Ann. Henri Poincaré 25 (2024), 1511–1543

1424-0637/24/021511-33
published online September 13, 2023
https://doi.org/10.1007/s00023-023-01356-4

Annales Henri Poincaré

Lagrangian Reduction by Stages in Field Theory

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Abstract. We propose a category of bundles in order to perform Lagrangian reduction by stages in covariant Field Theory. This category plays an analogous role to Lagrange–Poincaré bundles in Lagrangian reduction by stages in Mechanics and includes both jet bundles and reduced covariant configuration spaces. Furthermore, we analyze the resulting reconstruction condition and formulate the Noether theorem in this context. Finally, a model of a molecular strand with rotors is seen as an application of this theoretical frame.

Mathematics Subject Classification. Primary 58E15; Secondary 37J37, 58D19, 70S05, 70S10.

1. Introduction

Symmetry represents the core of many (and probably the most important) tools developed to tackle dynamical systems. In particular, since the geometric formulation of Mechanics (for example, Arnold [2], Marsden [1], Moser [20] and the references therein, among others), special attention has been always given to those systems endowed with a group of symmetries. When the system is modeled on a manifold as the configuration space, the symmetry is expressed in terms of smooth actions of Lie groups. One natural procedure is thus the construction of the quotient of the manifold by the group in the case of actions satisfying certain good properties. This is the so-called reduction procedure, which can be performed both in the Lagrangian and in the Hamiltonian geometric pictures of systems and that has attired and still attires the attention of many papers and books (a good reference can be [18]). With the word *system*, we can include the evolution of particles governed by variational principles, symplectic forms, Poisson brackets, or time evolution of sections of bundles (of which Mechanics is a particular case) in a variational or multisymplectic approach. The latter are known as (classical) Field Theory. In full generality, systems describing section of bundles with no-prescribed time evolution (for example, covariant fields on space-times or geometric theories as harmonic maps) are also included in this versatile panorama.

Restricting ourselves to the Lagrangian or variational case, the Lagrangian functions are defined in the phase space of the system, a manifold including independent variables (as positions) together with their derivatives. The tangent bundle is the paradigmatic example in the case of Mechanics, which is generalized to jet spaces for Field Theories [15]. The constructions of the variations, the variational principle and the equations for critical solutions are perfectly described in terms of geometric objects. In parallel, when reduction is performed, the new variations, the new variational principle as well as the new equations are now written in the reduced phase spaces that is not a tangent or jet space. This is the so-called Lagrange–Poincaré reduction first introduced in Mechanics (see [9] for a historical account) and generalized for arbitrary bundles in [8] and [12]. An important consequence of the new nature of the reduced phase space arises when one needs to concatenate consecutive reductions. Indeed, there are many situations where the symmetry group is split in two or more parts entailing completely different properties. This difference may require a separate reduction for each part, a procedure called *reduction by stages*. The work of Cendra, Marsden and Ratiu [9] gave for Mechanics the convenient setting for this recursive reductions scheme that has been extensively used in the literature. (Just to mention some, the reader can go to [4,10,13,14,16].) For that, a new category of phase spaces is introduced: the Lagrange–Poincaré category in Mechanics. See also [3] for the complete description of this category that closes some of the issues left open in [9].

The goal of this work is the construction of the Lagrange–Poincaré category for Field Theories. This includes the definition of the new phase manifolds, the variations, the variational principle and the equations for the critical sections of the configuration bundles. As we already learned from Lagrange–Poincaré reduction, these equations are split into two groups known as horizontal and vertical equations. Everything fits in a reduction program so that the reduced objects and principles by the action of groups remain in the category and hence they can be object of repetitive reductions. This new category includes the Lagrange–Poincaré category in Mechanics as a particular case. The structure of the paper is as follows. In Sect. 2, we provide the required preliminaries. This section also recalls the Lagrange–Poincaré category in Mechanics. In Sect. 3, the Lagrange–Poincaré reduction principle for Field Theories is reviewed. In Sects. 4 and 5, the construction of the Lagrange–Poincaré category for Field Theories is introduced together with the detailed description of the variational principle and equations. As we mentioned before, all this follows the spirit and generalizes the particular case of Mechanics but, interestingly, this particularization can be pushed back and some properties of the general case can be directly derived from Mechanic as we show below. Section 6 analyzes and confirms the correct behavior of the theory when successive reductions are performed. Section 7 studies the reconstruction process from solution of the variational problem after reduction, to solutions of the unreduced problem. One gets the characteristic trait in Field Theories that one does not find in the Lagrange–Poincaré category in Mechanics: A compatibility condition is needed to perform the reconstructions. This is already present since the first works on reduction for Field theories (see [7,8]) and continues in more recent works (see [5,6,12]). As usual in these papers, we describe it as the vanishing of the curvature of certain connection.

One of the most intrinsic concepts attached to the notion of symmetry is that of Noether current. In Sect. 8, we explore this object in the new category and analyze its conservation. In fact, it is proved that the Noether current is not conserved in general, but it satisfies a specific drift law. The interesting property of this law is that is makes part of the vertical equation when reduction is performed. Roughly speaking, reductions make the vertical equation involve more and more variables (and hence, the horizontal equation becomes successively smaller) by adjoining the successive Noether drift laws to it.

We complete the work in Sect. 9 with an example. One paradigmatic instance of reduction by stages in Mechanics is the rigid body with rotors. Here, we analyze the geometric setting describing a molecular strand composed by a chain of rigid bodies (as it is done in [11]) such that each body has one or more rotors. This could be regarded as a model of linked molecules with a rotating side chain(s). Simple proteins of amino acids could fit in this context. Future work will include further applications of the theory to other models which can be inspired by the numerous applications of the reduction by stages in Mechanics or can be taken from new systems of purely covariant nature.

2. Preliminaries

2.1. Principal Bundles

Let G be a Lie group acting freely and properly on the left on a manifold Q. Then, the quotient Q/G is also a manifold and the projection $\pi_{Q/G,Q}$: $Q \to Q/G$ is a principal G-bundle. We recall that a *principal connection* $\mathcal{A}$ on $Q \to Q/G$ is a 1-form on Q taking values on $\mathfrak{g}$, the Lie algebra of G, such that $\mathcal{A}(\xi_q^Q) = \xi$, for any $\xi \in \mathfrak{g}$, $q \in Q$, and $\rho_g^*\mathcal{A} = \mathrm{Ad}_g \circ \mathcal{A}$, where $\rho_g : Q \to Q$ denotes the action by $g \in G$, Ad denotes the adjoint action of G on $\mathfrak{g}$, and

$$\xi_q^Q = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi) \cdot q \in T_qQ.$$

Such a principal connection splits the tangent space as $T_qQ = H_qQ \oplus V_qQ$, for all $q \in Q$, where

$$V_qQ = \ker T_q\pi_{Q/G,Q} = \{v \in T_qQ | T_q\pi_{Q/G,Q}(v) = 0\}, \qquad q \in Q,$$
$$H_qQ = \ker \mathcal{A}_q = \{v \in T_qQ | \mathcal{A}_q(v) = 0\}, \qquad q \in Q,$$

are, respectively, called *vertical* and *horizontal subspace*. In fact, H_qQ is isomorphic to $T_x(Q/G)$, $x = \pi_{Q/G,Q}(q)$, through $T_q\pi_{Q/G,Q}$. The inverse of this isomorphism is called *horizontal lift* and is denoted by $\mathrm{Hor}_q^{\mathcal{A}}$. The *curvature* of a connection $\mathcal{A}$ is the $\mathfrak{g}$-valued 2-form

$$B(v, w) = d\mathcal{A}(\mathrm{Hor}(v), \mathrm{Hor}(w)),$$

where $v, w \in T_qQ$ and $\mathrm{Hor}(v)$ is the projection of $v \in T_qQ$ to H_qQ.

The *adjoint bundle* to $Q \to Q/G$ is the associated bundle $(Q \times \mathfrak{g})/G$ by the adjoint action of G on $\mathfrak{g}$. We shall denote it by $\mathrm{Ad}Q \to Q/G$ and its elements by $[q, \xi]_G$, $q \in Q$, $\xi \in \mathfrak{g}$. Remarkably, $\mathrm{Ad}Q \to Q/G$ is a Lie algebra bundle with a fiberwise Lie bracket given by

$$[[q, \xi_1]_G, [q, \xi_2]_G] = [q, [\xi_1, \xi_2]]_G, \qquad [q, \xi_1]_G, [q, \xi_2]_G \in \mathrm{Ad}Q_x, \ \ x = \pi_{Q/G,Q}(q).$$

The principal connection $\mathcal{A}$ on $Q \to Q/G$ defines a linear connection on the vector bundle $\mathrm{Ad}Q \to Q/G$, denoted $\nabla^{\mathcal{A}}$ and given by the covariant derivative along curves

$$\frac{\mathrm{D}[q(t), \xi(t)]_G}{\mathrm{D}t} = \Big[q(t), \dot{\xi}(t) - [\mathcal{A}(\dot{q}(t)), \xi(t)]\Big]_G .$$

In addition, the curvature of $\mathcal{A}$ can be seen as a 2-form on Q/G with values in the adjoint bundle as for any $X, Y \in T_x(Q/G)$

$$\tilde{B}(X, Y) = [q, B(\mathrm{Hor}_q^{\mathcal{A}} X, \mathrm{Hor}_q^{\mathcal{A}} Y)]_G.$$

The connection $\mathcal{A}$ induces a well-known vector bundle isomorphism

$$\begin{aligned} \alpha_{\mathcal{A}} : TQ/G &\longrightarrow T(Q/G) \oplus \mathrm{Ad}Q \\ [v_q]_G &\ \mapsto\ T_q\pi_{Q/G,Q}(v_q) \oplus [q, \mathcal{A}(v_q)]_G \end{aligned} \tag{1}$$

used in Mechanics to reduce G-invariant Lagrangians defined on TQ. This is the so-called Lagrange–Poincaré reduction.

2.2. Quotient of Vector Bundles

Given a vector bundle $V \to Q$, we say that an action ρ of a Lie group G on V is a vector bundle action if for all $g \in G$, $\rho_g : V \to V$ are vector bundle isomorphisms and the action induced on Q is free and proper. Then, there is a vector bundle structure on $V/G \to Q/G$ with operations

$$[v_q]_G + [w_q]_G = [v_q + w_q]_G \text{ and } \lambda[v_q]_G = [\lambda v_q]_G,$$

where $[v_q]_G, [w_q]_G \in V/G$ stand for the equivalence classes of $v_q, w_q \in V_q$ and $\lambda \in \mathbb{R}$. In the diagram,

$$\begin{array}{ccc} V & \xrightarrow{\pi_{V/G,V}} & V/G \\ \Big\downarrow{\scriptstyle \pi_{Q,V}} & & \Big\downarrow{\scriptstyle \pi_{Q/G,V/G}} \\ Q & \xrightarrow{\pi_{Q/G,Q}} & Q/G \end{array}$$

$\pi_{Q,V}$ and $\pi_{Q/G,V/G}$ are projections of vector bundles, $\pi_{Q/G,Q}$ is the projection of a G-principal bundle and $\pi_{V/G,V}$ is a surjective vector bundle homomorphism.

Suppose that $V \to Q$ has an affine connection ∇ or, equivalently, a covariant derivative $\mathrm{D}v(t)/\mathrm{D}t$ of curves $v(t)$ in V. A curve $v(t) : I \subset \mathbb{R} \to V$ is horizontal whenever $\mathrm{D}v(t)/\mathrm{D}t = 0$, for every $t \in I$. Let $q(t)$ be a curve in Q and denote $q_0 = q(t_0)$ a fixed value of the curve, then the *horizontal lift* of $q(t)$ through $v \in V_{q_0}$ at t_0 is an horizontal curve, $q_v^h(t)$, in V such that $\pi_{Q,V} \circ q_v^h = q$ and $q_v^h(t_0) = v$.

Let $v(t)$ be a curve in V, $q(t) = \pi_{Q,V}(v(t))$, $x(t) = \pi_{Q/G,Q}(q(t))$ and $q_0 = q(t_0)$. Consider $x_{q_0}^h(t)$ the horizontal lift through $q_0 \in Q$ of $x(t)$ with respect to the principal connection $\mathcal{A}$, we define $g_{q_0}(t)$ in G such that

$$q(t) = g_{q_0}(t) x_{q_0}^h(t),$$

as well as the curve $v_h(t) = g_{q_0}^{-1}(t) v(t)$ in V, and then,

$$\begin{aligned}\left.\frac{\mathrm{D}}{\mathrm{D}t}\right|_{t=t_0} v(t) &= \left.\frac{\mathrm{D}}{\mathrm{D}t}\right|_{t=t_0} g_{q_0}(t) v_h(t) = \left.\frac{\mathrm{D}}{\mathrm{D}t}\right|_{t=t_0} g_{q_0}(t) v_h(t_0) + \left.\frac{\mathrm{D}}{\mathrm{D}t}\right|_{t=t_0} g_{q_0}(t_0) v_h(t) \\ &= \left.\frac{\mathrm{D}}{\mathrm{D}t}\right|_{t=t_0} g_{q_0}(t) v(t_0) + \left.\frac{\mathrm{D}}{\mathrm{D}t}\right|_{t=t_0} v_h(t) = \dot{g}_{q_0}(t_0)^V_{v(t_0)} + \left.\frac{\mathrm{D}}{\mathrm{D}t}\right|_{t=t_0} v_h(t)\end{aligned}$$

Thus, the covariant derivative $\mathrm{D}v(t)/\mathrm{D}t$ at t_0 can be decomposed in horizontal and vertical components

$$\left.\frac{\mathrm{D}^{(\mathcal{A},H)}}{\mathrm{D}t}\right|_{t=t_0} v(t) = \left.\frac{\mathrm{D}}{\mathrm{D}t}\right|_{t=t_0} v_h(t), \qquad \left.\frac{\mathrm{D}^{(\mathcal{A},V)}}{\mathrm{D}t}\right|_{t=t_0} v(t) = \dot{g}_{q_0}(t_0)^V_{v(t_0)}.$$

Consequently, given $X \in \mathfrak{X}(Q)$ and $v \in \Gamma(Q,V)$ we can define

$$\nabla_X^{(\mathcal{A},H)} v(q_0) = \left.\frac{\mathrm{D}^{(\mathcal{A},H)}}{\mathrm{D}t}\right|_{t=t_0} v(t), \qquad \nabla_X^{(\mathcal{A},V)} v(q_0) = \left.\frac{\mathrm{D}^{(\mathcal{A},V)}}{\mathrm{D}}\right|_{t=t_0} v(t), \tag{2}$$

where $v(t) = v(q(t))$ and $q(t)$ is an integral curve of X in Q such that $q_0 = q(t_0)$.

Let $X = Y \oplus \bar{\xi} \in \mathfrak{X}(Q/G) \oplus \Gamma(\mathrm{Ad}Q) \simeq \Gamma(TQ/G)$, using the identification (1). There is a unique G-invariant vector field $\bar{X} \in \Gamma^G(TQ)$ on Q projecting to X. Furthermore, $\bar{X} = Y^h \oplus W$ with $Y^h \in \mathfrak{X}(TQ)$ the horizontal lift of Y and W the unique vertical G-invariant vector field such that for all $x \in Q/G$, $\bar{\xi}(x) = [q, \mathcal{A}(W(q))]_G$ with $q \in \pi_{Q/G,Q}^{-1}(x)$. Then, for $[v]_G \in \Gamma(Q/G, V/G)$ with $v \in \Gamma^G(Q,V)$ a G-invariant section, we define the *quotient connection* by

$$\left[\nabla^{(\mathcal{A})}\right]_{G,Y\oplus\bar{\xi}} [v]_G = [\nabla_{\bar{X}} v]_G,$$

the *horizontal quotient connection* is defined by

$$\left[\nabla^{(\mathcal{A},H)}\right]_{G,Y\oplus\bar{\xi}} [v]_G = [\nabla_{Y^h} v]_G = [\nabla_{\bar{X}}^{(\mathcal{A},H)} v]_G$$

and the *vertical quotient connection* is defined by

$$\left[\nabla^{(\mathcal{A},V)}\right]_{G,Y\oplus\bar{\xi}} [v]_G = [\nabla_W v]_G = [\xi_v^V]_G,$$

where ξ satisfies $\bar{\xi} = [\pi_{Q,V}(v), \xi]_G$. Note that these are not connections in the usual sense as derivation is performed with respect to sections of TQ/G instead of sections of $T(Q/G)$. Only the horizontal quotient connection can be thought as a usual connection since it only depends on $T(Q/G) \subset TQ/G$.

2.3. The $\mathfrak{LP}$ Category

In Lagrange–Poincaré reduction, the original Lagrangian L is defined on TQ, the tangent bundle of the configuration space Q. However, the reduced Lagrangian is defined on $TQ/G \cong T(Q/G) \oplus \mathrm{Ad}Q$ which needs not be a tangent bundle. To iterate Lagrange–Poincaré reduction, the category $\mathfrak{LP}$ of Lagrange–Poincaré bundles was introduced in [9]. Of course, this category includes TQ/G and is stable under reduction.

The *objects* of $\mathfrak{LP}$ are vector bundles $TQ \oplus V \to Q$ where $TQ \to Q$ is the tangent bundle of a manifold Q, and $V \to Q$ is a vector bundle with the following additional structure:

(a) A Lie bracket $[,]$ in the fibers of V;
(b) A V-valued 2-form ω on Q;
(c) A linear connection ∇ on V;
(d) The bilineal operator defined by

$$[X_1 \oplus w_1, X_2 \oplus w_2] = [X_1, X_2] \oplus (\nabla_{X_1} w_2 - \nabla_{X_2} w_1 - \omega(X_1, X_2) + [w_1, w_2]),$$

where $[X_1, X_2]$ denotes the Lie bracket of vector fields and $[w_1, w_2]$ denotes the Lie bracket in the fibers of V, is a Lie bracket on sections $X \oplus w \in \Gamma(TQ \oplus V)$.

The *morphisms* between two Lagrange–Poincaré bundles $TQ_i \oplus V_i$, $i = 1, 2$ with structures $[,]_i$, ω_i and $\mathrm{D}_i/\mathrm{D}t$ are vector bundle morphisms $f : TQ_1 \oplus V_1 \to TQ_2 \oplus V_2$ such that:

(a) $f(TQ_1) \subset TQ_2$ and $f|_{TQ_1} = Tf_0$, where $f_0 : Q_1 \to Q_2$ is the function induced by f in the base spaces;
(b) $f(V_1) \subset V_2$ and $f|_{V_1}$ commutes with the additional structure. That is, given $v, v' \in (\pi_{Q_1,V_1})^{-1}(q)$, $X, X' \in (\pi_{Q_1,TQ_1})^{-1}(q)$ and a curve $v(t)$ in V_1:

$$f([v, v']_1) = [f(v), f(v')]_2, \tag{3}$$

$$f(\omega_1(X, X')) = \omega_2(f(X), f(X')), \tag{4}$$

$$f\left(\frac{\mathrm{D}_1 v(t)}{\mathrm{D}t}\right) = \frac{\mathrm{D}_2 f(v(t))}{\mathrm{D}t}. \tag{5}$$

Definition 1. An allowed variation of a curve $\dot{q}(t) \oplus v(t)$, $t \in [t_0, t_1]$, on the Lagrange–Poincaré bundle $TQ \oplus V$ is a smooth map $\dot{q}_\varepsilon(t) \oplus v_\varepsilon(t) : [t_0, t_1] \times I \to TQ \oplus V$ such that $\delta\dot{q}$ is the lifted variation of a free variation δq of $q(t)$ and

$$\delta v = \frac{\mathrm{D}w}{\mathrm{D}t} + [v, w] + \omega_q(\delta q, \dot{q}), \tag{6}$$

where $w(t)$ is a curve in V with $w(t_0) = w(t_1) = 0$ and $\pi_{Q,V}(w(t)) = q(t)$.

Given a Lagrangian $L : TQ \oplus V \to \mathbb{R}$ defined on an element of $\mathfrak{LP}$, a curve $\dot{q}(t) \oplus v(t) : [t_0, t_1] \to TQ \oplus V$ is said to be critical if and only if

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_{t_0}^{t_1} L(\dot{q}_\varepsilon(t) \oplus v_\varepsilon(t))\mathrm{d}t,$$

for every allowed variation $\dot{q}_\varepsilon(t) \oplus v_\varepsilon(t)$ of $\dot{q}(t) \oplus v(t)$. As seen in [9], this variational principle is equivalent to the Lagrange–Poincaré equations in Mechanics. The following result is generalized to the context of field theory in Theorem 17 within Sect. 5.

Proposition 2. *Given a Lagrangian $L : TQ \oplus V \to \mathbb{R}$, a curve $\dot{q}(t) \oplus v(t)$ is critical if and only if the well-known Lagrange–Poincaré equations:*

$$\frac{\delta L}{\delta q} - \frac{\mathrm{D}}{\mathrm{D}t}\frac{\delta L}{\delta \dot{q}} = \left\langle \frac{\delta L}{\delta v}, \omega_q(\dot{q}, \cdot) \right\rangle, \tag{7}$$

$$\mathrm{ad}^*_v \frac{\delta L}{\delta v} = \frac{\mathrm{D}}{\mathrm{D}t}\frac{\delta L}{\delta v}, \tag{8}$$

where ad^* *stands for the coadjoint action in $V^* \to Q$.*

2.4. Reduction of $\mathfrak{LP}$ Bundles

Lagrange–Poincaré bundles can be reduced as follows.

Proposition 3 [9, §6.2]. *Let $TQ \oplus V \to Q$ be an object of $\mathfrak{LP}$ with additional structure $[,]$, ω and ∇. Let $\rho : G \times (TQ \oplus V) \to TQ \oplus V$ be a free and proper action in the category $\mathfrak{LP}$ (for all $g \in G$, ρ_g is an isomorphism in $\mathfrak{LP}$) and $\mathcal{A}$ a principal connection on $Q \to Q/G$. Then, the vector bundle*

$$T(Q/G) \oplus \mathrm{Ad}Q \oplus (V/G)$$

with additional structures $[,]^{\tilde{\mathfrak{g}}}$, $\omega^{\tilde{\mathfrak{g}}}$ and $\nabla^{\tilde{\mathfrak{g}}}$ in $\mathrm{Ad}Q \oplus (V/G)$ given by

$$\begin{aligned}
\nabla^{\tilde{\mathfrak{g}}}_X(\bar{\xi} \oplus [v]_G) =& \nabla^{\mathcal{A}}_X \bar{\xi} \oplus \left([\nabla^{(\mathcal{A},H)}]_{G,X}[v]_G - [\omega]_G(X, \bar{\xi}) \right), \\
\omega^{\tilde{\mathfrak{g}}}(X_1, X_2) =& \tilde{B}(X_1, X_2) \oplus [\omega]_G(X_1, X_2), \\
[\bar{\xi}_1 \oplus [v_1]_G, \bar{\xi}_2 \oplus [v_2]_G]^{\tilde{\mathfrak{g}}} =& [\bar{\xi}_1, \bar{\xi}_2] \oplus \Big([\nabla^{(\mathcal{A},V)}]_{G,\bar{\xi}_1}[v_2]_G \\
& - [\nabla^{(\mathcal{A},V)}]_{G,\bar{\xi}_2}[v_1]_G - [\omega]_G(\bar{\xi}_1, \bar{\xi}_2) + [[v_1]_G, [v_2]_G]_G \Big)
\end{aligned}$$

is an object of the $\mathfrak{LP}$ category called the reduced bundle with respect to the group G and the connection $\mathcal{A}$.

The variational principles set by invariant Lagrangians on Lagrange–Poincaré bundles can also be reduced.

Proposition 4 [3, §3]. *Let $L : TQ \oplus V \to \mathbb{R}$ be a G-invariant, and let π_G be the projection of $TQ \oplus V \to (TQ \oplus V)/G$ and $\alpha^{TQ\oplus V}_{\mathcal{A}}$ the identification between $(TQ \oplus V)/G$ and $T(Q/G) \oplus \mathrm{Ad}Q \oplus (V/G)$ provided by Proposition (3). Consider*

$$l : T(Q/G) \oplus \mathrm{Ad}Q \oplus (V/G) \to \mathbb{R},$$

the reduced Lagrangian induced by L. Then, a curve $\dot{q}(t) \oplus v(t)$ is critical for the variational problem set by L if and only if the curve

$$\dot{x}(t) \oplus \bar{\xi}(t) \oplus [v]_G(t) = \alpha^{TQ\oplus V}_{\mathcal{A}} \circ \pi_G(\dot{q}(t) \oplus v(t)),$$

is critical for the variational problem set by l (see [3]). Equivalently, $\dot{q}(t) \oplus v(t)$ solves the Lagrange–Poincaré equations given by L in $TQ \oplus V$ if and only *if* $\dot{x}(t) \oplus \bar{\xi}(t) \oplus [v]_G(t)$ *solves the Lagrange–Poincaré equations given by* l *in* $T(Q/G) \oplus \mathrm{Ad}Q \oplus (V/G)$.

Remark 5. Reduction of variational problems set in $\mathfrak{LP}$ bundles is a process than can be iterated. This allows to do reduction by stages: If we reduce by N, a normal subgroup of G and afterward by $K = G/N$, the final result is equivalent to a direct reduction by G, provided that the auxiliary connections used along the process are conveniently chosen.

3. Lagrange–Poincaré Field Equations

Let $\pi_{X,P} : P \to X$ be a (non-necessarily principal) fiber bundle. Two local sections $\rho : U \to P$ and $\rho' : U' \to P$ represent the same 1-jet, $j^1_x\rho$ at $x \in U \cap U'$ if and only if $\rho(x) = \rho'(x)$ and $T_x\rho = T_x\rho'$. This defines an equivalence relation, and we denote by $J^1_x P$ the space of such classes. The *1-jet bundle* is the space $J^1P = \cup_{x\in X} J^1_x P$ equipped with a natural smooth structure of fiber bundle over P with projection $j^1_x\rho \in J^1P \mapsto \rho(x) \in P$. The bundle $J^1P \to P$ is affine and modeled over the vector bundle $T^*X \otimes_P VP$, where the abuse of notation $T^*X = \pi^*_{X,P}T^*X$ has been used.

A first-order Lagrangian density is a smooth fiber map $\mathcal{L} : J^1P \to \bigwedge^n T^*X$, where $n = \dim(X)$. Suppose that X is oriented and $\mathrm{Vol} \in \Gamma(\bigwedge^n T^*X)$ is a volume form, then the function $L : J^1P \to \mathbb{R}$ such that $\mathcal{L} = L\mathrm{Vol}$ is called a Lagrangian. A section ρ of $P \to X$ is a critical section for the variational problem defined by $\mathcal{L} = L\mathrm{Vol}$ if

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_X \mathcal{L}(j^1\rho_\varepsilon) = 0$$

for all compactly supported variations ρ_ε of ρ that are vertical, that is for all $x \in X$, $\mathrm{d}\rho_\varepsilon(x)/\mathrm{d}\varepsilon|_{\varepsilon=0} \in V_{\rho(x)}P$. This variational principle is equivalent to the fact that $\rho(x)$ satisfies the Euler–Lagrange equations, which can be written in an implicit way as

$$\frac{\delta L}{\delta\rho} - \mathrm{div}^P \frac{\delta L}{\delta j^1\rho} = 0,$$

where $\delta L/\delta j^1\rho \in TX \otimes (VP)^*$ is the fiber derivative in J^1P, div^P is defined for $(VP)^*$-valued fields using a connection ∇^P in $(VP \subset TP) \to P$ and $\delta L/\delta\rho$ the horizontal differential with respect to ∇^P.

Let $\Phi : G \times P \to P$ be a free and proper action such that for all $g \in G$, $\pi_{X,P} \circ \Phi_g = \pi_{X,P}$. Then, $P \to \Sigma = P/G$ is a G-principal bundle and the action in P can be lifted to J^1P. This defines $\pi_G : J^1P \to (J^1P)/G$, and according to [12], once fixed a connection $\mathcal{A}$ in $P \to \Sigma$, there is an identification

$$\begin{aligned} \alpha_{\mathcal{A}} : (J^1P)/G &\to J^1\Sigma \oplus (T^*X \otimes_\Sigma \mathrm{Ad}P) \\ j^1\rho &\mapsto j^1\sigma \oplus [\rho, \rho^*\mathcal{A}]_G \end{aligned} \tag{9}$$

where $\sigma = \pi_{\Sigma,P} \circ \rho = [\rho]_G$ and $\oplus$ denotes the Whitney product of an affine bundle and a vector bundle. Observe that $J^1\Sigma \oplus (T^*X \otimes_\Sigma \mathrm{Ad}P)$ is in turn an affine bundle. We shall denote $\bar{\rho} = [\rho, \rho^*\mathcal{A}]_G$. Given a G-invariant Lagrangian $L : J^1P \to \mathbb{R}$ and its reduced Lagrangian $l : J^1\Sigma \oplus (T^*X \otimes_\Sigma \mathrm{Ad}P) \to \mathbb{R}$, the main result in [12] states that the variational principle defined by L is equivalent to the fact that the reduced section $j^1\sigma \oplus \bar{\rho}(x)$ satisfies the Lagrange–Poincaré equations;

$$\mathrm{ad}^* \frac{\delta l}{\delta \bar{\rho}} - \mathrm{div}^\nabla \frac{\delta l}{\delta \bar{\rho}} = 0,$$
$$\frac{\delta l}{\delta \sigma} - \mathrm{div}^\Sigma \frac{\delta l}{\delta j^1\sigma} = \left\langle \frac{\delta l}{\delta \bar{\rho}}, i_{T_\sigma} \tilde{B} \right\rangle.$$

This is, in turn, equivalent to the variational principle

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_X l(j^1\sigma_\varepsilon \oplus \bar{\rho}_\varepsilon)\mathrm{Vol} = 0$$

for variations $j^1\sigma_\varepsilon \oplus \bar{\rho}_\varepsilon$ such that $\delta\bar{\rho} = \nabla^{\mathcal{A}}\bar{\eta} - [\bar{\eta}, \bar{\rho}] - \tilde{B}(\delta\sigma, T\sigma)$, where $\delta\sigma$ is an arbitrary vertical variation of σ, $\bar{\eta}$ is an arbitrary section of $\mathrm{Ad}P \to X$ such that $\pi_{\Sigma,\mathrm{Ad}P}\bar{\eta} = \sigma$ and $\nabla^{\mathcal{A}}$ is the connection on $\mathrm{Ad}P$ induced by $\mathcal{A}$ and defined in §2. This procedure is called Lagrange–Poincaré reduction for field theoretical covariant Lagrangians.

The attentive reader may have noticed that $\nabla^{\mathcal{A}}$ is a connection on $\mathrm{Ad}P \to \Sigma$ and, consequently, acts on sections of $\mathrm{Ad}P \to \Sigma$, while $\bar{\eta}$ is a section of $\mathrm{Ad}P \to X$. We shall now explain how to extend a connection to derive this kind of sections. Let $V \to P$ be a vector bundle with connection ∇ and $P \to X$ a fiber bundle, given $f : X \to V$ a section of $V \to X$, define the ∇-derivative of f with respect to $u_x \in T_xX$ as

$$\tilde{\nabla}_{u_x} f = \left.\frac{\mathrm{D}^\nabla}{\mathrm{D}t}\right|_{t=0} f(c(t)) \in V_{\pi_{P,V} f(x)},$$

where $c(t)$ is a curve in X such that $\dot{c}(0) = u_x$, $\mathrm{D}^\nabla/\mathrm{D}t$ is the usual covariant derivative associated with ∇, and $t \mapsto f(c(t))$ is a curve in V. As f is a section on $V \to X$,

$$\mathrm{id}_X = \pi_{X,V} \circ f = \pi_{X,P} \circ \pi_{P,V} \circ f$$

and $\rho = \pi_{P,V} \circ f$ is a section of $P \to X$. Then, $f(c(t)) = f \circ \pi_{X,P} \circ \rho(c(t))$ projects to curve $\rho(c(t))$ in P with $\mathrm{d}\rho(c(t))/\mathrm{d}t|_{t=0} = T_x\rho(u_x)$ and

$$\tilde{\nabla}_{u_x} f = \left.\frac{\mathrm{D}^\nabla}{\mathrm{D}}\right|_{t=0} f(c(t)) = \nabla_{T_x\rho(u_x)}\bar{f},$$

where $\bar{f}$ is a local section of $V \to P$ around $\rho(x)$ such that $\bar{f}|_{\mathrm{im}(\rho(X))} = f \circ \pi_{X,P}$. Sometimes we will use the abuse of notation $f = \bar{f}$.

4. The FT𝔏𝔓 Category

We now define the category of bundles to perform reduction by stages of covariant Lagrangian Field Theories.

Definition 6. Given X a manifold called base space, the category $FT\mathfrak{LP}(X)$ of field theoretical Lagrange–Poincaré bundles over X is defined as follows:

(i) The *objects* of $FT\mathfrak{LP}(X)$ are bundles of the form $J^1P \oplus (T^*X \otimes_P V) \to P$, where $\pi_{XP} : P \to X$ is a fiber bundle (not necessarily principal), $T^*X \to P$ is an abuse of notation for the pullback $\pi_{XP}^* T^*X \to P$, and $V \to P$ is a vector bundle which is the vectorial part of an $\mathfrak{LP}$-bundle. In other words, $TP \oplus V \to P$ is an $\mathfrak{LP}$-bundle, which in turn is equivalent to the existence of a

  (a) Lie bracket, $[,]$, in the fibers of V;
  (b) V-valued 2-form ω on P;
  (c) Linear connection, ∇, on $V \to P$;
  (d) Lie bracket operation on the sections $Z \oplus u \in \Gamma(TP \oplus V)$ defined by

$$[Z_1 \oplus u_1, Z_2 \oplus u_2] = [Z_1, Z_2] \oplus \nabla_{Z_1} u_2 - \nabla_{Z_2} u_1 - \omega(Z_1, Z_2) + [u_1, u_2]$$

(ii) Let $J^1P_1 \oplus (T^*X \otimes_{P_1} V_1) \to P_1$ and $J^1P_2 \oplus (T^*X \otimes_{P_2} V_2) \to P_2$ be two field theoretical Lagrange–Poincaré bundles over X with structures $[,]_i$,∇_i and ω_i on $V_i \to P_i$, $i = 1, 2$. A *morphism*, $f : J^1P_1 \oplus (T^*X \otimes_{P_1} V_1) \to J^1P_2 \oplus (T^*X \otimes_{P_2} V_2)$ is a bundle map covering $f_0 : P_1 \to P_2$ that satisfies

  (a) $f_0 : P_1 \to P_2$ is a bundle map between $P_1 \to X$ and $P_2 \to X$ covering the identity on X,
  (b) f can be written as

$$f = (j^1 f_0, \mathrm{id} \otimes \bar{f}),$$

where $j^1 f_0$ is the 1-jet extension of f_0, and $\bar{f} : V_1 \to V_2$ is a vector bundle morphism that covers f_0 and commutes with the structures on V_i given by $[,]_i$,∇_i and ω_i on $V_i \to P_i$, $i = 1, 2$. More explicitly, given $u, u' \in (V_1)_{p_1}$, $Z, Z' \in (TP_1)_{p_1}$ and a curve $v(t)$ in V_1;

$$\bar{f}([u, u']_1) = [\bar{f}(u), \bar{f}(u')]_2, \tag{10}$$

$$\bar{f}(\omega_1(Z, Z')) = \omega_2(Tf_0(Z), Tf_0(Z')), \tag{11}$$

and

$$\bar{f}\left(\frac{\mathrm{D}_1 v(t)}{\mathrm{D}t}\right) = \frac{\mathrm{D}_2 \bar{f}(v(t))}{\mathrm{D}t}, \tag{12}$$

are satisfied.

Remark 7. There are several special cases of objects in $FT\mathfrak{LP}(X)$ appearing in the present bibliography. For $V = 0$, we obtain 1-jet bundles used in Lagrangian covariant Field Theory (for example, see [15]). Another instance of object in $FT\mathfrak{LP}(X)$ is the quotient of a 1-jet bundle J^1P by a proper and free lifted action of a Lie group G on P, found to be isomorphic to $J^1(P/G) \oplus (T^*X \otimes_{(P/G)} \mathrm{Ad}P)$ in [12]. In the case where $P \to X$ is G-principal bundle, the quotient J^1P/G is a $FT\mathfrak{LP}(X)$ bundle of the form $T^*X \otimes \mathrm{Ad}P$ which is the vector bundle underlying the affine bundle of connections used in [7] to perform Euler–Poincaré reduction.

Finally, the particular case when $X = \mathbb{R}$ and $P = \mathbb{R} \times Q$, with Q a manifold, gives the $\mathfrak{LP}$ bundle

$$T(\mathbb{R} \times Q) \oplus \mathrm{Ad}(\mathbb{R} \times Q) \simeq \mathbb{R} \times (TQ \oplus \mathrm{Ad}P)$$

appearing when reducing time-dependent Lagrangians in classical Mechanics.

There exists a way of thinking bundles in $FT\mathfrak{LP}(X)$ as $\mathfrak{LP}$ bundles used in Mechanics. First, we start by defining relevant subcategories of $\mathfrak{LP}$ bundles.

Definition 8. Given X a manifold called base space, we define the subcategory $\mathfrak{LP}(X)$ of $\mathfrak{LP}$ whose objects are $\mathfrak{LP}$ bundles $TP \oplus V$, such that P is the total space of a fiber bundle $P \to X$, and whose morphisms are $\mathfrak{LP}$ morphisms, $f = Tf_0 \oplus \bar{f} : TP_1 \oplus V_1 \to TP_2 \oplus V_2$, such that $f_0 : P_1 \to P_2$ is a bundle map over X covering id_X.

Proposition 9. *The applications*

$$\begin{aligned} \mathcal{F} : \mathfrak{LP}(X) &\to FT\mathfrak{LP}(X) \\ TP \oplus V &\mapsto J^1 P \oplus (T^*X \otimes V), \end{aligned}$$

$$\begin{aligned} \mathcal{F} : \mathrm{Hom}(\mathfrak{LP}(X)) &\to \mathrm{Hom}(FT\mathfrak{LP}(X)) \\ Tf_0 \oplus \bar{f} &\mapsto j^1(f_0) \oplus (\mathrm{id}_{f_0} \otimes \bar{f}), \end{aligned}$$

define a covariant functor with inverse. Thence, the categories $\mathfrak{LP}(X)$ and $FT\mathfrak{LP}(X)$ are isomorphic.

Proof. Let $TP \oplus V \in \mathfrak{LP}(X)$,

$$\mathcal{F}(\mathrm{id}_{TP\oplus V}) = \mathcal{F}(T\mathrm{id}_P \oplus \mathrm{id}_V) = j^1(\mathrm{id}_P) \oplus (\mathrm{id}_{\mathrm{id}_P} \otimes \mathrm{id}_V) = \mathrm{id}_{\mathcal{F}(TP\oplus V)}.$$

On the other hand, given $f, g \in \mathrm{Hom}(\mathfrak{LP}(X))$ it is easy to see that $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ as $T(g \circ f) = Tg \circ Tf$ and $j^1(g \circ f) = j^1 g \circ j^1 f$. These two properties prove that $\mathcal{F}$ is a functor. It has an inverse since

$$\begin{aligned} \mathcal{G} : FT\mathfrak{LP}(X) &\to \mathfrak{LP}(X) \\ J^1 P \oplus (T^*X \otimes V) &\mapsto TP \oplus V, \end{aligned}$$

$$\begin{aligned} \mathcal{G} : \mathrm{Hom}(FT\mathfrak{LP}(X)) &\to \mathrm{Hom}(\mathfrak{LP}(X)) \\ j^1(f_0) \oplus (\mathrm{id}_{f_0} \otimes \bar{f}) &\mapsto Tf_0 \oplus \bar{f}, \end{aligned}$$

is well defined, $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathfrak{LP}(X)}$, and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{FT\mathfrak{LP}(X)}$. □

Corollary 10. *The following three statements are equivalent: $j^1(f_0) \oplus (\mathrm{id}_{f_0} \otimes \bar{f})$ is an isomorphism; $Tf_0 \oplus \bar{f}$ is an isomorphism; and $\bar{f}$ is an isomorphism.*

Proof. The first two statements are equivalent since $\mathcal{F}$ is an isomorphism of categories. The third statement is equivalent to the others since $\bar{f}$ fully determines both $j^1(f_0) \oplus (\mathrm{id}_{f_0} \otimes \bar{f})$ and $Tf_0 \oplus \bar{f}$. □

We shall define the notion of an action of a group G on an object of $FT\mathfrak{LP}(X)$.

Definition 11. An action in the category $FT\mathfrak{LP}(X)$ of a group G on an object $J^1P \oplus (T^*X \otimes V)$ of $FT\mathfrak{LP}(X)$ is a differentiable action

$$\Phi : G \times (J^1P \oplus (T^*X \otimes V)) \to J^1P \oplus (T^*X \otimes V)$$

such that for each $g \in G$, $\Phi_g : J^1P \oplus (T^*X \otimes V) \to J^1P \oplus (T^*X \otimes V)$ belongs to $\mathrm{Hom}(FT\mathfrak{LP}(X))$. We will say that this action is free and proper if the induced action on P by the functions $(\Phi_g)_0$ is free and proper.

Proposition 12. *Let $J^1P \oplus (T^*X \otimes_P V)$ be an object of $FT\mathfrak{LP}(X)$, and let $[,]$, ∇ and ω be the structure in V. Let G be a Lie group acting freely and properly on $J^1P \oplus (T^*X \otimes_P V)$ and $\mathcal{A}$ a connection in the principal bundle $P \to \Sigma = P/G$. The bundle*

$$J^1\Sigma \oplus (T^*X \otimes_\Sigma (\mathrm{Ad}P \oplus (V/G)))$$

with the structure $[,]^{\tilde{\mathfrak{g}}}$, $\nabla^{\tilde{\mathfrak{g}}}$ and $\omega^{\tilde{\mathfrak{g}}}$ on $\mathrm{Ad}P \oplus V$ as in Proposition 3 is an object in $FT\mathfrak{LP}(X)$ diffeomorphic to $(J^1P \oplus (T^*X \otimes_P V))/G$ via the bundle diffeomorphism

$$\begin{aligned} \beta_{\mathcal{A}} : (J^1P \oplus (T^*X \otimes V))/G &\longrightarrow J^1\Sigma \oplus (T^*X \otimes (adP \oplus (V/G))) \\ [j^1_x s \oplus ((p,\alpha), v)]_G &\mapsto (T_{\pi_{\Sigma,P}} \circ j^1_x s, [p, s^*\mathcal{A}]_G + ((\sigma,\alpha), [v]_G)), \end{aligned} \tag{13}$$

where $p \in P$, $\sigma = \pi_{\Sigma,P}(p) \in \Sigma$, $x = \pi_{X,P}(p) \in X$ and $j^1_x s \in J^1_p P$.

Proof. As G acts on $J^1P \oplus (T^*X \otimes V)$, for each $g \in G$ there exists an isomorphism

$$\Phi_g = j^1((\Phi_g)_0) \oplus (\mathrm{id}_{(\Phi_g)_0} \otimes \bar{\Phi}_g) : J^1P \oplus (T^*X \otimes V) \to J^1P \oplus (T^*X \otimes V)$$

of the category $FT\mathfrak{LP}(X)$. Therefore, by Corollary 10, G acts on $TP \oplus V = \mathcal{F}^{-1}(J^1P \oplus (T^*X \otimes V))$ via isomorphisms

$$\mathcal{F}^{-1}(\Phi_g) = T(\Phi_g)_0 \oplus \bar{\Phi}_g : TP \oplus V \to TP \oplus V.$$

The quotient of $TP \oplus V$ by this action using connection $\mathcal{A}$ is isomorphic to another $\mathfrak{LP}$ bundle, $T\Sigma \oplus \mathrm{Ad}P \oplus (V/G)$ with structures $[,]^{\tilde{\mathfrak{g}}}$, $\nabla^{\tilde{\mathfrak{g}}}$ and $\omega^{\tilde{\mathfrak{g}}}$. This bundle happens to be in $\mathfrak{LP}(X)$ since $\Sigma \to X$ is a fiber bundle, and then, from Proposition 9

$$\mathcal{F}(T\Sigma \oplus \mathrm{Ad}P \oplus (V/G)) = J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G)))$$

is isomorphic to $(J^1P \oplus (T^*X \otimes_P V))/G$. Finally, it can be checked that $\beta_{\mathcal{A}}$ is well defined and that

$$\begin{aligned} \beta_{\mathcal{A}}^{-1} : J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G))) &\to (J^1P \oplus (T^*X \otimes V))/G \\ \delta_\sigma \oplus \ell_\sigma \oplus ((\sigma,\alpha), [v]_G) &\mapsto [(\mathrm{Hor}^{\mathcal{A}}_p \circ \delta_\sigma + \kappa_p \circ \ell_\sigma) \oplus ((p,\alpha), v_p)]_G, \end{aligned}$$

where $v_p \in \pi^{-1}_{P,V}(p) = V_p$ such that $[v_p]_G = [v]_G$ and $\kappa_p : (\mathrm{Ad}P)_\sigma \to V_pP$ defined by $\kappa_p([p,\xi]_G) = \xi^P_p$. □

5. Variational Problems in FT$\mathfrak{LP}$ Bundles

Let $J^1P \oplus (T^*X \otimes_P V)$ be a FT$\mathfrak{LP}$ bundle. A *Lagrangian density* is a smooth fiber map $\mathcal{L} : J^1P \oplus (T^*X \otimes_P V) \to \bigwedge^n TX$ where n is the dimension of X. We will assume that X is orientable and we choose a volume form $\mathrm{Vol} \in \bigwedge^n TX$, and then, the Lagrangian density can be expressed as $\mathcal{L} = L\mathrm{Vol}$ with $L : J^1P \oplus (T^*X \otimes_P V) \to \mathbb{R}$.

Let $U \subset X$ be an open subset whose closure $\bar{U}$ is compact. We will only consider smooth sections $j^1\rho \oplus \nu : \bar{U} \to J^1P \oplus (T^*X \otimes_P V)$ such that $\nu \in \Gamma(\bar{U}, T^*X \otimes_P V)$ projects to a section $\rho = \pi_{P,T^*X\otimes V} \circ \nu \in \Gamma(\bar{U}, P)$, the 1-jet extension of which is $j^1\rho$. These sections are called *allowed sections*. We say that $j^1\rho \oplus \nu \in \Gamma(\bar{U}, J^1P \oplus (T^*X \otimes_P V))$ is a *critical section* for the variational problem defined by $\mathcal{L}$ if

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_U \mathcal{L}(j^1\rho_\varepsilon \oplus \nu_\varepsilon) = 0 \tag{14}$$

for all smooth *allowed variations* $j^1\rho_\varepsilon \oplus \nu_\varepsilon$ of $j^1\rho \oplus \nu$. Observe that we still need to define the set of allowed variations which plays within the realm of FT$\mathfrak{LP}$ bundles an analogous role to Definition 1 with respect to category $\mathfrak{LP}$. To do this, we will first introduce a connection in $T^*X \otimes_P V$ from a connection ∇^X in $TX \to X$ and a connection ∇ on $V \to P$. In fact, given $\nu \in \Gamma(P, T^*X \otimes_P V)$, $Z \in \mathfrak{X}(P)$ and $u \in \mathfrak{X}(U)$, the connection ∇^L on $T^*X \otimes_P V \to P$ is given by

$$(\nabla^L_Z \nu)(u) = \nabla_Z(\nu(u)) - \nu(\nabla^X_{T\pi_{X,P}(Z)} u). \tag{15}$$

Definition 13. An allowed variation of an allowed section, $j^1\rho \oplus \nu : \bar{U} \to J^1P \oplus (T^*X \otimes_P V)$, is a smooth map $j^1\rho(x, \varepsilon) \oplus \nu(x, \varepsilon) : \bar{U} \times I \to J^1P \oplus (T^*X \otimes_P V)$, where I is an open interval with $0 \in I$, such that:

1. For all $\varepsilon \in I$, $j^1\rho_\varepsilon(x) \oplus \nu_\varepsilon(x) : \bar{U} \to J^1P \oplus (T^*X \otimes_P V)$ is an allowed section and $j^1\rho_\varepsilon \oplus \nu_\varepsilon|_{\partial U} = j^1\rho \oplus \nu|_{\partial U}$;
2. For $\varepsilon = 0$, $j^1\rho_\varepsilon(x) \oplus \nu_\varepsilon(x) = j^1\rho(x) \oplus \nu(x)$;
3. The variation of ν is of the form

   $$\delta\nu \equiv \left.\frac{\mathrm{D}^{\nabla^L}\nu_\varepsilon}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0} = \tilde{\nabla}\mu - [\mu, \nu] + \rho^*(i_{\delta\rho}\omega),$$

   where ω is the 2-form in the additional structure of V,$\tilde{\nabla}$ is the ∇-derivative of $(V, \nabla) \to P \to X$ and $\mu \in \Gamma(\bar{U}, V)$ an arbitrary section with $\pi_{P,V} \circ \mu = \rho$ and $\mu|_{\partial U} = 0$.

Remark 14. Let $j^1\rho_\varepsilon \oplus \nu_\varepsilon$ be an allowed variation of $j^1\rho \oplus \nu$, the variation of $\rho_\varepsilon = \pi_{P,T^*X\otimes_P V} \circ \nu_\varepsilon$ is vertical in the sense that

$$\delta\rho(x) \equiv \left.\frac{\mathrm{d}\rho_\varepsilon(x)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \in V_{\rho(x)}P = \ker(T_{\rho(x)}\pi_{X,P}).$$

Consequently, $\delta j^1\rho(x) \in V_{\rho(x)}J^1P = \ker(T_{j^1\rho(x)}\pi_{X,J^1P})$.

Remark 15. It is very important to realize that since $\rho = \pi_{P,T^*X\otimes_P V}\nu$, an allowed section $(j^1\rho, \nu)$ is completely defined by $\nu \in \Gamma(\bar{U}, T^*X \otimes_P V)$. In a similar way, an allowed variation is completely determined by $\mathrm{d}\nu_\varepsilon/\mathrm{d}\varepsilon|_{\varepsilon=0}$. Indeed,

$$\begin{aligned}\left.\frac{\mathrm{d}\nu_\varepsilon}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} &= \mathrm{Hor}\left(\left.\frac{d\nu_\varepsilon}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\right) + \mathrm{Ver}\left(\left.\frac{d\nu_\varepsilon}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\right)\\ &= \mathrm{Hor}_\nu^{\nabla^L}\left(\left.\frac{d\rho_\varepsilon}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\right) + \left.\frac{\mathrm{D}^{\nabla^L}\nu_\varepsilon}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0} = \mathrm{Hor}_\nu^{\nabla^L}(\delta\rho) + \delta\nu.\end{aligned}$$

Thus,

$$\delta\rho = T\pi_{P,T^*X\otimes V}\left(\left.\frac{\mathrm{d}\nu_\varepsilon}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\right).$$

Observe that throughout this paper $\delta\nu$ denotes only the vertical component of the variation of ν.

Remark 16. Given $u_x \in T_xX$,

$$\delta\nu(u_x) \equiv \left.\frac{\mathrm{D}^{\nabla^L}\nu_\varepsilon}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}(u_x) = \left.\frac{\mathrm{D}^{\nabla}\nu_\varepsilon(u_x)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} - \nu(\nabla^X_{T\pi_{X,P}(\delta\rho)}u) = \left.\frac{\mathrm{D}^{\nabla}\nu_\varepsilon(u_x)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0},$$

where u is a local section of $TX \to X$ around x such that $u(x) = u_x$ and the last equivalence is a consequence of $\delta\rho$ being vertical. Thus, the variation $\delta\nu$ does not depend on the connection ∇^X on $TX \to X$.

We now find the variational equations defined by these set of allowed sections and variations. For any smooth function $L : J^1P \oplus (T^*X \otimes_P V) \to \mathbb{R}$, the fiber derivatives are defined as

$$\left\langle \frac{\delta L}{\delta j^1\rho}(j^1\rho \oplus \nu), \alpha \right\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} L((j^1\rho + \epsilon\alpha) \oplus \nu); \tag{16}$$

$$\left\langle \frac{\delta L}{\delta \nu}(j^1\rho \oplus \nu), \beta \right\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} L(j^1\rho \oplus (\nu + \epsilon\beta)), \tag{17}$$

where $\alpha \in T^*X \otimes VP$ and $\beta \in T^*X \otimes V$. Therefore,

$$\frac{\delta L}{\delta j^1\rho}(j^1\rho \oplus \nu) \in TX \otimes (VP)^*, \qquad \frac{\delta L}{\delta \nu}(j^1\rho \oplus \nu) \in TX \otimes V^*,$$

and if we compose them with a section $j^1\rho(x) \oplus \nu(x) \in \Gamma(\bar{U}, J^1P \oplus (T^*X \otimes V))$, we obtain $\delta L/\delta j^1\rho(x)$ in $\Gamma(\bar{U}, TX \otimes (VP)^*)$ and $\delta L/\delta\nu(x)$ in $\Gamma(\bar{U}, TX \otimes V^*)$.

Given ∇^P a linear connection on $TP \to P$ and ∇^X a linear connection in $TX \to X$, using the dual and the product connections, there is a lineal connection in $T^*X \otimes_P TP$. According to [17], this linear connection induces a general connection ∇^{J^1P} on $J^1P \to P$ that does not depend on the choice of ∇^X. Furthermore, we can choose ∇^P to be projectable to ∇^X and the connection in ∇^{J^1P} will be affine. In addition, provided ∇^{J^1P} and ∇^L defined using the same connection ∇^X, we define an affine connection, $\nabla^{J^1P} \oplus \nabla^L$, on $J^1P \oplus (T^*X \otimes_P V)$ such that

$$\nabla^{J^1P} \oplus \nabla^L(x, \nu) = (\nabla^{J^1P}s, \nabla^L\nu),$$

for any section (s,ν) of $J^1P\oplus(T^*X\oplus V)\to X$. Therefore, we define an horizontal derivative,

$$\left\langle \frac{\delta L}{\delta\rho}(j^1\rho\oplus\nu), Z_p\right\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} L(\zeta^h_{j^1\rho\oplus\nu}(\epsilon)); \tag{18}$$

where $Z_p\in T_pP$ and $\zeta(\epsilon)$ is a curve in P such that $\dot\zeta(0)=Z_p$ and $\zeta^h_{j^1\rho\oplus\nu}(\epsilon)$ is the horizontal lift of $\zeta(\epsilon)$ to $J^1P\oplus(T^*X\otimes V)$ through $j^1\rho\oplus\nu$ using the connection $\nabla^{J^1P}\oplus\nabla^L$. Thus,

$$\frac{\delta L}{\delta\rho}(j^1\rho\oplus\nu)\in T^*P.$$

We will also need a general notion of divergence of fields with values in a vector bundle. Let $E\to P$ be a vector bundle with affine connection ∇ and $P\to X$ a fiber bundle, we define for any $\chi\in\Gamma(X,TX\otimes E^*)$ the divergence $\operatorname{div}^\nabla\chi\in\Gamma(X,E^*)$ such that, for any $\eta\in\Gamma(X,E)$,

$$\operatorname{div}\langle\chi,\eta\rangle=\langle\operatorname{div}^\nabla\chi,\eta\rangle+\langle\chi,\tilde\nabla\eta\rangle,$$

where div is the usual divergence of a vector field in X (with respect to the volume form Vol). In our exposition, we will use the operators

$$\begin{aligned}\operatorname{div}^\nabla &: \Gamma(\bar U, TX\otimes V^*)\to\Gamma(\bar U,V^*);\\ \operatorname{div}^P &: \Gamma(\bar U, TX\otimes (VP)^*)\to\Gamma(\bar U,(VP)^*);\end{aligned}$$

induced by the connection ∇ in $V\to P$, and the restriction of the connection ∇^P to $VP\subset TP$.

Finally, we define the coadjoint operator in this context as

$$\begin{aligned}\mathrm{ad}^*:\Gamma(\bar U,T^*X\otimes V)\times\Gamma(\bar U,TX\otimes V^*) &\to \Gamma(\bar U,V^*)\\ (\nu_1,\nu_2) &\mapsto (\mu\mapsto \mathrm{ad}_{\nu_1}\nu_2(\mu)=\langle\nu_2,[\nu_1,\mu]\rangle)\end{aligned}$$

for all $\mu\in\Gamma(\bar U,V)$.

Theorem 17. *Let $J^1P\oplus(T^*X\otimes V)$ be a FT$\mathfrak{LP}$ bundle with a Lagrangian density $\mathcal{L}:J^1P\oplus(T^*X\otimes V)\to\bigwedge^n TX$ and a volume form $\mathrm{Vol}\in\bigwedge^n TX$, such that $\mathcal{L}=L\mathrm{Vol}$. Let ∇^P be a linear connection in $TP\to P$. Then, an allowed section $j^1\rho\oplus\nu\in\Gamma(\bar U,J^1P\oplus(T^*X\otimes V))$ is critical for the variational problem defined by $\mathcal{L}$ if and only if it satisfies the Lagrange–Poincaré equations:*

$$\mathrm{ad}^*\frac{\delta L}{\delta\nu}-\operatorname{div}^\nabla\frac{\delta L}{\delta\nu}=0, \tag{19}$$

$$\frac{\delta L}{\delta\rho}-\operatorname{div}^P\frac{\delta L}{\delta j^1\rho}-\left\langle\frac{\delta L}{\delta j^1\rho}, i_{T_\rho}T^P\right\rangle=\left\langle\frac{\delta L}{\delta\nu}, i_{T_\rho}\omega\right\rangle, \tag{20}$$

where T^P is the torsion tensor of connection ∇^P. Since this connection is arbitrary, we can always choose a connection without torsion and remove this term.

Proof. Using the derivatives defined by Eqs. (16), (17) and (18), we rewrite the derivative of the action as;

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\int_U \mathcal{L}(j^1\rho_\varepsilon\oplus\nu_\varepsilon) &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\int_U L(j^1\rho_\varepsilon\oplus\nu_\varepsilon)\mathrm{Vol}\\ &= \int_U\left\langle\frac{\delta L}{\delta\rho}(x),\left.\frac{\mathrm{d}\rho_\varepsilon(x)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\right\rangle\mathrm{Vol}\\ &\quad+\int_U\left\langle\frac{\delta L}{\delta j^1\rho}(x),\left.\frac{\mathrm{D}^{\nabla^{J^1P}}j^1\rho_\varepsilon(x)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}\right\rangle\mathrm{Vol}\\ &\quad+\int_U\left\langle\frac{\delta L}{\delta\nu}(x),\left.\frac{\mathrm{D}^L\nu_\varepsilon(x)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}\right\rangle\mathrm{Vol}.\end{aligned}$$

We know from the definition of allowed variation that

$$\left.\frac{\mathrm{D}^{\nabla^L}\nu_\varepsilon}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}=\tilde{\nabla}\mu-[\mu,\nu]+\rho^*(i_{\delta\rho}\omega). \tag{21}$$

On the other hand, since the variation of ρ is vertical and ∇^P is projectable,

$$\left.\frac{\mathrm{D}^{\nabla^{J^1P}}j^1\rho_\varepsilon(x)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}(u_x)=\left.\frac{\mathrm{D}^{\nabla^P}j^1\rho_\varepsilon(x)(u_x)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0},$$

for all $u_x\in T_xX$. Consider $\rho_\varepsilon(\gamma(t))$ where $\gamma(t)$ is a curve such that $\dot\gamma(0)=u_x$. From the formula

$$\frac{\mathrm{D}^{\nabla^P}}{\mathrm{D}\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\rho_\varepsilon(\gamma(t))-\frac{\mathrm{D}^{\nabla^P}}{\mathrm{D}t}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\rho_\varepsilon(\gamma(t))=T^P\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\rho_\varepsilon(\gamma(t)),\frac{\mathrm{d}}{\mathrm{d}t}\rho_\varepsilon(\gamma(t))\right),$$

we get that

$$\left.\frac{\mathrm{D}^{\nabla^{J^1P}}j^1\rho_\varepsilon(x)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}(u_x)=\tilde{\nabla}^P_{u_x}\delta\rho(x)+T^P(\delta\rho(x),T_x\rho(u_x)) \tag{22}$$

After substitution of equations (21) and (22), the derivative of the action is

$$\begin{aligned}&\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\int_U\mathcal{L}(j^1\rho_\varepsilon\oplus\nu_\varepsilon)\\ &=\int_U\left\langle\frac{\delta L}{\delta\rho}(x),\delta\rho(x)\right\rangle\mathrm{Vol}+\int_U\left\langle\frac{\delta L}{\delta j^1\rho}(x),\tilde{\nabla}^P\delta\rho(x)+T^P(\delta\rho(x),T\rho)\right\rangle\mathrm{Vol}\\ &\quad+\int_U\left\langle\frac{\delta L}{\delta\nu}(x),\tilde{\nabla}\mu-[\mu,\nu]+\omega(\delta\rho(x),T\rho)\right\rangle\mathrm{Vol}\\ &=\int_U\left\langle\frac{\delta L}{\delta\rho}(x)-\mathrm{div}^P\frac{\delta L}{\delta j^1\rho}(x)-\left\langle\frac{\delta L}{\delta j^1\rho},i_{T\rho}T^P\right\rangle-\left\langle\frac{\delta L}{\delta\mu},i_{T\rho}\omega\right\rangle,\delta\rho(x)\right\rangle\mathrm{Vol}\\ &\quad+\int_U\left\langle-\mathrm{div}^\nabla\frac{\delta L}{\delta\nu}(x)+\mathrm{ad}^*_\nu\frac{\delta L}{\delta\nu}(x),\mu\right\rangle\mathrm{Vol},\end{aligned}$$

where for the second identity it has been used that $\delta\rho|_{\partial U}=0$ and $\mu|_{\partial U}=0$. Finally, from the last expression it is clear that $j^1\rho(x)\oplus\nu(x)$ is critical if and only if the Lagrange–Poincaré equations are satisfied. □

6. Reduction by Stages

In this section, we shall see that the reduction procedure can be performed in the category FT$\mathfrak{LP}$. Let $J^1P \oplus (T^*X \otimes V)$ be an object in FT$\mathfrak{LP}$, G a Lie group acting freely and properly on $J^1P \oplus (T^*X \otimes V)$ and $\mathcal{A}$ a connection on $P \to \Sigma = P/G$. We recall from Sect. 2.2 that the connection ∇ on $V \to P$ induces an affine connection $\left[\nabla^{(\mathcal{A},H)}\right]_{G,Y}$ on $V/G \to \Sigma$, for $Y \in \mathfrak{X}(\Sigma)$. Hence, there is a $\left[\nabla^{(\mathcal{A},H)}\right]_G$-derivative on $V/G \to \Sigma \to X$ denoted by $\left[\tilde{\nabla}^{(\mathcal{A},H)}\right]_G$. On the other hand, the vertical component of the reduced connection, $\left[\nabla^{(\mathcal{A},V)}\right]_{G,\bar{\xi}}$, is not a connection since we derive with respect to $\bar{\xi} \in \Gamma(\mathrm{Ad}P)$. However, given a section $[w]$ of $V/G \to X$, we define for all $x \in X$

$$\left[\nabla^{(\mathcal{A},V)}\right]_{G,\bar{\xi}}[w](x) = [\xi_v^V]_G(x),$$

where $v \in V$ such that $\pi_{V/G,V}(v) = [w](x)$, and $\bar{\xi} = [p,\xi]_G$ with $p = \pi_{V,P}(v)$. This is well defined since $\bar{\xi} = [gp, \mathrm{Ad}_g\xi]_G$ and $g\xi_v^V = (\mathrm{Ad}_g\xi)_{gv}^V$.

Theorem 18. *Let $J^1P \oplus (T^*X \otimes V)$ be an object in FT$\mathfrak{LP}$, G a Lie group acting freely and properly on $J^1P \oplus (T^*X \otimes V)$,* Vol *a volume form on X and $L : J^1P \oplus (T^*X \otimes V) \to \mathbb{R}$ a G-invariant Lagrangian.*

Consider $\mathcal{A}$ a connection on $P \to \Sigma = P/G$ and

$$l : J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G))) \to \mathbb{R}$$

the reduced Lagrangian induced in the quotient via the identification (9). Given a smooth local section $j^1\rho \oplus \nu \in \Gamma(\bar{U}, J^1P \oplus (T^*X \otimes V))$, where $\nu = ((\rho,\alpha), v)$, we define the reduced local section

$$j^1\sigma \oplus \bar{\rho} \oplus [\nu]_G = \beta_{\mathcal{A}} \circ \pi_G(j^1\rho \oplus \nu) = (T\pi_{P,\Sigma} \circ j^1\rho) \oplus [\rho, \rho^*\mathcal{A}]_G \oplus ((\sigma,\alpha), v),$$

where $\sigma \in \Gamma(\bar{U}, \Sigma)$ is $\pi_{\Sigma,P} \circ \rho$. Then, the following statements are equivalent

(i) Section $j^1\rho \oplus \nu \in \Gamma(\bar{U}, J^1P \oplus (T^*X \otimes V))$ is a critical section for the variational problem defined by L in $J^1P \oplus (T^*X \otimes V)$.
(ii) Section $j^1\rho \oplus \nu \in \Gamma(\bar{U}, J^1P \oplus (T^*X \otimes V))$ satisfies the Lagrange–Poincaré equations given by L in $J^1P \oplus (T^*X \otimes V)$.
(iii) Section $j^1\sigma \oplus \bar{\rho} \oplus [\nu]_G \in \Gamma(\bar{U}, J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G))))$ is a critical section for the variational problem defined by l in $J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G)))$.
(iv) Section $j^1\sigma \oplus \bar{\rho} \oplus [\nu]_G \in \Gamma(\bar{U}, J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G))))$ satisfies the Lagrange–Poincaré equations given by l in $J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G)))$.

Proof. From Theorem 17, statements (i) and (ii) are equivalent. In an analogue way, statements (iii) and (iv) are equivalent. We will prove the result by checking that statements (i) and (iii) are equivalent.

Let $j^1\rho_\varepsilon \oplus \nu_\varepsilon$ be an allowed variation of $j^1\rho \oplus \nu$ and

$$j^1\sigma_\varepsilon \oplus \bar{\rho}_\varepsilon \oplus [\nu]_{G,\varepsilon} = \beta_{\mathcal{A}} \circ \pi_G(j^1\rho_\varepsilon \oplus \nu_\varepsilon),$$

the projection of the allowed variation in $J^1P \oplus (T^*X \otimes V)$. Since

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\int_U L(j^1\rho_\varepsilon \oplus \nu_\varepsilon)\mathrm{Vol} &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\int_U l(j^1\sigma_\varepsilon \oplus \bar{\rho}_\varepsilon \oplus [\nu]_{G,\varepsilon})\mathrm{Vol}\\ &= \int_U \left\langle \frac{\delta l}{\delta\sigma}(x), \left.\frac{\mathrm{d}\sigma_\varepsilon(x)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\right\rangle \mathrm{Vol}\\ &\quad + \int_U \left\langle \frac{\delta l}{\delta j^1\sigma}(x), \left.\frac{\mathrm{D}^{\nabla^{J^1\Sigma}} j^1\sigma_\varepsilon(x)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}\right\rangle \mathrm{Vol}\\ &\quad + \int_U \left\langle \frac{\delta l}{\delta(\bar{\rho}\oplus[\nu]_G)}(x), \left.\frac{\mathrm{D}^L \bar{\rho}_\varepsilon \oplus [\nu]_{G,\varepsilon}(x)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}\right\rangle \mathrm{Vol},\end{aligned}$$

the variational problem defined by L on $J^1P \oplus (T^*X \otimes V)$ is equivalent to consider variations of $j^1\sigma \oplus \bar{\rho} \oplus [\nu]_G$ obtained by projecting allowed variations of $j^1\rho \oplus \nu$. Then, for all $u_x \in T_xX$,

$$\begin{aligned}\left.\frac{\mathrm{D}^L \bar{\rho}_\varepsilon \oplus [\nu]_{G,\varepsilon}}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}(u_x) &= \left.\frac{\mathrm{D}^{\tilde{\mathfrak{g}}} \bar{\rho}_\varepsilon \oplus [\nu]_{G,\varepsilon}(u_x)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}\\ &= \nabla^{\tilde{\mathfrak{g}}}_{\delta\sigma}\bar{\rho}_\varepsilon(u_x) \oplus \left([\nabla^{(\mathcal{A},H)}]_{G,\delta\sigma}[v]_G(u_x) - [\omega]_G(\delta\sigma, \bar{\rho}_\varepsilon(u_x))\right)\\ &= \nabla^{\tilde{\mathfrak{g}}}_{\delta\sigma}\bar{\rho}_\varepsilon(u_x) \oplus \Big([\nabla^{(\mathcal{A})}]_{G,\delta\rho}[\nu]_G(u_x)\\ &\quad -[\nabla^{(\mathcal{A},V)}]_{G,\bar{\eta}}[v]_G(u_x) - [\omega]_G(\delta\sigma, \bar{\rho}_\varepsilon(u_x))\Big),\end{aligned}$$

where $\bar{\eta}$ is such that $\delta\rho = \delta\sigma \oplus \bar{\eta}$. From Lagrange–Poincaré reduction in jet bundles (see [12, Corollary 3.2]), we know that

$$\nabla^{\tilde{\mathfrak{g}}}_{\delta\sigma}\bar{\rho}_\varepsilon(u_x) = \tilde{\nabla}_{u_x}\bar{\eta} - [\bar{\eta}, \bar{\rho}(u_x)] + \tilde{B}(\delta\sigma, T\sigma(u_x)).$$

On the other hand, as $\delta\nu$ is an allowed variation

$$[\nabla^{(\mathcal{A})}]_{G,\delta\rho}[\nu]_G(u_x) = [\delta\nu(u_x)]_G = [\tilde{\nabla}_{u_x}\mu - [\mu, \nu(u_x)] + \omega(\delta\rho, T\rho(u_x))]_G.$$

We rewrite this class in an alternative way. First,

$$\begin{aligned}[\tilde{\nabla}_{u_x}\mu]_G = [\nabla_{T\rho(u_x)}\mu]_G &= [\nabla_{T\sigma(u_x)\oplus\bar{\rho}(u_x)}\mu]_G\\ &= [\nabla^{(\mathcal{A},H)}]_{G,T\sigma(u_x)}[\mu]_G + [\nabla^{(\mathcal{A},V)}]_{G,\bar{\rho}(u_x)}[\mu]_G\\ &= [\tilde{\nabla}^{(\mathcal{A},H)}]_{G,u_x}[\mu]_G + [\tilde{\nabla}^{(\mathcal{A},V)}]_{G,u_x}[\mu]_G,\end{aligned}$$

where we have defined $[\tilde{\nabla}^{(\mathcal{A},V)}]_{G,u_x}[\mu]_G = [\nabla^{(\mathcal{A},V)}]_{G,\bar{\rho}(u_x)}[\mu]_G$. We also have that $[[\mu, \nu(u_x)]]_G = [[\mu]_G, [\nu(u_x)]_G]$, and

$$\begin{aligned}[\omega(\delta\rho, T\rho(u_x))]_G &= [\omega(\delta\sigma \oplus \bar{\eta}, T\sigma(u_x) \oplus \bar{\rho}(u_x))]_G\\ &= [\omega]_G(\delta\sigma, T\sigma(u_x)) + [\omega]_G(\bar{\eta}, T\sigma(u_x)) + [\omega]_G(\delta\sigma, \bar{\rho}(u_x))\\ &\quad + [\omega]_G(\bar{\eta}, \bar{\rho}(u_x)).\end{aligned}$$

In conclusion, for projected allowed variations

$$\begin{aligned}\left.\frac{\mathrm{D}^L\bar\rho_\varepsilon\oplus[\nu]_{G,\varepsilon}}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}(u_x) &= \left(\tilde\nabla_{u_x}\bar\eta-[\bar\eta,\bar\rho(u_x)]+\tilde B(\delta\sigma,T\sigma(u_x))\right)\\ &\quad\oplus[\tilde\nabla^{(\mathcal{A},H)}]_{G,u_x}[\mu]_G+[\tilde\nabla^{(\mathcal{A},V)}]_{G,u_x}[\mu]_G\\ &\quad-[[\mu]_G,[\nu(u_x)]_G]-[\nabla^{(\mathcal{A},V)}]_{G,\bar\eta}[v]_G(u_x)\\ &\quad+[\omega]_G(\delta\sigma,T\sigma(u_x))+[\omega]_G(\bar\eta,T\sigma(u_x))\\ &\quad+[\omega]_G(\bar\eta,\bar\rho(u_x)),\end{aligned}$$

where $\bar\eta\in\Gamma(\bar U,\mathrm{Ad}P)$ such that $\pi_{\Sigma,\mathrm{Ad}P}(\bar\eta)=\sigma$, $\bar\eta|_{\partial U}=0$ and $[\mu]_G\in\Gamma(\bar U,V/G)$ such that $\pi_{\Sigma,V/G}([\mu]_G)=\sigma$, $[\mu]_G|_{\partial U}=0$. We now see that these variations coincide with the allowed variations in $J^1\Sigma\oplus(T^*X\otimes(\mathrm{Ad}P\oplus(V/G)))$. An allowed variation of section $j^1\sigma\oplus\bar\rho\oplus[\nu]_G$ satisfies

$$\frac{D^L\bar\rho_\varepsilon\oplus[\nu]_{G,\varepsilon}}{\mathrm{D}\varepsilon|_{\varepsilon=0}}(u_x)=\tilde\nabla^{\tilde{\mathfrak g}}_{u_x}(\bar\eta\oplus[\mu]_G)-[\bar\eta\oplus[\mu]_G,\bar\rho\oplus[\nu]_G(u_x)]^{\tilde{\mathfrak g}}_G+\omega^{\tilde{\mathfrak g}}(\delta\sigma,T\sigma u_x)$$

where $\bar\eta\oplus[\mu]_G\in\Gamma(\bar U,\mathrm{Ad}P\oplus(V/G))$ such that $\pi_{\Sigma,\mathrm{Ad}P\oplus(V/G)}(\bar\eta\oplus[\mu]_G)=\sigma$, and $\bar\eta\oplus[\mu]_G|_{\partial U}=0$. The additional structure in $\mathrm{Ad}P\oplus(V/G)$ is the one detailed in Proposition 3. Hence,

$$\begin{aligned}\tilde\nabla^{\tilde{\mathfrak g}}_{u_x}(\bar\eta\oplus[\mu]_G)&=\nabla^{\tilde{\mathfrak g}}_{T\sigma(u_x)}(\bar\eta\oplus[\mu]_G)\\ &=\nabla_{T\sigma(u_x)}\bar\eta\oplus\left([\nabla^{(\mathcal{A},H)}]_{G,T\sigma(u_x)}[\mu]_G-[\omega]_G(T\sigma(u_x),\bar\eta)\right)\\ &=\tilde\nabla_{u_x}\bar\eta\oplus\left([\tilde\nabla^{(\mathcal{A},H)}]_{G,u_x}[\mu]_G+[\omega]_G(\bar\eta,T\sigma(u_x))\right).\end{aligned}$$

In addition,

$$\begin{aligned}[\bar\eta\oplus[\mu]_G,\bar\rho\oplus[\nu]_G(u_x)]^{\tilde{\mathfrak g}}_G&=[\bar\eta,\bar\rho(u_x)]\oplus[\nabla^{(\mathcal{A},V)}]_{G,\bar\eta}[\nu]_G(u_x)\\ &\quad-[\nabla^{(\mathcal{A},V)}]_{G,\bar\rho(u_x)}[\mu]_G-[\omega]_G(\bar\eta,\bar\rho(u_x))+[[\mu]_G,[\nu(u_x)]_G],\end{aligned}$$

and

$$\omega^{\tilde{\mathfrak g}}(\delta\sigma,T\sigma(u_x))=\tilde B(\delta\sigma,T\sigma(u_x))\oplus\omega(\delta\sigma,T\sigma(u_x)).$$

These last three expressions prove that the allowed variations in $J^1\Sigma\oplus(T^*X\otimes(\mathrm{Ad}P\oplus(V/G)))$ are the same as the projection of the allowed variations of the original space and, consequently, (i) and (iii) are equivalent statements. □

Remark 19. Reduction of variational problems set in objects of FT$\mathfrak{LP}$ is a process that can be iterated in a similar way as stated in Remark 5 for variational problems in $\mathfrak{LP}$ bundles. Let N be a normal subgroup of G, and $K=G/N$ the quotient group. We can reduce L by N and afterward by K. Let $\mathcal{A}_N$ be a principal connection on $P\to P/N$ and $\mathcal{A}_{G/N}$ a principal connection on $P/N\to(P/N)/K$. These connections are said to be compatible with respect to $\mathcal{A}$ if for all $u\in TP$,

$$\mathcal{A}(u)=0\Leftrightarrow\mathcal{A}_N(u)=0\text{ and }\mathcal{A}_{G/N}(T\pi_{P/N,P}(u))=0.$$

In this case, there exists a $\mathfrak{LP}$-isomorphism from $T(P/G)\oplus\mathrm{Ad}P\oplus(V/G)$ to

$$T((P/N)/K)\oplus\tilde{\mathfrak k}\oplus(\tilde{\mathfrak n}\oplus(V/N))/(G/N),$$

where $\mathfrak{n}$ is the Lie algebra of N, $\mathfrak{k}$ is the Lie algebra of K and $\tilde{\mathfrak{n}}, \tilde{\mathfrak{k}}$ their respective adjoint bundles. Hence, it is equivalent to perform reduction directly than by stages. This is exemplified in §9 below. For more details, see [9, §6.3] and [3, §3.4].

7. Reconstruction in FT$\mathfrak{LP}$

Given a critical section $j^1\sigma \oplus \bar{\rho} \oplus [\nu]_G$ in $\Gamma(\bar{U}, J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G))))$ for l, we investigate the existence of a critical section $j^1\rho \oplus \nu$ in $\Gamma(\bar{U}, J^1P \oplus (T^*X \otimes V))$ for the unreduced Lagrangian L.

Let $\sigma(x) = \pi_{\Sigma,V/G}([\nu]_G(x)) = \pi_{\Sigma,\mathrm{Ad}P}(\bar{\rho}(x))$. This section in $\Gamma(\bar{U}, \Sigma)$ defines the G-principal pullback bundle $\sigma^*P \to X$

$$\sigma^*P = \{(x,p) \in X \times P | \pi_{\Sigma,P}(p) = \sigma(x)\}$$

of the G-principal bundle $P \to \Sigma$. In addition, σ^*P can be identified with $P^\sigma = \{p \in P | \pi_{\Sigma,P}(p) \in \sigma(X)\}$ by $p \in P^\sigma \mapsto (\pi_{X,P}(p), p) \in \sigma^*P$. As $\sigma(x) = \pi_{\Sigma,\mathrm{Ad}P}(\bar{\rho}(x))$, the section $\bar{\rho}(x)$ can be interpreted as a section of $\Gamma(\bar{U}, \mathrm{Ad}P^\sigma)$ and there is an equivariant horizontal 1-form $\omega^{\bar{\rho}} \in \Omega^1(P^\sigma, \mathfrak{g})$ such that for all $x \in X$ and $u_x \in T_xX$,

$$\bar{\rho}(u_x) = [p, \omega^{\bar{\rho}}(u_p)]_G,$$

where $p \in P^\sigma$ and $u_p \in T_pP$ such that $T\pi_{X,P}(u_p) = u_x$. The connection $\mathcal{A}$ on $P \to \Sigma$ induces a connection $\mathcal{A}^\sigma$ on $P^\sigma \to X$, and recalling that the space of connections of a principal bundle is an affine space modeled over the space of equivariant 1-forms taking values in the adjoint bundle, we define a connection $\mathcal{A}^{\bar{\rho}}$ as:

$$\mathcal{A}^{\bar{\rho}} = \mathcal{A}^\sigma - \omega^{\bar{\rho}}.$$

Theorem 20. *Let $\mathcal{A}$ be a principal connection on the principal bundle $P \to \Sigma$, and let $L : J^1P \oplus (T^*X \otimes V) \to \mathbb{R}$ be a G-invariant Lagrangian defined in a FT$\mathfrak{LP}$ bundle. Finally, let $l : J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G))) \to \mathbb{R}$ be the reduced Lagrangian.*

Then, if $j^1\rho \oplus \nu$ in $\Gamma(\bar{U}, J^1P \oplus (T^*X \otimes V))$ satisfies the Lagrange–Poincaré equations given by L in $J^1P \oplus (T^*X \otimes V)$, the reduced section $j^1\sigma \oplus \bar{\rho} \oplus [\nu]_G \in \Gamma(\bar{U}, J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G))))$ satisfies the Lagrange–Poincaré equations given by l in $J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G)))$ and connection $\mathcal{A}^{\bar{\rho}}$ on $P^\sigma \to \Sigma$ is flat.

Conversely, given a solution $j^1\sigma \oplus \bar{\rho} \oplus [\nu]_G \in \Gamma(\bar{U}, J^1\Sigma \oplus (T^*X \otimes (\mathrm{Ad}P \oplus (V/G))))$ of the Lagrange–Poincaré equations given by l such that $\mathcal{A}^{\bar{\rho}}$ is flat and has trivial holonomy over an open set containing $\bar{U}$, there is a family $\Phi_g(j^1\rho \oplus \nu)$, $g \in G$, of solutions of the Lagrange–Poincaré equations given by L projecting to $j^1\sigma \oplus \bar{\rho} \oplus [\nu]_G$. If the connection $\mathcal{A}^{\bar{\rho}}$ is flat, one can always restrict it to an open simply connected set contained in U so that its holonomy on U is automatically zero.

Proof. Suppose that $j^1\sigma \oplus \bar{\rho} \oplus [\nu]_G$ is the projection of $j^1\rho \oplus \nu$, in particular, $\bar{\rho} = [\rho, \rho^*\mathcal{A}]_G$, where ρ is a section of $P \to X$. Observe that for all $p = \rho(x) \in P^\sigma$, $T_{\rho(x)}P^\sigma = T_x\rho(T_xX) \oplus \ker T_{\rho(x)}\pi_{\Sigma,P}$ and any $v_p \in T_pP^\sigma$ can be written as $v_p = T_x\rho(v_x) + \xi_p^P$, where $v_x \in T_xX$ and $\xi \in \mathfrak{g}$. Then,

$$\begin{aligned}\mathcal{A}^{\bar{\rho}}(v_p) &= \mathcal{A}^\sigma(v_p) - \omega^{\bar{\rho}}(v_p) = \mathcal{A}(v_p) - \mathcal{A}(T_x\rho(T_p\pi_{X,P}(v_p))) \\ &= \mathcal{A}(T_x\rho(v_x)) + \mathcal{A}(\xi_p^P) - \mathcal{A}(T_x\rho(v_x)) = \xi.\end{aligned}$$

Consequently, the horizontal subbundle defined by $\mathcal{A}^{\bar{\rho}}$ is given by

$$H^{\mathcal{A}^{\bar{\rho}}}_{\rho(x)} = T_x\rho(T_xX),$$

the horizontal distribution is integrable, the integral leaves are given by

$$\{\Phi_g(\rho(x)) | x \in X, g \in G\} = \Phi_g(\mathrm{Im}\rho),$$

and $\mathcal{A}^{\bar{\rho}}$ is a flat connection on $P^\sigma \to X$.

Conversely, given $j^1\sigma \oplus \bar{\rho} \oplus [\nu]_G$ and $\sigma(x) = \pi_{\Sigma,V/G}([\nu]_G(x)) = \pi_{\Sigma,\mathrm{Ad}P}(\bar{\rho}(x))$, suppose that $\mathcal{A}^{\bar{\rho}}$ is flat and has trivial holonomy over an open set containing $\bar{U}$. The horizontal distribution of $\mathcal{A}^{\bar{\rho}}$ is integrable and the leaves cover the base. Since the holonomy is trivial each fiber intersects the leaf exactly once, that is, they are sections of $P^\sigma \to X$. Thus, there is a family $\Phi_g(\rho(x))$ of sections of $P \to X$ that projects to σ via $\pi_{X,\Sigma}$ and such that

$$[\rho, \rho^*\mathcal{A}]_G = [\rho, \rho^*\mathcal{A}^\rho + \omega^{\bar{\rho}}]_G = [\rho, \omega^{\bar{\rho}}]_G = \bar{\rho}.$$

Furthermore, there is a unique section $\nu(x)$ of $T^*X \oplus V \to X$ such that

$$\pi_{T^*X\oplus(V/G),T^*X\oplus V}(\nu(x)) = [\nu](x), \qquad \pi_{P,T^*X\oplus V}(\nu(x)) = \rho(x).$$

In addition, $\Phi_g(\nu(x))$ is the unique section of $T^*X \oplus V \to X$ such that

$$\pi_{T^*X\oplus(V/G),T^*X\oplus V}\Phi_g(\nu(x)) = [\nu](x), \qquad \pi_{P,T^*X\oplus V}\Phi_g(\nu(x)) = \Phi_g(\rho(x)).$$

Thus, the family of sections $\Phi_g(j^1\rho \oplus \nu) = j^1\Phi_g(\rho) \oplus \Phi_g(\nu)$, $g \in G$, projects to $j^1\sigma \oplus \bar{\rho} \oplus [\nu]_G$ and, by equivalence (iv)⇒ (ii) in Theorem 18, they are solutions of the Lagrange–Poincaré equations given by L. □

The curvature of connection $\mathcal{A}^{\bar{\rho}}$ can be rewritten in terms of the curvature B of $\mathcal{A}$. Then, the flatness of $\mathcal{A}^{\bar{\rho}}$ gives the following *reconstruction condition*

$$B - \mathrm{d}^{\nabla^{\mathcal{A}}}\omega^{\bar{\rho}} - \omega^{\bar{\rho}} \wedge \omega^{\bar{\rho}} = 0, \tag{23}$$

where $\mathrm{d}^{\nabla^{\mathcal{A}}}$ is the exterior derivative of $\mathfrak{g}$-valued forms induced by the covariant derivative on $\mathrm{Ad}P$, $\nabla^{\mathcal{A}}$, and the Cartan formula.

8. The Noether Drift Law in the FT$\mathfrak{LP}$ and Its Reduction

In this section, we define a Noether current for symmetries in FT$\mathfrak{LP}$ bundles and prove that is not a constant of motion. Instead there is a drift of this current that reduces to the new vertical equation appearing in each step of the reduction.

Definition 21. Let $L : J^1P \oplus (T^*X \otimes V) \to \mathbb{R}$ be a Lagrangian and G a Lie group acting freely and properly on $J^1P \oplus (T^*X \otimes V)$ by isomorphisms in the category $\mathrm{FT}\mathfrak{LP}$. We define the *Noether current* as the function

$$J : J^1P \oplus (T^*X \otimes_P V) \to TX \otimes_P \mathfrak{g}^*$$

such that for all $j^1\rho \oplus \nu \in J^1P \oplus (T^*X \otimes_P V)$

$$\left\langle J(j^1\rho \oplus \nu), (\rho,\alpha) \otimes (\rho,\eta) \right\rangle = \frac{\delta L}{\delta j^1\rho}(j^1\rho \oplus \nu)((\rho,\alpha) \otimes \eta^P_\rho),$$

where $(\rho,\alpha) \otimes (\rho,\eta) \in T^*X \otimes_P \mathfrak{g}$ and $\langle \cdot,\cdot \rangle$ is the natural duality pairing.

Proposition 22. *Suppose that L is G-invariant and let $j^1\rho \oplus \nu(x)$ be a section satisfying the Lagrange–Poincaré equations. Then, the Noether current satisfies*

$$\operatorname{div}\left(\left\langle J(j^1\rho \oplus \nu(x)), (\rho,\eta)\right\rangle\right) = -\left\langle \frac{\delta L}{\delta \nu}, \omega(T_x\rho(\bullet), \eta^P_p) + \eta^V_{\nu(\bullet)} \right\rangle, \qquad (24)$$

where $\omega(T_x\rho(\bullet), \eta^P_p) + \eta^V_{\nu(\bullet)} \in T^*X \otimes V$ acts by replacing the slot $\bullet$ by an arbitrary element in TX. This is called the Noether drift law *of L.*

Proof. As the Lagrangian is G-invariant, for all $\eta \in \mathfrak{g}$,

$$\begin{aligned} 0 =& \mathrm{d}L\left(\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \exp(\varepsilon\eta)(j^1\rho \oplus \nu)\right) = \left\langle \frac{\delta L}{\delta \rho}, \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \exp(\varepsilon\nu)\rho \right\rangle \\ &+ \left\langle \frac{\delta L}{\delta j^1\rho}, \left.\frac{\mathrm{D}^{\nabla^{J^1P}}}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0} \exp(\varepsilon\eta) j^1\rho \right\rangle + \left\langle \frac{\delta L}{\delta \nu}, \left.\frac{\mathrm{D}^L}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0} \exp(\varepsilon\eta)\nu \right\rangle. \end{aligned}$$

Since $\exp(\varepsilon\eta)j^1\rho = j^1(\exp(\varepsilon\eta)\rho)$, it is the lifted variation of the vertical variation $\exp(\varepsilon\eta)\rho$ of ρ. Then, for all $u_x \in TX$;

$$\begin{aligned} \left.\frac{\mathrm{D}^{\nabla^{J^1P}} \exp(\varepsilon\eta) j^1\rho}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}(u_x) &= \left.\frac{\mathrm{D}^{\nabla^P} \exp(\varepsilon\eta) j^1\rho(u_x)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0} \\ &= \tilde{\nabla}^P_{u_x}\eta^P_\rho + T^P(\eta^P_\rho, T_x\rho(u_x)) \end{aligned}$$

Furthermore, $\exp(\varepsilon\eta)\nu$ is a vertical variation and

$$\left.\frac{\mathrm{D}^L \exp(\varepsilon\eta)\nu}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0}(u_x) = \left.\frac{\mathrm{D}^\nabla \exp(\varepsilon\eta)\nu(u_x)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0} = \eta^V_{\nu(u_x)}.$$

Then, the G-invariance of L can be written as,

$$0 = \left\langle \frac{\delta L}{\delta \rho}, \eta^P_\rho \right\rangle + \left\langle \frac{\delta L}{\delta j^1\rho}, \tilde{\nabla}^P_{u_x}\eta^P_\rho + T^P(\eta^P_\rho, T_x\rho(u_x)) \right\rangle + \left\langle \frac{\delta L}{\delta \nu}, \eta^V_{\nu(u_x)} \right\rangle. \quad (25)$$

Finally,

$$
\begin{aligned}
\operatorname{div}\left(\left\langle J(j^1\rho\oplus\nu(x)),(\rho,\eta)\right\rangle\right) &= \operatorname{div}\left\langle \frac{\delta L}{\delta j^1\rho}(j^1\rho\oplus\nu(x)),\eta^P_{\rho(x)}\right\rangle\\
&= \left\langle \operatorname{div}^P\frac{\delta L}{\delta j^1\rho},\eta^P_\rho\right\rangle + \left\langle \frac{\delta L}{\delta j^1\rho},\tilde{\nabla}^P\eta^P_\rho\right\rangle\\
&= \left\langle \operatorname{div}^P\frac{\delta L}{\delta j^1\rho},\eta^P_\rho\right\rangle - \left\langle \frac{\delta L}{\delta \rho},\eta^P_\rho\right\rangle\\
&\quad - \left\langle \frac{\delta L}{\delta j^1\rho},T^P(\eta^P_\rho,T_x\rho(\bullet))\right\rangle - \left\langle \frac{\delta L}{\delta \nu},\eta^V_{\nu(\bullet)}\right\rangle\\
&= -\left\langle \frac{\delta L}{\delta \nu},\omega(T\rho(\bullet),\eta^P_p)+\eta^V_{\nu(\bullet)}\right\rangle,
\end{aligned}
$$

where we have used relation (25) and Lagrange–Poincaré equations. □

The Noether current is G-equivariant so that it defines a bundle map in the quotient by G

$$j : J^1\Sigma\oplus(T^*X\otimes_\Sigma(\mathrm{Ad}P\oplus(V/G)))\to TX\otimes_P\mathrm{Ad}^*P$$

such that for all $j^1\sigma\oplus\bar{\rho}\oplus[\nu]_G\in J^1\Sigma\oplus(T^*X\otimes_\Sigma(\mathrm{Ad}P\oplus(V/G)))$, and all $(\sigma,\alpha)\otimes\bar{\eta}\in T^*X\otimes_\Sigma\mathrm{Ad}P$,

$$
\begin{aligned}
&j(j^1\sigma\oplus\bar{\rho}\oplus[\nu]_G)((\sigma,\alpha)\otimes\bar{\eta})\\
&= J(j^1\rho\oplus\nu)((\rho,\alpha)\otimes(\rho,\eta)) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(j^1\rho+\epsilon((\rho,\alpha)\otimes\eta^P_\rho)\oplus\nu)\\
&= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} l(j^1\sigma\oplus(\bar{\rho}+\epsilon((\sigma,\alpha)\otimes\bar{\eta})\oplus[\nu]_G) = \frac{\delta l}{\delta\bar{\rho}}(j^1\sigma\oplus\bar{\rho}\oplus[\nu]_G)((\sigma,\alpha)\otimes\bar{\eta}).
\end{aligned}
$$

Consequently, the reduced Noether current j coincides with $\frac{\delta l}{\delta\bar{\rho}}$.

Proposition 23. *The drift of the Noether current along critical sections $j^1\sigma\oplus\bar{\rho}\oplus[\nu]_G$ given by equation (24) projects to the equation*

$$\left\langle \operatorname{div}^{\mathcal{A}}\frac{\delta l}{\delta\bar{\rho}},\bar{\eta}\right\rangle = \left\langle \operatorname{ad}^*_{\bar{\rho}}\frac{\delta l}{\delta\bar{\rho}},\bar{\eta}\right\rangle - \left\langle \frac{\delta l}{\delta[\nu]_G},[\nabla^{(\mathcal{A},V)}]_{G,\bar{\eta}}[\nu]_G+[\omega]_G(T\sigma\oplus\bar{\rho},\bar{\eta})\right\rangle, \tag{26}$$

where $\operatorname{div}^{\mathcal{A}}$ is the divergence of Ad^*P-valued vector fields induced by connection $\nabla^{\mathcal{A}}$ in $\mathrm{Ad}P$ and $\bar{\eta}(x)=[\rho(x),\eta]_G$

Proof. We rewrite the left-hand side of equation (24):

$$
\begin{aligned}
\operatorname{div}\left\langle J(j^1\rho\oplus\nu),(\rho,\eta)\right\rangle &= \operatorname{div}\left\langle j(j^1\sigma\oplus\bar{\rho}\oplus[\nu]_G),\bar{\eta}\right\rangle\\
&= \left\langle \operatorname{div}^{\mathcal{A}} j(j^1\sigma\oplus\bar{\rho}\oplus[\nu]_G),\bar{\eta}\right\rangle + \left\langle j(j^1\sigma\oplus\bar{\rho}\oplus[\nu]_G),\tilde{\nabla}^{\mathcal{A}}\bar{\eta}\right\rangle\\
&= \left\langle \operatorname{div}^{\mathcal{A}}\frac{\delta l}{\delta\bar{\rho}}(j^1\sigma\oplus\bar{\rho}\oplus[\nu]_G),\bar{\eta}\right\rangle - \left\langle \frac{\delta l}{\delta\bar{\rho}}(j^1\sigma\oplus\bar{\rho}\oplus[\nu]_G),[\bar{\rho},\bar{\eta}]\right\rangle\\
&= \left\langle \operatorname{div}^{\mathcal{A}}\frac{\delta l}{\delta\bar{\rho}}(j^1\sigma\oplus\bar{\rho}\oplus[\nu]_G) - \operatorname{ad}^*_{\bar{\rho}}\frac{\delta l}{\delta\bar{\rho}}(j^1\sigma\oplus\bar{\rho}\oplus[\nu]_G),\bar{\eta}\right\rangle.
\end{aligned}
$$

On the other hand, the right-hand side of equation (24) projects to

$$-\left\langle \frac{\delta l}{\delta[\nu]_G}, [\omega]_G(T\sigma(\bullet)\oplus\bar{\rho}(\bullet),\bar{\eta}) + [\eta^V_{\nu(\bullet)}]_G \right\rangle,$$

and we conclude by observing that $[\eta^V_{\nu(\bullet)}]_G = [\nabla^{(\mathcal{A},V)}]_{G,\bar{\eta}}[\nu]_G(\bullet)$. □

The vertical Lagrange–Poincaré equations,

$$\operatorname{div}^{\tilde{\mathfrak{g}}}\left(\frac{\delta l}{\delta\bar{\rho}}\oplus\frac{\delta l}{\delta[\nu]_G}\right) - \operatorname{ad}^*_{\bar{\rho}\oplus[\nu]_G}\left(\frac{\delta l}{\delta\bar{\rho}}\oplus\frac{\delta l}{\delta[\nu]_G}\right) = 0,$$

associated with the reduced bundle $J^1\Sigma\oplus(T^*X\oplus(\mathrm{Ad}P\oplus(V/G)))$ are an equation in $\mathrm{Ad}^*P\oplus(V/G)^*$ acting on vectors $\bar{\eta}\oplus[u]_G\in\mathrm{Ad}P\oplus(V/G)$. We can decompose these equations restricting to each of the factors on $\mathrm{Ad}P\oplus(V/G)$. First,

$$\begin{aligned}
&\left\langle \operatorname{div}^{\tilde{\mathfrak{g}}}\left(\frac{\delta l}{\delta\bar{\rho}}\oplus\frac{\delta l}{\delta[\nu]_G}\right), \bar{\eta}\oplus[u]_G\right\rangle \\
&\quad= \operatorname{div}\left\langle\left(\frac{\delta l}{\delta\bar{\rho}}\oplus\frac{\delta l}{\delta[\nu]_G}\right), \bar{\eta}\oplus[u]_G\right\rangle - \left\langle\left(\frac{\delta l}{\delta\bar{\rho}}\oplus\frac{\delta l}{\delta[\nu]_G}\right), \tilde{\nabla}^{\tilde{\mathfrak{g}}}(\bar{\eta}\oplus[u]_G)\right\rangle \\
&\quad= \operatorname{div}\left\langle\left(\frac{\delta l}{\delta\bar{\rho}}\oplus\frac{\delta l}{\delta[\nu]_G}\right), \bar{\eta}\oplus[u]_G\right\rangle - \left\langle\frac{\delta l}{\delta\bar{\rho}}, \tilde{\nabla}^{\mathcal{A}}\bar{\eta}\right\rangle - \left\langle\frac{\delta l}{\delta[\nu]_G}, [\tilde{\nabla}^{(\mathcal{A},H)}]_G[u]_G\right\rangle \\
&\qquad+ \left\langle\frac{\delta l}{\delta[\nu]_G}, [\omega]_G(T\sigma,\bar{\eta})\right\rangle \\
&\quad= \operatorname{div}\left\langle\left(\frac{\delta l}{\delta\bar{\rho}}\oplus\frac{\delta l}{\delta[\nu]_G}\right), \bar{\eta}\oplus[u]_G\right\rangle - \operatorname{div}\left\langle\frac{\delta l}{\delta\bar{\rho}}, \bar{\eta}\right\rangle + \left\langle\operatorname{div}^{\mathcal{A}}\frac{\delta l}{\delta\bar{\rho}}, \bar{\eta}\right\rangle \\
&\qquad- \operatorname{div}\left\langle\frac{\delta l}{\delta[\nu]_G}, [u]_G\right\rangle + \left\langle\operatorname{div}^{(\mathcal{A},H)}\frac{\delta l}{\delta[\nu]_G}, [u]_G\right\rangle + \left\langle\frac{\delta l}{\delta[\nu]_G}, [\omega]_G(T\sigma,\bar{\eta})\right\rangle \\
&\quad= \left\langle\operatorname{div}^{\mathcal{A}}\frac{\delta l}{\delta\bar{\rho}}, \bar{\eta}\right\rangle + \left\langle\operatorname{div}^{(\mathcal{A},H)}\frac{\delta l}{\delta[\nu]_G}, [u]_G\right\rangle + \left\langle\frac{\delta l}{\delta[\nu]_G}, [\omega]_G(T\sigma,\bar{\eta})\right\rangle
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle\operatorname{ad}^*_{\bar{\rho}\oplus[\nu]_G}\left(\frac{\delta l}{\delta\bar{\rho}}\oplus\frac{\delta l}{\delta[\nu]_G}\right), \bar{\eta}\oplus[u]_G\right\rangle = \left\langle\frac{\delta l}{\delta\bar{\rho}}\oplus\frac{\delta l}{\delta[\nu]_G}, [\bar{\rho}\oplus[\nu]_G, \bar{\eta}\oplus[u]_G]\right\rangle \\
&\quad= \left\langle\frac{\delta l}{\delta\bar{\rho}}, [\bar{\rho},\bar{\eta}]\right\rangle\left\langle\frac{\delta l}{\delta[\nu]_G}, [\nabla^{(\mathcal{A},V)}]_{G,\bar{\rho}}[u]_G - [\nabla^{(\mathcal{A},V)}]_{G,\bar{\eta}}[\nu]_G - [\omega]_G(\bar{\rho},\bar{\eta}) + [[\nu]_G,[u]_G]\right\rangle \\
&\quad= \left\langle\operatorname{ad}^*_{\bar{\rho}}\frac{\delta l}{\delta\bar{\rho}}, \bar{\eta}\right\rangle + \left\langle\operatorname{ad}^*_{[\nu]_G}\frac{\delta l}{\delta[\nu]_G}, [u]_G\right\rangle \\
&\qquad+ \left\langle\frac{\delta l}{\delta[\nu]_G}, [\nabla^{(\mathcal{A},V)}]_{G,\bar{\rho}}[u]_G - [\nabla^{(\mathcal{A},V)}]_{G,\bar{\eta}}[\nu]_G - [\omega]_G(\bar{\rho},\bar{\eta})\right\rangle.
\end{aligned}$$

Thus, the vertical Lagrange–Poincaré equations restricted to $\mathrm{Ad}P$ are

$$\left\langle\operatorname{div}^{\mathcal{A}}\frac{\delta l}{\delta\bar{\rho}}, \bar{\eta}\right\rangle = \left\langle\operatorname{ad}^*_{\bar{\rho}}\frac{\delta l}{\delta\bar{\rho}}, \bar{\eta}\right\rangle - \left\langle\frac{\delta l}{\delta[\nu]_G}, [\nabla^{(\mathcal{A},V)}]_{G,\bar{\eta}}[\nu]_G + [\omega]_G(T\sigma\oplus\bar{\rho},\bar{\eta})\right\rangle,$$

which according to Proposition 23 coincides with the drift of the Noether current, while the vertical Lagrange–Poincaré equations restricted to V/G are

$$\left\langle \operatorname{div}^{(\mathcal{A},H)} \frac{\delta l}{\delta [\nu]_G}, [u]_G \right\rangle = \left\langle \operatorname{ad}^*_{[\nu]_G} \frac{\delta l}{\delta [\nu]_G}, [u]_G \right\rangle + \left\langle \frac{\delta l}{\delta [\nu]_G}, [\nabla^{(\mathcal{A},V)}]_{G,\bar{\rho}} [u]_G \right\rangle,$$

obtained from the projections of the vertical Lagrange–Poincaré equations induced by L.

Remark 24. In the special case $V = 0$, the drift law becomes a conservation law expressed as a vanishing of a divergence. Indeed, we recover the Noether theorem for covariant invariant Lagrangians from equation (24). In addition, the conservation of this current is equivalent to the vertical Lagrange–Poincaré equation of the reduced Lagrangian:

$$\operatorname{div}^{\mathcal{A}} \frac{\delta l}{\delta \bar{\rho}} - \operatorname{ad}^*_{\bar{\rho}} \frac{\delta l}{\delta \bar{\rho}} = 0.$$

9. The Molecular Strand with Rotors

In this section, we will discuss a problem of strand dynamics as an example of the theory of reduction by stages. Our model is called the molecular strand with rotors and consists of a mobile base strand repeating the same configuration with different orientation. In turn, this configuration will have a moving piece that can rotate, called rotor. In this example, we will suppose that each configuration has three rotors, each in the principal axes. However, the case with only one rotor or multiple rotors non-necessarily along the principal axes follows directly from our description. The particular case of this model when there are no rotors has been the object of research, for example, in [11] following the same covariant Lagrangian approach.

The principal bundle that we shall use has $X = \mathbb{R}^2$ as the base space and $P = X \times \mathbb{R}^3 \times SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ as the total space. The variables in $\mathbb{R}^2$ will be denoted by (s, t), and the first one can be thought as the parameter of the strand, whereas the second is the time. The Lie group $SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ acts on P since for any $(x, r, \Lambda, \theta) \in X \times \mathbb{R}^3 \times SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, where $x = (s, t) \in \mathbb{R}^2$ and $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, the element $(\Gamma, \alpha) \in SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ acts by

$$(\Gamma, \alpha) \cdot (x, r, \Lambda, \theta) = (x, \Gamma r, \Gamma\Lambda, \theta + \alpha).$$

We will first reduce by the normal subgroup $SO(3)$ and then by $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$.

9.1. First Reduction

We denote $\Sigma = P/SO(3) = X \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, the first quotient space, and $(x, \rho, \theta) \in \Sigma$. The projection is given by

$$\begin{aligned} \pi_{\Sigma,P} : X \times \mathbb{R}^3 \times SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 &\to X \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \\ (x, r, \Lambda, \theta) &\mapsto (x, \rho = \Lambda^{-1} r, \theta) \end{aligned}$$

, and its derivative is

$$\begin{aligned} T\pi_{\Sigma,P} : TP &\rightarrow T\Sigma \\ (x, r, \Lambda, \theta, v_x, v_r, v_\Lambda, v_\theta) &\mapsto (x, \rho = \Lambda^{-1} r, \theta, v_x, v_\rho = \Lambda^{-1} v_r - \Lambda^{-1} v_\Lambda \rho, v_\theta). \end{aligned}$$

To identify $J^1P/SO(3)$ as a FT$\mathfrak{LP}$ bundle, we need a connection $\mathcal{A}$ on $P \to \Sigma$. In terms of the Maurer–Cartan connection, any connection $\mathcal{A}$ can be written as

$$\mathcal{A}(x, r, \Lambda, \theta, v_x, v_r, v_\Lambda, v_\theta) = v_\Lambda \Lambda^{-1} + \tilde{\mathcal{A}}(x, r, \Lambda, \theta, v_x, v_r, v_\Lambda, v_\theta)$$

such that for any $\eta \in \mathfrak{so}(3)$, $\tilde{\mathcal{A}}(\eta^P_{x,r,\Lambda,\theta}) = 0$ and for all $\Gamma \in SO(3)$, $v \in TP$, $\tilde{\mathcal{A}}(\Gamma v) = \mathrm{Ad}_\Gamma \tilde{\mathcal{A}}(v)$. In addition, since $P \to X$ is trivial, we can trivialize $\mathrm{Ad}P$ as $\Sigma \times \mathfrak{so}(3)$ via the identification

$$[(x, r, \Lambda, \theta), \zeta]_{SO(3)} = [(x, \Lambda^{-1} r, e, \theta), \mathrm{Ad}_{\Lambda^{-1}} \zeta]_{SO(3)} = (x, \Lambda^{-1} r, \theta, \mathrm{Ad}_{\Lambda^{-1}} \zeta).$$

Then, the covariant derivative in $\mathrm{Ad}P$

$$\begin{aligned} \frac{\mathrm{D}^{\mathcal{A}}}{\mathrm{D}\tau}[x(\tau), r(\tau), \Lambda(\tau), \theta(\tau), \zeta(\tau)]_{SO(3)} = [&x(\tau), r(\tau), \Lambda(\tau), \theta(\tau), \dot{\zeta}(\tau) - (\Lambda_\tau \Lambda^{-1} \\ &+ \tilde{\mathcal{A}}(x, r, \Lambda, \theta, x_\tau, r_\tau, \Lambda_\tau, \theta_\tau)) \times \dot{\zeta}(\tau)]_{SO(3)} \end{aligned}$$

induces in $\Sigma \times \mathfrak{so}(3)$ the covariant derivative

$$\begin{aligned} \frac{\mathrm{D}^{\mathcal{A}}}{\mathrm{D}\tau}(x(\tau), \rho(\tau), \theta(\tau), \zeta(\tau)) = (&x(\tau), \rho(\tau), \theta(\tau), \dot{\zeta}(\tau) \\ &- \tilde{\mathcal{A}}(x, \rho, e, \theta, x_\tau, \rho_\tau, 0, \theta_\tau) \times \dot{\zeta}(\tau)). \end{aligned}$$

We lift the section $(x, r(x), \Lambda(x), \theta(x))$ of $P \to X$ to the section in $J^1P \to X$,

$$(x, r(x), \Lambda(x), \theta(x)), \mathrm{d}s + \mathrm{d}t, r_s \mathrm{d}s + r_t \mathrm{d}t, \Lambda_s \mathrm{d}s + \Lambda_t \mathrm{d}, \theta_s \mathrm{d}s + \theta_t \mathrm{d}t),$$

where subindices on r, Λ and θ denote partial derivatives. To project this section to $J^1\Sigma \oplus (T^*X \otimes (\Sigma \times \mathfrak{so}(3)))$, we evaluate

$$\begin{aligned} &\mathcal{A}(x, r, \Lambda, \theta, \mathrm{d}s + \mathrm{d}t, r_s \mathrm{d}s + r_t \mathrm{d}t, \Lambda_s \mathrm{d}s + \Lambda_t \mathrm{d}t, \theta_s \mathrm{d}s + \theta_t \mathrm{d}t) \\ &\quad = \Lambda_s \Lambda^{-1} \mathrm{d}s + \Lambda_t \Lambda^{-1} \mathrm{d}t + \tilde{\mathcal{A}}(x, r, \Lambda, \theta, 1, r_s, \Lambda_s, \theta_s) \mathrm{d}s \\ &\quad\quad + \tilde{\mathcal{A}}(x, r, \Lambda, \theta, 1, r_t, \Lambda_t, \theta_t) \mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Ad}_{\Lambda^{-1}} \mathcal{A}(x, r, \Lambda, \theta, \mathrm{d}s + \mathrm{d}t, r_s \mathrm{d}s + r_t \mathrm{d}t, \Lambda_s \mathrm{d}s + \Lambda_t \mathrm{d}t, \theta_s \mathrm{d}s + \theta_t \mathrm{d}t) \\ &\quad = \Omega \mathrm{d}s + \omega \mathrm{d}t + \bar{\mathcal{A}}_s \mathrm{d}s + \bar{\mathcal{A}}_t \mathrm{d}t = \Xi \mathrm{d}s + \xi \mathrm{d}t, \end{aligned}$$

where $\Omega = \Lambda^{-1} \Lambda_s$, $\omega = \Lambda^{-1} \Lambda_t$,

$$\bar{\mathcal{A}}_s = \mathrm{Ad}_{\Lambda^{-1}} \tilde{\mathcal{A}}(x, r, \Lambda, \theta, 1, r_s, \Lambda_s, \theta_s) = \tilde{\mathcal{A}}(x, \rho, e, \theta, 1, \rho_s, 0, \theta_s),$$

$\bar{\mathcal{A}}_t = \tilde{\mathcal{A}}(x, \rho, e, \theta, 1, \rho_t, 0, \theta_t)$, $\Xi = \Omega + \bar{\mathcal{A}}_s$ and $\xi = \omega + \bar{\mathcal{A}}_t$. Consequently, the bundle $J^1P/SO(3)$ is identified with FT$\mathfrak{LP}$ bundle $J^1\Sigma \oplus (T^*X \otimes (\Sigma \times \mathfrak{so}(3)))$ via

$$\begin{aligned} \pi_{\mathfrak{so}(3)} : J^1P/G &\rightarrow J^1\Sigma \oplus (T^*X \otimes (\Sigma \times \mathfrak{so}(3))) \\ j^1(x, r(x), \Lambda(x), \theta(x)) &\mapsto j^1(x, \rho(x), \theta(x)) \oplus (x, \rho(x), \theta(x), \Xi(x) \mathrm{d}s + \xi(x) \mathrm{d}t) \end{aligned}$$

and the reduced variables are ρ, ρ_s, ρ_t, θ, θ_s, θ_t, Ξ and ξ. The vertical Lagrange–Poincaré equations are:

$$\mathrm{ad}^*_{\Xi \mathrm{d}s+\xi \mathrm{d}t} \frac{\delta l}{\delta(\Xi \mathrm{d}s+\xi \mathrm{d}t)} - \mathrm{div}^{\nabla} \frac{\delta l}{\delta(\Xi \mathrm{d}s+\xi \mathrm{d}t)} = 0.$$

Hence, since

$$\mathrm{ad}^*_{\Xi \mathrm{d}s+\xi \mathrm{d}t} \frac{\delta l}{\delta(\Xi \mathrm{d}s+\xi \mathrm{d}t)} = -\Xi \times \frac{\delta l}{\delta \Xi} - \xi \times \frac{\delta l}{\delta \xi},$$

and

$$\mathrm{div}^{\nabla} \frac{\delta l}{\delta(\Xi \mathrm{d}s+\xi \mathrm{d}t)} = \partial_s \frac{\delta l}{\delta \Xi} - \bar{\mathcal{A}}_s \times \frac{\delta l}{\delta \Xi} + \partial_t \frac{\delta l}{\delta \xi} - \bar{\mathcal{A}}_t \times \frac{\delta l}{\delta \xi},$$

we conclude that the vertical Lagrange–Poincaré equations are

$$0 = \partial_s \frac{\delta l}{\delta \Xi} + \partial_t \frac{\delta l}{\delta \xi} + \Omega \times \frac{\delta l}{\delta \Xi} + \omega \times \frac{\delta l}{\delta \xi}. \tag{27}$$

On the other hand, the horizontal Lagrange–Poincaré equations applied to variation $\delta\rho \oplus \delta\theta$ are

$$\left\langle \frac{\delta l}{\delta(\rho \oplus \theta)} - \partial_s \frac{\delta l}{\delta(\rho_s \oplus \theta_s)} - \partial_t \frac{\delta l}{\delta(\rho_t \oplus \theta_t)}, \delta\rho \oplus \delta\theta \right\rangle$$
$$= \left\langle \frac{\delta l}{\delta(\Xi \mathrm{d}s+\xi \mathrm{d}t)}, \tilde{B}((\mathrm{d}s+\mathrm{d}t, \rho_s \mathrm{d}s+\rho_t \mathrm{d}t, \theta_s \mathrm{d}s+\theta_t \mathrm{d}t), (0, \delta\rho, \delta\theta)) \right\rangle.$$

We shall note that $\delta l/\delta(\rho \oplus \theta)$ is not a partial derivative; instead, we saw in § 5 that

$$\left\langle \frac{\delta l}{\delta(\rho \oplus \theta)}, \delta\rho \oplus \delta\theta \right\rangle = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} l(u^h_{j^1\rho \oplus j^1\theta, \Xi \mathrm{d}s+\xi \mathrm{d}t}),$$

where $u(\epsilon)$ is a curve in $\Sigma = X \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ with $u'(0) = (0, \delta\rho, \delta\theta)$ and $u^h_{j^1\rho \oplus j^1\theta, \Xi \mathrm{d}s+\xi \mathrm{d}t}(\epsilon) = (u(\epsilon), j^1\rho(0), j^1\theta(0), \nu(\epsilon))$ the horizontal lift to $J^1\Sigma \oplus (T^*X \otimes (\Sigma \times \mathfrak{so}(3)))$ with $\nu(\epsilon) = \nu_1(\epsilon)\mathrm{d}s + \nu_2(\epsilon)\mathrm{d}t$ such that $v(0) = \Xi \mathrm{d}s + \xi \mathrm{d}t$ and $\nu(\epsilon)$ is horizontal. From the connection in $\Sigma \times \mathfrak{so}(3)$ this means that

$$\dot{\nu}_1(\epsilon)\mathrm{d}s + \dot{\nu}_2(\epsilon)\mathrm{d}t = \mathcal{A}(0, \delta\rho, 0, \delta\theta) \times (\nu_1(\epsilon)\mathrm{d}s + \nu_2(\epsilon)\mathrm{d}t).$$

In particular, for $\epsilon = 0$;

$$\dot{\nu}_1(0)\mathrm{d}s + \dot{\nu}_2(0)\mathrm{d}t = \mathcal{A}(0, \delta\rho, 0, \delta\theta) \times (\Xi \mathrm{d}s + \xi \mathrm{d}t),$$

and

$$\left\langle \frac{\delta l}{\delta(\rho \oplus \theta)}, \delta\rho \oplus \delta\theta \right\rangle = \left\langle \frac{\partial l}{\partial \rho}, \delta\rho \right\rangle + \left\langle \frac{\partial l}{\partial \theta}, \delta\theta \right\rangle$$
$$+ \left\langle \frac{\delta l}{\delta \Xi}, \tilde{\mathcal{A}}(0, \delta\rho, 0, \delta\theta) \times \Xi \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{\mathcal{A}}(0, \delta\rho, 0, \delta\theta) \times \xi \right\rangle.$$

It is possible to split the horizontal equations in Σ in two parts. One related to $\mathbb{R}^3$ for which we make $\delta\theta = 0$;

$$\left\langle \frac{\partial l}{\partial \rho} - \partial_s \frac{\delta l}{\partial \rho_s} - \partial_t \frac{\delta l}{\partial \rho_t}, \delta\rho \right\rangle + \left\langle \frac{\delta l}{\delta \Xi}, \tilde{\mathcal{A}}(0, \delta\rho, 0, 0) \times \Xi \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{\mathcal{A}}(0, \delta\rho, 0, 0) \times \xi \right\rangle = \left\langle \frac{\delta l}{\delta \Xi}, \tilde{B}((1, \rho_s, \theta_s), \delta\rho) \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{B}((1, \rho_t, \theta_t), \delta\rho) \right\rangle, \tag{28}$$

and one related to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ for which $\delta\rho = 0$;

$$\left\langle \frac{\partial l}{\partial \theta} - \partial_s \frac{\delta l}{\partial \theta_s} - \partial_t \frac{\delta l}{\partial \theta_t}, \delta\theta \right\rangle + \left\langle \frac{\delta l}{\delta \Xi}, \tilde{\mathcal{A}}(0, 0, 0, \delta\theta) \times \Xi \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{\mathcal{A}}(0, 0, 0, \delta\theta) \times \xi \right\rangle = \left\langle \frac{\delta l}{\delta \Xi}, \tilde{B}((1, \rho_s, \theta_s), \delta\theta) \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{B}((1, \rho_t, \theta_t), \delta\theta) \right\rangle \tag{29}$$

In conclusion, the equations of motion of the molecular strand with rotors after reduction by $SO(3)$ are equations (27), (28) and (29). Yet, they are only equivalent to the unreduced equations if considered together with the reconstruction equation studied in §7 above. Observe that in $\mathrm{Ad}P$, $\omega^{\bar{\rho}} = \Lambda\Xi\Lambda^{-1}\mathrm{d}s + \Lambda\xi\Lambda^{-1}\mathrm{d}t$. Therefore,

$$\mathrm{d}^{\mathcal{A}}\omega^{\bar{\rho}} = \mathrm{Ad}_\Lambda(\Xi_t - \bar{\mathcal{A}}_t \times \Xi - \xi_s + \bar{\mathcal{A}}_s \times \xi)\mathrm{d}t \wedge \mathrm{d}s,$$

and

$$\omega^{\bar{\rho}} \wedge \omega^{\bar{\rho}} = (\Lambda\Xi\Lambda^{-1}\mathrm{d}s + \Lambda\xi\Lambda^{-1}\mathrm{d}t) \wedge (\Lambda\Xi\Lambda^{-1}\mathrm{d}s + \Lambda\xi\Lambda^{-1}\mathrm{d}t) = -\mathrm{Ad}_\Lambda(\Xi \times \xi)\mathrm{d}t \wedge \mathrm{d}s.$$

The reconstruction equation (23) is then written as

$$\xi_s - \bar{\mathcal{A}}_s \times \xi - \Xi_t + \bar{\mathcal{A}}_t \times \Xi - \Xi \times \xi + \mathrm{Ad}_\Lambda \tilde{B}((0, \rho_t, \theta_t, 0, \rho_s, \theta_s)) = 0. \tag{30}$$

9.2. Second Reduction

The group $S = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ acts on $J^1\Sigma \oplus (T^*X \otimes (\Sigma \times \mathfrak{so}(3)))$ by

$$\begin{aligned}\alpha \cdot (x, \rho, \theta, \mathrm{d}s + \mathrm{d}t, \rho_s\mathrm{d}s + \rho_t\mathrm{d}t, \theta_s\mathrm{d}s + \theta_t\mathrm{d}t) \oplus (x, \rho, \theta, \Xi\mathrm{d}s + \xi\mathrm{d}t) \\ = (x, \rho, \theta + \alpha, \mathrm{d}s + \mathrm{d}t, \rho_s\mathrm{d}s + \rho_t\mathrm{d}t, \theta_s\mathrm{d}s + \theta_t\mathrm{d}t) \oplus (x, \rho, \theta + \alpha, \Xi\mathrm{d}s + \xi\mathrm{d}t)\end{aligned}$$

for all $\alpha \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Let $R = \Sigma/(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) = X \times \mathbb{R}^3$. Since $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ is abelian, the identification $\mathrm{Ad}\Sigma = R \times \mathbb{R}^3$ is immediate and the principal connection in $\Sigma \to R$ is trivial. Consequently, we have the identification

$$\begin{aligned}\pi_S : J^1\Sigma \oplus (T^*X \otimes (\Sigma \times \mathfrak{so}(3)))/S &\to J^1R \oplus (T^*X \otimes (R \times \mathbb{R}^3 \times \mathfrak{so}(3))) \\ [j^1(x, \rho(x), \theta(x)) \oplus (x, \rho, \theta, \Xi\mathrm{d}s + \xi\mathrm{d}t)]_S &\mapsto j^1(x, \rho(x)) \oplus (x, \rho, \theta_s\mathrm{d}s + \theta_t\mathrm{d}t, \Xi\mathrm{d}s + \xi\mathrm{d}t),\end{aligned}$$

where $J^1R \oplus (T^*X \otimes (R \times \mathbb{R}^3 \times \mathfrak{so}(3)))$ is a FT$\mathfrak{LP}$ bundle with extra structure given by;

$$\begin{aligned}&\nabla^{\mathfrak{s}}_{v_x,v_\rho}(x,\rho,\beta(x,\rho),\zeta(x,\rho))\\ &\quad = (x,\rho,\beta_{v_x,v_\rho},\zeta_{v_x,v_\rho} - \tilde{B}((v_x,v_\rho,0),(0,0,\beta)) - \tilde{\mathcal{A}}(v_x,v_\rho,0,0)\times\zeta);\end{aligned}$$

the $(\mathbb{R}^3 \times \mathfrak{so}(3))$-valued 2-form $\omega^{\mathfrak{s}} = 0 \oplus [\tilde{B}]_S$; and the Lie bracket

$$\begin{aligned}[(\beta_1,\zeta_1),(\beta_2,\zeta_2)]^{\mathfrak{s}} = 0 \oplus (&-\mathcal{A}(0,0,0,\beta_1)\times\zeta_2 + \mathcal{A}(0,0,0,\beta_2)\times\zeta_1\\ &- \tilde{B}((0,0,\beta_1),(0,0,\beta_2)) + \zeta_1\times\zeta_2).\end{aligned}$$

The vertical Lagrange–Poincaré equations in the second step of reduction are

$$\mathrm{ad}^*_{(\theta_s,\Xi)\mathrm{d}s+(\theta_t,\xi)\mathrm{d}t}\frac{\delta l}{\delta(\theta_s\mathrm{d}s+\theta_t\mathrm{d}t,\Xi\mathrm{d}s+\xi\mathrm{d}t)} - \mathrm{div}^{\mathfrak{s}}\frac{\delta l}{\delta(\theta_s\mathrm{d}s+\theta_t\mathrm{d}t,\Xi\mathrm{d}s+\xi\mathrm{d}t)} = 0.$$

Given $(\delta\theta,\delta\eta)\in\mathbb{R}^3\times\mathfrak{so}(3)$, we obtain that, on the one hand,

$$\begin{aligned}&\left\langle \mathrm{ad}^*_{(\theta_s,\Xi)\mathrm{d}s+(\theta_t,\xi)\mathrm{d}t}\frac{\delta l}{\delta(\theta_s\mathrm{d}s+\theta_t\mathrm{d}t,\Xi\mathrm{d}s+\xi\mathrm{d}t)},(\delta\theta,\delta\eta)\right\rangle\\ &\quad = \left\langle \frac{\delta l}{\delta(\theta_s\mathrm{d}s+\theta_t\mathrm{d}t)}\oplus\frac{\delta l}{\delta(\Xi\mathrm{d}s+\xi\mathrm{d}t)},[(\theta_s\mathrm{d}s+\theta_t\mathrm{d}t,\Xi\mathrm{d}s+\xi\mathrm{d}t),(\delta\theta,\delta\eta)]^{\mathfrak{s}}\right\rangle\\ &\quad = \left\langle \frac{\delta l}{\delta\Xi},-\mathcal{A}(0,0,0,\theta_s)\times\delta\eta + \mathcal{A}(0,0,0,\delta\theta)\times\Xi - \tilde{B}((0,0,\theta_s),(0,0,a)) + \Xi\times\delta\eta\right\rangle\\ &\qquad + \left\langle \frac{\delta l}{\delta\xi},-\mathcal{A}(0,0,0,\theta_t)\times\delta\eta + \mathcal{A}(0,0,0,\delta\theta)\times\Xi - \tilde{B}((0,0,\theta_t),(0,0,a)) + \xi\times\delta\eta\right\rangle\end{aligned}$$

On the other hand,

$$\begin{aligned}&\left\langle \mathrm{div}^{\mathfrak{s}}\frac{\delta l}{\delta(\theta_s\mathrm{d}s+\theta_t\mathrm{d}t,\Xi\mathrm{d}s+\xi\mathrm{d}t)},(\delta\theta,\delta\eta)\right\rangle = \left(\left\langle \partial_s\frac{\delta l}{\delta\theta_s}+\partial_t\frac{\delta l}{\delta\theta_t},\delta\theta\right\rangle\right.\\ &\quad\left. + \left\langle\frac{\delta l}{\delta\Xi},\tilde{B}((0,\rho_s,0),(0,0,\delta\theta))\right\rangle + \left\langle\frac{\delta l}{\delta\xi},\tilde{B}((0,\rho_t,0),(0,0,\delta\theta))\right\rangle\right)\\ &\quad \oplus \left\langle \partial_s\frac{\delta l}{\delta\Xi}+\partial_t\frac{\delta l}{\delta\xi}+\mathcal{A}(0,\rho_s,0,0)\times\frac{\delta l}{\delta\Xi}+\mathcal{A}(0,\rho_t,0,0)\times\frac{\delta l}{\delta\xi},\delta\eta\right\rangle.\end{aligned}$$

From these two expressions, it can be proven that the vertical Lagrange–Poincaré equations can be split into two parts related to $\mathbb{R}^3$ and $\mathfrak{so}(3)$. These, respectively, coincide with Eqs. (29) and (27), except for the term $\delta l/\delta\theta$ which is zero from the symmetry of the Lagrangian with respect to $\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1$. The horizontal Lagrange–Poincaré equations are

$$\left\langle \frac{\delta l}{\delta\rho} - \partial_s\frac{\delta l}{\delta\rho_s} - \partial_t\frac{\delta l}{\delta\rho_t},\delta\rho\right\rangle = \left\langle\frac{\delta l}{\delta\Xi},\tilde{B}((0,\rho_s,0),\delta\rho)\right\rangle + \left\langle\frac{\delta l}{\delta\xi},\tilde{B}((0,\rho_t,0),\delta\rho)\right\rangle.$$

In an analogous way as we did in the first step of reduction, we express the horizontal derivative in terms of partial and fiber derivatives as

$$\left\langle\frac{\delta l}{\delta\rho},\delta\rho\right\rangle = \left\langle\frac{\partial l}{\partial\rho},\delta\rho\right\rangle + \left\langle\frac{\delta l}{\delta\Xi},\mathcal{A}(0,\delta\rho,0,0)\times\Xi\right\rangle + \left\langle\frac{\delta l}{\delta\xi},\mathcal{A}(0,\delta\rho,0,0)\times\xi\right\rangle.$$

Therefore, the horizontal Lagrange–Poincaré equation of step two coincides with equation (28). In conclusion, Eqs. (27), (28) and (29) are obtained as well in the second step of reduction. However, in the first step of reduction Eq. (29) is an horizontal equation, whereas in step two it is vertical. Furthermore, the reconstruction equation from the second step to the first one is easily seen to be

$$\partial_t \theta_s = \partial_s \theta_t. \tag{31}$$

Remark 25. The trivial Maurer–Cartan connection may not always be the more convenient for a particular Lagrangian. In Mechanics, the study of a rigid body with rotors in [19] is an example where the appropriate connection adapted to the Lagrangian is the mechanical connection (the connection defined by a Riemannian metric). Future work will study this example using the techniques of reduction by stages developed in [3].

9.3. A Particular Lagrangian

We now choose the particular Lagrangian in J^1P

$$\begin{aligned} L(r, r_s, r_t, \Lambda, \Lambda_s, \Lambda_t, \theta, \theta_s, \theta_t) &= \frac{1}{2}\langle r_t, r_t\rangle + \frac{1}{2}\langle \Lambda^{-1}\Lambda_t, I\Lambda^{-1}\Lambda_t\rangle \\ &\quad + \frac{1}{2}\langle \Lambda^{-1}\Lambda_t + \theta_t, K(\Lambda^{-1}\Lambda_t + \theta_t)\rangle \\ &\quad - E(\Lambda^{-1}\Lambda_s, \theta_s, \langle r, r\rangle), \end{aligned}$$

where I is the inertia tensor of the configuration of the strand, K is the inertia tensor of the rotors, and E is a function called potential energy. This Lagrangian combines terms appearing in the Lagrangian given for the rigid body with rotors in [19] and the Lagrangian proposed in [11] for the molecular strand. It is invariant by $SO(3) \times \mathbb{S} \times \mathbb{S} \times \mathbb{S}$. If the first step of reduction is performed by group $SO(3)$ with the Maurer–Cartan connection, that is $\tilde{\mathcal{A}} = 0$, then the reduced Lagrangian is:

$$\begin{aligned} l(\rho, \rho_s, \rho_t, \theta, \theta_s, \theta_t, \Omega, \omega) &= \frac{1}{2}\langle \rho_t + \omega \times \rho, \rho_t + \omega \times \rho\rangle + \frac{1}{2}\langle \omega, I\omega\rangle \\ &\quad + \frac{1}{2}\langle \omega + \theta_t, K(\omega + \theta_t)\rangle - E(\Omega, \theta_s, \langle \rho, \rho\rangle), \end{aligned}$$

For this Lagrangian, the vertical Lagrange–Poincaré equations (27) take the form

$$\begin{aligned} 0 &= \rho \times (\rho_{tt} + 2\omega \times \rho_t + \omega_t \times \rho + \langle \omega, \rho\rangle\omega) + (I + K)\omega_t + K\theta_{tt} \\ &\quad + \omega \times ((I + K)\omega + K\theta_t) - \partial_s \frac{\delta E}{\delta \Omega} + \Omega \times \frac{\delta E}{\delta \Omega}, \end{aligned} \tag{32}$$

while the horizontal Lagrange–Poincaré equations (28) and (29) particularize as

$$\rho_{tt} + 2\omega \times \rho_t + \omega_t \times \rho + \omega \times (\omega \times \rho) = -\frac{\partial E}{\partial \rho}\frac{\rho}{\|\rho\|}, \tag{33}$$

$$K\omega_t + K\theta_{tt} = \partial_s \frac{\partial E}{\partial \theta_s}. \tag{34}$$

The reconstruction equation (30) becomes

$$\omega_s - \Omega_t - \Omega \times \omega = 0. \tag{35}$$

Observe that on the left-hand side of equation (33), the acceleration R_{tt} in the inertial fixed frame is reinterpreted as the sum of the acceleration ρ_{tt} in a non-inertial frame and additional acceleration terms. These additional terms have been historically interpreted as fictitious forces: $2\omega \times \rho_t$ is the Coriolis force, $\omega_t \times \rho$ is the Euler force and $\omega \times (\omega \times \rho)$ is the centrifugal force. Thus, Eq. (33) states that the time evolution of ρ is the same as the evolution of particle in a potential E seen from a non-inertial frame with angular rotation $\omega(s,t)$. As the potential E is radial, Eq. (32) can be rewritten as

$$0 = (I+K)\omega_t + K\theta_{tt} + \omega \times ((I+K)\omega + K\theta_t) - \partial_s \frac{\partial E}{\partial \Omega} + \Omega \times \frac{\partial E}{\partial \Omega},$$

after substitution of Eq. (33). Equation (34) is related to the momentum of the rotors and will appear as a new vertical equation in the next stage of reduction. In fact, the second stage of reduction defines the reduced Lagrangian

$$\begin{aligned} l(\rho, \rho_s, \rho_t, a, b, \Omega, \omega) = &\frac{1}{2}\langle \rho_t + \omega \times \rho, \rho_t + \omega \times \rho \rangle + \frac{1}{2}\langle \omega, I\omega \rangle \\ &+ \frac{1}{2}\langle \omega + b, K(\omega + b)\rangle - E(\Omega, a, \langle \rho, \rho \rangle), \end{aligned}$$

whose vertical Lagrange–Poincaré equations are

$$\begin{aligned} 0 &= \rho \times (\rho_{tt} + 2\omega \times \rho_t + \omega_t \times \rho + \langle \omega, \rho \rangle \omega) + (I+K)\omega_t + K b_t \\ &\quad + \omega \times ((I+K)\omega + K b) - \partial_s \frac{\delta E}{\delta \Omega} + \Omega \times \frac{\delta E}{\delta \Omega}, \\ K\omega_t &+ K b_t = \partial_s \frac{\delta E}{\delta a}, \end{aligned}$$

the horizontal Lagrange–Poincaré equations are;

$$\omega \times (\rho \times \omega - 2\rho_t) - \rho_{tt} - \omega_t \times \rho = 2\rho \frac{\partial E}{\partial \langle \rho, \rho \rangle},$$

and reconstruction equation $a_t = b_s$, which makes this equations equivalent to the ones in the previous step.

Acknowledgements

Both authors have been partially supported by MICINN (Spain) under grants no. PGC2018-098321-B-I00 and PID2021-126124NB-I00. MAB has been partially supported by Ministerio de Universidades (Spain) under grant no. FPU17-02701. MCL has been partially supported by Junta Castilla Leon (Spain) under grant no. SA090G19.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

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Communicated by Dmitry Dolgopyat.

Received: December 23, 2020.

Accepted: August 14, 2023.