Adiabatic regularization of primordial perturbations generated during inflation

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Abstract

Recently, L. Parker *et al.* claimed that standard predictions of the slow-roll inflationary scenario of the early Universe regarding primordial spectra of scalar and tensor metric perturbations generated from quantum vacuum fluctuations during inflation should be radically changed because of the necessity of renormalization of the perturbations. Here we prove that, contrary, these basic predictions are robust with respect to any renormalization, if the transition from quantum to effective stochastic classical observables is made consistently. The crucial point is the necessity to follow the behaviour of terms subtracted in the process of the renormalization long after the moment of the first Hubble radius crossing during inflation, up to a period that the Hubble parameter H becomes much less than its value at that moment.

1 Introduction

Unambiguous predictions of the power spectrum and statistics of primordial (post-inflationary) scalar and tensor perturbations play the central role in the whole inflationary scenario of the early universe since only they provide us with a possibility to test and confirm/falsify any concrete inflationary model (the prediction of the approximate spatial flatness of the present Universe follows from a prediction about the amplitude of a monopole, $l = 0$, scalar perturbation on a spatially flat FRW background). Indeed, by comparing these predictions with numerous existing observational data, large amount of inflationary models have been already falsified, while many of them (including the pioneer ones) still remain viable. That is why if consistency of these predictions would be put under question, or it would be shown that they are based on additional and doubtful assumptions, this would have dramatic consequences for the fate of the inflationary scenario as a whole and all its concrete realizations (models). Recently such an attempt was undertaken in the series of papers by L. Parker and his collaborators [1–4]. Their key statement is the following. Since the origin of post-inflationary metric and matter perturbations are quantum fluctuations of the gravitational field and other light scalar fields during inflation, so that this process has quantum and even quantum-gravitational nature, a renormalization is needed, as usually in quantum field theory, to obtain final observable predictions. Applying the standard procedure of the adiabatic (n-wave) regularization [5–9] to the power spectra of both scalar (curvature) perturbations $\mathcal{R}_{\rm c}(x)$ and tensor ones (gravitational waves) $h_{ij}(x)$, they claimed to obtain completely different results for these quantities. In this paper, we revisit this problem and discuss whether any UV renormalization may change the standard inflationary predictions.

2 Quantization and adiabatic regularization of perturbations

First of all, we describe the setup of the problem. As in Ref. [3], we consider slow-roll inflation driven by a single scalar (inflaton) field, whose action is given by

$$S = \frac{1}{2} \int \sqrt{-g}\, [M_{\rm pl}^2 R - g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 2V(\phi)]{\rm d}^4x. \qquad (1)$$

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Assuming that background geometry is described by a spatially flat FRW metric, the background field equations take the form

$$6M_{\rm pl}^2\mathcal{H}^2=\phi'^{\,2}+2a^2V(\phi),\qquad \phi''+2\mathcal{H}\phi'+a^2V_\phi(\phi)=0\ , \tag{2}$$

where $\mathcal{H}:=a'/a=aH$ and the dash denotes the derivative with respect to the conformal time η. In the comoving gauge, the spatial part of the metric takes the form

$$g_{ij}=a^2(\eta)[(1+2\mathcal{R}_{\rm c}(x))\delta_{ij}+h_{ij}(x)]\ , \tag{3}$$

where $\mathcal{R}_{\rm c}(x)$ and $h_{ij}(x)$ denote a scalar (curvature) curvature perturbation and tensor perturbations (gravitational waves), respectively. Here we neglect vector perturbations which rapidly decays during expansion. Equations for the Fourier mode functions of $\mathcal{R}_{\rm c}$ and h_{ij} have the form

$$\mathcal{R}''_{{\rm c},k}(\eta)+2\frac{z'}{z}\mathcal{R}'_{{\rm c},k}(\eta)+k^2\mathcal{R}_{{\rm c},k}(\eta)=0,\quad h''_k(\eta)+2\frac{a'}{a}h'_k(\eta)+k^2h_k(\eta)=0\ , \tag{4}$$

where $z:=a\phi'/\mathcal{H}$. Using the positive frequency solutions $\mathcal{R}_{{\rm c},k}(\eta)$ and $h_k(\eta)$ for the initial adiabatic vacuum, two-point functions of the curvature perturbation and the gravitational waves in this vacuum are expressed as

$$\langle\mathcal{R}_{\rm c}(\eta,\boldsymbol{x})\mathcal{R}_{\rm c}(\eta,\boldsymbol{y})\rangle=\int\frac{{\rm d}k}{k}\frac{\sin k|\boldsymbol{x}-\boldsymbol{y}|}{k|\boldsymbol{x}-\boldsymbol{y}|}\Delta^2_{\mathcal{R}_{\rm c},k}(\eta),\quad \langle h_{ij}(\eta,\boldsymbol{x})h^{ij}(\eta,\boldsymbol{y})\rangle=\int\frac{{\rm d}k}{k}\frac{\sin k|\boldsymbol{x}-\boldsymbol{y}|}{k|\boldsymbol{x}-\boldsymbol{y}|}\Delta^2_{h,k}(\eta)\ . \tag{5}$$

where dimensional variances are defined as

$$\Delta^2_{\mathcal{R}_{\rm c},k}(\eta):=\frac{k^3}{2\pi^2}|\mathcal{R}_{{\rm c},k}(\eta)|^2\ ,\qquad \Delta^2_{h,k}(\eta):=\frac{k^3}{\pi^2}|h_k(\eta)|^2\ . \tag{6}$$

The adiabatic condition implies that $|\mathcal{R}_{{\rm c},k}(\eta)|^2$ and $|h_k(\eta)|^2$ scale as $1/k$ in the UV limit. Therefore, due to this UV behaviour, the two-point functions diverge in the coincidence limit $\boldsymbol{x}\to\boldsymbol{y}$.

Next, we briefly review the method of adiabatic (n-wave) regularization which detailed explanation can be found in Refs. [6–9]. Adiabatic regularization is a convenient framework to remove all UV divergences from average values of quantum fields in an expanding universe. In particular, the regularized two-point function for a scalar field ϕ is given by

$$\langle\,\phi(\eta,\,\boldsymbol{x})\phi(\eta,\,\boldsymbol{y})\,\rangle_{\rm R}:=\langle\,\phi(\eta,\,\boldsymbol{x})\phi(\eta,\,\boldsymbol{y})\,\rangle-\langle\,\phi(\eta,\,\boldsymbol{x})\phi(\eta,\,\boldsymbol{y})\,\rangle\big|^{(A)}, \tag{7}$$

where $|^{(A)}$ indicates that cross terms from field products which are of an adiabatic order greater than A have to be neglected. It is remarkable that the adiabatic regularization yields the same result as the point-splitting regularization, which is conceived to be an efficient regularization technique in curved space-time. Among advantages of the adiabatic regularization are that subtraction terms thus introduced preserve the conservation property of the average energy-momentum tensor of quantum fields and that the vacuum polarization given by Eq. (7) vanishes in Minkowski limit.

Since the adiabatic expansion coincides with the expansion in powers of $1/k$ (and with the n-wave expansion in terminology of [5]), by using it for the modes ${}^{(A)}\tilde{\mathcal{R}}_{{\rm c},k}(\eta)$ and ${}^{(A)}\tilde{h}_k(\eta)$ where the index (A) denotes the adiabatic order, we can subtract divergent parts of integrals over momentum in Eq. (5). Using Eq. (7), the regularized dimensionless power spectra are obtained as

$$\Delta^{({\rm R})}_{\mathcal{R}_{\rm c},k}{}^2(\eta)\ :=\ \frac{k^3}{2\pi^2}\Big[\ |\mathcal{R}_{{\rm c},k}(\eta)|^2-|{}^{(A)}\tilde{\mathcal{R}}_{{\rm c},k}(\eta)|^2\,\big|^{(A)}\ \Big], \tag{8}$$

$$\Delta^{({\rm R})}_{h,k}{}^2(\eta)\ :=\ \frac{k^3}{\pi^2}\Big[\ |h_k(\eta)|^2-|{}^{(A)}\tilde{h}_k(\eta)|^2\,\big|^{(A)}\ \Big], \tag{9}$$

where $|{}^{(A)}\tilde{\mathcal{R}}_{{\rm c},k}(\eta)|^2\,\big|^{(A)}$ and $|{}^{(A)}\tilde{h}_k(\eta)|^2\,\big|^{(A)}$ denote subtraction terms for a curvature perturbation and gravitational waves, respectively. In order not to violate the fundamental properties of the adiabatic regularization described above, we have to introduce the subtraction terms not only to the UV modes, but to all modes.

3 Effects of UV regularization

In this section, we discuss whether a UV renormalization can modify power spectra of primordial perturbations. Making use of the WKB expansion of solutions, the subtraction term for $\mathcal{R}_{\rm c}$ is given by

$$|{}^{(2)}\tilde{\mathcal{R}}_{{\rm c},k}(\eta)|^2\,\Big|^{(2)} = \frac{1}{2z^2k}\left[\,1+(1+\delta\varepsilon)\left(\frac{aH}{k}\right)^2\,\right] , \tag{10}$$

where $\delta\varepsilon$ is expressed in terms of the Hubble (horizon) flow functions ε_i as follows:

$$\delta\varepsilon := -\frac{1}{2}\varepsilon_1 + \frac{3}{4}\varepsilon_2 + \frac{1}{8}\varepsilon_2^2 - \frac{1}{4}\varepsilon_1\varepsilon_2 + \frac{1}{4}\varepsilon_2\varepsilon_3 \; . \tag{11}$$

Fluctuations from which the present large-scale structure of the Universe has originated cross the Hubble radius during inflation long before inflation ends. In the long wavelength limit $-k\eta \ll 1$, the mode function $\mathcal{R}_{{\rm c},k}$ approaches the constant value

$$\mathcal{R}_{{\rm c},k}(\eta) \simeq e^{i(2\nu+3)\pi/4}\,\frac{2^{\nu-\frac{3}{2}}}{\sqrt{2k}}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\frac{(1-\varepsilon_1)^{\nu-\frac{1}{2}}}{z(\eta_*)} \; , \tag{12}$$

where η_* is the Hubble radius crossing time. The existence of the constant solution at large scales is assured by the very structure of the Einstein equations and remains valid in the non-linear regime, too. Quantum fluctuations generated deep inside the Hubble radius become indistinguishable from classical stochastic ones at super-Hubble scales after neglecting their decaying mode. Thus, we obtain the average squared value of $\mathcal{R}_{{\rm c},k}$ as

$$|\mathcal{R}_{{\rm c},k}(\eta)|^2 \simeq \frac{1}{4\varepsilon_1(\eta_*)}\frac{1}{k^3}\left(\frac{H(\eta_*)}{M_{\rm pl}}\right)^2(1+\delta\varepsilon_s) \; , \tag{13}$$

in the large scale limit, where $\delta\varepsilon_s := (2\varepsilon_1+\varepsilon_2)(2-\log 2-\gamma)-2\varepsilon_1$. Taking the large scale limit in Eq. (10), the adiabatic subtraction term is approximated as

$$|{}^{(2)}\tilde{\mathcal{R}}_{{\rm c},k}(\eta)|^2\,\Big|^{(2)} \simeq \frac{1+\delta\varepsilon}{2k^3}\left(\frac{aH}{z}\right)^2 = \frac{1+\delta\varepsilon}{4\varepsilon_1}\frac{1}{k^3}\left(\frac{H}{M_{\rm pl}}\right)^2 . \tag{14}$$

Since the Hubble parameter H and the Hubble flow functions are time dependent, the amplitude of the subtraction term changes even after the moment of the Hubble radius crossing. This reflects the fact that, while the adiabatic expansion of the modes ${}^{(A)}\tilde{\mathcal{R}}_{{\rm c},k}(\eta)$ well approximates the exact solution at sub-Hubble scales, it does not do so at super-Hubble ones. For instance, during the chaotic inflation, the Hubble parameter H decreases and the Hubble flow function ε_1 increases. The amplitude of the subtraction term then becomes smaller and smaller.

Substituting Eqs. (13) and (14) into Eq. (8), we obtain the regularized dimensionless variance of the curvature perturbation as

$$\Delta^{({\rm R})}_{\mathcal{R}_{\rm c},k}{}^2(\eta) = \frac{1}{2M_{\rm pl}^2\varepsilon_1(\eta_*)}\left(\frac{H(\eta_*)}{2\pi}\right)^2\left[1+\delta\varepsilon_s-(1+\delta\varepsilon)\,\frac{\varepsilon_1(\eta_*)}{\varepsilon_1(\eta)}\left(\frac{H(\eta)}{H(\eta_k)}\right)^2\right] . \tag{15}$$

Through the inflationary and the post inflationary evolution, the Hubble parameter H and the Hubble flow function ε_1 scale as $H(N)/H(N_*) = \exp[-\int_{N_*}^N {\rm d}N'\varepsilon_1(N')]$ and $\varepsilon_1(N)/\varepsilon_1(N_*) = \exp[\int_{N_*}^N {\rm d}N'\varepsilon_2(N')]$. Now it is clear that the subtraction term is suppressed by the factor:

$$\frac{\varepsilon_1(N_*)}{\varepsilon_1(N)}\left(\frac{H(N)}{H(N_*)}\right)^2 = e^{-\int_{N_*}^N {\rm d}N'[2\varepsilon_1(N')+\varepsilon_2(N')]} \; . \tag{16}$$

Therefore, for each Fourier mode, terms subtracted in the course of the adiabatic regularization are negligibly small when $H(\eta)$ becomes much less than $H(\eta_*)$ as the Universe evolves. Note that the bare power spectrum is completely canceled by the adiabatic subtraction terms in the exact de Sitter space-time. However, this does not necessarily mean a problem, because in this case generated fluctuations, stretched beyond the future event horizon, cannot be observed.

The regularized amplitude of gravitational waves can be obtained in a manner similar to curvature perturbations. At large scales, their regularized dimensionless variance is evaluated as

$$\Delta_{h,k}^{(\mathrm{R})\ 2}(\eta) \simeq \frac{8}{M_{\mathrm{pl}}^2}\left(\frac{H(\eta_*)}{2\pi}\right)^2\left[1+\delta\varepsilon_t-\left(1-\frac{\varepsilon_1}{2}\right)\left(\frac{H(\eta)}{H(\eta_*)}\right)^2\right], \tag{17}$$

where $\delta\varepsilon_t = 2\varepsilon_1(1-\log 2-\gamma)$. Now it is clear that the subtraction term is suppressed by $(H(N)/H(N_*))^2 = \exp[-2\int_{N_*}^{N} \mathrm{d}N'\ \varepsilon_1(N')]$. Through the time evolution after the Hubble radius crossing, influence from the adiabatic regularization become negligibly small. This suppression is missed in the computations by L. Parker *et al* where the regime $H \ll H(\eta_*)$ has not been reached. Note that in the case of the exact Sitter space-time, the regularized amplitude for gravitational waves vanishes, too.

4 Summary

We have shown that even if we remove UV divergences in two-point correlation functions of metric perturbations during inflation using the adiabatic regularization, this does not leave any detectable imprints on the final power spectra for both curvature perturbations and gravitational waves after the end of inflation, or even during it but when H becomes much less than $H(\eta_*)$. This happens because any consistent UV renormalization modifies only local quantities (small scale inhomogeneities), ensuring that this regularization may be justified by introducing generally covariant counter terms constituted from local quantities.

Note that subtraction terms have to be introduced at all times, not during inflation only. Here our considerations differ somewhat from the recent paper [10] which otherwise arrives to the same conclusion regarding the papers [1–4]. It is also important that regularization of UV modes should be performed prior to the transition from quantum to classical description of fluctuations. While the bare quantum amplitudes given by the first terms of Eqs. (8) and (9) become indistinguishable from classical stochastic fluctuations after the first Hubble radius crossing, their UV regulators given by the second terms remain quantum up to the present time. This requires us to perform the adiabatic regularization before the consideration of quantum-to-classical transition for metric fluctuations.

In this paper, we have only considered regularization of the UV divergence which appears in the coincidence limit of two-point correlation functions in the one-loop approximation. However, a UV divergence can appear also from higher loop corrections. We leave regularization of these corrections for further study.

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