 w} \cos\left[\frac{6n+1}{2}\left(w - \frac{\pi}{2}\right)\right]\right\}. \quad (35)$$

For a particle (along the extra-dimension) interpretation, let us perform the change of function $Z_+(w) = s_+(w)/\cos^2 w$, so that equation (28) casts in the Schrödinger-like form

$$\frac{d^2 s_+}{dw^2} + \left[\left(4 - \varepsilon^2\right) + \frac{\frac{1}{4} - 9n^2}{\cos^2 w}\right] s_+ = 0\,, \quad (36)$$

where the positive potential

$$V_S(w) = \frac{9n^2 - \frac{1}{4}}{\cos^2 w}$$

is always above the energy parameter $4 - \varepsilon^2$, for $n \neq 0$. Thus, one may conclude by stating that the decaying states on the brane, described by the wave functions (23), are not traveling in the bulk between the branes.

Finally, for a negative value of the separation constant, $C = -k$, in equation (20.b), the equivalent of equation (36) is

$$\frac{d^2 s_-}{dw^2} + \left[\left(4 - \varepsilon^2\right) - \frac{2 - \frac{k}{\omega^2}}{\cos^2 w}\right] s_- = 0\,, \quad (37)$$

highlighting the potential

$$V_S(w) = \frac{2 - \frac{k}{\omega^2}}{\cos^2 w}\,,$$

which is negative, in view of condition (26).

The solution is expressed in terms of hypergeometric functions as

$$s_- \sim (\cos w)^{\frac{1}{2} \mp i\sigma}\, {}_2F_1\left[\frac{1}{4} - \frac{\sqrt{4 - \varepsilon^2}}{2} \mp \frac{i\sigma}{2}\,, \frac{1}{4} + \frac{\sqrt{4 - \varepsilon^2}}{2} \mp \frac{i\sigma}{2}\,, 1 \mp i\sigma\,; \cos^2 w\right],$$

where the quantity

$$\sigma \equiv \sqrt{\frac{k}{\omega^2} - \frac{9}{4}}$$

is positive, in view of (26), and there are no restrictions on the parameter ε^2.

The definition range of the solution is a union of domains of the typical form $w \in \left(-\frac{\pi}{2}\,, \frac{\pi}{2}\right)$.

For the inflection value $k_0 = 9\omega^2/4$, the hypergeometric function is periodic and real ($\sigma = 0$), for ε either smaller or greater than 2.

Finally, let us notice that the Schrödinger equation (37), by the duality switch $4 - \varepsilon^2 \to \varepsilon^2 - 4$ of the formal eigenvalue $\lambda = \pm\left(\varepsilon^2 - 4\right)$, can be written in the generic form of the Schrödinger-like equation in the bulk,

$$\frac{d^2 s}{dw^2} + \left[\varepsilon^2 - 4 + \frac{\frac{k}{\omega^2} - 2}{\cos^2 w}\right] s = 0\,,$$

which can be identified with that for the Pöschl-Teller potential [26]

$$V_{PT}(w) = -A^2 + \frac{A(A-1)}{\cos^2 w}\,, \quad (38)$$

i.e.

$$\frac{d^2\psi}{dw^2} + \left[E_n - V_{PT}(w)\right]\psi = 0\,,$$

once the constant A is the root of the equation

$$A^2 - A + \frac{k}{\omega^2} - 2 = 0\,.$$

In order to obtain the first non-negative value for A, one has to impose $k = k_0 = 9\omega^2/4$ so that $A = 1/2$. Therefore, the bound state "energy" eigenvalues of the potential (38), generally given by [27]

$$E_n = -A^2 + (2n + A)^2 \quad (39)$$

lead to the following quantization law

$$\varepsilon_n^2 = 4\left[\left(n + \frac{1}{4}\right)^2 + 1\right].$$

For large values of n, the above relation becomes $\varepsilon_n \approx 2n$, so that the mass parameter (corresponding to the effective mass-squared on the brane $k_0 = m_0^2 = 3\Lambda_0/4$) is a multiple of de Sitter quanta,

$$\mu_n = \sqrt{\frac{\Lambda}{3}}\, n\,. \tag{40}$$

5. Conclusions

For a five-dimensional de Sitter bulk with an induced $k = 0$—FRW brane, the system of Einstein's equations with a massless scalar field depending on time as its matter source, is satisfied by the warp factor (9.a), highlighting the typical domain $Q_0\zeta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the scale function (10).

The massive scalar evolving in this given background, seen as a test particle, with no backreaction on the geometry, is described by the Klein–Gordon equation (18). For the wave function depending on time and on extra-dimension, we have derived the massive scalar modes, both in the bulk and on the brane.

Special attention has been given to their spectrum and its interpretation, particularly with respect to their localization.

We have found general massive bound states, whose mass parameter, corresponding to the effective four-dimensional mass-squared $m_0^2 = 3\Lambda_0/4$, is a multiple of de Sitter quanta.

When $k > 9\omega^2/4$, the corresponding five-dimensional Pöschl–Teller energy parameter (39) has an imaginary part, meaning that the corresponding states are decaying back into the brane.

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