

A dissertation on General Covariance and its application in particle physics

Luciano Petruzzello^{1,2,3}

¹Dipartimento di Ingegneria, Università di Salerno, Via Giovanni Paolo II, 132 I-84084 Fisciano (SA), Italy.

²INFN, Sezione di Napoli, Gruppo collegato di Salerno, I-84084 Fisciano (SA), Italy.

³Dipartimento di Fisica, Università di Salerno, Via Giovanni Paolo II, 132 I-84084 Fisciano (SA), Italy.

E-mail: lpetruzzello@na.infn.it

Abstract. In this paper, we provide a concise overview on the principle of General Covariance, one of the fundamental cornerstones of Einstein's General Relativity. We retrace all the steps that led to the final settlement of a generally covariant theory of gravitation, dwelling specifically on the significance of the well-known "hole argument". In addition, we discuss about the importance of General Covariance in connection with some recent claims in literature revolving around particle physics. In particular, we summarize the results associated with the decay of accelerated protons.

1. Introduction

The principle of General Covariance (GC) is one of the essential building blocks that led Einstein to the implementation of General Relativity (GR) [1]. In its standard formulation (cf. Ref. [2]), GC states that *all physical laws retain the same form under any arbitrary differentiable coordinate transformation (diffeomorphisms)*. On the other hand, it can also be addressed with a different statement, as the one reported in the book by Wald [3]: "*the principle of general covariance*" [...] "*states that the metric of space is the only quantity pertaining to space that can appear in the laws of physics*".

General Covariance has represented a constant guide for Einstein throughout the development of GR, even though he himself was on the verge of discarding it for a short period of time. A similar occurrence is due to the fact that, although simple in its formulation, the full extent of GC implications is not completely obvious. Indeed, as claimed in Ref. [2], the ground-breaking impact withheld by GC *fathered half a century of confusion*. A famous episode of such bewilderment is represented by the criticism towards GR raised by Kretschmann [4], who recognized no physical motivations behind the adoption of General Covariance, which in his opinion could be introduced ad hoc in any theory.

Moreover, the requirement of having the mathematical apparatus dictated by GC fulfillment has been object of several reappraisals also in recent years. For instance, in Ref. [5] it is argued that GC needs to be reformulated in a proper way so as to clarify and to better expound the doubts driven against it in the last century. A more radical viewpoint is contemplated by the authors of Ref. [6], in which GC is considered a "dogma" that must be revisited. According to their reasoning, in all the works published after the settlement of GR, physicists have always

preferred a given coordinate system instead of other equivalent ones, with the aim of both defining quantities of physical interest and simplifying computations. Such a tendency suggests the possibility that GC can either be overcome in favor of a different principle or be regarded as a negligible requisite. Without delving further into the literature, by relying on the previous examples it should be clear that the most delicate issue about GC is to treat the set of chosen coordinates as nothing else but *gauge functions* [7, 8].

In order to clarify the last statement, in Sec. 2 we briefly recall the steps taken by Einstein for the development of GR. In so doing, we mainly follow the path traced in Refs. [7, 8]. Sec. 3 is devoted to a summary of results related to a direct enforcement of GC in the context of particle physics. Specifically, we review the so-called inverse β -decay, by virtue of which an accelerated proton cannot be regarded as a stable particle anymore. By requiring GC fulfillment, we show that we can theoretically prove the existence of the Unruh effect [9] and exhibit some intriguing features concerning neutrino physics. Sec. 4 contains discussions and conclusions.

2. The advent of General Covariance

In 1912, Einstein moved to Zurich and started his collaboration with Marcel Grossmann, who introduced him to the mathematical field of absolute differential calculus. In these years, the seminal papers aiming at the emergence of a general theory of relativity were published. Specifically, several works contain hints that clearly indicate a set of equations for the gravitational field of the kind

$$G_{\mu\nu}(g, \partial g, \partial^2 g) = k T_{\mu\nu}, \quad (1)$$

in which $T_{\mu\nu}$ is the stress-energy tensor of the source of gravity, whereas $G_{\mu\nu}$ is a function of the metric tensor and its field derivatives only¹.

Although the physical intuition was flawless, Einstein believed that $G_{\mu\nu} = R_{\mu\nu}$. However, such a choice would not return the Newtonian limit, and this occurrence was seen as the first signal of a premature reappraisal of GC role. Moreover, the requirement that the same metric solution of (1) is still a solution after a change of coordinates, namely

$$g'_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta}(x), \quad (2)$$

led Einstein to further question the requirement of imposing General Covariance [10]. The idea underlying the previous hypothesis can be summarized in the *hole argument*².

2.1. The hole argument

Suppose to consider a portion of spacetime where no matter is present (i.e. $T_{\mu\nu} = 0$), which will be addressed as ‘‘hole’’. The solution of the field equations (1) provides a metric tensor $g_{\mu\nu}(x)$ that determines the gravitational field in a given coordinate system. For the sake of clarity, we restrict the attention only on two points belonging to the hole, A and B, and suppose that the former is located in a flat region whereas the latter is not (see Fig. 1, left part).

Let us now perform a change of coordinate system $x^\mu \rightarrow y^\mu(x)$, which means that $g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(y)$ according to (2). In doing so, we demand $y^\mu(x)$ to behave such that $x^\mu = y^\mu$ outside the hole while smoothly changing inside of it. In particular, we want to switch the positions of the aforementioned points A and B. We then introduce a *new* metric $g'_{\mu\nu}(x)$, which is the starting metric g written in the new coordinate system y^μ , but expressed in terms of the old coordinates x^μ instead of y^μ . By virtue of this step, we now have two *distinct* gravitational fields expressed

¹ As we know, later on the quantity $G_{\mu\nu}$ was discovered to be the Einstein tensor we are familiar with.

² For a pedagogical explanation involving the Schwarzschild solution explicitly, see Ref. [11].

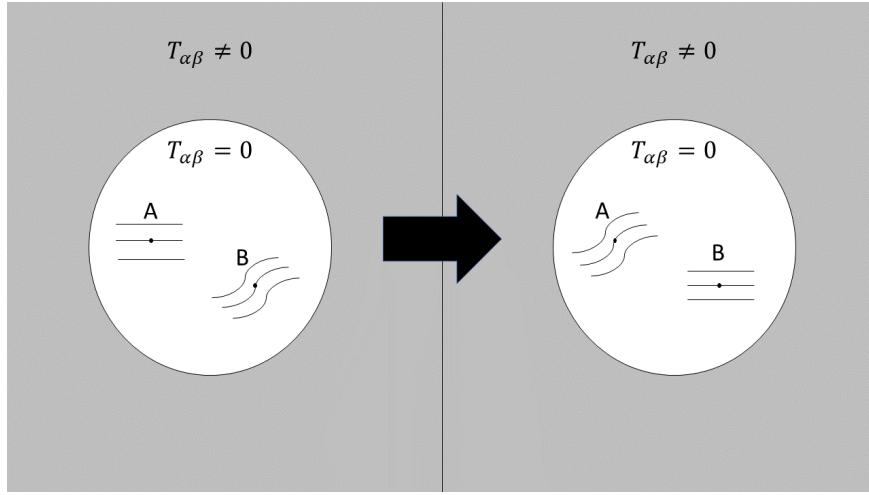


Figure 1. In this picture, the grey portion is where the stress-energy tensor is non-vanishing, whilst no matter is present in the white part. The straight lines specify a flat region of spacetime while the wiggly ones denote the presence of curvature.

in the *same* coordinate system. However, GC states that $g'_{\mu\nu}(x)$ is still a solution for (1), but as such it produces a radically different interpretation. As a matter of fact, because of the choice made for the set of y^μ , we know for sure that inside the hole there are two solutions of the same field equations that behave differently. Indeed, we note that, according to the setting given by $g'_{\mu\nu}(x)$, the flat region is now occupied by the point B whereas A is placed in the curved one (see Fig. 1, right part).

The above reasoning conveys that (1) is not capable of describing physics at the spacetime points A and B, thus resulting in a lack of determinism. Actually, there are two conclusions one can come up with by means of the above analysis:

- General Covariance is not a necessary requisite for the theory;
- points belonging to the spacetime manifold have no physical meaning.

2.2. Point-coincidence argument

The key to solve the controversy is hidden within the second option: spacetime manifold has no *per se* physical interpretation. In the complete formulation of GR, by resorting to Einstein's words [1]: “*That this requirement of general covariance, which takes away from space and time the last remnant of physical objectivity, is a natural one, will be seen from the following reflexion. All our space-time verifications invariably amount to a determination of spacetime coincidences*”.

Such a claim finally settles the misunderstanding revolving around the crucial role played by GC in the framework of General Relativity, and it is often regarded as the spacetime coincidence argument [8], but also as point-coincidence argument [7, 12]. The concept behind these names is simple, but at the same time astonishing, and in order to illustrate it we refer to the configuration already used for Fig. 1.

As shown before, we have realized that the points A and B on the spacetime manifold do not possess a prominent role from a physical perspective; to cast it in a different fashion, we can safely assume that they are *not* associated to any observable quantity. At this point, we now introduce two point-like test particles inside the hole whose world lines intersect in B in the reference frame where the metric tensor is $g_{\mu\nu}(x)$ (see Fig. 2, left part). Such intersection represents an interaction that occurs in B, and therefore it is an event which can be detected. Consequently, if we perform the same description according to the metric tensor $g'_{\mu\nu}(x)$, we

note that the interaction does not occur in B anymore, but rather in A (see Fig. 2, right part). Therefore, it is now meaningful to ask whether the gravitational field is vanishing or not *in the spacetime point where test particles interact*. The answer is the same both for $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x)$, which thus implies a preservation of determinism. In order to achieve this result, we have had to require *background independence*, which can be loosely explained by assuming that the spacetime structure is relevant only when dynamical entities (physical fields) are present [8].

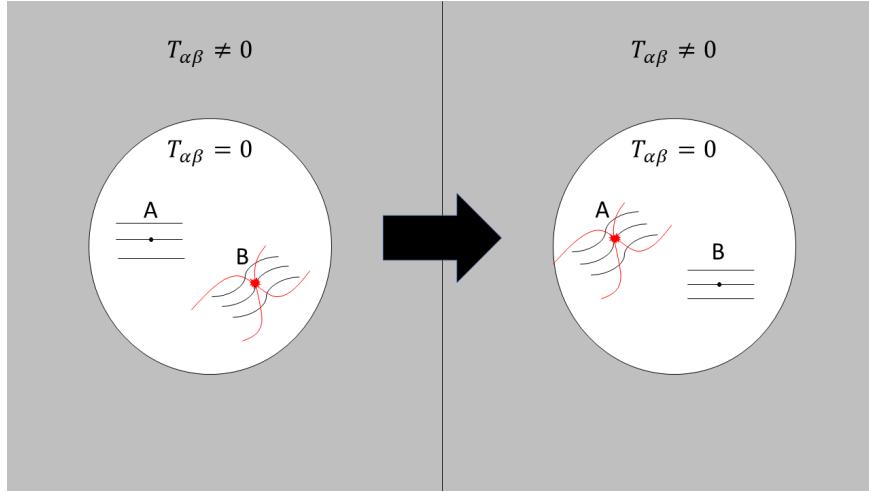


Figure 2. The difference between this figure and Fig. 1 consists in the world lines of point-like test particles appearing here and sketched in red. Scribbles denote interactions between particles.

Starting from the outlined scenario, we conclude that both $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x)$ describe the same field, as it should correctly be. Another rephrasing to express the aforesaid concept conveys that the localization on the manifold is merely a *gauge*. A diffeomorphism acting on a field simply changes its position on the spacetime manifold (i.e. the redefinition of the metric tensor), but such a freedom has no consequences, since the physical properties and events whose description should remain invariant for any observer (i.e. the interaction of the test particles in Fig. 2) are “dragged” along. Therefore, in Rovelli’s words [8]: “*A state of the universe does not correspond to a configuration of fields on M* [...] “*It corresponds to an equivalence class of field configurations under active diffeomorphisms*”.

In light of the previous considerations, it is licit to wonder whether there exists a fundamental and straightforward physical application stemming from General Covariance, given that it appears to be only a principle any reasonable gravitational model should observe. In the next Section, we answer the above question by summarizing a variety of results in the context of Quantum Field Theory (QFT) that can be theoretically achieved only by requiring GC.

3. An application of General Covariance: inverse β -decay

Before the beginning, it is opportune to chronicle a series of crucial results appeared in literature which help us to show how GC may be used as a lighthouse in developing a consistent theory. First and foremost, we start from the brilliant idea due to Muller [13] which deals with the decay properties of particles that are constantly accelerating due to an external source (i.e. an electric field for charged particles). In his simplified analysis, the author clearly exhibits that the decay rate of several physical processes acquires an extra term that depends on the acceleration the particle is subject to. In particular, one of the examples turns out to be extremely illustrative,

since it shows that also a supposedly stable particle such as the proton may decay via a channel that is typically addressed as *inverse β -decay*, namely

$$p \rightarrow n + e^- + \bar{\nu}_e, \quad (3)$$

with n being the neutron, e^- the electron and $\bar{\nu}_e$ the electron antineutrino. Finally, the author envisages a profound connection between such processes and the Unruh effect [9], which is still a vibrant subject of investigation, both at theoretical and phenomenological level [14].

In view of the aforesaid sharp intuition, several remarkable papers [15] have demonstrated Muller's hypothesis to hold true. As a matter of fact, by means of a thorough investigation on the inverse β -decay in two dimensions with massless neutrinos and by enforcing General Covariance, the authors of Refs. [15] have explicitly proven the *absolute need* of the Unruh effect for the internal consistency of QFT. The same considerations can be straightforwardly extended in four dimensions and for massive neutrinos [16]. In a nutshell, General Covariance is fulfilled in the above scenario by requiring that the mean proper lifetime of the proton τ_p must be the same from the point of view of an inertial observer and a comoving one which sees the proton at rest. In order to evaluate the lifetime, one must study the decay rate Γ , since

$$\Gamma \sim \tau_p^{-1}. \quad (4)$$

To this aim, we consider the interacting action [15]

$$S_I = \int d^4x \sqrt{-g} J_\mu^{(h)} (\bar{\Psi}_{\nu_e} \gamma^\mu \Psi_e + \bar{\Psi}_e \gamma^\mu \Psi_{\nu_e}), \quad (5)$$

where $g = \det(g_{\mu\nu})$, γ^μ are the Dirac matrices [20], Ψ_e and Ψ_{ν_e} the electron and neutrino field, respectively, whilst $J_\mu^{(h)}$ is the semi-classical current associated with the nucleon two-level system, where the proton is the ground state and the neutron represents the excited level [15].

To evaluate the tree-level decay rate for all the required calculations, we need to compute the following quantity [15]:

$$\Gamma \sim \frac{1}{T} \sum_{\sigma_e, \sigma_{\nu_e}} |\langle f | S_I | i \rangle|^2, \quad (6)$$

where the sum is to be intended over all possible polarizations of the leptonic fields, T is the nucleon proper time whereas $|i\rangle$ and $|f\rangle$ denote the initial and final state, respectively. For the case of the inertial observer, the process that should be studied is the one in Eq. (3). Therefore, we have $|i\rangle = |p\rangle \otimes |0\rangle_e \otimes |0\rangle_{\nu_e}$ and $|f\rangle = |n\rangle \otimes |e\rangle \otimes |\bar{\nu}_e\rangle$, by virtue of which it is possible to compute Γ_{in} . On the other hand, the observer comoving with the proton experiences the inertial vacuum as a thermal bath of particles due to the Unruh effect [9], in particular electrons, neutrinos and antineutrinos. Consequently, the possible processes that can occur in this reference frame are

$$(i) \ p + \bar{\nu}_e \rightarrow n + e^+, \quad (ii) \ p + e^- \rightarrow n + \nu_e, \quad (iii) \ p + e^- + \bar{\nu}_e \rightarrow n, \quad (7)$$

each of them with the opportune probability distribution $n_F = [\exp(2\pi\omega/a) + 1]^{-1} (1 - n_F)$ for the proton to absorb (emit) a particle with a given frequency ω from (to) the thermal bath that must be taken into account when evaluating (6) and that depends on the magnitude of the acceleration a . Finally, the three decay rates obtained at the end of calculations return the total decay rate for the accelerated observer $\Gamma_{acc} = \Gamma_{(i)} + \Gamma_{(ii)} + \Gamma_{(iii)}$. At this point, one can prove both numerically [15] and analytically [16] that $\Gamma_{in} = \Gamma_{acc}$. This is crucial, since Γ_{acc} can only be determined by requiring the existence of the Unruh effect, thus yielding a theoretical demonstration of its occurrence which exclusively relies on GC fulfillment.

Moreover, apart from the aforementioned results, there are further intriguing achievements that can be obtained by noting that neutrinos are mixed particles. Indeed, we want to stress that the treatment of the inverse β -decay with mixed neutrinos (which seems a harmless generalization of the above formalism) is actually the main source of disagreement between different controversial approaches recently appeared in literature. The first studies on the accelerated proton decay with neutrino mixing have been developed in a couple of papers [17] in which the authors encounter several theoretical problems. Later on, such complications have been cured with distinct methods in Refs. [18] and [19]. However, a way to discriminate between all these approaches is represented by the consistency of the employed formalism with neutrino flavor transition. As a matter of fact, such a feature has only been introduced in Refs. [18], in which GC fulfillment unambiguously shows that the Unruh thermal radiation is made up of flavor neutrinos which do oscillate.

For the sake of clarity, in the following table we summarize all the relevant aspects of Refs. [17, 18, 19].

| | Ref. [17] approach | Ref. [18] approach | Ref. [19] approach |
|--|-----------------------|-----------------------|-----------------------|
| Asympt. neutrinos in the laboratory frame | Flavor | Flavor | Mass |
| Asympt. neutrinos in the comoving frame | Mass | Flavor | Mass |
| Agreement between the decay rates | No | Yes | Yes |
| Consistency with neutrino oscillations | No | Yes | No |

4. Conclusions

In this paper, we have outlined the main features related to the principle of General Covariance. In particular, we have emphasized the relevant role it covered for the development of General Relativity by resorting to the “hole argument”, whose explanation laid the foundations for one of the most successful physical models both from an experimental and a theoretical perspective. Furthermore, we would like to stress that GC appears to be a guiding concept also in the framework of extended theories of gravity, since almost all of them are still based on a generally covariant formulation. Indeed, as for example it can be deduced from the analysis of quadratic models of gravity [21] in the context of neutrino oscillations [22], there are many attempts that try to go beyond GR by relaxing the equivalence principle rather than GC³.

In conjunction with the above considerations, we have furnished an example of how the requirement of a generally covariant theory is also capable of yielding a series of relevant achievements. To this aim, we have briefly recalled the most important steps that led to the theoretical check of the Unruh effect via the study of the inverse β -decay [15]. Additionally, the very same formalism also allows to exhibit that the Unruh thermal radiation is constituted by flavor (rather than mass) neutrinos which are subject to flavor oscillations [18]. It must be pointed out that neutrino oscillations in accelerated frames have already been analyzed before (i.e. see Refs. [24]), but the point of view raised in Refs. [18] is a completely different one.

In conclusion, General Covariance fulfillment not only embodies a crucial principle for any reasonable theory of gravitation, but it also opens new perspectives towards the theoretical

³ For the sake of completeness, it must be said that the equivalence principle is violated also in the framework of GR when there exists a non-vanishing temperature [23].

check of formal aspects of modern physics. In fact, the reasoning carried out for accelerated protons can also be employed to explain some quantum field theoretical features related to bremsstrahlung [25]. Thus, it appears quite natural to state that similar procedures may still be potentially exploited in other different scenarios.

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