HOLOGRAPHY AND KOSZUL DUALITY IN QUANTUM FIELD THEORY

by

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ABSTRACT

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In this thesis, we investigate mathematical constructions related to holography principle from physics, organized into three main parts. Firstly, we introduce the concept of quadratic duality for chiral algebras, extending the construction from associative algebras. We establish its relationship with the Maurer-Cartan equation, bridging it with physical intuition. Secondly, we define the notion of a vertex operator algebra (VOA) in a (pseudo)-tensor category. Specifically, we study a $\beta\gamma$ VOA in the Deligne category. This construction provides a rigorous mathematical definition for the large N vertex algebra relevant to holography. Thirdly, we analyze the structure of the higher dimensional Laurent series, which are analog of the $1d$ Laurent series $\mathbb{C}((z))$. Here, the derived structure becomes crucial, distinguishing it from the $1d$ case. We compute the A_∞ structure on the cohomology and explore various variations of this model. These A_∞/L_∞ algebraic structures can define certain (vertex) Poisson algebra. As a consequence of the holography conjecture, the vertex Poisson algebra is isomorphic to the one constructed from the $\beta\gamma$ system in the Deligne Category. We provide several checks of this conjecture.

To James Simons,
who did something nice.

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PUBLICATIONS

This thesis consists of material which I authored or co-authored.

Chapter 2 is based on the publication:
[GLZ22] *Quadratic Duality for Chiral Algebras.* Zhengping Gui, Si Li, **Keyou Zeng**. arXiv:2212.11252.

Chapter 3 is based on my on-going and unpublished work.

Chapter 4 and 5 is based on the publication:
[Zen24] *Twisted Holography and Celestial Holography from Boundary Chiral Algebra.* **Keyou Zeng**. Commun.Math.Phys. 405 (2024) 1, 19

CONTENTS

1 INTRODUCTION

The holography principle, initially proposed by 't Hooft [tH93] and elaborated by Susskind [Sus95], asserts that within a quantum gravity framework, "given any closed surface, we can represent all that happens inside it by degrees of freedom on this surface itself."[tH93] This insightful concept implies that all information contained in some region of space can be represented as a 'Hologram' - a theory which lives on the boundary of that region.

Despite appearing mysterious and difficult to conceive at first glance, physicist have accumulated increasing evidence about the holographic principle. The AdS/CFT correspondence, first discovered by Maldacena [Mal98], is perhaps the most successful realization of the holographic principle. Generally speaking, it is a conjectured relationship between two kinds of theories. On one side of the correspondence are quantum gravity theories in anti-de Sitter spaces (AdS) formulated in terms of string theory or M-theory. On the other side are conformal quantum field theories originated as the world volume theories on D branes. Since its discovery, the AdS/CFT correspondence has significantly influenced numerous branches of theoretical physics, extending far beyond the realm of string theory.

However, a bridge between the AdS/CFT correspondence and mathematics had been missing due to the rather mysterious nature of this duality, and the lack of mathematical formulation of the theories on both sides. Simplification is needed if we wish to extract any mathematical meaningful statement out of the AdS/CFT duality.

A procedure known as "twisting" is an ideal tool to perform such simplification for supersymmetric gauge theories. Since its introduction by Witten [Wit88a], this technique has been widely explored and has had a profound impact on quantum field theory and on related areas of mathematics. Classical examples includes A and B model topological string [Wit88b] in $2d$, Rozansky-Witten theory [RW97] in $3d$, Donaldson-Witten theory[Wit88a], Vafa-Witten theory[VW94] and Kapustin-Witten theory[KW07] in $4d$. On the one hand, twisted theories have particularly nice properties that often admit mathematical rigorous formulation and exact computations. On the other hand, they know about information of certain BPS sector in the original physical theories. Though the twisting techniques has been successfully applied to a variety of supersymmetric field theories, the original construction does not apply to supergravity. Only until recently the proper definition of twisted supergravity is proposed, in [CL16]. In that paper, Costello and Li also proposed a twisted form of AdS/CFT correspondence. This program throws light upon the mathematical structure hidden behind (part of) the AdS/CFT correspondence.

The notion of Koszul duality is shown to play a crucial part in this story. In [Cos17], Costello formulated the following twisted form of holography principle (or conjecture):

Conjecture 1.1. *Consider a stack of branes in twisted string theory or M-theory. We can consider the following two algebras*

- *The algebras $\mathcal{A}_N$ of local operators of the twisted supersymmetric gauge theory living on a stack of N branes, after sending $N \to \infty$.*
- *The algebra $\mathcal{B}$ of local operators of the twisted supergravity restricted along the location of the branes.*

Then, these two algebras are Koszul dual:

$$\lim_{N\to\infty} \mathcal{A}_N = \mathcal{B}^!. \tag{1.1}$$

In this direction, there has been a surge of recent works exploring different aspects of twisted supergravity and twisted holography. Various twisted holography models are proposed [IFMZ20, CG18, CP21]. More examples of twisted supergravity are found [EH21, RSW21]. Operators that create D-branes with non-trivial geometrical shapes called giant graviton in twisted holography are studied in [BG21].

While the twisting procedure has significantly simplified the physical theories, the conjecture above is still a few steps away from being a mathematical conjecture. In this thesis, we aim to bridge this gap as much as possible, and try to make some mathematical rigorous statement/conjecture of holography.

1.1 KOSZUL DUALITY IN QUANTUM FIELD THEORY

We first briefly review the central concept that appears in the twisted version of holography, which is Koszul duality. Koszul duality is a ubiquitous concept in homological algebra, which has also found many applications in representation theory, algebraic geometry, and topology (see, e.g., [BGS96, BGG78, GK94, GKM97]). In this introduction, we mainly focus on the interaction between Koszul duality and quantum field theory. In fact, the concept of Koszul duality, though quite new to physicists, is plentiful in the structure of quantum field theory, especially in the study of defects and boundaries [PW21]. In this introduction, we briefly review two main sources of Koszul duality in field theory.

One important aspect of Koszul duality in quantum field theory arises from defects. Given a field theory $\mathcal{T}$ on some manifold M, one can consider coupling some other system along a submanifold $S \subset M$. In the simplest case, we consider one dimensional defect, so that the algebra alone the defect is given by (dg) associative algebra. Suppose the algebra of the original field theory and the defect theory are given by A and B respectively. Then the uncoupled theory has the algebra $A \otimes B$. A coupling between the two theories can be understood as an element $\alpha \in A \otimes B$ which satisfies the Maurer-Cartan equation. Then we identify the space of coupling between the two systems as the Maurer-Cartan elements $MC(A \otimes B)$. Among all possible defect couplings, we are particularly interested in the universal one. The universal defect, if exist, is defined as an algebra $A^!$ together with a Maurer-Cartan element $\alpha_{uni} \in MC(A \otimes A^!)$ that satisfy the universal property: For any other algebra B and a coupling $\alpha \in MC(A \otimes B)$, there exist a unique morphism $\phi : A^! \to B$ such that $\alpha = \phi(\alpha_{uni})$. We have

$$MC(A \otimes B) \cong \mathrm{Hom}(A^!, B), \text{ for any } B, \tag{1.2}$$

We see that this is the well known properties satisfied by the Koszul duality for associative algebra. In this thesis, we will also study this property for a version of Koszul duality for chiral algebra [GLZ22]. This give us the interpretation of the algebra of the universal defect that we can couple to the theory $\mathcal{T}$ as the Koszul dual to the algebra of the theory $\mathcal{T}$ along S. We refer to [CP21, PW21] for more examples of Koszul duality from universal defect in quantum field theory.

Another perspective of Koszul duality comes from considering the boundary. For simplification, we consider boundary condition of two dimensional topological field theory, which consist of a dg category $\mathcal{C}$. Consider two generators $\mathcal{B}, \mathcal{B}^!$ of the category. The significance of the condition of being a generator is that the category $\mathcal{C}$ is equivalent to the (derived) category of modules of the algebra $A_\partial := \mathrm{End}_{\mathcal{C}}(\mathcal{B})$. The algebra A_∂ (resp. $A^!_\partial := \mathrm{End}_{\mathcal{C}}(\mathcal{B}^!)$) is also called the boundary algebra associated with the boundary condition $\mathcal{B}$ (resp. $\mathcal{B}^!$). Now suppose that the two boundary conditions $\mathcal{B}, \mathcal{B}^!$ are transversal to each other, which means that

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{B}, \mathcal{B}^!) \approx \mathbb{C}\,. \tag{1.3}$$

Physically, this correspond to the following situation. We consider the theory to be placed on $[0,1] \times N$ satisfying boundary conditions $\mathcal{B}$ and $\mathcal{B}^!$ on the two sides of the interval respectively. Then the condition 1.3 is equivalent to the bulk theory on $[0, 1] \times \mathbb{C}$ being cohomologically trivial. More explicitly, if we try to solve the equation of motion on the interval with the corresponding boundary condition, then there is a unique solution modulo gauge transformation.

The condition that $\mathcal{B}, \mathcal{B}^!$ are generators guarantees that the two algebras A_∂, $A_\partial^!$ generate the whole commutant of each other. More explicitly, we use the equivalence of categories $\mathcal{C} \to A_\partial - \text{mod}$. This functor sends $\mathcal{B}^!$ to $\text{Hom}(\mathcal{B}, \mathcal{B}^!) \cong \mathbb{C}$. Computing Hom in the two equivalent categories leads to

$$A_\partial^! = \text{Hom}(\mathcal{B}^!, \mathcal{B}^!) \cong \text{REnd}_{A_\partial}(\mathbb{C}) \tag{1.4}$$

This turns out to be one definition of Koszul duality for associative algebra. We refer to [PW21, Zen21] for more examples of Koszul duality in quantum field theory from boundary algebra.

1.2 KOSZUL DUALITY AND OPEN CLOSED COUPLING

As mentioned earlier, Koszul duality can emerge in quantum field theory through universal defects. An important question arises: how can we find or construct examples of universal defects? While constructing a defect in a quantum field theory is straightforward, such as Wilson line, identifying a universal one can be challenging. String theory offers abundant examples of bulk/defect systems – branes are defect objects in string theory. In fact, the holography conjecture 1.1 predicts that a stack of N branes becomes a universal defect system in the limit as $N \to \infty$.

The relationship between closed and open strings have been a central problem in string theory [KR04]. A string world sheet with boundaries may typically be read in two different ways as either a closed string or an open string scattering process. For example, a cylinder worldsheet stretching between two D-brane can represent either as an open string with ends on both branes going around a 1-loop vacuum diagram or a closed string emitted from one of the branes and absorbed by the other. The fact that string scattering amplitude should be the same for the two perspectives exhibits the so called open-closed duality. In the TCFT setup, it is shown in [Cos07] that an open closed TCFT is fully encoded in the Calabi-Yau A_∞ category of its boundary conditions (D branes).

The relationship between open closed coupling and Koszul duality becomes more transparent if we look at the space-time string field theory of topological string. The open-string field theory for the topological B-model is the Holomorphic Chern-Simons, and the closed string field theory is the BCOV theory. It was proved in [CL15] that there exists a unique perturbative quantization for the coupled open-closed theory.

Furthermore, according to the well-known Loday-Quillen-Tsygan theorem [Qui84, Tsy83], the large N algebra of the open string theory is represented by cyclic cohomology, which is dual to the closed string states. Therefore, these consideration naturally lead to conjecture that open-closed coupling is an instance of Koszul duality.

1.3 LARGE N ALGEBRA VIA DELIGNE CATEGORY

As we have introduced, on one side of the duality we consider the algebra $\mathcal{A}_N$ of local operators on the stack of N branes, after sending $N \to \infty$. In fact, it is a mathematical nontrivial task to make sense of the limit $\lim_{N\to\infty} \mathcal{A}_N$, and part of this thesis is devoted to solve this problem.

A twisted field theory that correspond to a stack of N branes is typically described by a (dg) Lie algebra of the form [Wit95, CS15]

$$\mathfrak{gl}_N(A) = \mathfrak{gl}_N \otimes A \tag{1.5}$$

for some (dg) associative algebra or an A_∞ algebra A. At the tree level, the space of local operators can be computed via the Lie algebra cohomology of $\mathfrak{gl}_N(A)$. In the limit when N goes to infinity, we consider the sequence of inclusion

$$\mathfrak{gl}_1(A) \hookrightarrow \mathfrak{gl}_2(A) \hookrightarrow \cdots \hookrightarrow \mathfrak{gl}_N(A) \hookrightarrow \dots \tag{1.6}$$

which gives rise to a sequence of the Lie algebra cohomology

$$H^\bullet(\mathfrak{gl}_1(A)) \leftarrow H^\bullet(\mathfrak{gl}_2(A)) \leftarrow \dots H^\bullet(\mathfrak{gl}_N(A)) \leftarrow \dots \tag{1.7}$$

Then the space of local operators in the large N limit can be defined as the limit of the above diagram. One can then use the Loday-Quillen-Tsygan theorem to simplify the results.

However, the above definition has a major draw back that the space of local operators is not just a graded vector space, but also inherits some algebraic structures from field theory computation. The morphisms induced by the inclusions $\mathfrak{gl}_N(A) \hookrightarrow \mathfrak{gl}_{N+1}(A)$ does not preserve the algebraic structure of local operators in general. Therefore, the sequence 1.6 does not provide us a definition of the algebra of local operators in many cases.

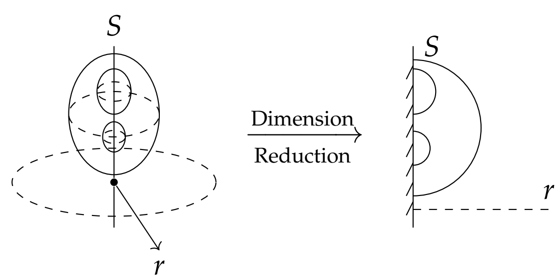
To address this issue, let's revisit how we compute the large N algebra in practice. Essentially, physicists compute the algebra as if the fields reside in a "space" resembling matrices of rank N, treated as an arbitrary parameter. Furthermore, no additional trace relations are imposed on these matrices, unlike in the case of finite N matrices.

Deligne category, defined in [Del07, DMO+82], is the mathematical gadget that rigorously realize the desired properties. It is a category that consists of objects that are sums and summands of $V^{\otimes r} \otimes (V^*)^{\otimes s}$, where V is regarded as the fundamental representation of the general linear group. There isn't any further relation among those objects, and we set $\mathrm{Tr}(1) = N$ by hand with N an arbitrary parameter.

Our strategy to define the large N algebra is to define the algebra or vertex algebra in Deligne category first. In fact, we go further and define the notion of vertex algebra in more general symmetric monoidal category. To obtain the large N vertex algebra as a vertex algebra in vector space, we apply the functor $\mathrm{Hom}(\mathbb{1}, -)$. This also allows us to define a family of vertex algebra as we vary N. Certain specialization of the parameter will also be of interest to us.

1.4 DIMENSIONAL REDUCTION AND DERIVED LAURENT SERIES

Koszul duality possesses the property that it returns us to the original algebra when applied twice. In the setup of quantum field theory, this results in a nontrivial isomorphism if we can realize Koszul duality in two distinct ways within a single system. In the string theory context, Koszul duality can be realized as the algebra on a stack of N branes in the limit as N tends to infinity. Alternatively, one can incorporate the other definition of Koszul duality, as the transversal boundary conditions. This can be achieved in a general setup by transforming a bulk-defect system into a bulk-boundary system."



Suppose we are considering a d dimensional defect $S = \mathbb{R}^d \times \{0\}$ in $\mathbb{R}^d \times \mathbb{R}^{n-d}$. We are interested in the algebra of operators restricted along S. Instead of studying the algebra directly, we first remove the defect locus and perform dimensional reduction on the unit sphere of $\mathbb{R}^{n-d}$ normal to the defect. Namely, we consider the projection

$$\pi^{S^{n-d-1}} : \mathbb{R}^d \times (\mathbb{R}^{n-d} \backslash \{0\}) \to \mathbb{R}^d \times \mathbb{R}_{>0}, \tag{1.8}$$

where we send $(\mathbf{x}, \mathbf{y})$ to $(\mathbf{x}, r = |\mathbf{y}|)$.

After this dimensional reduction, we get a theory on $\mathbb{R}^d \times \mathbb{R}_{>0}$. Since we have removed the defect locus, fields of this theory correspond to fields of the original theory with arbitrary poles at $r = 0$. Now, we can add a boundary condition to this theory at $\mathbb{R}^d \times \{r = 0\}$. There is a natural candidate for the boundary condition, given by requiring that fields of the original theory have no pole at $r \to 0$. We find that this requirement leads to a valid boundary condition in most examples. Moreover, we expect that the boundary algebra after imposing this boundary condition is the same as the algebra of the original theory restricted along the defect locus.

Then both pictures of Koszul duality enter this field theory construction. On the one hand, we can consider the algebra on a stack of N branes in $N \to \infty$, which is conjectured to be the universal defect algebra. On the other hand, we consider the boundary algebra with the boundary condition transversal to the one we mentioned. Both are Koszul dual to the same algebra, so they should be isomorphic.

As is often the case, the transversal boundary condition can also be realized as the boundary condition that requires fields of the original theory to have no pole at $r \to \infty$. Therefore, we can add one more piece of ingredient to the twisted holography conjecture

- The boundary algebra $\mathcal{A}_\partial$ of the theory obtained by performing KK compactification of the twisted supergravity along the unit sphere of the normal direction to the defect. The boundary condition is chosen so that fields have no pole at $r \to \infty$.

Then the twisted holography conjecture predicts that

$$\lim_{N \to \infty} \mathcal{A}_N \cong \mathcal{A}_\partial. \tag{1.9}$$

In this thesis, we will be considering a special class of quantum field theory called holomorphic theory. After dimensional reduction, these theories have an algebraic model closely related to the (higher dimensional) Laurent series. For example, a $2d$ CFT can be considered as a $2d$ holomorphic theory. The associated mode algebra can be considered as the algebra after dimensional reduction along the unit circle. The description of such mode algebra is typically related to the formal Laurent series $\mathbb{C}((z))$. For example, the mode algebra of Kac Moody VOA is given by the universal enveloping algebra of some central extension of $\mathfrak{g}((z))$.

Higher dimensional analog of Laurent series exist, but with some essential difference. For $d \geq 2$, Hartogs' theorem implies that the space of holomorphic functions on the punctured affine space is the same as the space of holomorphic functions on the affine space

$$\mathcal{O}(\mathring{\mathbb{A}}^d) = \mathcal{O}(\mathbb{A}^d) \cong k[z_1, \ldots, z_d] \tag{1.10}$$

Instead, we should replace the "classical" algebra of functions $\mathcal{O}(\mathring{\mathbb{A}}^d) = H^0(\mathring{\mathbb{A}}^d, \mathcal{O})$ by the derived space of functions $\mathrm{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$

$$H^i(\mathring{\mathbb{A}}^d, \mathcal{O}) = \begin{cases} k[z_1, \ldots, z_d], & i = 0, \\ z_1^{-1} \ldots z_d^{-1} k[z_1^{-1}, \ldots, z_d^{-1}], & i = d-1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.11}$$

We see that the singular data is restored, but in different cohomological degree.

Still, the cohomology 1.11 forgets the information about the dg algebra structure on the derived space of functions $\mathrm{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$. To restore this part of information, we need to work with a proper dg algebra model for $\mathrm{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$. Such a model is introduced in [FHK19] called the Jouanolou model. By the homotopy transfer theorem, the dg algebra of $\mathrm{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$ induces an A_∞ algebra structure on the cohomology $H^\bullet(\mathring{\mathbb{A}}^d, \mathcal{O})$. In this thesis, we will analyze this A_∞ structure in detail.

In this thesis, we mainly focus on the case when $d = 2$. Later we will see that the A_∞ structure on $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O})$ and its many variants define a class of (vertex) Poisson algebra structures. These (vertex) Poisson algebras can be regarded as certain semi classical limit of the algebra A_∂. From "corollary" 1.9 of the Holography conjecture, we propose the main conjectures of this thesis. We conjecture that the (vertex) Poisson algebra defined through the derived Laurent series $H^\bullet(\mathring{\mathbb{A}}^d, \mathcal{O})$ are isomorphic to the semi classical limit of certain large N algebra $\mathcal{A}_N$ defined through Deligne category.

2 KOSZUL DUALITY

The notion of Koszul duality is ubiquitous in mathematics. It is a simple, yet powerful principle that has found many applications in representation theory, algebraic geometry, homological algebra, and topology (see, e.g., [BGS96, BGG78, GK94, GKM97]).

Given an augmented associative algebra A, we can construct a new algebra

$$A^! = \mathrm{RHom}_A(k,k)$$

equipped with the Yoneda product. Under certain conditions, performing this construction twice bring us back to the original algebra A. If this is the case, we call such an algebra A Koszul and the functor $A \to A^!$ Koszul duality. While simple in its definition, this construction yields significant implications, including the equivalence of certain derived categories of modules. Its deep connection with quantum field theory is also discussed in the introduction.

For associative algebra that admits a presentation as a quadratic algebra, its Koszul dual has particularly nice construction. In this chapter, we first review the classical construction of quadratic Koszul duality for associative algebra.

Our goal in this chapter is to generalize this construction to chiral algebra. We also discuss generalization to the quadratic-linear-scalar case. Finally we present some simple example that parallel the well-know examples of associative algebra.

2.1 BAR CONSTRUCTION AND TWISTING MORPHISM

We first introduce the bar and cobar functor for algebra and coalgebra. We follow the discussion in [LV12]. Let (A, d_A, μ_A) be an differential graded algebra with an augmentation $\varepsilon : A \to k$. We denote $\bar{A} = \ker \varepsilon$, then $A \cong k \oplus \bar{A}$. We define the bar construction of A to be the following differential graded coalgebra

$$BA := (T^c(s\bar{A}), d_{BA} = d_1 + d_2), \tag{2.1}$$

where $T^c(s\bar{A})$ is the cofree coalgebra generated by $s\bar{A}$. The differential d_1 is induced by the differential d_A and the differential d_2 is the Hochschild differential induced by the product μ_A.

Similarly, given a differential graded coalgebra (C, d_C, Δ) with a coaugmentation $C = k \oplus \bar{C}$. We define the cobar construction of C to be the following differential graded algebra

$$\Omega C := (T(s^{-1}\bar{C}), d_{\Omega C} = d_1, d_2) \tag{2.2}$$

where $T(s^{-s}\bar{C})$ is the free algebra generated by $s^{-1}\bar{C}$. The differential d_1 is induced by the differential d_C and the differential d_2 is induced by the coproduct Δ_C.

Given a dg algebra A and a dg coalgebra C, we can equip the space $\mathrm{Hom}(C, A)$ a dg algebra structure $(\mathrm{Hom}(C, A), \partial, \star)$. The differential is naturally induced from the differential of A and C. The product is given by the convolution product

$$f \star g := \mu \circ (f \otimes g) \circ \Delta \tag{2.3}$$

One can check by definition that the differential ∂ is indeed a derivation with respect to the convolution product $\star$.

Definition 2.1. A twisting morphism is a degree -1 map $\alpha : C \to A$ that satisfy the Maurer-Cartan equation

$$\partial\alpha + \alpha \star \alpha = 0 \tag{2.4}$$

We denote $\mathrm{Tw}(C, A)$ the space of twisting morphism from C to A.

Using the space of twisting morphism, we can show that the bar and cobar construction form a pair o adjoint functors

$$\Omega : \{\text{conil. dg coalgebras}\} \rightleftarrows \{\text{aug. dg algebras}\} : B \tag{2.5}$$

Theorem 2.2. *For every augmented dg algebra A and every conilpotent dg coalgebra C, there exist natural bijections*

$$\mathrm{Hom}_{\mathrm{dg-Alg}}(\Omega C, A) \cong \mathrm{Tw}(C, A) \cong \mathrm{Hom}_{\mathrm{dg-coAlg}}(C, BA) \tag{2.6}$$

Given the bar and cobar construction, we can construct the standard bar resolution as a twisted tensor product complex. We first define the twisted tensor product of a coalgebra C and a algebra A. A map $\alpha : C \to A$ also induces a derivation d^r_α on $C \otimes A$, which is defined by

$$d^r_\alpha : C \otimes A \xrightarrow{\Delta \otimes \mathrm{id}_A} C \otimes C \otimes A \xrightarrow{\mathrm{id}_C \otimes \alpha \otimes \mathrm{id}_A} C \otimes A \otimes A \xrightarrow{\mathrm{id}_C \otimes \mu} C \otimes A \quad (2.7)$$

Direct calculation shows that

Lemma 2.3. *The coderivation $d_\alpha := d_{C \otimes A} + d^r_\alpha$ satisfy*

$$d^2_\alpha = d^r_{\partial\alpha + \alpha \star \alpha} \quad (2.8)$$

As a result, when $\alpha \in \mathrm{Tw}(C, A)$, we obtain a chain complex

$$C \otimes_\alpha A := (C \otimes A, d_\alpha) \quad (2.9)$$

which is called the twisted tensor product.

From the identity map on BA we obtain the universal twisting map $\pi : BA \to A$. Explicitly, this map is given by $\pi : BA \to s\bar{A} \to \bar{A} \to A$. We consider the corresponding twisted tensor product $BA \otimes_\pi A$. We see that this is the (reduced) bar resolution of A, which is an acyclic complex. Similarly, we have the universal twisting map $\iota : C \to \Omega C$, given by $\iota : C \to \bar{C} \to s^{-1}\bar{C} \to \Omega C$. It gives rise to a acyclic complex $C \otimes_\iota \Omega C$.

2.2 KOSZUL DUALITY FOR QUADRATIC ALGEBRA

First we introduce the notion of quadratic data. A quadratic data (V, R) is a graded vector space V together with a graded subspace $R \subset V^{\otimes 2}$. A morphism of quadratic data $f : (V, R) \to (W, S)$ is a graded linear map $f : V \to W$ such that $(f \otimes f)(R) \subset S$.

The quadratic algebra $A(V, R)$ is defined by the quotient of free associative (tensor) algebra $T(V)$ by the two sided ideal (R) generated by R.

$$A(V, R) = T(V)/(R) \quad (2.10)$$

Since (R) is a homogeneous ideal, $A(V, R)$ is graded and we call this degree weight. By definition, an element a in the image of $V^{\otimes n} \to T(V)/(R)$ has weight degree $\omega(a) = n$. Explicitly, $A(V, R)$ is given by

$$A(V, R) = k \oplus V \oplus (V^{\otimes 2}/R) \oplus \cdots \oplus \left(V^{\otimes n} / \sum_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right) \oplus \cdots \quad (2.11)$$

It has the universal property that for any associative algebra A with a algebra map $T(V) \to A$ such that (R) maps to 0, then there exists a unique morphism $A(V,R) \to A$ which makes the diagram commutes

$$\begin{array}{ccc} T(V) & \longrightarrow & A \\ & \searrow \quad \nearrow & \\ & A(V,R) & \end{array} \tag{2.12}$$

Similarly, we define the quadratic coalgebra $C(V,R)$ as the sub-coalgebra of the cofree coassociative coalgebra $T^c(V)$, cogenerated by R. Explicitly, $C(V,R)$ is given by

$$C(V,R) = k \oplus V \oplus R \oplus \cdots \oplus \left(\bigcap_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right) \oplus \cdots \tag{2.13}$$

It has the universal property that for any coalgebra C with coalgebra maps $C \to T^c(V)$ such that the composition with the projection $T^c(V) \to V^{\otimes 2}/R$ is 0, then there exist a coalgebra map $C \to C(V,R)$ such that the following diagram commute

$$\begin{array}{ccc} & C(V,R) & \\ & \nearrow \quad \searrow & \\ C & \longrightarrow & T^c(V) \end{array}$$

Let (V,R) be a quadratic data (V,R),. By definition the Koszul dual coalgebra of the quadratic algebra $A(V,R)$ is the coalgebra

$$A^{¡} = C(sV, s^2R) \tag{2.14}$$

where s^2R is the image of R in $(sV)^{\otimes 2}$.

We define the Koszul dual algebra of the quadratic algebra as the linear dual of $A^{¡}$. Dualizing the exact sequence

$$0 \to R \to V^{\otimes 2} \to V^{\otimes 2}/R \to 0 \tag{2.15}$$

gives us the exact sequence

$$0 \to R^{\perp} \to (V^*)^{\otimes 2} \to R^* \to 0 \tag{2.16}$$

where $R^{\perp}$ is the image of $(V^{\otimes 2}/R)^*$ in $(V^*)^{\otimes 2}$. In other words, $R^{\perp}$ is the subspace of $(V^*)^{\otimes 2}$ that consists of elements f that vanish on R, $f(R) = 0$.

Therefore, we find that the linear dual of $C(V,R)$ is given by

$$A^{!} = A(s^{-1}V^{*}, s^{-2}R^{\perp}) \tag{2.17}$$

Now we consider the bar construction for a quadratic algebra. There are two natural gradings on BA for $A = A(V,R)$: a homological degree and a weight degree. An element $sa_1 \otimes \cdots \otimes sa_n \in (s\bar{A})^n$ is of homological degree n and weight degree $\omega(a_1) + \cdots + \omega(a_n)$. We define a syzygy degree to be the weight degree minus the homological degree. For example, the syzygy degree of $sa_1 \otimes \cdots \otimes sa_n \in (s\bar{A})^n$ is $\omega(a_1) + \cdots + \omega(a_n) - n$. The complex BA is now bigraded by syzygy degree and the weight, and the differential raises the syzygy degree by 1. The syzygy degree d component of BA is denoted $B^d A$, and the weight n component of of BA is denoted $(BA)^{(n)}$. We illustrate this bi-grading in the following diagram

$$\begin{array}{ccccccccl}
\cdots & & \cdots & & \cdots & & \cdots & & (4) \\
0 & \longleftarrow & \frac{V^{\otimes 3}}{V\otimes R + R\otimes V} & \longleftarrow & (\frac{V^{\otimes 2}}{R} \otimes V) \oplus (V \otimes \frac{V^{\otimes 2}}{R}) & \longleftarrow & V^{\otimes 3} & & (3) \\
 & & 0 & \longleftarrow & \frac{V^{\otimes 2}}{R} & \longleftarrow & V^{\otimes 2} & & (2) \\
 & & & & 0 & \longleftarrow & V & & (1) \\
 & & & & & & k & & (0) \\
3 & & 2 & & 1 & & 0 & &
\end{array}$$

We indicate the syzygy degrees on the bottom row and weight degrees are indicated on the rightmost column.

We see that the syzygy degree 0 column forms the cofree coalgebra $T^c(sV)$. Therefore the Koszul dual coalgebra $A^{¡} = C(sV, s^2R)$ is a subspace of this column, and we get an inclusion map

$$i : A^{¡} \to BA \tag{2.18}$$

which factors through the syzygy degree 0 component B^0A. Moreover, we can show that this inclusion is an isomorphism to the degree 0 component of the homology.

Proposition 2.4. *The natural coalgebra inclusion $i : A^{¡} \to BA$ induces an isomorphism of graded coalgebras*

$$i : A^{¡} \cong H^0(B^{\bullet}A) \tag{2.19}$$

Under the inclusion i, the universal twisting morphism $\pi : BA \to A$ becomes the following map

$$\kappa : C(sV, s^2R) \to sV \cong V \to A(V, R) \tag{2.20}$$

Lemma 2.5. *We have $\kappa \in \mathrm{Tw}(A^{¡}, A)$.*

We see that the Koszul dual coalgebra $A^{¡}$ is a good candidate to replace the very big bar coalgebra BA. The main theorem of (quadratic) Koszul duality theory is the following criterion that characterize this property.

Theorem 2.6. *(Koszul criterion). Let (V, R) be a quadratic data. Let $A = A(V, R)$ the associated quadratic algebra and $A^{¡} = C(sV, s^2R)$ its quadratic dual coalgebra. Then the following assertions are equivalent*

1. *The right Koszul complex $A^{¡} \otimes_\kappa A$ is acyclic.*
2. *The inclusion $i : A^{¡} \to BA$ is a quasi-isomorphism.*

Let $A = A(V, R)$ be a finitely generated quadratic algebra. For A a Koszul algebra, we can use the Koszul complex to compute the derived Ext functor by

$$\mathrm{Ext}_A(k, k) = H^\bullet(\mathrm{Hom}_A(A \otimes_\kappa A^{¡}, k)) \tag{2.21}$$

Corollary 2.7. *If $A = A(V, R)$ be a finitely generated quadratic algebra and is Koszul, we have an isomorphism*

$$\mathrm{Ext}_A(k, k) \cong A^! \tag{2.22}$$

In the finitely generated case, we can also identify the space of twisting morphism as the space of solutions to the Maurer-Cartan equation. We can reformulate Theorem 2.2 using the Koszul dual algebra

Corollary 2.8. *Let A be a finitely generated Koszul algebra. Then we have a quasi isomorphism*

$$\mathrm{MC}(A \otimes B) \cong \mathrm{Hom}(A^!, B) \tag{2.23}$$

2.3 CHIRAL AND FACTORIZATION ALGEBRA

Before we consider the quadratic duality theory for chiral algebra, we first give a brief review of some basic definitions of chiral algebras and factorization algebras. For more details, see [BD04, FBZ04, Gai98]. Throughout this section, X stands for a smooth curve over $\mathbb{C}$.

Definition 2.9. Let $\mathcal{A}$ be a $\mathbb{Z}$-graded $\mathcal{D}_X$-module. A chiral algebra structure on $\mathcal{A}$ is a degree 0 $\mathcal{D}_{X^2}$-module map [BD04]:

$$\mu : j_* j^* \mathcal{A} \boxtimes \mathcal{A} \to \Delta_*(\mathcal{A}),$$

where $\Delta : X \to X^2$ is the diagonal embedding and $j : U = X^2 - \Delta \hookrightarrow X^2$ is the open embedding.

The map μ satisfies the following two conditions:

- Antisymmetry:

  If $f(z_1, z_2) \cdot a \boxtimes b$ is a local section of $j_* j^* \mathcal{A} \boxtimes \mathcal{A}$, then

  $$\mu(f(z_1, z_2) \cdot a \boxtimes b) = -(-1)^{|a||b|} \sigma_{1,2} \mu(f(z_2, z_1) \cdot b \boxtimes a), \qquad (2.24)$$

  where $\sigma_{1,2}$ acts on $\Delta_* \mathcal{A}$ by permuting two factors of X^2.

- Jacobi identity:

  If $a \boxtimes b \boxtimes c \cdot f(z_1, z_2, z_3)$ is a local section of $j_* j^* \mathcal{A}^{\boxtimes 3}$ where j is the open embedding of the complement of the big diagonal in X^3. Then

  $$\mu(\mu(f(z_1, z_2, z_3) \cdot a \boxtimes b) \boxtimes c) + (-1)^{|a|(|b|+|c|)} \sigma_{1,2,3} \mu(\mu(f(z_2, z_3, z_1) \cdot b \boxtimes c) \boxtimes a) + (-1)^{|c|(|a|+|b|)} \sigma_{1,2,3}^{-1} \mu(\mu(f(z_3, z_1, z_2) \cdot c \boxtimes a) \boxtimes b) = 0,$$

  here $\sigma_{1,2,3}$ denotes the cyclic permutation action on $\Delta_*^{X \to X^3} \mathcal{A}$ and $\Delta^{X \to X^3} : X \to X^3$ is the diagonal embedding.

Now we give the definition of the factorization algebra. Later we will discuss the relationship between chiral algebras and factorization algebras.

We use the following conventions in the definition. For a surjective map $\pi : J \twoheadrightarrow I$ between two finite sets I and J, let $j^{[J/I]} : U^{[J/I]} \hookrightarrow X^J$ be the complement to all the diagonals that are transversal to $\Delta^{(J/I)} : X^I \hookrightarrow X^J$. Therefore one has

$$U^{[J/I]} = \left\{ (x_j) \in X^J : x_{j_1} \neq x_{j_2} \text{ if } \pi(j_1) \neq \pi(j_2) \right\}.$$

Definition 2.10. A *factorization algebra* on X consists of the following datum:

(1) A graded quasicoherent sheaf B_{X^I} over X^I for any finite set I, which has no non-zero local sections supported at the union of all partial diagonals.

(2) Isomorphisms of graded quasicoherent sheaves

$$\nu^{(\pi)} = \nu^{(J/I)} : \Delta^{(\pi)*} B_{X^J} \xrightarrow{\sim} B_{X^I}$$

for every surjection $\pi : J \twoheadrightarrow I$ and compatible with the composition of the π's.

(3) (*factorization*) For every surjection $J \twoheadrightarrow I$, there is an isomorphism of $\mathcal{O}_{U^{[J/I]}}$-modules

$$c_{[J/I]} : j^{[J/I]*}(\boxtimes_{i\in I} B_{X^{J_i}}) \xrightarrow{\sim} j^{[J/I]*} B_{X^J}.$$

We require that c's are mutually compatible: for $K \twoheadrightarrow J$ the isomorphism $c_{[K/J]}$ coincides with the composition $c_{[K/I]}(\boxtimes_{i\in I} c_{[K_i/J_i]})$. And c should be compatible with ν: for every $J \twoheadrightarrow J' \twoheadrightarrow I$ one has

$$\nu^{(J/J')}\Delta^{(J/J')*}(c_{[J/I]}) = c_{[J'/I]}(\boxtimes_{i\in I} \nu^{(J_i/J_i')}).$$

(4)(*unit*) There exists a global section 1 of B_X such that for every $f \in B_X$ one has $1 \boxtimes f \in B_{X^2} \subset j_*j^*B_X^{\boxtimes 2}$ and $\Delta^*(1 \boxtimes f) = f$.

There is an equivalence between the category of factorization algebras and that of chiral algebras [BD04]. More precisely, we can obtain a chiral algebra from a factorization algebra B as follows. For each surjection $J \twoheadrightarrow I$ we have a natural isomorphism of left $\mathcal{D}-$modules

$$\Delta^{(J/I)*} B_{X^J} \xrightarrow{\sim} B_{X^I}.$$

We can rewrite it as an isomorphism of right $\mathcal{D}_{X^J}$-modules

$$\Delta_*^{(J/I)} \omega_X^{\boxtimes I} \otimes_{\mathcal{O}_{X^J}} B_{X^J} \xrightarrow{\sim} \Delta_*^{(J/I)}(\omega_X^{\boxtimes I} \otimes_{\mathcal{O}_{X^I}} B_{X^I}).$$

In particular for $\Delta : X \hookrightarrow X^2$, we have

$$\Delta_* \omega_X \otimes_{\mathcal{O}_{X^2}} B_{X^2} \xrightarrow{\sim} \Delta_*(\omega_X \otimes_{\mathcal{O}_X} B_X).$$

Then we have

$$j_*j^*B^{r\boxtimes 2} = j_*j^*\omega_X^{\boxtimes 2} \otimes_{\mathcal{O}_{X^2}} B_{X^2} \to \Delta_*\omega_X \otimes_{\mathcal{O}_{X^2}} B_{X^2} = \Delta_*(\omega_X \otimes_{\mathcal{O}_X} B) = \Delta_* B^r. \tag{2.25}$$

One can verify that the above binary operation makes the right $\mathcal{D}_X$-module B^r into a chiral algebra.

Now we explain the inverse direction. Suppose we have a chiral algebra $\mathcal{A}$, then we define $\mathcal{F}_{X^I} = \mathcal{A}^l_{X^I} := \mathcal{A}_{X^I} \otimes_{\mathcal{O}_{X^I}} \omega_{X^I}^{-1}$ on X^I. Here $\mathcal{A}_{X^I}$ is the intersections of the kernels of all the chiral operations on $j_*j^*\mathcal{A}^{\boxtimes I}$. Then we have

$$\Delta^{(J/I)*}\mathcal{F}_{X^J} \simeq \mathcal{F}_{X^I}$$

and $\mathcal{F}$ is a factorization algebra. See [BD04, Section 3.4] for more details.

2.4 CHIRAL QUADRATIC DUALITY

We would like to generalize the previous construction of Koszul duality to the world of chiral algebra. On the one hand, possible construction of chiral algebra is limited due to the axiom of locality. For example, there is no chiral counterpart of tensor algebra. On the other hand, we still have the chiral analog of quadratic algebra, and a proper quadratic relation is required for locality to hold. There is also a vertex algebra version of this construction [Roi01]. We use the chiral algebra language as it is more convenient to prove some theorem. We use this construction to formulate the chiral quadratic duality and extend it to non-homogeneous cases [GLZ22].

Throughout this section, X denotes a smooth complex algebraic curve and $j : U \hookrightarrow X \times X$ denotes the complement of the diagonal.

Quadratic constructions

We first introduce the chiral algebra freely generated by a given non-empty set of sections subject to quadratic relations, constructed in [BD04, Section 3.4.14,pp184].

Definition 2.11. A *chiral quadratic datum* is a pair (N, P) where N is a locally free $\mathbb{Z}$-graded $\mathcal{O}_X$-module of finite rank and $P \subset j_*j^*N \boxtimes N$ is a locally free $\mathcal{O}_{X\times X}$-submodule such that $P|_U = N \boxtimes N|_U$.

Remark 2.12. In the original construction [BD04], N can be any quasi-coherent $\mathcal{O}_X$-module and P can be any quasi-coherent submodule of $j_*j^*N \boxtimes N$. Here for simplicity, we will only consider the case when both N and P are locally free.

Remark 2.13. The condition $P|_U = N \boxtimes N|_U$ is the main difference from the quadratic algebra case. It corresponds to the locality axiom in the definition of vertex algebras. This simply means that for every local section $a \boxtimes b \in N \boxtimes N$, we can find an integer $n >> 0$ sufficiently large such that $(z_1 - z_2)^n a \boxtimes b \in P$ where z_1, z_2 are local coordinates on X^2. It translates to the locality axiom: for every pair of generators (a, b) of a vertex algebra, we have

$$a_{(n)}b = 0, \quad n >> 0,$$

here $-_{(n)}-$ is the standard n-th product notation in vertex algebras.

Suppose we have a chiral quadratic datum (N, P). We present the universal property satisfied by the quadratic chiral algebra that correspond to (N, P). This is simply a generalization of the universal property of quadratic algebra 2.12.

Consider a functor on the category of chiral algebras $\mathcal{CA}(X)$ which assigns to a chiral algebra $\mathcal{A}$ the set of all $\mathcal{O}_X$-linear morphisms

$$\phi : N_\omega = N \otimes_{\mathcal{O}_X} \omega_X \to \mathcal{A}$$

such that the chiral product $\mu_{\mathcal{A}}$ annihilates the submodule $\phi^{\boxtimes 2}(P \otimes_{\mathcal{O}_{X^2}} \omega_{X^2}) \subset j_* j^* \mathcal{A}^{\boxtimes 2}$. Beilinson and Drinfeld prove that this functor is representable. They refer to the corresponding universal chiral algebra as the *chiral algebra freely generated by* (N, P). We will denote this chiral algebra by $\mathcal{A}(N, P)$.

Definition 2.14. The *quadratic chiral algebra* associated to a chiral quadratic datum (N, P) is defined to be $\mathcal{A}(N, P)$.

We do not present the detail of the construction of $\mathcal{A}(N, P)$ here. The reader can refer to [BD04, 3.4.14,pp184] for detail.

Motivated by the construction of the quadratic duality for quadratic associated algebra

$$A = A(V, R) \to A^! = A(s^{-1}V^*, s^{-2}P^\perp), \tag{2.26}$$

we introduce the quadratic dual relation $P^\perp$ as follows.

Definition 2.15. Let (N, P) be a chiral quadratic datum. Define a $\mathcal{O}_{X\times X}$-submodule $P^\perp$ of $j_* j^* s^{-1} N^\vee_{\omega^{-1}} \boxtimes s^{-1} N^\vee_{\omega^{-1}}$ as follows. Consider the following sequence of maps

$$\begin{aligned} j_* j^* s^{-1} N^\vee_{\omega^{-1}} \boxtimes s^{-1} N^\vee_{\omega^{-1}} &\xrightarrow{\langle -,-|\rangle}_P \mathrm{Hom}_{\mathcal{O}_{X^2}}(P, j_* j^* s^{-2} \omega^{-1}_{X^2}) \\ &\longrightarrow \mathrm{Hom}_{\mathcal{O}_{X^2}}(P, s^{-2} \Delta_! \omega_X), \end{aligned} \tag{2.27}$$

where the first map is given by the restriction of the natural pairing

$$\langle -, - \rangle : (j_* j^* N \boxtimes N) \otimes_{\mathcal{O}_{X^2}} (j_* j^* s^{-1} N^\vee_{\omega^{-1}} \boxtimes s^{-1} N^\vee_{\omega^{-1}}) \to j_* j^* s^{-2} \omega^{-1}_{X^2}$$

to P. Let $P^\perp$ be the kernel of the composition. In other words, we have $P^\perp$ is the sheaf of sections whose pairing with P is regular. We have

$$P^\perp|_V = \{t | \forall p \in P|_V,\ \langle t, p \rangle \in s^{-2} \omega^{-1}_{X^2}|_V\}$$

for any open subset V of $X \times X$.

In general, the pair $(s^{-1} N^\vee_{\omega^{-1}}, P^\perp)$ is not a chiral quadratic datum. For example, we can take P to be $j_* j^* N \boxtimes N$ itself. Then $P^\perp$ will be the zero sheaf and does not satisfy the condition $P|_U = N \boxtimes N|_U$. This leads to the following definition.

Definition 2.16. A chiral quadratic datum (N, P) is called *dualizable* if

$$P^{\perp}|_U = s^{-1}N^{\vee}_{\omega^{-1}} \boxtimes s^{-1}N^{\vee}_{\omega^{-1}}|_U.$$

From the following proposition, we can obtain a dual chiral quadratic datum from a dualizable chiral quadratic datum.

Proposition 2.17. *If a chiral quadratic datum (N, P) is dualizable, then*

$$P^{\perp} \simeq s^{-2}P^{\vee} \otimes_{\mathcal{O}_{X^2}} \omega^{-1}_{X^2},$$

here $P^{\vee} = \mathrm{Hom}_{\mathcal{O}_{X^2}}(P, \mathcal{O}_{X^2})$ is the dual of P. This implies that $P^{\perp}$ is also locally free and $(N^{\vee}_{\omega^{-1}}, P^{\perp})$ is a dualizable chiral quadratic datum.

Proof. Assume that $\mathrm{rank}(N) = r$. We first show that there exists a positive integer $k > 0$ such that

$$P \subset N \boxtimes N(k\Delta).$$

We prove this by contradiction. Suppose that for any positive integer $k > 0$, P is not contained in $N \boxtimes N(k\Delta)$. Then we can find an open subset $V \subset X$ such that N and $N^{\vee}_{\omega^{-1}}$ can be trivialized on V (and we denote a basis of $N|_V$ by $\{e_i\}_{1=1,\dots,r}$) and a sequence of sections

$$\{\sum_{1\le i,j\le r} f^n_{ij} e_i \boxtimes e_j\}_{n\ge 1}, \quad \sum_{1\le i,j\le r} f^n_{ij} e_i \boxtimes e_j \in \Gamma(V \times V, P|_{V\times V})$$

which satisfies that

$$\{\mathrm{ord}_{\Delta}(f^n_{ij})\}^{1\le i,j\le r}_{n\ge 1}$$

is unbounded below. Here the notation ord_{Δ} means the pole order along the diagonal. This means that we can find $(i_0, j_0) \in \{1, \dots, r\} \times \{1, \dots r\}$ and $n_1 < n_2 < n_3 < \cdots$ such that $\{\mathrm{ord}_{\Delta}(f^{n_i}_{i_0 j_0})\}_{i\ge 1}$ is unbounded below. Then we conclude that for $k \in \mathbb{Z}$, $\frac{e^{\vee}_{i_0} dz_1^{-1} \boxtimes e^{\vee}_{j_0} dz_2^{-1}}{(z_1 - z_2)^k} \notin \Gamma(V \times V, P^{\perp}|_{V\times V})$. This implies that P is not dualizable, we get a contradiction.

We conclude that P is a locally free sheaf of rank r^2. Then the obvious map

$$P^{\perp} \to s^{-2}P^{\vee} \otimes_{\mathcal{O}_{X^2}} \omega^{-1}_{X^2}$$

is an isomorphism. In fact, we can construct an inverse as follows. We work locally as above, suppose $\{e_i \boxtimes e_j\}$ (resp. $\{p_k\}$) is a local basis of $N \boxtimes N$ (resp. P). We can find local functions $\{f^k_{ij}\}, \{f^{-1\ k}_{ij}\}$ regular away from the diagonal such that

$$p_k = \sum_{1\le i,j\le r} f^k_{ij} e_i \boxtimes e_j, \quad e_i \boxtimes e_j = \sum_{k=1}^{r^2} f^{-1\ k}_{ij} p_k.$$

Define

$$p_k^\vee \mapsto \sum_{1\le i,j\le r} f_{ij}^{-1\ k} e_i^\vee \boxtimes e_j^\vee.$$

This defines the desired inverse $s^{-2}P^\vee \otimes_{\mathcal{O}_{X^2}} \omega_{X^2}^{-1} \to P^\perp$. □

Now we are ready to introduce the notion of quadratic dual chiral algebra.

Definition 2.18. Let $\mathcal{A}(N,P)$ be a quadratic chiral algebra associated to a dualizable quadratic datum (N,P). We define $\mathcal{A}^!$ to be

$$\mathcal{A}(s^{-1}N^\vee_{\omega^{-1}}, P^\perp).$$

We call $\mathcal{A}^!$ the *quadratic dual chiral algebra* of $\mathcal{A}$.

Since P is locally free, we have $P^{\vee\vee} = P$ which implies that $(\mathcal{A}^!)^! = \mathcal{A}$. This explains the name of "quadratic dual chiral algebra".

Non-homogeneous constructions

In this subsection, we modify the construction in the previous discussion to study the non-homogeneous cases. Namely, we introduce a duality notion that can be viewed as a chiral analogue of non-homogeneous quadratic duality for associative algebras [LV12, Pos93].

Let $\mathbf{1}^\circ \simeq \mathcal{O}_X$ be a copy of the trivial line bundle.

Definition 2.19. A chiral *quadratic-linear-scalar* (QLS) datum is a chiral quadratic datum in the form of $(N \oplus \mathbf{1}^\circ, P^\circ)$, such that

$$j_*j^*(N \boxtimes \mathbf{1}^\circ \oplus \mathbf{1}^\circ \boxtimes N \oplus \mathbf{1}^\circ \boxtimes \mathbf{1}^\circ) \cap P^\circ = N \boxtimes \mathbf{1}^\circ \oplus \mathbf{1}^\circ \boxtimes N \oplus \mathbf{1}^\circ \boxtimes \mathbf{1}^\circ$$

The QLS chiral algebra associated to a QLS datum $(N \oplus \mathbf{1}^\circ, P^\circ)$ is defined to be

$$\mathcal{A}(N,P^\circ)_{\mathrm{QLS}} := \frac{\mathcal{A}(N\oplus \mathbf{1}^\circ, P^\circ)}{\langle \mathbf{1}^\circ_\omega - 1_\omega \rangle}$$

where $1_\omega = \omega_X$ is the unit and $\langle \mathbf{1}^\circ_\omega - 1_\omega \rangle$ is the ideal generated by $\mathbf{1}^\circ_\omega - 1_\omega$.

For a chiral quadratic datum $(N \oplus \mathbf{1}^\circ, P^\circ)$, we denote $\mathbf{q}P^\circ \subset j_*j^*N \boxtimes N$ to be the image of

$$P^\circ \hookrightarrow j_*j^*(N \oplus \mathbf{1}^\circ) \boxtimes (N \oplus \mathbf{1}^\circ) \to j_*j^*N \boxtimes N,$$

where the first arrow is the inclusion and the second arrow is the projection. Using the fact that $(\mathbf{q}P^\circ)^\perp \subset P^{\circ\perp}$, we have the following lemma.

Lemma 2.20. *Assume that the chiral quadratic datum $(N \oplus \mathbf{1}^\circ, P^\circ)$ is dualizable. Then the identity map* $\mathbf{id} : s^{-1}N^\vee_{\omega^{-1}} \to s^{-1}N^\vee_{\omega^{-1}}$ *induces a injective morphism of chiral algebras*

$$i : \mathcal{A}(s^{-1}N^\vee_{\omega^{-1}}, (\mathbf{q}P^\circ)^\perp) \to \mathcal{A}(s^{-1}N^\vee_{\omega^{-1}} \oplus s^{-1}\mathbf{1}^\circ_{\omega^{-1}}, P^{\circ\perp}).$$

Retain the same notations, we introduce the notion of dualizable chiral QLS datum.

Definition 2.21. We call a chiral QLS datum $(N \oplus \mathbf{1}^\circ, P^\circ)$ *dualizable* if $(N \oplus \mathbf{1}^\circ, P^\circ)$ is dualizable as a chiral quadratic datum and

1) The inner derivation

$$\begin{aligned} d :=& \mu(\underline{s^{-1}\mathbf{1}^\circ} \boxtimes -) : \\ & \mathcal{A}(s^{-1}N^\vee_{\omega^{-1}} \oplus s^{-1}\mathbf{1}^\circ_{\omega^{-1}}, P^{\circ\perp}) \to \Delta_*\mathcal{A}(s^{-1}N^\vee_{\omega^{-1}} \oplus s^{-1}\mathbf{1}^\circ_{\omega^{-1}}, P^{\circ\perp}) \end{aligned}$$

preserves $\mathrm{Im}(i)$. More precisely, $d(a)$ is in the image of $\Delta_* i$ if a is in the image of i;

2) The element $\mu(\underline{s^{-1}\mathbf{1}^\circ} \boxtimes \underline{s^{-1}\mathbf{1}^\circ}) \in \Delta_*\mathcal{A}(s^{-1}N^\vee_{\omega^{-1}} \oplus s^{-1}\mathbf{1}^\circ_{\omega^{-1}}, P^{\circ\perp})$ is in the image of $\Delta_* i$.

Here, the notation $\underline{s^{-1}\mathbf{1}^\circ}$ means the identity global section $\underline{s^{-1}\mathbf{1}^\circ} \in \Gamma(X, s^{-1}\mathcal{O}_X)$.

We introduce the notion of twisted pair which later will serve as the "dual chiral algebra" of a QLS chiral algebra.

Definition 2.22. A twisted pair is a triple $(\mathcal{B}, \mathcal{B}^\circ, \mathbf{S})$, where $\mathcal{B}^\circ$ is a graded chiral algebra and $\mathcal{B} \subset \mathcal{B}^\circ$ is a subalgebra. And $\mathbf{S} \in \Gamma(X, \mathcal{B}^\circ)$ is a global section of degree -1 such that

1) the map $(h \boxtimes \mathrm{id})\mu(\mathbf{S} \boxtimes -) : \mathcal{B}^\circ \to \mathcal{B}^\circ$ preserves the subalgebra $\mathcal{B}$. Here $h(M) := M \otimes_{\mathcal{D}_X} \mathcal{O}_X$ denotes the de Rham sheaf for any right $\mathcal{D}_X$-module M,

2) the element $\mu(\mathbf{S} \boxtimes \mathbf{S})$ belongs to $\Delta_*\mathcal{B}$.

The following proposition is just a reformulation of previous definitions.

Proposition 2.23. *Let $(N \oplus \mathbf{1}^\circ, P^\circ)$ be a dualizable chiral QLS datum. Then the triple*

$$(\mathcal{A}(s^{-1}N^\vee_{\omega^{-1}}, (\mathbf{q}P^\circ)^\perp), \mathcal{A}(s^{-1}N^\vee_{\omega^{-1}} \oplus s^{-1}\mathbf{1}^\circ_{\omega^{-1}}, P^{\circ\perp}), \underline{s^{-1}\mathbf{1}^\circ})$$

is a twisted pair.

We define the quadratic dual of the chiral QLS algebra $\mathcal{A}(N, P^\circ)_{\mathrm{QLS}}$ to be the above twisted pair.

We now introduce the notion of chiral CDG-algebra (curved DG-algebra) which will appear in Section 2.6.

Definition 2.24. A chiral CDG-algebra is a triple $(\mathcal{B}, d, \iota)$, where $\mathcal{B}$ is a graded chiral algebra, $d : \mathcal{B} \to \mathcal{B}$ is a derivation of $\mathcal{B}$ of degree -1, that is, d satisfies

$$d(\mu(a \boxtimes b)) = \mu(da \boxtimes b) + \mu(a \boxtimes db).$$

And $\iota \in \Gamma(X, \mathcal{B})$ is a global section of degree -2 which is called *curving*. It satisfies the following
1) $d^2(-) = (h \boxtimes \mathrm{id})\mu(\iota \boxtimes -)$,
2) $d(\iota) = 0$.

We can obtain a chiral CDG-algebra from a twisted pair.

Proposition 2.25. *Let $(\mathcal{B}, \mathcal{B}^\circ, \mathbf{S})$ be a twisted pair. Define*

$$d := (h \boxtimes \mathrm{id})\mu(\mathbf{S} \boxtimes -) : \mathcal{B} \to \mathcal{B},$$

$$\iota := (h \boxtimes \mathrm{id})\mu(\mathbf{S} \boxtimes \mathbf{S}) \in \Gamma(X, \mathcal{B}),$$

then $(\mathcal{B}, d, \iota)$ is a CDG chiral algebra.

Proof. It follows directly from the definition. □

Remark 2.26. In the case of associated algebra, Positselski [Pos93] defines the dual of a QLS algebra to be a CDG algebra constructed from the QLS data. However, in the context of chiral algebras, passing from the twisted pair to the CDG-algebra loses information. Also, the twisted pair is more suitable to construct the curved chiral chain complex which serves as the chiral analogue of the curved Hochschild chain complex in [Pos93].

2.5 MAURER-CARTAN EQUATION AND QUADRATIC DUALITY

In this section, we study the relationship between chiral quadratic duality and the Maurer-Cartan equations. In the associative algebra case, we have shown that if an algebra A is Koszul, then the space $\mathrm{MC}(A \otimes B) := \{\alpha \in A \otimes B | [\alpha, \alpha] = 0, |\alpha| = 1\}$ of solutions of the Maurer-Cartan equation has a one-to-one correspondence with the space $\mathrm{Hom}(A^!, B)$ of algebra homomorphisms. We study similar correspondence for chiral algebras. However, it not clear to us how to define the Koszulness for chiral algebras at this stage. Nevertheless, we establish the chiral analogue of this connection for some special cases.

We first introduce the Maurer-Cartan equation for chiral algebras.

Definition 2.27. Let $\mathcal{A}$ be a graded chiral algebra. The Maurer-Cartan equation is defined to be

$$\mu(\alpha \boxtimes \alpha) = 0, \quad \alpha \in \Gamma(X, \mathcal{A}),\ |\alpha| = 1.$$

The set of the solutions is denoted by $\mathrm{MC}(\mathcal{A})$.

Remark 2.28. Sometimes, one encounters a weaker form of the Maurer-Cartan equation. It has the form of $h(\mu(\alpha \boxtimes \alpha)) = 0$, where $h(-) = - \otimes_{\mathcal{D}_X} \mathcal{O}_X$ is the de Rham sheaf. For example, in [Li23] the author established an correspondence between renormalized quantum master equations and this form of Maurer Cartan equations of vertex algebras. In the language of vertex algebras (suppose that $X = \mathbb{C}$), a constant section vdz satisfies the equation in Definition 2.27 is equivalent to $v_{(n)}v = 0$ for $n \geq 0$. While the latter equation is equivalent to $v_{(0)}v = 0$.

We recall the definition of tensor products of chiral algebras. Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are chiral algebras. We denote the corresponding factorization algebras by $\mathcal{F}(\mathcal{A}_i), i = 1, 2$. Then

$$\mathcal{F}_{X^I} := \mathcal{F}(\mathcal{A}_1)_{X^I} \otimes_{\mathcal{O}_{X^I}} \mathcal{F}(\mathcal{A}_2)_{X^I}$$

is also a factorization algebra. The tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ is defined to be the chiral algebra that corresponds to $\mathcal{F}$.

Remark 2.29. Suppose X is the complex plane and $\mathcal{A}_i = X \times (V_i)_\omega, i = 1, 2$, where V_i are vertex algebras. Then the above tensor product is the same as the usual vertex algebra tensor product.

In the context of quadratic associative algebras, the tensor product of a quadratic algebra and its dual contains a canonical element that satisfies the usual Maurer-Cartan equation. Here we have the chiral algebra version of this.

Proposition 2.30. *If we take $\mathcal{A}$ to be $\mathcal{A}(N, P)$ and $\mathcal{A}^!$ to be $\mathcal{A}(s^{-1}N^\vee_{\omega^{-1}}, P^\perp)$ then the canonical element $\phi(\underline{s^{-1}\mathbf{Id}}) \in \Gamma(X, \phi(s^{-1}N^\vee \otimes_{\mathcal{O}_X} N)) \subset \Gamma(X, \mathcal{A}^! \otimes \mathcal{A})$ is a solution to the Maurer-Cartan equation. Here $\phi : s^{-1}N^\vee \otimes_{\mathcal{O}_X} N \to \mathcal{A}^! \otimes \mathcal{A}$ is the natural map.*

Proof. Suppose that $\mathrm{rank}(N) = r$. To simplify the notation, we omit the symbol ϕ and pretend that $s^{-1}N^\vee \otimes N$ is a submodule of $\mathcal{A}^! \otimes \mathcal{A}$. We can cover $X \times X$ by open subsets, such that we can find a collection of sections

$$\{P_\alpha\}_{\alpha=1,\dots,r^2}, P_\alpha \in P|_V,$$

and

$$\{P_\alpha^\vee\}_{\alpha=1,\dots,r^2}, P_\alpha^\vee \in P^\perp \otimes_{\mathcal{O}_{X^2}} \omega_{X^2}|_V,$$

such that

$$\underline{s^{-1}\mathbf{Id}} \boxtimes \underline{s^{-1}\mathbf{Id}}|_V = \sum_{\alpha=1}^{r^2} P_\alpha^\vee \otimes P_\alpha \tag{2.28}$$

for each open subset V that belongs to the covering. By the definition of the tensor product of chiral algebras, we have

$$P^\perp \otimes_{\mathcal{O}_{X^2}} P \otimes_{\mathcal{O}_{X^2}} \omega_{X^2} \subset \ker \mu_{\mathcal{A}^! \otimes \mathcal{A}}.$$

This implies that $\mu(s^{-1}\mathbf{Id} \boxtimes s^{-1}\mathbf{Id})|_V = 0$ for every V. Therefore $\mu(s^{-1}\mathbf{Id} \boxtimes s^{-1}\mathbf{Id}) = 0$.

□

Parallel to the quadratic associative algebra case, we can characterize morphisms from a quadratic chiral algebra $\mathcal{A} = \mathcal{A}(N, P)$ to an arbitrary graded chiral algebra $\mathcal{B}$ as solutions of the Maurer-Cartan equations for $\mathcal{A}^! \otimes \mathcal{B}$, i.e., the tensor product of the chiral quadratic dual and the target chiral algebra.

Theorem 2.31. *Let $\mathcal{B}$ be a graded chiral algebra. There exists an injective map*

$$\mathrm{Hom}(\mathcal{A}(N, P), \mathcal{B}) \hookrightarrow \mathrm{MC}(\mathcal{A}(s^{-1}N^\vee_{\omega^{-1}}, P^\perp) \otimes \mathcal{B}).$$

Proof. Suppose that we have a morphism $\varphi : \mathcal{A}(N, P) \to \mathcal{B}$. We claim that the element

$$(\mathrm{id} \otimes \varphi)(\underline{s^{-1}\mathbf{Id}}) \in \Gamma(X, \mathcal{A}(s^{-1}N^\vee_{\omega^{-1}}, P^\perp) \otimes \mathcal{B})$$

is a solution of the Maurer-Cartan equation. This claim follows from Proposition 2.30 and the fact that $\mathrm{id} \otimes \varphi : \mathcal{A}^! \otimes \mathcal{A} \to \mathcal{A}^! \otimes \mathcal{B}$ is a morphism of chiral algebras. The injectivity follows from the construction – $(\mathrm{id} \otimes \varphi)(\underline{s^{-1}\mathbf{Id}})$ is a zero section if and only if $\varphi = 0$. □

We can show that the above injective map is bijective if we put more conditions. We introduce the notion of effective chiral quadratic datum.

Definition 2.32. A chiral quadratic datum (N, P) is called *effective* if the natural map $\phi : N_\omega \to \mathcal{A}(N, P)$ is injective and (for simplicity of notation, we will omit the symbol ϕ)

$$P \otimes_{\mathcal{O}_X} \omega_{X^2} = j_*j^*N_\omega \boxtimes N_\omega \cap \ker \mu_{\mathcal{A}(N,P)}.$$

Remark 2.33. It is easy to find effective chiral quadratic datum. We can start from an arbitrary chiral quadratic datum (N, P). If $P' \otimes_{\mathcal{O}_X} \omega_{X^2} = j_*j^*N_\omega \boxtimes N_\omega \cap \ker \mu_{\mathcal{A}(N,P)}$ is locally free, then we can take (N, P') to be our new chiral quadratic datum. From the construction in [BD04, 3.4.14,pp184], we have $\mathcal{A}(N,P) = \mathcal{A}(N,P')$ and (N,P') is effective.

Theorem 2.34. *Let $\mathcal{B}$ be a graded chiral algebra which concentrated in degree 0. Assume that N is in degree 0 and $(s^{-1}N^{\vee}_{\omega^{-1}}, P^{\perp})$ is effective, then there exists a bijection*

$$\mathrm{Hom}(\mathcal{A}(N,P),\mathcal{B}) \cong \mathrm{MC}(\mathcal{A}(s^{-1}N^{\vee}_{\omega^{-1}}, P^{\perp}) \otimes \mathcal{B}).$$

Proof. We omit the symbol ϕ as before. We use the notation $\mathcal{A} = \mathcal{A}(N,P)$, $\mathcal{A}^! = \mathcal{A}(s^{-1}N^{\vee}_{\omega^{-1}}, P^{\perp})$. Suppose that we have $\alpha \in \mathcal{A}^! \otimes \mathcal{B}, |\alpha| = 1$ satisfies the Maurer-Cartan equation. Since we assume that both $\mathcal{B}$ and N are in degree 0, we have

$$\alpha \in s^{-1}N^{\vee}_{\omega^{-1}} \otimes_{\mathcal{O}_X} \mathcal{B} \subset \mathcal{A}^! \otimes \mathcal{B}.$$

Then α defines a morphism of $\mathcal{O}_X$ modules

$$\phi_\alpha : N_\omega \to \mathcal{B},$$

$$\phi_\alpha(-) = \langle s\alpha, - \rangle.$$

Note that we have

$$(\mathrm{id} \otimes \phi_\alpha)(\underline{s^{-1}\mathbf{Id}}) = \alpha.$$

We can cover X^2 by open subsets $\cup V_i$. We can find $\{P^i_\alpha\}, \{P^{i\,\vee}_\alpha\}$such that the equation 2.28 holds on V_i. Now take $V = V_i$, we have

$$\begin{aligned} 0 = \mu(\alpha \boxtimes \alpha)|_{\mathbf{V}} &= \mu((\mathrm{id} \boxtimes \mathrm{id}) \otimes (\phi_\alpha \boxtimes \phi_\alpha)(\underline{s^{-1}\mathbf{Id}} \boxtimes \underline{s^{-1}\mathbf{Id}}))|_V \\ &= \mu((\mathrm{id} \boxtimes \mathrm{id}) \otimes (\phi_\alpha \boxtimes \phi_\alpha)(\sum_{\alpha \in S} P^{\vee}_\alpha \otimes P_\alpha))|_V \\ &= \mu(\sum_{\alpha \in S} P^{\vee}_\alpha \otimes (\phi_\alpha \boxtimes \phi_\alpha)(P_\alpha))|_V. \end{aligned}$$

We have

$$(\sum_{\alpha \in S} P^{\vee}_\alpha \otimes (\phi_\alpha \boxtimes \phi_\alpha)(P_\alpha))|_V = \sum_{\alpha \in S} P^{\vee}_\alpha \otimes Q_\alpha, \quad Q_\alpha \in \ker(\mu_{\mathcal{B}})|_V$$

since we assume that $(s^{-1}N^{\vee}_{\omega^{-1}}, P^{\perp})$ is effective. This implies that

$$\mu_{\mathcal{B}}((\phi_\alpha \boxtimes \phi_\alpha)(P_\alpha))|_V = \mu_{\mathcal{B}}(Q_\alpha) = 0, \quad \alpha \in S.$$

□

We can generalize the notion of Maurer-Cartan equation to twisted pairs.

Definition 2.35. Let $\mathcal{A}$ be a graded chiral algebra and $(\mathcal{B}, \mathcal{B}^\circ, \mathbf{S})$ be a twisted pair. The Maurer-Cartan equation is defined to be

$$\mu((\mathbf{S}+\alpha)\boxtimes(\mathbf{S}+\alpha)) = 0, \quad \alpha \in \Gamma(X, \mathcal{A}\otimes\mathcal{B}),\ |\alpha| = 1.$$

The set of the solutions is denoted by $\mathrm{MC}((\mathcal{B}, \mathcal{B}^\circ, \mathbf{S})\otimes\mathcal{A})$

Proposition 2.36. *If we take $\mathcal{A}$ to be $\frac{\mathcal{A}(N\oplus\mathbf{1}^\circ, P^\circ)}{\langle \mathbf{1}^\circ_\omega - 1_\omega\rangle}$ and $(\mathcal{B}, \mathcal{B}^\circ, \mathbf{S})$ to be*

$$(\mathcal{A}(s^{-1}N^\vee_{\omega^{-1}}, (\mathbf{q}P^\circ)^\perp), \mathcal{A}(s^{-1}N^\vee_{\omega^{-1}}\oplus s^{-1}\mathbf{1}^\circ_{\omega^{-1}}, P^{\circ\perp}), \underline{s^{-1}\mathbf{1}^\circ}) \tag{2.29}$$

the canonical element $\underline{s^{-1}\mathbf{Id}} \in \Gamma(X, s^{-1}N^\vee \otimes_{\mathcal{O}_X} N) \subset \Gamma(X, \mathcal{B}\otimes\mathcal{A})$ is a solution to the Maurer-Cartan equation.

Proof. The identity element $\underline{s^{-1}\mathbf{Id}}^\circ \in \Gamma(X, (s^{-1}N^\vee_{\omega^{-1}}\oplus s^{-1}\mathbf{1}^\circ_{\omega^{-1}})\otimes_{\mathcal{O}_X}(N\oplus\mathbf{1}^\circ)\otimes_{\mathcal{O}_X}\omega_X)$ satisfies the usual Maurer-Cartan equation in $\mathcal{B}^\circ\otimes\mathcal{A}(N\oplus\mathbf{1}^\circ, P^\circ)$

$$\mu(\underline{s^{-1}\mathbf{Id}}^\circ \boxtimes \underline{s^{-1}\mathbf{Id}}^\circ) = 0.$$

Note that $\underline{s^{-1}\mathbf{Id}}^\circ = \underline{s^{-1}\mathbf{Id}} + \mathbf{S} \in \Gamma(X, \mathcal{B}^\circ\otimes\mathcal{A})$, the proposition follows. □

Theorem 2.37. *Let $\mathcal{C}$ be a graded chiral algebra and $(\mathcal{B}, \mathcal{B}^\circ, \mathbf{S})$ be the twisted pair (2.29). Then there is a injection*

$$\mathrm{Hom}(\frac{\mathcal{A}(N\oplus\mathbf{1}^\circ, P^\circ)}{\langle \mathbf{1}^\circ_\omega - 1_\omega\rangle}, \mathcal{C}) \hookrightarrow \mathrm{MC}((\mathcal{B}, \mathcal{B}^\circ, \mathbf{S})\otimes\mathcal{C}).$$

Proof. Suppose there is a morphism of chiral algebras

$$\phi : \frac{\mathcal{A}(N\oplus\mathbf{1}^\circ, P^\circ)}{\langle \mathbf{1}^\circ_\omega - 1_\omega\rangle} \to \mathcal{C}.$$

Note that ϕ is induced by the following morphism

$$\tilde{\phi} : \mathcal{A}(N\oplus\mathbf{1}^\circ, P^\circ) \to \mathcal{C}$$

such that $\tilde{\phi}|_{N_\omega} = \phi|_{N_\omega}$ and $\tilde{\phi}|_{\mathbf{1}^\circ_\omega} = 1_\omega = \omega_X$. Then

$$\alpha = \tilde{\phi}(\underline{s^{-1}\mathbf{Id}}^\circ) - \mathbf{S} = \phi(\underline{s^{-1}\mathbf{Id}})$$

is the solution of the Maurer-Cartan equation.

□

Similarly, we have the following theorem.

Theorem 2.38. *Let $\mathcal{C}$ be a graded chiral algebra concentrated in degree 0 and $(\mathcal{B}, \mathcal{B}^\circ, \mathbf{S})$ be the twisted pair (2.29). Assume that N is degree 0 and $(s^{-1}N^\vee_{\omega^{-1}} \oplus s^{-1}\mathbf{1}^\circ_{\omega^{-1}}, P^{\circ\perp})$ is effective. Then there is a bijection*

$$\mathrm{Hom}(\frac{\mathcal{A}(N \oplus \mathbf{1}^\circ, P^\circ)}{\langle \mathbf{1}^\circ_\omega - 1_\omega \rangle}, \mathcal{C}) \cong \mathrm{MC}((\mathcal{B}, \mathcal{B}^\circ, \mathbf{S}) \otimes \mathcal{C}).$$

Proof. Suppose we have a solution α of the Maurer-Cartan equation. We can define a map $\tilde{\phi}_\alpha : N_\omega \oplus \mathbf{1}^\circ_\omega \to \mathcal{C}$ such that

$$\alpha = (\mathrm{id} \otimes \tilde{\phi}_\alpha|_{N_\omega})(\underline{s^{-1}\mathbf{Id}}),$$

and $\tilde{\phi}_\alpha|_{\mathbf{1}^\circ_\omega} : \mathbf{1}^\circ_\omega \to \mathcal{C}$ is equal to the unit map $\omega_X \to \mathcal{C}$. Then repeat the proof in Theorem 2.31, we have a morphism of chiral algebras

$$\tilde{\phi}_\alpha : \mathcal{A}(N \oplus \mathbf{1}^\circ, P^\circ) \to \mathcal{C},$$

and it factors through the ideal $\langle \mathbf{1}^\circ_\omega - 1_\omega \rangle$ by construction. Thus, we have a morphism

$$\phi_\alpha : \frac{\mathcal{A}(N \oplus \mathbf{1}^\circ, P^\circ)}{\langle \mathbf{1}^\circ_\omega - 1_\omega \rangle} \to \mathcal{C}.$$

The proof is complete. □

2.6 EXAMPLES

There are some classical examples of Koszul duality for associative algebra. The most famous example of Koszul dual algebras are the symmetric algebra $S(V)$ and the exterior algebra $\wedge V^\vee$. In the non-homogeneous case, we have the Koszul duality between the universal enveloping algebra $U(\mathfrak{g})$ and the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$. In this section we discuss examples of quadratic duality for chiral algebra that parallel the cases of associative algebra.

Commutative chiral algebra

First, we consider the simplest quadratic datum $(N, P = N \boxtimes N)$, with N locally free of finite rank. We have $\mathcal{A}(N, P) = \mathrm{Sym}(N_\mathcal{D})$, which is the commutative chiral algebra generated by $N_\mathcal{D} := N \otimes_{\mathcal{O}_X} \mathcal{D}_X$.

The dual quadratic datum is given by $(s^{-1}N^\vee_{\omega^{-1}}, P^\perp = s^{-1}N^\vee_{\omega^{-1}} \boxtimes s^{-1}N^\vee_{\omega^{-1}})$. It automatically satisfies $P^\perp|_U = s^{-1}N^\vee_{\omega^{-1}} \boxtimes s^{-1}N^\vee_{\omega^{-1}}|_U$, so this quadratic datum is dualizable. We have $\mathcal{A}(s^{-1}N^\vee_{\omega^{-1}}, P^\perp) = \mathrm{Sym}((s^{-1}N^\vee_{\omega^{-1}})_\mathcal{D})$, which is the graded commutative chiral algebra generated by $(s^{-1}N^\vee_{\omega^{-1}})_\mathcal{D}$.

Another pure quadratic example

Let N be the free $\mathcal{O}_X$-module $N = \bigoplus_{i=1}^4 \mathcal{O}_X$. We denote the corresponding basis by $\{\phi_i\}_{i=1,\dots 4}$. We define P to be the $\mathcal{O}_{X^2}$ module with basis

$$\phi_i \boxtimes \phi_j, \ \{i,j\} \neq \{1,2\},$$
$$\phi_1 \boxtimes \phi_2 - \frac{1}{z_1 - z_2}\phi_3 \boxtimes \phi_4,$$
$$\phi_2 \boxtimes \phi_1 + \frac{1}{z_1 - z_2}\phi_4 \boxtimes \phi_3.$$

For the dual datum, we have $s^{-1}N^\vee_{\omega^{-1}} = \bigoplus_{i=1}^4 s^{-1}\omega_X^{-1}$. We denote the corresponding basis by $\{\psi_i = s^{-1}\phi_i^\vee\}_{i=1,\dots 4}$. Then $P^\perp$ has the following basis

$$\psi_i \boxtimes \psi_j, \ \{i,j\} \neq \{3,4\},$$
$$\psi_3 \boxtimes \psi_4 + \frac{1}{z_1 - z_2}\psi_1 \boxtimes \psi_2,$$
$$\psi_4 \boxtimes \psi_3 - \frac{1}{z_1 - z_2}\psi_2 \boxtimes \psi_1.$$

$P^\perp$ defined above satisfies $P^\perp|_U = s^{-1}N^\vee_{\omega^{-1}} \boxtimes s^{-1}N^\vee_{\omega^{-1}}|_U$, so this quadratic datum is dualizable.

Affine Kac-Moody chiral algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra with an invariant pairing κ. We take a basis $\{x_a\}_{1\le a\le n}$ of $\mathfrak{g}$. Let $N = \mathfrak{g} \otimes \omega_X^{-1}$. We let $P^\circ \subset j_*j^*(N \oplus \mathbf{1}^\circ) \boxtimes (N \oplus \mathbf{1}^\circ)$ to be the $\mathcal{O}_{X^2}$-module defined by the following basis

$$\mathbf{1}^\circ \boxtimes \mathbf{1}^\circ,$$
$$\mathbf{1}^\circ \boxtimes x_a, \quad x_a \boxtimes \mathbf{1}^\circ, \quad 1 \le a \le n,$$
$$x_a \boxtimes y_b - \frac{1}{2}\sum_{c=1}^n (\frac{f_{ab}^c}{z_1 - z_2})\,(\mathbf{1}^\circ \boxtimes x_c + x_c \boxtimes \mathbf{1}^\circ) - \frac{\kappa_{ab}}{(z_1-z_2)^2}\mathbf{1}^\circ \boxtimes \mathbf{1}^\circ, ; 1 \le a, b \le n, \tag{2.30}$$

where $\kappa_{ab} = \kappa(x_a, x_b)$.

As a more familiar construction, we consider the affine Kac-Moody Lie* algebra $\mathfrak{g}^\kappa_\mathcal{D} = \mathfrak{g}_\mathcal{D} \oplus \omega_X$. It gives rise to the twisted chiral enveloping algebra $U(\mathfrak{g}_\mathcal{D})^\kappa$ [BD04, Section 3.7.25, pp227].

Proposition 2.39. *We have an isomorphism of chiral algebras*

$$\frac{\mathcal{A}(N \oplus \mathbf{1}^\circ, P^\circ)}{\langle \mathbf{1}^\circ_\omega - 1_\omega \rangle} = U(\mathfrak{g}_\mathcal{D})^\kappa.$$

Proof. On the one hand, we have a map $N_\omega \oplus \mathbf{1}^\circ_\omega \to \mathfrak{g}^\kappa_\mathcal{D} \to U(\mathfrak{g}_\mathcal{D})^\kappa$. By the universal property, we get a map of chiral algebra $\mathcal{A}(N \oplus \mathbf{1}^\circ, P) \to U(\mathfrak{g}_\mathcal{D})^\kappa$. By construction, $\mathbf{1}^\circ_\omega$ is mapped to the unit of $U(\mathfrak{g}_\mathcal{D})^\kappa$. Therefore we have a map of chiral algebra $\frac{\mathcal{A}(N\oplus\mathbf{1}^\circ,P^\circ)}{\langle\mathbf{1}^\circ_\omega-1_\omega\rangle} \to U(\mathfrak{g}_\mathcal{D})^\kappa$.

On the other hand, we consider the map $N_\omega \oplus \mathbf{1}^\circ_\omega \to \frac{\mathcal{A}(N\oplus\mathbf{1}^\circ,P^\circ)}{\langle\mathbf{1}^\circ_\omega-1_\omega\rangle}$, which extends to a $\mathcal{D}_X$-module map $\mathfrak{g}^\kappa_\mathcal{D} \to \frac{\mathcal{A}(N\oplus\mathbf{1}^\circ,P^\circ)}{\langle\mathbf{1}^\circ_\omega-1_\omega\rangle}$. Using the relation 2.30, we find that the image of this map has the same Lie* bracket as $\mathfrak{g}^\kappa_\mathcal{D}$. Therefore we get a map of Lie* algebra $\mathfrak{g}^\kappa_\mathcal{D} \to \frac{\mathcal{A}(N\oplus\mathbf{1}^\circ,P^\circ)}{\langle\mathbf{1}^\circ_\omega-1_\omega\rangle}$. By the universal property of (twisted) chiral envelope, we have a map of chiral algebra $U(\mathfrak{g}_\mathcal{D})^\kappa \to \frac{\mathcal{A}(N\oplus\mathbf{1}^\circ,P^\circ)}{\langle\mathbf{1}^\circ_\omega-1_\omega\rangle}$.

The composition $\mathfrak{g}^\kappa_\mathcal{D} \to \frac{\mathcal{A}(N\oplus\mathbf{1}^\circ,P^\circ)}{\langle\mathbf{1}^\circ_\omega-1_\omega\rangle} \to U(\mathfrak{g}_\mathcal{D})^\kappa$ is the canonical map $\mathfrak{g}^\kappa_\mathcal{D} \to U(\mathfrak{g}_\mathcal{D})^\kappa$. Therefore the composition $U(\mathfrak{g}_\mathcal{D})^\kappa \to \frac{\mathcal{A}(N\oplus\mathbf{1}^\circ,P^\circ)}{\langle\mathbf{1}^\circ_\omega-1_\omega\rangle} \to U(\mathfrak{g}_\mathcal{D})^\kappa$ is the identity. Similarly the composition $\frac{\mathcal{A}(N\oplus\mathbf{1}^\circ,P^\circ)}{\langle\mathbf{1}^\circ_\omega-1_\omega\rangle} \to U(\mathfrak{g}_\mathcal{D})^\kappa \to \frac{\mathcal{A}(N\oplus\mathbf{1}^\circ,P^\circ)}{\langle\mathbf{1}^\circ_\omega-1_\omega\rangle}$ also gives the identity. □

Now we analyze the quadratic dual datum. We find that $P^{\circ\perp}$ is given by the following basis

$$\begin{aligned}
&s^{-1}\mathbf{1}^\circ_{\omega^{-1}} \boxtimes s^{-1}\mathbf{1}^\circ_{\omega^{-1}} + \sum_{1\le a,b\le n} \frac{\kappa_{ab}}{(z_1-z_2)^2} s^{-1}x_a^\vee \boxtimes s^{-1}x_b^\vee,\\
&s^{-1}\mathbf{1}^\circ_{\omega^{-1}} \boxtimes s^{-1}x_c^\vee + \frac{1}{2}\sum_{1\le a,b\le n} \frac{f^c_{ab}}{z_1-z_2} s^{-1}x_a^\vee \boxtimes s^{-1}x_b^\vee, \quad 1\le c\le n,\\
&s^{-1}x_c^\vee \boxtimes s^{-1}\mathbf{1}^\circ_{\omega^{-1}} + \frac{1}{2}\sum_{1\le a,b\le n} \frac{f^c_{ab}}{z_1-z_2} s^{-1}x_a^\vee \boxtimes s^{-1}x_b^\vee, \quad 1\le c\le n,\\
&s^{-1}x_a^\vee \boxtimes s^{-1}x_b^\vee, \quad 1\le a,b\le n.
\end{aligned}$$

We see that $(N \oplus \mathbf{1}^\circ, P^\circ)$ is dualizable as qudratic datum. The quadratic projection $(\mathbf{q}P^\circ)^\perp$ is given by the following basis

$$s^{-1}x_a^\vee \boxtimes s^{-1}x_b^\vee, \quad 1\le a,b\le n.$$

Therefore, the chiral algebra $\mathcal{B} = \mathcal{A}(s^{-1}N^\vee_{\omega^{-1}}, (\mathbf{q}P^\circ)^\perp) = \mathrm{Sym}((s^{-1}N^\vee_{\omega^{-1}})_\mathcal{D})$ is the graded commutative chiral algebra generated by $s^{-1}N^\vee_{\omega^{-1}}$.

We denote $\mathcal{B}^\circ = \mathcal{A}(s^{-1}N^\vee_{\omega^{-1}} \oplus s^{-1}\mathbf{1}^\circ_{\omega^{-1}}, P^{\circ\perp})$. To prove that $(\mathcal{B}, \mathcal{B}^\circ, \underline{s^{-1}\mathbf{1}^\circ})$ is indeed a twisted pair, we analyze the differential and the curving element.

Proposition 2.40. *The differential defined by* $d = (h \boxtimes \mathrm{id})\mu(\underline{s^{-1}\mathbf{1}^\circ} \boxtimes -)$ *preserves* $\mathcal{B}$*. Moreover, the DG chiral algebra* $(\mathcal{B}, d)$ *is isomorphic to the Chevalley DG algebra* $(\mathcal{C}(\mathfrak{g}_\mathcal{D}), d_{\mathrm{CE}})$ *for the Lie* algebra* $\mathfrak{g}_\mathcal{D}$ *(see [BD04, Section 4.7, pp348] for details, where they use the name "de Rham-Chevalley algebra" as the construction is for general Lie* algebroids).*

Proof. $\mathcal{B}$ is a commutative chiral algebra, which coincides with $\mathcal{C}(\mathfrak{g}_\mathcal{D})$ as plain graded chiral algebra. The corresponding left $\mathcal{D}$-module $\mathcal{B}^l$ is a commutative $\mathcal{D}_X$-algebra.

We denote the image of $s^{-1}x_c^\vee$ under $s^{-1}N_{\omega^{-1}}^\vee \to \mathcal{B}^l$ by the same symbol $s^{-1}x_c^\vee$. Using the dual relation we can compute d restricted to $s^{-1}N_{\omega^{-1}}^\vee \otimes \omega_X$ as follows

$$\begin{aligned} d(s^{-1}x_c^\vee dz) &= (h \boxtimes \mathrm{id})\mu(s^{-1}\mathbf{1}^\circ \boxtimes s^{-1}x_c^\vee dz_2) \\ &= \frac{1}{2}\sum_{1\le a,b\le n}(h\boxtimes \mathrm{id})\mu(\frac{f_{ab}^c}{z_1 - z_2}s^{-1}x_a^\vee dz_1 \boxtimes s^{-1}x_b^\vee dz_2). \end{aligned}$$

The chiral map μ restricted to $\mathcal{B}$ is given by the commutative product on $\mathcal{B}^l$. We can simplify the above map as follows

$$d(s^{-1}x_c^\vee dz) = \frac{1}{2}\sum_{1\le a,b\le n} f_{ab}^c(s^{-1}x_a^\vee \cdot s^{-1}x_b^\vee)dz.$$

Since d is a $\mathcal{D}$-module map, the above result extend to a map $d : (s^{-1}N_{\omega^{-1}}^\vee)_\mathcal{D} \to \mathcal{B}$. We see that d restricted to $(s^{-1}N_{\omega^{-1}}^\vee)_\mathcal{D}$ is given by the composition $(s^{-1}N_{\omega^{-1}}^\vee)_\mathcal{D} \overset{[-,-]^*}{\to} (s^{-1}N_{\omega^{-1}}^\vee)_\mathcal{D} \otimes (s^{-1}N_{\omega^{-1}}^\vee)_\mathcal{D} \to \mathrm{Sym}^2((s^{-1}N_{\omega^{-1}}^\vee)_\mathcal{D})$, which coincide with d_{CE}.

The Jacobi identity of chiral map implies that d satisfies Leibniz rule with respect to the chiral map. We thus complete the proof.

□

The final ingredient is the curving. Using the dual relation we find that it is given by

$$\begin{aligned} \iota &= (h \boxtimes \mathrm{id})\mu(s^{-1}\mathbf{1} \boxtimes s^{-1}\mathbf{1}) \\ &= -\sum_{1\le a,b\le n}(h\boxtimes \mathrm{id})\mu(\frac{\kappa_{ab}}{(z_1 - z_2)^2}s^{-1}x_a^\vee dz_1 \boxtimes s^{-1}x_b^\vee dz_2). \end{aligned}$$

We see that $(h \boxtimes \mathrm{id})\mu(\frac{\kappa_{ab}}{(z_1-z_2)^2}s^{-1}x_a^\vee dz_1 \boxtimes s^{-1}x_b^\vee dz_2)$ is indeed an element of $\mathcal{B}$. Therefore the triple $(\mathcal{B}, \mathcal{B}^\circ, \underline{s^{-1}\mathbf{1}^\circ})$ is a twisted pair and serves as the quadratic dual of $U(\mathfrak{g}_\mathcal{D})^\kappa$.

From the vertex algebra point of view, the vertex algebra corresponding to the twisted chiral envelope $U(\mathfrak{g}_\mathcal{D})^\kappa$ is the affine Kac-Moody VOA $V_\kappa(\mathfrak{g})$.

The quadratic dual vertex algebra can be identified with the graded commutative vertex algebra $V^{CE}(\mathfrak{g}) := CE(L\mathfrak{g})$ equipped with the Chevalley-Eilenberg differential and a curving. Explicitly, we denote

$$\{J_a(z) = \sum_{n\in\mathbf{Z}} J_{a,(n)} z^{-n-1}\}_{1\le a\le n}$$

the set of generating fields of $V_\kappa(\mathfrak{g})$. The quadratic dual vertex algebra $V^{CE}(\mathfrak{g})$ is generated by fields $\{c^a(z) = \sum_{n\in\mathbf{Z}} c^a_{(n)} z^{-n-1}\}_{1\le a\le n}$. The differential can be expressed as follows

$$d(\partial^m c^a) = -\frac{1}{2}\sum_{1\le b,c\le n}\sum_{r+s=m} f^a_{bc}\binom{m}{r}(\partial^r c^b)(\partial^s c^c),$$

where we define $\partial^m c^a = \partial^m c^a(0)|0\rangle$. Using VOA axiom, $\partial^m c^a$ can also be identified with $T^m c^a$. The curving element can be identified with

$$\iota = -\sum_{1\le a,b\le n} \kappa_{ab}(\partial c^a)c^b.$$

The canonical element $s^{-1}\mathbf{Id} \in \Gamma(X, s^{-1}N \otimes_{\mathcal{O}_X} N)$ corresponds to the following element in the vertex algebra $V_\kappa(\mathfrak{g}) \otimes V^{CE}(\mathfrak{g})$

$$\mathbf{I} := \sum_{a=1}^{n} J_a \otimes c^a.$$

We can verify the corresponding Maurer-Cartan equation using vertex algebra operation. Note that $\mathbf{I}_{(0)} = \sum_{1\le a\le n}\sum_{l+m=-1} J_{a,(l)} \otimes c^a_{(m)}$. We find the following

$$\mathbf{I}_{(0)}\mathbf{I} = \sum_{1\le a,b,c\le n} f^c_{ab} J_c \otimes c^a c^b + \sum_{1\le a,b\le n} |0\rangle \otimes \kappa_{ab}(\partial c^a)c^b.$$

We also have

$$d\mathbf{I} = -\frac{1}{2}\sum_{1\le a,b,c\le n} f^a_{bc} J_a \otimes c^b c^c.$$

Therefore, the following Maurer-Cartan equation is satisfied

$$d\mathbf{I} + \frac{1}{2}\mathbf{I}_{(0)}\mathbf{I} + \frac{1}{2}\iota = 0. \tag{2.31}$$

We can use $\mathbf{I}_{(m)} = \sum_{1\le a\le n} \sum_{l+k=m-1} J_{a,(l)} \otimes c^a_{(k)}$ to check that the stronger form of Maurer-Cartan equation (see the Remark 2.28) is also satisfied

$$\mathbf{I}_{(m)}\mathbf{I} = 0, \text{ for } m \ge 1. \tag{2.32}$$

As a consequence, for any vertex algebra V and a homomorphism $\varphi : V_\kappa(\mathfrak{g}) \to V$, $(\varphi \otimes \mathrm{id})(\mathbf{I})$ satisfies the Maurer-Cartan equation. On the other hand, for any vertex algebra V concentrated in degree 0, a degree 1 element of $V^{CE}(\mathfrak{g}) \otimes V$ takes the following form

$$\alpha = \sum_{a=1}^{n} c^a \otimes y_a, \; y_a \in V.$$

For the (strong form of) Maurer-Cartan equation 2.31,2.32 to hold for α, we must have

$$\begin{aligned} y_{a,(0)}y_b &= \sum_{c=0}^{n} f^c_{ab} y_c, \\ y_{a,(1)}y_b &= \kappa_{ab}|0\rangle, \\ y_{a,(m)}y_b &= 0, \text{ for } m \ge 2. \end{aligned}$$

Using Borcherds identities [FBZ04], we find

$$[y_{a,(l)}, y_{b,(m)}] = \sum_{c=1}^{n} f^c_{ab} y_{a,(l+m)} + \kappa_{ab}\delta_{n,-m}.$$

This implies that the following map

$$J_a \to y_a, \quad 1 \le a \le n,$$

defined a homomorphism of vertex algebra $V_\kappa(\mathfrak{g}) \to V$.

$\beta\gamma$ system

Let $\mathbf{L} = \bigoplus_{\alpha\in\mathbb{Q}} \mathbf{L}^\alpha$ be a finite dimensional $\mathbb{Q}$(conformal weight)-graded superspace. Suppose that $\mathbf{L}$ is equipped with an even symplectic pairing of conformal weight -1

$$\langle -, - \rangle : \mathbf{L}^\alpha \otimes \mathbf{L}^{1-\alpha} \to \mathbb{C}.$$

We define $N = \bigoplus_{\alpha \in \mathbb{Q}} \mathbf{L}^{\alpha} \otimes \omega_X^{1-\alpha}$. Let $\{x_a\}_{1 \leq a \leq n}$ be a basis of $\mathbf{L}$. We consider $P^\circ \subset j_* j^*(N \oplus \mathbf{1}^\circ) \boxtimes (N \oplus \mathbf{1}^\circ)$ defined by the following basis

$$\begin{aligned}
&\mathbf{1}^\circ \boxtimes \mathbf{1}^\circ, \\
&\mathbf{1}^\circ \boxtimes x_a, \quad x_a \boxtimes \mathbf{1}^\circ, \quad 1 \leq a \leq n, \\
&x_a \boxtimes y_b - \frac{\Omega_{ab}}{z_1 - z_2} \mathbf{1}^\circ \boxtimes \mathbf{1}^\circ, \quad 1 \leq a, b \leq n.
\end{aligned}$$

where $\Omega_{ab} = \langle x_a, x_b \rangle$.

Proposition 2.41. *The chiral algebra* $\frac{\mathcal{A}(N \oplus \mathbf{1}^\circ, P)}{\langle \mathbf{1}^\circ_\omega - 1_\omega \rangle}$ *defined as above is isomorphic to the chiral Weyl algebra* $\mathcal{W}(\mathbf{L}, \langle -, - \rangle)$ *defined in [BD04, Section 3.8.1, pp228]. The corresponding vertex algebra is the* $\beta\gamma - bc$ *system.*

Proof. This is a corollary of Proposition 2.39 □

We assume that the symplectic pairing is non-degenerate. Then the dual relation $P^\perp$ is given by the following basis

$$\begin{aligned}
&s^{-1}\mathbf{1}^\circ_{\omega^{-1}} \boxtimes s^{-1}\mathbf{1}^\circ_{\omega^{-1}} + \sum_{1 \leq a,b \leq n} \frac{\Omega_{ab}}{z_1 - z_2} s^{-1} x_a^\vee \boxtimes s^{-1} x_b^\vee, \\
&s^{-1}\mathbf{1}^\circ_{\omega^{-1}} \boxtimes s^{-1} x_a^\vee, \quad s^{-1} x_a^\vee \boxtimes s^{-1}\mathbf{1}^\circ_{\omega^{-1}}, \quad 1 \leq a \leq n, \\
&s^{-1} x_a^\vee \boxtimes s^{-1} x_b^\vee, \quad 1 \leq a, b \leq n,
\end{aligned}$$

The quadratic projection $(\mathbf{q}P^\circ)^\perp$ is given by the following basis

$$s^{-1} x_a^\vee \boxtimes s^{-1} x_b^\vee.$$

We get the graded commutative chiral algebra $\mathcal{A}(s^{-1}N^\vee_{\omega^{-1}}, (\mathbf{q}P^\circ)^\perp) = \mathrm{Sym}((s^{-1}N^\vee_{\omega^{-1}})_{\mathcal{D}})$. The differential d is zero. The curving element is given as follows

$$\begin{aligned}
\iota &= (h \boxtimes \mathrm{id})\mu(s^{-1}\mathbf{1}^\circ \boxtimes s^{-1}\mathbf{1}^\circ) \\
&= - \sum_{1 \leq a,b \leq n} (h \boxtimes \mathrm{id})\mu\left(\frac{\Omega_{ab}}{z_1 - z_2} s^{-1} x_a^\vee dz_1 \boxtimes s^{-1} x_b^\vee dz_2\right).
\end{aligned}$$

By identifying the chiral map with the commutative product of $\mathrm{Sym}((s^{-1}N^\vee_{\omega^{-1}})_{\mathcal{D}})^l$ as in the proof of 2.40, we see that the curving ι is an element of $\mathrm{Sym}((s^{-1}N^\vee_{\omega^{-1}})_{\mathcal{D}})$. Moreover, ι is given by

$$- \sum_{1 \leq a,b \leq n} \Omega_{ab} (s^{-1} x_a^\vee \cdot s^{-1} x_b^\vee) dz$$

3

VERTEX ALGEBRA IN DELIGNE CATEGORY

3.1 (pseudo-)tensor category

Symmetric monoidal category

In this section, we briefly review the definitions related to symmetric monoidal category. For more detail, see [EGNO15]

Definition 3.1. A monoidal category is a category $\mathcal{C}$ equipped with

1. A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the tensor product functor;
2. A distinguished object $\mathbb{1}$ called the unit object;
3. A natural isomorphism $\alpha : ((- \otimes -) \otimes -) \to (- \otimes (- \otimes -))$, with components:

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \tag{3.1}$$

   called the associativity isomorphism;

4. Two natural isomorphisms $l : (\mathbb{1} \otimes -) \to (-)$, $r : (- \otimes \mathbb{1}) \to (-)$,

such that the following two identity holds

1. triangle identity:

$$\begin{array}{ccc} (X \otimes \mathbb{1}) \otimes Y & \xrightarrow{\alpha_{X,\mathbb{1},Y}} & X \otimes (\mathbb{1} \otimes Y) \\ & {\scriptstyle r_X \otimes \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes l_X} & \\ & X \otimes Y & \end{array}$$

2. pentagon identity:

$$\begin{array}{ccc} ((X \otimes Y) \otimes Z) \otimes W & \xrightarrow{\alpha_{X,Y,Z} \otimes \mathrm{id}_W} & (X \otimes (Y \otimes Z)) \otimes W \\ \swarrow {\scriptstyle \alpha_{X \otimes Y,Z,W}} & & \searrow {\scriptstyle \alpha_{X,Y \otimes Z,W}} \\ (X \otimes Y) \otimes (Z \otimes W) & & X \otimes ((Y \otimes Z) \otimes W) \\ {\scriptstyle \alpha_{X,Y,Z \otimes W}} \searrow & & \swarrow {\scriptstyle \mathrm{id}_X \otimes \alpha_{Y,Z,W}} \\ & X \otimes (Y \otimes (Z \otimes W)) & \end{array}$$

Definition 3.2. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$, $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ be two monoidal categories. A (lax) monoidal functor between $\mathcal{C}$ and $\mathcal{D}$ is a triple (F, J, ϵ), where F is a functor $F : \mathcal{C} \to \mathcal{D}$, $J_{X,Y} : F(X) \otimes_{\mathcal{D}} F(Y) \xrightarrow{\sim} F(X \otimes_{\mathcal{C}} D)$ is a natural isomorphism and $\epsilon : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ is an isomorphism. These data are required to satisfy the following conditions

1. associativity:

$$\begin{array}{ccc}
(F(X) \otimes_{\mathcal{D}} F(Y)) \otimes_{\mathcal{D}} F(Z) & \xrightarrow{\alpha^{\mathcal{D}}_{F(X),F(Y),F(Z)}} & F(X) \otimes_{\mathcal{D}} (F(Y) \otimes_{\mathcal{D}} F(Z)) \\
\big\downarrow J_{X,Y} \otimes_{\mathcal{D}} \mathrm{id}_{F(Z)} & & \big\downarrow \mathrm{id}_{F(X)} \otimes_{\mathcal{D}} J_{Y,Z} \\
F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} F(Z) & & F(X) \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \\
\big\downarrow J_{X \otimes_{\mathcal{C}} Y, Z} & & \big\downarrow J_{X, Y \otimes_{\mathcal{C}} Z} \\
F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & \xrightarrow{F(\alpha^{\mathcal{C}}_{X,Y,Z})} & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z))
\end{array}$$

2. unitality:

$$\begin{array}{ccc}
\mathbb{1}_{\mathcal{D}} \otimes_{\mathcal{D}} F(X) & \xrightarrow{\epsilon \otimes \mathrm{id}_{F(X)}} & F(\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} F(X) \\
\big\downarrow l^{\mathcal{D}}_{F(X)} & & \big\downarrow J_{\mathbb{1}_{\mathcal{C}}, X} \\
F(X) & \xleftarrow[F(l^{\mathcal{C}}_X)]{} & F(\mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} X)
\end{array}$$

$$\begin{array}{ccc}
F(X) \otimes_{\mathcal{D}} \mathbb{1}_{\mathcal{D}} & \xrightarrow{\mathrm{id}_{F(X)} \otimes \epsilon} & F(X) \otimes_{\mathcal{D}} F(\mathbb{1}_{\mathcal{C}}) \\
\big\downarrow r^{\mathcal{D}}_{F(X)} & & \big\downarrow J_{X, \mathbb{1}_{\mathcal{C}}} \\
F(X) & \xleftarrow[F(r^{\mathcal{C}}_X)]{} & F(X \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}})
\end{array}$$

Given an object X in $\mathcal{C}$, an object X^* is called a left dual of X if there exist morphisms $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$ and $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$, called the evaluation and coevaluation maps, such that the compositions

$$\begin{gathered}
X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \\
X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^*
\end{gathered} \tag{3.2}$$

are the identity morphisms.

Similarly, we have the notion of right dual ${}^*X$ of an object X.

Proposition 3.3. *If an object X has a left (respectively, right) dual object, then it is unique up to a unique isomorphism.*

Definition 3.4.
1. An object X in $\mathcal{C}$ is called rigid if it has both a left dual and a right dual.
2. A monoidal category $\mathcal{C}$ is called rigid if every object of $\mathcal{C}$ is rigid.

Then we define the notions of braided monoidal category and symmetric monoidal category.

Definition 3.5. A braided monoidal category is a monoidal category $\mathcal{C}$ equipped with a natural isomorphism

$$\sigma_{X,Y} : X \otimes Y \to Y \otimes X \tag{3.3}$$

called the braiding, such that the following two hexagonal diagrams commute

$$\begin{array}{ccccccc}
 & & X \otimes (Y \otimes Z) & \xrightarrow{\sigma_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X & & \\
 & \overset{\alpha_{X,Y,Z}}{\nearrow} & & & & \overset{\alpha_{Y,Z,X}}{\searrow} & \\
(X \otimes Y) \otimes Z & & & & & & Y \otimes (Z \otimes X) \\
 & \underset{\sigma_{X,Y}\otimes \mathrm{id}_Z}{\searrow} & & & & \underset{\mathrm{id}_Y \otimes \sigma_{X,Z}}{\nearrow} & \\
 & & (Y \otimes X) \otimes Z & \xrightarrow[\alpha_{Y,X,Z}]{} & Y \otimes (X \otimes Z) & &
\end{array}$$

and

$$\begin{array}{ccccccc}
 & & (X \otimes Y) \otimes Z & \xrightarrow{\sigma_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) & & \\
 & \overset{\alpha^{-1}_{X,Y,Z}}{\nearrow} & & & & \overset{\alpha^{-1}_{Z,X,Y}}{\searrow} & \\
X \otimes (Y \otimes Z) & & & & & & (Z \otimes X) \otimes Y \\
 & \underset{\mathrm{id}_X\otimes \sigma_{Y,Z}}{\searrow} & & & & \underset{\sigma_{Z,X} \otimes \mathrm{id}_Y}{\nearrow} & \\
 & & X \otimes (Z \otimes Y) & \xrightarrow[\alpha^{-1}_{Y,X,Z}]{} & (X \otimes Z) \otimes Y & &
\end{array}$$

Definition 3.6. Let $\mathcal{C}$, $\mathcal{D}$ be two braided monoidal categories. A monoidal functor (F, J, ϵ) between $\mathcal{C}$ and $\mathcal{D}$ is called braided if the following diagram commutes

$$\begin{array}{ccc}
F(X) \otimes_{\mathcal{D}} F(Y) & \xrightarrow{\sigma^{\mathcal{D}}_{F(X),F(Y)}} & F(Y) \otimes_{\mathcal{D}} F(X) \\
\Big\downarrow J_{X,Y} & & \Big\downarrow J_{Y,X} \\
F(X \otimes_{\mathcal{C}} Y) & \xrightarrow{F(\sigma^{\mathcal{C}}_{X,Y})} & F(Y \otimes_{\mathcal{C}} X)
\end{array}$$

Definition 3.7. A symmetric monoidal category is a braided monoidal category for which the braiding satisfy the condition

$$\sigma_{Y,X} \circ \sigma_{X,Y} = \mathrm{id}_{X\otimes Y} \tag{3.4}$$

Karoubian category

In this section, we briefly review definitions of the Karoubian category, also called the pseudo-abelian category.

Given a category $\mathcal{C}$, an idempotent of $\mathcal{C}$ is an endomorphism $e : X \to X$, such that

$$e \circ e = e$$

An idempotent e is said to split if there is an object Y and morphisms $p : X \to Y$, $i : Y \to X$ such that $e = i \circ p$ and $\mathrm{id}_Y = p \circ i$.

Definition 3.8. A category $\mathcal{C}$ is called Karoubian if every idempotent splits.

Definition 3.9. A Karoubi envelope of a category $\mathcal{C}$ is a tuple $(\mathcal{C}^{kar}, \iota)$ where $\mathcal{C}^{kar}$ is Karoubian and $\iota : \mathcal{C} \to \mathcal{C}^{kar}$ is a fully-faithful functor such that for any Karoubian category $\mathcal{D}$, the restriction function

$$\mathrm{Fun}(\mathcal{C}^{kar}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \tag{3.5}$$

is an equivalence of categories.

Given a category, one can always construct its Karoubi envelope $\mathcal{C}^{kar}$ as follows. The objects of $\mathcal{C}^{kar}$ are pairs (X, e) where X is an object in $\mathcal{C}$ and $e : X \to X$ is an idempotent. The morphisms between (X, e) and (X', e') is given by

$$\mathrm{Hom}((X, e), (X', e')) = \{f \in \mathrm{Hom}(X, X') | f \circ e = e' \circ f\} \tag{3.6}$$

We use the following terminology in our paper.

Definition 3.10.
1. A pseudo-abelian category is a additive Karoubian category.
2. A pseudo-tensor category over a field k is a pseudo-abelian, rigid, symmetric monoidal category over k.
3. A tensor category over a field k is an abelian, rigid, symmetric monoidal category over k.

Ind completion and compact objects

Definition 3.11. A category I is filtered if it is nonempty and satisfies

1. For every two objects j, j' in I, there exists an object k and two morphisms $j \to k$ and $j' \to k$.

2. For every two parallel morphisms $f, g : j \to k$, there exists an object l and a morphism $h : k \to l$ such that $h \circ f = h \circ g$.

A filtered colimit in a category $\mathcal{C}$ is the colimit of a diagram $F : I \to \mathcal{C}$, where I is filtered. Such a diagram is also called an ind object in $\mathcal{C}$.

Example 3.12. Colimits in **Set** are easy to describe. We have

$$\operatorname*{colim}_{i \in I} F(i) = \left(\coprod_{i \in I} F(i) \right) / \sim \tag{3.7}$$

where $\sim$ is the equivalence relation identifying $a \in F(i) \sim b \in F(j)$ if for the object k with morphisms $f : i \to k$, $g : j \to k$, we have $F(f)(a) = F(g)(b)$.

Ind objects in $\mathcal{C}$ form a category $\mathrm{Ind}(\mathcal{C})$ called the ind completion of $\mathcal{C}$. Given two ind objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$, the space of morphisms from X to Y is defined to be

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, Y) = \lim_{i \in I} \operatorname*{colim}_{j \in J} \mathrm{Hom}(X_i, Y_j) \tag{3.8}$$

Definition 3.13. An object X in a locally small category $\mathcal{C}$ is called compact if the functor

$$\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{Set} \tag{3.9}$$

preserves filtered colimits.

We denote $\mathcal{C}^c$ the sub-category of compact objects in $\mathcal{C}$.

Definition 3.14. A category $\mathcal{C}$ is compactly generated if $\mathcal{C} \cong \mathrm{Ind}(\mathcal{C}^c)$.

By definition, objects in $\mathcal{C}$ are compact in $\mathrm{Ind}(\mathcal{C})$. The converse is not necessarily true in general. However, in this paper, we will focus on the case when $\mathcal{C}$ is Karoubian. In this case, compact objects in $\mathrm{Ind}(\mathcal{C})$ are all isomorphic to objects in $\mathcal{C}$.

Proposition 3.15. *The functor $\mathcal{C} \to (\mathrm{Ind}(\mathcal{C}))^c$ is an equivalence of category if and only if $\mathcal{C}$ is Karoubian.*

Proof. First, we show that $A \in \mathrm{Ind}(\mathcal{C})$ is a compact object of $\mathrm{Ind}(\mathcal{C})$ if and only if there exists an object $X \in \mathcal{C}$ and morphisms $i : A \to X$ and $p : X \to A$ such that $p \circ i = \mathrm{id}_A$.

For $A \in \mathrm{Ind}(\mathcal{C})$, let $A = \operatorname*{colim}_{i \in I} A_i$. Since A is compact, we have

$$\mathrm{Hom}(A, A) = \operatorname*{colim}_{i \in I} \mathrm{Hom}(A, A_i)$$

Therefore, there exist an object $X = A_i \in \mathcal{C}$ for some $i \in I$ with morphisms $i : A \to X$ such that $\mathrm{id}_A = p \circ i$.

On the other hand, if there exist an object $X \in \mathcal{C}$ and morphisms $i : A \to X$ and $p : X \to A$ such that $p \circ i = \mathrm{id}_A$. For any filtered colimit $Y = \mathrm{colim}_i Y_i$, we have a map

$$\mathrm{Hom}(A, Y) \to \mathrm{Hom}(X, Y) \xrightarrow{\cong} \mathrm{colim}_i \mathrm{Hom}(X, Y_i) \to \mathrm{colim}_i \mathrm{Hom}(A, Y_i)$$

We can check that this map is an isomorphism. Therefore A is compact.

$\mathcal{C} \to (\mathrm{Ind}(\mathcal{C}))^c$ induces an equivalence of categories if and only if any compact object A of $\mathrm{Ind}(\mathcal{C})$ is isomorphic to an object $A' \in \mathcal{C}$. We have shown that this condition is equivalent to the condition that $\mathcal{C}$ being Karoubian. □

Corollary 3.16. *Let $\widetilde{\mathcal{C}} = \mathrm{Ind}(\mathcal{C})$ be the Ind completion of a Karoubian category. Then the tensor product preserves compact objects of $\widetilde{\mathcal{C}}$.*

3.2 ENVELOPING ALGEBRA IN PSEUDO-TENSOR CATEGORY

The usual notion of Lie algebra and associative algebra extend to any symmetric monoidal category $\mathcal{C}$

Definition 3.17. A Lie algebra object in $\mathcal{C}$ is an object $\mathfrak{g}$ in $\mathcal{C}$ together with a morphism

$$[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \tag{3.10}$$

We require the following axioms

1. (Skew-symmetry)
$$[-,-] \circ (\mathrm{id}_{\mathfrak{g}}^{\otimes 2} + \sigma) = 0 \tag{3.11}$$

2. (Jacobi identity)
$$[-,[-,-]] \circ (\mathrm{id}_{\mathfrak{g}}^{\otimes 3} + (\mathrm{id}_{\mathfrak{g}} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}_{\mathfrak{g}}) + (\sigma \otimes \mathrm{id}_{\mathfrak{g}}) \circ (\mathrm{id}_{\mathfrak{g}} \otimes \sigma)) = 0 \tag{3.12}$$
where we denote $[-,[-,-]] = [-,-] \circ (\mathrm{id}_{\mathfrak{g}} \otimes [-,-])$.

Definition 3.18. An associative algebra object in $\mathcal{C}$ is an object A in $\mathcal{C}$ together with a morphism $m : A \otimes A \to A$, such that

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m) \tag{3.13}$$

A unital algebra in $\mathcal{C}$ is an associative algebra (A, m) together with a map $\eta : \mathbb{1} \to A$ called unit, such that

$$m \circ (\eta \otimes \mathrm{id}_A) = l_A, \quad m \circ (\mathrm{id}_A \otimes \eta) = r_A \tag{3.14}$$

A commutative algebra in $\mathcal{C}$ is an associative algebra (A, m) such that

$$m = m \circ \sigma \tag{3.15}$$

An important construction related to a Lie algebra $\mathfrak{g}$ in vector space is its universal enveloping algebra $U(\mathfrak{g})$. It is defined by the quotient

$$U(\mathfrak{g}) = T(\mathfrak{g})/I \tag{3.16}$$

where I is the ideal given by

$$I = \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle \tag{3.17}$$

This construction extend naturally to Lie algebra in a tensor category. However, if $\mathcal{C}$ is only a pseudo-tensor category, i.e. without arbitrary quotient, the existence of the universal enveloping algebra is not obvious. In this section, we present a construction of universal enveloping algebra as a deformation of the symmetric algebra.

For any object X in $\mathcal{C}$, the tensor algebra $T(X)$ is defined as

$$T(X) = \bigoplus_{k \geq 0} T^k(X) = \bigoplus_{k \geq 0} X^{\otimes k} \tag{3.18}$$

It has a natural associative product given by the tensor product $T^k(X) \otimes T^l(X) \to T^{k+l}(X)$. The symmetric group S_k acts on $T^k(X)$. We have an idempotent

$$\begin{gathered} e_{\text{Sym}} = \frac{1}{k!} \sum_{\tau \in S_n} \tau : T^k(X) \to T^k(X) \\ e_{\text{Sym}}^2 = e_{\text{Sym}} \end{gathered} \tag{3.19}$$

Since $\mathcal{C}$ is Karoubian, we get the symmetric tensor $S^k(X)$ as an object in $\mathcal{C}$. Then we define

$$S(V) = \bigoplus_{k \geq 0} S^k(X). \tag{3.20}$$

$S(V)$ inherit an product m from the tensor algebra $T(X)$, defined as

$$m : S(V) \otimes S(V) \xrightarrow{i \otimes i} T(V) \otimes T(V) \xrightarrow{\otimes} T(V) \xrightarrow{p} S(V) \tag{3.21}$$

Lemma 3.19. *For any object X and Z, the Hom space $\text{Hom}_{\mathcal{C}}(Z, T^n X)$ is naturally a S_n module. We have*

$$\text{Hom}_{\mathcal{C}}(Z, S^n X) = \text{Hom}_{\mathcal{C}}(Z, T^n X)^{S_n} \tag{3.22}$$

Proof. By definition, $\mathrm{Hom}_{\mathcal{C}}(Z, S^n X) = \{f \in \mathrm{Hom}_{\mathcal{C}}(Z, T^n X) | e_{\mathrm{Sym}} \circ f = f\}$. For any such f, we have $\sigma \circ f = \sigma \circ e_{\mathrm{Sym}} \circ f = e_{\mathrm{Sym}} \circ f = f$ for any $\sigma \in S_n$. On the other hand, suppose $f \in \mathrm{Hom}_{\mathcal{C}}(Z, T^n X)$ that satisfy $\sigma \circ f = f$ for any $\sigma \in S_n$. Clearly $e_{\mathrm{Sym}} \circ f = f$. Thus we have proved $\mathrm{Hom}_{\mathcal{C}}(Z, S^n X) = \{f \in \mathrm{Hom}_{\mathcal{C}}(Z, T^n X) | \sigma \circ f = f, \text{ for any } \sigma \in S_n\}$. □

Weyl algebra and Weyl quantization

Before we present the construction of universal enveloping algebra, it will be helpful to look at the easier example of Weyl algebra.

We assume that X is equipped with a antisymmetric two form

$$\begin{aligned} \omega : X \otimes X \to \mathbb{1} \\ \omega \circ \sigma = -\Omega \end{aligned} \tag{3.23}$$

ω extends to a series of maps

$$\omega_{ij} : T^k(X) \otimes T^l(X) \to T^{k-1}(X) \otimes T^{-1}(X) \tag{3.24}$$

by applying ω to the i-th and j-th X and identities on the others. Explicitly, we can write ω_{ij} as the composition

$$T^k(X) \otimes T^l(X) \overset{\sigma_{(k,k-1,\dots,i)} \otimes \sigma_{(1,2,\dots,j)}}{\longrightarrow} T^k(X) \otimes T^l(X) \overset{\mathrm{id}_X^{\otimes k-1} \otimes \Omega \otimes \mathrm{id}_X^{l-1}}{\longrightarrow} T^{k-1}(X) \otimes T^{l-1}(X) \tag{3.25}$$

where $\sigma_{(k,k-1,\dots,i)}$ is the braiding map that correspond to the permutation $(k, k-1, \dots, i)$.

We define $\boldsymbol{\omega} : T^k(X) \otimes T^l(X) \to T^{k-1}(X) \otimes T^{-1}(X)$ by

$$\boldsymbol{\omega} = \sum_{i=1}^{k} \sum_{j=1}^{l} \omega_{ij} \tag{3.26}$$

It induces a map on the symmetric tensor, that we also denoted $\boldsymbol{\omega}$

$$\boldsymbol{\omega} : S(X) \otimes S(X) \to S(X) \otimes S(X) \tag{3.27}$$

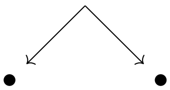
We define the Weyl algebra as the Moyal-Weyl deformation of the symmetric algebra.

Definition 3.20. The Weyl algebra $\mathcal{W}(X, \omega)$ associated to (X, ω) is the associative algebra $(S(V), \star)$, where $\star$ is given by

$$- \star - = m \circ \exp(\frac{\boldsymbol{\omega}}{2})(- \otimes -) \tag{3.28}$$

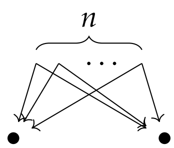
We can pictorially represent the map $\boldsymbol{\omega}$ by a digram

$\boldsymbol{\omega}$:



Then the $\star$ product can be pictorially represented by the following

$\sum_{n\geq 0} \frac{1}{n!}$



In fact, the star product has the property that shifting the antisymmetric map ω by a symmetric map $\alpha : S^2(X) \to \mathbb{1}$ leads to an isomorphic star product. We first extend α to a map $S(X) \to S(X)$. Then The isomorphism is given by exponentiate α

$$\exp(\frac{1}{4}\alpha) : S(X) \to S(X) \tag{3.29}$$

Proposition 3.21. *Let $\alpha : X \otimes X \to \mathbb{1}$ be a symmetric two form, i.e. $\alpha \circ \sigma = \alpha$. Denote $\star_\omega$ the star product defined by ω and $\star_{\omega+\alpha}$ the star product defined by $\omega + \alpha$. Then we have the following commutative diagram*

$$\begin{array}{ccc} S(X) \otimes S(X) & \xrightarrow{e(\frac{1}{4}\alpha)\otimes e(\frac{1}{4}\alpha)} & S(X) \otimes S(X) \\ \downarrow{\star_\omega} & & \downarrow{\star_{\omega+\alpha}} \\ S(X) & \xrightarrow{\exp(\frac{1}{4}\alpha)} & S(X) \end{array} \tag{3.30}$$

Shifting by a symmetric form α is related to the choice of ordering in the identification of symmetric algebra and the non-commutative Weyl algebra. The standard Moyal-Weyl product $\star_\omega$ is associated with the symmetric ordering. Sometimes a different ordering, called normal ordering, is also used. This ordering is more convenient in defining the module associated with a polarization.

We call a polarization of (X, ω) a decomposition $X = L_+ \oplus L_-$, so that the symplectic form also decomposes as $\omega = \omega_+ + \omega_-$, where

$$\omega_+ \in \mathrm{Hom}(L_- \otimes L_+, \mathbb{1}),\ \omega_- \in \mathrm{Hom}(L_+ \otimes L_-, \mathbb{1}) \tag{3.31}$$

and $\omega_+ = -\omega_- \circ \sigma$. Given a polarization, we can consider the star product associated with the two form $2\omega_-$

$$\star_{2\omega_-} = \exp(\boldsymbol{\omega}_-) \in \mathrm{Hom}(S(X) \otimes S(X), \mathbb{1}) \tag{3.32}$$

This product corresponds to the so called normal ordering. It differs from the standard star product by the symmetric two form $\alpha = \omega_+ - \omega_-$. The advantage of $\star_{2\omega_-}$ is that we can naturally define a $(S(X), \star_{2\omega_-})$ left module structure on $S(L_-)$. The module map is given by the same formula as $\star_{2\omega_-}$, composed with π_{L_-}, where π_{L_-} is the natural projection

$$\pi_{L_-} : S(L_+ \oplus L_-) \to S(L_-) \tag{3.33}$$

Using the isomorphism 3.29, we obtain a left $\mathcal{W}(X, \omega)$ module structure on $S(L_-)$, which we denote by $\star_l$:

$$\star_l : \mathcal{W}(X, \omega) \otimes S(L_-) \to S(L_-) \tag{3.34}$$

This map can be written as follows:

$$- \star_l - = \pi_{L_-} \circ m \circ \exp(\boldsymbol{\omega}_-) \circ (\exp(\frac{\omega_+ - \omega_-}{4}) - \otimes -) \tag{3.35}$$

Universal enveloping algebra

Now we can move on to define the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ in $\mathcal{C}$. The Lie structure is given by a map

$$[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

Such a map extend by Leibniz rule to a map on the symmetric algebra $S(\mathfrak{g})$

$$S(\mathfrak{g}) \otimes S(\mathfrak{g}) \to S(\mathfrak{g}) \tag{3.36}$$

We construct the universal enveloping algebra $U(\mathfrak{g})$ as the deformation of $S(\mathfrak{g})$ equipped with the (CBH) star product. Recall that the Campbell-Baker-Hausdorff (CBH) formula provides us with an expression for the exponential of two Lie algebra elements X, Y

$$\exp(X)\exp(Y) = \exp(X + Y + CBH(X, Y))$$

The first few terms in $CBH(X, Y)$ reads

$$CBH(X, Y) = \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [[X, Y], Y]) + \dots$$

For any Lie words of length $n + 1$ that appears in the CBH formula, we attach a directed graph Γ, so that

1. Γ has $n + 2$ nodes denoted $\{1, 2, \dots, n\} \cup \{L, R\}$;

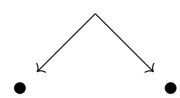
2. Γ has $2n$ edges. For each node $k \in \{1, 2, \dots, n\}$, there are two edges start from k;

3. There is no self loop edge of the form (k, k).

The graph Γ is constructed as follows. Recall that any Lie words of two symbol X, Y can be represented as a binary rooted tree with leaves label by X or Y. For each internal node (a node that is not a leaf), we associate a node of the graph, and the leaves labeled by X is associated with node L and the leaves labeled by Y is associated with the node R. Then edges of Γ are associated with edges of the tree, with the same orientation as from root to leaves.

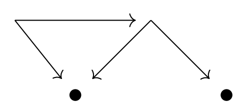
Note that the same set of graphs is used by Kontsevich [Kon03] in his construction of general star product. In the same way, we further assign a map $B_\Gamma : S(\mathfrak{g}) \otimes S(\mathfrak{g}) \to S(\mathfrak{g})$ to each graph Γ. For example, for the Lie word $[X, Y]$ we attach the graph

$[X, Y]$:



The corresponding map on $S(\mathfrak{g})$ is the map 3.36. As another example, for $[X, [X, Y]]$ we attach the graph

$[X, [X, Y]]$:



The corresponding map on $S(\mathfrak{g})$ can be constructed by symmetrizing the following composition of maps

$$\begin{aligned} &T^k(\mathfrak{g}) \otimes T^l(\mathfrak{g}) \xrightarrow{\mathrm{id}_X^{\otimes k-1} \otimes [-,-] \otimes \mathrm{id}_X^{l-1}} T^{k-1}(\mathfrak{g}) \otimes \mathfrak{g} \otimes T^{l-1}(\mathfrak{g}) \\ &\xrightarrow{\mathrm{id}_X^{\otimes k-2} \otimes [-,-] \otimes \mathrm{id}_X^{l-1}} T^{k-2}(\mathfrak{g}) \otimes \mathfrak{g} \otimes T^{l-1}(\mathfrak{g}) = T^{k+l-2}(\mathfrak{g}) \end{aligned} \tag{3.37}$$

Suppose that we can write the CBH formula as

$$\exp(X) \exp(Y) = \exp(X + Y + CBH(X, Y)) = \exp(X + Y + \sum_\Gamma c_\Gamma \Gamma)$$

Then we define the universal enveloping algebra $U(\mathfrak{g})$ as the following CBH quantization of the symmetric algebra $S(\mathfrak{g})$

$$\star = \exp(\sum_\Gamma c_\Gamma B_\Gamma) \tag{3.38}$$

3.3 VERTEX ALGEBRA IN PSEUDO TENSOR CATEGORY

Basic definitions

First, we recall the definition of a vertex algebra. A vertex algebra consist of a vector space V together wih the following data:

1. The vacum vector: $|0\rangle \in V$.
2. The translation map: $T : V \to V$.
3. An infinite collection of bilinear maps: $\cdot_n : V \otimes V \to V$ for $n \in \mathbb{Z}$. We require that for any $u, v \in V$, there exist an integer N such that $u \cdot_n v = 0$ for any $n \geq N$.

   We usually collect these maps into a power series and write $Y(u,z)v = \sum_{n \in \mathbb{Z}} u \cdot_n v\, z^{-n-1}$.

These data satisfy the following axioms

1. Vacuum. $Y(|0\rangle, z)u = u$, and $Y(u,z)|0\rangle \in u + zV[[z]]$ for any $u \in V$.
2. Translation. $T|0\rangle = 0$. Further more, $[T, Y(u,z)] = \partial_z Y(u,z)$
3. Locality. For any $u, v \in V$, there exist an integer N such that

$$(z-w)^N [Y(u,z), Y(v,w)] = 0 \tag{3.39}$$

We observe that most part of the definition applies seamlessly to any braided monoidal category, requiring no modification. The aspect demanding particular attention arises when we must consider specific elements in V, as we cannot talk about elements in an object of an arbitrary category. The essence of this problem lies in recognizing that such structures concern the compact objects within the category. Hence, in our definition, we consider a compactly generated symmetric monoidal category $\mathcal{C}$. To work in a setup where we have a reasonable notion of vertex algebra, in this and the following sections, we consider symmetric monoidal category $\mathcal{C}$ that satisfies the following conditions

1. $\mathcal{C}$ is compactly generated. I.e. $\mathcal{C} \cong \mathrm{Ind}(\mathcal{C}_0)$ for some symmetric monoidal category $\mathcal{C}_0$.
2. $\mathcal{C}_0$ is a pseudo tensor category, i.e. a rigid symmetric monoidal category and is idempotent complete.
3. $\mathbb{Q} \subset \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$

As we have seen, such a category has the nice properties that compact objects are closed under the monoidal structure, and the universal enveloping construction works.

We define vertex algebra in $\mathcal{C}$ as follows:

Definition 3.22. A vertex algebra in $\mathcal{C}$ consist of the data of an object V in $\mathcal{C}$ together with morphisms

1. The vacum vector: $|0\rangle \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, V)$.
2. The "translation" map: $T \in \mathrm{Hom}_{\mathcal{C}}(V, V)$.
3. An infinite collection of maps: $\cdot_n \in \mathrm{Hom}_{\mathcal{C}}(V \otimes V, V)$ for $n \in \mathbb{Z}$. We usually collect these maps together and get a map $Y(z) = \sum_{n\in\mathbb{Z}} \cdot_n z^{-n-1} \in \mathrm{Hom}(V \otimes V, V)[[z, z^{-1}]]$. We also require that for any compact object X in $\mathcal{C}$ and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(X, V)$, there exist an integer N such that
$$\cdot_n \circ (\alpha \otimes \beta) = 0 \tag{3.40}$$
for any $n \geq N$.

These data satisfy the following axioms

1. Vacuum. $Y(z) \circ (|0\rangle \otimes \mathrm{id}_V) = l_V$. Further more, $Y(z) \circ (\mathrm{id}_V \otimes |0\rangle) \in \mathrm{Hom}_{\mathcal{C}}(V \otimes \mathbb{1}, V)[[z]]$, so that $Y(z) \circ (\mathrm{id}_V \otimes |0\rangle)|_{z=0} = r_V$.
2. Translation. $T|0\rangle = 0$. Further more, $T \circ Y(z) - Y(z) \circ (\mathrm{id}_V \otimes T) = \partial_z Y(z)$
3. Locality. For any compact object X and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(X, V)$, we can find a integer N such that
$$\begin{aligned}(z-w)^N \big(&Y(w) \circ (\mathrm{id}_V \otimes Y(z)) \circ (\alpha \otimes \beta \otimes \mathrm{id}_V) \\ &- Y(z) \circ (\mathrm{id}_V \otimes Y(w)) \circ (\sigma \otimes \mathrm{id}_V) \circ (\alpha \otimes \beta \otimes \mathrm{id}_V)\big) = 0\end{aligned} \tag{3.41}$$

Remark 3.23. Since we have assumed that our category $\mathcal{C}$ is compactly generated by a pseudo tensor category, tensor product of compact objects is still compact. We can show that the collection of maps $Y(z)$ satisfying condition 3.40 is equivalent to say that it is an elements in
$$\mathrm{Hom}_{\mathcal{C}}(V \otimes V, V((z))) \tag{3.42}$$

To see this, we let $V = \underset{i\in I}{\mathrm{colim}} V_i$ for V_i compact. To simplify the discussion, we focus on the singular part of $Y(z)$. By definition, we have
$$\mathrm{Hom}_{\mathcal{C}}(V \otimes V, Vz^{-1}[z^{-1}]) = \lim_{(i,j)\in I\times I} \underset{k\in\mathbb{Z}}{\mathrm{colim}}\, \mathrm{Hom}_{\mathcal{C}}(V_i \otimes V_j, \oplus_{i=0}^{k} Vz^{-i-1}) \tag{3.43}$$

This gives us, for each $i, j \in I \times I$, an integer $N_{i,j}$ and maps $\mu_{i,j;n} : V_i \otimes V_j \to Vz^{-n-1}$ for $n \leq N_{i,j}$. These maps are compatible in the obvious way. So we get a series of maps μ_n, and on each $V_i \otimes V_j$, μ_n vanishes for $n > N_{i,j}$. This is equivalent to the condition we give in our definition, because any map $X \to V$ from a compact object X must factor through a map $X \to V_i$ for some i. Our definition has the advantage of being independent of the presentation of V as a colimit,

Remark 3.24. Compact objects in Vect_k are finite dimensional vector spaces. Therefore, for $\mathcal{C} = \mathrm{Vect}_k$, the above definition gives us the usual definition of vertex operator algebra. Moreover, we can let $\mathcal{C} = \mathrm{Vect}_k^{\mathbb{Z}}$ (or $\mathrm{Vect}_k^{\mathbb{Z}_2}$), then the above construction gives us the usual notion of graded (or super) vertex algebra

The following statement is a simple corollary of the vacuum and translation axioms.

Lemma 3.25. *We have the following identity*

$$Y(z) \circ (\mathrm{id}_V \otimes |0\rangle) = e^{zT} \circ r_V$$

in $\mathrm{Hom}_{\mathcal{C}}(V \otimes \mathbb{1}, V)[[z]]$.

Proof. By the translation axiom, we have

$$\partial_z Y(z) \circ (\mathrm{id}_V \otimes |0\rangle) = T \circ Y(z) \circ (\mathrm{id}_V \otimes |0\rangle)$$

This implies, by induction, that

$$\partial_z^n Y(z) \circ (\mathrm{id}_V \otimes |0\rangle) = T^n \circ Y(z) \circ (\mathrm{id}_V \otimes |0\rangle)$$

By the vacuum axiom, we have

$$\partial_z^n Y(z) \circ (\mathrm{id}_V \otimes |0\rangle)|_{z=0} = T^n \circ r_V$$

Since $Y(z) \circ (\mathrm{id}_c \otimes |0\rangle) \in \mathrm{Hom}_{\mathcal{C}}(V, V)[[z]]$, we have

$$Y(z) \circ (\mathrm{id}_V \otimes |0\rangle) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \partial_z^n Y(0) \circ (\mathrm{id}_V \otimes |0\rangle) = e^{zT} \circ r_V$$

□

Remark 3.26. There are several equivalent formulations of the locality axiom for vertex algebra in Vect_k. An often used one is the so called Borcherds identity. We discuss these equivalent definition in $\mathcal{C}$ in the next section.

Associativity

In section, we study associativity property of vertex algebra in $\mathcal{C}$ and different formulation of the locality axiom. Most part of this section follows from the same results as vertex algebra in vector space.

Lemma 3.27. *We have the following identity*

$$e^{wT} \circ Y(z) \circ (\mathrm{id}_V \otimes e^{-wT}) = Y(z+w)$$

in $\mathrm{Hom}_{\mathcal{C}}(V \otimes V, V)[[z^{\pm 1}, w^{\pm 1}]]$.

Proof. By the translation axiom, we have

$$e^{wT} \circ Y(z) \circ (\mathrm{id}_V \otimes e^{-wT}) = \sum_{n=0}^{\infty} \frac{w^n}{n!} \partial_z^n Y(z) = Y(z+w)$$

□

Proposition 3.28. *(skew-symmetry) For any compact object X and morphisms $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(X, V)$, the following identity hold*

$$Y(z) \circ (\alpha \otimes \beta) = e^{zT} \circ Y(-z) \circ \sigma \circ (\alpha \otimes \beta) \tag{3.44}$$

in $\mathrm{Hom}_{\mathcal{C}}(X \otimes X, V)((z))$

Proof. We apply the Locality axiom and find

$$\begin{aligned}(z-w)^N \Big(& Y(w) \circ (\mathrm{id}_V \otimes Y(z)) \circ (\alpha \otimes \beta \otimes |0\rangle) \\ & - Y(z) \circ (\mathrm{id}_V \otimes Y(w)) \circ (\sigma \otimes \mathrm{id}_V) \circ (\alpha \otimes \beta \otimes |0\rangle) \Big) = 0\end{aligned}$$

for large enough N. Using Lemma 3.25, we find

$$(z-w)^N \Big(Y(w) \circ (\mathrm{id}_V \otimes e^{zT}) - Y(z) \circ (\mathrm{id}_V \otimes e^{wT}) \circ \sigma \Big) \circ (\alpha \otimes \beta) = 0$$

Then we apply Lemma 3.27 and find

$$(z-w)^N Y(w) \circ (\mathrm{id}_V \otimes e^{zT}) \circ (\alpha \otimes \beta) = (z-w)^N e^{wT} \circ Y(z-w) \circ \sigma \circ (\alpha \otimes \beta) = 0$$

We can choose N large enough so that the right hand side does not contain any negative power of $(z-w)$. Then we set $z = 0$ and the identity becomes $w^N Y(w) \circ (\alpha \otimes \beta) = w^N e^{wT} \circ Y(-w) \circ \sigma \circ (\alpha \otimes \beta) = 0$. We divide both side by w^N, which gives the formula 3.44. □

Theorem 3.29. *For any compact object X and morphisms $\alpha, \beta, \gamma \in \mathrm{Hom}_{\mathcal{C}}(X, V)$, the three expansions*

$$\begin{aligned}
&Y(z) \circ (\mathrm{id}_V \otimes Y(w)) \circ (\alpha \otimes \beta \otimes \gamma) \in \mathrm{Hom}_{\mathcal{C}}(X^{\otimes 3}, V)((z))((w)) \\
&Y(w) \circ (\mathrm{id}_V \otimes Y(z)) \circ (\sigma \otimes \mathrm{id}_V) \circ (\alpha \otimes \beta \otimes \gamma) \in \mathrm{Hom}_{\mathcal{C}}(X^{\otimes 3}, V)((w))((z)) \\
&Y(w) \circ (Y(z-w) \otimes \mathrm{id}_V) \circ (\alpha \otimes \beta \otimes \gamma) \in \mathrm{Hom}_{\mathcal{C}}(X^{\otimes 3}, V)((w))((z-w))
\end{aligned} \tag{3.45}$$

are the expansion of the same element of

$$\mathrm{Hom}_{\mathcal{C}}(X^{\otimes 3}, V)[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}] \tag{3.46}$$

Proof. By the locality axiom, $Y(z) \circ (\mathrm{id}_V \otimes Y(w)) \circ (\alpha \otimes \beta \otimes \gamma)$ and $Y(w) \circ (\mathrm{id}_V \otimes Y(z)) \circ (\sigma \otimes \mathrm{id}_V) \circ (\alpha \otimes \beta \otimes \gamma)$ are the expansions of the same element. Therefore we only need to prove that the first and last expression are the expansions of the same element.

By the skew symmetry property, we have the following identity

$$\begin{aligned}
&Y(z) \circ (\mathrm{id}_V \otimes Y(w)) \circ (\alpha \otimes \beta \otimes \gamma) \\
&= Y(z) \circ (\mathrm{id}_V \otimes e^{wT} \circ Y(-w) \circ \sigma) \circ (\alpha \otimes \beta \otimes \gamma)
\end{aligned} \tag{3.47}$$

Then we use Lemma 3.27 and find

$$\begin{aligned}
&Y(z) \circ (\mathrm{id}_V \otimes Y(w)) \circ (\alpha \otimes \beta \otimes \gamma) \\
&= e^{wT} \circ Y(z-w) \circ (\mathrm{id}_V \otimes Y(-w) \circ \sigma) \circ (\alpha \otimes \beta \otimes \gamma)
\end{aligned} \tag{3.48}$$

On the other hand, $Y(z-w) \circ (\alpha \otimes \beta) = \sum_{n \in \mathbb{Z}} (z-w)^{-n-1} \cdot_n \circ (\alpha \otimes \beta)$ by definition. The composition $\cdot_n \circ (\alpha \otimes \beta)$ is still a map from a compact object $X^{\otimes 2}$. Therefore we can apply the skew symmetry property to $Y(w) \circ (\cdot_n \circ (\alpha \otimes \beta) \otimes \gamma)$, which gives us

$$\begin{aligned}
&Y(w) \circ (Y(z-w) \otimes \mathrm{id}_V) \circ (\alpha \otimes \beta \otimes \gamma) \\
&= e^{wT} \circ Y(-w) \circ \sigma \circ (Y(z-w) \otimes \mathrm{id}_V) \circ (\alpha \otimes \beta \otimes \gamma)
\end{aligned} \tag{3.49}$$

By applying the locality axiom again, we find that $Y(z) \circ (\mathrm{id}_V \otimes Y(w)) \circ (\alpha \otimes \beta \otimes \gamma)$ and $Y(w) \circ (Y(z-w) \otimes \mathrm{id}_V) \circ (\alpha \otimes \beta \otimes \gamma)$ are expansions of the same elements. $\square$

Theorem 3.30. *(Borcherds identity) For any compact object X and morphisms $\alpha, \beta, \gamma \in \mathrm{Hom}_{\mathcal{C}}(X, V)$, we have the following identity*

$$\begin{aligned}&\sum_{n\geq 0}\binom{m}{n}\cdot_{m+k-n}\circ(\cdot_{n+l}\otimes \mathrm{id}_V)\circ(\alpha\otimes\beta\otimes\gamma)=\\&\sum_{j\geq 0}\binom{l}{j}(-1)^j\Big(\cdot_{m+l-j}\circ(\mathrm{id}_L\otimes\cdot_{k+j})-\\&\quad(-1)^l\cdot_{k+l-j}\circ(\mathrm{id}_V\otimes\cdot_{m+j})\circ(\sigma\otimes\mathrm{id}_V)\Big)\circ(\alpha\otimes\beta\otimes\gamma)\end{aligned} \tag{3.50}$$

for any integers m, k, l.

Proof. By Theorem 3.29, the three expressions $Y(z)\circ(\mathrm{id}_V\otimes Y(w))(\alpha\otimes\beta\otimes\gamma)$, $Y(w)\circ(\mathrm{id}_V\otimes Y(z))\circ(\sigma\otimes\mathrm{id}_V)(\alpha\otimes\beta\otimes\gamma)$ and $Y(w)\circ(Y(z-w)\otimes\mathrm{id}_V)(\alpha\otimes\beta\otimes\gamma)$ are the expansion of the same elements $X(z,w)$ in $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes 3}, V)[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$. Let $f(z,w)$ be a rational function which has poles only at $z=0, w=0$ and $z=w$. Let $R>\rho>r>0$, we consider the contour integral

$$\begin{aligned}&\oint_{C_w^\rho}\oint_{C_z^R}Y(z)\circ(\mathrm{id}_V\otimes Y(w))(\alpha\otimes\beta\otimes\gamma)f(z,w)dzdw\\&-\oint_{C_w^\rho}\oint_{C_z^r}Y(w)\circ(\mathrm{id}_V\otimes Y(z))\circ(\sigma\otimes\mathrm{id}_V)(\alpha\otimes\beta\otimes\gamma)f(z,w)dzdw\end{aligned} \tag{3.51}$$

This contour integral can be written as $\oint_{C_w^\rho}\oint_{C_z^R-C_z^r}X(z,w)f(z,w)dzdw$. We can further replace $C_z^R - C_z^r$ by a circle $C_z^\delta(w)$ of radius $\delta<\rho$ around w. In this region, $X(z,w)$ is expanded as $Y(w)\circ(Y(z-w)\otimes\mathrm{id}_V)(\alpha\otimes\beta\otimes\gamma)$. We find that 3.51 is equal to

$$\oint_{C_w^\rho}\oint_{C_z^\delta(w)}Y(w)\circ(Y(z-w)\otimes\mathrm{id}_V)(\alpha\otimes\beta\otimes\gamma)f(z,w)dzdw \tag{3.52}$$

If we choose $f(z,w)=z^m w^k(z-w)^l$, the above identity gives us 3.50. □

OPE and normally ordered product

Given a compact object X, we call a collection of maps $A(z)=\sum_n A_n z^{-n-1}$

$$A_n : X\otimes V\to V \tag{3.53}$$

a field labeled by X if for any compact object X' and morphism $\beta : X' \to V$, there exist an integer N such that $A_n \circ (\mathrm{id}_X \otimes \beta) = 0$ for all $n > N$. In other words, a field $A(z)$ labeled by X is a morphism

$$A(z) \in \mathrm{Hom}_{\mathcal{C}}(X \otimes V, V((z))) \tag{3.54}$$

By construction, for any morphism $\alpha : X \to V$, the map $Y(z)(\alpha \otimes \mathrm{id}_V)$ is a field labeled by X.

Theorem 3.31. *(Goddard's uniqueness theorem) Let V be a vertex algebra in $\mathcal{C}$, $A(z)$ a field on V labeled by X. Suppose there exist a map $\alpha : X \to V$ such that*

$$A(z) \circ (\mathrm{id}_X \otimes |0\rangle) = Y(z) \circ (\alpha \otimes |0\rangle) \tag{3.55}$$

and $A(z)$ is local with respect to the field $Y(z) \circ (\beta \otimes \mathrm{id}_{X'})$ for any $\beta : X' \to V$. Then $A(z) = Y(z) \circ (\alpha \otimes \mathrm{id}_V)$.

Proof. Let $V = \operatorname{colim}_{i \in I} V_i$ for V_i compact, and denote $s_i : V_i \to V$ the inclusion map. By the locality, we have, for large enough N

$$\begin{aligned}(z-w)^N A(z) &\circ (\mathrm{id}_X \otimes (Y(w) \circ (s_i \otimes |0\rangle))) \\ &= (z-w)^N Y(w) \circ (s_i \otimes A(z)) \circ (\sigma_{X,X'} \otimes |0\rangle)\end{aligned} \tag{3.56}$$

Since $A(z)(\mathrm{id}_X \otimes |0\rangle) = Y(z) \circ (\alpha \otimes |0\rangle)$, we further have

$$\begin{aligned}(z-w)^N A(z) &\circ (\mathrm{id}_X \otimes (Y(w) \circ (s_i \otimes |0\rangle))) \\ &= (z-w)^N Y(w) \circ (s_i \otimes Y(z)) \circ (\alpha \otimes |0\rangle) \circ (\sigma_{X,X'} \otimes id_V) \\ &= (z-w)^N Y(z) \circ (\mathrm{id}_X \otimes Y(w)) \circ (\alpha \otimes s_i \otimes |0\rangle)\end{aligned} \tag{3.57}$$

By the vacuum axiom, both sides of the above equation are well-defined at $w = 0$, and $Y(w) \circ (s_i \otimes |0\rangle) = s_i$. Setting $w = 0$, and divide both sides by z^N, we obtain

$$A(z)(\mathrm{id}_X \otimes s_i) = Y(z) \circ (\alpha \otimes s_i) \tag{3.58}$$

This equation hold for any $i \in I$, therefore we have

$$A(z) = Y(z) \circ (\alpha \otimes \mathrm{id}_V) \tag{3.59}$$

□

Corollary 3.32. *We have the identity*

$$Y(z) \circ (T \otimes \mathrm{id}_V) = \partial_z Y(z) \tag{3.60}$$

Proof. Let $V = \operatorname{colim}_{i\in I} V_i$ for V_i compact, and denote $s_i : V_i \to V$ the inclusion map. We define the field $A(z) = \partial_z Y(z) \circ (s_i \otimes \mathrm{id}_V)$. Since $Y(z) \circ (s_i \otimes \mathrm{id}_V)$ satisfies the locality condition with any other $Y(z) \circ (\beta \otimes \mathrm{id}_V)$, $A(z)$ also satisfy the locality condition with any other $Y(z) \circ (\beta \otimes \mathrm{id}_V)$. We also have

$$A(z) \circ (\mathrm{id}_{V_i} \otimes |0\rangle) = \partial_z Y(z) \circ (s_i \otimes |0\rangle) = \partial_z e^{zT} s_i = e^{zT} T s_i = Y(z) \circ (Ts_i \otimes |0\rangle) \tag{3.61}$$

By the Goddard's uniqueness theorem, we have $\partial_z Y(z) \circ (s_i \otimes \mathrm{id}_V) = Y(z) \circ (Ts_i \otimes \mathrm{id}_V)$. This holds for any $i \in I$, which implies $Y(z) \circ (T \otimes \mathrm{id}_V) = \partial_z Y(z)$. □

We define the notion of normally ordered product

Definition 3.33. Let X_1, X_2 be two compact objects, and $A(z) = \sum_n A_n z^{-n-1}$, $B(z) = \sum_n B_n z^{-n-1}$ two fields labeled X_1 and X_2 respectively. The normally ordered product $: A(z)B(w) :$ is defined as the formal power series

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \Bigg(& \sum_{m<0} A_m \circ (\mathrm{id}_X \otimes B_n) z^{-m-1} \\ & + \sum_{m\geq 0} B_n \circ (\mathrm{id}_{X'} \otimes A_m) \circ (\sigma \otimes \mathrm{id}_V) z^{-m-1} \Bigg) w^{-n-1} \end{aligned} \tag{3.62}$$

as an element in $\mathrm{Hom}_{\mathcal{C}}(X \otimes X' \otimes V, V)[[z^{\pm}, w^{\pm}]]$. Equivalently, we have

$$: A(z)B(w) := A(z)_+ \circ (\mathrm{id}_X \otimes B(w)) + B(w) \circ (\mathrm{id}_{X'} \otimes A(z)_-) \circ (\sigma \otimes \mathrm{id}_V) \tag{3.63}$$

Lemma 3.34. *The specialization of* $: A(z)B(w) :$ *at* $w = z$ *is a well defined field labeled by* $X \otimes X'$. *Moreover,*

$$\begin{aligned} : A(w)B(w) := \mathrm{Res}_{z=0}(&\delta(z-w)_- A(z) \circ (\mathrm{id}_X \otimes B(w)) \\ &+ \delta(z-w)_+ B(w) \circ (\mathrm{id}_{X'} \otimes A(z)) \circ (\sigma \otimes \mathrm{id}_V)) \end{aligned} \tag{3.64}$$

Proof. Let us denote $C(z) =: A(z)B(z) :$. As a formal expression, $C(z) = \sum_{n\in\mathbb{Z}} C_n z^{-n-1}$ with

$$C_l = \sum_{n>l-1} A_{l-1-n} \circ (\mathrm{id}_X \otimes B_n) + \sum_{n\leq l-1} B_n \circ (\mathrm{id}_{X'} \otimes A_{l-1-n}) \circ (\sigma \otimes \mathrm{id}_V) \tag{3.65}$$

To show that each C_l is a well defined map $\mathrm{Hom}_{\mathcal{C}}(X \otimes X' \otimes V, V)$, we write $V = \operatorname{colim}_{i\in I} V_i$. We show that each $C_l \circ (\mathrm{id}_{X\otimes X'} \otimes s_i)$ is well defined.

Since both $A(z), B(w)$ are fields, we can find an integer N such that $A_n(\mathrm{id}_X \otimes s_i) = 0$, $B_n(\mathrm{id}_{X'} \otimes s_i) = 0$ for $n > N$. Therefore,

$$\begin{aligned} C_l \circ (\mathrm{id}_{X \otimes X'} \otimes s_i) = & \sum_{n=l}^{N} A_{l-1-n} \circ (\mathrm{id}_X \otimes B_n \circ (\mathrm{id}_{X'} \otimes s_i)) \\ & + \sum_{n=l-1-N}^{l-1} B_n \circ (\mathrm{id}_{X'} \otimes A_{l-1-n}) \circ (\sigma \otimes s_i) \end{aligned} \tag{3.66}$$

which is a well-defined finite sum.

For $l > N$, the summation in the first line vanishes. For the second line, we find that each $A_m \circ (\mathrm{id}_X \otimes s_i)$ is a map from the compact object $X \otimes V_i$ to V. We can further find another integer N_m such that $B_n \circ (\mathrm{id}_{X'} \otimes A_m \circ (\mathrm{id}_X \otimes s_i)) = 0$ for $n > N_m$. If we take $\bar{N} = \max\{N_m\}_{0 \leq m \leq N}$, we find that $B_n \circ (\mathrm{id}_{X'} \otimes A_{l-1-n}) \circ (\sigma \otimes s_i) = 0$ for $n > \bar{N}$ in the summation range $l - 1 - N \leq n \leq l - 1$. As a result, we find that $C_l \circ (\mathrm{id}_{X \otimes X'} \otimes s_i) = 0$ for $l > N + \bar{N} + 1$. This prove that $C(z)$ is a field.

The last identity is a simple computation

$$\mathrm{Res}_{z=0} \delta(z-w)_- A(z) \circ (\mathrm{id}_X \otimes B(w)) = A(w)_+ \circ (\mathrm{id}_X \otimes B(w)) \tag{3.67}$$

Similarly

$$\mathrm{Res}_{z=0} \delta(z-w)_+ B(w) \circ (\mathrm{id}_{X'} \otimes A(z)) = B(w) \circ (\mathrm{id}_{X'} \otimes A(z)_-) \tag{3.68}$$

□

Corollary 3.35. *We have the identity*

$$\partial_w : A(w)B(w) :=: \partial_w A(w) B(w) : + : A(w) \partial_w B(w) : \tag{3.69}$$

An important result that allows us to construct vertex algebra from generators is the Dong's lemma.

Lemma 3.36. *(Dong's Lemma) If $A(z), B(z), C(z)$ are mutually local fields, then $: A(z)B(z) :$ and $C(z)$ are also mutually local.*

Proof. The proof is the same as the Dong's Lemma for ordinary vertex algebra [FBZ04]. □

Basic examples

In this section we consider some basic examples of vertex algebra in $\mathcal{C}$.

COMMUTATIVE VERTEX ALGEBRA The easiest example of a vertex algebra is the commutative vertex algebra.

Definition 3.37. A vertex algebra $(V, |0\rangle, T, Y)$ in $\mathcal{C}$ is called commutative if the morphisms $\cdot_n : V \otimes V \to V$ vanish for all $n \geq 0$.

Definition 3.38. A differential algebra in $\mathcal{C}$ is a commutative algebra (A, m) equipped with a derivation T, i.e. a map $T : A \to A$ that satisfy the Leibniz rule $T \circ m = m \circ (T \otimes \mathrm{id} + \mathrm{id} \otimes T)$.

Proposition 3.39. *There is a one to one correspondence between commutative vertex algebra in $\mathcal{C}$ and unital differential algebra in $\mathcal{C}$.*

Proof. Given a commutative vertex algebra $(V, |0\rangle, T, Y)$, we define a unital commutative algebra structure on V. We define $m = \cdot_{-1} : V \otimes V \to V$. Let $V = \operatorname{colim}_{i \in I} V_i$ for V_i compact, and denote $s_i : V_i \to V$ the canonical inclusion map. By the skew symmetry property 3.44, we have

$$m \circ (s_i \otimes s_j) = m \circ \sigma \circ (s_i \otimes s_j) \tag{3.70}$$

for any $i, j \in I$. Notice that the map m is defined as a element in

$$m \in \mathrm{Hom}_{\mathcal{C}}(V \otimes V, V) = \lim_{(i,j) \in I \times I} \mathrm{Hom}_{\mathcal{C}}(V_i \otimes V_j, V) \tag{3.71}$$

Therefore, $m \circ (s_i \otimes s_j)$ for all $i, j \in I$ determines m. We have $m = m \circ \sigma$.

By the Borcherds identity, we have

$$m \circ (m \otimes \mathrm{id}_V) \circ (s_i \otimes s_j \otimes s_k) = m \circ (\mathrm{id}_V \otimes m) \circ (s_i \otimes s_j \otimes s_k) \tag{3.72}$$

for any $i, j, k \in I$. This implies the associativity of m. Therefore, m defines a commutative algebra structure on V. $|0\rangle$ is a unit follows from the vacuum axiom.

On the other hand, given a unital commutative algebra $(V, |0\rangle, m)$ equipped with a derivation T. We define the vertex algebra structure by

$$Y(z) = m \circ (e^{zT} \otimes \mathrm{id}_V) \tag{3.73}$$

Then we can check that all axioms of vertex algebra are satisfied. □

Given a compact object X in $\mathcal{C}$, we explicitly construct the commutative vertex algebra "generated" by X.

First we define $L_{-}X = \bigoplus_{n<0} Xt^n$ and define a operator $T : L_{-}X \to L_{-}X$ by $-\partial/\partial_t$. More precisely, $L_{-}X = \operatorname{colim}_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{i=0}^{n} X_{-i-1}$, where each

$X_{-n-1} \cong X$ is a copy of X. The operator T by definition is an element in $\prod_{n=0}^{\infty} \mathrm{Hom}(X_{-n-1}, X_{-n-2})$, which is given by

$$T = ((n+1) \cdot \mathrm{id}_X \in \mathrm{Hom}(X_{-n-1}, X_{-n-2}))_{n \in \mathbb{Z}_{\geq 0}} \tag{3.74}$$

Then we consider the symmetric algebra $S(L_-X)$ and extend T to $S(L_-X)$ by Leibniz rule. More precisely, we first define $T : T^k(L_-X) \to T^k(L_-X)$ by

$$\sum_{i=0}^{k-1} \mathrm{id}_{L_-X}^{\otimes i} \otimes T \otimes \mathrm{id}_{L_-X}^{\otimes k-i-1} : (L_-X)^{\otimes k} \to (L_-X)^{\otimes k} \tag{3.75}$$

We also define T to be 0 for $k = 0$. By abuse of notation we used the same symbol T here. It is easy to check that T commute with the symmetric idempotent e_{Sym} 3.19, $e_{\mathrm{Sym}} \circ T = T = T \circ e_{\mathrm{Sym}}$. Therefore, T defines an map $S(L_-X) \to S(L_-X)$ and it is a derivation with respect to the commutative product.

We have constructed $(S(L_-X), \cdot, T)$ as a unital commutative algebra with a derivation T, which is equivalent to a commutative vertex algebra structure.

SYMPLECTIC BOSON Next, we consider a class of vertex algebra called symplectic boson, which is also called the chiral Weyl algebra. We start with a compact symplectic object X, i.e. a compact object equipped with a symplectic form

$$\begin{gathered} \Omega : X \otimes X \to \mathbb{1} \\ \Omega \circ \sigma = -\Omega \end{gathered} \tag{3.76}$$

We define $LX = X[t^{\pm}] := \operatorname{colim}_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{i=-n-1}^{n} X_i$, where as before each $Xt^n = X_n \cong X$ is a copy of X. We can equip LX with a symplectic form, given by

$$\delta_{n,-m-1} \Omega \in \mathrm{Hom}(X_n \otimes X_m, \mathbb{1}) \tag{3.77}$$

We can use the construction in Section 3.3 and define the Weyl algebra $\mathcal{W}(LX) = (S(LX), \star)$. The Lagrangian decomposition $LX = L_+X \oplus L_-X$ induces a left $\mathcal{W}(LX)$ module structure on $S(L_-X)$.

As a object in $\mathcal{C}$ we set $V = S(L_-X)$ and define the map T in a same way as 3.74, 3.75. The vacuum is the natural map $|0\rangle : \mathbb{1} \to S(L_-X)$. The only nontrivial part is to define the vertex algebra map $Y(z)$. By definition, it suffice to construct a series of map

$$Y_k(z) : S^k(L_-X) \otimes V \to V((z)) \text{ for any } k \geq 0 \tag{3.78}$$

For $k = 0$, we define it by the vacuum axiom $Y_0(z) = \mathrm{id}_V \in \mathrm{Hom}(\mathbb{1} \otimes V, V)$.

For $k = 1$, we can define it as a collection of fields $Y(X_n, z) \in \mathrm{Hom}(X_n \otimes V, V((z)))$ for $n \leq -1$. We denote $X(z) := Y(X_{-1}, z) \in \mathrm{Hom}(X_{-1} \otimes V, V((z)))$. Let $t^k = \mathrm{id}_X \in \mathrm{Hom}(X_n, X_{n+k})$ be the identity map that only shift the index by k. We define $X(z)$ as follows

$$X(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} \star_l \circ (t^{n+1} \otimes \mathrm{id}_V) \in \mathrm{Hom}(X_{-1} \otimes V, V((z))) \tag{3.79}$$

where $\star_l$ is the module map defined in 3.34. To check that $X(z)$ is indeed an element in $\mathrm{Hom}(X_{-1} \otimes V, V((z)))$, we notice that $\star_l$ restricted to $\mathrm{Hom}(X_n \otimes S^k(\bigoplus_{i=-m-1}^{m} X_i), V)$ vanish when k, m is fixed and n large enough. Then we define the fields $Y(X_{-n-1}, z)$ by

$$Y(X_{-n-1}, z) = \frac{1}{n!}\partial_z^n X(z) \circ (t^n \otimes \mathrm{id}_V) \in \mathrm{Hom}(X_{-n-1} \otimes V, V((z))) \tag{3.80}$$

For higher k, we first define a collection of fields

$$Y(X_{-n_1-1} \dots X_{-n_k-1}, z) \in \mathrm{Hom}(X_{-n_1-1} \otimes \cdots \otimes X_{-n_k-1} \otimes V, V((z))) \tag{3.81}$$

by normally ordered product. Namely, we set

$$Y(X_{-n_1-1} \dots X_{-n_k-1}, z) = \frac{1}{n_1! \dots n_k!} :\partial_z^{n_1} X(z) \dots \partial_z^{n_k} X(z): \tag{3.82}$$

Such collection of fields defines a map $T^k(L_- X) \otimes V \to V((z))$. We use the inclusion $S^k(L_- X) \to T^k(L_- X)$ and find a map $S^k(L_- X) \otimes V \to V((z))$.

Proposition 3.40. *The data $(V, |0\rangle, T, Y)$ defined above is a vertex algebra in $\mathcal{C}$.*

Proof. First we check the vacuum axiom. $Y(z) \circ (|0\rangle \otimes \mathrm{id}_V) = \mathrm{id}_V$ holds by definition. For the remaining part $Y(z) \circ (\mathrm{id}_V \otimes |0\rangle)|_{z=0} = \mathrm{id}_V$, we check this on each $S^k(L_- X)$. For $k = 1$, we find by definition of $X(z)$ 3.79 that $X(z) \circ (\mathrm{id}_{X_{-1}} \otimes |0\rangle)|_{z=0}$ is well defined and gives $\mathrm{id}_{X_{-1}}$. Similarly, $Y(X_{-n-1}, z) \circ (\mathrm{id}_{X_{-n-1}} \otimes |0\rangle)|_{z=0} = \mathrm{id}_{X_{-n-1}}$. Then we can prove by induction that

$$Y(X_{-n_1-1} \dots X_{-n_k-1}, z) \circ (\mathrm{id}_{X_{-n_1-1} \otimes \dots X_{-n_k-1}} \otimes |0\rangle)|_z = \mathrm{id}_{X_{-n_1-1} \otimes \dots X_{-n_k-1}} \tag{3.83}$$

Then we check the translation axiom. $T \circ |0\rangle = 0$ by definition. For the remaining part of the translation axiom, we also check it on each $S^k(L_- X)$.

Notice that $T|_{X_0} = 0$, so $T \circ \pi = \pi \circ T$. Also T is a derivation of the Weyl algebra $\mathcal{W}(LX)$. For $k = 1$, we find

$$\begin{aligned} T \circ X(z) - X(z) \circ (\mathrm{id}_{X_{-1}} \otimes T) &= \sum_{n\in\mathbb{Z}} z^{-n-1} \star_l \circ (T \circ t^{n+1} \otimes \mathrm{id}_V) \\ &= \sum_{n\in\mathbb{Z}} -nz^{-n-1} \star_l \circ (T \circ t^n \otimes \mathrm{id}_V) \\ &= \partial_z X(z) \end{aligned} \tag{3.84}$$

Similarly, we can check that $T \circ Y(X_{-n-1}, z) - Y(X_{-n-1}, z) \circ (\mathrm{id}_{X_{-n-1}} \otimes T) = \partial_z Y(X_{-n-1}, z)$. We prove by induction and that ∂_z satisfy the Leibniz rule with respect to the normally ordered product.

Finally, we check the locality axiom. As implied by the Dong's lemma, we only need to check that the field $X(z)$ is local with respect to itself. We compute

$$\begin{aligned} X(z) \circ (\mathrm{id}_{X_{-1}} \otimes X(w)) &= \sum_{n,m\in\mathbb{Z}} z^{-n-1}w^{-m-1} \star_l \circ (t^{n+1} \otimes \star_l \circ (t^{m+1} \otimes \mathrm{id}_V)) \\ &= \sum_{n,m\in\mathbb{Z}} z^{-n-1}w^{-m-1} \star_l \circ (\star \circ (t^{n+1} \otimes t^{m+1}) \otimes \mathrm{id}_V) \end{aligned} \tag{3.85}$$

Therefore, the commutator is given by

$$\begin{aligned} &X(z) \circ (\mathrm{id}_{X_{-1}} \otimes X(w)) - X(w) \circ (\mathrm{id}_{X_{-1}} \otimes X(z)) \circ (\sigma \otimes \mathrm{id}_V) \\ = &\sum_{n,m\in\mathbb{Z}} z^{-n-1}w^{-m-1} \star_l \circ ([-,-]_\star \circ (t^{n+1} \otimes t^{m+1}) \otimes \mathrm{id}_V) \end{aligned} \tag{3.86}$$

where $[-,-]_\star = \star - \star \circ \sigma$. We have, by definition,

$$[-,-]_\star \circ (t^{n+1} \otimes t^{m+1}) = \delta_{n,-m-1}\omega \in \mathrm{Hom}(X_{-1} \otimes X_{-1}, \mathbb{1}) \tag{3.87}$$

This implies

$$\begin{aligned} &X(z) \circ (\mathrm{id}_{X_{-1}} \otimes X(w)) - X(w) \circ (\mathrm{id}_{X_{-1}} \otimes X(z)) \circ (\sigma \otimes \mathrm{id}_V) \\ =&\delta(z-w) l_V \circ (\omega \otimes \mathrm{id}_V) \end{aligned} \tag{3.88}$$

Hence $X(z)$ is local with respect to itself. □

Remark 3.41. From 3.85 we can easily check that

$$X(z) \circ (\mathrm{id}_{X_{-1}} \otimes X(w)) = \frac{1}{z-w} l_V \circ (\omega \otimes \mathrm{id}_V) + :X(z)X(w): \tag{3.89}$$

The singular part $\frac{1}{z-w} l_V \circ (\omega \otimes \mathrm{id}_V)$ is also called the OPE between the fields $X(z)$ and $X(w)$.

Vertex Lie algebra and universal enveloping construction

An important method to construct vertex algebra is through the universal envelope of vertex Lie algebra. Firstly, we define the notion of vertex Lie algebra in $\mathcal{C}$.

Definition 3.42. A vertex Lie algebra in $\mathcal{C}$ is an object $L \in \mathcal{C}$ equipped with a map $T \in \mathrm{Hom}_{\mathcal{C}}(L, L)$ and maps

$$Y_-(z) = \sum_{n \geq 0} \cdot_n z^{-n-1} \in \mathrm{Hom}_{\mathcal{C}}(L \otimes L, L) \otimes z^{-1}\mathbb{C}[[z^{-1}]] \tag{3.90}$$

We require that for any compact object $X \in \mathcal{C}$ and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(X, L)$, there exist an integer N such that $\cdot_n(\alpha \otimes \beta) = 0$ for $n \geq N$.

These structures are required satisfy the following axioms

1. Translation. $Y_-(z) \circ (T \otimes \mathrm{id}_L) = \partial_z Y_-(z)$.
2. Skew-symmetry. For any compact object $X \in \mathcal{C}$ and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(X, L)$, $Y_-(z) \circ (\alpha \otimes \beta) = (e^{zT} \circ Y_-(-z) \circ \sigma)_- \circ (\alpha \otimes \beta)$.
3. Commutator. For any compact object X in $\mathcal{C}$ and $\alpha, \beta, \gamma \in \mathrm{Hom}_{\mathcal{C}}(X, L)$, we have

$$\sum_{n \geq 0} \binom{m}{n} \cdot_{m+n-k} \circ (\cdot_n \otimes \mathrm{id}_L) \circ (\alpha \otimes \beta \otimes \gamma) =$$
$$\cdot_m \circ (\mathrm{id}_L \otimes \cdot_k) \circ (\alpha \otimes \beta \otimes \gamma) - \cdot_k \circ (\mathrm{id}_L \otimes \cdot_m) \circ (\sigma \otimes \mathrm{id}_L)(\alpha \otimes \beta \otimes \gamma) \tag{3.91}$$

A vertex Poisson algebra is a commutative vertex algebra together with a compatible vertex Lie algebra structure.

Definition 3.43. A vertex Poisson algebra in $\mathcal{C}$ is a unital commutative algebra in $\mathcal{C}$ with a derivation $(V, |0\rangle, m, T)$, together with a vertex Lie algebra structure $(V, |0\rangle, Y_-)$ such that

$$Y_-(z) \circ (\mathrm{id} \otimes m) = m \circ (Y_-(z) \otimes \mathrm{id}) + m \circ (\mathrm{id} \otimes Y_-(z)) \circ (\sigma \otimes \mathrm{id})$$

Given a vertex algebra $(V, |0\rangle, T, Y)$ in $\mathcal{C}$, we can construct a vertex Lie algebra simply by forgetting the operations $\cdot_n$ for $n < 0$.

Lemma 3.44. *Let $(V, |0\rangle, T, Y)$ be a vertex algebra in $\mathcal{C}$. Then $V_{\mathrm{Lie}} = (V, T, Y_-)$ is a vertex Lie algebra in $\mathcal{C}$, where*

$$Y_-(z) = \sum_{n \geq 0} \cdot_n z^{-n-1} \tag{3.92}$$

Proof. The translation axiom follows from the translation axiom of the vertex algebra V. The skew-symmetry axiom is a consequence of 3.44. The commutator axiom follows from the Borcherds identity 3.50. □

This construction defines a functor from the category of vertex algebra to the category of vertex Lie algebra. This functor actually have an left adjoint functor, whose image is called the enveloping vertex algebra.

Theorem 3.45. *Given a vertex Lie algebra L in $\mathcal{C}$, there is a vertex algebra $\mathcal{U}(L)$, such that for any vertex algebra V, there is a canonical isomorphism*

$$\mathrm{Hom}(\mathcal{U}(L), V) = \mathrm{Hom}(L, V_{\mathrm{Lie}}) \tag{3.93}$$

Some other constructions

In this section, we analyze how the notion of vertex algebra behaves under symmetric monoidal functor. We show that the image of a vertex algebra is still a vertex algebra if the functor is braided monoidal and preserve filtered colimit, i.e.

$$F(\operatorname*{colim}_{i\in I} X_i) = \operatorname*{colim}_{i\in I} F(X_i)$$

Example 3.46. Let $\mathcal{C}$ and $\mathcal{D}$ two categories as per our requirements, i.e. $\mathcal{C} \cong \mathrm{Ind}(\mathcal{C}_0), \mathcal{D} \cong \mathrm{Ind}(\mathcal{D}_0)$ and $\mathcal{C}_0, \mathcal{D}_0$ are pseudo tensor category. Let (F_0, J, ϵ) be a braided monoidal functor between $\mathcal{C}_0$ and $\mathcal{D}$. Then, F_0 has an extension

$$F : \mathrm{Ind}(\mathcal{C}_0) \to \mathcal{D} \tag{3.94}$$

that preserve filtered colimmit.

Proposition 3.47. *Let $(F, J, \epsilon) : \mathcal{C} \to \mathcal{D}$ be a braided monoidal functor that preserve filtered colimit. Given a vertex algebra $(V, |0\rangle, T, Y(z))$ in $\mathcal{C}$, we define*

1. $\widetilde{V} = F(V)$.
2. $\widetilde{|0\rangle} = F(|0\rangle) \circ \epsilon$.
3. $\widetilde{T} = F(T)$.
4. $\widetilde{Y}(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} F(\cdot_n) \circ J_{V,V}$.

Then $(\widetilde{V}, \widetilde{|0\rangle}, \widetilde{T}, \widetilde{Y}(z))$ is a vertex algebra in $\mathcal{D}$.

Proof. First we show that $\widetilde{Y}(z)$ is indeed a map to the formal Laurent series. Let $V = \operatorname*{colim}_{i\in I} V_i$ with $V_i \in \mathcal{C}_0$ and denote $s_i : V_i \to V$ the inclusion map. Take any compact object X in $\mathcal{D}$ and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{D}}(X, F(V))$. We have

$$\mathrm{Hom}_{\mathcal{D}}(X, F(V)) = \mathrm{Hom}_{\mathcal{D}}(X, \operatorname*{colim}_{i\in I} F(V_i)) = \operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathcal{D}}(X, F(V_i)) \tag{3.95}$$

As a result, we can always find a $i \in I$ and a map $\alpha' \in \mathrm{Hom}_{\mathcal{D}}(X, F(V_i))$ such that $\alpha = F(s_i) \circ \alpha'$. Similarly we can find a $j \in I$ and a map $\beta' \in \mathrm{Hom}_{\mathcal{D}}(X, F(V_j))$ such that $\beta = F(s_j) \circ \beta'$. Therefore, to check that $F(\cdot_n) \circ J_{V,V} \circ (\alpha \otimes \beta) = 0$ for large enough n, we only need to check that $F(\cdot_n) \circ J_{V,V} \circ (F(s_i) \otimes F(s_j)) = 0$ for large enough n.

Since J is a natural transformation, we have $J_{V,V} \circ (F(s_i) \otimes F(s_j)) = F(s_i \otimes s_j) \circ J_{V_i,V_j}$. Therefore $F(\cdot_n) \circ J_{V,V} \circ (F(s_i) \otimes F(s_j)) = F(\cdot_n \circ (s_i \otimes s_j)) \circ J_{V_i,V_j}$, which vanish for large enough n.

Next, we check the vacuum axiom. By abuse of notation, we write $\widetilde{Y}(z) = F(Y(z)) \circ J_{V,V}$. We have

$$\begin{aligned}\widetilde{Y}(z) \circ (\widetilde{|0\rangle} \otimes \mathrm{id}_{\widetilde{V}}) &= F((z)) \circ J_{V,V} \circ (F(|0\rangle) \circ \epsilon \otimes \mathrm{id}_{\widetilde{V}}) \\ &= F(Y(z) \circ (|0\rangle \otimes \mathrm{id}_V)) \circ J_{\mathbb{1},V} \circ (\epsilon \otimes \mathrm{id}_{\widetilde{V}}) \\ &= F(l_V) \circ J_{\mathbb{1},V} \circ (\epsilon \otimes \mathrm{id}_{\widetilde{V}}) \\ &= l_{\widetilde{V}}\end{aligned} \tag{3.96}$$

Similarly, $\widetilde{Y}(z) \circ (\mathrm{id}_{\widetilde{V}} \otimes \widetilde{|0\rangle}) = F(Y(z) \circ (\mathrm{id}_V \otimes |0\rangle)) \circ J_{V,\mathbb{1}} \circ (\mathrm{id}_{\widetilde{V}} \otimes \epsilon)$, which has no pole and specialize to $F(r_V) \circ J_{V,\mathbb{1}} \circ (\mathrm{id}_{\widetilde{V}} \otimes \epsilon) = r_{\widetilde{V}}$ at $z = 0$.

Then we check the translation axiom. $\widetilde{T} \circ \widetilde{|0\rangle} = F(T \circ |0\rangle) \circ \epsilon = 0$. By definition, $\widetilde{T} \circ \widetilde{Y}(z) = F(T \circ Y(z)) \circ J_{V,V}$. On the other hand,

$$\begin{aligned}\widetilde{Y}(z) \circ (\mathrm{id}_{\widetilde{V}} \otimes \widetilde{T}) &= F(Y(z)) \circ J_{V,V} \circ (\mathrm{id}_{\widetilde{V}} \otimes F(T)) \\ &= F(Y(z) \circ (\mathrm{id}_V \otimes T)) \circ J_{V,V}\end{aligned} \tag{3.97}$$

Therefore, $\widetilde{T} \circ \widetilde{|0\rangle} - \widetilde{Y}(z) \circ (\mathrm{id}_{\widetilde{V}} \otimes \widetilde{T}) = F(\partial_z Y(z)) \circ J_{V,V} = \partial_z \widetilde{Y}(z)$.

Finally, we check the locality axiom. As before, it suffice to check the locality for maps $F(s_i), F(s_j)$. We have

$$\begin{aligned}&\widetilde{Y}(w) \circ (\mathrm{id}_{\widetilde{V}} \otimes \widetilde{Y}(z)) \circ (F(s_i) \otimes F(s_j) \otimes \mathrm{id}_{\widetilde{V}}) \\ =&F(Y(w)) \circ J_{V,V} \circ (\mathrm{id}_{\widetilde{V}} \otimes F(Y(z)) \circ J_{V,V}) \circ (F(s_i) \otimes F(s_j) \otimes \mathrm{id}_{\widetilde{V}}) \\ =&F(Y(w)) \circ J_{V,V} \circ (F(s_i) \otimes F(Y(z) \circ (s_j \otimes \mathrm{id}_V)) \circ (\mathrm{id}_{F(V_i)} \otimes J_{V_j,V}) \\ =&F(Y(w) \circ (\mathrm{id}_V \otimes Y(z)) \circ (s_i \otimes s_j \otimes \mathrm{id}_V))) \circ J_{V_i,V_j \otimes V} \circ (\mathrm{id}_{F(V_i)} \otimes J_{V_j,V})\end{aligned} \tag{3.98}$$

Similarly,

$$\begin{aligned}&\widetilde{Y}(z) \circ (\mathrm{id}_{\widetilde{V}} \otimes \widetilde{Y}(w)) \circ (\sigma \otimes \mathrm{id}_{\widetilde{V}}) \circ (F(s_i) \otimes F(s_j) \otimes \mathrm{id}_{\widetilde{V}}) \\ =&F(Y(w) \circ (\mathrm{id}_V \otimes Y(z)) \circ (s_j \otimes s_i \otimes \mathrm{id}_V))) \circ J_{V_j,V_i \otimes V} \circ (\mathrm{id}_{F(V_j)} \otimes J_{V_i,V}) \circ (\sigma \otimes \mathrm{id}_{\widetilde{V}}) \\ =&F(Y(w) \circ (\mathrm{id}_V \otimes Y(z)) \circ (s_j \otimes s_i \otimes \mathrm{id}_V))) \circ J_{V_j \otimes V_i,V} \circ (J_{V_j,V_i} \otimes \mathrm{id}_{\widetilde{V}}) \circ (\sigma \otimes \mathrm{id}_{\widetilde{V}})\end{aligned} \tag{3.99}$$

Since F is braided, $J_{V_j,V_i} \circ \sigma = F(\sigma) \circ J_{V_i,V_j}$. Moreover, $J_{V_j \otimes V_i,V} \circ (F(\sigma) \otimes F(\mathrm{id}_V)) = F(\sigma \otimes \mathrm{id}_V) \circ J_{V_i \otimes V_j,V}$. We find

$$\begin{aligned} &\widetilde{Y}(z) \circ (\mathrm{id}_{\widetilde{V}} \otimes \widetilde{Y}(w)) \circ (\sigma \otimes \mathrm{id}_{\widetilde{V}}) \circ (F(s_i) \otimes F(s_j) \otimes \mathrm{id}_{\widetilde{V}}) \\ =&F(Y(w) \circ (\mathrm{id}_V \otimes Y(z)) \circ (s_j \otimes s_i \otimes \mathrm{id}_V) \circ (\sigma \otimes \mathrm{id}_V)) \circ J_{V_i,V_j \otimes V} \circ (\mathrm{id}_{F(V_i)} \otimes J_{V_j,V}) \end{aligned} \tag{3.100}$$

In summary, $\widetilde{[\ldots]} = F([\ldots]) \circ J_{V_i,V_j \otimes V} \circ (\mathrm{id}_{F(V_i)} \otimes J_{V_j,V})$, which vanish when we multiply it by $(z-w)^N$ for N large enough. □

For a category $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ as above, and suppose that $R = \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is a commutative ring. The functor

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, -) : \mathcal{C} \to R\text{−mod} \tag{3.101}$$

extend to a braided monoidal functor. The natural isomorphism $J_{X,Y}$: $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X) \otimes_R \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, Y) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes Y)$ is given by tensor of the two maps $\alpha \otimes \beta$ composed with the isomorphism $\mathbb{1} \cong \mathbb{1} \otimes \mathbb{1}$. It commute with the permutation σ by construction.

As a corollary, we have an easy way to construct a vertex algebra over the ring R from a vertex algebra in $\mathcal{C}$.

Corollary 3.48. *Let $(V, |0\rangle, T, Y)$ be a vertex algebra in $\mathcal{C}$. Suppose* $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) =$ *R for some commutative ring R. Then $\widetilde{V} := \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, V)$ have the structure of vertex algebra over R, defined as follows*

1. *The vacuum vector is simply $|0\rangle \in \widetilde{V}$.*
2. *The translation operator $\widetilde{T} : \widetilde{V} \to \widetilde{V}$ is constructed as composition with T: for any $\alpha \in \widetilde{V}$, $\widetilde{T}\alpha$ is the composite map $\mathbb{1} \xrightarrow{\alpha} V \xrightarrow{T} V$.*
3. *$\widetilde{Y}(z) : \widetilde{V} \otimes \widetilde{V} \to \widetilde{V}((z))$ is constructed as composition with $Y(z)$: for any $\alpha, \beta \in \widetilde{V}$, $\alpha \cdot_n \beta$ is the composite map $\mathbb{1} \xrightarrow{\alpha \otimes \beta} V \otimes V \xrightarrow{\cdot_n} V$.*

It turns out that we also have a family of vertex algebra modules over $\widetilde{V}$ labeled by compact objects of $\mathcal{C}$.

Proposition 3.49. *The functor* $\mathrm{Hom}_{\mathcal{C}}(-, V) : (\mathcal{C}^c)^{op} \to R - \mathrm{mod}$ *factor through the forgetful functor* $\widetilde{V} - \mathrm{mod} \to R - \mathrm{mod}$.

Proof. For any compact object X in $\mathcal{C}$, we denote $M_X = \mathrm{Hom}_{\mathcal{C}}(X, V)$. We construct a $\widetilde{V}$ module structure $Y_M : \widetilde{V} \otimes M_X \to M_X((z))$ on M_X. For any $\alpha \in \widetilde{V}$ and $v \in M_X$, we define $\alpha \cdot_{n,M} v$ as the composition

$$\alpha \cdot_{n,M} v : X \cong \mathbb{1} \otimes X \xrightarrow{\alpha \otimes v} V \otimes V \xrightarrow{\cdot_n} V \tag{3.102}$$

And we set $Y_M(\alpha, z)v = \sum_{n\in\mathbb{Z}} \alpha \cdot_{n,M} v\, z^{-n-1}$. Since X is a compact object, for any $\alpha \in \widetilde{V}$ and $v \in M_X$, we can find a N such that $\alpha \cdot_{n,M} v = 0$ for $n > N$. Therefore, $Y_M(\alpha, z)v$ is indeed a Laurent series.

By the identity axiom for V, we have $|0\rangle \cdot_{n,M} v = \delta_{n,-1} v$. This implies $Y_M(|0\rangle, z)v = v$ for any $v \in M_X$.

Notice that in the Borcherds identity 3.50 for V, we can choose α, β, γ to be maps from different compact objects to V (as finite sum of compact objects is compact). It immediately implies the Jacobi identity for Y_M.

For any two compact object X_1, X_2 and a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$. We can check that the map $f^* : M_{X_2} \to M_{X_1}$, defined by $f^*(v) = v \circ f$, intertwine with the module map, i.e. $\alpha \cdot_{n,M_1} f^*(v) = f^*(\alpha \cdot_{n,M_2} v)$.

□

Given a vertex algebra V in $\mathcal{C}$, we call a map $D : V \to V$ a derivation of the vertex algebra if D satisfy

$$D \circ Y(z) = Y(z) \circ (D \otimes \mathrm{id}_V) + Y(z) \circ (\mathrm{id}_V \otimes D) \tag{3.103}$$

By definition and Corollary 3.32. T is a derivation for the vertex algebra V.

Proposition 3.50. *For any map $\alpha : \mathbb{1} \to V$, the map $\alpha_{(0)} := \cdot_0 \circ (\alpha \otimes \mathrm{id}_V) \circ l_V^{-1} : V \cong \mathbb{1} \otimes V \to V$ is a derivation.*

It becomes a differential, i.e. $\alpha_{(0)}^2 = 0$ if and only if $\cdot_0(\alpha \otimes \alpha) = 0$.

Proof. By the Borcherds identity, we have

$$\alpha_{(0)} \circ \cdot_k \circ (s_i \otimes s_j) = \cdot_k \circ (\alpha_{(0)} \circ s_i \otimes s_j) + \cdot_k \circ (s_i \otimes \alpha_{(0)} \circ s_j) \tag{3.104}$$

where $s_i : V_i \to V$ is the inclusion. This identity hold for any s_i, s_j, which implies that $\alpha_{(0)}$ is a derivation. □

3.4 DELIGNE CATEGORY

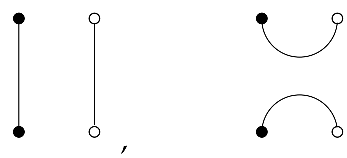
Basic construction

In this section, we review the definition of the Deligne category $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_\delta)$. We refer to [CW12, Eti16] for more detail. The Deligne category is constructed out of a "skeleton category" $\mathrm{Rep}_0(\mathrm{GL}_\delta)$ and then followed by additive envelope and Karoubi envelope. We describe $\mathrm{Rep}_0(\mathrm{GL}_\delta)$ first.

We fix $\delta \in k$. Objects of $\mathrm{Rep}_0(\mathrm{GL}_\delta)$ consist of (possibly empty) words w of two symbols $\bullet, \circ$. We denote $\mathbb{1}$ the empty word.

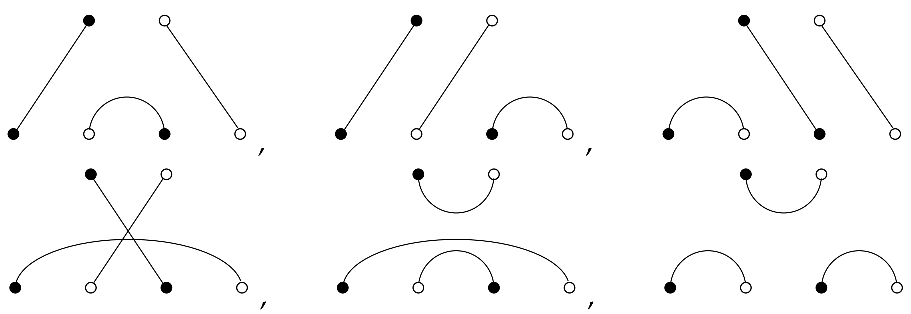
Given two words w, w', a (w, w') diagram is a graph with two rows of vertices where we set w as the first row of vertices and w' as the second row of vertices. We require that each vertices is connected to exactly one edge. An edge is connected to both a • and a ∘ vertices if and only if the two vertices are in the same row. We define the Morphism space between two words w, w' to be the k-linear space on basis $\{(w, w') -$ diagrams$\}$.

Example 3.51. We have the following two $(\bullet\circ, \bullet\circ)$ diagrams.



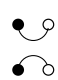
We have the following two $(\bullet \circ \bullet\circ, \mathbb{1})$ diagram

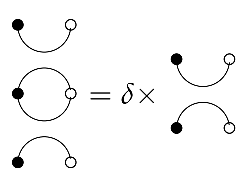
There are six $(\bullet \circ \bullet\circ, \bullet\circ)$ diagrams

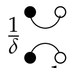


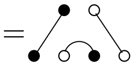
Given a (w, w') diagram X and a (w', w'') diagram Y. We let $Y * X$ be the graph obtained by stacking Y atop X, and $Y \cdot X$ the graph obtained from $Y * X$ by forgetting the middle row of vertices. Thus $Y \cdot X$ is a (w, w'')-diagram. We denote $l(X, Y)$ the number of cycles in the graph $Y * X$. Then composition of morphism in $\mathrm{Rep}_0(\mathrm{GL}_\delta)$ is defined by

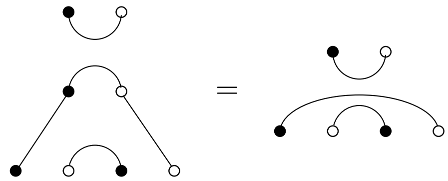
$$\begin{aligned} \mathrm{Hom}(w', w'') \times \mathrm{Hom}(w, w') &\to \mathrm{Hom}(w, w'') \\ (Y, X) &\mapsto \delta^{l(X,Y)} Y \cdot X \end{aligned} \tag{3.105}$$

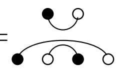
Example 3.52. Let $X =$ , we compute X^2

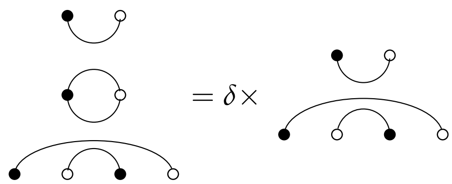


Therefore, if we let $e = \frac{1}{\delta}$ , then $e^2 = e$.

As another example, we let $Y_1 =$ . Then $X \cdot Y_1$ is given by



If we let $Y_2 =$ , then $X \cdot Y_1$ is given by



Remark 3.53. It is easy to check that two words w, w' are isomorphic if and only if they have the same number of $\bullet$ and $\circ$. Therefore, each object in $\mathrm{Rep}_0(\mathrm{GL}_\delta)$ is isomorphic to an object of the form

$$[r,s] := \underbrace{\bullet, \dots, \bullet}_{r} \underbrace{\circ, \dots, \circ}_{s} \tag{3.106}$$

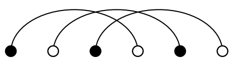
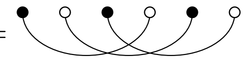
Remark 3.54. $\mathrm{End}([r,s]) = B_{r,s}(\delta)$ is the walled Brauer algebra.

Now we equip $\mathrm{Rep}_0(\mathrm{GL}_\delta)$ with the structure of a rigid symmetric monoidal category. The tensor functor $- \otimes -$ is defined as follows. On objects, $w_1 \otimes w_2 = w_1 w_2$ is simply concatenation of words. The tensor product of morphisms $X_1 \otimes X_2$ is simply the diagram obtained by placing the diagram X_1 to the left of the diagram X_2. The braiding $\sigma_{w_1,w_2} : w_1 \otimes w_2 \to w_2 \otimes w_1$ is the $(w_1 w_2, w_2 w_1)$ diagram that connect each letter of w_i in the first row to the same letter of w_i in the second row.

It is easy to check that the above definitions gives $\mathrm{Rep}_0(\mathrm{GL}_\delta)$ the structure of symmetric monoidal category.

Next, we show that $\mathrm{Rep}_0(\mathrm{GL}_\delta)$ is rigid. For any word w, we define w^* the word obtained from w by replacing all $\bullet$ with $\circ$ and vice versa. We define the morphism $\mathrm{ev}_w : w^* \otimes w \to \mathbb{1}$ the $(w^* w, \mathbb{1})$ diagram that connect the i-th letter in w^* with the i-th letter in w. Similarly, we define $\mathrm{coev}_w : \mathbb{1} \to w \otimes w^*$ the $(\mathbb{1}, ww^*)$ the $(\mathbb{1}, ww^*)$ diagram that connect the i-th letter in w with the i-th letter in w^*.

Example 3.55. For $w = \bullet \circ \bullet$, $\mathrm{ev}_{\bullet\circ\bullet}$ and $\mathrm{coev}_{\bullet\circ\bullet}$ are given by

$\mathrm{ev}_{\bullet\circ\bullet} =$  , $\mathrm{coev}_{\bullet\circ\bullet} =$ 

Finally, we can define the Deligne category $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta})$

Definition 3.56. For $\delta \in \mathbf{k}$, the Deligne category $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta})$ is the Karoubi envelope of the additive envelope of the category $\mathrm{Rep}_{0}(\mathrm{GL}_{\delta})$. The tensor structure $\mathrm{Rep}_{0}(\mathrm{GL}_{\delta})$ extend to $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta})$ in the natrual way.

The most important properties of $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta})$ are listed below:

Proposition 3.57. *(i) For $\delta \notin \mathbb{Z}$, the category $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta})$ is a semisimple abelian category.*

(ii) If $\delta \in \mathbb{Z}$ and if p, q are nonnegative integers with $p - q = \delta$, then the category $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta})$ (which is not abelian) admits a non-faithful symmetric tensor functor $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta}) \to \mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{p|q})$ to the (finite dimensional) representation category of the supergroup $GL_{p|q}$, which sends $[1,0]$ to the supervector space $V = k^{1|0}$.

(iii) The category $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta})$ has the following universal property: if $\mathcal{D}$ is a rigid symmetric monoidal category then isomorphism classes of (possibly non-faithful) symmetric tensor functors $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta}) \to \mathcal{D}$ are in bijection with isomorphism classes of objects X in $\mathcal{D}$ of dimension δ, via $F \to F([1,0])$.

Some variants

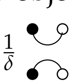
In this paper, we mostly work with the ind completion of the Deligne category $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta})$. We denote $\mathrm{Rep}(\mathrm{GL}_{\delta}) = \mathrm{Ind}(\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta}))$.

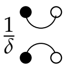
We would also like to work with a category where δ is not a number, but an indeterminate formal parameter.

Definition 3.58. We define $\mathrm{Rep}_{0}(\mathrm{GL}_{[[\delta]]})$ the $k[[\delta]]$-linear category, whose objects are still given by words w of symbols $\bullet, \circ$. The space of morphisms between two words w, w' is the space of $k[[\delta]]$-linear span of (w, w') diagrams. The composition of morphisms is defined as before, except that a loop now contribute a factor of δ.

We define $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{[[\delta]]})$ as the Karoubi envelope of additive envelope of $\mathrm{Rep}_{0}(\mathrm{GL}_{[[\delta]]})$. We also define $\mathrm{Rep}(\mathrm{GL}_{[[\delta]]}) = \mathrm{Ind}(\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{[[\delta]]}))$.

Similarly, we define the $k((\delta))$-linear version of the Deligne category $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{((\delta))})$, $\mathrm{Rep}(\mathrm{GL}_{((\delta))})$.

Remark 3.59. Although the construction for $\mathrm{Repf}(\mathrm{GL}_{[[\delta]]})$ is basically the same as $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta})$, $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{[[\delta]]})$ has far fewer objects than $\mathrm{Rep}_{\mathrm{f}}(\mathrm{GL}_{\delta})$. As a simple example, we have seen that $1 - \frac{1}{\delta}$  is an idempotent.

Therefore $\left(\bullet\circ, 1 - \frac{1}{\delta}\right.$  $\left.\right)$ is an object in $\mathrm{Rep_f}(\mathrm{GL}_\delta)$ but not an object in $\mathrm{Rep_f}(\mathrm{GL}_{[[\delta]]})$.

However, $\mathrm{Rep}(\mathrm{GL}_{[[\delta]]})$ is still compactly generated by pseudo tensor category by construction, and it make sense to define vertex algebra in it. It also satisfy the following universal property

Proposition 3.60. *The category* $\mathrm{Rep_f}(\mathrm{GL}_\delta)$ *has the following universal property: if* $\mathcal{D}$ *is a rigid symmetric monoidal category over the ring* $k[[\delta]]$*, then isomorphism classes of (possibly non-faithful) symmetric tensor functors* $\mathrm{Rep_f}(\mathrm{GL}_\delta) \to \mathcal{D}$ *are in bijection with isomorphism classes of objects* X *in* $\mathcal{D}$ *of dimension* δ*, via* $F \to F([1,0])$.

We also define the $\mathbb{Z}$ graded version of the Deligne category $\mathrm{Rep}^{\mathbb{Z}}(\mathrm{GL}_\delta)$ (or $\mathrm{Rep}^{\mathbb{Z}}(\mathrm{GL}_{[[\delta]]})$). Objects of $\mathrm{Rep}^{\mathbb{Z}}(\mathrm{GL}_\delta)$ are given by direct sum of objects in Deligne category $X = \bigoplus_{n\in\mathbb{Z}} X_n$. The tensor product is defined so that $(X \otimes Y)_n = \bigoplus_{p+q=n} X_p \otimes Y_q$. The braiding follows the Koszul sign rule, where σ_{w_1,w_2} pick up a sign $(-1)^{pq}$ for w_1 in degree p and w_2 in degree q.

3.5 VERTEX ALGEBRA IN DELIGNE CATEGORY

Motivation

In the remarkable work [BLL+15], Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees have constructed a map

$$\mathbb{V} : \{4d\ \mathcal{N} = 2 \text{ SCFTs}\} \to \{\text{Vertex algebras}\}$$

called $4d/2d$ duality. The vertex algebra $\mathbb{V}(\mathcal{T})$ can be understood as a protected sub-sector of the corresponding $4d\ \mathcal{N} = 2$ superconformal field theory $\mathcal{T}$. It contains a wealth of information and invariant of the theory $\mathcal{T}$. For example, the character of the vertex algebra $\mathbb{V}(\mathcal{T})$ coincides with the Schur index of $\mathcal{T}$. Another important conjecture is that the associated variety $X_{\mathbb{V}(\mathcal{T})}$ is isomorphic to the Higgs branch $\mathrm{Higgs}(\mathcal{T})$.

Given a $4d\ \mathcal{N} = 2$ SCFT $\mathcal{T}_{G,R}$ with vector multiplets in a gauge group G and hypermultiplets transforming in the representation R of G, the vertex algebra $\mathbb{V}(\mathcal{T}_{G,R})$ can be constructed as a BRST reduction of a $\beta\gamma$ bc system. Roughly speaking, it is given by a $\beta\gamma$ system valued in R together with a bc system valued in the adjoint $\mathfrak{g}$, subject to the BRST reduction. In this thesis, we focus on the case of $\mathcal{N} = 4$ SYM theory with gauge group G, which is the same as a $\mathcal{N} = 2$ theory with hypermultiplets transforming in the adjoint representation $\mathfrak{g}$, $\mathrm{SYM}_{\mathfrak{g}} = \mathcal{T}_{G,\mathfrak{g}}$. The corresponding VOA $\mathbb{V}(\mathrm{SYM}_{\mathfrak{g}})$ consists of a $\beta\gamma$ system

$$Z_1^a(z)Z_2^b(0) \sim \frac{\eta^{ab}}{z}$$

and a bc system

$$b^a(z)c^b(0) \sim \frac{\eta^{ab}}{z}$$

The corresponding BRST operator is given by

$$Q_{BRST} = \oint dz f_{abc} : b^a c^b c^c : (z) + f_{abc} : c^a Z_1^b Z_2^c : (z) \tag{3.107}$$

When we take $\mathfrak{g} = \mathfrak{gl}_N$, such a vertex algebra has another interpretation as the algebra of local operators on a stack of N $D1$ branes in the B model topological string.

While offering a comprehensive definition of the VOA, this description poses practical challenges due to the inherent difficulty of computing the BRST cohomology in general. Only a few examples can be explicitly computed. In this thesis, we consider a simplification by considering the "large N limit" of the VOA $\mathbb{V}(\mathrm{SYM}_{\mathfrak{gl}_N})$ and its generalization. As discussed earlier in the introduction, defining the "large N limit" straightforwardly as a limit in N is not feasible. It is also straightforward to illustrate why this is the case. First we see that the two field $\mathrm{Tr}\, Z_i(z)$, $i = 1, 2$, are both in the BRST cohomology. If we compute their OPE we find

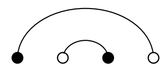
$$\mathrm{Tr}\, Z_1(z)\, \mathrm{Tr}\, Z_2(0) \sim \frac{N}{z} \tag{3.108}$$

Therefore, the map $\mathcal{A}_N \to \mathcal{A}_{N+1}$ cannot be a morphism of vertex algebra. Instead, we define the corresponding VOA in the Deligne category and then apply the functor $\mathrm{Hom}(\mathbb{1}, -)$. We'll demonstrate that the generator of the BRST cohomology can be explicitly described using cyclic homology, and the corresponding OPE possess a rich structure related to the topological string.

Main example

In this section, we use Deligne category to construct the large N vertex algebra that correspond to the $\mathcal{N} = 4$ SYM with gauge group GL_N. We work with the $\mathbb{Z}$ graded version of the Deligne category. It doesn't matter whether we use $\mathrm{Rep}^{\mathbb{Z}}(\mathrm{GL}_{[[N]]})$ or $\mathrm{Rep}^{\mathbb{Z}}(\mathrm{GL}_N)$ as our construction works in both cases.

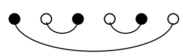
Through a slight abuse of notation, we denote c, Z_1, Z_2, b as four copies of the object $[1,1]$, in degrees $-1, 0, 0, 1$ respectively. We define a symplectic form on the sum $X = c \oplus Z_1 \oplus Z_2 \oplus b$ by (anti-)symmetrizing the following pairing on $[1,1]$


$$\in \mathrm{Hom}([1,1] \otimes [1,1], \mathbb{1}) \tag{3.109}$$

Using our previous construction, we define the chiral Weyl algebra $\mathcal{W}(X)$ in the Deligne category $\mathrm{Rep}(\mathrm{GL}_N)$. The corresponding vertex algebra $\mathrm{Hom}_{\mathrm{GL}_N}(\mathbb{1}, \mathcal{W}(X))$ is our main object of studides. Recall from Section 3.3 that $\mathcal{W}(X)$ can be identified with the object $S(\bigoplus_{n<0}(c_n \oplus Z_{1,n} \oplus Z_{2,n} \oplus b_n))$. Using the same notation as in Section Section 3.3, we denote $c(z) \in \mathrm{Hom}(c_{-1} \otimes \mathcal{W}(X), \mathcal{W}(X)(z))$ the field that correspond to the inclusion $c_{-1} \to \mathcal{W}(X)$. Similarly, we define the field $b(z), Z_1(z), Z_2(z)$.

As in 3.107, we define the BRST charge as follows

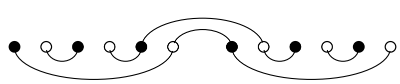
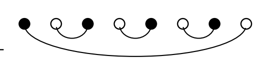
$$Q = \oint dz \, \mathrm{Tr}(: b(z)c(z)c(z) :) + \mathrm{Tr}(: c(z)Z_1(z)Z_2(z) :) \tag{3.110}$$

Remark 3.61. In the above expression, $\mathrm{Tr}(\dots)$ should be understood as the morphism  in $\mathrm{Hom}(\mathbb{1}, \dots)$.

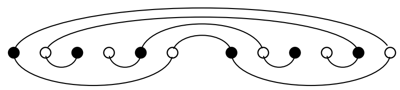
Proposition 3.62. *For Q given as above, we have $Q^2 = 0$.*

Proof. Let us denote $J(z) = \mathrm{Tr}(: b(z)c(z)c(z) :) + \mathrm{Tr}(: c(z)Z_1(z)Z_2(z) :)$. To prove $Q^2 = 0$, it suffice to show that the OPE between $J(z)$ and itself vanish. As an illustration, we compute, in a diagrammatic way, the OPE between $\mathrm{Tr}(: b(z)c(z)c(z) :)$ and itself. This is done by considering all possible Wick contractions, and each Wick contraction is represented by the diagram 3.109.

For example, the Wick contraction $\mathrm{Tr}(: bc\overbrace{c : (z))\ \mathrm{Tr}(: b}cc : (w))$ gives us


$$= \frac{1}{z-w}$$

$$= \frac{1}{z-w} \mathrm{Tr}(: bccc : (w))$$

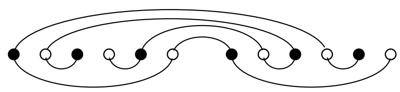
We can check that the sum of all possible single Wick contractions cancel each other and gives us 0. There are also contributions from double Wick contractions. For example, $\mathrm{Tr}(:bcc:(z))\ \mathrm{Tr}(:bcc:(w))$ gives us



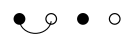
$$= \frac{N}{(z-w)^2}$$

$$= \frac{N}{(z-w)^2}\,\mathrm{Tr}(:c(z)c(w):)$$

On the other hand, $\mathrm{Tr}(:bcc:(z))\ \mathrm{Tr}(:bcc:(w))$ gives us



$$= -\frac{1}{(z-w)^2}$$



$$= -\frac{1}{(z-w)^2} :\mathrm{Tr}(c(z))\,\mathrm{Tr}(c(w)):$$

Summing all contribution together, we find

$$\mathrm{Tr}(:bcc:(z))\,\mathrm{Tr}(:bcc:(w)) \sim 2\frac{N\,\mathrm{Tr}(:c(z)c(w):) - :\mathrm{Tr}(c(z))\,\mathrm{Tr}(c(w)):}{(z-w)^2}$$

Similarly, we can compute

$$\begin{aligned}\mathrm{Tr}(:c[Z_1,Z_2]:(z))\,\mathrm{Tr}(:c[Z_1,Z_2]:(w)) \\ \sim -2\frac{N\,\mathrm{Tr}(:c(z)c(w):) - :\mathrm{Tr}(c(z))\,\mathrm{Tr}(c(w)):}{(z-w)^2} \\ -2\frac{\mathrm{Tr}(:cc[Z_1,Z_2]:(z))}{(z-w)}\end{aligned}$$

$$\mathrm{Tr}(:bcc:(z))\,\mathrm{Tr}(:c[Z_1,Z_2]:(w)) \sim \frac{\mathrm{Tr}(:cc[Z_1,Z_2]:(z))}{(z-w)}$$

It follows that different contribution cancel, and we have $J(z)J(w) \sim 0$. □

As a consequence, Q defines a differential on $\mathrm{Hom}(\mathbb{1}, \mathcal{W}(X))$. Our vertex algebra is then defined as the BRST cohomology, $(\mathrm{Hom}(\mathbb{1}, \mathcal{W}(X)), Q)$. We will explore this vertex algebra in more detail in the following sections.

An important extension of the above construction is to add a $\beta\gamma$ system valued in the (anti-)fundamental representation. We denote $I = [0,1] \otimes \mathbb{C}^{K|K}$, which is $K+K$ copies of $\circ = [0,1]$ with K of them in degree 0 and K of them in degree 1. Similarly, we define $J = [1,0] \otimes (\mathbb{C}^{K|K})^*$ as $K+K$ copies of $\bullet = [1,0]$ with K of them in degree 0 and K of them in degree -1.

We define a symplectic form on $I \oplus J$ by (anti-)symmetrizing the following morphism

 $\in \mathrm{Hom}([0,1] \otimes [1,0], \mathbb{1})$

Then we consider the corresponding chiral Weyl algebra $\mathcal{W}(X \oplus I \oplus J)$. For the new system, we add the following element to the BRST charge

$$Q_M = \oint dz \, \mathrm{Tr}_F(: I(z) c(z) J(z) :)$$

Here $\mathrm{Tr}_F(\dots)$ denote the morphism  in $\mathrm{Hom}(\mathbb{1}, \dots)$, composed with the natural pairing $(\mathbb{C}^{K|K}) \otimes (\mathbb{C}^{K|K})^* \to \mathbb{C}$.

Proposition 3.63. *We have* $(Q + Q_M)^2 = 0$.

A more general construction

In this section, we consider a generalization of the construction in the last section. Construction in this section follows a unpublished work with D. Gaiotto, A. Lopez and H. Silverans.

Let A be a compact $2d$ Calabi Yau algebra. Due to [KS06], we take A to be a unital cyclic A_∞ algebra, i.e. an A_∞ algebra $(A, m_1, m_2, \dots)$ equipped with a symmetric and non degenerate pairing $(-,-) : A \otimes A \to \mathbb{C}[2]$, such that the expression $(a_0, m_n(a_1, \dots, a_n))$ is cyclically symmetric in the graded sense.

We denote the inverse of the pairing by $\eta \in A \otimes A$. We can choose a basis $\{a_1, \dots, a_n\}$ of A and express η as $\sum \eta^{ij} a_i \otimes a_j$. By definition, we have

$$(a a_i) \eta^{ij} a_j = a$$

This element also defines a symmetric pairing on the linear dual A^*, which is given by

$$(f, g) = (f \otimes g)(\eta)$$

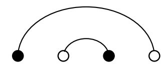
If we choose the dual basis $\mathcal{B} = \{f^1, \dots, f^n\}$ of A^*, then $(f^i, f^j) = \eta^{ij}$. We define the following object in $\mathrm{Rep}(\mathrm{GL}_{[[N]]})$

$$X_{A^*} = \bigoplus_{f_i \in \mathcal{B}} \phi^i \tag{3.111}$$

where each ϕ^i is a copy of $[1,1] = \bullet\circ$ in degree $|f_i| - 1$. X_{A^*} is equipped with a symplectic form

$$\Omega = \sum_{ij} \Omega_{ij} \in \bigoplus_{i,j} \mathrm{Hom}(\phi^i \otimes \phi^j, \mathbb{1}) \tag{3.112}$$

given by

$$\Omega_{ij} = \eta^{ij}$$  (3.113)

Then one can use our general construction to define the symplectic boson system $\mathcal{W}(X_{A^*})$ generated by the object X_{A^*}.

Lemma 3.64. *We have an isomorphism*

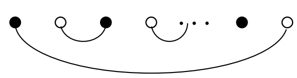
$$\mathrm{Hom}(\mathbb{1}, \mathcal{W}(X_{A^*})) \cong \mathrm{Sym}(CC^\bullet_\lambda(A[[t]])[1]) \tag{3.114}$$

Proof. Recall that we can identify $\mathcal{W}(X_A) = S([1,1] \otimes t^{-1}A^*[t^{-1}])$. Using Lemma 3.19 we have

$$\begin{aligned}\mathrm{Hom}(\mathbb{1}, \mathcal{W}(X_A)) &\cong \bigoplus_{n\geq 0}(\mathrm{Hom}(\mathbb{1}, T^n([1,1] \otimes t^{-1}A^*[t^{-1}]))^{S_n} \\ &= \bigoplus_{n\geq 0}(k[S_n] \otimes (t^{-1}A^*[t^{-1}])^{\otimes n})^{S_n} \\ &= \bigoplus_{n\geq 0}(k[S_n] \otimes \mathrm{Hom}(A[[t]]^{\otimes n}, \mathbb{C}))^{S_n}\end{aligned}$$

The rest of the proof follows from the proof of Loday-Quillen-Tsygan Theorem [Qui84, Tsy83]. □

Under the above isomorphism, a cyclic cochain $f : A[[t]]^{\otimes n} \to \mathbb{C}$ of the form $f^{i_1}t^{-k_1} \otimes f^{i_2}t^{-k_2} \otimes \dots f^{i_n}t^{-k_n}$ is mapped to the element

 $\in \mathrm{Hom}(\mathbb{1}, \phi^{i_1}_{-k_1} \otimes \dots \phi^{i_n}_{-k_n})$

We denote $\mathcal{A}(A) = \mathrm{Hom}(\mathbb{1}, \mathcal{W}(X_{A^*}))$ the corresponding vertex algebra. To define the BRST reduction, we specify an element $Q \in \mathrm{Hom}(\mathbb{1}, \mathcal{W}(X_{A^*}))$. By the previous lemma, we can define it as an element in $CC^\bullet_\lambda(A[[t]])$. We consider the following maps

$$\sum_{n\geq 1}(a_0(t) \otimes a_1(t) \otimes \dots \otimes a_n(t) \to \frac{1}{(n+1)!}(a_0(0), m_n(a_1(0), \dots a_n(0))))$$

The corresponding BRST charges can be written as follows

$$Q = \sum_{n\geq 1}\sum_{i_0,\dots,i_n} \frac{1}{(n+1)!}\oint dz (a_{i_0}, m_n(a_{i_1}, \dots, a_{i_n}))\,\mathrm{Tr}(:\phi^{i_0}(z)\dots\phi^{i_n}(z):) \tag{3.115}$$

Remark 3.65. We can let $A = \mathbb{C}[\epsilon_1, \epsilon_2]$, with both ϵ_i in degree 1. Then A is a $2d$ Calabi-Yau algebra with the map $(-) : A \to \mathbb{C}$ given by $(\epsilon_1\epsilon_2) = 1$

and 0 otherwise. It is easy to check that the construction in this section for $A = \mathbb{C}[\epsilon_1, \epsilon_2]$ reproduces the example introduced in the last section.

We can also add (anti)-fundamental representation into this construction. Let M (resp. $\widetilde{M}$) be a finite dimensional left (resp. right) A module. Suppose we have a non-degenerate pairing

$$(-,-) : \widetilde{M} \otimes M \to \mathbb{C}.$$

Using this non-degenerate pairing, we can identify $\widetilde{M}$ as the dual M^* of M and vice versa. We also assume that the identification is as A-module. This means that

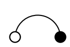
$$(\tilde{m}, m_n^M(a_1, \ldots, a_n, m)) = (m_n^{\widetilde{M}}(\tilde{m}, a_1, \ldots, a_n), m) \tag{3.116}$$

where m_n^M and $m_n^{\widetilde{M}}$ are the A module map on M and $\widetilde{M}$ respectively. We can choose a basis $\{m_j\} = \mathcal{B}_M$ of M and the dual basis $\{m^j\}$ of $\widetilde{M}$. Then the pairing becomes $(m^i, m_j) = \delta^i_j$.

We define the following objects in the Deligne category

$$X_M = \bigoplus_{m_j \in \mathcal{B}_M} I_j \oplus J^j$$

where each I_j is a copy of $[0,1]$ in degree $|m_j|$ and J^j is a copy of $[1,0]$ in degree $|m_j|$. We can define symplectic form on X_M as follows

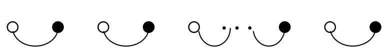
$\sum \delta^j_i$  $\in \mathrm{Hom}(I_i \otimes J^j, \mathbb{1}),$

Given this symplectic structure, we define the symplectic boson system $\mathcal{W}(X_{A^*} \oplus X_M)$.

Lemma 3.66. *We have an isomorphism (as a graded vector space)*

$$\mathrm{Hom}(\mathbb{1}, \mathcal{W}(X_{A^*} \oplus X_M)) \cong \mathrm{Sym}(CC^\bullet_\lambda(A[[t]])[1] \oplus B^\bullet(\widetilde{M}[[t]], A[[t]], M[[t]])^*) \tag{3.117}$$

Under the above isomorphism, a element in $(M[[t]] \otimes A[[t]]^{\otimes n} \otimes \widetilde{M}[[t]])$ of the form $m_j t^{-k_0} \otimes f^{i_1} t^{-k_1} \otimes \ldots f^{i_n} t^{-k_n} \otimes m^i t^{-k_{n+1}}$ is mapped to the element

 $\in \mathrm{Hom}(\mathbb{1}, I_{j,-k_0} \otimes \phi^{i_1}_{-k_1} \otimes \ldots \phi^{i_n}_{-k_n} \otimes J^i_{-k_{n+1}})$

We also add a new term Q_M into the BRST differential. By the previous lemma, we can define it as an element in $B^\bullet(\widetilde{M}[[t]], A[[t]], M[[t]])^*$. We consider the following collection of maps for $n \geq 1$

$$\tilde{m}(t) \otimes a_1(t) \cdots \otimes a_n(t) \otimes m(t) \to \frac{1}{n!}(\tilde{m}(0), m_n^M(a_1(0), \dots a_n(0), m(0)))$$

The corresponding BRST charges can be written as follows

$$Q_M = \sum_{n\geq 1} \sum_{i_0,\dots,i_n} \frac{1}{n!} \oint dz (\tilde{m}^j, m_n^M(a_{i_1} \dots, a_{i_n}, m_i)) \operatorname{Tr}(: I_j(z)\phi^{i_1}(z) \dots \phi^{i_n}(z) J^i :) \tag{3.118}$$

3.6 VERTEX POISSON ALGEBRA STRUCTURES

In this thesis, we will consider various vertex Poisson structures as different classical limits of the algebra $(\mathcal{A}(A), Q)$. First, we consider the easiest one, which corresponds to the tree level planar limit in physics terminology.

First, we add a formal parameter $\hbar$ and consider the version of Weyl algebra $\mathcal{W}_\hbar(X_A)$ with symplectic form $\hbar\omega$. We denote $\mathcal{A}_\hbar = \operatorname{Hom}(\mathbb{1}, \mathcal{W}_\hbar(X_A))$. In this section, we consider the classical limit with $\hbar \to 0$:

$$\mathcal{W}_{\hbar=0}(X_A) = \mathcal{W}_\hbar(X_A)/\hbar\mathcal{W}_\hbar(X_A)$$

And similarly define $\mathcal{A}_{\hbar=0}$.

Since the $\hbar \to 0$ limit of the Weyl algebra is simply the commutative algebra, $\mathcal{W}_{\hbar=0}(X_A)$ is a commutative vertex algebra. Then by [FBZ04], $\mathcal{W}_{\hbar=0}(X_A)$ acquires the structure of vertex Poisson algebra (in Deligne category). As a result, $\mathcal{A}_{\hbar=0}$ has the structure of Poisson vertex algebra. Denote Q_0 the differential on $\mathcal{A}_{\hbar=0}$ induced by Q. We have the following proposition that improves the results of 3.64

Proposition 3.67. *1.* $Q_0^2 = 0$.

2. We have an isomorphism of chain complexes

$$(\mathcal{A}_{\hbar=0}, Q_0) \cong (\operatorname{Sym}(CC_\lambda^\bullet(A[[z]])[1]), b) \tag{3.119}$$

where b is the Hochschild differential on $CC_\lambda^\bullet(A[[z]])$ that computes+ the cyclic cohomology.

Proof. Denote $J(z) = \sum_{n\geq 1}\sum_{i_0,\dots,i_n} \frac{1}{(n+1)!}(a_{i_0}, m_n(a_{i_1}, \dots, a_{i_n})) \operatorname{Tr}(: \phi^{i_0} \dots \phi^{i_n} : (z))$. To prove that $Q_0^2 = 0$ we only need to prove that a single Wick contraction between $J(z)$ and itself vanishes. It is easy to check that computing the Wick contraction gives us the A_∞ relation on m_n. □

We proceed to consider a more complicated classical limit. We define a three parameter family version $\mathcal{A}_{d,\hbar,N}(A)$ of the algebra. We work with the Deligne category $\mathrm{Rep}(\mathrm{GL}_{[[N]]})$, this gives us the parameters $\hbar$ and N. We consider the following Rees construction

$$\mathcal{A}_{d,\hbar,N}(A) = \bigoplus_{n\geq 0} d^{\frac{n}{2}}\mathrm{Sym}^n(CC^\bullet_\lambda(A[[z]])[1]) \subset \mathrm{Sym}(CC^\bullet_\lambda(A[[z]])[1])[[d^{\frac{1}{2}}]] \tag{3.120}$$

Then we consider the following re-parametrization

$$\mathcal{A}_{d,\lambda} = \mathcal{A}_{d,\hbar=d^{\frac{1}{2}},N=\lambda/d^{\frac{1}{2}}} \tag{3.121}$$

Proposition 3.68. *$\mathcal{A}_{d,\lambda}$ lifts to a module over $\mathbb{C}[[d,\lambda]]$. Moreover,*

$$\mathcal{A}_{d=0,\lambda} = \mathcal{A}_{d,\lambda}/d\mathcal{A}_{d,\lambda} \tag{3.122}$$

is a commutative vertex algebra.

Proof. For the first statement to hold, we need to show that the OPE coefficients for the algebra $\mathcal{A}_{d,\hbar,N}(A)$ has the following form

$$d^{\frac{a}{2}}\hbar^b N^c \tag{3.123}$$

where all a, b, c are all integers and satisfy the conditions

$$\begin{aligned} &c \geq 0, \\ &b - c \geq 0, \\ &a + b - c \text{ is a positive even integer} \end{aligned} \tag{3.124}$$

We prove this by induction on the number of Wick contractions. When there is a single Wick contraction, it only connects two single trace operators and produce one single trace operator. Thus the OPE coefficient is $d^{\frac{1}{2}}\hbar$, which satisfy the above conditions.

Now suppose we already have n wick contraction and the coefficient take the form 3.123. Now we add one more Wick contraction, which always add an extra $\hbar$ factor. It might also introduce another factor of N or $d^{\frac{1}{2}}$, which we now discuss.

When this new Wick contraction is adjacent to a previous one, we have an extra factor of N, but the number of output trace is not changed. In this case, the coefficient becomes

$$d^{\frac{a}{2}}\hbar^{b+1}N^{c+1}$$

which satisfy the conditions 3.124.

When the new Wick contraction is not adjacent to any previous one, it will not add a extra factor of N. But in this case, we either have one more trace or one less trace in the output. Then the coefficient becomes

$$d^{\frac{a+1}{2}}\hbar^{b+1}N^c \quad \text{or } d^{\frac{a-1}{2}}\hbar^{b+1}N^c$$

which also satisfy the condition 3.124. □

As a result, $\mathcal{A}_{d=0,\lambda}$ has the structure of vertex Poisson algebra. In fact, the specialization $\lambda = 0$ of the Poisson vertex algebra $\mathcal{A}_{d=0,\lambda}$ is the same as the $\mathcal{A}_{\hbar=0}$ we defined earlier. We will study the vertex Poisson algebra $\mathcal{A}_{d=0,\lambda}$ in more detail for $A = \mathbb{C}[\epsilon_1, \epsilon_2]$ later.

4 DERIVED LAURENT SERIES

The Laurent polynomial $\mathcal{O}(\mathring{\mathbb{A}}^1) = k[z, z^{-1}]$ is a fundamental object in the local structures of chiral algebra/conformal field theory over curves. For $d \geq 2$, Hartogs' theorem implies that the space of holomorphic functions on a punctured affine space is the same as the space of holomorphic functions on affine space

$$\mathcal{O}(\mathring{\mathbb{A}}^d) = \mathcal{O}(\mathbb{A}^d) \cong k[z_1, \ldots, z_d] \tag{4.1}$$

Naively, information about the singular behavior when points collide is lost in dimension $d \geq 2$. To overcome this problem, we replace the "classical" algebra of functions $\mathcal{O}(\mathring{\mathbb{A}}^d) = H^0(\mathring{\mathbb{A}}^d, \mathcal{O})$ by the derived space of functions $\mathrm{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$. We have the following well known answer

$$H^i(\mathring{\mathbb{A}}^d, \mathcal{O}) = \begin{cases} k[z_1, \ldots, z_d], & i = 0, \\ z_1^{-1} \ldots z_d^{-1} k[z_1^{-1}, \ldots, z_d^{-1}], & i = d-1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

We see that the singular data is restored, but in different cohomological degree.

As we learn from homological algebra, instead of the cohomology, we should keep track of the actual complex that gives rise to the cohomology, up to quasi-isomorphism. Indeed, the cohomology 4.2 forgets the information about the dg algebra structure on the derived space of functions $\mathrm{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$. To restore this part of information, we either need to work with a proper dg algebra model for $\mathrm{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$, or consider the A_∞ algebra structure on the cohomology $H^\bullet(\mathring{\mathbb{A}}^d, \mathcal{O})$.

In this chapter, we introduce the dg algebra model for $\mathrm{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$ called the Jouanolou model following [FHK19]. We also compute explicitly the A_∞ algebra structure on the cohomology $H^\bullet(\mathring{\mathbb{A}}^d, \mathcal{O})$ for $d = 2$ using representation theory. The latter structure can be used to construct the minimal model of the higher Kac-Moody algebra $\mathrm{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O} \otimes \mathfrak{g})$. These structure, and many of their variants, are used to define certain Poisson algebra and vertex Poisson algebra structures that are important in the twisted holography context.

4.1 THE JOUANOLOU MODEL

In this section, we introduce a dg algebra model for $\mathrm{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$, which is called the Jouanolou model. We follow the discussion in [FHK19]. First we introduce another copy of $\mathbb{A}^d$ and denote its coordinates by $(z_1^*, \dots, z_d^*)$. We write

$$zz^* := \sum_{i=1}^{d} z_i z_i^* \tag{4.3}$$

Let J denote the closed subscheme of $\mathbb{A}^{2d}$ cut out by the equation $zz^* = 1$.

$$J = \{(z_i, z_i^*) \in \mathbb{A}^{2d} \mid zz^* = 1\} \tag{4.4}$$

The affine scheme J is called the Jouanolou torsor of $\mathring{\mathbb{A}}^d$.

Lemma 4.1. *The projection onto the first factor*

$$\pi : J \to \mathring{\mathbb{A}}^d \tag{4.5}$$

is Zariski locally trivial with fibers isomorphic to $\mathbb{A}^{d-1}$.

Proof. We have an affine open cover $\{D(z_i) = \operatorname{Spec} k[z_1, \dots, z_d]_{z_i}\}_i$ of $\mathring{\mathbb{A}}^d$. For each $D(z_i)$, we have

$$\begin{aligned} \pi^{-1}(D(z_i)) &= \operatorname{Spec}\left(k[z_1, \dots, z_d, z_1^*, \dots, z_d^*]/zz^* - 1\right)_{z_i} \\ &\cong \operatorname{Spec} k[z_1, \dots, z_d]_{z_i} \otimes k[z_1^*, \dots, \hat{z_i^*}, \dots, z_d^*] \\ &= D(z_i) \times \mathbb{A}^{d-1} \end{aligned} \tag{4.6}$$

□

For any quasi-coherent sheaf E on $\mathring{\mathbb{A}}^d$ we consider the global relative de Rham complex

$$A^\bullet_{[d]}(E) = \Gamma(J, \Omega^\bullet_{J\backslash\mathring{\mathbb{A}}^d} \otimes \pi^* E) \tag{4.7}$$

Proposition 4.2. *(a)* $A^\bullet_{[d]}(E)$ *is a model for* $\mathrm{R}\Gamma(\mathring{\mathbb{A}}^d, E)$.

(b) If E is a commutative $\mathcal{O}_{\mathring{\mathbb{A}}^d}$ *algebra,* $A^\bullet_{[d]}(E)$ *is a commutative dg algebra.*

Proof. (a) Since J is affine, we have a quasi-isomorphism

$$A^\bullet_{[d]}(E) \cong \mathrm{R}\Gamma(J, \Omega^\bullet_{J\backslash\mathring{\mathbb{A}}^d} \otimes \pi^* E) \cong \mathrm{R}\Gamma(\mathring{\mathbb{A}}, R\pi_*(\Omega^\bullet_{J\backslash\mathring{\mathbb{A}}^d} \otimes \pi^* E)) \tag{4.8}$$

Moreover, since π is Zariski locally trivial with fibers isomorphic to $\mathbb{A}^{d-1}$, the map

$$E \to R\pi_*(\Omega^\bullet_{J\backslash\mathring{\mathbb{A}}^d} \otimes \pi^* E) \tag{4.9}$$

is a quasi-isomorphism of complexes of sheaves. The statement follows immediately. □

We can give a explicit description of the Jouanolou model $A^\bullet_{[d]} := A^\bullet_{[d]}(\mathcal{O}^d_{\mathring{\mathbb{A}}})$. Let $k[z,z^*] = k[z_1,\dots,z_d,z_1^*,\dots,z_d^*]$ be the algebra of regular function on $\mathring{\mathbb{A}}^{2d}$. Let

$$k[z,z^*][(zz^*)^{-1}] \tag{4.10}$$

be the localization of the polynomial algebra $k[z,z^*]$ with respect to $zz^* = \sum_{i=1}^d z_i z_i^*$.

Proposition 4.3. *The m-th graded component $A^m_{[d]}$ is identified with the space of differential forms*

$$\omega = \sum f_{i_1,\dots,i_m}(z,z^*)dz^*_{i_1}\dots dz^*_{i_m} \tag{4.11}$$

where the coefficients $f_{i_1,\dots,i_m}(z,z^*) \in k[z,z^*][(zz^*)^{-1}]$ satisfy the following two conditions

1. *The coefficients $f_{i_1,\dots,i_m}(z,z^*)$ has degree $-m$ with respect to the z^* variables.*
2. *The contraction $\iota_\xi(\omega)$ with the Euler vector field $\xi = \sum_i z_i^* \frac{\partial}{\partial z_i^*}$ vanishes.*

Under the above identification, the de Rham differential is given by

$$\bar\partial = \sum_{i=1}^d dz_i^* \frac{\partial}{\partial z_i^*} \tag{4.12}$$

Another useful presentation of the algebra $A^\bullet_{[d]}$ is through the tangential Cauchy-Riemann (CR) complex $\Omega_b^{0,\bullet}(S^{2d-1})$ of S^{2d-1}. We adopt the definition using the embedding of S^{2d-1} into $\mathbb{C}^{2d}$ (defined by the equation $r = 1$)[Fol72]. We define $\Omega_b^{0,\bullet}(S^{2d-1})$ as the quotient of $\Omega^{0,\bullet}(\mathbb{C}^{2d})|_{S^{2d-1}}$ by the ideal $I(\bar\partial r)$ generated by $\bar\partial r$. By choosing a metric $\langle -,- \rangle$ on $\mathbb{C}^{2d}$, the CR complex $\Omega_b^{0,\bullet}(S^{2d-1})$ can also be identified with the orthogonal complement of $I(\bar\partial r)$ in $\Omega^{0,\bullet}(\mathbb{C}^{2d})|_{S^{2d-1}}$.

The CR differential $\bar\partial$ can be defined as follows. Let $f \in C^\infty(S^{2d-1})$ be a function on S^{2d-1}, and f' be an extension of f to $\mathbb{C}^{2d}$. Then $\bar\partial f$ is the restriction to S^{2d-1} of

$$\bar\partial f' - \frac{\langle \bar\partial f', \bar\partial r\rangle}{\langle \bar\partial r, \bar\partial r\rangle}\bar\partial r. \tag{4.13}$$

Notice that we have a map of dg algebra from $A^\bullet_{[d]}$ to the Dolbeault complex $\Omega^{0,\bullet}(\mathbb{C}^d - \{0\})$, given by

$$\sum f_{i_1,\dots,i_m}(z,z^*)dz^*_{i_1}\dots dz^*_{i_m} \mapsto f_{i_1,\dots,i_m}(z,\bar z)d\bar z_{i_1}\dots d\bar z_{i_m} \tag{4.14}$$

By the construction of the CR complex, we have a map of dg algebra $\Omega^{0,\bullet}(\mathbb{C}^d - \{0\}) \to \Omega_b^{0,\bullet}(S^{2d-1})$. Thus we have a map $A^\bullet_{[d]} \to \Omega_b^{0,\bullet}(S^{2d-1})$. This map identifies $A^\bullet_{[d]}$ as the space of polynomial sections of $\Omega_b^{0,\bullet}(S^{2d-1})$.

Proposition 4.4. *The image of $A^\bullet_{[d]}$ under the above map is dense in the L_2 completion of $\Omega_b^{0,\bullet}(S^{2d-1})$.*

Finally, we define a higher dimensional analogue of the residue map. First we denote

$$A^{p,\bullet}_{[d]} = A^\bullet_{[d]} \otimes \Omega^p_{\mathring{\mathbb{A}}^d} \tag{4.15}$$

Explicitly, we can write

$$A^{\bullet,\bullet}_{[d]} = A^\bullet_{[d]}[dz_1, \dots, dz_d] \tag{4.16}$$

The map $A^\bullet_{[d]} \to \Omega^{0,\bullet}(\mathbb{C}^d - \{0\})$ extent to a map $A^{\bullet,\bullet}_{[d]} \to \Omega^{\bullet,\bullet}(\mathbb{C}^d - \{0\})$. We define the residue map Res : $A^{d,d-1}_{[d]}$ as

$$\mathrm{Res}(\omega) = \frac{(d-1)!}{(2\pi i)^d} \int_{S^{2d-1}} \omega \tag{4.17}$$

This residue is normalized so that the Bochner–Martinelli kernel

$$\Omega_B = \frac{\sum_{j=1}^d (-1)^{d+1} z_j^* dz_1^* \wedge \dots \widehat{dz_j^*} \wedge \cdots \wedge dz_d^*}{(zz^*)^d} \tag{4.18}$$

has residue 1

$$\mathrm{Res}(\Omega_B dz_1 \dots dz_d) = 1 \tag{4.19}$$

In this and the following sections, we also consider the d dimensional formal disk $\mathbb{D}^d = k[[z_1, \dots, z_d]]$ and formal punctured disk $\mathring{\mathbb{D}}^d = \mathbb{D}^d - \{0\}$. We denote $A^\bullet_d$ the Jouanolou model for $R\Gamma(\mathring{\mathbb{D}}^d, \mathcal{O}_{\mathring{\mathbb{D}}^d})$. The cohomology of $(A^\bullet_d, \partial)$ gives

$$H^i(\mathring{\mathbb{D}}^d, \mathcal{O}) = \begin{cases} k[[z_1, \dots, z_d]], & i = 0, \\ z_1^{-1} \dots z_d^{-1} k[z_1^{-1}, \dots, z_d^{-1}], & i = d-1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.20}$$

This is the higher dimensional analogue of the formal Laurent series $\mathbb{C}((z))$.

4.2 REPRESENTATION THEORETIC ANALYSIS

In this and the following sections, we focus on the case when $d = 2$. In this case, the SL_2 (and GL_2) group acts on $\mathbb{A}^2$ and $\mathbb{D}^2$, which induces an action on the Jouanolou torsor so that zz^* is GL_2 invariant. Therefore, the dg algebras $A^\bullet_{[2]}$ and $A^\bullet_2$ inherit an action of SL_2 (and GL_2).

It will be convenient to introduce another set of coordinates

$$\begin{aligned} w_i &= z_i, \quad \bar{w}_i = \frac{z_i^*}{zz^*} \\ \xi_i &= \frac{dz_i^*}{zz^*} \end{aligned} \tag{4.21}$$

Then we can identify

$$\begin{aligned} A^0_{[2]} &\cong \mathbb{C}[w_i, \bar{w}_i]/(w\bar{w} = 1), \\ A^1_{[2]} &\cong A^0_{[2]}\Omega_B, \end{aligned} \tag{4.22}$$

where Ω_B is the class in $A^1_{[2]}$ generated by the Bochner–Martinelli kernel

$$\Omega_B = \frac{z_1^* dz_2^* - z_2^* dz_1^*}{zz^*} = \bar{w}_1\xi_2 - \bar{w}_2\xi_1 \tag{4.23}$$

Recall that $A^0_{[2]}$ can be identified as the space of polynomial functions on S^3, therefore we have the following harmonic decomposition

$$A^0_{[2]} = \bigoplus_{j,\bar{j} \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathcal{H}_{j,\bar{j}} \tag{4.24}$$

where

$$\mathcal{H}_{j,\bar{j}} = \left\{ \begin{array}{l} \text{Harmonic polynomials that are homogeneous} \\ \text{of degree } 2j \text{ in } (w_1, w_2) \text{ and degree } 2\bar{j} \text{ in } (\bar{w}_1, \bar{w}_2) \end{array} \right\}_{j,\bar{j} \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \tag{4.25}$$

Our labeling of the space $\mathcal{H}_{j,\bar{j}}$ using half integers instead of integers seems unnatural. We chose this labeling to be compatible with the usual quantum mechanical notation of spin. In fact, $\mathcal{H}_{j,\bar{j}}$ is a spin $j + \bar{j}$ representation of SL_2.

The residue map introduced in the last section gives us an SL_2 invariant pairing $(A^0_{[2]})^{\otimes 2} \to \mathbb{C}$:

$$(\alpha, \beta) \mapsto \text{Res}(\alpha\beta\Omega_B dz_1 dz_2) \tag{4.26}$$

We can check that this pairing is non-degenerate. It will be convenient to use an orthonormal basis for each space $\mathcal{H}_{j,\bar{j}}$. Under the SL_2 action, there is a canonical choice of basis generated by the highest weight vector. For the space $\mathcal{H}_{j,0}$, we choose the orthonormal basis to be

$$\left\{ e_m^{(j)} := \sqrt{\frac{(2j+1)!}{(j+m)!(j-m)!}} w_1^{j+m} w_2^{j-m} \mid -j \le m \le j \right\}, \tag{4.27}$$

with $e_j^{(j)} = \sqrt{2j+1} w_1^{2j}$ the highest weight vector. For the space $\mathcal{H}_{0,\bar{j}}$, we choose the orthonormal basis to be

$$\left\{ \bar{e}_{\bar{m}}^{(\bar{j})} := \sqrt{\frac{(2\bar{j}+1)!}{(\bar{j}+\bar{m})!(\bar{j}-\bar{m})!}} \bar{w}_2^{\bar{j}+\bar{m}} (-\bar{w}_1)^{\bar{j}-\bar{m}} \mid -\bar{j} \le \bar{m} \le \bar{j} \right\}. \tag{4.28}$$

For $\mathcal{H}_{j,\bar{j}}$ with $j, \bar{j} \neq 0$, we denote the corresponding orthonormal basis by

$$\left\{ e_m^{(j,\bar{j})} \mid -(j+\bar{j}) \le m \le j+\bar{j} \right\}. \tag{4.29}$$

we choose the highest weight vector to be

$$e_{j+\bar{j}}^{(j,\bar{j})} = \sqrt{\frac{(2j+2\bar{j}+1)!}{(2j)!(2\bar{j})!}} w_1^{2j} \bar{w}_2^{2\bar{j}}. \tag{4.30}$$

Other elements of this basis are uniquely determined by the SL_2 action. Later we will write down explicitly this orthonormal basis in terms of harmonic polynomials.

An important structure of $A_{[2]}^{\bullet}$ is its commutative product. We want to understand how this product behaves under the decomposition $A_{[2]}^0 = \bigoplus_{j,\bar{j} \in \frac{1}{2}\mathbb{Z}_{\ge 0}} \mathcal{H}_{j,\bar{j}}$. Note that harmonic polynomials are just polynomials of the variable w_i and $\bar{w}_i$. The product in the subalgebra $\mathbb{C}[w_i]$ and $\mathbb{C}[\bar{w}_i]$ respectively is easy and is the commutative product of polynomial. The only difficult part is the product between $\mathbb{C}[w_i]$ and $\mathbb{C}[\bar{w}_i]$ after harmonic decomposition. Therefore, we first compute the product M restricted to $\mathcal{H}_{j,0} \otimes \mathcal{H}_{0,\bar{j}}$:

$$M : \mathcal{H}_{j,0} \otimes \mathcal{H}_{0,\bar{j}} \to \mathcal{H}_{j,\bar{j}} \oplus \mathcal{H}_{j-\frac{1}{2},\bar{j}-\frac{1}{2}} \oplus \cdots \oplus \begin{cases} \mathcal{H}_{j-\bar{j},0} & j > \bar{j} \\ \mathcal{H}_{0,\bar{j}-j} & j \le \bar{j} \end{cases}. \tag{4.31}$$

Denote $\pi_j = \mathcal{H}_{j,0}$. Note that $\{\pi_j\}_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}}$ enumerates all irreducible representations of SL_2. We have an isomorphism

$$\phi_{j,\bar{j}} : \pi_{j+\bar{j}} \xrightarrow{\cong} \mathcal{H}_{j,\bar{j}} \tag{4.32}$$

as SL_2 representation. Using the orthogonal basis of π_j and $\mathcal{H}_{j,\bar{j}}$ defined earlier, we have $e_m^{(j,\bar{j})} = \phi_{j,\bar{j}}(e_m^{(j+\bar{j})})$.

Note that we have the well known tensor product rule of SL_2 representations

$$CG : \pi_j \otimes \pi_{\bar{j}} \cong \pi_{j+\bar{j}} \oplus \pi_{j+\bar{j}-1} \oplus \cdots \oplus \pi_{|j-\bar{j}|}. \tag{4.33}$$

The matrix elements of the above isomorphism in the orthogonal basis are given by the SL_2 ($SU(2)$) Clebsch-Gordan coefficients $C^{j_1,j_2;j_3}_{m_1,m_2;m_3}$. Then we consider the map

$$\begin{aligned} &M \circ CG^{-1} : \\ &\pi_{j+\bar{j}} \oplus \pi_{j+\bar{j}-1} \oplus \cdots \oplus \pi_{|j-\bar{j}|} \to \mathcal{H}_{j,\bar{j}} \oplus \mathcal{H}_{j-\frac{1}{2},\bar{j}-\frac{1}{2}} \oplus \cdots \oplus \begin{cases} \mathcal{H}_{j-\bar{j},0} & j > \bar{j} \\ \mathcal{H}_{0,\bar{j}-j} & j \leq \bar{j} \end{cases}. \end{aligned} \tag{4.34}$$

Since both M and CG intertwine the SL_2 action, by Schur's lemma $M \circ CG^{-1}$ must be a constant multiple of identity on each irreducible subspace. Therefore, we have the following diagram

$$\begin{array}{ccc} & H_{j,0} \otimes \mathcal{H}_{0,\bar{j}} & \\ \swarrow^{CG} & & \searrow^{M} \\ \pi_{j+\bar{j}} & \xrightarrow{\lambda_{j,\bar{j},0}} & \mathcal{H}_{j,\bar{j}} \\ \oplus & & \oplus \\ \pi_{j+\bar{j}-1} & \xrightarrow{\lambda_{j,\bar{j},1}} & \mathcal{H}_{j-\frac{1}{2},\bar{j}-\frac{1}{2}} \\ \oplus & & \oplus \\ \vdots & & \vdots \\ \pi_{j+\bar{j}-k} & \xrightarrow{\lambda_{j,\bar{j},k}} & \mathcal{H}_{j-\frac{k}{2},\bar{j}-\frac{k}{2}} \\ \vdots & & \vdots \end{array}$$

We see that the map M is completely characterized by the constant $\lambda_{j,\bar{j},k}$. To compute each constant $\lambda_{j,\bar{j},k}$, it suffices to compute the map $M \circ CG^{-1}$ on each highest weight vector. We have

$$\begin{aligned} M \circ CG^{-1}(e^{(j+\bar{j}-k)}_{j+\bar{j}-k}) &= \sum_{m=-j}^{j} C^{j,\bar{j};j+\bar{j}-k}_{j-m,\bar{j}-k+m;j+\bar{j}-k} e^{(j)}_{j-m} \bar{e}^{(\bar{j})}_{\bar{j}-k+m} \\ &= (-1)^k \sqrt{\frac{(2j+1)!(2\bar{j}+1)!}{k!(2j+2\bar{j}-k+1)!}} e^{(j-\frac{k}{2},\bar{j}-\frac{k}{2})}_{j+\bar{j}-k}. \end{aligned} \tag{4.35}$$

Therefore,

$$\lambda_{j,\bar{j},k} = (-1)^k \sqrt{\frac{(2j+1)!(2\bar{j}+1)!}{k!(2j+2\bar{j}-k+1)!}}. \tag{4.36}$$

This gives us the following

Proposition 4.5. *The product map M on the subspace $\mathcal{H}_{j,0} \otimes \mathcal{H}_{0,\bar{j}}$ is computed by the following formula*

$$M(e^{(j)}_m, \bar{e}^{(\bar{j})}_{\bar{m}}) = \sum_{k=0}^{\min(2j,2\bar{j})} \lambda_{j,\bar{j},k} C^{j,\bar{j};j+\bar{j}-k}_{m,\bar{m};m+\bar{m}} e^{(j-\frac{k}{2},\bar{j}-\frac{k}{2})}_{m+\bar{m}}. \tag{4.37}$$

In fact, we should understand the product M as an identity in the ring $\mathbb{C}[w_i, \bar{w}_i]/(w_1\bar{w}_1 + w_2\bar{w}_2 - 1)$, that expresses a monomial of $w_i, \bar{w}_i$ as a linear combination of harmonic polynomials. As a result, the inverse of M on $\mathcal{H}_{j,\bar{j}}$:

$$M^{-1} : \mathcal{H}_{\bar{j},\bar{j}} \to \mathcal{H}_{j,0} \otimes \mathcal{H}_{0,\bar{j}} \tag{4.38}$$

should be understood as an identity that expresses the orthonormal basis of $\mathcal{H}_{j,\bar{j}}$ as a polynomial of $w_i, \bar{w}_i$. Since the Clebsch-Gordan coefficients are real and form a unitary matrix, we can easily write down the matrix elements of M^{-1}. This leads us to the following

Proposition 4.6. *The orthonormal basis of the space of harmonic polynomials $\mathcal{H}_{j,\bar{j}}$ can be written as follows*

$$\begin{aligned} e^{(j,\bar{j})}_l &= \sum_m \lambda^{-1}_{j,\bar{j},0} C^{j,\bar{j};j+\bar{j}}_{l-m,m;l} e^{(j)}_{l-m} \bar{e}^{(\bar{j})}_m \\ &= \sum_m (-1)^{\bar{j}-m} \frac{\sqrt{(2j+2\bar{j}+1)(2j)!(2\bar{j})!(j+\bar{j}+l)!(j+\bar{j}-l)}}{(j+l-m)!(j-l+m)!(\bar{j}-m)!(\bar{j}+m)!} \\ &\quad \times w_1^{j+l-m} w_2^{j-l+m} \bar{w}_1^{\bar{j}-m} \bar{w}_2^{\bar{j}+m}. \end{aligned} \tag{4.39}$$

Given the above results, one can compute the product M of any two harmonic polynomials. First, we write the harmonic polynomials into polynomials of $w_i, \bar{w}_i$ using the above formula. Then we can perform the product in the polynomial ring $\mathbb{C}[w_i, \bar{w}_i]$. Finally, we use $M|_{\mathcal{H}_{j,0}\otimes\mathcal{H}_{0,\bar{j}}}$ to decompose the polynomial into harmonic polynomials, which gives us the desired product map. Following this idea, we show in Appendix B.1 that the product of two arbitrary harmonic polynomials is given by

$$M(e_{m_1}^{(j_1,\bar{j}_1)}, e_{m_2}^{(j_2,\bar{j}_2)}) = \sum_k \begin{Bmatrix} j_1 & j_2 & j_1+j_2 \\ \bar{j}_1 & \bar{j}_2 & \bar{j}_1+\bar{j}_2 \\ j_1+\bar{j}_1 & j_2+\bar{j}_2 & j_1+j_2+\bar{j}_1+\bar{j}_2-k \end{Bmatrix}$$
$$\times \sqrt{(2j_1+1)(2j_2+1)(2\bar{j}_1+1)(2\bar{j}_2+1)(2j_1+2\bar{j}_1+1)(2j_2+2\bar{j}_2+1)}$$
$$\times \lambda^{-1}_{j_1,\bar{j}_1,0}\lambda^{-1}_{j_2,\bar{j}_2,0}\lambda_{j_1+j_2,\bar{j}_1+\bar{j}_2,k} C^{j_1+\bar{j}_1,j_2+\bar{j}_2;j_1+j_2+\bar{j}_1+\bar{j}_2-k}_{m_1,m_2;m_1+m_2} e_{m_1+m_2}^{(j_1+j_2-\frac{k}{2},\bar{j}_1+\bar{j}_2-\frac{k}{2})}, \tag{4.40}$$

where $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}$ is the Wigner $9j$ symbol.

4.3 A SPECIAL DEFORMATION RETRACT

In this section, we start to analyze the cohomology of Jouanolou model for $d = 2$. We would like to compute the A_∞ algebra structure on the cohomology $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O})$. To do this we first construct a special deformation retract on $A^\bullet_{[2]}$ and then apply the homotopy transfer theorem.

The easiest way to construct such a deformation retract is to utilize the SL_2 representation studied in previous section. We need to understand how the differential $\bar{\partial}$ of the Jouanolou model behaves under the harmonic decomposition. The operator $\bar{\partial}$ commute with SL_2 action. By Schur's lemma, on each SL_2 irreducible subspace, $\bar{\partial}$ is either zero or a scalar multiple of the identity onto an irreducible subspace of the same SL_2 representation. Therefore, it suffices to look at the action of $\bar{\partial}$ on the highest weight vector on each irreducible subspace. Using the reparametrization in 4.21, we have

$$\bar{\partial}(w_1^{2j}\bar{w}_2^{2\bar{j}}) = 2\bar{j}w_1^{2j}\bar{w}_2^{2\bar{j}-1}\xi_2 = 2\bar{j}w_1^{2j+1}\bar{w}_2^{2\bar{j}-1}\Omega_B. \tag{4.41}$$

We find that

$$\bar{\partial}: \begin{array}{l} \mathcal{H}_{j,0} \to 0 \\ \mathcal{H}_{j,\bar{j}} \xrightarrow{\cong} \mathcal{H}_{j+\frac{1}{2},\bar{j}-\frac{1}{2}}\Omega_B \end{array}. \tag{4.42}$$

Under the orthonormal basis, the differential $\bar{\partial}$ is given by the following constant

$$\bar{\partial}|_{\mathcal{H}_{j,\bar{j}}\to\mathcal{H}_{j+\frac{1}{2}}\Omega_B,\bar{j}-\frac{1}{2}} = \sqrt{2\bar{j}(2j+1)}. \tag{4.43}$$

Given this knowledge about $\bar{\partial}$, it is easy to reproduce the known result 4.20

$$H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathcal{H}_{j,0},$$
$$H^1(\mathring{\mathbb{A}}^2, \mathcal{O}) = \bigoplus_{\bar{j} \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathcal{H}_{0,\bar{j}}\Omega_B. \tag{4.44}$$

We define the homotopy operator $h : A^1_{[2]} \to A^0_{[2]}$ by "inverse" of $\bar{\partial}$ as follows

$$h : \begin{array}{l} \mathcal{H}_{0,\bar{j}}\Omega_B \to 0 \\ \mathcal{H}_{j,\bar{j}}\Omega_B \xrightarrow{\simeq} \mathcal{H}_{j-\frac{1}{2},\bar{j}+\frac{1}{2}} \end{array}, \tag{4.45}$$

so that h acts on the highest weight vector by

$$h(w_1^{2j}\bar{w}_2^{2\bar{j}}\Omega_B) = \frac{1}{2\bar{j}+1} w_1^{2j-1}\bar{w}_2^{2\bar{j}+1}. \tag{4.46}$$

Under the orthonormal basis 4.30, h is given by the following constant

$$h_{j,\bar{j}} := h|_{\mathcal{H}_{j,\bar{j}}\Omega_B \to \mathcal{H}_{j-\frac{1}{2},\bar{j}+\frac{1}{2}}} = \frac{1}{\sqrt{2j(2\bar{j}+1)}}. \tag{4.47}$$

We can verify that

$$i \circ p - 1 = \bar{\partial} \circ h + h \circ \bar{\partial}, \tag{4.48}$$

where i and p are the standard inclusion and projection between $A^\bullet_{[2]}$ and $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O})$.

We can also verify that

$$h \circ i = 0, \ p \circ h = 0, \ h \circ h = 0. \tag{4.49}$$

As a result, we have constructed a special deformation retract (SDR)

$$h \circlearrowright (A^\bullet_{[2]}, \bar{\partial}) \underset{i}{\overset{p}{\rightleftarrows}} (H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O}), 0). \tag{4.50}$$

We give more details about the definition of SDR in Appendix A, where we also provide its connection with homotopy transfer theorem.

It is important to note that the SDR we constructed is compatible with the pairing on $A^\bullet_{[2]}$. One easily check that $\mathrm{Res}(d^2z h(a) b) = \mathrm{Res}(d^2z a h(b))$. Under this extra condition, it is shown in [Kaj07] that the transferred A_∞ structure on the cohomology is also a cyclic A_∞ algebra. The bilinear pairing on the cohomology is simply given by the restriction of the original bilinear pairing.

4.4 A_∞ STRUCTURE ON THE COHOMOLOGY: m_2 AND m_3

In this and the next section, we analyze the algebraic structure on the cohomology

$$H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O}) \tag{4.51}$$

induced from the dg algebra structure on $A^\bullet_{[2]}$. We first look at the induced product structure m_2 on $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O})$. For the degree 0 part, the product on $H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) = \mathbb{C}[w_1, w_2]$ is simply the commutative product of the polynomial algebra. By degree reason, the only other nonzero product is $H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) \otimes H^1(\mathring{\mathbb{A}}^2, \mathcal{O}) \to H^0(\mathring{\mathbb{A}}^2, \mathcal{O})$, which is given by

$$m_2 = p \circ M : \ \mathcal{H}_{j,0} \otimes \mathcal{H}_{0,\bar{j}}\Omega_B \to \begin{cases} 0 & \text{if } j > \bar{j}, \\ \mathcal{H}_{0,\bar{j}-j}\Omega_B & \text{if } j \leq \bar{j}. \end{cases} \tag{4.52}$$

Using the formula for M from the last section, we find that

$$m_2(w_1^p w_2^q, \frac{(r+s+1)!}{r!s!} \bar{w}_1^r \bar{w}_2^s \Omega_B) = \frac{(r+s-p-q+1)!}{(r-p)!(s-q)!} \bar{w}_1^{r-p} \bar{w}_2^{s-q} \Omega_B. \tag{4.53}$$

Remark 4.7. Alternatively, we can identify $H^1(\mathring{\mathbb{A}}^2, \mathcal{O}) = \mathbb{C}[\bar{w}_1, \bar{w}_2]\Omega_B$ with the (degree shifted) dual of $H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) = \mathbb{C}[w_1, w_2]$ via the residue pairing 4.26. Then the commutative product of $\mathbb{C}[w_1, w_2]$ induces a dual map

$$m_2' : H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) \otimes H^1(\mathring{\mathbb{A}}^2, \mathcal{O}) \to H^1(\mathring{\mathbb{A}}^2, \mathcal{O}), \tag{4.54}$$

which is defined by $m_2'(f, g\Omega_B)(h) = (f \cdot h, g\Omega_B)$ for $f, h \in H^0(\mathring{\mathbb{A}}^2, \mathcal{O})$ and $g\Omega_B \in H^1(\mathring{\mathbb{A}}^2, \mathcal{O})$.

The fact that m_2' and m_2 are the same is a consequence of the cyclic property of m_2: $(h, m_2(f, g\Omega_B)) = (g\Omega_B, m_2(f, h))$. This fact will be particularly useful when we consider the higher products m_n on the cohomology.

Now we can proceed to consider the higher structure on the cohomology $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O})$.

Proposition 4.8. *There exist a nontrivial A_∞ structure (actually a C_∞ structure) $\{m_n\}_{n\geq 2}$ on $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O})$, such that the A_∞ algebra $(H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O}), \{m_n\}_{n\geq 2})$ is A_∞ quasi-isomorphic to the dg commutative algebra $(A^\bullet_{[2]}, \bar{\partial}, \cdot)$*

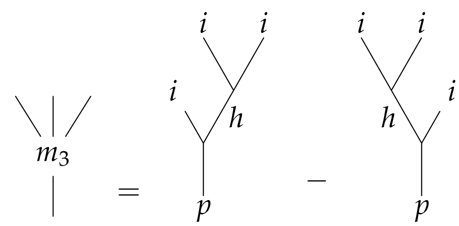
The existence of this A_∞ structure is a corollary of the homotopy transfer theorem. Since $(A^\bullet_{[2]}, \bar{\partial}, \cdot)$ is a dg commutative algebra, the transferred structure is also a C_∞ algebra [ZG06]. The fact that this A_∞ structure is nontrivial is shown in [Pol03] in a more general context.

The A_∞ operation m_n can be construed as follows

$$m_n = \sum_{T \in PBT_n} (\pm) m_T. \tag{4.55}$$

Here the summation is taken over all rooted planar binary trees T with n leaves. The map m_T is construed by assigning the product map M on the vertices, h on the internal edges, i on the leaves and p on the roots.

In this section, we warm up by computing the product m_3 on $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O})$. It is given by the following trees



Explicitly, we have

$$m_3(a,b,c) = pM(a, hM(b,c)) - pM(hM(a,b),c), \tag{4.56}$$

where we omit the inclusion i for simplicity.

Since $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O})$ is concentrated in degree 0 and 1, and m_3 is of degree -1 by definition, we have that m_3 is non zero only in the following subspace of $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O})^{\otimes 3}$

$$\begin{aligned} &\bigoplus_{\text{perm}} H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) \otimes H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) \otimes H^1(\mathring{\mathbb{A}}^2, \mathcal{O}), \\ &\bigoplus_{\text{perm}} H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) \otimes H^1(\mathring{\mathbb{A}}^2, \mathcal{O}) \otimes H^1(\mathring{\mathbb{A}}^2, \mathcal{O}), \end{aligned} \tag{4.57}$$

where we sum over all permutations of the tensor factors. Due to the cyclic structure, m_3 on the different subspaces are related by

$$(a_0, m_3(a_1, \bar{a}_0\Omega_B, \bar{a}_1\Omega_B)) = -(\bar{a}_1\Omega_B, m_3(a_0, a_1, \bar{a}_0\Omega_B)) \tag{4.58}$$

Therefore it suffices to only compute m_3 on $\bigoplus_{\text{perm}} H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) \otimes H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) \otimes H^1(\mathring{\mathbb{A}}^2, \mathcal{O})$.

First we consider m_3 restricted on $\mathcal{H}_{j_1,0} \otimes \mathcal{H}_{j_2,0} \otimes \mathcal{H}_{0,\bar{j}}\Omega_B$. Because $h = 0$ restricted on $H^0(\mathring{\mathbb{A}}^2, \mathcal{O})$, $pM(hM(a,b),c) = 0$ for $a, b \in H^0(\mathring{\mathbb{A}}^2, \mathcal{O})$. Therefore, m_3 is given by $pM(a, hM(b,c))$ in this case.

If $\bar{j} < j_2$, $pM(-, hM(-,-))$ is given by the following composition of maps

$$\begin{aligned}\mathcal{H}_{j_1,0}\otimes\mathcal{H}_{j_2,0}\otimes\mathcal{H}_{0,\bar{j}}\Omega_B &\overset{1\otimes M}{\cong} \mathcal{H}_{j_1,0}\otimes(\mathcal{H}_{j_2,\bar{j}}\Omega_B\oplus\mathcal{H}_{j_2-\frac{1}{2},\bar{j}-\frac{1}{2}}\Omega_B\oplus\cdots\oplus\mathcal{H}_{j_2-\bar{j},0}\Omega_B)\\ &\overset{1\otimes h}{\to} \mathcal{H}_{j_1,0}\otimes(\mathcal{H}_{j_2-\frac{1}{2},\bar{j}+\frac{1}{2}}\Omega_B\oplus\mathcal{H}_{j_2-1,\bar{j}}\Omega_B\oplus\cdots\oplus\mathcal{H}_{j_2-\bar{j}-\frac{1}{2},\frac{1}{2}}\Omega_B)\\ &\overset{pM}{\to}\mathcal{H}_{j_1+j_2-\bar{j}-1,0}.\end{aligned} \tag{4.59}$$

If $\bar{j} \geq j_2$, the formula for computing m_3 takes the same form. However, there is a slight difference in the last summand

$$\mathcal{H}_{j_1,0}\otimes\mathcal{H}_{j_2,0}\otimes\mathcal{H}_{0,\bar{j}}\Omega_B \overset{1\otimes M}{\cong} \mathcal{H}_{j_1,0}\otimes(\mathcal{H}_{j_2,\bar{j}}\Omega_B\oplus\mathcal{H}_{j_2-\frac{1}{2},\bar{j}-\frac{1}{2}}\Omega_B\oplus\cdots\oplus\mathcal{H}_{0,\bar{j}-j_2}\Omega_B). \tag{4.60}$$

Note that $\mathcal{H}_{0,\bar{j}-j_2}$ is sent to 0 after we apply h.

We compute $m_3(e^{(j_1)}_{m_1}, e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j})}_{\bar{m}}\Omega_B)$ following these steps. According to the above formula, we first need to compute $hM(e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j})}_{\bar{m}}\Omega_B)$. We have

$$h(M(e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j})}_{\bar{m}}\Omega_B)) = \sum_{i=0}^{\min(2\bar{j},2j_2-1)} h_{j_2-\frac{i}{2},\bar{j}-\frac{i}{2}}\lambda_{j_2,\bar{j},i}C^{j_2,\bar{j};j_2+\bar{j}-i}_{m_2,\bar{m},m_2+\bar{m}}e^{(j_2-\frac{i+1}{2},\bar{j}-\frac{i-1}{2})}_{m_2+\bar{m}}. \tag{4.61}$$

To compute the product $pM(e^{(j_1)}_{m_1}, e^{(j_2-\frac{i+1}{2},\bar{j}-\frac{i-1}{2})}_{m_2+\bar{m}})$, we can use a variation of the formula 4.39 of M^{-1} to write $e^{(j_2-\frac{i+1}{2},\bar{j}-\frac{i-1}{2})}_{m_2+\bar{m}}$ as a polynomial

$$e^{(j_2-\frac{i+1}{2},\bar{j}-\frac{i-1}{2})}_{m_2+\bar{m}} = \sum_{m'}\lambda^{-1}_{j_2-\frac{1}{2},\bar{j}+\frac{1}{2},i}C^{j_2-\frac{1}{2},\bar{j}+\frac{1}{2};j_2+\bar{j}-i}_{m',m_2+\bar{m}-m';m_2+\bar{m}}e^{(j_2-\frac{1}{2})}_{m'}\bar{e}^{(\bar{j}+\frac{1}{2})}_{m_2+\bar{m}-m'}. \tag{4.62}$$

Then we find that

$$\begin{aligned}pM(e^{(j_1)}_{m_1}, e^{(j_2-\frac{i+1}{2},\bar{j}-\frac{i-1}{2})}_{m_2+\bar{m}}) &= \sum_{m'}(-1)^{2\bar{j}+1}\lambda^{-1}_{j_2-\frac{1}{2},\bar{j}+\frac{1}{2},i}\sqrt{\frac{2j_2(2j_1+1)(2\bar{j}+2)}{2j_1+2j_2}}\\ &\quad\times C^{j_2-\frac{1}{2},\bar{j}+\frac{1}{2};j_2+\bar{j}-i}_{m',m_2+\bar{m}-m';m_2+\bar{m}}C^{j_1,j_2-\frac{1}{2};j_1+j_2-\frac{1}{2}}_{m_1,m';m_1+m'}C^{j_1+j_2-\frac{1}{2},\bar{j}+\frac{1}{2};j_1+j_2-\bar{j}-1}_{m_1+m',m_2+\bar{m}-m';m_1+m_2+\bar{m}}e^{(j_1+j_2-\bar{j}-1)}_{m_1+m_2+\bar{m}}\\ &=(-1)^{2j_1+2j_2-i+1}\lambda^{-1}_{j_2-\frac{1}{2},\bar{j}+\frac{1}{2},i}\sqrt{2j_2(2j_1+1)(2\bar{j}+2)(2j_2+2\bar{j}-2i+1)}\\ &\quad\times C^{j_2+\bar{j}-i,j_1;j_1+j_2-\bar{j}-1}_{m_2+\bar{m},m_1;m_1+m_2+\bar{m}}\begin{Bmatrix} j_2-\frac{1}{2} & \bar{j}+\frac{1}{2} & j_2+\bar{j}-i\\ j_1+j_2-\bar{j}-1 & j_1 & j_1+j_2-\frac{1}{2}\end{Bmatrix}e^{(j_1+j_2-\bar{j}-1)}_{m_1+m_2+\bar{m}},\end{aligned} \tag{4.63}$$

where $\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}$ is the Wigner $6j$-Symbol.

Combining the above results, we find that $m_3(e^{(j_1)}_{m_1}, e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j})}_{\bar{m}}\Omega_B)$ is given by

$$m_3(e^{(j_1)}_{m_1}, e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j})}_{\bar{m}}\Omega_B) = \sum_{i=0}^{\min(2\bar{j}, 2j_2-1)} (-1)^{2j_1+2j_2-i+1}$$
$$\times \begin{Bmatrix} j_2 - \frac{1}{2} & \bar{j} + \frac{1}{2} & j_2 + \bar{j} - i \\ j_1 + j_2 - \bar{j} - 1 & j_1 & j_1 + j_2 - \frac{1}{2} \end{Bmatrix} C^{j_2,\bar{j};j_2+\bar{j}-i}_{m_2,\bar{m},m_2+\bar{m}} C^{j_2+\bar{j}-i,j_1;j_1+j_2-\bar{j}-1}_{m_2+\bar{m},m_1;m_1+m_2+\bar{m}}.$$
$$\times \sqrt{\frac{(2j_1+1)2j_2(2j_2+1)(2j_2+2\bar{j}-2i+1)}{(2j_2-i)(2\bar{j}-i+1)}} e^{(j_1+j_2-\bar{j}-1)}_{m_1+m_2+\bar{m}} \tag{4.64}$$

In fact, the above result is sufficient to determine all values of m_3. We have

$$m_3(\bar{e}^{(\bar{j})}_{\bar{m}}\Omega_B, e^{(j_1)}_{m_1}, e^{(j_2)}_{m_2}) = -m_3(e^{(j_2)}_{m_2}, e^{(j_1)}_{m_1}, \bar{e}^{(\bar{j})}_{\bar{m}}\Omega_B), \tag{4.65}$$
$$m_3(e^{(j_1)}_{m_1}, \bar{e}^{(\bar{j})}_{\bar{m}}\Omega_B, e^{(j_2)}_{m_2}) = m_3(e^{(j_1)}_{m_1}, e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j})}_{\bar{m}}\Omega_B) - m_3(e^{(j_2)}_{m_2}, e^{(j_1)}_{m_1}, \bar{e}^{(\bar{j})}_{\bar{m}}\Omega_B). \tag{4.66}$$

Then we use the cyclic property:

$$\left(\bar{e}^{(\bar{j}_2)}_{\bar{m}_2}\Omega_B, m_3(e^{(j_1)}_{m_1}, e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j}_1)}_{\bar{m}_1}\Omega_B)\right) = \left(e^{(j_1)}_{m_1}, m_3(e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j}_1)}_{\bar{m}_1}\Omega_B, \bar{e}^{(\bar{j}_2)}_{\bar{m}_2}\Omega_B)\right). \tag{4.67}$$

This determines the value of m_3 on $H^0(\mathring{A}^2, \mathcal{O}) \otimes H^1(\mathring{A}^2, \mathcal{O}) \otimes H^1(\mathring{A}^2, \mathcal{O})$.

For our later application, it will be more convenient to use the value of m_3 in a different basis. We make the following change of variable

$$\begin{cases} p = j_1 + m_1 \\ q = j_1 - m_1 \end{cases}, \begin{cases} r = j_2 + m_2 \\ s = j_2 - m_2 \end{cases}, \begin{cases} u_1 = \bar{j}_1 - \bar{m}_1 \\ v_1 = \bar{j}_1 + \bar{m}_1 \end{cases}, \begin{cases} u_2 = \bar{j}_2 - \bar{m}_2 \\ v_2 = \bar{j}_2 + \bar{m}_2 \end{cases}, \tag{4.68}$$

with the constraint that

$$u_1 + u_2 = p + r - 1, \quad v_1 + v_2 = q + s - 1. \tag{4.69}$$

This constraint is equivalent to $\bar{j}_1 + \bar{j}_2 = j_1 + j_2 - 1$, $\bar{m}_1 + \bar{m}_2 = -(m_1 + m_2)$. Then we define the constant $(m_3)^{p,q;r,s}_{u_1,v_1;u_2,v_2}$ by the following

$$(m_3)^{p,q;r,s}_{u_1,v_1;u_2,v_2} := \frac{(-1)^{\bar{j}_1-\bar{m}_1} N(j_1,m_1) N(j_2,m_2)}{N(\bar{j}_1,\bar{m}_1) N(\bar{j}_2,\bar{m}_2)} \left(\bar{e}^{(\bar{j}_2)}_{\bar{m}_2}\Omega_B, m_3(e^{(j_1)}_{m_1}, e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j}_1)}_{\bar{m}_1}\Omega_B)\right). \tag{4.70}$$

where $N(j,m) = \sqrt{\frac{(j-m)!(j+m)!}{(2j+1)!}}$. This expression is non zero given the constraint 4.69.

The constant $(m_3)^{p,q;r,s}_{u_1,v_1;u_2,v_2}$ can be regarded as the value of m_3 in a unnormalized basis. We have

$$m_3(w_1^p w_2^q, w_1^r w_2^s, \frac{(u_1+v_1+1)!}{u_1!v_1!}\bar{w}_1^{u_1}\bar{w}_2^{v_1}\Omega_B) = (m_3)^{p,q;r,s}_{u_1,v_1;u_2,v_2} w_1^{u_2} w_2^{v_2}. \quad (4.71)$$

Using the relation 4.65 and the cyclic property, we find that the constant $(m_3)^{p,q;r,s}_{u_1,v_1;u_2,v_2}$ satisfy the following relation

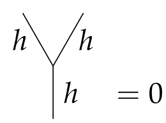
$$(m_3)^{p,q;r,s}_{u_1,v_1;u_2,v_2} = -(m_3)^{r,s;p,q}_{u_2,v_2;u_1,v_1}. \quad (4.72)$$

4.5 A_∞ STRUCTURE ON THE COHOMOLOGY: m_n

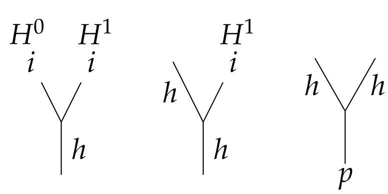
In this section, we analyze all the higher products m_n on the cohomology $H^\bullet(\mathring{A}^2, \mathcal{O})$. First, by degree reason, the n-th product is only non zero on the following subspace of $H^\bullet(\mathring{A}^2, \mathcal{O})^{\otimes n}$

$$\begin{aligned} &\bigoplus_{\text{perm}} H^0(\mathring{A}^2, \mathcal{O})^{\otimes 2} \otimes H^1(\mathring{A}^2, \mathcal{O})^{\otimes n-2}, \\ &\bigoplus_{\text{perm}} H^0(\mathring{A}^2, \mathcal{O}) \otimes H^1(\mathring{A}^2, \mathcal{O})^{\otimes n-1}. \end{aligned} \quad (4.73)$$

The fact that $H^\bullet(\mathring{A}^2, \mathcal{O})$ is concentrated in degree $0, 1$ and the homotopy operator h decreases the degree by 1 strongly restricts possible trees that contribute to the higher product m_n. A tree that contains the following vertex must be zero



As a result, for any tree that gives a non-zero map, all vertices must be directly connected to a leaf or the root. Moreover, the product map is zero on $(A^1_{[2]})^{\otimes 2}$. Therefore, for $n \geq 3$, a tree that gives a non-zero map must only consist of the following vertices:



Only a few trees survive under this condition.

First we consider the map m_n on $H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) \otimes H^1(\mathring{\mathbb{A}}^2, \mathcal{O})^{\otimes n-1}$. In this case, only one tree contribute, which is given by

$$\begin{aligned} &m_n(a_0, \bar{a}_1\Omega_B, \ldots, \bar{a}_{n-1}\Omega_B) \\ &= pM(hM(\ldots hM(hM(a_0, \bar{a}_1\Omega_B), \bar{a}_2\Omega_B), \ldots, \bar{a}_{n-1}\Omega_B)), \end{aligned} \tag{4.74}$$

for $a_0 \in H^0(\mathring{\mathbb{A}}^2, \mathcal{O})$ and $\bar{a}_1\Omega_B, \ldots, \bar{a}_{n-1}\Omega_B H^1(\mathring{\mathbb{A}}^2, \mathcal{O})$.

We emphasize that the order of input elements does matter in a higher operation m_n. Therefore, the above formula does not directly apply to other cases when we insert $a_0 \in H^0(\mathring{\mathbb{A}}^2, \mathcal{O})$ in a different slot of the map m_n. However, since since $(H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O}), \{m_n\}_{n\geq 2})$ is a C_∞ algebra, all other cases can be derived from the result of $m_n(a_0, \bar{a}_1\Omega_B, \ldots, \bar{a}_{n-1}\Omega_B)$. We have the following

$$\begin{aligned} &m_n(\bar{a}_{n-1}\Omega_B, \ldots, \bar{a}_{k+1}\Omega_B, a_0, \bar{a}_1\Omega_B, \ldots, \bar{a}_k\Omega_B) \\ &= \sum_{\sigma \in Sh(k, n-1-k)} (\pm) m_n(a_0, \bar{a}_{\sigma^{-1}(1)}\Omega_B, \ldots, \bar{a}_{\sigma^{-1}(n-1)}\Omega_B), \end{aligned} \tag{4.75}$$

where $Sh(k, n-1-k)$ the subset of $(k, n-1-k)$-shuffles in S_{n-1}.

For $n \geq 3$, let us denote

$$\begin{aligned} &\mu_n(a_0; \bar{a}_1\Omega_B, \ldots, \bar{a}_{n-1}\Omega_B) \\ &= M(hM(\ldots hM(hM(a_0, \bar{a}_1\Omega_B), \bar{a}_2\Omega_B), \ldots, \bar{a}_{n-1}\Omega_B)). \end{aligned} \tag{4.76}$$

Then $m_n(a_0, \bar{a}_1\Omega_B, \ldots, \bar{a}_{n-1}\Omega_B) = p\mu_n(a_0, \bar{a}_1\Omega_B, \ldots, \bar{a}_{n-1}\Omega_B)$. The purpose of defining μ_n is that it can be computed iteratively as follows

$$\mu_n(a_0; \bar{a}_1\Omega_B, \ldots, \bar{a}_{n-1}\Omega_B) = M(h\mu_{n-1}(a_0; \bar{a}_1\Omega_B, \ldots, \bar{a}_{n-2}\Omega_B), \bar{a}_{n-1}\Omega_B). \tag{4.77}$$

Computation of μ_n is similar to the computation of m_3 in the last section. First we compute $\mu_3(e^{(j_0)}_{m_0}, \bar{e}^{(\bar{j}_1)}_{m_1}\Omega_B, \bar{e}^{(\bar{j}_2)}_{m_2}\Omega_B)$. Using B.7, we have

$$\begin{aligned} &\mu_3(e^{(j_0)}_{m_0}, \bar{e}^{(\bar{j}_1)}_{m_1}\Omega_B, \bar{e}^{(\bar{j}_2)}_{m_2}\Omega_B) = \sum_{i_1, i_2} h_{j_0 - \frac{i_1}{2}, \bar{j}_1 - \frac{i_1}{2}} \lambda_{j_0, \bar{j}_1, i_1} \lambda^{-1}_{j_0 - \frac{1}{2}, \bar{j}_1 + \frac{1}{2}; i_1} \lambda_{j_0 - \frac{1}{2}, \bar{j}_1 + \bar{j}_2 + \frac{1}{2}; i_2} \\ &\sqrt{(2\bar{j}_1 + 2)(2\bar{j}_2 + 1)(2j_0 + 2\bar{j}_1 - 2i_1 + 1)} \begin{Bmatrix} \bar{j}_1 + \frac{1}{2} & j_0 - \frac{1}{2} & j_0 + \bar{j}_1 - i_1 \\ j_0 + \bar{j}_1 + \bar{j}_2 - i_2 & \bar{j}_2 & \bar{j}_1 + \bar{j}_2 + \frac{1}{2} \end{Bmatrix} \\ &\times C^{j_0, \bar{j}_1; j_0 + \bar{j}_1 - i_1}_{m_0, m_1; m_0 + m_1} C^{j_0 + \bar{j}_1 - i_1, \bar{j}_2; j_0 + \bar{j}_1 + \bar{j}_2 - i_0}_{m_0 + m_1, m_2; m_0 + m_1 + m_2} e^{(j_0 - \frac{1}{2} - \frac{i_2}{2}, \bar{j}_1 + \bar{j}_2 + \frac{1}{2} - \frac{i_2}{2})}_{m_0 + m_1 + m_2} \Omega_B. \end{aligned} \tag{4.78}$$

In general, $\mu_n(a_0, \bar{a}_1\Omega_B, \ldots, \bar{a}_{n-1}\Omega_B)$ takes the following form

$$\mu_n(e^{(j_0)}_{m_0}, \bar{e}^{(\bar{j}_1)}_{m_1}\Omega_B, \ldots \bar{e}^{(\bar{j}_{n-1})}_{m_{n-1}}\Omega_B) = \sum_{i_1,\ldots,i_{n-1}} (\mu_n)^{j_0,\bar{j}_1\ldots,\bar{j}_{n-1}}_{i_1,\ldots,i_{n-1}} \left(\prod_{l=0}^{n-2} C^{J_l-i_l,\bar{j}_{l+1},J_{l+1}-i_{l+1}}_{M_l,m_{l+1},M_{l+1}}\right) e^{(j_0-\frac{n-2}{2}-\frac{i_{n-1}}{2},\bar{J}_{n-1}+\frac{n-2}{2}-\frac{i_{n-1}}{2})}_{M_{n-1}}\Omega_B, \tag{4.79}$$

where $i_0 = 0$ and we define

$$\begin{aligned} J_l &= j_0 + \bar{j}_1 + \cdots + \bar{j}_l, \quad l \geq 0, \\ \bar{J}_l &= J_l - j_0, \quad l \geq 1, \\ M_l &= m_0 + \cdots + m_l, \quad l \geq 0. \end{aligned} \tag{4.80}$$

Using the recursion relation 4.77 and the formula B.7, we find that

$$\begin{aligned} &(\mu_n)^{j_0,\bar{j}_1\ldots,\bar{j}_{n-1}}_{i_1,\ldots,i_{n-1}} \\ &= \prod_{l=2}^{n-1} h_{j_0-\frac{l-2}{2}-\frac{i_{l-1}}{2},\bar{J}_{l-1}+\frac{l-2}{2}-\frac{i_{l-1}}{2}} \sqrt{(2\bar{J}_{l-1}+l)(2\bar{j}_l+1)(2J_{l-1}-2i_{l-1}+1)} \\ &\times \left(\prod_{l=1}^{n-2} \lambda_{j_0-\frac{l-1}{2},\bar{J}_l+\frac{l-1}{2};i_l} \lambda^{-1}_{j_0-\frac{l}{2},\bar{J}_l+\frac{l}{2};i_l}\right) \lambda_{j_0-\frac{n-2}{2},\bar{J}_{n-1}+\frac{n-2}{2};i_{n-1}} \\ &\times \prod_{l=2}^{n-1} \begin{Bmatrix} \bar{J}_{l-1}+\frac{l_1}{2} & j_0-\frac{l-1}{2} & J_{l-1}-i_{l-1} \\ J_l-i_l & \bar{j}_l & \bar{J}_l+\frac{l-1}{2} \end{Bmatrix}. \end{aligned} \tag{4.81}$$

The above expression simplifies to

$$(\mu_n)^{j_0,\bar{j}_1\ldots,\bar{j}_{n-1}}_{i_1,\ldots,i_{n-1}} = \lambda_{j_0-\frac{n-2}{2},\bar{J}_{n-1}+\frac{n-2}{2};i_{n-1}} \prod_{l=2}^{n-1} \sqrt{\frac{(2j_0-l+3)(2\bar{j}_l+1)(2J_{l-1}-2i_{l-1}+1)}{(2j_0-l+2-i_{l-1})(2\bar{J}_{l-1}+l-1-i_{l-1})}} \begin{Bmatrix} \bar{J}_{l-1}+\frac{l-1}{2} & j_0-\frac{l-1}{2} & J_{l-1}-i_{l-1} \\ J_l-i_l & \bar{j}_l & \bar{J}_l+\frac{l-1}{2} \end{Bmatrix}. \tag{4.82}$$

We have the following formula for $m_n(e^{(j_0)}_{m_0}, \bar{e}^{(\bar{j}_1)}_{m_1}\Omega_B, \ldots \bar{e}^{(\bar{j}_{n-1})}_{m_{n-1}}\Omega_B)$

$$\begin{aligned} m_n(e^{(j_0)}_{m_0}, \bar{e}^{(\bar{j}_1)}_{m_1}\Omega_B, \ldots \bar{e}^{(\bar{j}_{n-1})}_{m_{n-1}}\Omega_B) &= \sum_{i_1,\ldots,i_{n-2}} (\mu_n)^{j_0,\bar{j}_1\ldots,\bar{j}_{n-1}}_{i_1,\ldots,i_{n-1}} \\ \times \left(\prod_{l=0}^{n-2} C^{J_l-i_l,\bar{j}_{l+1},J_{l+1}-i_{l+1}}_{M_l,m_{l+1},M_{l+1}}\right)\Bigg|_{\substack{i_0=0;\\ i_{n-1}=2j_0-n+2}} &\bar{e}^{(\bar{J}_{n-1}-j_0+n-2)}_{M_{n-1}}\Omega_B. \end{aligned} \tag{4.83}$$

Here, the range of summation is taken to be

$$0 \leq i_l \leq \min\{2j_0 - l, 2\bar{J}_l + l - 1\}, \quad \text{for } l = 1, \ldots n-2. \tag{4.84}$$

However, the actual range of summation is much smaller due to the constraint of the Wigner $6j$ symbol and the Clebsch–Gordan coefficients. For example, the summand is nonzero only when

$$i_l \geq i_{l-1}, \quad \text{for } l = 2, \dots n-1. \tag{4.85}$$

Using the cyclic structure, the above result is sufficient to determine the whole m_n. For example, we have the following

$$\begin{aligned} &\left(e^{(j_1)}_{m_1}, m_n(e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j}_1)}_{m_1}\Omega_B, \dots \bar{e}^{(\bar{j}_{n-1})}_{m_{n-1}}\Omega_B)\right) \\ &= \left(\bar{e}^{(\bar{j}_{n-1})}_{m_{n-1}}\Omega_B, m_n(e^{(j_1)}_{m_1}, e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j}_1)}_{m_1}\Omega_B, \dots \bar{e}^{(\bar{j}_{n-2})}_{m_{n-2}}\Omega_B)\right) \end{aligned} \tag{4.86}$$

This determines the value of m_n on $H^0(\mathbb{\mathring{A}}^2, \mathcal{O})^{\otimes 2} \otimes H^1(\mathbb{\mathring{A}}^2, \mathcal{O})^{\otimes n-2}$. We can use the C_∞ property to determine all the remaining values of m_n.

It will be convenient to define the constant

$$\begin{aligned} &(m_n)^{p,q;r,s}_{u_1,v_1;\dots;u_{n-1},v_{n-1}} \\ &= \left(\prod_{i=1}^{n-1} \frac{(u_i+v_i+1)!}{u_i! v_i!}\right) \left(w_1^p w_2^q, m_n(w_1^r w_2^s, \bar{w}_1^{u_1}\bar{w}_2^{v_1}\Omega_B, \dots, \bar{w}_1^{u_{n-1}}\bar{w}_2^{v_{n-1}}\Omega_B)\right) \\ &= \left(\prod_{l=1}^{2} \sqrt{\frac{(j_l+m_l)!(j_l-m_l)!}{(2j_l+1)!}}\right) \left(\prod_{l=1}^{n-1} \sqrt{\frac{(2\bar{j}_l+1)!}{(\bar{j}_l+\bar{m}_l)!(\bar{j}_l-\bar{m}_l)!}}\right) \\ &\quad \times \left(e^{(j_1)}_{m_1}, m_n(e^{(j_2)}_{m_2}, \bar{e}^{(\bar{j}_1)}_{m_1}\Omega_B, \dots \bar{e}^{(\bar{j}_{n-1})}_{m_{n-1}}\Omega_B)\right) \end{aligned} \tag{4.87}$$

The variables p, q, r, s, u_i, v_i and $j_i, m_i, \bar{j}_i, \bar{m}_i$ are related as follows

$$\begin{cases} p = j_1 + m_1 \\ q = j_1 - m_1 \end{cases}, \quad \begin{cases} r = j_2 + m_2 \\ s = j_2 - m_2 \end{cases}, \quad \begin{cases} u_i = \bar{j}_i - \bar{m}_i \\ v_i = \bar{j}_i + \bar{m}_i \end{cases}. \tag{4.88}$$

We also have the constraint $u_1 + \dots u_{n-1} = p + r - n + 2$, $v_1 + \dots v_{n-1} = q + s - n + 2$ on the variable, which is equivalent to

$$\bar{j}_1 + \bar{j}_2 + \dots + \bar{j}_{n-1} = j_1 + j_2 - n + 2, \quad \bar{m}_1 + \bar{m}_2 + \dots \bar{m}_n = -(m_1 + m_2). \tag{4.89}$$

4.6 A NON-COMMUTATIVE DEFORMATION

In this section, we consider the Moyal deformation of the structure sheaf $\mathcal{O}_{\mathbb{\mathring{A}}^2}$ with respect to the standard Poisson bivector

$$\Pi = \partial_{z_1} \wedge \partial_{z_2} \tag{4.90}$$

We denote $\mathcal{O}_{\mathring{\mathbb{A}}^2,c}$ the non-commutative deformation $(\mathcal{O}_{\mathring{\mathbb{A}}^2}[[c]], *_c)$, where c is the formal deformation parameter and the star product $*_c$ can be expressed by

$$\begin{aligned} f *_c g &= m \circ e^{\frac{c\Pi}{2}}(f \otimes g) \\ &= fg + \frac{c}{2}\epsilon_{ij}\partial_{z_i} f \partial_{z_j} + (\frac{c}{2})^2 \epsilon_{i_1 j_1}\epsilon_{i_2 j_2}\partial_{z_{i_1}}\partial_{z_{i_2}} f \partial_{z_{j_1}}\partial_{z_{j_2}} g + \dots \end{aligned} \tag{4.91}$$

This non-commutative deformation induces a non-commutative deformation to the Jouanolou model. We denote $A^\bullet_{[2],c} = A^\bullet_{[d]}(\mathcal{O}_{\mathring{\mathbb{A}}^2,c})$. As a dg associative algebra, $A^\bullet_{[2],c}$ has the same underlying dg vector space as $A^\bullet_{[2]}[[c]]$, but equipped with a deformed product M_c given by the same formula 4.91. In the remainder of this section, we analyze the deformed A_∞ structure on the cohomology $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O}_{\mathring{\mathbb{A}}^2,c})$.

As a first step, we analyze the deformed product M_c under the Harmonic decomposition. In the easiest case, we consider $M_c|_{\mathcal{H}_{j,0}\otimes\mathcal{H}_{j',0}}$

$$M_c(w_1^p w_2^q, w_1^r w_2^s) = \sum_{k\geq 0} (\frac{c}{2})^k \frac{1}{k!} R_k(p,q,r,s) w_1^{p+r-k} w_2^{q+s-k} \tag{4.92}$$

where the constant $R_k(p,q,r,s)$ is defined as

$$R_k(p,q,r,s) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} [p]_{k-i}[q]_i[r]_i[s]_{k-i} \tag{4.93}$$

We also use the descending Pochhammer symbol $[a]_n = \frac{a!}{(a-n)!}$.

We can also write it using the orthonormal basis

$$M_c(e^{(j_1)}_{m_1}, e^{(j_2)}_{m_2}) = \sum_{k\geq 0} \frac{(c/2)^k C^{j_1,j_2;j_1+j_2-k}_{m_1,m_2;m_1+m_2}}{(2j_1+2j_2-2k+1)} \sqrt{\frac{[2j_1+1]_{k+1}[2j_2+1]_{k+1}}{k![2j_1+2j_2-2k]_{k-1}}} e^{(j_1+j_2-k)}_{m_1+m_2} \tag{4.94}$$

To obtain the full A_∞ structure on $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O}_{\mathring{\mathbb{A}}^2,c})$, we need more information about the product M_c. For example, we also need to compute $M_c(e^{(j_1)}_{m_1}, \bar{e}^{(\bar{j}_2)}_{\bar{m}_2}\Omega_B)$. Recall that $\bar{e}^{(\bar{j}_2)}_{\bar{m}_2}\Omega_B \propto \frac{\bar{z}_1^{\bar{j}_2-\bar{m}_2}\bar{z}_2^{\bar{j}_2+\bar{m}_2}}{(z_1\bar{z}_1+z_2\bar{z}_2)^{2\bar{j}_2+1}})(z_1^* dz_2^* - z_2^* dz_1^*)$. Therefore

$$\epsilon_{ij}\partial_{z_i} e^{(j_1)}_{m_1}\partial_{z_j}\bar{e}^{(\bar{j}_2)}_{\bar{m}_2}\Omega_B = \frac{2\bar{j}_2+1}{z_1\bar{z}_1+z_2\bar{z}_2}(\bar{z}_2\partial_{z_1} - \bar{z}_1\partial_{z_2})(e^{(j_1)}_{m_1})\bar{e}^{(\bar{j}_2)}_{\bar{m}_2}\Omega_B \tag{4.95}$$

We can define an operator $\eta : \mathcal{H}_{j,\bar{j}} \to \mathcal{H}_{j-\frac{1}{2},\bar{j}-\frac{1}{2}}$ by the formula $\eta = \bar{w}_2\partial_{w_1} - \bar{w}_1\partial_{w_2}$. The action of η on $A^\bullet_{[2]}[[c]]$ is completely determined by its action on the highest weight vector. We have $\eta(w_1^{2j}\bar{w}_2^{2\bar{j}}) = 2jw_1^{2j-1}\bar{w}_2^{2\bar{j}+1}$. Therefore,

$$\eta(e_m^{(j,\bar{j})}) = \sqrt{(2j)(2\bar{j}+1)}e_m^{(j-\frac{1}{2},\bar{j}+\frac{1}{2})} \tag{4.96}$$

Then we can simplify $M_c(e_{m_1}^{(j_1)}, \bar{e}_{\bar{m}_2}^{(\bar{j}_2)}\Omega_B)$ as follows

$$M_c(e_{m_1}^{(j_1)}, \bar{e}_{\bar{m}_2}^{(\bar{j}_2)}\Omega_B) = \sum_{k\geq 0}(\frac{c(2\bar{j}_2+1)}{2})^k M(\eta^k e_{m_1}^{(j_1)}, \bar{e}_{\bar{m}_2}^{(\bar{j}_2)}\Omega_B) \tag{4.97}$$

More generally, we have

$$\begin{aligned} M_c(e_{m_1}^{(j_1,\bar{j}_1)}, \bar{e}_{\bar{m}_2}^{(\bar{j}_2)}\Omega_B) &= \sum_{k\geq 0}(\frac{c(2\bar{j}_2+1)}{2})^k M(\eta^k e_{m_1}^{(j_1,\bar{j}_1)}, \bar{e}_{\bar{m}_2}^{(\bar{j}_2)}\Omega_B) \\ &= \sum_{k\geq 0}(\frac{c(2\bar{j}_2+1)}{2})^k \sqrt{[2j_1]_k[2\bar{j}_1+k]_k} M(e_{m_1}^{(j_1-\frac{k}{2},\bar{j}_1+\frac{k}{2})}, \bar{e}_{\bar{m}_2}^{(\bar{j}_2)}\Omega_B) \end{aligned} \tag{4.98}$$

while the $M(e_{m_1}^{(j_1-\frac{k}{2},\bar{j}_1+\frac{k}{2})}, \bar{e}_{\bar{m}_2}^{(\bar{j}_2)}\Omega_B)$ can be computed by the formula B.7. Then we can, in principle, use these formula to compute the A_∞ structure $\{m_2^c, m_3^c, \dots\}$on $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O}_{\mathring{\mathbb{A}}^2,c})$.

4.7 MINIMAL MODEL FOR HIGHER KAC-MOODY ALGEBRA

Given a Lie algebra $\mathfrak{g}$, the formal current algebra $\mathfrak{g}((z))$ and its central extension plays a fundamental role in the study of Kac-Moody vertex algebra. The generalization of $\mathfrak{g}((z))$ to higher dimension is developed in [FHK19], where the authors considered the current algebra

$$\mathfrak{g}_d^\bullet := \mathfrak{g} \otimes A_d^\bullet \tag{4.99}$$

and its central extension. $\mathfrak{g}_d^\bullet$ is a dg Lie algebra, and its natural to consider its minimal model. Namely, we consider the transferred L_∞ algebra structure on the cohomology

$$H^\bullet(\mathring{\mathbb{A}}^2, \mathfrak{g} \otimes \mathcal{O}) \tag{4.100}$$

One way to obtain the L_∞ structure on the cohomology is to directly apply the homotopy transfer theorem to the dg Lie algebra $\mathfrak{g}_d^\bullet$. However, as we have already computed the C_∞ structure on $H^\bullet(\mathring{\mathbb{A}}^2, \mathcal{O})$, it would save us a lot of work if we can construct the L_∞ structure by considering the tensor product. This lead us to the more general question of the existence of L_∞ algebra structure on the tensor product between a Lie algebra and a C_∞ algebra.

This is proved in [Rob17] with a more general statement. Let $\Psi : \mathcal{P} \to \mathcal{L}$ be a morphism between two dg operad. Suppose $\mathcal{P}$ is augmented and Koszul, and each $\mathcal{P}(n)$ is finite dimensional, then the main theorem in [Rob17] state that there is a morphism of operad

$$\mathcal{L}ie_\infty \to \mathcal{L} \otimes \mathcal{P}^!_\infty \tag{4.101}$$

where $\mathcal{P}^!_\infty$ is the resolution of the Koszul dual operad of $\mathcal{P}$. Moreover, such construction is compatible with the homotopy transfer theorem.

If we let $\mathcal{L} = \mathcal{P} = \mathcal{L}ie$ and Ψ the identity morphism of $\mathcal{L}ie$ operad. This result implies that there is a canonical L_∞ structure on the tensor product between a Lie algebra and a C_∞ algebra.

In our cases, an explicit formula can be easily obtained by thinking about the homotopy transfer of the Lie algebra $A^\bullet_{[2]} \otimes \mathfrak{g}$. The homotopy transfer from a Lie algebra to a L_∞ algebra is similar to the homotopy transfer of associative algebra, except that we consider all (not necessarily planar) binary rooted trees and we replace the product map with the Lie bracket on each vertex. In our cases, the condition on non zero trees discussed in the last section still hold. Therefore, up to a permutation of the leaves, the trees that contribute to the L_∞ operations are the same as in the last section. As a result, we have the following formula for the higher bracket l_n restricted on $(H^0(\mathring{\mathbb{A}}^2, \mathcal{O}) \otimes \mathfrak{g}) \otimes S^{n-1}(H^1(\mathring{\mathbb{A}}^2, \mathcal{O}) \otimes \mathfrak{g})$

$$\begin{aligned} & l_n(a_0 \otimes x_0, \bar{a}_1\Omega_B \otimes x_1, ..., \bar{a}_{n-1}\Omega_B \otimes x_{n-1}) \\ = & \sum_{\sigma \in S_{n-1}} m_n(a_0, \bar{a}_{\sigma(1)}\Omega_B, ..., \bar{a}_{\sigma(n-1)}\Omega_B)[...[[x_0, x_{\sigma(1)}], x_{\sigma(2)}], ..., x_{\sigma(n-1)}], \end{aligned} \tag{4.102}$$

where $a_0 \in H^0(\mathring{\mathbb{A}}^2, \mathcal{O}), \bar{a}_i\Omega_B \in H^1(\mathring{\mathbb{A}}^2, \mathcal{O})$ and $x_i \in \mathfrak{g}$. We have that $|S_{n-1}| = (n-1)! = \dim \mathcal{L}ie(n)$. In fact, the set of elements

$$\{[...[[x_0, x_{\sigma(1)}], x_{\sigma(2)}], ..., x_{\sigma(n-1)}], \sigma \in S_{n-1}\}$$

form a basis of the Lie operad $\mathcal{L}ie(n)$ and are called Dynkin elements.

Similar construction applies if we replace $A^\bullet_2$ by its non commutative generalization $A^\bullet_{2,c}$. However, since $A^\bullet_{2,c}$ is no longer commutative, we cannot choose arbitrary Lie algebra $\mathfrak{g}$ in the tensor product $A^\bullet_{2,c} \otimes \mathfrak{g}$. In this case, we can only choose $\mathfrak{g}$ to be $\mathfrak{gl}_K$, which comes from the associative algebra $\mathrm{Mat}_{K\times K}$. We apply 4.101 to $\mathcal{L} = \mathcal{P} = \mathcal{A}ss$ and Ψ the identity. Since $\mathcal{A}ss^! = \mathcal{A}ss$, 4.101 implies that there is an L_∞ structure on tensor product between an associative algebra and a A_∞ algebra. Let us denote m^c_n the A_∞

structure map on $H^\bullet(A^\bullet_{2,c})$, then the L_∞ structure on $H^\bullet(A^\bullet_{2,c}) \otimes \mathrm{Mat}_{K\times K}$ can be written as follows

$$l^c_n(a_0 \otimes x_0, a_1 \otimes x_1, ..., a_{n-1} \otimes x_{n-1}) \\ = \sum_{\sigma \in S_n} m^c_n(a_{\sigma^{-1}(0)}, a_{\sigma^{-1}(1)}, ..., a_{\sigma^{-1}(n-1)}) \otimes (x_{\sigma^{-1}(0)} x_{\sigma^{-1}(1)} \dots x_{\sigma^{-1}(n-1)}) \tag{4.103}$$

where $a_i \in H^\bullet(A^\bullet_{2,c})$ and $x_i \in \mathfrak{gl}_K$

4.8 polyvector fields

Another important construction is the algebra of polyvector fields on $\mathring{\mathbb{A}}^d$

$$\mathrm{PV}^{p,\bullet}_{[d]} = A^\bullet_{[d]}(\wedge^p T_{\mathring{\mathbb{A}}^d}), \quad \mathrm{PV}^{p,\bullet}_{d} = A^\bullet_{d}(\wedge^p T_{\mathring{\mathbb{D}}^d}) \tag{4.104}$$

Explicitly, we have

$$\mathrm{PV}^{p,\bullet}_{[d]} = A^\bullet_{[d]}[\partial_{z_1}, \dots, \partial_{z_d}], \quad \mathrm{PV}^{p,\bullet}_{d} = A^\bullet_{d}[\partial_{z_1}, \dots, \partial_{z_d}] \tag{4.105}$$

with ∂_{z_i} in degree $(1,0)$.

One can equip the dg algebra $\mathrm{PV}^{p,\bullet}_{[d]}$ (and $\mathrm{PV}^{p,\bullet}_{d}$) a dg BV algebra structure. To do this we choose a holomorphic volume form $\Omega_{\mathring{\mathbb{A}}^d}$, which is typically chosen as the standard one $\Omega_{\mathring{\mathbb{A}}^d} = dz_1 \dots dz_d$. Then we can identify $\mathrm{PV}^{j,i}_{[d]}$ with $A^{d-j,i}_{[d]}$ via contraction with $\Omega_{\mathring{\mathbb{A}}^d}$:

$$\begin{aligned} \mathrm{PV}^{j,i}_{[d]} &\cong A^{d-j,i}_{[d]} \\ \alpha &\mapsto \alpha \vee \Omega_{\mathring{\mathbb{A}}^d} \end{aligned} \tag{4.106}$$

The differential $\partial : A^{j,i}_{[d]} \to A^{j+1,i}_{[d]}$ then induce a differential on $\mathrm{PV}^{j,i}_{[d]}$ via the above isomorphism

$$\partial : \mathrm{PV}^{j,i}_{[d]} \to \mathrm{PV}^{j-1,i}_{[d]} \tag{4.107}$$

Explicitly, given the identification 4.105, ∂ can be expressed as follows

$$\partial = \sum_{i=1}^{d} \frac{\partial}{\partial(\partial_{z_i})} \frac{\partial}{\partial z_i} \tag{4.108}$$

The ∂ operator on polyvector fields is not a derivation with respect to the product structure. The failure of it being a derivation can be measured by a bracket

$$\{\alpha, \beta\} := \partial(\alpha \wedge \beta) - (\partial\alpha) \wedge \beta - (-1)^{|\alpha|} \alpha \wedge (\partial\beta), \tag{4.109}$$

which coincides with the Schouten-Nijenhuis bracket on polyvector fields (up to a sign). Explicitly, this bracket is given by the following expression

$$\{\alpha, \beta\} = \sum_{i=1}^{d} \frac{\partial \alpha}{\partial(\partial_{z_i})} \frac{\partial \beta}{\partial z_i} + (-1)^{|\alpha|} \frac{\partial \alpha}{\partial z_i} \frac{\partial \beta}{\partial(\partial_{z_i})} \tag{4.110}$$

The fundamental algebraic structures of polyvector fields on Calabi-Yau geometry can be summarized by saying that the tuple $\{\mathrm{PV}^{\bullet,\bullet}_{[d]}, \bar{\partial}, \wedge, \partial, \{-,-\}\}$ defines a differential graded Batalin-Vilkovisky algebra.

Using the isomorphism 4.106, it is easy to find that the cohomology $H^\bullet(\mathrm{PV}^{\bullet,\bullet}_{[d]}, \bar{\partial})$ is given as follows

$$H^i(\mathrm{PV}^{\bullet,\bullet}_{[d]}, \bar{\partial}) = \begin{cases} k[z_1, \ldots, z_d][\partial_{z_1}, \ldots, \partial_{z_d}], & i = 0, \\ z_1^{-1} \ldots z_d^{-1} k[z_1^{-1}, \ldots, z_d^{-1}][\partial_{z_1}, \ldots, \partial_{z_d}], & i = d-1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.111}$$

In the original formulation, the classical Kodaira-Spencer theory is related to the dg Lie algebra $(\ker \partial, \bar{\partial}, \{-,-\})$. In the case when $d = 2$, the L_∞ structure on the cohomology $H^\bullet(\ker \partial, \bar{\partial})$ is analyzed in [Zen24]. In [CL12], a different formulation of Kodaira-Spencer theory, called BCOV theory, is proposed, where the constraint $\partial \mu = 0$ is imposed homologically. The derived version of the kernel of ∂ is the homotopy fixed points for the corresponding action of $\mathbb{C}[1]$

$$\mathrm{PV} \otimes^{\mathbb{L}}_{\mathbb{C}[\varepsilon]} \mathbb{C} \tag{4.112}$$

In our case, we consider the complex

$$(\mathrm{PV}^{\bullet,\bullet}_{[d]}[[u]], \bar{\partial} - u\partial) \tag{4.113}$$

equipped with the bracket $\{-,-\}$. It induces an L_∞ structure on the cohomology $H^\bullet(\mathrm{PV}^{\bullet,\bullet}_{[d]}[[u]], \bar{\partial} - u\partial)$.

We can couple the Kodaira-Spencer theory with the Holomorphic Chern-Simons theory. For the coupled theory, we consider the complex

$$\mathrm{PV}^{\bullet,\bullet}_{d}[[u]] \oplus A^\bullet_d \otimes \mathfrak{g} \tag{4.114}$$

where we take $\mathfrak{g} = \mathfrak{gl}_K$ or $\mathfrak{gl}_{K|K}$. The coupling between polyvector fields and holomorphic Chern-Simons theory in the classical theory is encoded by a L_∞ map

$$\mathrm{PV}^{\bullet,\bullet}_{d}[[u]] \to (D_{\mathrm{poly}}(A^\bullet_d))^\sigma \tag{4.115}$$

which is a cyclic version [Wil11] of the Kontsevich's formality map [Kon03]. This defines a L_∞ structure on the complex 4.115, and induces an L_∞ structure on the cohomology.

4.9 L_∞ ALGEBRA AND POISSON STRUCTURE

Poisson structure

In all of our above examples, we obtain a cyclic L_∞ algebra L with the decomposition

$$L = L_+ \oplus L_-[-1] \tag{4.116}$$

Under this decomposition, the cyclic pairing becomes a non-degenerate pairing $L_+ \otimes L_- \to \mathbb{C}$ and induces an isomorphism $L_- \cong L_+^*$. The L_∞ structure $\{l_n : L^{\otimes n} \to L[n-2]\}_{n\geq 2}$ can be reduced to maps:

$$\begin{aligned} l_n &: S^{n-1}(L_-) \otimes L_+ \to L_- \\ l_n &: S^{n-2}(L_-) \otimes \wedge^2 L_+ \to L_+ \end{aligned} \tag{4.117}$$

Using the cyclic pairing, we observe that the above two maps comes from the same elements in $S^{n-1}(L_-^*) \otimes \wedge^2(L_+^*)$. Collecting them together, we find that the L_∞ structure $L_+ \oplus L_-[-1]$ is the same as an element in

$$\mathcal{O}(L_-) \otimes \wedge^2 L_-$$

This is the same as the bivector field on L_-. In fact, the L_∞ condition guarantees that this bivector is a Poisson bivector.

Lemma 4.9. *Let $L = L_+ \oplus L_-[-1]$ equipped with a non-degenerate pairing $L_+ \otimes L_-$, then we have an bijection between the set of cyclic L_∞ structures on L with the paring and the set of Poisson structures on $\mathcal{O}(L_-)$*

Proof. Given a bivector $\pi \in \mathcal{O}(L_-) \otimes \wedge^2 L_-$, the corresponding bracket $\{-,-\}$ satisfies the Jacobi identity if and only if π satisfies $[\pi, \pi] = 0$, with respect to the Schouten–Nijenhuis bracket. Then its easy to check that this condition translates to the L_∞ condition as in (26) of [ASZK97]. □

Therefore, we obtained various Poisson algebra $\mathcal{O}(L_-)$ which correspond to the various L_∞ algebras that come from the derived Laurent series.

Promoting to vertex Poisson structure

We wish to promote the above Poisson algebra into a vertex Poisson algebra. The easiest way is to consider the associated infinite jet space

$$J_\infty L_- = L_-[[t]]$$

Then according to [AM21], there is a Poisson vertex algebra structure on $\mathcal{O}(J_\infty L_-)$.

In most of our situations, the Poisson vertex algebra is a deformation of $L_-[[t]]$, which comes from a deformation of the L_∞ structure of $L[[t]]$.

Lemma 4.10. *For L as in Lemma 4.9. Then there is a bijection between the set of L_∞ structure on $L[[t]]$, ..., and the set of Poisson vertex algebra structures on $\mathcal{O}(J_\infty L_-)$.*

As a example, we consider a deformation of $J_\infty(H^\bullet(\mathring{\mathbb{D}}^2, \mathfrak{g}\otimes\mathcal{O})) \cong H^\bullet(\mathbb{D}\times\mathring{\mathbb{D}}^2, \mathfrak{g}\otimes\mathcal{O})$ constructed as follows. Recall that the Jouanolou model for $H^\bullet(\mathbb{D}\times\mathring{\mathbb{D}}^2, \mathfrak{g}\otimes\mathcal{O})$ is given by

$$A_2^\bullet[[t]]\otimes\mathfrak{g}$$

equipped with the differential $\bar\partial$ 4.12 and Lie bracket $[-,-]$. In the context of twisted holography, we introduce a deformed differential

$$\bar\partial + \lambda\Omega_B\partial_t \tag{4.118}$$

Then there will be a new L_∞ structure on the cohomology $H^\bullet(A_2^\bullet[[t]]\otimes\mathfrak{g}, \bar\partial+\lambda\Omega_B\partial_t)$. This new L_∞ structure can be computed via homological perturbation lemma and the homotopy transfer. Recall that we have the following special deformation retract (SDR) from $A_2^\bullet$ to $H^\bullet(\mathring{\mathbb{D}}^2, \mathcal{O})$

$$h \circlearrowright (A_2^\bullet[[t]], \bar\partial) \underset{i}{\overset{p}{\rightleftarrows}} (H^\bullet(\mathbb{D}\times\mathring{\mathbb{D}}^2, \mathcal{O}), 0). \tag{4.119}$$

After adding $\lambda\Omega_B\partial_t$ to the differential, we can apply the homological perturbation lemma and find the following new SDR

$$h' \circlearrowright (A_2^\bullet[[t]], \bar\partial+\lambda\Omega_B\partial_t) \underset{i'}{\overset{p'}{\rightleftarrows}} (H^\bullet(\mathbb{D}\times\mathring{\mathbb{D}}^2, \mathcal{O}), D). \tag{4.120}$$

The new differential D is given by

$$D = p(1-\lambda\Omega_B\partial_t h)^{-1}\lambda\Omega_B\partial_t i = p(\lambda\Omega_B\partial_t + \lambda^2\Omega_B\partial_t h\Omega_B\partial_t + \dots)i \tag{4.121}$$

The other maps i', p', h' in the SDR are given by

$$\begin{aligned} p' &= p + p(1-\lambda\Omega_B\partial_t h)^{-1}\lambda\Omega_B\partial_t h = p + p\lambda\Omega_B\partial_t h + \dots \\ i' &= i + h(1-\lambda\Omega_B\partial_t h)^{-1}\lambda\Omega_B\partial_t i = i + h\lambda\Omega_B\partial_t i + \dots \\ h' &= h + h(1-\lambda\Omega_B\partial_t h)^{-1}\lambda\Omega_B\partial_t h = h + h\lambda\Omega_B\partial_t h + \dots \end{aligned} \tag{4.122}$$

Given this new SDR, we obtain a new A_∞ structure on $H^\bullet(\mathbb{D}\times\mathring{\mathbb{D}}^2,\mathcal{O}) = J_\infty H^\bullet(\mathring{\mathbb{D}}^2,\mathcal{O})$. It can be computed by the same method as in Section 4.5, but with the new maps i', p', h'.

As an example, we compute the new product m_2' on $H^\bullet(\mathring{\mathbb{D}}^2,\mathcal{O})[[t]]$. It will be convenient to introduce a degree 0 map $\tilde{h}: A_2^\bullet \to A_2^\bullet$:

$$\tilde{h}: \begin{array}{l} \mathcal{H}_{0,\bar{j}} \to 0 \\ \mathcal{H}_{j,\bar{j}} \overset{\cong}{\to} \mathcal{H}_{j-\frac{1}{2},\bar{j}+\frac{1}{2}} \end{array}, \tag{4.123}$$

so that $\tilde{h}(-\Omega_B)$ acts the same as $h(-)$. We have

$$\tilde{h}(w_1^{2j}\bar{w}_2^{2\bar{j}}) = \frac{1}{2\bar{j}+1} w_1^{2j-1}\bar{w}_2^{2\bar{j}+1}. \tag{4.124}$$

Using $\tilde{h}$, we can simplify the expression for i', p', h' and D. We have

$$D = \sum_{k\geq 1} p\Omega_B\lambda^k\tilde{h}^{k-1}\partial_t^k i \tag{4.125}$$

and

$$\begin{aligned} p' &= p + \sum_{k\geq 1} p\lambda^k\tilde{h}^k\partial_t^k|_{A_2^1} \\ i' &= i + \sum_{k\geq 1} \lambda^k\tilde{h}^k\partial_t^k i|_{A_2^0} \\ h' &= \sum_{k\geq 0} \lambda^k\tilde{h}^k\partial_t^k h \end{aligned} \tag{4.126}$$

The new product m_2' can be computed as $m_2'(-,-) = p'M(i'-,i'-)$. We focus on m_2' on $H^0(\mathring{\mathbb{D}}^2,\mathcal{O})[[t]]^{\otimes 2}$. In this case, m_2' simplifies to

$$m_2'(f_1a_1, f_2a_2) = \sum_{k_1,k_2\geq 0} \lambda^{k_1+k_2}(\partial_t^{k_1}f_1)(\partial_t^{k_2}f_2)pM(\tilde{h}^{k_1}a_1,\tilde{h}^{k_2}a_2), \tag{4.127}$$

for $f_1, f_2 \in \mathbb{C}[[t]]$ and $a_1, a_2 \in H^0(\mathring{\mathbb{D}}^2,\mathcal{O})$. To simplify the notation, we denote the maps $m_2^{(k_1,k_2)}(a_1,a_2) = pM(\tilde{h}^{k_1}a_1,\tilde{h}^{k_2}a_2)$, then we can write m_2' as $m_2'(f_1a_1,f_2a_2) = \sum_{k_1,k_2\geq 0}\lambda^{k_1+k_2}(\partial_t^{k_1}f_1)(\partial_t^{k_2}f_2)m_2^{(k_1,k_2)}(a_1,a_2)$.

In order to compute the maps $m_2^{(k_1,k_2)}$, one can employ the formula 4.40 for the product of arbitrary two harmonic polynomial. Here, we provide a different method based on SL_2 representation. Note that $m_2^{(k_1,k_2)}$ is compatible with the SL_2 action and send $\mathcal{H}_{j_1,0} \otimes \mathcal{H}_{j_2,0}$ to $\mathcal{H}_{j_1+j_2-k_1-k_2,0}$. Therefore the corresponding matrix elements must be proportional to the Clebsch-Gordan coefficients in the orthonormal basis. To determine the constant of proportionality it suffices to compute one non-zero value of $m_2^{(k_1,k_2)}$. Using the definition 4.124 of $\tilde{h}$, we find that

$$\tilde{h}^k(e_m^{(j)}) = \sqrt{\frac{(2j-k)!}{(2j)!k!}} e_m^{(j-\frac{k}{2},\frac{k}{2})}. \tag{4.128}$$

The easiest case of a non-zero product is the product between the highest weight vector and the lowest weight vector. We have

$$\begin{aligned} m_2^{(k_1,k_2)}(e_{j_1}^{(j_1)}, e_{-j_2}^{(j_2)}) &= \sqrt{\frac{(2j_1-k_1)!(2j_2-k_2)!}{k_1!k_2!(2j_1)!(2j_2)!}} m_2(e_{j_1}^{(j_1-\frac{k_1}{2},\frac{k_1}{2})}, e_{-j_2}^{(j_2-\frac{k_2}{2},\frac{k_2}{2})}) \\ &= (-1)^{k_2} \frac{\sqrt{(2j_1+1)(2j_2+1)}(2j_1-k_1)!(2j_2-k_2)!}{k_1!k_2!(2j_1+2j_2-k_1-k_2+1)!} \\ &\times \sqrt{\frac{(2j_1+2j_2-2k_1-2k_2+1)!}{(2j_1-k_1-k_2)!(2j_2-k_1-k_2)!}} e_{j_1-j_2}^{(j_1+j_2-k_1-k_2)}. \end{aligned} \tag{4.129}$$

This implies

$$\begin{aligned} & m_2^{(k_1,k_2)}(e_{m_1}^{(j_1)}, e_{m_2}^{(j_2)}) \\ =& (-1)^{k_2} \sqrt{\frac{(2j_1+1)(2j_2+1)(2j_1+2j_2-2k_1-2k_2)!}{(2j_1+2j_2-k_1-k_2+1)!(2j_1)!(2j_2)!(2j_1-k_1-k_2)!(2j_2-k_1-k_2)!}} \\ & \times \frac{(2j_1-k_1)!(2j_2-k_2)!}{k_1!k_2!} C_{m_1,m_2,m_1+m_2}^{j_1,j_2,j_1+j_2-k_1-k_2} e_{m_1+m_2}^{(j_1+j_2-k_1-k_2)}. \end{aligned} \tag{4.130}$$

We can also rewrite this result into the unnormalized basis $\{w_1^p w_2^q\}$ (see formula (2.18) in [PRS90]). We have

$$m_2^{(k_1,k_2)}(w_1^p w_2^q, w_1^r w_2^s) = \frac{(-1)^{k_2} R_k(p,q,r,s) w_1^{p+r-k} w_2^{q+s-k}}{k_1!k_2![p+q]_{k_1}[r+s]_{k_2}[p+q+r+s-k+1]_k} \tag{4.131}$$

where $k = k_1 + k_2$.

Later, we will observe that the holography conjecture predicts that the Poisson vertex algebra structures constructed in this section are isomorphic to the Poisson vertex algebra constructed from the Deligne category. Various tests of this conjecture will be conducted in the next chapter.

5

HOLOGRAPHY IN QUANTUM FIELD THEORY

5.1 Koszul duality in quantum field theory

In this section, we explore in examples the relationship between Koszul duality and quantum field theory. As discussed in the introduction, Koszul duality in quantum field theory arises primarily from two sources: universal defects and transversal boundary conditions. We illustrate these two points with explicit examples.

Universal defect Let us explain how universal defect relates to Koszul duality in the simplest example: that of Chern-Simons theory with Lie algebra $\mathfrak{g}$. Fields of Chern Simons theory consist of the gauge field $A \in \Omega^1(X) \otimes \mathfrak{g}$. The action functional is

$$CS(A) = \int \mathrm{Tr}(AdA + \frac{2}{3}A^3), \tag{5.1}$$

which is invariant under the gauge transformation

$$\delta A = dc + [c, A] \tag{5.2}$$

The local operators of Chern-Simons theory are entirely generated by the ghost, given by the Chevalley-Eilenberg (CE) dg algebra $\mathrm{CE}^\bullet(\mathfrak{g})$. When coupling Chern-Simons theory with topological quantum mechanics along the $\mathbb{R}_t$ axis, we demonstrate that the universal defect produces the universal enveloping algebra $U(\mathfrak{g})$, which is Koszul dual to $\mathrm{CE}^\bullet(\mathfrak{g})$.

The most general coupling we can have involves coupling the Chern-Simons gauge field A_t^a to local operators in the algebra, which we denote ρ_a. The action functional of the coupled system take the form

$$CS(A) + \mathrm{P}\exp(\int_{\mathbb{R}_t} A_t^a \rho_a) + S_{1d} \tag{5.3}$$

We ask the action functional to be gauge invariant under the gauge transformation $\delta A^a = dc^a + f^a_{bc} c_b A_c$. We find that the path ordered exponential transforms as follows

$$\sum_{n\geq 1} \sum_{i=1}^{n} \int_{t_1 \leq t_2 \leq \cdots \leq t_n} A_t^{a_1}(t_1)\rho(t_1) \ldots (d_t c^{a_i} + f^{a_i}_{bc} c_b(t_i) A_c(_i)) \ldots A_t^{a_n}(t_n)\rho_{a_n} \tag{5.4}$$

Integrating by parts the terms with $d_t c$ picks up boundary terms, which comes from t_i colliding with other points. We find that the boundary contribution of the term with $d_t c$ is given by

$$\sum_{n\geq 1} \sum_{i=1}^{n} \int_{t_1 \leq t_2 \leq \cdots \leq t_n} A_t^{a_1}(t_1)\rho(t_1) \ldots (-c^b A^c_t \rho_b \rho_c + c^b A^c_t \rho_c \rho_b) \ldots A_t^{a_n}(t_n)\rho_{a_n} \tag{5.5}$$

We find that the path-ordered exponential is gauge invariant if we require

$$\rho_b \rho_c - \rho_c \rho_b = f^a_{bc} \rho_a \tag{5.6}$$

This defines the universal enveloping algebra $U(\mathfrak{g})$.

TRANSVERSAL BOUNDARY CONDITION We consider the transversal boundary conditions in the Poisson sigma model [Kon03, CF01]. For different types of Poisson tensor, we can reproduce many classical examples of pairs of Koszul dual algebras 2.2.

In the BV formalism, the space of fields of Poisson sigma model is given by

$$(\mathbf{X}^i, \boldsymbol{\eta}_i) \in \Omega^\bullet(\Sigma) \otimes T^*[1]V \tag{5.7}$$

The vector space V is equipped with a Poisson structure $\alpha = \alpha^{ij}(x)\partial_i \wedge \partial_j$, satisfying $[\alpha, \alpha] = 0$ under the Schouten-Nijenhuis bracket. The BV action functional is given as follows

$$S = \int \boldsymbol{\eta}_i d\mathbf{X}^i + \frac{1}{2}\alpha^{ij}(\mathbf{X})\boldsymbol{\eta}_i \boldsymbol{\eta}_j \tag{5.8}$$

The BV-BRST differential can be defined by bracketing with the BV action $Q = \{S, -\}$. We have

$$\begin{aligned} Q\mathbf{X}^i &= d\mathbf{X}^i + \alpha^{ij}(\mathbf{X})\boldsymbol{\eta}_j \\ Q\boldsymbol{\eta}_i &= d\boldsymbol{\eta}_i + \frac{1}{2}\frac{\partial \alpha^{jk}}{\partial x^i}(\mathbf{X})\boldsymbol{\eta}_j \boldsymbol{\eta}_k \end{aligned} \tag{5.9}$$

In the AKSZ formalism, this BV theory can be formulated as the following mapping space

$$\mathrm{Maps}(\Sigma_{\mathrm{dR}}, T^*[1]V) \tag{5.10}$$

It is equipped with the symplectic structure induced from the Poisson structure α.

A boundary condition is defined to be a Lagrangian of the phase space of the theory on the boundary. We mainly consider two types of boundary conditions, Dirichlet and Neumann boundary conditions. The Dirichlet boundary condition is given by setting $\boldsymbol{\eta}_i = 0$. It corresponds to the Lagrangian $\mathrm{Maps}(M^1_{\mathrm{dR}}, V) \subset \mathrm{Maps}(M^1_{\mathrm{dR}}, T^*[1]V)$. The Neumann boundary condition is given by setting $\boldsymbol{X}^i = 0$. It correspond to the Lagrangian $\mathrm{Maps}(M^1_{\mathrm{dR}}, V^*[1]) \subset \mathrm{Maps}(M^1_{\mathrm{dR}}, T^*[1]V)$

Now we discuss the boundary algebras for different types of Poisson tensor:

- For vanishing Poisson tensor, we get a free theory. Choosing Dirichlet boundary condition set $\boldsymbol{\eta}_i = 0$. The boundary algebra is generated by the fields $\mathbf{X}^i$ with boundary BRST differential

  $$Q\mathbf{X}^i = d\mathbf{X}^i \tag{5.11}$$

  By passing to the cohomology, we see that the boundary algebra is generated by the lowest component x^i of $\mathbf{X}^i$. There is no boundary OPE and the boundary algebra is the commutative algebra $\mathrm{Sym}(V)$. Choosing Neumann boundary condition set $\mathbf{X}^i = 0$. We find that the boundary algebra is given by $\mathrm{Sym}(V^*[1])$, which is Koszul dual to $\mathrm{Sym}(V)$.

- For a linear Poisson structure $\alpha^{ij}(x) = f^{ij}_k x^k$, the condition $[\alpha, \alpha] = 0$ is equivalent to the Jacobi identity of the structure constants f^{ij}_k. It therefore defines a Lie algebra that we denote as $\mathfrak{g}$. For the Dirichlet boundary condition, the boundary algebra is generated by x^i. The boundary algebra in this case is essentially the deformation quantization of the Poisson algebra $(\mathrm{Sym}(\mathfrak{g}), [-,-])$, which is the universal enveloping algebra $U(\mathfrak{g})$ [Kon03].

  For the Neumann boundary condition, the boundary algebra is generated by η_i. There is no OPE but the algebra is equipped with a boundary differential

  $$d\eta_k = \frac{1}{2} f^{ij}_k \eta_i \eta_k \tag{5.12}$$

  We see that this boundary algebra is the Chevalley-Eilenberg algebra $C^\bullet(\mathfrak{g})$. This is another classical examples of Koszul duality pair.

- We can also consider a constant Poisson tensor. First we discuss the Dirichlet boundary condition. The boundary algebra is the deformation quantization of the Poisson algebra $(\mathrm{Sym}(V), \alpha)$, and in this case gives us the Weyl algebra.

  For the Neumann boundary condition, we can observe the following boundary anomaly

  $$\{S, S\} = \{\int \boldsymbol{\eta}_i d\mathbf{X}^i, \alpha^{ij}\boldsymbol{\eta}_i\boldsymbol{\eta}_j\} = \int_{\mathbb{R}\times\mathbb{R}_{>0}} \alpha^{ij} d\boldsymbol{\eta}_i\boldsymbol{\eta}_j = \int_{\mathbb{R}} (\alpha^{ij}\eta_i\eta_j)^{(1)} \quad (5.13)$$

  It is generally inconsistent to quantize a theory when the system exhibit an anomaly. However, at least in this example, we can define a consistent boundary algebra because there is no possible nontrivial OPE at all. The boundary algebra is simply the exterior algebra $\mathbb{C}[\eta_i]$, and we need to "remember" the distinguished element $\alpha^{ij}\eta_i\eta_j$ as the anomaly.

  Recall that the anomaly measures the failure of the quantum master equation to hold. This naturally lead us to the definition of curved algebra, in which the Maurer-Cartan equation is modified to incorporate the curved element. In fact, the curved algebra $(\mathbb{C}[\eta_i], d = 0, c = \alpha^{ij}\eta_i\eta_j)$ is exactly the Koszul dual of the Weyl algebra.

- For a quadratic Poisson tensor, we get two quadratic algebras for both Dirichlet and Neumann boundary conditions. Their Koszul duality has being studied in [Sho10]. We can also study Koszul duality of boundary algebras for more general Poisson tensors. We expect the Dirichlet boundary condition to give rise to the Kontsevich's deformation quantization. However, the Neumann boundary will lead to A_∞ algebra in general.

5.2 TWISTED HOLOGRAPHY: EXAMPLE OF B MODEL ON C^3

In this thesis, we study the example of twisted holography proposed in [CG18]. This model studies the B model topological string on $\mathbb{C}^3$. We put a stack of N branes at $\mathbb{C} \times \{0\} \subset \mathbb{C}^3$ and consider the open string theory on the brane and closed string theory on the target $\mathbb{C}^3$.

The open string theory on the stack of N branes lead to the vertex algebra of gauged $\beta\gamma$ system defined in Section 3.5. Let us recall that it is defined by the symplectic boson $\mathcal{W}_{SYM}$ in the Deligne category generated by four felds $c(z)$, $Z_1(z)$, $Z_2(z)$, $b(z)$. The BRST charge is defined as

$$Q = \oint dz\, \mathrm{Tr}(: b(z)c(z)c(z) :) + \mathrm{Tr}(: c(z)Z_1(z)Z_2(z) :) \quad (5.14)$$

The vertex algebra in Deligne category produces for us a (dg) vertex algebra in vector space, defined as $(\mathcal{A} = \mathrm{Hom}(\mathbb{1}, \mathcal{W}_{SYM}), Q)$.

On the other hand, the B model topological string on a Calabi-Yau threefold can be formulated as a quantum field theory called Kodaira-Spencer gravity [BCOV94]. This theory is further studied in [CL12], where the authors used a slight variant of this theory called BCOV theory. As in [GV99], adding a stack of N branes wrapping $\mathbb{C} \subset \mathbb{C}^3$ leads to a gravitational backreaction. A non-trivial Beltrami differential is turned on, deforming $\mathbb{C}^3$ into the deformed conifold $SL_2(\mathbb{C})$. This is the reason we study the deformation 4.118 in Section 4.9. Perturbative quantization of BCOV theory is studied in [CL12, CL15]. In principle, we can define the factorization algebra (in the sense of [CG17]) that correspond to the BCOV theory Obs_{BCOV}.

The twisted holography conjecture predicts that the chiral algebra that corresponds to the theory on the brane $\mathcal{A}$ is "Koszul dual" to the factorization algebra of the BCOV theory Obs_{BCOV} restricted to $\mathbb{C}$

$$\mathcal{A} = (\mathrm{Obs}_{BCOV}|_{\mathbb{C}})^! \tag{5.15}$$

However, the above 'conjecture' still has some issues. Although we attempt to define Koszul duality for chiral algebras in Section 2 ([GLZ22]), our definition only applies to a small class of chiral algebras called quadratic (or quadratic-linear-scalar). It is challenging for us to understand what the Koszul dual would be for a chiral algebra as complicated as $\mathcal{A}$. On the BCOV theory side, although in principle there exists a factorization algebra that corresponds to the perturbative quantization of BCOV theory, it will be very difficult to study it. An even more challenging problem is that we need to translate the factorization algebra structure in the sense of [CG17] to the factorization algebra in the sense of [BD04] to connect with the chiral algebra structure of $\mathcal{A}$.

To formulate a rigorous and testable conjecture, we make the following simplifications. First, we transform the statement about Koszul duality into a statement of isomorphism. This is achieved by applying "Koszul duality" again to the factorization algebra of BCOV theory. As we have introduced, by realizing the algebra along $\mathbb{C}$ as a boundary algebra, its Koszul dual is given by the boundary algebra of the transversal boundary condition. As a result, we conjecture that the large N chiral algebra is isomorphic to this boundary algebra of the transversal boundary condition. Second, instead of studying the full quantum theory, we consider the semiclassical limit. This reduces the factorization algebra/chiral algebra structure on both sides to the much more tractable vertex Poisson algebra structures.

Recall from section 3.6 that we also defined a three parameters family $\mathcal{A}_{d,\hbar,N}$ version of the vertex algebra. After a parametrization, we defined a Poisson vertex algebra $\mathcal{A}_{d=0,\lambda}$. This is the Poisson vertex algebra on the one side of the duality.

We also constructed various Poisson vertex algebra structure obtained from the various construction related to the derived Laurent series. The phase space for the BCOV theory on $\mathbb{C}^3\backslash\mathbb{C}$ is given by

$$\mathrm{PV}_2^{\bullet,\bullet}[[t,\partial_t]][[u]] \tag{5.16}$$

This complex is equipped with the differential $\bar{\partial}+u\partial$ before the back-reaction. After the back reaction, the differential deforms to $\bar{\partial}+u\partial+\{\lambda\Omega_B\partial_t,-\}$. The structure of the classical field theory is encoded in the L_∞ structure on it. By homotopy transfer theorem, we obtain an L_∞ structure on the cohomology $H^\bullet(\mathrm{PV}_2^{\bullet,\bullet}[[t,\partial_t]][[u]],Q)$. By the construction in Section 4.9, the L_∞ structure on the cohomology induces a vertex Poisson structure on $L_-\subset H^\bullet(\mathrm{PV}_2^{\bullet,\bullet}[[t,\partial_t]][[u]],Q)$. As we have discussed in 4.8, we can identify the dual of L_- as the cyclic cohomology $HC^\bullet(\mathbb{C}[\epsilon_1,\epsilon_2,t])$.

Conjecture 5.1. *The Poisson vertex algebra $\mathcal{A}_{d=0,\lambda}$ defined in 3.6 is isomorphic to the Poisson vertex algebra structure on $HC^\bullet(\mathbb{C}[\epsilon_1,\epsilon_2,t])$ defined by the L_∞ structure on the cohomology of* $\mathrm{PV}_2^{\bullet,\bullet}[[t,\partial_t]][[u]]$.

One can generalize the construction to include space filling branes in this setup. To cancel the anomaly, we add K space filling branes and K anti branes. On the large N chiral algebra side, we consider the new gauged $\beta\gamma$ system defined in Section 3.5. Recall that we have new fields $I(z),J(z)$ valued in $\mathbb{C}^{K|K}\otimes[1,0]$ and its dual. A new term is added to the BRST differential

$$Q_M=\oint dz\,\mathrm{Tr}_F(:I(z)c(z)J(z):)$$

The string field theory of the $K|K$ space filling brane on $\mathbb{C}^3$ is given by the holomorphic Chern Simons theory valued in the Lie algebra $\mathfrak{gl}_{K|K}$. The phase space for the holomorphic Chern-Simons theory on $\mathbb{C}^3\backslash\mathbb{C}$ is given by

$$A_2^\bullet[[t]]\otimes\mathfrak{gl}_{K|K}$$

We consider the BCOV theory coupled with the holomorphic Chern-Simons theory. The coupled theory is given by the L_∞ algebra

$$\mathrm{PV}_2^{\bullet,\bullet}[[t,\partial_t]][[u]]\oplus A_2^\bullet[[t]]\otimes\mathfrak{gl}_{K|K}$$

By the construction in Section 4.9, we obtain a vertex Poisson algebra structure on the degree 1 part of it cohomology.

Then the twisted holography conjecture predicts that the above two vertex Poisson algebra are isomorphic.

5.3 TEST OF THE CONJECTURE

Poisson vertex algebra from derived Laurent series

In this section, we outline the process of deriving the Poisson vertex algebra structure from the L_∞ structure defined in Section 4.7. We focus on the case of $H^\bullet(\mathring{\mathbb{D}}^2, \mathfrak{g}\otimes\mathcal{O})$. Recall that we denoted $L_-[1] = H^1(\mathring{\mathbb{D}}^2, \mathfrak{g}\otimes\mathcal{O})) = \mathfrak{g}[\bar{w}_1, \bar{w}_2]\Omega_B$. First, we consider L_∞ algebra before the deformation 4.118 and the Poisson structure on $\mathcal{O}(L_-)$. We denote $\{B_a[n,m]\}$ a basis of L_-^* dual to the basis $\{\frac{(n+m+1)!}{n!m!} t^a \bar{w}_1^n \bar{w}_2^m\}$.

As a first step, we consider the Lie bracket $[-,-]$ on $H^\bullet(\mathring{\mathbb{D}}^2, \mathfrak{g}\otimes\mathcal{O})$. Composing with the cyclic pairing we find a map

$$\wedge^2 L_+ \otimes L_- \to \mathbb{C} \tag{5.17}$$

given by

$$(t_a w_1^p w_2^q, t_b w_1^r w_2^s, \frac{(p+r+q+s+1)!}{(p+r)!(q+s)!} t^c \bar{w}_1^{p+r} \bar{w}_2^{q+s}) \to f_{ab}^c$$

Using the dual pairing between L_+ and L_-, we find that the above map gives us a linear two form $(L_-)^* \otimes \wedge^2 L_-$. This two form lead to the following Poisson bracket

$$\{B_a[p,q], B_b[r,s]\} = f_{ab}^c B_c[p+q, r+s] \tag{5.18}$$

This Poisson structure on $\mathcal{O}(L_-)$ induces a Poisson vertex algebra structure on $\mathcal{O}(J_\infty L_-) = \mathbb{C}[\partial^n B_a[p,q] \mid n \geq 0, p,q \geq 0]$. Using the prescription in [Ara12], we have

$$B_a[p,q]_{(n)} B_b[r,s] = \delta_{n,0} f_{ab}^c B_c[p+q, r+s] \tag{5.19}$$

Remark 5.2. We can also write the above structure in terms of the map $Y_- : \mathcal{O}(L_-) \otimes \mathcal{O}(L_-) \to \mathcal{O}(L_-) z^{-1}[z^{-1}]$

$$Y_-(B_a[p,q], z) B_b[r,s] = \frac{1}{z} f_{ab}^c B_c[p+q, r+s] \tag{5.20}$$

We can proceed to consider the higher bracket l_n. By composing with the cyclic pairing we obtain a map

$$\wedge^2 L_+ \otimes S^{n-2}(L_-) \to \mathbb{C} \tag{5.21}$$

Recall that we define the constant $(m_n)^{p,q;r,s}_{u_1,v_1;\ldots;u_{n-1},v_{n-1}}$ as follows

$$\begin{aligned}&(m_n)^{p,q;r,s}_{u_1,v_1;\ldots;u_{n-1},v_{n-1}}\\ &= \left(\prod_{i=1}^{n-1} \frac{(u_i+v_i+1)!}{u_i!v_i!}\right) \left(w_1^p w_2^q, m_n(w_1^r w_2^s, \bar{w}_1^{u_1}\bar{w}_2^{v_1}\Omega_B, \ldots, \bar{w}_1^{u_{n-1}}\bar{w}_2^{v_{n-1}}\Omega_B)\right)\end{aligned} \tag{5.22}$$

By the prescription 4.102 in Section 4.7 on the L_∞ structure on the tensor product $H^\bullet(\mathring{\mathbb{D}}^2, \mathcal{O}) \otimes \mathfrak{g}$, we find the following expression for the maps 5.21

$$\begin{aligned}&(t_a w_1^p w_2^q, t_b w_1^r w_2^s, \frac{(p+r+q+s+1)!}{(p+r)!(q+s)!} t^c \bar{w}_1^{p+r} \bar{w}_2^{q+s})\\ &\to \frac{1}{(n-1)!} \sum_{\sigma} \sum_{\substack{u_1+\ldots u_{n-1}=p+r-n+2\\ v_1+\ldots v_{n-1}=q+s-n+2}} \sum_{c_1,\ldots,c_{n-1}} \sum_{\sigma\in S_{n-1}} (m_n)^{p,q;r,s}_{u_{\sigma(1)},v_{\sigma(1)};\ldots;u_{\sigma(n-1)},v_{\sigma(n-1)}}\\ &K(t_a, [\ldots[[t_b, t^{c_{\sigma(1)}}], t^{c_{\sigma(2)}}], \ldots, t^{c_{\sigma(n-1)}}])\end{aligned} \tag{5.23}$$

This correspond to a two form $S^{n-1}(L_-)^* \otimes \wedge^2 L_-$, and gives us the following vertex Poisson structure

$$\begin{aligned}&B_a[p,q]_{(m)} B_b[r,s]\\ &= \frac{\delta_{m,0}}{(n-1)!} \sum_{\sigma} \sum_{\substack{u_1+\ldots u_{n-1}=p+r-n+2\\ v_1+\ldots v_{n-1}=q+s-n+2}} \sum_{c_1,\ldots,c_{n-1}} \sum_{\sigma\in S_{n-1}} (m_n)^{p,q;r,s}_{u_{\sigma(1)},v_{\sigma(1)};\ldots;u_{\sigma(n-1)},v_{\sigma(n-1)}}\\ &K(t_a, [\ldots[[t_b, t^{c_{\sigma(1)}}], t^{c_{\sigma(2)}}], \ldots, t^{c_{\sigma(n-1)}}]) B_{c_1}[u_1, v_1] B_{c_2}[u_2, v_2] \cdots B_{c_{n-1}}[u_{n-1}, v_{n-1}].\end{aligned} \tag{5.24}$$

We consider the special case when $n = 3$, which gives

$$\begin{aligned}B_a[p,q]_{(m)} B_b[r,s] &= \delta_{m,0} \sum_{\substack{u_1+u_2=p+r-1\\ v_1+v_2=q+s-1}} \sum_{c,d,e,f} K^{fc}\\ &\times \left((m_3)^{p,q;r,s}_{u_1,v_1} f^e_{bf} f^d_{ae} - (m_3)^{r,s;p,q}_{u_1,v_1} f^e_{af} f^d_{be}\right) B_c[u_1,v_1] B_d[u_2,v_2].\end{aligned} \tag{5.25}$$

where we used the shorthand $(m_3)^{p,q;r,s}_{u,v} := (m_3)^{p,q;r,s}_{u,v;p+r-1-u,q+s-1-v}$. Though the above formula is for general Lie algebra, in the holographic setting we need to use the Lie algebra $\mathfrak{gl}(K|K)$ or $\mathfrak{gl}(K)$. Strictly speaking, the bulk anomaly of the open-closed coupled theory cancels only for the super Lie algebra $\mathfrak{gl}(K|K)$. Here we works with $\mathfrak{gl}(K)$ for simplicity. The result can be easily generalized to $\mathfrak{gl}(K|K)$ by taking care of the $\pm$ sign. Note that $\mathfrak{gl}(K)$ has a canonical basis given by the elementary matrices $\{E_{a_1a_2}\}_{1\leq a_1,a_2\leq K}$. Therefore we replace the indices a by a_1a_2 in the above formula. The Killing form $K^{a_1a_2;b_1b_2}$ is given by

$$K^{a_1a_2;b_1b_2} = \delta_{a_1b_2}\delta_{a_2b_1}. \tag{5.26}$$

The structure constant can be extracted from the commutation relation

$$[E_{a_1a_2}, E_{b_1b_2}] = \delta_{a_2b_1}E_{a_1b_2} - \delta_{a_1b_2}E_{b_1a_2}. \tag{5.27}$$

Then the formula 5.24 can be expanded as

$$\begin{aligned} &B_{a_1a_2}[p,q]_{(m)}B_{b_1b_2}[r,s] \\ &= \delta_{m,0}\sum_{\substack{u_1+u_2=p+r-1\\ v_1+v_2=q+s-1}}((m_3)^{r,s;p,q}_{u_1,v_1} - (m_3)^{p,q;r,s}_{u_1,v_1})B_{a_1b_2}[u_1,v_1]B_{b_1a_2}[u_2,v_2](0) \\ &+\sum_c\big((m_3)^{p,q;r,s}_{u_1,v_1}\delta_{a_1b_2}B_{b_1c}[u_1,v_1]B_{ca_2}[u_2,v_2] \\ &-(m_3)^{r,s;p,q}_{u_1,v_1}\delta_{a_2b_1}B_{a_1c}[u_1,v_1]B_{cb_2}[u_2,v_2]\big)\,. \end{aligned} \tag{5.28}$$

By computing m_3 using 4.64, we find that $(m_3)^{1,0;0,1}_{0,0} = -\frac{1}{2}$ and $(m_3)^{0,1;1,0}_{0,0} = \frac{1}{2}$. We find the following

$$\begin{aligned} &B_{a_1a_2}[1,0]_{(m)}B_{b_1b_2}[0,1] = \delta_{m,0}\Big(B_{a_1b_2}[0,0]B_{b_1a_2}[0,0](0) \\ &-\frac{1}{2}\sum_c \delta_{a_2b_1}B_{a_1c}[0,0]B_{cb_2}[0,0](0) - \delta_{a_1b_2}B_{b_1c}[0,0]B_{ca_2}[0,0](0)\Big). \end{aligned} \tag{5.29}$$

As a more nontrivial example, we can use the formula 4.64 to compute the following value of m_3

$$\begin{aligned} (m_3)^{1,0;r,s}_{u,v} &= -\frac{r!s!(u+v+1)!(r+s-u-v-1)!}{(r+s+1)!u!v!(r-u)!(s-v-1)!}, \\ (m_3)^{r,s;1,0}_{u,v} &= \frac{r!s!(u+v)!(r+s-u-v)!}{(r+s+1)!u!v!(r-u)!(s-v-1)!}. \end{aligned} \tag{5.30}$$

We obtain the following

$$\begin{aligned}
&B_{a_1a_2}[1,0]_{(m)}B_{b_1b_2}[r,s]\\
\sim&\delta_{m,0}\sum_{u,v}\binom{r+s}{r}^{-1}\binom{u+v}{u}\binom{r+s-u-v-1}{r-u}\\
&\Big(B_{a_1b_2}[u,v]B_{b_1a_2}[r-u,s-v-1](0)\\
-&\frac{1}{2}\sum_c\frac{r+s-u-v}{r+s+1}\delta_{a_2b_1}B_{a_1c}[u,v]B_{cb_2}[r-u,s-v-1](0)\\
-&\frac{1}{2}\sum_c\frac{u+v+1}{r+s+1}\delta_{a_1b_2}B_{b_1c}[u,v]B_{ca_2}[r-u,s-v-1](0)\Big).
\end{aligned} \tag{5.31}$$

Next, we consider the Poisson vertex algebra structure that correspond to the deformed A_∞ structure on $A_2^\bullet[[t]]$ defined in Section 4.9. The whole collection of A_∞ maps is deformed starting from the differential. Recall that the new differential is given by $D=\sum_{k\geq 1}p\Omega_B N^k\tilde{h}^{k-1}\partial_t^k i$ on $H^\bullet(\mathring{\mathbb{D}}^2,\mathcal{O})[[t]]$. By composing with the cyclic pairing we obtain a map

$$L_+[[t]]\otimes L_+[[t]]\to\mathbb{C}[[t]] \tag{5.32}$$

Using 4.128, we find that the above map is given by

$$(w_1^pw_2^qt^n,w_1^rw_2^st^m)=\sum_{k\geq 1}N^k\delta_{k,p+q+1}\delta_{r,q}\delta_{s,p}\frac{(-1)^pp!q!}{k!(k-1)!}t^n\partial_t^k(t^m) \tag{5.33}$$

This correspond to the following Poisson vertex algebra structure

$$B_a[p,q]_{(m)}B_b[q,p]=\delta_{m,p+q+1}K_{ab}(-1)^qN^{p+q+1}\frac{p!q!}{(p+q)!}, \tag{5.34}$$

Remark 5.3. We can also write the above structure in terms of the map Y_-, we have

$$Y_-(B_a[p,q],z)B_b[r,s]=\frac{K_{ab}}{z^{p+q+2}}(-1)^qN^{p+q+1}\frac{p!q!}{(p+q)!} \tag{5.35}$$

We also computed the deformation to the product structure: $m_2'(f_1a_1,f_2a_2)=\sum_{k_1,k_2\geq 0}N^{k_1+k_2}(\partial_t^{k_1}f_1)(\partial_t^{k_2}f_2)m_2^{(k_1,k_2)}(a_1,a_2)$, with

$$m_2^{(k_1,k_2)}(w_1^pw_2^q,w_1^rw_2^s)=\frac{(-1)^{k_2}R_k(p,q,r,s)w_1^{p+r-k}w_2^{q+s-k}}{k_1!k_2![p+q]_{k_1}[r+s]_{k_2}[p+q+r+s-k+1]_k} \tag{5.36}$$

where $k=k_1+k_2$.

To obtain the corresponding Poisson vertex algebra structure, we compute the following summation

$$
\begin{aligned}
&k! \sum_{k_1+k_2=k} \frac{R_k(p,q,r,s)}{k_1!k_2![p+q]_{k_1}[r+s]_{k_2}[p+q+r+s-k+1]_k} \\
&= \frac{R_k(p,q,r,s)}{[p+q]_k[r+s]_k},
\end{aligned} \tag{5.37}
$$

where we used the formula B.17 derived in the Appendix. We also have

$$
\begin{aligned}
&(k-l)! \sum_{k_1+k_2=k} \binom{k_2}{l} \frac{R_k(p,q,r,s)}{k_1!k_2![p+q]_{k_1}[r+s]_{k_2}[p+q+r+s-k+1]_k} \\
&= \frac{R_k(p,q,r,s)}{l![p+q]_{k-l}[r+s]_k[p+q+r+s-2k+l+1]_l},
\end{aligned} \tag{5.38}
$$

The above results give us

$$
\begin{aligned}
&B_a[p,q]_{(m)}B_b[r,s] \\
&= f_{ab}^c \sum_{l=0}^{k} \delta_{m,k-l} N^k \frac{R_k(p,q,r,s) \times \partial_z^l B_c[p+r-k,q+s-k](0)}{l![p+q]_{k-l}[r+s]_k[p+q+r+s-2k+l+1]_l}.
\end{aligned} \tag{5.39}
$$

OPE from large N chiral algebra

In this section, we analyze the OPE from the large N chiral algebra defined via the Deligne category. We focus on the part of the algebra that correspond to $\mathbb{C}[t] \otimes_{\mathbb{C}[\epsilon_1,\epsilon_2,t]} \mathbb{C}[t] \cong \mathbb{C}[x_1,x_2,t]$ according to 3.66. We can choose the following BRST representative:

$$
E^{(p,q)}(z) :=: IZ^{(i_1}Z^{i_2}\dots Z^{i_n)}J : (z), \quad p,q \in \mathbb{Z}_{\geq 0}
$$

where $i_1,\dots,i_n = \{\overbrace{1,\dots,1}^{p},\overbrace{2,\dots,2}^{q}\}$. It is convenient to organize this tower of fields into generating function. Let

$$
Z(\lambda;z) := Z^1(z)\lambda_1 + Z^2(z)\lambda_2.
$$

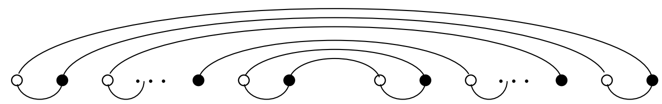
We define

$$
E^{(n)}(\lambda;z) =: IZ(\lambda)^n J : (z) \tag{5.40}
$$

It has the expansion

$$
E^{(n)}(\lambda;z) = \sum_{p+q=n} \binom{n}{p} E^{(p,q)}(z)\lambda_1^p\lambda_2^q. \tag{5.41}
$$

We consider different terms in the OPE between $E^{(n)}(z)$ and $E^{(n')}(z')$. Firstly, we consider the constant term in the OPE. This correspond to the full wick contraction depicted as follows



Counting loops and pole, we find that the above contraction gives the following OPE

$$E^{(n)}(\lambda, z)E^{(n')}(\lambda', 0) \sim \delta_{n,n'} \frac{N^{n+1}[\lambda\lambda']^n}{z^{n+2}}. \tag{5.42}$$

where we denote $[\lambda\lambda'] = \lambda_1\lambda'_2 - \lambda_2\lambda'_1$. Expanding $E^{(n)}(\lambda, z)$ into the symmetrized operator $E^{(p,q)}$, we find

$$E^{(p,q)}(z)E^{(q,p)}(0) \sim (-1)^q N^{p+q+1} \frac{p!q!}{(p+q)!} \frac{1}{z^{p+q+2}}. \tag{5.43}$$

Then we consider the linear terms in the OPE. We wish to organize the results into a series of N. The first term, which is of zeroth order in N, corresponds to a wick contraction of a single $I - J$ pair. For example,

$$(IZ^{i_1} \dots \overbracket{J)(z)(I}Z^{j_1} \dots J)(0) = \frac{1}{z} IZ^{i_1} \dots Z^{j_1} \dots J. \tag{5.44}$$

Remark 5.4. We also have the contraction $(\overbracket{IZ^{i_1} \dots J)(IZ^{j_1} \dots J})$. However, this term gives us the same combinatorial factors and poles and should be attributed to the Lie algebra factor that we omitted.

Thus the corresponding OPE is

$$E^{(p,q)}(z)E^{(r,s)}(0) \sim \frac{1}{z} E^{(p+r,q+s)}(0) + \dots. \tag{5.45}$$

This OPE can also be written as

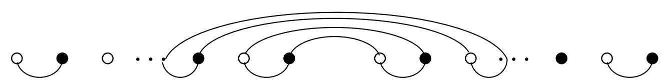
$$E^{(n)}(\lambda; z)E^{(n')}(\lambda'; 0) \sim \frac{1}{z} \frac{n'!}{(n+n')!} (\lambda \cdot \partial_{\lambda'})^n E^{(n+n')}(\lambda'; 0) + \dots \tag{5.46}$$

where $\lambda \cdot \partial_{\lambda'} = \lambda_1 \partial_{\lambda'_1} + \lambda_2 \partial_{\lambda'_2}$.

We emphasize that there are secretly other terms in this contraction. In fact, after the I, J contraction of the symmetrized operators, we obtain $IZ^{(i_1} \dots Z^{i_n)} Z^{(j_1} \dots Z^{j_n)} J$. This is, in general, not the symmetrized operator $IZ^{(i_1} \dots Z^{i_n} Z^{j_1} \dots Z^{j_n)} J$. However, we can always manipulate the final expression into a sum of the symmetrized operator $IZ^{(i_1} \dots Z^{i_n} Z^{j_1} \dots Z^{j_n)} J$ and other operators in the BRST representative.

For example, an I, J contraction of $IZ^1 J$ and $IZ^2 J$ is $IZ^1 Z^2 J$, which is not symmetrized. Using the BRST relation, we find that $I[Z^1, Z^2]J$ is cohomologous to $IJIJ$. Therefore the term $IZ^1 Z^2 J$ is cohomologous to $IZ^{(1} Z^{2)} J + \frac{1}{2} IJIJ$. The remaining terms like $IJIJ$ are also important, and correspond to the quadratic terms in the OPE that we will analyze later.

We also expect to find linear OPE's of higher order in N. The part of k-th order should correspond to a single $I - J$ contraction together with k adjacent $Z - Z$ contractions.



Such a contraction produce a N^k factor and a pole $\frac{1}{z^{k+1}}$. We have

$$\frac{N^k}{z^{k+1}} I(z) Z(\lambda; z)^{n-k} Z(\lambda'; 0)^{n'-k} J(0). \tag{5.47}$$

By symmetrizing the Z operators, we can rewrite this OPE as follows

$$E^{(n)}(\lambda; z) E^{(n')}(\lambda'; 0) \sim \frac{N^k [\lambda\lambda']^k}{z^{k+1}} \frac{(n'-k)!}{(n+n'-2k)!} (\lambda \cdot \partial_{\lambda'})^{n-k} E^{(n+n'-2k)}(\lambda'; 0) \dots, \tag{5.48}$$

where we omitted terms with derivatives of E. Expanding the operators into $E^{(p,q)}$, we can compute the constant coefficient by

$$\sum_{i=0}^{k} \frac{(-1)^i \binom{k}{i} \binom{p+q-k}{q-i} \binom{r+s-k}{r-i}}{\binom{p+q}{p} \binom{r+s}{r}} = \frac{R_k(p,q,r,s)}{[p+q]_k [r+s]_k}, \tag{5.49}$$

which follows from the definition 4.93 of the constant $R_k(p,q,r,s)$. This matches the leading term of 5.39. To obtain the coefficients of the derivative terms, we use the planar three point function

$$\begin{aligned} &\langle E^{(n)}(\lambda, z) E^{(n')}(\lambda', z') E^{(n'')}(\lambda'', z'') \rangle \\ &= \frac{N^{\frac{n+n'+n''}{2}+1} [\lambda\lambda']^{\frac{n+n'-n''}{2}} [\lambda'\lambda'']^{\frac{n'+n''-n}{2}} [\lambda\lambda'']^{\frac{n+n''-n'}{2}}}{(z-z')^{\frac{n+n'-n''}{2}+1} (z'-z'')^{\frac{n'+n''-n'}{2}+1} (z-z'')^{\frac{n+n''-n'}{2}+1}}, \end{aligned} \tag{5.50}$$

which is computed via full planar wick contraction. Expanding this three point function we find that

$$\begin{aligned}
&\langle E^{(n)}(\lambda,z)E^{(n')}(\lambda',z')E^{(n'')}(\lambda'',z'')\rangle \\
=&\frac{N^{\frac{n+n'+n''}{2}+1}[\lambda\lambda']^{\frac{n+n'-n''}{2}}[\lambda'\lambda'']^{\frac{n'+n''-n}{2}}[\lambda\lambda'']^{\frac{n+n''-n'}{2}}}{(z-z')^{\frac{n+n'-n''}{2}+1}} \\
&\times\sum_{l\geq 0}\frac{(\frac{n+n''-n'}{2}+l)!}{(\frac{n+n''-n'}{2})!l!}(-1)^l\left(\frac{z-z'}{z'-z''}\right)^l\frac{1}{(z'-z'')^{n''+2}} \\
=&\sum_{l\geq 0}\frac{N^{\frac{n+n'+n''}{2}+1}[\lambda\lambda']^{\frac{n+n'-n''}{2}}}{(z-z')^{\frac{n+n'-n''}{2}-l+1}}\frac{[\frac{n+n''-n'}{2}+l]_l(\frac{n'+n''-n}{2})!}{[n''+l+1]_l n''!} \\
&\times(\lambda\cdot\partial_{\lambda'})^{\frac{n+n''-n'}{2}}[\lambda'\lambda'']^{n''}\frac{\partial_{z'}^l}{l!}\frac{1}{(z'-z'')^{n''+2}}
\end{aligned} \tag{5.51}$$

Using the two-point function, we can extract from the above expression the following OPE

$$\begin{aligned}
E^{(n)}(\lambda;z)E^{(n')}(\lambda';0)\sim\sum_{l=0}^{k}\frac{N^k[\lambda\lambda']^k}{z^{k-l+1}}&\frac{[n-k+l]_l(n'-k)!}{[n+n'-2k+l+1]_l(n+n'-2k)!} \\
&\times(\lambda\cdot\partial_{\lambda'})^{n-k}\frac{\partial_z^l}{l!}E^{(n+n'-2k)}(\lambda';0).
\end{aligned} \tag{5.52}$$

We can check that this matches exactly with 5.39.

In the rest of this section, we consider the quadratic terms in the OPE. Naively, we expect these terms can be produced by a wick contraction of a single $Z-Z$ pair

$$\begin{aligned}
&(I...\overbrace{Z^{i_k}...J)(z)(I...Z^{j_l}}...J)(0) \\
=&\frac{1}{z}(IZ^{j_1}...Z^{j_{l-1}}Z^{i_k+1}...J)(0)(IZ^{i_1}...Z^{i_{k-1}}Z^{j_{l+1}}Z^{j_1}...J)(0).
\end{aligned} \tag{5.53}$$

However, as we have mentioned in the last section, after a single $I-J$ contraction, we need to manipulate the final expression into a BRST representative. In this process, we can trade the commutator $[Z^1,Z^1]$ with $\sum J_a I_a$. Therefore, OPE of the form $(I\dots J)(z)(I\dots J)(0)\sim\frac{1}{z}(I\dots J)(I\dots J)$ can also be generated in the single $I-J$ wick contraction studied in the last section.

It is a tedious work to analyze the combinatorial factor of the wick contractions in the general situation. To simplify the discussion, we study some special cases.

The simplest example we can look at is the OPE of $I_{a_1}Z^1J_{a_2}$ and $I_{b_1}Z^2J_{b_2}$. A single $Z-Z$ contraction gives us

$$(I_{a_1}Z^1J_{a_2})(z)(I_{b_1}Z^2J_{b_2})(0) \sim \frac{1}{z}(I_{a_1}J_{b_2})(I_{b_1}J_{a_2})(0). \tag{5.54}$$

A single $I-J$ contraction gives us

$$\begin{aligned}&(I_{a_1}Z^1J_{a_2})(z)(I_{b_1}Z^2J_{b_2})(0) + (I_{a_1}Z^1J_{a_2})(z)(I_{b_1}Z^2J_{b_2})(0)\\ &\sim -\frac{1}{2z}\sum_c(\delta_{a_2b_1}(I_{a_1}J_c)(I_cJ_{b_2})(0)+\delta_{a_1b_2}(I_{b_1}J_c)(I_cJ_{a_2})),\end{aligned} \tag{5.55}$$

where we omitted the operator $IZ^{(1}Z^{2)}J$ that is analyzed in the last section.

More generally, we consider the OPE between $E^{(1)}$ and $E^{(n')}$. A wick contraction of a single $Z-Z$ pair gives us

$$E^{(1)}_{a_1a_2}(\lambda;z)E^{(n')}_{b_1b_2}(\lambda',0) \sim \sum_{n''}\frac{[\lambda\lambda']}{z}E^{(n'')}_{a_1b_2}(\lambda',0)E^{(n'-n''-1)}_{b_1a_2}(\lambda',0). \tag{5.56}$$

Expanding the above formula with 5.41, we find

$$\begin{aligned}E^{(1,0)}_{a_1a_2}(z)B^{(r,s)}_{b_1b_2}(0) \sim \frac{1}{z}\sum_{u,v}\binom{r+s}{r}^{-1}\binom{u+v}{u}\binom{r+s-u-v-1}{r-u}\\ \times E^{(u,v)}_{a_1b_2}E^{(r-u,s-v-1)}_{b_1a_2}(0).\end{aligned} \tag{5.57}$$

This matches with the first term of the boundary OPE of 5.31. To match the remaining terms, we consider the single $I-J$ contraction. As in previous discussion, we use the BRST relation and find that $I\ldots[Z(\lambda),Z(\lambda')]\ldots J$ is cohomologous to $[\lambda\lambda']I\ldots JI\ldots J$. Therefore, after dropping the symmetrized operator that we have discussed, we find the following OPE

$$\begin{aligned}&E^{(1)}_{a_1a_2}(\lambda;z)E^{(n')}_{b_1b_2}(\lambda',0)\\ &\sim -\frac{[\lambda\lambda']}{z}\sum_{n''}\sum_c\left(\frac{n'-n''}{n'+1}\delta_{a_2b_1}E^{(n'')}_{a_1c}(\lambda',0)E^{(n'-n''-1)}_{cb_2}(\lambda',0)\right.\\ &\left.+\frac{n''+1}{n'+1}\delta_{a_1b_2}E^{(n'')}_{b_1c}(\lambda',0)E^{(n'-n''-1)}_{ca_2}(\lambda',0)\right).\end{aligned} \tag{5.58}$$

Expanding this formula, we find

$$E^{(1,0)}_{a_1a_2}(z)B^{(r,s)}_{b_1b_2}(0) \sim -\frac{1}{z}\sum_{u,v}\sum_{c}\binom{r+s}{r}^{-1}\binom{u+v}{u}\binom{r+s-u-v-1}{r-u}$$
$$\times\left(\frac{r+s-u-v}{r+s+1}\delta_{a_2b_1}E^{(u,v)}_{a_1c}B^{(r-u,s-v-1)}_{cb_2}(0)\right.$$
$$\left.+\frac{u+v+1}{r+s+1}\delta_{a_1b_2}E^{(u,v)}_{b_1c}B^{(r-u,s-v-1)}_{ca_2}(0)\right). \tag{5.59}$$

This gives the remaining terms of the boundary OPE of 5.31.

As we have discussed, the deformed geometry will introduce deformation to the higher operations and gives us deformed interaction vertices. We expect that the deformed quartic action $I^{(3,k)}_{O}$ corresponds to the wick contraction of $k+1$ adjacent $Z-Z$ fields, and also the wick contraction of a single $I-J$ together with k adjacent $Z-Z$ fields. It will be interesting to explore these OPE.

A

HOMOTOPY ALGEBRA AND HOMOTOPY TRANSFER

Since this paper heavily uses techniques from homotopy algebra. We briefly review this topic in this appendix. We recommend the survey [Val14] for a detailed review.

Convention and Koszul sign rule

First, we fix the convention for our discussion. We work with $\mathbb{Z}$-graded $\mathbb{C}$-vector space

$$V = \bigoplus_{n\in\mathbb{Z}} V_n. \tag{A.1}$$

The grading n is related to the ghost number in physics. The degree of an element $v \in V_n$ is denoted by $|v| = n$, and such a v is called a homogeneous element.

For V and W two graded vector spaces, the tensor product $V \otimes W$ and the Hom space $\mathrm{Hom}(V, W)$ has the following grading

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j, \quad \mathrm{Hom}(V, W)_n = \bigoplus_{i} \mathrm{Hom}(V_i, W_{i+n}).$$

We denote the Koszul sign braiding on tensor products to be

$$\begin{aligned}\tau_{V,W} : V \otimes W &\to W \otimes V, \\ v \otimes w &\mapsto (-1)^{|v||w|} w \otimes v.\end{aligned}$$

The above sign rule induces naturally a sign rule for the action of the symmetric group S_n on the n-th tensor product $V^{\otimes n}$

$$\sigma : \; v_1 \otimes v_2 \otimes \cdots \otimes v_n \to \epsilon(\sigma, v) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots v_{\sigma(n)},$$

where $\epsilon(\sigma, v)$ is called the Koszul sign.

For V a $\mathbb{Z}$ graded vector space, we denote $V[n]$ the degree n-shifted space such that

$$V[n]_m := V_{n+m}. \tag{A.2}$$

We also use the notation of suspension sV and desuspension $s^{-1}V$ as follows

$$sV := V[1], \quad s^{-1}V := V[-1]. \tag{A.3}$$

We can also regard s as a degree -1 linear map $s : V \to V[1]$. For a homogeneous $a \in V$, we have $sa \in V[1]$ and $|sa| = |a| - 1$. Similarly, s^{-1} can be regarded as a degree 1 linear map, such that $s^{-1}s = ss^{-1} = 1$.

Homotopy algebra

In this appendix, we review the definition of various homotopy algebras including A_∞, C_∞ and L_∞ algebras.

A_∞ ALGEBRA

Definition A.1. An A_∞ algebra is a graded vector space $A = \{A_n\}_{n\in\mathbb{Z}}$ with a collection of multi-linear operations

$$m_n : A^{\otimes n} \to A \text{ of degree } n-2 \text{ for all } n \geq 1, \tag{A.4}$$

which satisfy the following relations:

$$\sum_{k=1}^{n}\sum_{j=0}^{n-k}(-1)^{jk+(n-j-k)}m_{n-k+1}\circ(\mathrm{id}^{\otimes j}\otimes m_k\otimes \mathrm{id}^{\otimes n-j-k}) = 0. \tag{A.5}$$

Let's demonstrate the above relations for small values of n:

1. $n = 1$. We have $m_1 \circ m_1 = 0$, which means that m_1 is a differential on A. We also denote $d = m_1$.
2. $n = 2$. We have

$$dm_2(x_1, x_2) = m_2(dx_1, x_2) + (-1)^{|x_1|}m_2(x_1, dx_2). \tag{A.6}$$

   This relation implies m_1 is a derivation with respect to the binary product m_2.
3. $n = 3$. The relation yields

$$\begin{aligned} &m_2(m_2(x_1, x_2), x_3) - m_2(x_1, m_2(x_2, x_3)) = \\ &\quad dm_3(x_1, x_2, x_3) + m_3(dx_1, x_2, x_3) + m_3(x_1, dx_2, x_3) + m_3(x_1, x_2, dx_3). \end{aligned} \tag{A.7}$$

An A_∞ algebra with $m_k = 0$ for $k \geq 3$ is also called a differential graded associative (dga) algebra. For example, the tangential Cauchy-Riemann complex $(\Omega_b^{0,\bullet}(S^3), \bar{\partial}_b, \cdot)$ is a dga algebra

There is an equivalent definition of A_∞ algebra in terms of coderivation. We introduce the reduced tensor coalgebra

$$\bar{T}^c(V) = \bigoplus_{n\geq 1} V^{\otimes n}, \tag{A.8}$$

with comultiplication given by

$$\bar{\Delta}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n-1} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n). \tag{A.9}$$

Recall that a coderivation on a coalgebra (C, Δ) is a map $L : C \to C$ such that $\Delta \circ L = (L \otimes 1 + 1 \otimes L)\Delta$.

For the (reduced) tensor coalgebra $\bar{T}^c(V)$, a coderivation on it is completely determined by its projection $p_V \circ L : \bar{T}^c(V) \to \bar{T}^c(V) \to V$. To see this, we first notice that $p_V \circ L$ is given by a set of maps $L_k \in \mathrm{Hom}(V^{\otimes k}, V), k \geq 1$. Given this set of maps, the coderivation is uniquely given by

$$L = \sum_{i\geq 1}^{n} \sum_{j=0}^{n-i} \mathbb{1}^{\otimes j} \otimes L_i \otimes \mathbb{1}^{n-i-j}. \tag{A.10}$$

The structure of an A_∞ algebra on A can be compactly organized into the structure of a square zero coderivation on $\bar{T}^c(sA)$.

Proposition A.2. *The following data are equivalent*

- *A collection of linear maps $m_k : A^{\otimes k} \to A$ of degree $2-k$ satisfying A_∞ relation.*
- *A degree 1 coderivation b on $\bar{T}^c(A[1])$ satisfying $b^2 = 0$.*

Proof. We only sketch the proof here and refer to [GJ+90] for more details. Given linear maps $m_k : A^{\otimes k} \to A$, we define maps $b_k : (sA)^{\otimes k} \to sA$ by

$$b_k = s \circ m_k \circ (s^{-1})^{\otimes k}. \tag{A.11}$$

The maps b_k further define a coderivation b on $\bar{T}^c(A[1])$ through A.10. One can check that the requirement $b^2 = 0$ is equivalent to the A_∞ relations A.5. □

C_∞ ALGEBRA In this paper, the dga algebras that we studied satisfy additional properties of being graded commutative.

$$m_2(a, b) = (-1)^{|a||b|} m_2(b, a). \tag{A.12}$$

Such algebras are called differential graded commutative (dgc) algebra. The homotopy version of dgc algebra is called C_∞ algebra, which we now define.

A (p,q)-shuffle is a permutation $\sigma \in S_{p+q}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p), \ \ \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q). \tag{A.13}$$

We denote by $Sh(p,q)$ the subset of (p,q)-shuffles in S_{p+q}.

We have introduced the reduced tensor coalgebra $\bar{T}^c(V) = \bigoplus_{n\geq 1} V^{\otimes n}$. It becomes a Hopf algebra when equipped with the multiplication map called shuffle product

$$sh((a_1,\dots,a_p)\otimes(a_{p+1},\dots a_{p+q})) = \sum_{\sigma\in Sh(p,q)} \epsilon(\sigma,a)(a_{\sigma^{-1}(1)},a_{\sigma^{-1}(2)},\dots a_{\sigma^{-1}(p+q)}). \tag{A.14}$$

Definition A.3. A C_∞-algebra structure on a graded vector space $A = \{A_n\}_{n\in\mathbb{Z}}$ is an A_∞ structure $(A,\{m_n\}_{n\geq 1})$ such that the set of maps $\{b_k = s\circ m_k\circ(s^{-1})^{\otimes k}, k\geq 1\}$ vanish on the image of the shuffle product $sh : T^c(sA)\otimes T^c(sA)\to T^c(sA)$.

For example, the element $sa\otimes sb + (-1)^{(|a|+1)(|b|+1)} sb\otimes sa$ is in the image of the shuffle product. Vanishing of b_2 on this element is the same as the graded commutativity of m_2.

L_∞ ALGEBRA We also introduce the notion of L_∞ algebra.

Definition A.4. Let $\mathfrak{g} = \{\mathfrak{g}^n\}_{n\in\mathbb{Z}}$ be a graded vector space. An L_∞ structure on $\mathfrak{g}$ is a collection of multi-linear maps

$$l_n : \mathfrak{g}^{\otimes n} \to \mathfrak{g} \ \text{ of degree } n-2 \text{ for all } n\geq 1, \tag{A.15}$$

that are graded skew-symmetric:

$$l_n(x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(n)}) = (-1)^\sigma \epsilon(\sigma,x) l_n(x_1,\dots,x_n), \quad \text{for all } \sigma\in S_n, \tag{A.16}$$

and satisfy the following relations:

$$\begin{aligned}&\sum_{k=1}^n (-1)^k \sum_{\sigma\in Sh(k,n-k)} (-1)^\sigma \epsilon(\sigma,x)\\ &l_{n-k-1}(l_k(x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(k)}),x_{\sigma^{-1}(k+1)},\dots,x_{\sigma^{-1}(n)}) = 0.\end{aligned} \tag{A.17}$$

Let us analyze the defining relations for small values of n:

1. $n=1$. The relation is $l_1\circ l_1 = 0$, which means that l_1 is a differential on $\mathfrak{g}$.

2. $n = 2$. We have

$$l_1(l_2(x_1, x_2)) = l_2(l_1(x_1), x_2) + (-1)^{|x_1|} l_2(x_1, l_1(x_2)) \tag{A.18}$$

which says that l_1 is a derivation with respect to the binary map l_2.

3. $n = 3$. The relations yields

$$\begin{aligned} & l_2(l_2(x_1, x_2), x_3) + (-1)^{(|x_1|+|x_2|)|x_3|} l_2(l_2(x_3, x_1), x_2) \\ & + (-1)^{(|x_2|+|x_3|)|x_1|} l_2(l_2(x_2, x_3), x_1) = l_1 l_3(x_1, x_2, x_3) + l_3(l_1(x_1), x_2, x_3) \\ & + (-1)^{|x_1|} l_3(x_1, l_1(x_2), x_3) + (-1)^{|x_1|+|x_2|} l_3(x_1, x_2, l_1(x_3)). \end{aligned} \tag{A.19}$$

which says that l_2 satisfies Jacobi identities up to homotopy given by l_3.

There is a similar characterization of L_∞ algebra in terms of a coderivation. Instead of the tensor coalgebra, we consider the reduced symmetric coalgebra $\bar{S}^c(V)$ where

$$\bar{S}^c(V) = \bigoplus_{n \geq 1} \mathrm{Sym}^n(V).$$

The coproduct $\bar{\Delta} : \bar{S}^c(V) \to \bar{S}^c(V) \otimes \bar{S}^c(V)$ is defined by

$$\begin{aligned} & \bar{\Delta}(v_1 \cdot v_2 \dots v_n) \\ & = \sum_{i=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}(i,n-i)} \epsilon(\sigma, v)(v_{\sigma^{-1}(1)} \cdot v_{\sigma^{-1}(2)} \dots v_{\sigma^{-1}(i)}) \otimes (v_{\sigma^{-1}(i+1)} \cdots v_{\sigma^{-1}(n)}). \end{aligned} \tag{A.20}$$

Then we have

Proposition A.5. *The following data are equivalent*

- *A collection of linear maps $l_k : \mathfrak{g}^{\otimes k} \to \mathfrak{g}$ of degree $2 - k$ satisfying L_∞ relation.*
- *A degree 1 coderivation Q on $\bar{S}^c(\mathfrak{g}[1])$ satisfying $Q^2 = 0$.*

Homological perturbation lemma

We introduce an important technical tool called the homological perturbation lemma. We refer to [Cra04] for a more detailed discussion.

Let us first consider the following homotopy data of chain complexes.

Definition A.6. A special deformation retract (SDR) from a cochain complex (A, d_A) to (H, d_H) consists of the following data

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H), \tag{A.21}$$

where i, p are cochain maps and h is a degree -1 map on A, such that

$$i \circ p - \mathbb{1}_A = d_A \circ h + h \circ d_A, \quad p \circ i = \mathbb{1}_H, \tag{A.22}$$

and

$$h \circ i = 0, \; p \circ h = 0, \; h \circ h = 0. \tag{A.23}$$

Consider a perturbation δ to the differential on A:

$$d'_A = d_A + \delta, \;\; d'^2_A = 0 \tag{A.24}$$

The perturbation is called small if $(1 - \delta h)$ is invertible.

Lemma A.7. *(Homological perturbation lemma) Given a SDR data as A.25 and a small perturbation, there is a new SDR:*

$$h \circlearrowright (A, d'_A) \underset{i'}{\overset{p'}{\rightleftarrows}} (H, d'_H) \tag{A.25}$$

where the maps above are defined by

$$\begin{aligned} d'_H &= d_H + p(1 - \delta h)^{-1} \delta i, \\ h' &= h + h(1 - \delta h)^{-1} \delta h, \\ p' &= p + p(1 - \delta h)^{-1} \delta h, \\ i' &= i + h(1 - \delta h)^{-1} \delta i. \end{aligned} \tag{A.26}$$

The homological perturbation lemma can be regarded as a substitution of the spectral sequence techniques, which provides explicit formulae.

Homotopy transfer

Given a dga algebra (or an A_∞ algebra in general) and a chain complex quasi-isomorphic to it, homotopy transfer theorem [Kad80] gives the complex an A_∞ structure. In particular, one gets an A_∞ structure on the cohomology of a dga algebra. We emphasize that there are different approaches to construct this A_∞ structure. In this appendix, we take the approach using homological perturbation lemma [Ber14].

Given a dga algebra $(A, d, \cdot)$. Suppose we can find a SDR to its cohomology $H = H^\bullet(A)$

$$h \circlearrowright (A, d) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H = 0). \tag{A.27}$$

Recall that the dga algebra structure on A is equivalent to a differential b on $\bar{T}^c(sA)$. Therefore, we first extend the above SDR to the corresponding tensor coalgebra

Proposition A.8. *The following is a SDR*

$$Th^s \circlearrowright (\bar{T}^c(sA), Td^s) \underset{Ti^s}{\overset{Tp^s}{\rightleftarrows}} (\bar{T}^c(sH), 0), \tag{A.28}$$

where the differential Td^s is defined by $Td^s = \sum_{n\geq 1}\sum_{i=0}^{n-1} \mathbb{1}^i \otimes (s \circ d \circ s^{-1}) \otimes \mathbb{1}^{n-i-1}$. The projection and inclusion maps are defined by $Tp^s = \sum_{n\geq 1}(s \circ p \circ s^{-1})^{\otimes n}$ and $Ti^s = \sum_{n\geq 1}(s \circ i \circ s^{-1})^{\otimes n}$. The deformation retract is defined as

$$Th^s = \sum_{n\geq 1}\sum_{i=0}^{n-1} \mathbb{1}^{\otimes i} \otimes (s \circ h \circ s^{-1}) \otimes (s \circ i \circ p \circ s^{-1})^{\otimes n-i-1}.$$

The product $\cdot$ on the dga algebra A defined a map $b_2 : (sA)^{\otimes 2} \to sA$ and extend to a map $\delta : \bar{T}^c(sA) \to \bar{T}^c(sA)$. Together with the differential Td^s, the sum $b = Td^s + \delta : \bar{T}^c(sA) \to \bar{T}^c(sA)$ encode the dga algebra structure A in the sense of Proposition A.2. Now we can regard δ as a perturbation to the differential and apply the homological perturbation lemma. We have the following new SDR

$$h' \circlearrowright (\bar{T}^c(sA), Td^s + \delta) \underset{i'}{\overset{p'}{\rightleftarrows}} (\bar{T}^c(sH), b'). \tag{A.29}$$

The homological perturbation lemma provides us a formula for all the maps h', p', i'. However, only the differential b_H matter to us as it encodes the transferred A_∞ structure on the cohomology H. We have

$$b' = Tp^s \circ (1 - \delta \circ Th^s)^{-1} \circ \delta \circ Ti^s = \sum_{n\geq 0} Tp^s \circ (\delta \circ Th^s)^n \circ \delta \circ Ti^s. \tag{A.30}$$

If we further expand the above formula into components, we find the usual tree description of the transferred A_∞ structure on H. Let $\mathrm{PBT_n}$ be the set of planar binary rooted trees with n leaves. We consider the following construction that assigns each $T \in \mathrm{PBT}_n$ an n array operation

m_T on H. The operation m_T is obtained by putting i on the leaves, m on the vertices, h on the internal edges and p on the root. Then we consider

$$m_n = \sum_{T \in PBT_n} (\pm) m_T, \tag{A.31}$$

where the $(\pm)$ sign can be tracked by a careful analysis of the Koszul sign rule in A.30.

Theorem A.9. *The operations $\{m_n\}_{n\geq 2}$ defined on H by the formulae A.31 form an A_∞-algebra structure on H.*

Moreover, the transferred A_∞-algebra $(H, \{m_n\}_{n\geq 2})$ is A_∞ quasi-isomorphic to the dg algebra $(A, d_A, \cdot)$.

In the example of our study, the tangential Cauchy-Riemann complex $(\Omega_b^{0,\bullet}(S^3), \bar{\partial}_b, \cdot)$ is graded commutative. We are interested in the transferred structure for dgc algebra. This scenario is analyzed in [ZG06]. For $(A, d, \cdot)$ a dgc algebra, if we regard it as a dga algebra, the A_∞ structure constructed by A.31 actually defines a C_∞ structure.

For homotopy transfer of dg Lie algebra and L_∞ algebra, a similar result can be established. We start with a dg Lie algebra $(L, d, [-,-])$ and consider the transferred structure on its cohomology $\mathfrak{g} = H^\bullet(L)$. Suppose we are given the following SDR

$$h \circlearrowright (L, d) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H = 0). \tag{A.32}$$

The tensor trick can be extended to the symmetric case

$$Sh^s \circlearrowright (\bar{S}^c(sL), Sd^s) \underset{Si^s}{\overset{Sp^s}{\rightleftarrows}} (\bar{S}^c(s\mathfrak{g}), 0), \tag{A.33}$$

where the differential Sd^s is defined by $Sd^s = \sum_{n\geq 1} \sum_{i=0}^{n-1} \mathbb{1}^i \otimes (s \circ d \circ s^{-1}) \otimes \mathbb{1}^{n-i-1}$. The projection and inclusion maps are defined by $Sp^s = \sum_{n\geq 1} (s \circ p \circ s^{-1})^{\otimes n}$ and $Si^s = \sum_{n\geq 1} (s \circ i \circ s^{-1})^{\otimes n}$. The deformation retract is defined as

$$Sh^s = \sum_{n\geq 1} \frac{1}{n!} \sum_{\sigma \in S_n} \sigma^{-1} \left(\sum_{i=0}^{n-1} \mathbb{1}^{\otimes i} \otimes (s \circ h \circ s^{-1}) \otimes (s \circ i \circ p \circ s^{-1})^{\otimes n-i-1} \right) \sigma.$$

The Lie bracket $[-,-]$ on L defined a map $Q_2 : (sL)^{\otimes 2} \to sL$ and extend to a map $\delta : \bar{S}^c(sL) \to \bar{T}^c(sL)$. We add this differential to the above SDR as a perturbation. Then we have a new SDR, with a new differential on $\bar{S}^c(s\mathfrak{g})$ given by the following

$$Q' = Sp^s \circ (1 - \delta \circ Sh^s)^{-1} \circ \delta \circ Si^s = \sum_{n \geq 0} Sp^s \circ (\delta \circ Sh^s)^n \circ \delta \circ Si^s. \quad \text{(A.34)}$$

We can expand the above formula into components. This gives us the usual tree description of the transferred L_∞ structure on $\mathfrak{g}$. Let $\mathrm{BT_n}$ be the set of binary rooted trees with n leaves. In this case, we need to consider trees not necessarily planar, which means edges can cross each other. We consider the following construction that assigns each $T \in \mathrm{BT}_n$ an n array operation l_T on H. The operation l_T is obtained by putting i on the leaves, $[-,-]$ on the vertices, h on the internal edges and p on the root. We consider

$$l_n = \sum_{T \in BT_n} (\pm) l_T. \quad \text{(A.35)}$$

Then the operations $\{l_n\}_{n \geq 2}$ defined an L_∞-algebra structure on $\mathfrak{g}$. Moreover, the L_∞ algebra $(\mathfrak{g}, l_2, l_3, \dots)$ is L_∞ quasi-isomorphic to the dg Lie algebra $(L, d, [-,-])$.

B SOME COMPUTATIONS

B.1 PRODUCT OF S^3 HARMONIC POLYNOMIALS

In this section, we compute the product of two arbitrary S^3 harmonics. We first recall the formula 4.39 that decomposes a harmonic polynomial into sum of monomials

$$e_m^{(j,\bar{j})} = \sum_l \lambda^{-1}_{j,\bar{j},0} C^{j,\bar{j};j+\bar{j}}_{m-l,l;m} e^{(j)}_{m-l} \bar{e}^{\bar{j}}_l, \tag{B.1}$$

where

$$\lambda_{j,\bar{j},k} = (-1)^k \sqrt{\frac{(2j+1)!(2\bar{j}+1)!}{k!(2j+2\bar{j}-k+1)!}}. \tag{B.2}$$

Then we can write

$$M(e^{(j_1,\bar{j}_1)}_{m_1}, e^{(j_2,\bar{j}_2)}_{m_2}) = \sum_{l_1,l_2} \lambda^{-1}_{j_1,\bar{j}_1,0} \lambda^{-1}_{j_2,\bar{j}_2,0} C^{j_1,\bar{j}_1;j_1+\bar{j}_1}_{m_1-l_1,l_1;m_1} C^{j_2,\bar{j}_2;j_2+\bar{j}_2}_{m_2-l_2,l_2;m_2} M(e^{(j_1)}_{m_1-l_1} \bar{e}^{(\bar{j}_1)}_{l_1}, e^{(j_2)}_{m_2-l_2} \bar{e}^{(\bar{j}_2)}_{l_2}). \tag{B.3}$$

To compute $M(e^{(j_1)}_{m_1-l_1} \bar{e}^{(\bar{j}_1)}_{l_1}, e^{(j_2)}_{m_2-l_2} \bar{e}^{(\bar{j}_2)}_{l_2})$, we consider the product $e^{(j_1)}_{m_1-l_1} e^{(j_2)}_{m_2-l_2}$ and $\bar{e}^{(\bar{j}_1)}_{l_1} \bar{e}^{(\bar{j}_2)}_{l_2}$ separately. We find

$$\begin{aligned} M(e^{(j_1)}_{m_1-l_1} \bar{e}^{(\bar{j}_1)}_{l_1}, e^{(j_1)}_{m_1-l} \bar{e}^{(\bar{j}_1)}_{l_2}) &= \sqrt{\frac{(2j_1+1)(2j_2+1)(2\bar{j}_1+1)(2\bar{j}_2+1)}{(2j_1+2j_2+1)(2\bar{j}_1+2\bar{j}_2+1)}} \\ &\times C^{j_1,j_2;j_1+j_2}_{m_1-l_1,m_2-l_2,m_1+m_2-l_1-l_2} C^{\bar{j}_1,\bar{j}_2;\bar{j}_1+\bar{j}_2}_{l_1,l_2,l_1+l_2} M(e^{(j_1+j_2)}_{m_1+m_2-l_1-l_2}, \bar{e}^{(\bar{j}_1+\bar{j}_2)}_{l_1+l_2}) \\ = \sum_k \lambda_{j_1+j_2,\bar{j}_1+\bar{j}_2,k} &\sqrt{\frac{(2j_1+1)(2j_2+1)(2\bar{j}_1+1)(2\bar{j}_2+1)}{(2j_1+2j_2+1)(2\bar{j}_1+2\bar{j}_2+1)}} \\ &\times C^{j_1,j_2;j_1+j_2}_{m_1-l_1,m_2-l_2;m_1+m_2-l_1-l_2} C^{\bar{j}_1,\bar{j}_2;\bar{j}_1+\bar{j}_2}_{l_1,l_2;l_1+l_2} C^{j_1+j_2,\bar{j}_1+\bar{j}_2;j_1+j_2+\bar{j}_1+\bar{j}_2-k}_{m_1+m_2-l_1-l_2,l_1+l_2;m_1+m_2} e^{(j_1+j_2-\frac{k}{2},\bar{j}_1+\bar{j}_2-\frac{k}{2})}_{m_1+m_2}. \end{aligned} \tag{B.4}$$

Therefore

$$
\begin{aligned}
M(e^{(j_1,\bar{j}_1)}_{m_1}, e^{(j_2,\bar{j}_2)}_{m_2}) &= \sum_k \sum_{l_1,l_2} \sqrt{\frac{(2j_1+1)(2j_2+1)(2\bar{j}_1+1)(2\bar{j}_2+1)}{(2j_1+2j_2+1)(2\bar{j}_1+2\bar{j}_2+1)}} \\
&\times \lambda^{-1}_{j_1,\bar{j}_1,0}\lambda^{-1}_{j_2,\bar{j}_2,0}\lambda_{j_1+j_2,\bar{j}_1+\bar{j}_2,k} C^{j_1,\bar{j}_1;j_1+\bar{j}_1}_{m_1-l_1,l_1;m_1} C^{j_2,\bar{j}_2;j_2+\bar{j}_2}_{m_2-l_2,l_2;m_2} C^{j_1,j_2;j_1+j_2}_{m_1-l_1,m_2-l_2;m_1+m_2-l_1-l_2} \\
&\times C^{\bar{j}_1,\bar{j}_2;\bar{j}_1+\bar{j}_2}_{l_1,l_2;l_1+l_2} C^{j_1+j_2,\bar{j}_1+\bar{j}_2;j_1+j_2+\bar{j}_1+\bar{j}_2-k}_{m_1+m_2-l_1-l_2,l_1+l_2;m_1+m_2} e^{(j_1+j_2-\frac{k}{2},\bar{j}_1+\bar{j}_2-\frac{k}{2})}_{m_1+m_2} \\
&= \sum_k \lambda^{-1}_{j_1,\bar{j}_1,0}\lambda^{-1}_{j_2,\bar{j}_2,0}\lambda_{j_1+j_2,\bar{j}_1+\bar{j}_2,k} \begin{Bmatrix} j_1 & j_2 & j_1+j_2 \\ \bar{j}_1 & \bar{j}_2 & \bar{j}_1+\bar{j}_2 \\ j_1+\bar{j}_1 & j_2+\bar{j}_2 & j_1+j_2+\bar{j}_1+\bar{j}_2-k \end{Bmatrix} \\
&\times \sqrt{(2j_1+1)(2j_2+1)(2\bar{j}_1+1)(2\bar{j}_2+1)(2j_1+2\bar{j}_1+1)(2j_2+2\bar{j}_2+1)} \\
&\times C^{j_1+\bar{j}_1,j_2+\bar{j}_2;j_1+j_2+\bar{j}_1+\bar{j}_2-k}_{m_1,m_2;m_1+m_2} e^{(j_1+j_2-\frac{k}{2},\bar{j}_1+\bar{j}_2-\frac{k}{2})}_{m_1+m_2},
\end{aligned}
\tag{B.5}
$$

where $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}$ is the Wigner $9-j$ symbol.

In our study of the higher product on the CR cohomology, a constantly appearing computation is the product of the form $M(e^{(j_1-\frac{i}{2},\bar{j}_1-\frac{i}{2})}_{m_1}, \bar{e}^{(\bar{j}_2)}_{m_2})$. One can use the above general formula to compute this. Here, we derive an alternative formula that is more succinct. The key is that we use a variation of 4.39 to expand the harmonics polynomial $e^{(j_1-\frac{i}{2},\bar{j}_1-\frac{i}{2})}_{m_1}$

$$
e^{(j_1-\frac{i}{2},\bar{j}_1-\frac{i}{2})}_{m_1} = \sum_l \lambda^{-1}_{j_1,\bar{j}_1,i} C^{j_1,\bar{j}_1;j_1+\bar{j}_1-i}_{m_1-l,l;m_1} e^{(j_1)}_{m_1-l}\bar{e}^{(\bar{j}_1)}_l. \tag{B.6}
$$

Using this, we find

$$
\begin{aligned}
M(e^{(j_1-\frac{i}{2},\bar{j}_1-\frac{i}{2})}_{m_1}, \bar{e}^{(\bar{j}_2)}_{m_2}) &= \sum_l \lambda^{-1}_{j_1,\bar{j}_1,i} \sqrt{\frac{(2\bar{j}_1+1)(2\bar{j}_2+1)}{(2\bar{j}_1+2\bar{j}_2+1)}} \\
&\times C^{j_1,\bar{j}_1;j_1+\bar{j}_1-i}_{m_1-l,l;m_1} C^{\bar{j}_1,\bar{j}_2;\bar{j}_1+\bar{j}_2}_{l,m_2;l+m_2} M(e^{(j_1)}_{m_1-l}, \bar{e}^{(\bar{j}_1+\bar{j}_2)}_{l+m_2}) \\
&= \sum_{k\geq 0}\sum_l \lambda^{-1}_{j_1,\bar{j}_1,i}\lambda_{j_1,\bar{j}_1+\bar{j}_2;k} \sqrt{\frac{(2\bar{j}_1+1)(2\bar{j}_2+1)}{(2\bar{j}_1+2\bar{j}_2+1)}} \\
&\times C^{j_1,\bar{j}_1;j_1+\bar{j}_1-i}_{m_1-l,l;m_1} C^{\bar{j}_1,\bar{j}_2;\bar{j}_1+\bar{j}_2}_{l,m_2;l+m_2} C^{j_1,\bar{j}_1+\bar{j}_2;j_1+\bar{j}_1+\bar{j}_2-k}_{m_1-l,m_2+l;m_1+m_2} e^{(j_1-\frac{k}{2},\bar{j}_1+\bar{j}_2-\frac{k}{2})}_{m_1+m_2} \\
&= \sum_{k\geq 0} (-1)^{2(j_1+\bar{j}_1+\bar{j}_2)-k} \lambda^{-1}_{j_1,\bar{j}_1,i}\lambda_{j_1,\bar{j}_1+\bar{j}_2;k} \sqrt{(2\bar{j}_1+1)(2\bar{j}_2+1)(2j_1+2\bar{j}_1-2i+1)} \\
&\times \begin{Bmatrix} \bar{j}_1 & j_1 & j_1+\bar{j}_1-i \\ j_1+\bar{j}_1+\bar{j}_2-k & \bar{j}_2 & \bar{j}_1+\bar{j}_2 \end{Bmatrix} C^{j_1+\bar{j}_1-i,\bar{j}_2;j_1+\bar{j}_1+\bar{j}_2-k}_{m_1,m_2;m_1+m_2} e^{(j_1-\frac{k}{2},\bar{j}_1+\bar{j}_2-\frac{k}{2})}_{m_1+m_2}.
\end{aligned}
\tag{B.7}
$$

Though we write the summation range as $k \geq 0$, the Wigner $6j$ symbol actually constraint it such that $k \geq i$ and $k \leq \min\{2j_1, 2\bar{j}_1 + 2\bar{j}_2\}$.

B.2 SOME IDENTITIES INVOLVING POCHHAMMER SYMBOLS

In this appendix, we review some identities involving Pochhammer symbols that are used in the calculation of holography chiral algebra. In the main text, we introduced the descending Pochhammer symbols

$$[a]_n := a(a-1)\ldots(a-n+1) = \frac{(a)!}{(a-n)!}. \tag{B.8}$$

We also introduce the ascending Pochhammer symbol

$$(a)^{(n)} := a(a+1)\ldots(a+n-1) = \frac{(a+n-1)!}{(a-1)!}. \tag{B.9}$$

The descending and ascending Pochhammer symbols are related to one another by

$$(a)^{(n)} = [a+n-1]_n. \tag{B.10}$$

The hypergeometric function ${}_2F_1$ is defined as a power series using the ascending Pochhammer symbol

$${}_2F_1(a,b,c,z) = \sum_{i=0}^{\infty} \frac{(a)^{(i)}(b)^{(i)}}{(c)^{(i)}} \frac{1}{i!} z^i. \tag{B.11}$$

The series terminates if either a or b is a nonpositive integer, in which case the function reduces to a polynomial:

$${}_2F_1(-k,b,c,z) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{(b)^{(i)}}{(c)^{(i)}} z^i. \tag{B.12}$$

The following result is important in obtaining various generalizations of the Chu–Vandermonde's identity.

Proposition B.1 ([FPW12]). *For any $k \geq 1$, $x, y \in \mathbb{R}_+$, and $a, b > 0$, we have*

$$\sum_{i=0}^{k} \binom{k}{i} x^i y^{n-i} (a)^{(i)} (b)^{(n-i)} = y^k (a+b)^{(k)} \, {}_2F_1(-k, a, a+b, 1-\frac{x}{y}). \tag{B.13}$$

Taking $x, y = 1$ in the above formula we obtain the Chu–Vandermonde's identity

$$\sum_{i=0}^{k} \binom{n}{i} (a)^{(i)} (b)^{(k-i)} = (a+b)^{(k)}. \tag{B.14}$$

Corollary B.2.

$$\sum_{i=l}^{n} \binom{i}{l} \binom{k}{i} (a)^{(i)} (b)^{(k-i)} = \binom{k}{l} \frac{(a+b)^{(k)}}{(a+b)^{(l)}} (a)^{(l)}. \tag{B.15}$$

Proof. Letting $x \to 1+x, y \to 1$ in the formula B.13, we obtain the following

$$\sum_{i=0}^{k} \binom{k}{i} (1+x)^i (a)^{(i)} (b)^{(k-i)} = (a+b)^{(k)} \, {}_2F_1(-n, a, a+b, -x). \tag{B.16}$$

Expanding both side into a series of x we obtain the formula B.15. □

Corollary B.3. *We have the following identity*

$$\sum_{i=0}^{k} \binom{k}{i} \frac{1}{[a]_i [b]_{k-i}} = \frac{[a+b-k+1]_k}{[a]_k [b]_k}. \tag{B.17}$$

Proof.

$$\begin{aligned} \sum_{i=0}^{k} \binom{k}{i} \frac{1}{[a]_i [b]_{k-i}} &= \sum_{i=0}^{k} \binom{k}{i} \frac{(a-k+k-i)!(b-k+i)!}{a!b!} \\ &= \sum_{i=0}^{k} \binom{k}{i} \frac{(a-k+1)^{(k-i)}(a-k)!(b-k+1)^{(i)}(b-k)!}{a!b!} \\ &= (a+b-2k+2)^{(k)} \frac{(a-k)!(b-k)!}{a!b!} \\ &= \frac{[a+b-k+1]_k}{[a]_k [b]_k}, \end{aligned} \tag{B.18}$$

where we used the Chu–Vandermonde's identity B.14 in the third line. □

Corollary B.4. *We have the following identity*

$$\sum_{i=l}^{k} \binom{i}{l} \binom{k}{i} \frac{1}{[a]_i [b]_{k-i}} = \binom{k}{l} \frac{[a+b-k+1]_k}{[a]_{k-l} [b]_k [a+b-2k+l+1]_l}. \tag{B.19}$$

Proof.

$$\begin{aligned}\sum_{i=l}^{k}\binom{i}{l}\binom{k}{i}\frac{1}{[a]_i[b]_{k-i}} &= \sum_{i=l}^{k}\binom{k}{i}\binom{i}{l}\frac{(a-k+1)^{(k-i)}(a-k)!(b-k+1)^{(i)}(b-k)!}{a!b!}\\ &= \binom{k}{l}\frac{(a-k)!(b-k)!}{a!b!}\frac{(a+b-2k+2)^{(k)}}{(a+b-2k+2)^{(l)}}(a-k+1)^{(l)}\\ &= \binom{k}{l}\frac{[a+b-k+1]_k}{[a]_{k-l}[b]_k[a+b-2k+l+1]_l}.\end{aligned} \tag{B.20}$$

□

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