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Article

Leading Logarithm Quantum Gravity

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Abstract

The continual production of long wavelength gravitons during primordial inflation endows graviton loop corrections with secular growth factors. During a prolonged period of inflation, these factors eventually overwhelm the small loop-counting parameter of GH^2, causing perturbation theory to break down. A technique was recently developed for summing the leading secular effects at each order in non-linear sigma models, which possess the same kind of derivative interactions as gravity. This technique combines a variant of Starobinsky's stochastic formalism with a variant of the renormalization group. Generalizing the technique to quantum gravity is a two-step process, the first of which is the determination of the gauge fixing condition that will allow this summation to be realized; this is the subject of this paper. Moreover, we briefly discuss the second step, which shall obtain the Langevin equation, in which secular changes in gravitational phenomena are driven by stochastic fluctuations of the graviton field.

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1. Prologue

The geometry of cosmology can be characterized by a scale factor $a(t)$ and its two first time derivatives, the Hubble parameter $H(t)$ and the first slow roll parameter $\epsilon(t)$:[1]

$$ds^2 = -dt^2 + a^2(t)\, d\mathbf{x} \cdot d\mathbf{x} \qquad \Longrightarrow \qquad H(t) \equiv \frac{\dot{a}}{a} \quad , \quad \epsilon(t) \equiv -\frac{\dot{H}}{H^2} \; . \qquad (1)$$

In the very early universe, primordial inflation is an era of accelerated expansion ($H > 0$ with $0 \leq \epsilon < 1$). During this era virtual particles are ripped out of the vacuum [1] and the phenomenon is largest for particles such as massless, minimally coupled (MMC) scalars and gravitons, because they are both massless and not conformally invariant [2,3]. This particle production is thought to be the physical mechanism causing the primordial tensor [4] and scalar [5] power spectra.

Many fascinating effects can be studied [6–8]. Of particular interest to us are those associated with the fact that, as inflation progresses, more and more quanta are created so that correlators which involve interacting MMC scalars and gravitons often show secular growth in the form of powers of $\ln[a(t)]$ [9].[2] An excellent example is provided by the theory of an MMC scalar with a quartic self-interaction and the study of the perfect fluid form the expectation value of its stress tensor takes in de Sitter background ($\epsilon = 0$):

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\, g^{\mu\nu}\sqrt{-g} - \frac{\lambda}{4!}\phi^4\sqrt{-g}\,, \tag{2}$$

$$\langle T_{\mu\nu}\rangle = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,. \tag{3}$$

The 2-loop dimensionally regulated and fully renormalized expectation value of the stress tensor equals [10–12]:

$$\rho(t) \;=\; \frac{\lambda H^4}{2^7\pi^4}\times \ln^2(a) + O(\lambda^2)\,, \tag{4}$$

$$p(t) \;=\; \frac{\lambda H^4}{2^7\pi^4}\Big\{-\ln^2(a) - \tfrac{2}{3}\ln(a)\Big\} + O(\lambda^2)\,. \tag{5}$$

In the correlators of this theory, for each factor of the coupling constant λ, up to two factors of $\ln(a)$ can be associated. When this bound is saturated, the contribution is known as *leading logarithm* (LLOG), for instance, in the pressure (5) the factor of $-\ln^2(a)$ is a leading logarithm. Contributions which have fewer factors of $\ln(a)$ are known as *subleading logarithm*, for instance, in the pressure (5) the factor of $-\frac{2}{3}\ln(a)$ is subleading.

Many similar effects have been studied [13,14]. During a long period of inflation, factors of $\ln[a(t)]$ can grow large enough to overwhelm even the smallest coupling constant. Obviously the most interesting particle to study is the carrier of the gravitational force, the graviton. Can the universally attractive nature of the gravitational interaction alter cosmological parameters, kinematical parameters and long-range forces? A preliminary study of this, albeit with "semi-primitive" for the intended purpose quantum field theoretic tools, indicated a positive answer [15,16].

After the subsequent development of the appropriate tools, we revisit pure quantum gravity and try step by step to re-sum its leading logarithms and hopefully obtain the late time limits of cosmological correlators. The dimensionless coupling constant of pure quantum gravity is GH^2, and at some time the secular increase by powers of $\ln[a(t)]$ will overwhelm GH^2, causing perturbation theory to break. It is always a formidable affair to decipher the dynamics of a theory after its perturbative analysis becomes invalid. While it is easy to state what is needed—a re-summation technique for the leading logarithms—its realization is very hard. It is also easy to perhaps contemplate that developing such a technique to sum up the series of leading logarithms may eventually be as important for cosmology as the renormalization group summation of leading momentum logarithms was to flat space quantum field theory. The re-summation technique we shall consider, and which, in our mind, has been adequately developed, is the stochastic technique pioneered by the late Alexei Starobinsky [17–19].

This paper consists of five sections and one appendix, of which this prologue is the first. In Section 2 we briefly present the relevant facts from pure quantum gravity in general, from its form in de Sitter spacetime, and from the stochastic re-summation technique. Section 3 contains the main result of this paper; it extends the quantum gravity perturbative setup to arbitrary constant H backgrounds, including the appropriate gauge fixing condition. Section 4 gives an example of how the results of Section 3 shall give the desired stochastic (Langevin) equations that pure quantum gravity implies. Section 5 is the epilogue where we discuss the physical implications and prospects. Finally, some useful identities are catalogued in Appendix A.

2. Quantum Gravity

Pure gravity defined by the Lagrangian:[3]

$$\mathcal{L}_{inv} = \frac{1}{\kappa^2}\left[R\sqrt{-g} - (D-2)(D-1)H^2\sqrt{-g}\,\right] , \tag{6}$$

is a two-parameter theory:[4]

$$\kappa^2 \equiv 16\pi G \quad , \quad \Lambda \equiv (D-1)H^2 , \tag{7}$$

with the following equations of motion:

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} + \frac{1}{2}(D-2)(D-1)H^2 g_{\mu\nu} = 0 . \tag{8}$$

In terms of the full metric $g_{\mu\nu}$, the conformally rescaled full metric $\widetilde{g}_{\mu\nu}$ and the graviton field $h_{\mu\nu}$ are defined thus:

$$g_{\mu\nu} \equiv a^2 \widetilde{g}_{\mu\nu} \equiv a^2\left[\eta_{\mu\nu} + \kappa h_{\mu\nu}\right] . \tag{9}$$

It is straightforward to express (6) in terms of the graviton field as follows:

$$\begin{aligned} \mathcal{L}_{inv} \;=\; & a^{D-2}\sqrt{-\widetilde{g}}\,\widetilde{g}^{\alpha\beta}\widetilde{g}^{\rho\sigma}\widetilde{g}^{\mu\nu}\Big\{\tfrac{1}{2}h_{\alpha\rho,\mu}h_{\nu\sigma,\beta} - \tfrac{1}{2}h_{\alpha\beta,\rho}h_{\sigma\mu,\nu} + \tfrac{1}{4}h_{\alpha\beta,\rho}h_{\mu\nu,\sigma} \\ & -\tfrac{1}{4}h_{\alpha\rho,\mu}h_{\beta\sigma,\nu}\Big\} + (\tfrac{D}{2}-1)a^{D-1}H\sqrt{-\widetilde{g}}\,\widetilde{g}^{\rho\sigma}\widetilde{g}^{\mu\nu}h_{\rho\sigma,\mu}h_{\nu 0} , \end{aligned} \tag{10}$$

which is the form of $\mathcal{L}_{inv}$ we shall use thereafter.

2.1. The de Sitter Case

The standard paradigm of a primordial inflationary spacetime is the de Sitter (dS) maximally symmetric geometry:

$$\widetilde{g}^{dS}_{\mu\nu} = \eta_{\mu\nu} . \tag{11}$$

For our purposes we only need the Feynman rules associated with this geometry, which allow successful computations. Not surprisingly, it seems that one gauge fixing choice[5] has almost always been used to successfully compute Feynman loop diagrams, starting with the first [20] and proceeding to ten dimensionally regularized and fully renormalized results [21]:[6]

$$F_\mu = \eta^{\rho\sigma}\left[h_{\mu\rho,\sigma} - \tfrac{1}{2}h_{\rho\sigma,\mu} + (D-2)\, a\, H h_{\mu\rho}\,\delta^0_\sigma\right] . \tag{12}$$

The gauge fixing Lagrangian is the usual one:

$$\mathcal{L}_{GF} = -\tfrac{1}{2}a^{D-2}\eta^{\mu\nu}F_\mu F_\nu , \tag{13}$$

and so is the ghost Lagrangian:

$$\mathcal{L}_{gh} = -a^{D-2}\eta^{\mu\nu}\,\bar{c}_\mu\,\delta F_\nu . \tag{14}$$

In terms of the ghost and antighost fermionic fields c_μ and $\bar{c}_\mu$:

$$\delta F_\nu \;=\; \eta^{\rho\sigma}\left[\delta h_{\nu\rho,\sigma} - \tfrac{1}{2}\delta h_{\rho\sigma,\nu} + (D-2)\, a\, H\,\delta h_{\nu\rho}\,\delta^0_{\ \sigma}\right] , \tag{15}$$

$$\delta h_{\mu\nu} \;=\; -c_{\mu,\nu} - c_{\nu,\mu} - 2aH(\eta_{\mu\nu}+\kappa h_{\mu\nu})c^0 - \kappa h_{\mu\nu,\alpha}\,c^\alpha - 2\kappa h_{\alpha(\mu}\,c^\alpha_{\ ,\nu)} . \tag{16}$$

In this gauge, the graviton propagator takes the form [22,23]:

$$
\begin{aligned}
i\Big[{}_{\alpha\beta}\Delta_{\rho\sigma}\Big](x;x') \;=\; & \Big[2\,\overline{\eta}_{\alpha(\rho}\overline{\eta}_{\sigma)\beta} - \tfrac{2}{D-3}\,\overline{\eta}_{\alpha\beta}\overline{\eta}_{\rho\sigma}\Big]\, i\Delta_A(x;x') \\
& -4\,\delta^0_{(\alpha}\overline{\eta}_{\beta)(\rho}\delta^0_{\sigma)}\, i\Delta_B(x;x') \\
+\tfrac{2}{(D-3)(D-2)}\Big[(D-3)\delta^0_{\alpha}\delta^0_{\beta} + \overline{\eta}_{\alpha\beta}\Big] & \Big[(D-3)\delta^0_{\rho}\delta^0_{\sigma} + \overline{\eta}_{\rho\sigma}\Big] i\Delta_C(x;x')\,, \qquad (17)
\end{aligned}
$$

while the ghost propagator equals [22,23]:

$$
i\Big[{}_{\alpha}\Delta_{\rho}\Big](x;x') = \overline{\eta}_{\alpha\rho}\, i\Delta_A(x;x') - \delta^0_{\alpha}\delta^0_{\rho}\, i\Delta_B(x;x')\,. \qquad (18)
$$

In the propagators (17) and (18) $i\Delta_A(x;x')$ is the massless minimally coupled scalar propagator in de Sitter spacetime [10,11]:[7]

$$
D_A\, i\Delta_A(x;x') = i\delta^D(x-x')\,, \qquad (19)
$$

while $i\Delta_B(x;x')$ is the massive scalar propagator with $m^2 = (D-2)H^2$ [24]:

$$
D_A\, i\Delta_B(x;x') = i\delta^D(x-x') + (D-2)H^2 a^D i\Delta_B(x;x')\,, \qquad (20)
$$

and $i\Delta_C(x;x')$ is the massive scalar propagator with $m^2 = 2(D-3)H^2$ [24]:

$$
D_A\, i\Delta_C(x;x') = i\delta^D(x-x') + 2(D-3)H^2 a^D i\Delta_C(x;x') \qquad (21)
$$

Moreover, the graviton and ghost kinetic operators are, respectively:

$$
\begin{aligned}
D^{\mu\nu\alpha\beta} &= \big[\tfrac{1}{2}\eta^{\mu(\alpha}\eta^{\beta)\nu} - \tfrac{1}{4}\eta^{\mu\nu}\eta^{\alpha\beta}\big] D_A + (D-2)H^2 a^D \delta^{(\mu}_{0}\eta^{\nu)(\alpha}\delta^{\beta)}_{0}\,, \qquad (22)\\
D^{\mu\alpha} &= \eta^{\mu\alpha} D_A + (D-2)H^2 a^D \delta^{\mu}_{0}\delta^{\alpha}_{0}\,, \qquad (23)
\end{aligned}
$$

and satisfy, respectively:

$$
\begin{aligned}
D^{\mu\nu\alpha\beta}\, i\Big[{}_{\alpha\beta}\Delta_{\rho\sigma}\Big](x;x') &= \delta^{\mu}_{(\rho}\delta^{\nu}_{\sigma)}\, i\delta^D(x-x')\,, \qquad (24)\\
D^{\mu\alpha}\, i\Big[{}_{\alpha}\Delta_{\rho}\Big](x;x') &= \delta^{\mu}_{\rho}\, i\delta^D(x-x')\,. \qquad (25)
\end{aligned}
$$

2.2. The Leading Logarithm Approximation

The leading logarithm approximation (LLOG) becomes a very essential field theoretic tool to face the simple fact that the presence of secular leading logarithms eventually causes the breakdown of perturbation theory. As a result the only reliable computing method has to be non-perturbative, and LLOG is such a method, which seems to be appropriate for the specific physical environment of interest.

In this physical environment the significance of the leading logarithms is a very slow process that requires a very long time evolution to become noticeable due to the smallness of the gravitational dimensionless parameter $G\Lambda$. Hence, the graviton field $h_{\mu\nu}$ changes significantly less than the scale factor $a(t)$ with time; it is *always* better to act derivatives on the scale factor $a(t)$ than to act them on $h_{\mu\nu}$.

The purpose of the LLOG technique is to sum the leading logarithms coming from all orders of perturbation theory. How does this goal translate into a specific set of practical steps for the theories at hand? It turns out that after the dust settles, it is only two operations that determine the simplified form the field equations take. The first of these—which we could call *"stochastic reduction"*—operates on the classical level, while the second—which we could call *"integrating out"*—operates on the quantum level.

We start by considering some theory of a quantum field in the presence of the cosmological background (1). Suppose the theory develops secular leading logarithms in its perturbative development.[8] The relevant question to ask is whether the interactions of the field possess derivatives.

- The case where they do *not* is the one Starobinsky successfully addressed in his original analysis of a single scalar field ϕ with a potential $V(\phi)$:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\,g^{\mu\nu}\sqrt{-g} - V(\phi)\sqrt{-g}\,, \tag{26}$$

for the particular case of $V(\phi) = \frac{\lambda}{4!}\phi^4$ [17]. Starobinsky's formalism was based on replacing the full field operator $\phi(t,\mathbf{x})$ with a stochastic field $\varphi(t,\mathbf{x})$, which commutes with itself $[\varphi(t,\mathbf{x}), \varphi(t',\mathbf{x}')] = 0$, and whose correlators are completely free of ultraviolet divergences. This stochastic field $\varphi(t,\mathbf{x})$ is constructed from the same free creation and annihilation operators that appear in $\phi(t,\mathbf{x})$ in such a way that the two fields produce the same leading logarithms at each order in perturbation theory. The Heisenberg field equation for ϕ gives rise to a Langevin equation for φ:

$$\frac{\delta S[\phi]}{\delta\phi(x)} = \partial_\mu\big[\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\big] - V'(\phi)\sqrt{-g} \tag{27}$$

$$\longrightarrow\quad 3Ha^3\big[\dot{\varphi} - \dot{\varphi}_0\big] - V'(\varphi)a^3\,. \tag{28}$$

Here $\varphi_0(t,\mathbf{x})$ is a truncation of the Yang–Feldman free field with the ultraviolet excised and the mode function taken to its limiting infrared form:

$$\varphi_0(t,\mathbf{x}) \equiv \int\frac{d^3k}{(2\pi)^3}\,\theta(aH-k)\,\frac{\theta(k-H)H}{\sqrt{2k^3}}\Big\{\alpha_{\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{x}} + \alpha^\dagger_{\mathbf{k}}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\Big\}\,. \tag{29}$$

We can derive (28) from QFT by first integrating the exact field equation to reach the Yang–Feldman form. We then note that reaching leading logarithm order requires each free field to contribute an infrared logarithm, so there will be no change to correlators, at leading logarithm order, if the full free field mode sum is replaced by (29). Differentiating this truncated Yang–Feldman equation gives Starobinsky's classical Langevin equation [25]. Furthermore, Starobinsky's technique can be proven to reproduce each order's leading logarithms [25] and—when $V(\phi)$ is bounded below—all orders can be summed up to give the late time limits of cosmological correlators in those cases for which a static limit is approached, as is the case for $V(\phi) = \frac{\lambda}{4!}\phi^4$ [26]. The "bottom line" of this analysis is the following *"stochastic reduction"* rule:

- *Rule for field with non-derivative interactions:*

$$\frac{\delta S[\phi]}{\delta\phi}\Big|_{LLOG} \equiv \frac{\delta S[\varphi]_{class}}{\delta\varphi}\Big|_{stoch} = 0\,. \tag{30}$$

The rule says that the equation of motion capturing the leading logarithms to all orders—the LHS of (30)—is tantamount to a classical Langevin equation—the RHS of (30)—derived from the full Heisenberg equation of motion $\frac{\delta S[\phi]}{\delta\phi}$ by:

(i) At each order in the field, retain only the terms with no derivatives and with the smallest number of derivatives,
(ii) For the linear terms in the field, each time derivative has a stochastic source subtracted.

Hence, according to our *"stochastic reduction"* rule, the full Heisenberg equation of motion (27) emanating from (26) becomes:

$$\frac{\delta S[\phi]}{\delta \phi} = \ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2}{a^2}\,\phi + V'(\phi) \tag{31}$$

$$\longrightarrow \frac{\delta S[\varphi]_{class}}{\delta \varphi}\Big|_{stoch} = 3H\big(\,\dot{\varphi} - \dot{\varphi}_0\,\big) + V'(\varphi)\,. \tag{32}$$

The enormous advantage of the method becomes apparent should we be interested in the all-orders LLOG re-summation; we must deal with a classical stochastic equation instead of the Heisenberg field equations of an interacting QFT.

- The extension to theories with interactions that possess derivatives was addressed in [27–30]. We shall concentrate on theories with field equations containing derivative interactions of a single field because it is the case relevant for pure gravity. From the point of view of LLOG, the field has a "dual role in the sense that:
  (i) when undifferentiated it can and does produce leading logarithms;
  (ii) when differentiated it does not produce leading logarithms due to the action of the derivatives.

Thus, we need a (simple) way to isolate only the gravitons that contribute leading logarithms. In other words, we need a (simple) way which distinguishes and separates undifferentiated from differentiated field bilinears. In the presence of a constant field background, the only bilinears that will survive are the undifferentiated ones; the differentiated ones contribute constants in time due to the action of the derivatives. Since only the undifferentiated bilinears furnish leading logarithms, we have our (simple) way at our disposal.

Therefore, in the case of a "dual role" field, the *"stochastic reduction"* rule gets augmented with the *"integrating out"* rule, which integrates out the differentiated field bilinears from the equations of motion and *adds* the induced result to the stochastically reduced equation of motion:

- *Rule for field with derivative interactions:*

$$\frac{\delta S[\Phi]}{\delta \Phi}\Big|_{LLOG} \equiv \frac{\delta S[\Psi]_{class}}{\delta \Psi}\Big|_{stoch} + T[\Psi]\Big|_{ind} = 0\,. \tag{33}$$

- Perhaps it would be appropriate, before embarking in quantum gravity, to review a simple and well-studied non-linear σ-model example for a single scalar Φ [28]:

$$\mathcal{L} = -\frac{1}{2}\big(1 + \tfrac{\lambda}{2}\Phi\big)^2\, \partial_\mu \Phi\, \partial_\nu \Phi\, g^{\mu\nu} \sqrt{-g}\,. \tag{34}$$

A single field non-linear σ-model can be reduced to a free theory by a local field redefinition. Nonetheless, although the S-matrix is unity, interactions can still cause changes to the kinematics of Φ particles and to the evolution of the Φ background.

The field equation obtained from (34) is:

$$\frac{\delta S[\Phi]}{\delta \Phi(x)} = \big(1 + \tfrac{\lambda}{2}\Phi\big)\partial_\mu \Big[\Big(1 + \tfrac{\lambda}{2}\Phi\Big)\sqrt{-g}g^{\mu\nu}\partial_\nu \Phi\Big]\,. \tag{35}$$

* <u>Dual Role I:</u> The *"stochastic reduction"*

The stochastic form of (35) as a homogeneous evolution equation in co-moving time is obtained in three steps:[9]

$$\frac{\delta S[\Phi]}{\delta \Phi(x)} \longrightarrow -\left(1+\tfrac{\lambda}{2}\Phi\right)\frac{d}{dt}\left[\left(1+\tfrac{\lambda}{2}\Phi\right)a^3\dot{\Phi}\right] \tag{36}$$

$$\longrightarrow -3Ha^3\left(1+\tfrac{\lambda}{2}\Phi\right)^2\dot{\Phi} \tag{37}$$

$$\longrightarrow -3Ha^3\left(1+\tfrac{\lambda}{2}\Psi\right)^2\left[\dot{\Psi}-\dot{\Psi}_0\right] \equiv \frac{\delta S[\Psi]_{class}}{\delta \Psi}\Big|_{stoch} . \tag{38}$$

* <u>Dual Role II</u>: The *"integrating out"*

We must add to the stochastic equation of motion (38) the induced effective force arising from the contribution of undifferentiated fields Φ in the constant background Φ_0:[10]

$$\frac{\delta S[\Phi]}{\delta \Phi(x)} \longrightarrow -\left(1+\tfrac{\lambda}{2}\Phi_0\right)\frac{d}{dt}\left[\tfrac{\lambda}{4}a^3\tfrac{d}{dt}\langle\Phi^2\rangle_{\Phi_0}\right] \tag{39}$$

$$\longrightarrow -\left(1+\frac{\lambda}{2}\Phi_0\right)\frac{d}{dt}\left[\tfrac{\lambda}{4}a^3\tfrac{H^3}{4\pi^2}\left(1+\tfrac{\lambda}{2}\Phi_0\right)^{-2}\right] \tag{40}$$

$$\longrightarrow -\frac{3\lambda H^4}{16\pi^2}\,\frac{a^3}{\left(1+\frac{\lambda}{2}\Psi\right)} \equiv T[\Psi]\Big|_{ind} , \tag{41}$$

where we have replaced the constant field Φ_0 with the spacetime field Ψ.

Adding the two contributions (38) and (41) gives the desired Langevin equation associated with (34):

$$\frac{\delta S[\Phi]}{\delta \Phi}\Big|_{LLOG} \equiv \frac{\delta S[\Psi]_{class}}{\delta \Psi}\Big|_{stoch} + T[\Psi]\Big|_{ind}$$

$$= -3Ha^3\left(1+\tfrac{\lambda}{2}\Psi\right)^2\left[\dot{\Psi}-\dot{\Psi}_0\right] - \frac{3\lambda H^4}{16\pi^2}\,\frac{a^3}{\left(1+\frac{\lambda}{2}\Psi\right)} = 0 \tag{42}$$

$$\Longrightarrow \dot{\Psi} = \dot{\Psi}_0 - \frac{\lambda H^3}{16\pi^2}\,\frac{1}{\left(1+\frac{\lambda}{2}\Psi\right)^3} . \tag{43}$$

We should mention that the above procedure has been thoroughly checked against perturbative computations up to 2-loop order and the highly non-trivial agreement is complete [28].

- We conclude by noting that to arrive at the elusive equations which describe LLOG pure quantum gravity, we simply have to effect the two operations which will allow us to do that:[11]

  (i) the *"stochastic reduction"* of the field equations to a classical Langevin equation;
  (ii) the *"integrating out"* of the differentiated fields in a constant background to obtain the induced stress tensor.

3. The Extension to Any Constant Graviton Background

In order to accomodate the LLOG approximation in pure gravity (6), we should like to integrate out the differentiated graviton fields in the presence of a constant graviton background. Thus, we first consider the general class of conformally rescaled backgrounds with constant H and arbitrary $\widetilde{g}_{\mu\nu}$ because the ultimate object of our study is after all the time evolution of constant H spacetimes:

$$g_{\mu\nu}(x) \equiv a^2\,\widetilde{g}_{\mu\nu}(x) \equiv a^2\left[\eta_{\mu\nu}+\kappa h_{\mu\nu}(x)\right] \quad , \quad a = -(H\eta)^{-1} . \tag{44}$$

When we restrict ourselves, for LLOG reasons, to constant $\widetilde{g}_{\mu\nu}$, the curvature tensor takes the form:

$$R^{\rho}{}_{\sigma\mu\nu}\Big|_{\widetilde{g}_{\mu\nu}=c} = -H^2\,\widetilde{g}^{00}\left(\delta^{\rho}{}_{\mu}\,g_{\sigma\nu} - \delta^{\rho}{}_{\nu}\,g_{\sigma\mu}\right)\,, \tag{45}$$

and we recognize a de Sitter geometry, albeit with a different cosmological constant:[12]

$$\widetilde{g}_{\mu\nu,\rho} = 0 \quad\Longrightarrow\quad H^2 \longrightarrow -\widetilde{g}^{00}H^2\,. \tag{46}$$

However, because gravitons have tensor indices, the gauge fixing procedure must be extended to accomodate the arbitrary constant graviton background. The most efficient way to extend the graviton Feynman rules from de Sitter to any spacetime such that $\widetilde{g}_{\mu\nu,\rho} = 0$ starts with the 3+1 decomposition.

3.1. The 3+1 Decomposition

The standard 3+1 decomposition of the metric tensor was pioneered by Arnowit, Deser, and Misner (ADM) to formulate the Hamiltonian dynamics of general relativity [31]. The full metric is expressed in terms of the lapse function N, the shift function N^i, and the spatial metric γ_{ij}:

$$\widetilde{g}_{\mu\nu} = \begin{pmatrix} -N^2+\gamma_{kl}N^kN^l & -\gamma_{jl}N^l \\ -\gamma_{ik}N^k & \gamma_{ij} \end{pmatrix} \tag{47}$$

$$= \begin{pmatrix} \gamma_{kl}N^kN^l & -\gamma_{jl}N^l \\ -\gamma_{ik}N^k & \gamma_{ij} \end{pmatrix} - \begin{pmatrix} -N \\ 0 \end{pmatrix}_{\mu} \begin{pmatrix} -N \\ 0 \end{pmatrix}_{\nu} \equiv \overline{\gamma}_{\mu\nu} - u_\mu u_\nu\,, \tag{48}$$

which implies the following form for its inverse:

$$\widetilde{g}^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & -\frac{N^j}{N^2} \\ -\frac{N^i}{N^2} & \gamma^{ij} - \frac{N^iN^j}{N^2} \end{pmatrix} \tag{49}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix} - \begin{pmatrix} \frac{1}{N} \\ \frac{N^i}{N} \end{pmatrix}^{\mu} \begin{pmatrix} \frac{1}{N} \\ \frac{N^j}{N} \end{pmatrix}^{\nu} \equiv \overline{\gamma}^{\mu\nu} - u^\mu u^\nu\,. \tag{50}$$

In relations (48) and (50), we have expressed the 3+1 decomposition in the form convenient for our purposes; $\overline{\gamma}^{\mu\nu}$ is the "spatial part" and u_μ is the "temporal part".

Finally, in the appendix some identities associated with the 3+1 decomposition just described are recorded (A1) and (A2).

3.2. The Gauge Fixing Extension

In analogy with (13), the extended gauge fixing Lagrangian term equals:

$$\widetilde{\mathcal{L}}_{GF} = -\tfrac{1}{2}a^{D-2}\sqrt{-\widetilde{g}}\,\widetilde{g}^{\mu\nu}\,\widetilde{F}_\mu\,\widetilde{F}_\nu\,, \tag{51}$$

and, similarly, in analogy with (12) and taking into account (A2), the extended gauge condition becomes:

$$\widetilde{F}_\mu = \widetilde{g}^{\rho\sigma}\left[h_{\mu\rho,\sigma} - \tfrac{1}{2}h_{\rho\sigma,\mu} - (D-2)a\widetilde{H}h_{\mu\rho}u_\sigma\right]\quad,\quad \widetilde{H} \equiv \frac{H}{N}\,. \tag{52}$$

Substituting (52) into (51) we arrive at the desired form for the gauge fixing Lagrangian term:

$$\begin{aligned}\widetilde{\mathcal{L}}_{GF} &= a^{D-2}\sqrt{-\widetilde{g}}\,\widetilde{g}^{\alpha\beta}\widetilde{g}^{\rho\sigma}\widetilde{g}^{\mu\nu}\Big\{-\tfrac{1}{2}h_{\mu\rho,\sigma}h_{\nu\alpha,\beta}+\tfrac{1}{2}h_{\mu\rho,\sigma}h_{\alpha\beta,\nu}-\tfrac{1}{8}h_{\rho\sigma,\mu}h_{\alpha\beta,\nu}\\ &\quad +(D-2)a\widetilde{H}h_{\mu\rho,\sigma}h_{\nu\alpha}u_{\beta}-(\tfrac{D}{2}-1)a\widetilde{H}h_{\rho\sigma,\mu}h_{\nu\alpha}u_{\beta}\\ &\quad -\tfrac{1}{2}(D-2)^2a^2\widetilde{H}^2h_{\mu\rho}h_{\nu\alpha}u_{\beta}u_{\sigma}\Big\}\end{aligned} \tag{53}$$

3.3. The Graviton Propagator

In analogy with (17), and using the 3+1 decomposition (48) and identities (A2), we deduce the extension for the graviton propagator:

$$\begin{aligned}i\Big[{}_{\alpha\beta}\widetilde{\Delta}_{\rho\sigma}\Big](x;x') &= \Big[2\,\overline{\gamma}_{\alpha(\rho}\overline{\gamma}_{\sigma)\beta}-\tfrac{2}{D-3}\,\overline{\gamma}_{\alpha\beta}\overline{\gamma}_{\rho\sigma}\Big]i\widetilde{\Delta}_A(x;x') \qquad (54)\\ &\quad -4\,u_{(\alpha}\overline{\gamma}_{\beta)(\rho}u_{\sigma)}\,i\widetilde{\Delta}_B(x;x')\\ &\quad +\tfrac{2}{(D-3)(D-2)}\Big[(D-3)u_\alpha u_\beta+\overline{\gamma}_{\alpha\beta}\Big]\Big[(D-3)u_\rho u_\sigma+\overline{\gamma}_{\rho\sigma}\Big]i\widetilde{\Delta}_C(x;x')\;.\end{aligned}$$

The corresponding quadratic operator equals:

$$\widetilde{\mathcal{D}}^{\mu\nu\alpha\beta}=\tfrac{1}{2}\Big[\widetilde{g}^{\mu(\alpha}\,\widetilde{g}^{\beta)\nu}-\tfrac{1}{2}\widetilde{g}^{\mu\nu}\,\widetilde{g}^{\alpha\beta}\Big]\widetilde{\mathcal{D}}_A+(D-2)\widetilde{H}^2a^D\sqrt{-\widetilde{g}}\,u^{(\mu}\,\widetilde{g}^{\nu)(\alpha}\,u^{\beta)}\;, \tag{55}$$

and allows us to check that indeed the proper condition is satisfied:

$$\widetilde{\mathcal{D}}^{\mu\nu\alpha\beta}\,i\Big[{}_{\alpha\beta}\widetilde{\Delta}_{\rho\sigma}\Big](x;x')=\delta^{\mu}_{\ (\rho}\delta^{\nu}_{\ \sigma)}\,i\delta^D(x-x')\;. \tag{56}$$

Finally, in the appendix we have catalogued the coincidence propagator limits (A6)–(A12) of potential interest and we should like to emphasize that of these, only $i\widetilde{\Delta}_A(x;x')$ contains a divergence.

3.4. The Ghost Contribution

The contribution to the action of the ghost and antighost fermionic fields c_μ and $\overline{c}_\mu$ is the usual one:

$$\widetilde{\mathcal{L}}_{gh}=-a^{D-2}\sqrt{-\widetilde{g}}\,\widetilde{g}^{\mu\nu}\overline{c}_\mu\,\delta F_\nu\;, \tag{57}$$

where in the infinitesimal variation δF_ν of the gauge fixing functional (52), the variation parameter is the ghost field c:

$$\begin{aligned}\delta F_\nu &= \widetilde{g}^{\rho\sigma}\Big[\delta h_{\nu\rho,\sigma}-\tfrac{1}{2}\delta h_{\rho\sigma,\nu}-(D-2)a\widetilde{H}\,\delta h_{\nu\rho}\,u_\sigma\Big]\\ &\quad +\delta\widetilde{g}^{\rho\sigma}\Big[h_{\nu\rho,\sigma}-\tfrac{1}{2}h_{\rho\sigma,\nu}-(D-2)a\widetilde{H}\,h_{\nu\rho}\,u_\sigma\Big]\;,\end{aligned} \tag{58}$$

so that:

$$\delta h_{\mu\nu} = -c^{\alpha}_{\ ,\mu}\,\widetilde{g}_{\alpha\nu}-c^{\alpha}_{\ ,\nu}\,\widetilde{g}_{\alpha\mu}+2a\widetilde{H}\,\widetilde{g}_{\mu\nu}u_\alpha c^\alpha-\kappa h_{\mu\nu,\alpha}\,c^\alpha \tag{59}$$

$$\delta\widetilde{g}^{\rho\sigma} = -\widetilde{g}^{\rho\alpha}\widetilde{g}^{\sigma\beta}\,\kappa\,\delta h_{\alpha\beta}\;. \tag{60}$$

The ghost propagator (18) generalizes to:

$$i\Big[{}_{\alpha}\Delta_{\rho}\Big](x;x')=\overline{\gamma}_{\alpha\rho}\,i\widetilde{\Delta}_A(x;x')-u_\alpha u_\rho\,i\widetilde{\Delta}_B(x;x')\;. \tag{61}$$

The action of the ghost kinetic operator:

$$\widetilde{D}^{\mu\alpha} = \widetilde{g}^{\mu\alpha}\widetilde{D}_A + (D-2)\widetilde{H}^2 a^D \sqrt{-\widetilde{g}}\, u^\mu u^\alpha \,, \tag{62}$$

on (61) gives:

$$\widetilde{D}^{\mu\alpha}\, i\Big[{}_\alpha\widetilde{\Delta}_\rho\Big](x;x') = \delta^\mu_\rho\, i\delta^D(x-x') \,. \tag{63}$$

4. What Follows

The LLOG methodology was extensively analyzed in Section 2.2, where the relevant rule for the pure gravitational case was identified:

$$\frac{\delta S[h]}{\delta h_{\mu\nu}}\Big|_{LLOG} \equiv \frac{\delta S[h]_{class}}{\delta h_{\mu\nu}}\Big|_{stoch} - a^4\sqrt{-\widetilde{g}}\; T[h]^{\mu\nu}\Big|_{ind} = 0 \,. \tag{64}$$

The main purpose of this paper—the complete set of the relevant Feynman rules—was achieved in Section 3. The natural next step is to use them and derive the appropriate dynamical equations for LLOG pure quantum gravity using (64). Although this is work in progress [32], we shall present herein an example of this process, since obtaining the LLOG gravitational equations and ultimately solving them is the primary objective.

An important simplification can be made a priori since it turns out that all coincidence limits that appear in the computations required by (64) are finite and the only one that is not does not appear.[13] Hence, we can take the $D = 4$ limit, which shall considerably simplify the intricate tensor algebra that must be done.[14]

It turns out that it is preferable to add the contributions of $\mathcal{L}_{inv}$ (10) and $\widetilde{\mathcal{L}}_{GF}$ (53) because some of their terms simplify against each other. The six terms comprising $\mathcal{L}_{inv} + \widetilde{\mathcal{L}}_{GF}$ can be grouped into three convenient parts:

$$\mathcal{L}_{inv} + \widetilde{\mathcal{L}}_{GF} \;\equiv\; \mathcal{L}_{1+2+3} + \mathcal{L}_{4+5} + \mathcal{L}_6 \,, \tag{65}$$

where these three convenient parts equal:

$$\mathcal{L}_{1+2+3} = a^2\sqrt{-\widetilde{g}}\,\widetilde{g}^{\alpha\beta}\widetilde{g}^{\rho\sigma}\widetilde{g}^{\gamma\delta} \times \Big\{-\tfrac{1}{4}h_{\alpha\rho,\gamma}h_{\beta\sigma,\delta} + \tfrac{1}{8}h_{\alpha\beta,\gamma}h_{\rho\sigma,\delta} + a^2\widetilde{H}^2 h_{\gamma\rho}u_\sigma h_{\delta\alpha}u_\beta\Big\} \,, \tag{66}$$

$$\mathcal{L}_{4+5} = \sqrt{-\widetilde{g}}\,\widetilde{g}^{\alpha\beta}\widetilde{g}^{\rho\sigma}\widetilde{g}^{\gamma\delta}\partial_\beta\Big[-\tfrac{1}{2}\partial_\sigma\big(a^2 h_{\gamma\rho}h_{\delta\alpha}\big) + a^2 h_{\gamma\rho}h_{\delta\alpha,\sigma}\Big] \,, \tag{67}$$

$$\mathcal{L}_6 = \kappa a^3 H\sqrt{-\widetilde{g}}\,\widetilde{g}^{\alpha\beta}\widetilde{g}^{\rho\sigma}\widetilde{g}^{\gamma\delta}h_{\rho\sigma,\gamma}h_{\delta\alpha}h_{0\beta} \,. \tag{68}$$

The reduction process described in Section 2.2 is a straightforward but cumbersome procedure. As an example we present the results coming from (66):

- The "*stochastic reduction*" of the full field equation from $\mathcal{L}_{1+2+3}$ furnishes the following classical Langevin form:

$$\frac{\delta S[h]^{class}_{(1+2+3)}}{\delta h_{\mu\nu}}\Big|_{stoch} = \tag{69}$$
$$a^4\sqrt{-\widetilde{g}}\Big\{\tfrac{3}{2}\,\widetilde{g}^{00}H\Big[\widetilde{g}^{\rho\mu}\widetilde{g}^{\sigma\nu} - \tfrac{1}{2}\widetilde{g}^{\rho\sigma}\widetilde{g}^{\mu\nu}\Big]\big[\dot{h}_{\rho\sigma} - \dot{\chi}_{\rho\sigma}\big] + 2\widetilde{H}^2 u^{(\mu}\widetilde{g}^{\nu)(\alpha}u^{\beta)}h_{\alpha\beta}$$
$$+\kappa\widetilde{H}^2\Big[\tfrac{1}{2}\,\widetilde{g}^{\mu\nu}u^{(\alpha}\widetilde{g}^{\beta)(\rho}u^{\sigma)} - 2u^{(\mu}\widetilde{g}^{\nu)(\alpha}\widetilde{g}^{\beta)(\rho}u^{\sigma)} - u^{(\alpha}\widetilde{g}^{\beta)(\mu}\widetilde{g}^{\nu)(\rho}u^{\sigma)}\Big]h_{\alpha\beta}h_{\rho\sigma}\Big\} \,,$$

where $\chi_{\mu\nu}$ denotes the stochastic jitter.

- The corresponding *"integrating out"* operation leads to the following induced stress tensor:

$$-a^4\sqrt{-\widetilde{g}}\; T[h]^{\mu\nu}_{(1+2+3)}\Big|_{ind} \;=\; a^4\sqrt{-\widetilde{g}}\,\frac{\kappa\widetilde{H}^4}{8\pi^2}\Big[-\tfrac{1}{2}\,\widetilde{g}^{\mu\nu}+6u^\mu u^\nu\Big]\,. \tag{70}$$

5. Epilogue

The continual production of inflationary gravitons tends to endow graviton loop corrections with secular growth factors. During a prolonged period of inflation, these factors eventually overwhelm the small loop-counting parameter of GH^2, which causes perturbation theory to break down. Developing a re-summation technique that permits one to evolve to late times has been a long struggle owing to the derivative interactions of gravity. The analogous problem for non-linear sigma models was recently solved by combining a variant of Starobinsky's stochastic formalism with a variant of the renormalization group [28]. The technique has recently been generalized to scalar corrections to gravity [30], and here we took the first essential step for its definitive extension to pure gravity.

The basic assumption in our analysis is that changes in the geometrical background are significantly slower than the changes in the scale factor. The basic goal is the derivation of the Langevin equation for pure quantum gravity on de Sitter background. The basic advantage of this equation is that the leading logarithm (LLOG) approximation it represents is valid to *all* orders of perturbation theory.

To achieve this entails doing two things:

(i) Generalizing the gauge fixing condition by replacing $\eta_{\mu\nu}$ with $\widetilde{g}_{\mu\nu}$.
(ii) Integrating out differentiated graviton fields to produce a leading logarithm stress tensor and stochastically simplifying the classical equation.

In this paper we completed the first of these, while the second is work in progress [32]. The extension of the gauge fixing condition—necessary since the graviton has tensor indices—was achieved by employing the 3+1 decomposition.

By far the most interesting physical applications are those where perturbative results exhibit a secular effect which leads to a breakdown of perturbation theory. In that case, a re-summation technique is necessary and the natural technique is to re-sum the leading logarithm effects in each order of coupling constant perturbation theory [15,16]. This problem has been much studied by other authors [33–36] and by ourselves [9].

We should close by commenting on the fascinating issue of the extent to which this stochastic formalism can be extended from de Sitter, and related cosmological backgrounds, to the anti-de Sitter (and related backgrounds), of great interest to fundamental theory. The simple answer is that we do not yet know. The physics of quantum fields on anti-de Sitter differs greatly from that on de Sitter. Whereas de Sitter fields experience a continual redshift of modes from the ultraviolet to the infrared, anti-de Sitter fields experience the opposite blueshift. It is not known if this injects time dependence into simple correlators the way it does in cosmology. Some things are known, for example, the first order formalism allows one to establish a connection between the braneworld and cosmology [37]. This has been much studied [38–40] but it is still not known if there is a stochastic realization.

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Appendix A. Useful Identities

* Some relations from the 3+1 decomposition:

$$\widetilde{H} = \frac{H}{N} \quad , \quad \widetilde{H}^2 = \frac{H^2}{N^2} = -\widetilde{g}^{00} H^2 \,, \tag{A1}$$

$$\delta^0_\mu = -\frac{1}{N} u_\mu \quad , \quad \widetilde{g}^{0\mu} = -\frac{1}{N} u^\mu \quad , \quad \widetilde{g}^{\mu\nu} u_\mu u_\nu = -1 = \widetilde{g}_{\mu\nu} u^\mu u^\nu \,. \tag{A2}$$

* Some tensor algebra identities:

$$\widetilde{g}^{\mu\nu} = \eta^{\mu\nu} - \kappa h^\mu_{\ \alpha} \, \widetilde{g}^{\alpha\nu} \,, \tag{A3}$$

$$\eta_{\mu\nu} = \overline{\eta}_{\mu\nu} - \delta^0_{\ \mu} \delta^0_{\ \nu} \quad , \quad \delta^\mu_\nu = \overline{\delta}^\mu_\nu + \delta^\mu_{\ 0} \delta^0_{\ \nu} \,, \tag{A4}$$

$$\delta^\mu_{\ \alpha} \delta^\nu_{\ \beta} = \overline{\delta}^\mu_{\ (\alpha} \overline{\delta}^\nu_{\ \beta)} + 2 \delta^{(\mu}_{\ \ 0} \overline{\delta}^{\nu)}_{\ (\alpha} \delta^0_{\ \beta)} + \delta^\mu_{\ 0} \delta^\nu_{\ 0} \delta^0_{\ \alpha} \delta^0_{\ \beta} \,. \tag{A5}$$

* The various propagator coincident limit identities:

$$i\widetilde{\Delta}_A(x;x')\big|_{x=x'} = \frac{\widetilde{H}^2}{4\pi^2} \ln a + \text{“}\infty\text{”} \quad , \quad i\widetilde{\Delta}_B(x;x')\big|_{x=x'} = -\frac{\widetilde{H}^2}{16\pi^2} \,, \tag{A6}$$

$$i\widetilde{\Delta}_C(x;x')\big|_{x=x'} = +\frac{\widetilde{H}^2}{16\pi^2} \,, \tag{A7}$$

$$\partial'_\sigma \, i\widetilde{\Delta}_A(x;x')\big|_{x=x'} = -\frac{\widetilde{H}^3}{8\pi^2} \, a \, u_\sigma \quad , \quad \partial'_\sigma \, i\widetilde{\Delta}_B(x;x')\big|_{x=x'} = 0 \,, \tag{A8}$$

$$\partial'_\sigma \, i\widetilde{\Delta}_C(x;x')\big|_{x=x'} = 0 \,, \tag{A9}$$

$$\partial_\rho \partial'_\sigma \, i\widetilde{\Delta}_A(x;x')\big|_{x=x'} = -\frac{3\widetilde{H}^4}{32\pi^2} a^2 \, \widetilde{g}_{\rho\sigma} \,, \tag{A10}$$

$$\partial_\rho \partial'_\sigma \, i\widetilde{\Delta}_B(x;x')\big|_{x=x'} = \frac{\widetilde{H}^4}{32\pi^2} a^2 \, \widetilde{g}_{\rho\sigma} \,, \tag{A11}$$

$$\partial_\rho \partial'_\sigma \, i\widetilde{\Delta}_C(x;x')\big|_{x=x'} = -\frac{\widetilde{H}^4}{32\pi^2} a^2 \, \widetilde{g}_{\rho\sigma} \,. \tag{A12}$$

Note: In the above relations, we first take the derivatives and then the coincidence limit $x' \to x$.

* The $\widetilde{D}_A$ operator equals:

$$\widetilde{D}_A \equiv \partial_\alpha \big[a^{D-2} \sqrt{-\widetilde{g}} \, \widetilde{g}^{\alpha\beta} \partial_\beta \big] \,, \tag{A13}$$

and its operation on the three kinds of scalar propagators gives:

$$\widetilde{D}_A \, i\widetilde{\Delta}_A(x;x') \;=\; i\delta^D(x-x') \,, \tag{A14}$$

$$\widetilde{D}_A \, i\widetilde{\Delta}_B(x;x') \;=\; i\delta^D(x-x') + (D-2)\widetilde{H}^2 a^D \sqrt{-\widetilde{g}} \, i\widetilde{\Delta}_B(x;x') \,, \tag{A15}$$

$$\widetilde{D}_A \, i\widetilde{\Delta}_C(x;x') \;=\; i\delta^D(x-x') + 2(D-3)\widetilde{H}^2 a^D \sqrt{-\widetilde{g}} \, i\widetilde{\Delta}_C(x;x') \,. \tag{A16}$$

Notes

1 It is more often than not convenient to employ conformal instead of co-moving coordinates: $ds^2 = -dt^2 + a^2(t)\, d\mathbf{x} \cdot d\mathbf{x} = a^2(\eta)\big[-d\eta^2 + a^2(\eta)\, d\mathbf{x} \cdot d\mathbf{x}\big]$, with t the co-moving time and η the conformal time.

2 Examples of this particular secular growth behavior can be found in citations [14,16,24,38,44,72,73,77,78] therein.

3 Hellenic indices take on spacetime values, while Latin indices take on space values. Our metric tensor $g_{\mu\nu}$ has spacelike signature $(-\,+\,+\,+)$ and our curvature tensor equals $R^{\alpha}{}_{\beta\mu\nu} \equiv \Gamma^{\alpha}{}_{\nu\beta,\mu} + \Gamma^{\alpha}{}_{\mu\rho}\, \Gamma^{\rho}{}_{\nu\beta} - (\mu \leftrightarrow \nu)$.

4 Notice that even for a cosmological mass scale $M \sim 10^{18}$ GeV close to the Planck scale $M_{\rm Pl}$, the dimensionless coupling constant is very small: $G\Lambda = \frac{M^4}{M_{\rm Pl}^4} \sim 10^{-4}$.

5 The graviton propagator in this particular gauge is simple in two essential ways: *(i)* it is the sum of three scalar propagators times spacetime *constant* tensor factors; *(ii)* in $D = 4$ the three propagators only have one or two terms.

6 Examples of such regularized and fully renormalized results can be found in citations [14,47–56] therein.

7 The de Sitter D_A operator is: $D_A \equiv \partial_\alpha \big[a^{D-2} \partial^{\alpha}\big]$.

8 Fields like gravitons and MMC scalars will do precisely that.

9 As described above, the first step (36) follows since only time derivatives matter, the second step (37) follows since the evolution of Φ is much slower than that of the scale factor $a = e^{Ht}$ so that the largest contribution comes from the external derivative acting on a^3, and the third step (38) follows from the stochastic rule whereby the full stochastic field Ψ has its associated stochastic jitter Ψ_0 subtracted.

10 Again only time derivatives matter, while $\partial_\nu \langle \Phi^2 \rangle|_{\Phi_0} = 2\delta^0_\nu H \frac{H^2}{8\pi^2} \big(1 + \frac{\lambda}{2}\Phi_0\big)^{-2}$. Moreover, the process of integrating out, for instance, singly or doubly differentiated scalar bilinears amounts to replacing them with singly or doubly differentiated scalar propagators in the presence of a spacetime constant scalar, and this is tantamount to changing the scalar field strength [28].

11 Although only theories with a single field were discussed, it is clear that the same operations apply to theories with many fields. Such analysis can be more intricate when some of the fields can produce leading logarithms, e.g., gravitons, MMC scalars, while others cannot, e.g., fermions, photons, conformally coupled scalars. Examples of such theories which were fully studied can be found in [27–30].

12 The process of integrating out, for instance, singly or doubly differentiated graviton bilinears amounts to replacing them with singly or doubly differentiated graviton propagators in the presence of a constant graviton background, and this is tantamount to replacing them with singly or doubly differentiated de Sitter graviton propagators with a different Hubble parameter.

13 All propagator coincidence limits are displayed in Appendix A (A6)–(A12). Of these, only $i\widetilde{\Delta}_A(x;x')\big|_{x=x'}$ is not finite.

14 Some of the identities used in the algebraic tensor steps can be found in Appendix A (A3)–(A5).

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