

STABILITY OF QUANTUM ELECTRODYNAMICS
WITH NON-RELATIVISTIC MATTER
AND MAGNETIC LIEB-THIRRING ESTIMATES

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Abstract

The subject of this thesis is the investigation of the stability of a system composed of an arbitrary number of non-relativistic electrons and an arbitrary number of static nuclei interacting through Coulomb forces, where in addition the electrons are coupled to the photon field, or more precisely, to the quantized ultraviolet-cutoff electromagnetic field. Stability means that the energy per particle in the system is bounded below uniformly in the number of particles (and in the nuclear configurations). For this system of matter and radiation we prove two results. The first one asserts that stability holds if the values of the fine structure constant, α , and of the product of its square with the largest nuclear charge, Z , are not too large. These constraints are of the same kind as the ones occurring in a system of nuclei and electrons coupled to a classical external field, which are caused first of all by the fact that the Pauli operator has zero-modes. The second result establishes stability for every value of α and $Z\alpha^2$, showing in this way that actually stability can no longer manifest, once an ultraviolet cutoff is imposed on the quantized electromagnetic field. Both results are presented in Section I.7.

The proofs of these two results rest on two magnetic Lieb-Thirring type inequalities. By this we mean a lower bound on the sum of the negative eigenvalues of a Pauli operator, that describes the motion of one electron in an external electromagnetic field, including also the interaction of its magnetic moment (i.e., spin) with the external magnetic field. The original inequality stems from Elliot H. Lieb and Walter Thirring, who 1975 derived such an estimate for the Schrödinger operator describing the motion of a non-relativistic spinless electron in an external potential. The proofs of our two Lieb-Thirring inequalities require the use of effective magnetic fields (i.e., smeared magnetic fields) in order to account for the semilocal cooperation between the physical magnetic field and the potential in creating new bound states, and to them are devoted the second and third chapter.

Zusammenfassung

Gegenstand dieser Dissertation ist das Studium der Stabilität eines Systems, das aus einer beliebigen Anzahl nicht-relativistischer Elektronen und einer beliebigen Anzahl statischer Kerne besteht, die durch Coulomb-Kräfte wechselwirken, wobei die Elektronen noch an das Photonenfeld gekoppelt sind, oder genauer, an das quantisierte elektromagnetische Feld mit Ultraviolet-Cutoff. Stabilität bedeutet, dass die Energie pro Teilchen im System nach unten beschränkt ist, und zwar gleichmässig in der Anzahl der Teilchen (und in den Kernkonfigurationen). Für dieses System von Materie und Strahlung beweisen wir zwei Resultate. Das erste besagt, dass Stabilität gilt, falls die Werte der Feinstrukturkonstante α und des Produkts deren Quadrats mit der höchsten Kernladung Z nicht zu gross sind. Diese Einschränkungen sind von derselben Art wie sie in einem System von Kernen und Elektronen, die an ein äusseres klassisches Feld gekoppelt sind, vorkommen, und die vor allem von der Tatsache herführen, dass der Pauli Operator Nullmoden besitzt. Das zweite Resultat liefert Stabilität für jeden Wert von α und $Z\alpha^2$, und zeigt damit, dass eigentlich Stabilität nicht mehr zum Vorschein kommen kann, wenn wir einen Ultraviolet-Cutoff am quantisierten elektromagnetischen Feld einführen. Beide Resultate werden in Abschnitt I.7 vorgezeigt.

Die Beweise dieser beiden Resultate beruhen auf zwei magnetischen Lieb-Thirring Ungleichungen. Damit meinen wir eine untere Schranke für die Summe der negativen Eigenwerte eines Pauli Operators, der die Bewegung eines Elektrons in einem äusseren elektromagnetischen Feld mitsamt der Wechselwirkung von seinem magnetischen Moment (d.h. Spin) mit dem äusseren Magnetfeld beschreibt. Die ursprüngliche Ungleichung stammt von Elliot H. Lieb und Walter Thirring, die 1975 eine analoge Abschätzung für den Schrödinger Operator hergeleitet haben, der die Bewegung eines nicht-relativistischen Elektrons ohne Spin in einem äusseren Potential beschreibt. Die Beweise unserer beiden Lieb-Thirring Ungleichungen fordern die Benutzung von effektiven magnetischen Feldern (d.h. ausgeschmierten magnetischen Feldern), um die halblokale Kooperation zwischen physikalischen Magnetfeld und Potential bei der Erzeugung von neuen gebundenen Zuständen zu berücksichtigen. Ihnen sind das zweite und dritte Kapitel gewidmet.

I. Stability of matter

1. The constitution of matter

The microscopical structure of matter is a topic man has been interested in since the ancient times of the Greek philosophers, among whom Leucippos and Democritos (5th-4th century B.C.) speculated that substances consist of small particles, which they called atoms (*ἀτομος*=indivisible), that cannot be split further. But for a detailed analysis of the structure of matter we had to wait for a long time, until the scientific method, first introduced by Galileo in the 17th century for the study of falling bodies, could be applied to the problem of the constitution of matter. Necessary for that was the industrial revolution that put at scientists' disposal the relevant technology they needed to look for the basic microscopic constituents of matter. It was only around 1805 that Dalton recognized that the atomic theory was compatible with experimental data, while the first evidence for subatomic particles was given 1897 by J.J. Thomson, who could discover the existence of particles with negative charge, whose mass was about 1/1800 of that of hydrogen, that were emitted by all substances he tested by putting them in a strong electric field. Thomson postulated that atoms were made up of these new particles, called electrons, surrounded by a continuous distribution of positive charge. This model turned out to be false when 1911 Rutherford, interpreting the data of his students Geiger and Marsden on the scattering of α -particles (discovered meanwhile by the Curies as radioactive radiation emitted by certain substances) off a thin gold plate, deduced from the presence of a significant part of large scattering angles that inside an atom the positive charge must be concentrated in a heavy particle (the so-called nucleus), which carries almost all the mass of the atom and occupies a very small fraction of its total volume. Electrons, attracted by the nucleus with a r^{-2} -force (Coulomb force), orbit like planets around a nuclear sun, forming an electric analogous of the solar system, whose laws were revealed by Kepler and Newton in the 18th century. Calculations yielded 10^{-8} cm for the radius of an atom and 10^{-13} cm for that of a nucleus, which means that given the fact that an electron can be taken to be pointlike, that is dimensionless, the atoms are prevalently empty.

But also this model, proposed by Rutherford in order to explain his scattering experiment, possessed some failures. Indeed, it could not explain why spectra emitted by atoms always exhibit a discrete structure, which for hydrogen for instance reads

$$\hbar\omega = Ry \left(\frac{1}{n^2} - \frac{1}{m^2} \right) , \quad (1.1)$$

where n and m are integers, Ry = Rydberg constant $\cong 13,6$ eV and \hbar = Planck's constant $\cong 6,58 \cdot 10^{-16}$ eV·s. Nor could it give account of the stability of atoms, that is, of the fact that electrons eventually do not crash onto the nucleus, nor of the reason why atoms are that big (while atoms themselves can be tightly packed together into molecules). Indeed, if we consider a simple atom, made up of one nucleus and one electron (a so-called hydrogenic atom), we know that according to Rutherford's model the electron can occupy orbits with

very negative energies, i.e., indefinitely close to the nucleus. Therefore, the collision of two such atoms could push an electron, revolving on an elliptical trajectory maybe at a large distance ($\sim 10^{-8}$ cm) from its nucleus, very close to it, in such a way that the atom itself would no longer be recognizable as an object with the extension of 1 Å. Every atom would be an (almost) infinite source of energy that could be transmitted to other atoms or to the electromagnetic radiation. Furthermore, we know that an accelerated particle emits electromagnetic radiation and, in this way, loses energy. This mechanism, that applies to this “electrical solar system model”, would cause the electron to move on ever-decreasing orbits and eventually to fall onto the nucleus, where only huge forces could then separate them from each other. Matter would tend to shrink into nothing or to diminish indefinitely in size.

A first tentative answer to these problems came 1913 from Bohr. His postulates state, roughly speaking, that among the classically allowed electronic trajectories those are selected that satisfy some orbital momentum quantization condition. These lucky trajectories are not permitted to radiate, while in the transition from one orbit with energy E_1 to another orbit with energy E_2 light is emitted or absorbed with frequency

$$\omega = \frac{|E_1 - E_2|}{\hbar}.$$

On one hand, this theory evades the problem of stability, on the other hand it reproduces exactly the experimental structure of the hydrogen spectrum, namely formula (1.1), and yields as minimal distance between nucleus and electron the so-called Bohr radius $a = 0,529 \cdot 10^{-8}$ cm, that would also explain the astonishing size of atoms.

A more detailed, adequate and satisfying description of atoms and molecules could eventually be found in quantum mechanics by Heisenberg (1925) and Schrödinger (1926). Nevertheless, also in this theory, which is an operator-theory with a probabilistic interpretation and where atoms and molecules held together by Coulomb forces have been shown to fit in mathematically (but only 1951, by Kato [40], who showed the self-adjointness of their Hamilton operators), the only case for which we have an exact solution of the problem of stability is hydrogen, that can be shown to possess a ground state with ground state energy $E_0 = -Ry$. For all the other atoms and molecules we have at our disposal another result of Kato [41], connected with the previous one, asserting that the energy of these quantum systems is bounded from below, i.e., they do not collapse and are in this sense stable.

Nevertheless, we would like to know more than that, namely whether the energy per particle present in the system is finite or not, i.e., we would like to find a lower bound for the total energy of the system that is linear in the total number of particles, and this is the topic of this introductory chapter and of this thesis. The importance for the energy of a system of N particles to be proportional to the particle number N can be seen for instance as follows. Suppose we have two half full glasses of wine. When they are far apart, their ground state energy is approximately $2E(N)$, where $E(N)$ is the energy of a closed system of N “wine particles”. If we now pour the content of one glass into the other one, the new

ground state energy is $E(2N)$. We denote by $\Delta E = 2E(N) - E(2N)$ the energy released in this process of putting the two wine glasses together. If the energy is proportional to N , then $\Delta E = o(N)$; if, on the other hand, $E(N) \propto -N^p$, $p > 1$, then $\Delta E = -(2^p - 2)E(N)$, and the energy released would be of the order of the energy contained in each glass of wine, thus priming a very violent explosion [44].

The plan of the thesis is as follows. In Section 2 we introduce the quantum mechanics of electrons and nuclei interacting through Coulomb forces and make the statement of stability of matter precise. In Section 3 we illustrate the basic quantum mechanical features that produce stability and explain some ingredients occurring in the first proofs of stability of matter (among which the Lieb-Thirring inequality), that we will also use in order to handle with the case of stability of matter in magnetic fields, whose rudiments are proposed in Section 5. Our classical stability results, together with two new Lieb-Thirring type estimates, can be found in Section 6, while in Section 7 we present our results on stability of non-relativistic matter coupled to the quantized ultraviolet-cutoff electromagnetic field (a topic which is also introduced in that section) and their derivation from our classical results from Section 6. Section 4 contains some general remarks on stability of matter, its limits and its importance. This is the content of the first chapter, while the following two are devoted to the proofs of our Lieb-Thirring type inequalities and of the classical stability results of Section 6.

2. Stability of matter

From the brief and sketchy history of atomic physics outlined above we have learned that negatively charged electrons and positively charged nuclei are fundamental constituents of matter, which is held together by Coulomb forces. In bulk matter their number can be very large, and the effects of Coulomb forces can be manifold and subtle: they are namely responsible for chemical binding (e.g. covalent and ionic bonds), metallic cohesion, Van der Waals forces, superconductivity and superfluidity, i.e., for many properties of atomic, molecular and condensed matter physics, and, in a broad sense, for biology. To fix things we consider an arbitrary number, N , of electrons of charge e , mass m , spin $1/2$ and positions x_i ($i = 1, \dots, N$) interacting with an arbitrary number, K , of nuclei with charges $-Z_j e$, masses M_j and positions R_j ($j = 1, \dots, K$). Electrons are fermions, while we don't fix the nature of nuclei. If both electrons and nuclei are taken to be non-relativistic (we can then forget about retardation and relativistic effects), their quantum mechanical Hamiltonian reads

$$H' = - \sum_{i=1}^N \frac{\hbar^2}{2m} \Delta_{x_i} - \sum_{j=1}^K \frac{\hbar^2}{2M_j} \Delta_{R_j} + V'_C, \\ V'_C = \sum_{\substack{i,j=1 \\ i < j}}^N \frac{e^2}{|x_i - x_j|} - \sum_{i,k=1}^{N,K} \frac{Z_k e^2}{|x_i - R_k|} + \sum_{\substack{k,l=1 \\ k < l}}^K \frac{Z_k Z_l e^2}{|R_k - R_l|}, \quad (2.1)$$

where $-\hbar^2 \Delta$ in Fourier space has the familiar “classical” form p^2 . The Hilbert space of

one electron is $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, where \mathbb{C}^2 accounts for its spin. For N electrons we have to give account of their fermionic nature by imposing the Pauli principle, that is by antisymmetrizing the wave functions: $\mathcal{H}_{el} = \wedge^N \mathcal{H} = N$ -fold antisymmetric tensor product of \mathcal{H} . We write the Hilbert space of nuclei generically as \mathcal{H}_{nuc} (it depends on the spin and statistics of the nuclei we don't want, and don't need, to specify). The total Hilbert space of the system is then $\mathcal{H}_{el} \otimes \mathcal{H}_{nuc}$. Since the Laplacian $-\Delta$ is a self-adjoint operator on $D(-\Delta) = H^{2,2}(\mathbb{R}^3)$, the Sobolev space of functions in $L^2(\mathbb{R}^3)$ with first and second weak derivatives also in $L^2(\mathbb{R}^3)$, and $V(x) = \frac{1}{|x|}$ belongs to $L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ (decompose $V(x)$ as $V(x)\chi_{B_R(0)}(x) + V(x)\chi_{\mathbb{R}^3 \setminus B_R(0)}(x)$, where χ_S is the characteristic function of the set S , and $B_R(0)$ is a ball of arbitrary radius $R < \infty$ around the origin), this Hamiltonian is self-adjoint on $H^{2,2}(\mathbb{R}^3)$ and semibounded ($H' \geq \text{const} > -\infty$) (Kato [40, 41]). In this model we neglect gravitational forces (they become important only for objects of the size of a star, due to their weakness as compared to electronic interactions), nuclear forces (they act between the hadrons that make up the nuclei and don't affect the stability of the atoms), and magnetic dipole interactions, since experimentally they all give only small corrections to the binding energies of atoms and molecules. The question we now ask is whether this system is stable, i.e., if the ground state energy per particle remains bounded, or, equivalently, if there exists a finite positive constant C (called henceforth the stability constant) depending at most on $\max\{Z_j | 1 \leq j \leq K\}$, and with the dimension of an energy, such that

$$H' \geq -C \cdot (N + K).$$

This type of stability is called H -stability, to distinguish it from thermodynamic stability (see Section 4.a)).

Classically, a necessary condition for stability would be $V'_C(x_1, \dots, x_N, R_1, \dots, R_K) \geq -C \cdot (N + K)$. But this is evidently false, unless we don't impose hard cores on the particles, i.e., unless V'_C also contains a term which is $+\infty$ if $|s_i - s_j| < R$, for $s_i \neq s_j \in \{x_1, \dots, x_N, R_1, \dots, R_K\}$. In this case we can think the charge of each particle to be distributed in any spherically symmetric way within a ball of radius $\frac{1}{2}R$ centered at the position of that particle, because, due to the fact that the particles cannot approach each other any closer than a distance R , the Coulomb interactions between them will remain the same. Electrostatics now tells us that

$$V'_C = \frac{1}{8\pi} \int_{\mathbb{R}^3} E(x)^2 d^3x - \sum_{i=1}^{N+K} (\text{self-energy of the } i\text{-th particle}) \geq -C \cdot (N + K),$$

where $E(x)$ is the electrostatic field generated by all the particles and C is the maximum self-energy of any of the balls. Note that classical stability implies quantum mechanical stability, but not conversely.

What prevents quantum mechanically the energy from growing arbitrarily negative? Before answering this question, we want to make some general considerations about the problem and to simplify notations.

a) As already mentioned, the electron mass is only a small fraction of that of a nucleus: the mass M_p of a proton, the smallest possible nucleus, is about 1800 times that of an electron. We can therefore consider the nuclei to be fixed (i.e., $M_p = \infty$), although their position will be eventually determined by the requirement that the total energy of the electrons-nuclei system be minimal. This means that we can neglect the kinetic energy of the nuclei, supported in this operation by the fact that their kinetic energy operator, $-\sum_{j=1}^K \frac{\hbar^2}{2M_j} \Delta_{R_j}$, is strictly positive: in order to find a lower bound for H' we can restrict ourselves to the study of the smaller Hamiltonian

$$-\sum_{i=1}^N \frac{\hbar^2}{2m} \Delta_{x_i} + V'_C$$

acting on \mathcal{H}_{el} , and look for a linear lower bound which is uniform in the positions R_j of the nuclei. We could argue that we may now further simplify the Hamiltonian by dropping the last term in V'_C , that is the mutual nuclear repulsion, which is a constant positive term, but it turns out that without this term stability does not hold, and furthermore we want to study the R_j -dependence of stability.

b) Let us take $0 < \eta < \infty$ and scale the positions of the particles as:

$$x_i = \eta y_i, \quad i = 1, \dots, N; \quad R_j = \eta Q_j, \quad j = 1, \dots, K.$$

By choosing the parameter η to have the dimension of a length, we can express our Hamiltonian as a function of the new dimensionless variables y_i and Q_j :

$$-\sum_{i=1}^N \frac{\hbar^2}{2m} \Delta_{x_i} + V'_C = \frac{\hbar^2}{2m\eta^2} \left(-\sum_{i=1}^N \Delta_{y_i} + \frac{2me^2\eta}{\hbar^2} V_C \right),$$

with

$$V_C = V_C(y_1, \dots, y_N, Q_1, \dots, Q_K) = \sum_{\substack{i,j=1 \\ i < j}}^N \frac{1}{|y_i - y_j|} - \sum_{i,k=1}^{N,K} \frac{Z_k}{|y_i - Q_k|} + \sum_{\substack{k,l=1 \\ k < l}}^K \frac{Z_k Z_l}{|Q_k - Q_l|}.$$

We choose $\eta = \hbar^2(2me^2)^{-1} = a/2 = (\text{Bohr radius})/2$ to be our unit length. If we then define $\hbar^2(2m\eta^2)^{-1} = 2me^4\hbar^{-2} = 4 \text{ Ry}$ to be our energy unit, we end up with the Hamilton operator

$$-\sum_{i=1}^N \Delta_{x_i} + V_C$$

after renaming the variables $y_i \rightarrow x_i$ and $Q_j \rightarrow R_j$.

c) Furthermore, we can set in V_C all nuclear charges Z_j equal to $Z = \max\{Z_j | j = 1, \dots, K\}$. This is no restriction, because the energy is concave in each Z_j and so when stability holds for $Z_j = Z$ ($j = 1, \dots, K$), it also holds for $\{0 \leq Z_j \leq Z\}_{j=1}^K$ [16].

Resuming, the Hamiltonian whose stability we investigate is

$$H = - \sum_{i=1}^N \Delta_{x_i} + V_C ,$$

$$V_C = \sum_{\substack{i,j=1 \\ i < j}}^N \frac{1}{|x_i - x_j|} - \sum_{i,k=1}^{N,K} \frac{Z}{|x_i - R_k|} + \sum_{\substack{k,l=1 \\ k < l}}^K \frac{Z^2}{|R_k - R_l|} , \quad (2.2)$$

acting on \mathcal{H}_{el} .

3. The pioneers

Among many other things, the ferment of the year 1968 produced also the first proof of stability of matter thanks to Dyson and Lenard [18, 19]. Their proof, which is lengthy and complicated and yields a stability constant of the order of 10^{14} Ry because of the successive use of many inequalities, begins by decomposing the physical space into cells, each containing one negatively charged particle, but an arbitrary number of positively charged particles. Then they show stability for each cell and finally reassemble the fragments and obtain the stability for the whole space and the whole system. The kernel of this proof is an estimate of the binding energy of an electron in a periodic Coulomb potential:

$$\int |\nabla \psi(x)|^2 d^3x - \int R(x)|\psi(x)|^2 d^3x > -4 \int |\psi(x)|^2 d^3x ,$$

where $R(x)$ is the Coulomb potential generated by a unit positive charge at each vertex of a cubic lattice with arbitrary lattice spacing L , with a constant negative background density to preserve neutrality. Though the appearance of this periodicity in the potential, which Dyson and Lenard conjectured not to be accidental (since “after all, the ground states of most forms of matter are crystals in which electrons are actually moving in periodic Coulomb potentials” [19, p.711]), gets lost in the following proofs, (despite their assertion that “the essence of a proof of stability of matter should be a demonstration that an aperiodic arrangement of particles cannot give greater binding than a periodic arrangement” [ibid.]), this estimate, representing a sort of uncertainty principle, a typical quantum mechanical feature, is a key point for stability, together with other fundamental aspects, and will be found in all the following proofs, maybe disguised under different appearances: it imposes a lower bound on the binding energy of an electron in the potential $-R(x)$ generated by the nuclei, or it tells us that there is a balance between quantum mechanical kinetic and potential energy according to which when the potential energy becomes more negative, that is when the electron tries to come closer to the nuclei, the kinetic energy must grow correspondingly to this decrease in potential energy in order for the sum of them to remain above that given limit. To illustrate this idea in a better way

we can apply another form of the uncertainty principle [59], known as Hardy's inequality, to an hydrogenic atom:

$$\int |\nabla \psi(x)|^2 d^3x \geq \frac{1}{4} \int \frac{|\psi(x)|^2}{|x|^2} d^3x, \quad \psi \in C_0^\infty(\mathbb{R}^3).$$

The energy of a nucleus of charge Z sitting in the origin and attracting an electron is then (take (2.2) with $N = 1$ and $K = 1$)

$$\begin{aligned} (\psi, H\psi) &= \int |\nabla \psi(x)|^2 d^3x - Z \int \frac{|\psi(x)|^2}{|x|} d^3x \\ &\geq \int |\psi(x)|^2 \left(\frac{1}{4|x|^2} - \frac{Z}{|x|} \right) d^3x \geq -Z^2 = -4Z^2 \text{ Ry}, \end{aligned} \quad (3.1)$$

since the expression in brackets has a minimum value $-Z^2$ and the wave function ψ (for simplicity we neglect spin) has unit L^2 -norm $\|\psi\|_2 = 1$ (by $\|\cdot\|_p$ we will always denote L^p -norms).

Another way to put the fact that we cannot compress a wave function without letting the kinetic energy increase, is provided by Sobolev's inequality [69]:

$$\int |\nabla \psi(x)|^2 d^3x \geq K_S \|\psi\|_6^2, \quad K_S = 3 \left(\frac{\pi}{2} \right)^{4/3}. \quad (3.2)$$

Setting $\rho_\psi(x) = |\psi(x)|^2$, the energy (3.1) is

$$\begin{aligned} (\psi, H\psi) &\geq K_S \left(\int \rho_\psi(x)^3 d^3x \right)^{1/3} - Z \int \frac{\rho_\psi(x)}{|x|} d^3x \equiv h(\rho_\psi) \\ &\geq \min\{h(\rho) : \rho(x) \geq 0, \int \rho = 1\} \geq -\frac{4}{3} Z^2 \text{ Ry}. \end{aligned}$$

A weaker form of Sobolev's inequality, which also yields a link to the proof of stability of matter by Lieb and Thirring [53], can be obtained from (3.2): if we apply Hölder's inequality to its right side,

$$\|\psi\|_6^2 \cdot 1 = \left(\int \rho_\psi(x)^3 d^3x \right)^{1/3} \left(\int \rho_\psi(x) d^3x \right)^{2/3},$$

we have

$$\int |\nabla \psi(x)|^2 d^3x \geq K_S \int \rho_\psi(x)^{5/3} d^3x. \quad (3.3)$$

(Note besides that K_S is here not the best constant, although it is the best constant in Sobolev's inequality (3.2).) The minimization of the energy (3.1) can be carried out also using this last estimate and we get only a slightly worse constant in the result. All these stability constants have to be compared with the exact result $-Z^2 \text{ Ry}$.

Lieb and Thirring succeeded 1975 in extending this kind of uncertainty principle to many electrons by means of their well-known estimate:

Theorem 1.1. [53, 55] *Let $-e_i < 0$ be the negative eigenvalues of the one-particle Schrödinger Hamiltonian $h = -\Delta - V$, acting on $L^2(\mathbb{R}^3)$, where $V(x) \geq 0$ is a multiplication operator. Then*

$$\sum e_i \leq \frac{4}{15\pi} \int V(x)^{5/2} d^3x . \quad (3.4)$$

Remark. If V is alternating, then (3.4) holds true if we substitute V with its positive part $V_+ \equiv \max(V, 0)$.

Proof. We express the sum of the negative eigenvalues of h as a function of $N_E(V)$, the number of such eigenvalues that are smaller than $-E$, $E > 0$:

$$\sum e_i = - \int_0^\infty E dN_E(V) = \int_0^\infty N_E(V) dE . \quad (3.5)$$

The Birman-Schwinger principle [63] states that $N_E(V)$ is smaller than the number of eigenvalues of

$$K_E(V) \equiv V^{1/2}(-\Delta + E)^{-1}V^{1/2}$$

that are bigger than 1. This comes from the fact that the Birman-Schwinger kernel $K_E(V)$, which is a compact positive semidefinite operator on $L^2(\mathbb{R}^3)$ for $E > 0$ and is monotone increasing in $-E$, has an eigenvalue 1 when E is an eigenvalue of h :

$$h\psi = E\psi \iff K_E\varphi = \varphi , \quad \varphi = V^{1/2}\psi .$$

If we further notice that

$$N_E(V) = N_{\frac{E}{2}}(V - \frac{E}{2}) \leq N_{\frac{E}{2}}((V - \frac{E}{2})_+) \leq \text{tr}[K_{\frac{E}{2}}((V - \frac{E}{2})_+)]^2 , \quad (3.6)$$

(the subscript + again means the positive part), then we have, by a change of variables,

$$\sum e_i \leq 2 \int_0^\infty dE \int v(x)v(y)\check{k}(x-y)^2 d^3x d^3y , \quad (3.7)$$

with $v = (V - E)_+$ and $k(p) = (2\pi)^{-3/2}(p^2 + E)^{-1}$. The Fourier transforms are defined by

$$\check{k}(x) = (2\pi)^{-3/2} \int k(p)e^{ipx} d^3p \quad \text{and} \quad \hat{k}(p) = (2\pi)^{-3/2} \int k(x)e^{-ipx} d^3x .$$

Rewriting the spatial integrations in (3.7) as the L^2 -scalar product of v and the convolution of $(\check{k})^2$ with v , and using Parseval, we come to

$$(v, (\check{k})^2 * v) = (\hat{v}, [(\check{k})^2 * v]^\wedge) = (\hat{v}, (2\pi)^{3/2}[(\check{k})^2]^\wedge \hat{v}) = (\hat{v}, (k * k)\hat{v}) ,$$

where the convolution of k with itself, $(k * k)(p)$, can be bounded from above by first applying Cauchy-Schwarz and then noticing that the maximum of the resulting expression occurs at $p = 0$:

$$(k * k)(p) \leq (2\pi)^{-3} \int d^3q (q^2 + E)^{-2} = (8\pi E^{1/2})^{-1}. \quad (3.8)$$

Using Parseval again, together with Fubini, we have the desired result:

$$\sum e_i \leq \frac{1}{4\pi} \int_0^\infty dE E^{-1/2} \int d^3x (V(x) - E)_+^2 = \frac{4}{15\pi} \int V(x)^{5/2} d^3x. \quad (3.9)$$

■

With the help of their inequality, Lieb and Thirring attained their aim of generalizing (3.3) to many fermions. The l.h.s. of (3.3) represents the kinetic energy of one particle, and in the N -particle case it must be substituted by

$$T_\psi = N \sum_{s_i=1}^2 \int |\nabla_{x_1} \psi(x_1, \dots, x_N; s_1, \dots, s_N)|^2 d^3x_1 \dots d^3x_N,$$

for $\psi \in \mathcal{H}_{el}$ (we now write spin explicitly). The new r.h.s. can be found by studying the fermionic ground state energy E_0 of the N -particle Hamiltonian $H_N = \sum_{i=1}^N h_i$, $h_i = -\Delta_{x_i} - V(x_i)$, acting on \mathcal{H}_{el} , with $V(x) = (\frac{3\pi}{4})^{2/3} \rho_\psi(x)^{2/3}$, where now

$$\rho_\psi(x) = N \sum_{s_i=1}^2 \int |\psi(x, x_2, \dots, x_N; s_1, \dots, s_N)|^2 d^3x_2 \dots d^3x_N$$

is the single particle density. On one hand, E_0 is bounded below by the sum of the negative eigenvalues $-e_i$ of H_N (that represents the energy of the non-interacting Fermi gas in the external potential V), for which they already had at their disposal a lower bound given by (3.4):

$$E_0 \geq -2 \sum e_i \geq -\frac{8}{15\pi} \left(\frac{3\pi}{4}\right)^{5/3} \int \rho_\psi(x)^{5/3} d^3x.$$

(The factor 2 comes from the inclusion of spin: each eigenvalue can be “occupied” by two electrons.) On the other hand, by the variational principle,

$$E_0 \leq (\psi, H_N \psi) = T_\psi - \left(\frac{3\pi}{4}\right)^{2/3} \int \rho_\psi(x)^{5/3} d^3x.$$

From these two bounds for E_0 they obtained

$$T_\psi \geq \frac{3}{5} \left(\frac{3\pi}{4}\right)^{2/3} \int \rho_\psi(x)^{5/3} d^3x, \quad (3.10)$$

that signifies the uncertainty and Pauli principles and that can be used to make a comparison between the quantum mechanical energy $(\psi, H\psi)$ and the Thomas-Fermi energy functional $\mathcal{E}_{TF}(\rho_\psi)$, where the kinetic energy term has the same form as the r.h.s. of (3.10), apart from the constant in front of it, which is smaller. Stability of matter then immediately follows from Teller's no-binding theorem [66, 43] (atoms do not bind in Thomas-Fermi theory), and the stability constant is of the right order of magnitude, namely 23 Ry.

We would like to mention here a nice inequality for the Coulomb potential, that Lieb and Yau [56] derived 1988 in order to prove relativistic stability of matter, for which the use of Thomas-Fermi theory is not appropriate (although a modified Thomas-Fermi theory, Thomas-Fermi-von Weizsäcker, can be used [48]). This reduction of the Coulomb potential, when combined with the Lieb-Thirring estimate, can also be exploited to prove stability of non-relativistic matter, a method we will profit by later in our proofs for stability with magnetic fields. We first introduce for each nucleus $j = 1, \dots, K$ the nearest neighbour, or Voronoi, cell $\Gamma_j = \{x \mid |x - R_j| \leq |x - R_k| \text{ for } k = 1, \dots, K\}$ and the smallest distance, D_j , of the j th nucleus to the boundary of its Voronoi cell Γ_j (consisting, by the way, of a finite number of planes): $D_j = \min\{|R_j - R_k| \mid j \neq k\}/2$.

Theorem I.2. [56] *For any $0 < \lambda < 1$ the Coulomb potential given in (2.2) satisfies*

$$V_C(x_1, \dots, x_N, R_1, \dots, R_K) \geq - \sum_{i=1}^N W(x_i) + \frac{Z^2}{8} \sum_{j=1}^K D_j^{-1}, \quad (3.11)$$

and, for x in the Voronoi cell Γ_j , $W(x) = W_j^\lambda(x) = Z|x - R_j|^{-1} + F_j^\lambda(x)$ with

$$F_j^\lambda(x) = \begin{cases} (2D_j)^{-1}(1 - D_j^{-2}|x - R_j|^2)^{-1} & \text{for } |x - R_j| \leq \lambda D_j \\ (\sqrt{2Z} + 1/2)|x - R_j|^{-1} & \text{for } |x - R_j| > \lambda D_j. \end{cases}$$

The content of this theorem (that reduces the number of terms in V_C from $O(N^2)$ to N) is that the Coulombic energy can be lowered if we neglect the electronic repulsion and keep only the electronic attractions to the nearest nucleus plus a small error term, F_j^λ . This is a manifestation of screening: locally each electron feels only the attraction to its nearest nucleus, while interactions with far off parts of the system almost mutually cancel. In this procedure up to a quarter of the nearest neighbour nuclear repulsions can also be kept (last term in (3.11)), and exactly this term will be responsible for stabilizing the system in the magnetic case. Here, however, it is superfluous, if we don't appeal to Thomas-Fermi theory, as we now show. First we partition \mathbb{R}^3 into Voronoi cells and reduce the Coulomb potential according to Theorem I.2. We choose a value for λ , say $\lambda = 8/9$, and notice that for $x \in \Gamma_j$ we have

$$W(x) \leq [Z + \max(\lambda(1 - \lambda^2)^{-1}/2, \sqrt{2Z} + 1/2)]|x - R_j|^{-1} \leq Q|x - R_j|^{-1}, \quad (3.12)$$

where $Q = Z + \sqrt{2Z} + 2.2$. For any $\nu > 0$ we have then

$$H \geq \sum_{i=1}^N h_i - \nu N + \frac{Z^2}{8} \sum_{j=1}^K D_j^{-1},$$

where $h = -\Delta - (W - \nu)_+$: we exploit the fact that $W \leq (W - \nu)_+ + \nu$ in order to deal with an operator, $(W - \nu)_+$, which is in $L^{5/2}(\mathbb{R}^3)$. In this way we cut off the long-range part of $W \sim |x|^{-1}$ and we must thereby just pay the acceptable price of a term which is linear in the number N of electrons. Application of (3.4) thus yields

$$\sum_{i=1}^N h_i \gtrsim - \int (W(x) - \nu)_+^{5/2} d^3x \gtrsim -Q^3 \nu^{-1/2} K = \text{const } Q^2 K \quad (3.13)$$

for $\nu = Q^2 \lesssim (Z+1)^2$. Here and in the following $X \lesssim Y$ means that there exists a constant C independent of the data such that $X \leq CY$. The above integral, unlike $\int W(x)^{5/2} d^3x$, is finite and has been evaluated decomposing \mathbb{R}^3 into the union of (disjoint) Voronoi cells and applying (3.12) in each cell. The result is

$$H \geq -C \cdot (Z+1)^2 (N+K).$$

Another proof based on an electrostatic inequality (asserting that Coulomb energies V_C are lowered as \mathbb{R}^3 is decomposed into simplices and the interaction is restricted to points belonging to the same simplex) has been given by Graf [36], while other proofs, based on different methods, have been given by Federbush [24] and Fefferman [29].

4. Applications, implications and annotations

a) The question of stability of matter, which has its own intrinsic meaning, has been raised also in relationship with the necessity of establishing a rigorous mathematical foundation for statistical mechanics [33]. Once we assume that the equilibrium properties of matter can be described by means of the canonical partition function $Z = \text{tr } e^{-\beta H}$, $\beta = (kT)^{-1}$, we still have to show that the resulting properties of matter (e.g. extensivity of the energy) are those postulated in thermodynamics. In particular, we have to prove the existence of the thermodynamic, or bulk, limit for the Helmholtz free energy derived from the partition function, and, when it is established, we still have to investigate its properties, hoping that it possesses the appropriate convexity, i.e., stability (thermodynamic stability). Stability here means non-negative specific heat and compressibility. More specifically, the free energy per unit volume of a neutral system of N_j charged particles contained in the domain Ω_j and interacting through Coulomb potentials is $F_j = -\beta^{-1}|\Omega_j|^{-1} \log Z(\beta, N_j, \Omega_j)$. We choose increasing sequences $(N_j)_{j=1}^\infty$ and $(\Omega_j)_{j=1}^\infty$ with $\lim_{j \rightarrow \infty} N_j/|\Omega_j| = \rho$, where ρ is the density in the thermodynamic limit. Of course, the Hamiltonian appearing in the partition function is now the total Hamiltonian (2.1), comprehensive of the nuclear kinetic energy, since we are interested in the study of the complete system. Lieb and Lebowitz [46] proved the existence of $\lim_{j \rightarrow \infty} F_j = F(\rho, \beta)$, its independence from the choice of the particular sequences and its convexity in ρ resp. concavity in β^{-1} , that is

$$\text{specific heat} = -\beta^2 \frac{\partial^2 \beta F(\rho, \beta)}{\partial \beta^2} \geq 0$$

and

$$(\text{compressibility})^{-1} = \frac{\partial (\text{pressure})}{\partial \rho} = \rho \frac{\partial^2 F(\rho, \beta)}{\partial \rho^2} \geq 0.$$

A basic ingredient in this proof is H -stability of matter, which is valid also for the total Hamiltonian, as noted above in Section 2. It allows easily to find a uniform lower bound on F and is namely a necessary condition for the existence of the thermodynamic limit, although it is not sufficient: H -stability solves the problem of the short range behavior (singularity) of the Coulomb potential (heuristically, it solves the problem of collapse), but it remains the difficulty of the long range behavior, i.e., the problem of explosion. Thermodynamic stability doesn't hold, for instance, for a collection of N particles all of the same sign, though H -stability is trivial, since the energy of such a system is positive and hence bigger than $-C \cdot (N + K)$.

In addition to the thermodynamic limit for the free energy density we can also consider, with the same notations as before, the limit $j \rightarrow \infty$ of the ground state energy per unit volume $E_j = E(N_j, \Omega_j) = |\Omega_j|^{-1} \inf_{\psi} (\psi, H\psi) / (\psi, \psi)$. With the same methods as for the free energy F the existence of the limiting function $e(\rho) = \lim_{j \rightarrow \infty} E_j$ and its convexity in ρ can be shown [46].

As a final remark, we point out that it is possible to define and prove the thermodynamic limit, discussed here only for the canonical ensemble, also for the microcanonical and grandcanonical ensembles. Their equivalence has also been shown [46].

b) We reproduce in this subsection a proof by Lieb [43] that matter is bulky, i.e., that the radius of a system of M particles is at least of the order $M^{1/3}$. For simplicity our system will consist of $M = 2N$ particles, N electrons and N nuclei, e.g. protons, but we could take an arbitrary number K of nuclei with arbitrary nuclear charge Z_j , $j = 1, \dots, K$. The proof works for any wave function ψ and any nuclear configuration for which $E_{\psi} = (\psi, H\psi) \leq 0$ (by compressing matter we could shrink the size of the system, but at the cost of raising the energy). An important fact will be stability of matter, which here takes the form $E_{\psi} \geq -C \cdot N$. We begin by defining what we mean by the radius of the system:

$$R(p) := \left(\frac{1}{N} \left(\psi, \sum_{i=1}^N |x_i|^p \psi \right) \right)^{1/p}, \quad p \geq 0,$$

where the x_i ($i = 1, \dots, N$) are the positions of the electrons measured from an arbitrary origin. Then we split the energy into two parts:

$$0 \geq E_{\psi} = \frac{1}{2} T_{\psi} + (\psi, \tilde{H}\psi),$$

where \tilde{H} is (2.2) but with a factor $1/2$ multiplying the kinetic energy term. Since $(\psi, \tilde{H}\psi) \geq -2C \cdot N$ (\tilde{H} is unitarily equivalent to $2H$), we have, taking also into account (3.10) (Pauli principle!):

$$\int \rho_{\psi}(x)^{5/3} d^3x \lesssim T_{\psi} \lesssim N.$$

Next, we mention that for any $p \geq 0$ there is a constant $C_p > 0$ such that for any non-negative $\rho(x)$

$$\left(\int \rho(x)^{5/3} d^3x \right)^{p/2} \int |x|^p \rho(x) d^3x \geq C_p \left(\int \rho(x) d^3x \right)^{1 + \frac{5p}{6}}. \quad (4.1)$$

But since $\|\rho_\psi\|_1 = 1$, we obtain

$$\left(\psi, \sum_{i=1}^N |x_i|^p \psi \right) = \int |x|^p \rho_\psi(x) d^3x \gtrsim N \cdot N^{p/3},$$

which yields $R(p) \gtrsim N^{1/3}$ for each value of $p \geq 0$ and each choice of the origin of the coordinate system (this means that by an awkward choice of the coordinate system we can only make $R(p)$ larger, but never smaller than $N^{1/3}$). The result holds for instance in the center of mass frame, where $R(p)$ should really acquire the physical meaning of a typical mean radius for the system.

We come back to the proof of (4.1), namely of

$$C_p \|\rho\|_1^{1+5p/6} \leq \|\rho\|_{5/3}^{5p/6} \|\rho|x|^p\|_1. \quad (4.2)$$

We first notice that, for every $\varepsilon > 0$, $|x|^{-p}$ can be split as $|x|^{-p} = (|x|^{-p} - \varepsilon)_+ + \min(|x|^{-p}, \varepsilon)$, where the second term is bounded above by ε and the first one is in $L^{5/2}(\mathbb{R}^3)$ for $0 \leq p < 6/5$:

$$\|(|x|^{-p} - \varepsilon)_+\|_{5/2}^{5/2} = 4\pi \int_0^{\varepsilon^{-1/p}} dr r^2 (r^{-p} - \varepsilon)^{5/2} \leq C_p \varepsilon^{5/2-3/p}, \quad C_p > 0.$$

By Hölder's inequality,

$$\int |x|^{-p} \rho(x) d^3x \leq \varepsilon \|\rho\|_1 + C_p \varepsilon^{1-6/5p} \|\rho\|_{5/3} \lesssim \|\rho\|_1^{1-5p/6} \|\rho\|_{5/3}^{5p/6},$$

after optimization over ε . Thus

$$\begin{aligned} \|\rho\|_1 &\leq \|\rho^{1/2} |x|^{p/2}\|_2 \|\rho^{1/2} |x|^{-p/2}\|_2 \\ &\lesssim \|\rho|x|^p\|_1^{1/2} (\|\rho\|_1^{1-5p/6} \|\rho\|_{5/3}^{5p/6})^{1/2}. \end{aligned}$$

Taking the square we obtain (4.1) for $0 \leq p < 6/5$. Estimating the third norm in (4.2) by

$$\|\rho|x|^p\|_1 \leq \|\rho^{1/2}\|_2 \|\rho^{1/2} |x|^p\|_2 = \|\rho\|_1^{1/2} \|\rho|x|^{2p}\|_1^{1/2},$$

we get then by squaring an inequality that is again (4.2) but with p replaced by $2p$, thus extending its validity to all $p \geq 0$.

c) There is an interesting result of Fefferman asserting that, at suitable temperature and density, electrons and protons in a box $\Omega \subset \mathbb{R}^3$ interacting through Coulomb forces form a gas of hydrogen atoms or molecules, provided stability of matter holds true. We now reproduce the statement of this result in better detail. The Hamiltonian of N electrons and K protons ($Z_j = 1$ for all $j = 1, \dots, K$), using our notations (see (2.2)), is

$$H^\Omega = -c_1 \sum_{i=1}^N \Delta_{x_i} - c_2 \sum_{j=1}^K \Delta_{R_j} + V_C$$

and acts on wave functions $\psi \in \wedge^N L^2(\Omega) \otimes \wedge^K L^2(\Omega)$ (the spin degrees of freedom are omitted here for simplicity) satisfying Dirichlet boundary conditions. (We pick units such that c_1 and c_2 ($\sim c_1/1800$) satisfy $c_1 + c_2 = 1$.) The mathematical translation of the concept of a gas of hydrogen atoms is as follows.

i) Take $R \gg 1$. An electron x_i and a proton y_j are said to form an R -atom if

$$|x_i - z|, |y_j - z| > R |x_i - y_j| \quad (4.3)$$

for any particle $z \neq x_i, y_j$, that is if x_i and y_j are much closer to each other than to any of the other particles. In a gas of hydrogen atoms almost all particles are expected to belong to R -atoms.

ii) We expect the displacement vectors $\xi = x_i - y_j$ of the R -atoms to behave like independent random variables with the probability distribution typical of an isolated hydrogen atom in its ground state: let $E \subset \mathbb{R}^3$ and $0 < \varepsilon \ll 1$, then

$$\left| \frac{\text{Number of } R\text{-atoms with } \xi \in E}{\text{Total number of atoms}} - c \int_E e^{-|\xi|} d\xi \right| < \varepsilon. \quad (4.4)$$

iii) Finally, we expect the various atoms, i.e., their positions and displacement vectors, to be nearly independent. Let ρ be the density of the system; we decompose Ω into disjoint cubes $\{Q_\alpha\}$ of volume comparable to $1/\rho$. Then we subdivide each Q_α into two halves, Q'_α and Q''_α . For $E \subset \mathbb{R}^3$ we study the events

- ε'_α : Q'_α contains a single atom and nothing else; and the displacement vector for that atom lies in E ;
- ε''_α : Q''_α contains a single atom and nothing else; and the displacement vector for that atom lies in E .

Let

$$p' = \frac{\text{Number of } \alpha \text{ for which } \varepsilon'_\alpha \text{ occurs}}{\text{Total number of } \alpha}, \quad p'' = \frac{\text{Number of } \alpha \text{ for which } \varepsilon''_\alpha \text{ occurs}}{\text{Total number of } \alpha},$$

$$p^* = \frac{\text{Number of } \alpha \text{ for which } \varepsilon'_\alpha, \varepsilon''_\alpha \text{ both occur}}{\text{Total number of } \alpha}.$$

The independence of different atoms is then reflected by

$$|p^* - p' p''| < \varepsilon. \quad (4.5)$$

Assumptions (stability of matter):

- $H^\Omega \geq -E_* \cdot (N + K - 1)$ with E_* independent of N, K, Ω ;
- $E_* < \frac{1}{4}$ for $N + K > 2$.

This last assumption is well established by experimental observation of hydrogen crystals, but a rigorous mathematical proof is not known yet. To get hydrogen molecules an even

sharper assumption is required. (Notice that $1/4$ corresponds to Ry in these units, that is to the binding energy of a hydrogen atom.) Under these assumptions the result is as follows:

Theorem I.3. [27, 28] *Given $0 < \varepsilon \ll 1$ and $R \gg 1$, there exist a density ρ and a temperature T so that on a large enough box we have (4.3) with a probability at least $(1 - \varepsilon)$. Moreover, for any $E \subset \mathbb{R}^3$, (4.4) and (4.5) hold with probability at least $(1 - \varepsilon)$.*

This topic has been also tackled and simplified by Conlon, Lieb, Yau [12], Macris, Martin [58], Graf, Schenker [37].

d) In order to study basic properties of heavy atoms, like the ground state energy and the electronic density, as given by non-relativistic quantum mechanics, semiclassical theories like Thomas-Fermi theory are often used. This last theory, despite its deficiencies, is found to be asymptotically exact (in the limit $Z \rightarrow \infty$), and Lieb-Thirring type inequalities enable us to make a comparison between Thomas-Fermi theory and quantum mechanics in this limit (see for example the review paper [45], where also other references can be found, and also at the end of Section 6, for a remark about magnetic Lieb-Thirring type inequalities in this context).

e) The fermionic nature of electrons is crucial for stability of matter. If electrons were bosons instead of fermions ($\mathcal{H}_{el} =$ symmetric tensor product of \mathcal{H}), the ground state energy of the system of N electrons and K static nuclei would not be bounded from below by an extensive quantity, namely N , but we would have [42]

$$H \geq -C \cdot (N + K)^{5/3},$$

while for dynamic nuclei the bound can be improved to [13]

$$H \geq -C \cdot (N + K)^{7/5}.$$

f) H -stability, that is the statement that the ground state energy is bounded from below (by a multiple of the total number of particles), doesn't tell us anything about the existence of a bound state, i.e., of a ground state that minimizes the energy. In order to make predictions about this fact we should find, by the variational principle, a wave function ψ such that the energy in this state ψ is smaller than the bottom, $\Sigma(H)$, of the essential spectrum of H : $(\psi, H\psi) < \Sigma(H)$. This task is unfortunately not easy (see, for instance, [34] or [61]).

g) Generalizations to relativistic electrons and fixed nuclei are not straightforward, but necessary in some cases. If we consider for instance an hydrogenic atom, we can easily see that the value of the velocity v of the electrons closest to the nucleus (which are the ones with the highest speed) is given, non-relativistically, by $v = Z\alpha c$, where $\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$ is the fine structure constant and c the speed of light. This means that when Z , the nuclear charge, is large, relativistic effects become important. A "rough" Hamiltonian, which is

meant to capture many of the relativistic effects of a correct theory, is given by

$$H' = \sum_{i=1}^N (\sqrt{p_i^2 c^2 + m^2 c^4} - mc^2) + V'_C, \quad p = \frac{\hbar}{i} \nabla. \quad (4.6)$$

The kinetic energy is clearly modelled on Einstein's relativistic expression for energy, but as an operator it is non-local and, as well as the instantaneous Coulomb interactions, it is not Lorentz invariant. Due to

$$|p|c \geq (p^2 c^2 + m^2 c^4)^{1/2} - mc^2 \geq |p|c - mc^2, \quad (4.7)$$

the stability of (4.6) is the same as the stability of

$$\sum_{i=1}^N |p_i|c + V'_C.$$

Rescaling this Hamiltonian, we cannot get rid of all the constants (as we succeeded in in (2.2)), because of the different scaling behavior of $|p|$ with respect to p^2 , or, better, because $|p|$ scales in the same way as the Coulomb potential; instead we are left with

$$H = \sum_{i=1}^N |p_i| + \alpha V_C,$$

where the fine structure constant α , which is dimensionless, must stay. For the same reason, i.e., the same scaling behavior of the relativistic kinetic energy and the potential, we see that the ground state energy E_{NK} of H is either 0 (relativistic stability) or $-\infty$ (relativistic instability). On the other hand, if $E_{NK}(m)$ denotes the ground state energy of the massive Hamiltonian

$$H(m) = \sum_{i=1}^N (\sqrt{p_i^2 + m^2} - m) + \alpha V_C,$$

a rescaled version of H' , then we have, thanks to (4.7), $E_{NK} \geq E_{NK}(m) \geq E_{NK} - mN$. Thus, the finiteness of E_{NK} is equivalent to stability for $H(m)$.

Relativistic stability is not trivial even in the hydrogenic case, $N = K = 1$:

$$H = |p| - \frac{\alpha Z}{|x|}.$$

A new “uncertainty principle” has been found in an inequality by Kato [41] and Herbst [38]:

$$(\psi, |\nabla| \psi) = \int |p| |\hat{\psi}(p)|^2 d^3 p \geq \frac{2}{\pi} \int |x|^{-1} |\psi(x)|^2 d^3 x.$$

Consequently, the hydrogenic atoms have the following ground state energy:

$$E_{11} = 0 \quad \text{if} \quad Z\alpha \leq \frac{2}{\pi},$$

$$E_{11} = -\infty \quad \text{if} \quad Z\alpha > \frac{2}{\pi}.$$

The one-electron relativistic molecule ($N = 1$, K arbitrary) has been tackled by Daubechies and Lieb [16], while Conlon [11] proved stability for arbitrary N and K , however under restrictive assumptions on $Z\alpha$ and α (that are not artifacts of the proof: see also Section 5), which yield $Z = 1$ and $\alpha \leq 10^{-200}$. He made use of a “relativistic generalization” of the Lieb-Thirring inequality given by Daubechies [15], namely

$$\text{tr}[\gamma(\mu|p| - U)] \geq -0.0258 q\mu^{-3} \int U(x)^4 d^3x$$

for a density matrix γ satisfying $0 \leq \gamma \leq q$ and for any $\mu > 0$.

Conlon’s constraints on $Z\alpha$ and α have been subsequently milder by Fefferman and de la Llave [31], Lieb and Yau [56] (including spin), and Lieb, Loss, Siedentop [48].

Investigating a related model, but in the context of gravitational interactions, Chandrasekhar [7] could predict collapse for neutron stars and white dwarfs and give a critical mass which is approximately correct.

From a trivial inequality, $0 \leq (|p| - \beta/2)^2$, $\beta \in \mathbb{R}$, we can derive an estimate [54] that gives an exact meaning to the expression that the relativistic kinetic energy operator $|p|$ is weaker than its non-relativistic counterpart p^2 :

$$|p|\beta \leq p^2 + \frac{\beta^2}{4}. \quad (4.8)$$

We are in a position to derive non-relativistic stability from relativistic stability. Indeed, relativistic stability yields a lower bound on $V_C \geq -\alpha^{-1} \sum_{i=1}^N |p_i|$ (for sufficiently small values of $Z\alpha$ and α) we can insert in the non-relativistic Hamiltonian (2.2) to obtain

$$\sum_{i=1}^N p_i^2 + V_C \geq \sum_{i=1}^N (p_i^2 - \alpha^{-1} |p_i|) \geq -\frac{1}{4\alpha^2} N,$$

after use of (4.8) with $\beta = \alpha^{-1}$. Unfortunately, this result is not as strong as the foregoing ones (see Section 3), because of the values of $Z\alpha$ and α imposed by relativistic stability.

5. External magnetic fields

In this section we couple matter to an external magnetic field, that could represent for example a sort of mean field generated by the motion of all the particles, in which case we would have to do with a more realistic model of matter. The question of stability when we plunge our collection of nuclei and electrons into an external magnetic field has also been raised in connection with the study of atomic properties in strong magnetic fields at the surface of white dwarfs (magnetic fields up to $10^8 - 10^9$ G) or neutron stars (which are believed to consist mostly of iron atoms in magnetic fields of $10^{11} - 10^{13}$ G) (see Channugam [8]).

We consider N non-relativistic electrons and K non-relativistic nuclei coupled to the external magnetic field $B = \nabla \wedge A$ (A is the vector potential and we choose to work in the Coulomb gauge $\nabla \cdot A = 0$) by minimal substitution $p \rightarrow p - \frac{q}{c}A$, where q is the charge of the particle in question. This takes into account the coupling of the orbital angular momentum of the particles with the magnetic field. Furthermore we have to consider the coupling of the magnetic moment $\mu = g \frac{q}{2mc} \hbar S$ (g is the gyromagnetic factor, while $\hbar S$ is the spin operator) of the particles with the magnetic field: $-\mu \cdot B$ (Zeeman term). However, we still neglect magnetic dipole interactions, i.e., spin-spin interactions. Since the electron has spin $1/2$, i.e., $S_{el} = \sigma/2$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices and these read

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the Hamiltonian H' of the system acting on $\mathcal{H}_{el} \otimes \mathcal{H}_{nuc}$ (see Section 2), is composed of the kinetic energy of electrons and nuclei,

$$\begin{aligned} T_{el} + T_{nuc} &= \sum_{i=1}^N \left[\frac{1}{2m} \left(p_i - \frac{e}{c} A(x_i) \right)^2 - g_{el} \frac{e\hbar}{2mc} \frac{\sigma_i}{2} \cdot B(x_i) \right] \\ &\quad + \sum_{j=1}^K \left[\frac{1}{2M_j} \left(p_j + \frac{Z_j e}{c} A(R_j) \right)^2 + g_{nuc,j} \frac{Z_j e\hbar}{2M_j c} S_{nuc,j} \cdot B(R_j) \right] \end{aligned}$$

($p = -i\hbar\nabla$), plus their Coulomb interactions V'_C (2.1), plus the energy of the magnetic field, $H'_f = (8\pi)^{-1} \int B(x)^2 d^3x$, which is positive and essential for the stability of the system, as we will show below, though it is constant:

$$H' = T_{el} + T_{nuc} + V'_C + H'_f.$$

As in the non-magnetic case we compare T_{el} with T_{nuc} . Since the ratio between the mass of an electron and that of a proton is about $1/1800$, we see that T_{nuc} can be considered small with respect to T_{el} , for the ratio of nuclear magnetic moment to electron magnetic moment is of the order

$$\frac{g_{nuc,j} \frac{Z_j e\hbar}{2M_j c}}{g_{el} \frac{e\hbar}{2mc}} \sim \frac{Z_j m}{M_j} \sim \frac{Z_j}{1800},$$

as well as the ratio of the first terms in T_{el} resp. T_{nuc} . We might therefore think that it should be possible to disregard the nuclear kinetic energy, as we did before in Section 2, in the spirit of the Born-Oppenheimer approximation. But this is not trivial now, since there are many nuclei with spin $\geq 1/2$ and/or with gyromagnetic factor $g_{nuc} > 2$ (the proton has, for instance, spin 1/2 and $g_{proton} = 5, 59$), and for these nuclei T_{nuc} is not a positive operator (as instead it was before the case for $-\sum_{j=1}^K \frac{\hbar^2}{2M_j} \Delta_{R_j}$) and stability would not hold. Fortunately, we can straighten the situation by taking into account the finite size of nuclei. Indeed, we introduce a form factor for the magnetic moment of the nuclei through a non-negative function ν on \mathbb{R}^3 with compact support, such that $\int \nu(x) d^3x = 1$ and $\nu(x) \leq \nu_0 e^{-|x|}$, for some $\nu_0 < \infty$, and we replace the Zeeman term $g_{nuc,j} \frac{Z_j e \hbar}{2M_j c} S_{nuc,j} \cdot B(R_j)$ by

$$\mathcal{Z}_{j,\nu} = g_{nuc,j} \frac{Z_j e \hbar}{2M_j c} \int \nu(R_j - x) S_{nuc,j} \cdot B(x) d^3x.$$

Its norm on the spin space of the nucleus (with R_j fixed) is bounded by

$$\|\mathcal{Z}_{j,\nu}\|_{spin} \leq g_{nuc,j} \frac{Z_j e \hbar}{2M_j c} \sqrt{S_j(S_j + 1)} \int \nu(R_j - x) \left(\frac{\beta}{2} B(x)^2 + \frac{1}{2\beta} \right) d^3x$$

for any $\beta > 0$ (Young's inequality). The r.h.s. can be estimated in terms of the magnetic field energy ($\varepsilon > 0$):

$$\|\mathcal{Z}_{j,\nu}\|_{spin} \leq \frac{\varepsilon}{8\pi} \int e^{-|R_j - x|} B(x)^2 d^3x + C_\varepsilon,$$

where $C_\varepsilon = (g_{nuc,j} \frac{Z_j e \hbar}{2M_j c})^2 (\frac{\varepsilon}{2})^{-1} S_j(S_j + 1) (\nu_0)^2$ is a finite constant, whose value we cannot however expect to be small (as compared to a typical atomic energy), despite of the factor M_j^{-2} , because of the presence in it of ν_0 , that should be approximately the inverse of the volume of a nucleus (i.e., more or less 10^{39} cm^{-3}). Thus, if we are willing to pay the price of a big stability constant, we are allowed to neglect the Zeeman energy in the definition of T_{nuc} (this is the case of spinless nuclei):

$$T_{nuc} = \sum_{j=1}^K \frac{1}{2M_j} \left(p_j + \frac{Z_j e}{c} A(R_j) \right)^2.$$

This “magnetic Schrödinger operator” is positive, so we can neglect it in our analysis of stability of matter, as we did in Section 2 for $-\sum_{j=1}^K \frac{\hbar^2}{2M_j} \Delta_{R_j}$. However, we must remember that whenever the spin of a nucleus exceeds 1/2 and/or its gyromagnetic factor is bigger than 2, our result will exhibit a dependence on the form factor ν_0 .

Always keeping in mind what we just said, and expanding our system of units to the electromagnetic case (in addition to the scaling parameters of Section 2.b) we introduce a new one related to the vector potential as $A(x) = \beta \mathcal{A}(y)$ and use the following units: $\hbar^2(2me^2)^{-1}$ for length, $2me^4\hbar^{-2}$ for energy and $2mech^{-1}$ for the magnetic vector potential

\mathcal{A} , which we then rename A), we limit ourselves to consider the problem in the Born-Oppenheimer approximation:

$$H = \sum_{i=1}^N \left[(p_i - A(x_i))^2 - \frac{g_{el}}{2} \sigma_i \cdot B(x_i) \right] + V_C + H_f, \quad (5.1)$$

where V_C is given in (2.2) and $H_f = (8\pi\alpha^2)^{-1} \int B(x)^2 d^3x$. This operator, defined as a quadratic form, is self-adjoint for $A \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$ and V_C in the Kato class K_3 [14]. Now stability means that H is bounded from below linearly in the total particle number and uniformly not only in the nuclear positions but also in the magnetic field.

Let us now have a look at the electronic kinetic energy operator. This operator, whose nuclear counterpart was so troublesome, for an electron (of spin 1/2) reads

$$\not{p}^2 \equiv D^2 - \frac{g_{el}}{2} \sigma \cdot B, \quad \text{where} \quad D = p - A.$$

As already mentioned before, for $g_{el} > 2$ this operator is not bounded below and stability doesn't hold (see [35]). The physical case, $g_{el} = 2$, is a borderline case: \not{p}^2 can in fact be written as $\not{p}^2 = [D \cdot \sigma]^2$ and is non-negative, still allowing for the existence of zero-modes [57, 23]. If we had $g_{el} < 2$, instead, \not{p}^2 would be positive: it can be decomposed as $\not{p}^2 = \frac{1}{2}g[D \cdot \sigma]^2 + (1 - \frac{1}{2}g)D^2$, where the first term is non-negative and the second one is positive. In the non-magnetic case the kinetic energy operator $p^2 = -\Delta$ was positive, for $p\psi = 0$ had no non-trivial solution for $\psi \in L^2(\mathbb{R}^3)$. There we could have even forgotten spin, provided we would have kept the antisymmetry of the wave function. In the magnetic case too, the case of spinless electrons is not problematic, since it can be easily brought back to the case where there is no magnetic field at all by means of the diamagnetic inequality,

$$(\psi, (p - A)^2 \psi) \geq (|\psi|, p^2 |\psi|), \quad (5.2)$$

from which it follows, by the variational principle, that

$$\inf_{\psi} (\psi, (p - A)^2 \psi) \geq \inf_{\psi} (\psi, p^2 \psi)$$

(while $(p - A)^2 \geq p^2$ is false) (see also Lemma III.1 and its Corollary). The case of Pauli electrons (i.e., with $g_{el} = 2$) with spin, on the contrary, is essentially different and this arises precisely from the fact that the Pauli operator \not{p} exhibits zero-modes for particular choices of A . This implies that there are wave functions for which the kinetic energy does not contribute with a positive term to the stability of matter, and this brings out new aspects with respect to the case of stability of matter without magnetic fields, since the uncertainty principle fails in this case, the same uncertainty principle that in the non-magnetic case prevented the electrons from getting too close to the nucleus. This fact already has repercussions on the stability of the one-electron atom, as Fröhlich, Lieb, Loss recognized in their paper [35], where the first result in the study of the stability of matter

in magnetic fields is established (see [1, 9] for earlier partial results). For $N = 1 = K$ the Hamiltonian (5.1) reads

$$H = \not{p}^2 - \frac{Z}{|x|} + \frac{1}{8\pi\alpha^2} \int B(x)^2 d^3x .$$

Let $\psi \in \mathcal{H}$ be a normalized zero-mode of \not{p} (i.e., $\not{p}\psi = 0$). By scaling $(\psi(x, s) \rightarrow \psi_\lambda(x, s) = \lambda^{-3/2}\psi(\frac{x}{\lambda}, s)$, $A(x) \rightarrow A_\lambda(x) = \lambda^{-1}A(\frac{x}{\lambda})$, $B(x) \rightarrow B_\lambda(x) = \lambda^{-2}B(\frac{x}{\lambda})$), the energy in this state is given by

$$E \equiv (\psi, H\psi) = -\frac{Z}{\lambda}(\psi, |x|^{-1}\psi) + \frac{1}{8\pi\alpha^2\lambda} \int B(x)^2 d^3x ,$$

since $[(p - A_\lambda) \cdot \sigma]\psi_\lambda = 0$. If $Z\alpha^2$ exceeds $(8\pi)^{-1} \int B(x)^2 d^3x / (\psi, |x|^{-1}\psi)$ we can drive the energy to $-\infty$ by letting λ go to zero [35]: in the magnetic case we have a constraint $Z\alpha^2 < \text{const}$ in order to have stability. This shows at the same time the importance of the magnetic energy in stabilizing the system. It is known even for constant magnetic fields [2] that in the absence of this term arbitrarily large values of the magnetic field would drive the ground state energy to $-\infty$ already for hydrogen.

Subsequently, Lieb and Loss [47] showed the stability of the one-electron molecule and of the many-electron atom. In that paper a new feature of magnetic stability appears, namely a bound also on α , besides that on $Z\alpha^2$. It is a many body effect and we can understand it by observing the one-electron molecule ($N = 1, K > 1$):

$$H = \not{p}^2 - \sum_{k=1}^K \frac{Z}{|x - R_k|} + \sum_{\substack{k, l=1 \\ k < l}}^K \frac{Z^2}{|R_k - R_l|} + \frac{1}{8\pi\alpha^2} \int B(x)^2 d^3x .$$

If we take again a zero-mode ψ of \not{p} and then rescale, the energy in this state is

$$E = (\psi, H\psi) = -\frac{Z}{\lambda} \sum_{k=1}^K \left(\psi, \frac{1}{|x - R_k|} \psi \right) + \frac{1}{\lambda} \sum_{\substack{k, l=1 \\ k < l}}^K \frac{Z^2}{|R_k - R_l|} + \frac{1}{8\pi\alpha^2\lambda} \int B(x)^2 d^3x .$$

The electron-nuclei Coulomb attractions and the nucleus-nucleus repulsions can be bounded from above by $-C_{e-n} \cdot ZK/\lambda$ resp. $C_{n-n} \cdot (ZK)^2/\lambda$, where both constants C_{e-n} and C_{n-n} are positive and independent of Z and K . Choosing the nuclear charge Z such that it minimizes the resulting expression for the electric interactions, i.e., $ZK = C_{e-n}(2C_{n-n})^{-1}$, we have

$$E \leq -\frac{C_{e-n}}{4C_{n-n}} \frac{1}{\lambda} + \frac{1}{8\pi\alpha^2\lambda} \int B(x)^2 d^3x .$$

But now it is manifest that for values of α large enough the first term will get the upper hand of the magnetic energy term and the energy will become negative, and then by letting $\lambda \rightarrow 0$ we can drive it to $-\infty$. The conclusion is that a bound on α is also needed.

This shows that both bounds on $Z\alpha^2$ and α (compare with the bounds on $Z\alpha$ and α for relativistic stability) are not artifacts of the proofs, but true physical features.

Only three years ago two proofs have been found by Fefferman [30] (but see [26] for an announcement) and Lieb, Loss, Solovej [49] for arbitrarily many electrons and nuclei. The Lieb, Loss, Solovej result covers $Z \leq 1050$ for $\alpha = 1/137$ and their stability constant is realistic (of the order of Ry). In that paper, they show non-relativistic magnetic stability in two different ways. The first one consists in linking it to relativistic magnetic stability, by first eliminating V_C in favour of $\alpha^{-1}|p - A|$ (see Section 4.g)), then comparing this last operator with its non-relativistic counterpart $(p - A)^2$ and finally using the Cwickel-Lieb-Rozenblum (CLR) bound [63]

$$N_E((p - A)^2 - V) \leq L_3 \int (V(x) - E)_+^{3/2} d^3x ,$$

where $L_3 = 0, 1156$ and N_E again denotes the number of eigenvalues below $-E$ ($E > 0$). (Notice by the way that it is possible to derive non-relativistic from relativistic stability also with another method, namely through a Birman-Koplienko-Solomyak inequality [48].) The second method they use is based on a new technique in order to handle the kinetic energy (they call it “running energy-scale renormalization of the kinetic energy”), together with the reduction of the Coulomb potential of Theorem I.2, the CLR bound, and another version of the uncertainty principle, that goes back to Dyson and Lenard [19], in order to control the Coulomb singularity:

$$\frac{\lambda}{4} \int_{\Omega} d^3x |(-i\nabla - A(x))\psi(x)|^2 - \int_{\Omega} d^3x \frac{1}{|x|} |\psi(x)|^2 \geq - \left(\frac{1}{\lambda} + \frac{3}{2R} \right) \int_{\Omega} d^3x |\psi(x)|^2 ,$$

where Ω is a domain containing a ball of radius R around the origin and λ is an arbitrary positive real number. Moreover, they prove a Lieb-Thirring type inequality for the negative eigenvalues $-e_i < 0$ of the one-particle Pauli operator $h = \not{p}^2 - V$, $V(x) \geq 0$, namely

$$\sum e_i \leq a \int V(x)^{5/2} d^3x + b \left(\int B(x)^2 d^3x \right)^{3/4} \left(\int V(x)^4 d^3x \right)^{1/4} , \quad (5.3)$$

with $a = a(\gamma) = (2^{3/2}/5)(1 - \gamma)^{-1}L_3$, $b = b(\gamma) = 3^{1/4}2^{-9/4}\pi\gamma^{-3/8}(1 - \gamma)^{-5/8}L_3$, for all $0 < \gamma < 1$. However, they could not use it directly because $W(x)$ (see (3.11)) is not integrable neither to the power $5/2$ nor to the power 4 (and not even $(W(x) - \nu)_+$, $\nu > 0$, to the 4th power is integrable: see (3.13)).

6. Two local magnetic results by means of two Lieb-Thirring type estimates

Our intentions are to modify H_f in such a way as to retain the energy of the magnetic field only in a neighbourhood of the nuclei, and not in all space. We namely suppose that the role played by the magnetic energy far away from the nuclei is uninfluential. The reason for this is twofold: first, we don't think that a magnetic field in a region which is well separated from the system of atoms or molecules we are studying can influence its stability; second, electrons that are "moving" far away from the nuclei have only a small (negative) interaction with the nuclei, precisely because of the large distance, that should not overwhelm the positivity of the kinetic energy and of the electrostatic repulsion from the other electrons. Stability of matter stated like that has thus even a better physical meaning. In addition, this local stability will enable us to show stability for a collection of static nuclei and non-relativistic electrons coupled to the quantized electromagnetic field (as noticed first by Fröhlich). The result is as follows:

Theorem I.4. *Let $\mathcal{R} = \{R_1, \dots, R_K\}$ be the collection of all the nuclei and $L, Z, \Gamma > 0$. Then there is a positive $C(Z, \Gamma)$, an $\varepsilon > 0$ and a function $\Phi_{\mathcal{R}}(x) \geq 0$ with*

$$\|\Phi_{\mathcal{R}}\|_{\infty} \lesssim 1, \quad \|\Phi_{\mathcal{R}}\|_1 \lesssim L^3 K,$$

uniformly in \mathcal{R} , Z , such that the Hamiltonian

$$H = \sum_{i=1}^N \not{p}_i^2 + V_C + \Gamma \int \Phi_{\mathcal{R}}(x) B(x)^2 d^3 x,$$

acting on $\mathcal{H}_{el} = \wedge^N \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, satisfies

$$H \geq -C(Z, \Gamma)(Z + 1)L^{-1}(N + K)$$

for arbitrary $L \leq (Z + 1)^{-1}$, provided $\Gamma^{-1}(Z + 1) \leq \varepsilon$, where

$$C(Z, \Gamma) = \text{const} [\Gamma(Z + 1)^{-1} + (Z + 1)].$$

This is however not the only result at our disposal. We also have a second local magnetic result, that yet can be interpreted exclusively as a mean to prove stability of non-relativistic quantum electrodynamics (QED).

Theorem I.5. *Let $\mathcal{R} = \{R_1, \dots, R_K\}$ be the collection of all the nuclei and $L, Z, \Gamma, \gamma > 0$. Then there is a positive $C(Z, \Gamma, \gamma)$ and a function $\Phi_{\mathcal{R}}(x) \geq 0$ with*

$$\|\Phi_{\mathcal{R}}\|_{\infty} \lesssim 1, \quad \|\Phi_{\mathcal{R}}\|_1 \lesssim L^3 K,$$

uniformly in \mathcal{R} , Z , such that the Hamiltonian

$$H = \sum_{i=1}^N \not{p}_i^2 + V_C + \Gamma \int \Phi_{\mathcal{R}}(x) (B(x)^2 + \gamma L^2 (\nabla \otimes B)(x)^2) d^3 x,$$

acting on $\mathcal{H}_{el} = \wedge^N \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, satisfies

$$H \geq -C(Z, \Gamma, \gamma)(Z+1)L^{-1}(N+K)$$

for arbitrary $L \leq (Z+1)^{-1}$. For $\Gamma \leq Z+1$ and $1 \leq \gamma \leq z^4$ one can take

$$C(Z, \Gamma, \gamma) = \text{const} [z^3 + z^5 \gamma^{-1/2} \log(z^5 \gamma^{-1/2})]$$

with $z = 1 + (Z+1)\Gamma^{-1}$.

We remark that the functions $\Phi_{\mathcal{R}}(x)$ appearing in both theorems have the meaning of approximate characteristic functions of the set $\Omega_L = \{x \mid \text{dist}(x, \mathcal{R}) < L\}$, which is the union of K balls of radius L , each one centered at one nuclear position. Note as well that $L \leq (Z+1)^{-1}$ means that we must retain the magnetic energy only in a neighbourhood of the nuclei of size at most that of a Bohr radius for nuclear charge Z .

The main difference between these two results is that the first one imposes an upper bound on the product $\Gamma^{-1}(Z+1)$, whereas the second one holds true for every value of the parameters involved (as long as $L \leq (Z+1)^{-1}$). In the physical case $\Gamma = (8\pi\alpha^2)^{-1}$, and Theorem I.4 then ensures stability only for sufficiently small values of α and $Z\alpha^2$. On the contrary, Theorem I.5 says that the system is always stable, but to this end we have to incorporate the second “magnetic energy term” where the magnetic field gradient enters.

Both these results are proved by first reducing the Coulomb potential V_C according to (3.11) and then applying new Lieb-Thirring estimates to the resulting one-particle Pauli Hamiltonian. The magnetic energy as well as the internuclear repulsions (see last term in (3.11)) will turn out to be necessary for stability.

Theorem I.4 has first been proved in [6] by adding a localization argument to [49] and by Fefferman using other methods [26, 30]. Theorem I.5 is a result (also obtained by other methods) of Fefferman [25], apart from a somewhat more explicit dependence on the parameters and from the presence in the lower bound for the Hamiltonian, besides the number, K , of nuclei (that is the only one appearing in [25]), also of the number, N , of electrons. Remark that we don't have explicit control on the numerical constants.

Let us now come back to these “new Lieb-Thirring estimates”. For the one-particle operator $h = \not{p}^2 - V$, $V(x) \geq 0$ a non-negative multiplication operator, acting on $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, we already have (see [49] and (5.3))

$$\sum e_i \leq a \int V(x)^{5/2} d^3x + b \left(\int B(x)^2 d^3x \right)^{3/4} \left(\int V(x)^4 d^3x \right)^{1/4},$$

where $-e_i < 0$ are the negative eigenvalues of h . This estimate contains the total energy of the magnetic field, $\sim \int B(x)^2 d^3x$, but since we want to employ it in the proof of a stability result where we retain the magnetic energy only in a neighbourhood of the nuclei, it is not well-suited to this end and we have to look for a local variant of it. We could conjecture the following estimate:

$$\sum e_i \leq C_1 \int V(x)^{5/2} d^3x + C_2 \int B(x)^{3/2} V(x) d^3x, \quad (6.1)$$

for some constants C_1 and C_2 . This inequality is local, since it couples B at the point x to V at the same point x , and represents a natural generalization of (5.3), since by Hölder's inequality we have from it

$$\int B(x)^{3/2}V(x)d^3x \leq \left(\int B(x)^2d^3x \right)^{3/4} \left(\int V(x)^4d^3x \right)^{1/4},$$

so that we obtain back estimate (5.3) (up to constants). Unfortunately, this estimate is false, unless we don't impose some restrictions on B . The first one who noticed it was Erdős [20], who found a counterexample for the validity of the related Lieb-Thirring type inequality

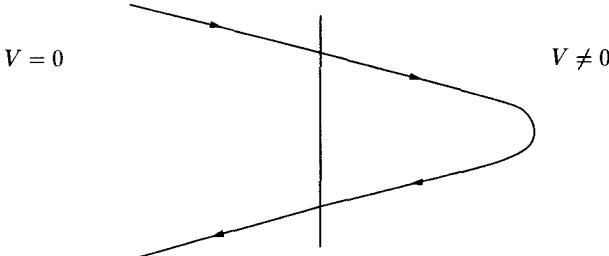
$$\sum e_i \leq C' \int V(x)^{5/2}d^3x + C'' \int B(x)V(x)^{3/2}d^3x, \quad (6.2)$$

that simultaneously explains why (6.1) is false, since the reasons are the same. We would like to rephrase here this counterexample in a very illuminating but also mathematically less involved way, following an idea of Graf. But before doing this, we observe that the word "related" used above for (6.2) and (6.1) is motivated by the fact that (6.1) follows from (6.2) (up to constants) by applying Hölder's and Young's inequalities:

$$\begin{aligned} \int B(x)V(x)^{2/3} \cdot V(x)^{5/6}d^3x &\leq \left(\int B(x)^{3/2}V(x)d^3x \right)^{2/3} \left(\int V(x)^{5/2}d^3x \right)^{1/3} \\ &\leq \frac{2}{3} \beta \int B(x)^{3/2}V(x)d^3x + \frac{1}{3} \beta^{-2} \int V(x)^{5/2}d^3x \end{aligned}$$

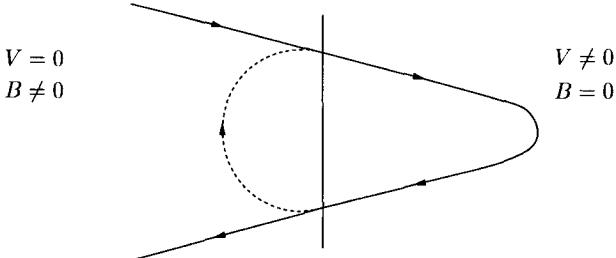
for every $\beta > 0$.

Let's come back to this counterexample. We look at the following semiclassical picture: we divide \mathbb{R}^3 into two parts by means of a plane; on one side of this plane, say on the left, we set $V = 0$, on the other side, say on the right, we set $V \neq 0$. We think now of a particle coming in towards the "cut" from the l.h.s., where it travels free. Let us assume that the potential on the r.h.s. bends the trajectory of the particle in such a way that it goes back into the first region, where it travels again free:



If we now turn on a constant magnetic field B on the l.h.s. only, with appropriate intensity $|B|$ and direction, it will bend the trajectory of the particle in such a way that we will get

a bound state spending a very little amount of energy, namely the energy of the lowest Landau level, which equals the kinetic energy in the direction parallel to the B -field.



But we see that estimate (6.1) cannot account for it, since the second integral vanishes, because V and B have disjoint supports. Therefore, the cooperation between V and B in producing bound states is only semilocal, and for this reason Erdős proposed the use of an effective (scalar) magnetic field b , i.e., of a smeared magnetic field, with a bigger support than that of the physical magnetic field. For a homogeneous magnetic field B the smearing should act on a length scale proportional to $|B|^{-1/2}$, i.e., proportional to the cyclotronic radius in the lowest Landau band. More generally, for non-constant magnetic fields, the smearing should be done over a length scale proportional to $b^{-1/2}$, i.e., in terms of the effective field itself. To translate these considerations into a mathematical expression we choose a smooth positive radially decreasing and symmetric function $\varphi(z)$ with $\int \varphi(z) d^3z = 1$, and we define

$$b(x)^2 = \int \varphi\left(\frac{y-x}{r(x)}\right) r(x)^{-3} B(y)^2 d^3y$$

for $r(x) = b(x)^{-1/2}$, that is, in terms of $r(x)$ only,

$$r(x) \int \varphi\left(\frac{y-x}{r(x)}\right) B(y)^2 d^3y = 1. \quad (6.3)$$

The length scale $r(x)$ is well-defined, as we will show in Section II.2, and through it we can prove

Theorem I.6. *There are constants $C', C'' > 0$ such that for any vector potential $A \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$*

$$\sum e_i \leq C' \int V(x)^{5/2} d^3x + C'' \int b(x)^{3/2} V(x) d^3x. \quad (6.4)$$

A. Sobolev [65, 64] has been the first one to collect the suggestion of Erdős and to use an effective magnetic field in order to prove this same Lieb-Thirring inequality (6.4). However, his choice for the effective field b is different from ours: his b -field dominates $|B|$ pointwise, while our b -field satisfies an L^2 -estimate:

$$\int b(x)^2 d^3x \leq C \int B(x)^2 d^3x,$$

that is, the energy of the effective field is bounded from above by the physical magnetic energy (up to a factor $C > 0$). By the way, this enables us to get back (5.3) (up to constants) by first applying Hölder and then this last energy estimate.

Generalizing our definition of r , Shen [62] also proved the same bound (6.4).

Inequality (6.4) is our main tool in the proof of the first local stability result (Theorem I.4). In order to prove Theorem I.5 we have to find a more sensitive Lieb-Thirring estimate, where the gradient of the magnetic field also enters. While our estimate (6.4) accounts for the effects of a non-vanishing magnetic field on top of the original Lieb-Thirring estimate (3.4), to which it reduces for $b = 0$, we now want to find an estimate that accounts for the effects of $\nabla \otimes B = (\partial_i B_j)_{i,j=1,2,3} \neq 0$ on top of the following Lieb-Thirring type estimate that holds for a homogeneous magnetic field [51]:

$$\sum e_i \leq L_1 \int V(x)^{5/2} d^3x + L_2 |B| \int V(x)^{3/2} d^3x , \quad (6.5)$$

where the constants above have the values $L_1 = 4(3\pi)^{-1}$, $L_2 = 8\sqrt{6}(5\pi)^{-1}$. Remark again that Erdős, in the context of deriving a generalization of this inequality, gave an argument which implies that (6.2) cannot hold true in general. Our method for estimating the effects of a non-vanishing magnetic field gradient on top of (6.5) is analogous to that explained before. First, we introduce a supplementary length scale l , that is related to $\nabla \otimes B$ in a similar way as r is related to B (but notice the different scaling properties of B and $\nabla \otimes B$):

$$l(x)^3 \int \varphi\left(\frac{y-x}{l(x)}\right) (\nabla \otimes B(y))^2 d^3y = 1 .$$

It is well-defined, as shown in Section III.2. Like r , l also satisfies a sort of energy estimate:

$$\int l(x)^{-6} d^3x \lesssim \int (\nabla \otimes B(x))^2 d^3x .$$

Then we have to define a new effective field, better a new effective field gradient, that involve r as well as l and vanish for homogeneous B -fields. Our choice is

$$P(x) = l(x)^{-1} (r(x)^{-1} + l(x)^{-1}) ,$$

(notice that the same gradient is more effective when it is superimposed on a strong magnetic field) and the result is as follows:

Theorem I.7. *For sufficiently small $\varepsilon > 0$ there are constants $C', C'' > 0$ such that for any vector potential $A \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$*

$$\sum e_i \leq C' \int V(x)^{3/2} (V(x) + \widehat{B}(x)) d^3x + C'' \int V(x) P(x)^{1/2} (P(x) + \widehat{B}(x)) d^3x , \quad (6.6)$$

where $\widehat{B}(x)$ is the average of $|B(y)|$ over a ball of radius $\varepsilon l(x)$ centered at x .

This estimate, which by the way reduces to (6.5) (up to constants) for $B = \text{const}$, since $l = \infty$ in this case, can be used, as noticed in [21], to give a bound on the density $n(x)$ of zero-modes of \not{p} , that we denote by ψ_1, ψ_2, \dots . By the variational principle

$$\sum_i (-e_i) \leq \sum_j (\psi_j, H\psi_j) = - \int V(x)n(x)d^3x ,$$

where $n(x) = \sum_j |\psi_j(x)|^2$. Combining this with (6.6), we have

$$\int V(x)n(x)d^3x \leq C' \int V(x)^{3/2}(V(x) + \widehat{B}(x))d^3x + C'' \int V(x)P(x)^{1/2}(P(x) + \widehat{B}(x))d^3x .$$

Scaling V as λV , we see, letting λ getting small, that it must hold

$$n(x) \lesssim P(x)^{1/2}(P(x) + \widehat{B}(x)) .$$

The right side vanishes in the case of a homogeneous magnetic field, consistently with the fact that \not{p} has in this case no zero-mode. Another estimate on $n(x)$ could have been obtained from (6.4) in the same way, but the result, $n(x) \lesssim b(x)^{3/2}$, while true, is much too rough.

An inequality related to (6.6) has been derived by Erdős and Solovej [21] under different technical conditions on the effective magnetic field and magnetic field gradient. They could employ it in the semiclassical analysis of atoms in strong magnetic fields [22], because it behaves like the corresponding semiclassical expression for the sum of the negative eigenvalues (see also [50, 51, 52]). These kind of studies, aiming at establishing basic properties of heavy atoms in strong magnetic fields (by means of semiclassical theories as for example magnetic Thomas-Fermi) are important for atoms on the surface of white dwarfs and neutron stars, as noticed already at the beginning of Section 5, but also for artificial atoms, so-called quantum dots [39, 52], that are two-dimensional atom-like systems confined within semiconductor heterostructures. The parameters of such artificial atoms (like mass and charge) can differ appreciably from their natural counterparts because of the interactions of the electrons with the crystal where they move in, and effects that for natural atoms require magnetic fields with astronomical strength can be studied for artificial atoms in “normal” terrestrial laboratories. In this case, however, two-dimensional Lieb-Thirring inequalities are needed.

As a final remark, we notice that since $\int B(x)^2 d^3x \gtrsim \int \Phi_{\mathcal{R}}(x)B(x)^2 d^3x$, our local result (Theorem I.4) implies up to constants the result of Lieb, Loss, Solovej [49], that however can also be recovered directly through Lieb-Thirring estimate (6.4) (see [4]).

7. Non-relativistic quantum electrodynamics

The primary importance of the previous results (Theorem I.4 and Theorem I.5) is that they enable us to show stability for a system composed of an arbitrary number, K , of static nuclei interacting via Coulomb forces with an arbitrary number, N , of non-relativistic Pauli electrons coupled to the quantized ultraviolet-cutoff electromagnetic field. We now introduce quantum electrodynamics.

The Hilbert space of photons is given by the bosonic Fock space $\mathcal{F}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)$ over $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. The \mathbb{C}^2 -factor accounts for the helicity of photons. On it acts the quantized magnetic vector potential $A(x)$, on which we impose an ultraviolet cutoff $0 < \Lambda < \infty$:

$$A_\Lambda(x) \equiv A(x) = A_-(x) + A_+(x) , \quad A_+(x) = A_-(x)^* ,$$

$$A_-(x) = \frac{\alpha^{1/2}}{2\pi} \int \kappa(k) |k|^{-1/2} \sum_{\lambda=\pm} a_\lambda(k) e_\lambda(k) e^{ikx} d^3k . \quad (7.1)$$

The cutoff function $\kappa(k)$ satisfies $|\kappa(k)| \leq 1$ and $\text{supp } \kappa \subset \{k \in \mathbb{R}^3 \mid |k| \leq \Lambda\}$. For each k , the direction of propagation $\hat{k} = k/|k|$ and the polarization (e.g. right and left circular polarization) vectors $e_\pm(k) \in \mathbb{C}^3$ are orthonormal. The operators $a_\lambda(k)^*$ and $a_\lambda(k)$ are creation and annihilation operators on \mathcal{F} and satisfy canonical commutation relations

$$[a_\lambda(k)^\#, a_{\lambda'}(k')^\#] = 0 , \quad [a_\lambda(k), a_{\lambda'}(k')^*] = \delta_{\lambda\lambda'} \delta(k - k') .$$

Furthermore, there exists a unitary vector $\Omega \in \mathcal{F}$, called the Fock vacuum, such that $a_\lambda(k)\Omega = 0$ for both values of λ and for all $k \in \mathbb{R}^3$. On one hand, the ultraviolet cutoff Λ we impose on $A(x)$ takes care that photons with energies large compared with typical atomic energies ($\propto mc^2(Z\alpha)^2$) are not coupled to the electrons, on the other hand it makes the vector potential $A(x)$ a well-defined operator-valued function in each point $x \in \mathbb{R}^3$ (since the domain of integration is a compact set, $A(x)$ is even an analytic function). Without the imposition of the ultraviolet cutoff, $A(x)$ would only make sense as an operator-valued distribution. With the energy of the free photon field given by

$$H_f = \alpha^{-1} \int |k| \sum_{\lambda=\pm} a_\lambda(k)^* a_\lambda(k) d^3k , \quad (7.2)$$

the Hamiltonian of our system is

$$H = \sum_{i=1}^N \not{p}_i^2 + V_C + H_f ,$$

where now the magnetic vector potential appearing in $\not{p} = (p - A) \cdot \sigma$ is the quantized one (7.1). The equations of motion stemming from this Hamiltonian H are the Lorentz equations for the electrons (including also a force arising from the interaction of the electron

magnetic moment with the magnetic field) and the Maxwell equations for the electromagnetic field, where the sources are given by matter, i.e., the correct equations for the coupled system of matter and radiation (see [3] for some interesting property of this model).

As a final remark about the Hamiltonian we notice that the energy of the free photon field is larger than the energy of the photons that are allowed to interact with the electrons, which is

$$H_{f,\Lambda} = \alpha^{-1} \int d^3k |\kappa(k)|^2 \sum_{\lambda=\pm} a_\lambda(k)^* a_\lambda(k) = \frac{1}{8\pi\alpha^2} \int d^3x :E(x)^2 + B(x)^2: \geq 0, \quad (7.3)$$

where $E(x) = -ia[H_f, A]$ is the transverse part of the electric field and $B = \nabla \wedge A$. The double colon indicates Wick ordering,

$$:E(x)^2 + B(x)^2: = E(x)^2 + B(x)^2 - C_\Lambda,$$

where $C_\Lambda = (\Omega, (E(x)^2 + B(x)^2)\Omega) = 2\alpha(2\pi)^{-2} \int d^3k |\kappa(k)|^2 |k| > 0$ is the zero-point energy density of the field. Although $H_{f,\Lambda}$ is a positive operator, the integrand in the last term of (7.3) can also take negative expectation values, and therefore we may say that this electromagnetic energy has a weaker positivity than the “classical” one. There is already a stability result by Bugliaro, Fröhlich, Graf [6] asserting that this system of nuclei and electrons coupled to the quantized ultraviolet-cutoff electromagnetic field is stable, provided α and $Z\alpha^2$ are small enough. Its proof makes use of a classical stability result that is very similar to Theorem I.4, apart from constants, and is proved by adding a localization argument to some results of [49]. Through Lieb-Thirring inequalities we can obtain two different results. The first one is a “remake” of [6]:

Theorem I.8. [6, 4] *There is a constant $\varepsilon > 0$ such that for any $\Lambda > 0$ the Hamiltonian H satisfies*

$$H \geq -C(Z, \alpha, \Lambda)(N + K),$$

where

$$C(Z, \alpha, \Lambda) = \text{const } (\alpha^{-2} + Z^*)(\Lambda + Z^*), \quad Z^* = Z + 1,$$

provided $\alpha^2 Z^* \leq \varepsilon$.

Our second result is a generalization of the first one and asserts that, once we impose an ultraviolet cutoff to the electromagnetic field, there is actually no critical value anymore for α and $Z\alpha^2$, that is, the instability explained in the previous section can no longer manifest.

Theorem I.9. [32, 5] *For any $\Lambda, \alpha, Z > 0$ the Hamiltonian H satisfies*

$$H \geq -C(Z, \alpha, \Lambda)(N + K),$$

where

$$C(Z, \alpha, \Lambda) = \text{const } z^{*5} \log(1 + z^*) Z^* (\Lambda + z^{*-2} Z^*)$$

with $z^* = 1 + Z^* \alpha^2$ and $Z^* = Z + 1$.

The fundamental difference between these two results is, as already remarked, the fact that in Theorem I.8 we have critical values for α and $Z\alpha^2$, while in Theorem I.9 we don't. On the contrary, the basic similarity between them resides in the linear dependence of both bounds on the ultraviolet cutoff Λ , that causes them to diverge for Λ going to $+\infty$. Yet we have to realize that we are dealing with the non-renormalized Hamiltonian and we should properly renormalize the theory in order to obtain bounds that are uniform in Λ . However, we did not do it. We remind in this place that all the parameters appearing in the theory represent bare quantities, like the electron mass m , the electric charge e and the gyromagnetic factor g_e .

Theorem I.9 is a statement of Fefferman, Fröhlich, Graf [32], starting from a classical stability result [25] analogous to Theorem I.5 that does not however involve Lieb-Thirring inequalities.

Once again we don't have explicit values for the numerical constants, but, at least as far as the second result, Theorem I.9, is concerned, we content ourselves by knowing that stability holds and let to nature the task of precisely arranging the numerical value of the stability constant in the way it is found to be.

As remarked already a couple of times, these results follow from the classical results mentioned in Section 6, namely Theorem I.4 resp. I.5, but with the help of two inequalities for the local "energy" of the free photon field, that are the subject of the next lemma.

Lemma I.10. [6, 32] *Let f be a non-negative real-valued function in $L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Then*

a)

$$\int f(x) B(x)^2 d^3x \leq 8\pi\alpha^2 \|f\|_\infty H_{f,\Lambda} + \frac{\alpha\Lambda^4}{\pi} \|f\|_1 ; \quad (7.4)$$

b)

$$\int f(x) (\nabla \otimes B)(x)^2 d^3x \leq 8\pi\alpha^2 \Lambda^2 \|f\|_\infty H_{f,\Lambda} + \frac{2\alpha\Lambda^6}{3\pi} \|f\|_1 . \quad (7.5)$$

Proof. Let $F(x)$ be either $B(x) = \nabla \wedge A(x)$ or $\nabla \otimes B(x)$. Notice that

$$B_-(x) = \frac{i\alpha^{1/2}}{2\pi} \int \kappa(k) |k|^{1/2} \sum_{\lambda=\pm} a_\lambda(k) (\hat{k} \wedge e_\lambda(k)) e^{ikx} d^3k ,$$

and

$$\nabla \otimes B_-(x) = -\frac{\alpha^{1/2}}{2\pi} \int \kappa(k) |k|^{3/2} \sum_{\lambda=\pm} a_\lambda(k) (\hat{k} \otimes (\hat{k} \wedge e_\lambda(k))) e^{ikx} d^3k .$$

As in (7.1), we may write $F(x) = F_-(x) + F_+(x)$. Since $X^* X \geq 0$ for any arbitrary

operator X , we obtain

$$\begin{aligned}
F(x)^2 &\leq F(x)^2 + (F_-(x) - F_+(x))^*(F_-(x) - F_+(x)) \\
&= 2(F_+(x)F_-(x) + F_-(x)F_+(x)) \\
&= 2(2F_+(x)F_-(x) + [F_-(x), F_+(x)]) .
\end{aligned} \tag{7.6}$$

The commutator is $(\Omega, F_-(x)F_+(x)\Omega) = \|F_+(x)\Omega\|^2$, where Ω is the Fock vacuum. Hence, it is a multiple of the identity and is independent of x (since Ω is translation invariant). In particular,

$$\begin{aligned}
\|B_+(0)\Omega\|^2 &= \frac{2\alpha}{(2\pi)^2} \int |\kappa(k)|^2 |k| d^3k \leq \frac{\alpha\Lambda^4}{2\pi} , \\
\|\nabla \otimes B_+(0)\Omega\|^2 &= \frac{2\alpha}{(2\pi)^2} \int |\kappa(k)|^2 |k|^3 d^3k \leq \frac{\alpha\Lambda^6}{3\pi} .
\end{aligned}$$

Integrating (7.6) against $f(x)d^3x$ and using $f(x) \leq \|f\|_\infty$ and Parseval in the first term the lemma follows. \blacksquare

Finally we can perform the proof of our main results.

Proof of Theorem I.8. We split the total Hamiltonian into two parts [32, 6]

$$H = H_{\text{I}} + H_{\text{II}}$$

with

$$\begin{aligned}
H_{\text{I}} &= \sum_{i=1}^N \not{p}_i^2 + V_C + \Gamma \int \Phi_{\mathcal{R}}(x) B(x)^2 d^3x , \\
H_{\text{II}} &= H_{\text{f}} - \Gamma \int \Phi_{\mathcal{R}}(x) B(x)^2 d^3x ,
\end{aligned}$$

where $\Phi_{\mathcal{R}}$ is the positive function appearing in Theorem I.4 and Γ is a parameter we will choose later. We prove stability for the two Hamiltonians separately. In H_{I} we added to the electronic kinetic energy and the Coulomb potential a local magnetic energy term containing the quantized magnetic field B . All the fields appearing in H_{I} commute with each other and are therefore multiplication operators in the same Schrödinger representation of \mathcal{F} [32]. This would not be true if we would have added to H_{I} a term containing also the quantized electric field $E = -i\alpha[H_{\text{f}}, A]$, but since the fields involved are only $A(y)$ and $B(x)$ this enables us to reduce the proof of the stability of H_{I} to a classical problem, namely to the problem of non-relativistic matter coupled to an external electromagnetic field, with the magnetic energy retained only in a neighbourhood of the nuclei. But this is exactly the content of Theorem I.4, that yields

$$H_{\text{I}} \geq -C(Z, \Gamma)(Z+1)L^{-1}(N+K) . \tag{7.7}$$

We are left therefore with the proof of stability of H_{II} , that concerns only photons. But this can also be easily carried out, because we have at our disposal inequality (7.4), that for $f = \Phi_{\mathcal{R}}$, recalling $H_{f,\Lambda} \leq H_f$, yields, for some positive constant C ,

$$\Gamma \int \Phi_{\mathcal{R}}(x) B(x)^2 d^3x \leq C \Gamma \alpha^2 (H_f + \alpha^{-1} \Lambda^4 L^3 K) .$$

The parameter Γ must be chosen within the ranges allowed by Theorem I.4 in such a way that the factor in front of H_f is less than 1. A possible choice is $\Gamma \leq C^{-1} \alpha^{-2}$, since it follows that the condition $\varepsilon \geq \Gamma^{-1} Z^* \gtrsim \alpha^2 Z^*$ for fixed $\varepsilon > 0$ (see Theorem I.4) can be fulfilled for $\alpha^2 Z^*$ sufficiently small. For $L = (\Lambda + Z^*)^{-1}$ we have $L \leq Z^{*-1}$ and

$$H_{\text{II}} \gtrsim -\alpha^{-1} \Lambda K .$$

These choices for Γ and L imply

$$H_{\text{I}} \gtrsim -(\alpha^{-2} + Z^*)(\Lambda + Z^*)(N + K) ,$$

and hence Theorem I.8, since $1 \leq \alpha^{-1} \leq \alpha^{-2}$. ■

Proof of Theorem I.9. We extend the strategy of the proof of Theorem I.8 to include also the magnetic field gradient. When we split the total Hamiltonian into two parts,

$$H = H_{\text{I}} + H_{\text{II}} ,$$

we also add the “local energy of the magnetic field gradient”:

$$\begin{aligned} H_{\text{I}} &= \sum_{i=1}^N \not{p}_i^2 + V_C + \Gamma \int \Phi_{\mathcal{R}}(x) (B(x)^2 + \gamma L^2 (\nabla \otimes B)(x)^2) d^3x , \\ H_{\text{II}} &= H_f - \Gamma \int \Phi_{\mathcal{R}}(x) (B(x)^2 + \gamma L^2 (\nabla \otimes B)(x)^2) d^3x , \end{aligned}$$

where $B = \nabla \wedge A$, and $\Phi_{\mathcal{R}}$ is the positive function appearing in Theorem I.5. Notice that besides Γ we have the possibility to play also with γ . Both are parameters that will be chosen later. The next arguments are parallel to those used before. Indeed, even if we add a term containing the magnetic field gradient $(\nabla \otimes B)(z)$, all the fields appearing in H_{I} still commute with each other and are therefore multiplication operators in the same Schrödinger representation of \mathcal{F} [32]. Thus, we can again restrict ourselves to classical fields, but then the stability of H_{I} is exactly the statement of Theorem I.5 (for $L \leq (Z+1)^{-1}$):

$$H_{\text{I}} \geq -C(Z, \Gamma, \gamma)(Z+1)L^{-1}(N+K) . \quad (7.8)$$

We now turn to H_{II} . Stability of this Hamiltonian concerns only photons, for which we have the two inequalities (7.4) and (7.5) that together yield, for $f = \Phi_{\mathcal{R}}$,

$$\Gamma \int \Phi_{\mathcal{R}}(x) (B(x)^2 + \gamma L^2 (\nabla \otimes B)(x)^2) d^3x \leq \text{const} \Gamma \alpha^2 (1 + \gamma(\Lambda L)^2) (H_f + \alpha^{-1} \Lambda^4 L^3 K) .$$

We may now optimize over Γ, γ, L , within the ranges allowed by Theorem I.5, in such a way that the factor in front of H_f is less than 1. The resulting choice is as follows: We pick $\Gamma \ll Z^*(1 + Z^*\alpha^2)^{-1}$ and $L = \gamma^{-1/2}(\Lambda + Z^*(Z^*\alpha^2)^{-2})^{-1}$. As a result, the factor in front of H_f is indeed less than 1 and

$$H_{II} \gtrsim -Z^* \alpha \gamma^{-3/2} \Lambda K. \quad (7.9)$$

We finally choose $\gamma = z^4$ with z as in Theorem I.5. Since $z \approx 1 + Z^*\alpha^2$ we have $L \leq Z^{*-1}$, so that (7.8) applies:

$$H_I \gtrsim -z^3(1 + \log z)Z^*L^{-1}(N + K) \gtrsim -z^5(1 + \log z)Z^*(\Lambda + Z^*(Z^*\alpha^2)^{-2})(N + K).$$

This is also a lower bound to (7.9), because of $\alpha \leq 1 + Z^*\alpha^2$: $\alpha \gamma^{-3/2} = \alpha z^{-6} \leq (1 + Z^*\alpha^2)^{-5} \leq 1 \leq z^5 \leq z^5(1 + \log z)$. \blacksquare

II. The first Lieb-Thirring estimate

1. The skeleton of the proof

The aim of almost the whole chapter is to establish the first Lieb-Thirring type estimate (Theorem I.6) for the negative eigenvalues $-e_i < 0$ of the one-particle Pauli operator $h = \not{p}^2 - V$, $V(x) \geq 0$ a non-negative multiplication operator, acting on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. We reproduce it here for convenience:

Theorem I.6. *There are constants $C', C'' > 0$ such that for any vector potential $A \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$*

$$\sum e_i \leq C' \int V(x)^{5/2} d^3x + C'' \int b(x)^{3/2} V(x) d^3x. \quad (1.1)$$

The effective scalar magnetic field $b(x) = r(x)^{-2}$ has been introduced in Section I.6:

$$\frac{1}{r(x)} = \int \varphi\left(\frac{y-x}{r(x)}\right) B(y)^2 d^3y. \quad (1.2)$$

As for the usual Lieb-Thirring estimate (I.3.4), the proof begins with an application of the Birman-Schwinger principle [63] to the Pauli operator h , but slightly disguised as another, more useful, form. We namely make use of the number $n(X, \mu)$ of singular values $\lambda \geq \mu > 0$ of a compact operator X . Since $n(X, \mu)$, by definition, equals the number of eigenvalues $\lambda^2 \geq \mu^2$ of X^*X , we can write the Birman-Schwinger principle as (see (I.3.5) and (I.3.6))

$$\sum e_i \leq 2 \int_0^\infty n((\not{p}^2 + E)^{-1/2} (V - E)_+^{1/2}, 1) dE. \quad (1.3)$$

The advantage of this equivalent form is that, if we decompose the operator in (1.3) as $K_>(E) + K_<(E)$, with

$$\begin{aligned} K_>(E) &= (\not{p}^2 + \varepsilon^{-3} b + E)^{-1/2} (V - E)_+^{1/2}, \\ K_<(E) &= [(\not{p}^2 + E)^{-1/2} - (\not{p}^2 + \varepsilon^{-3} b + E)^{-1/2}] (V - E)_+^{1/2}, \end{aligned} \quad (1.4)$$

we can study the contributions of the high and low modes of \not{p}^2 , which are supposed to be captured by $K_>(E)$ resp. $K_<(E)$, separately because of Weyl's inequality [17, 68]

$$n(K_> + K_<, s_1 + s_2) \leq n(K_>, s_1) + n(K_<, s_2), \quad (1.5)$$

(we take $s_1 = s_2 = 1/2$). The value of the parameter $\varepsilon > 0$ will be chosen during the proof (in order to improve the positivity of the operator $K_>(E)$). The first factor in $K_>(E)$, in contrast with the operator $(\not{p}^2 + E)^{-1/2}$ in (1.3), which could diverge for $E \rightarrow 0$ because of the existence of zero-modes of \not{p} , is no longer sensitive to these zero-modes (the new feature of the spin magnetic case), because of the positive term $\varepsilon^{-3} b$ we added, and yields therefore the same $\int V^{5/2}$ -term as the original Lieb-Thirring estimate (I.3.4), up to constants. The

second one, which is zero for $b = 0$ and sensitive to zero-modes, while for large values of \not{p} it gives no essential contribution, yields the corrections to it due to the magnetic field. The idea behind the estimates of the singular values of these two operators is the following. First we rewrite $K_<$ with the help of the second resolvent identity and a Combes-Thomas argument as

$$K_<(E) = E^{-1/2} R_1(E) (\not{p}^2 + \varepsilon^{-3}b)^{-1} \varepsilon^{-3}b V^{1/2} R_2(E), \quad (1.6)$$

where $\|R_i(E)\| \lesssim 1$, $(i = 1, 2)$ uniformly in $E > 0$. At this point we have to deal in both terms with the operator $\not{p}^2 + \varepsilon^{-3}b$ and we proceed by localization. Indeed, we localize to neighbourhoods of length scale $\varepsilon r(x)$ around each point $x \in \mathbb{R}^3$. The localization error arising from this procedure will be of the order $\varepsilon^{-2}r(x)^{-2} = \varepsilon^{-2}b(x)$ and we can control it by playing with ε in $\varepsilon^{-3}b(x)$. Then we find an appropriate gauge where locally we have $A(x) \sim B(x)\varepsilon r(x)$. This, together with the fact that $B(x)$ and $b(x)$ are comparable over the length scale $\varepsilon r(x)$ (this comparison is done in some L^p -norm and we make use of a Sobolev inequality and the definition of $b(x)$ itself), enables us to eliminate the electromagnetic fields A and B completely from $\not{p}^2 = (p - A)^2 - \sigma \cdot B$: $A^2 \sim B^2 \varepsilon^2 r^2 \sim \varepsilon^2 b^2 r^2 \sim \varepsilon^2 b$, $|B(x)| \sim b(x)$, and all these terms can be taken to be errors that we can absorb into $\varepsilon^{-3}b$ by making ε small enough. Now we have reduced the operator \not{p}^2 to the Schrödinger operator $p^2 = -\Delta$ and we can use the usual Lieb-Thirring techniques in order to finish the proof.

The complete proof, whose main lines are given above, is spread over the next five sections: in Section 2 we discuss the length scale r as well as the partition of unity we use in the localization process; in Section 3 we localize the Pauli operator, and in Section 4 we present the appropriate gauge for the electromagnetic field in \not{p}^2 ; in Sections 5 and 6 we carry out the estimates of the contributions of the high and low modes, respectively, that together yield the proof of our first Lieb-Thirring estimate. In Section 7, we present an estimate for higher moments of the negative eigenvalues $-\varepsilon_i < 0$ of the Pauli Hamiltonian $h = \not{p}^2 - V$, and finally, in Section 8, equipped with our Lieb-Thirring estimate (1.1) and with some other nice properties of $b(x)$, we prove our classical stability result, Theorem I.4 (see Section I.6), that is the only part we left unproved in the proof of stability of non-relativistic QED of Theorem I.8.

2. Localization tools

For the definition of the basic length scale r we need a smooth positive, radially decreasing and symmetric function $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$ with $\int \varphi(z) d^3z \sim 1$, e.g. $\varphi(z) = (1 + \frac{1}{2}z^2)^{-2}$. Important properties of this function are

$$z \cdot \nabla \varphi(z) \leq 0, \quad (2.1)$$

$$|D_1 \dots D_n \varphi| \lesssim \varphi, \quad (n \in \mathbb{N}) \quad (2.2)$$

where $D_j = \partial_i$, ($i = 1, 2, 3$) or $D_j = z \cdot \nabla$. Our length scale is defined as the solution $r = r(x) > 0$ of the equation

$$r \int \varphi\left(\frac{y-x}{r}\right) U(y) d^3y = 1,$$

where $U \equiv B^2 \geq 0$ belongs to $L^1(\mathbb{R}^3, \varphi(z) d^3z)$, since the magnetic energy $(8\pi\alpha^2)^{-1} \int B(x)^2 d^3x$ is finite and φ is uniformly bounded from above by 1: $\varphi(z) \leq 1$. The solution exists and is unique, except for the case $U \equiv 0$ (a.e.), where we set $r \equiv \infty$. In fact, the integral is finite for all $r > 0$ and $x \in \mathbb{R}^3$, and on the interval $r \in (0, +\infty)$ the l.h.s. is increasing from 0 to ∞ because of

$$\frac{d}{dr} r \int \varphi\left(\frac{y-x}{r}\right) U(y) d^3y = \int (\varphi(z) - z \cdot \nabla \varphi(z)) \Big|_{z=(y-x)/r} U(y) d^3y > 0,$$

due to (2.1). By the same reason and the implicit function theorem, $r(x)$ is smooth, and, together with the effective magnetic field $b(x) = r(x)^{-2}$, enjoys the following “handy” properties:

Lemma II.1.

a) For any multiindex $\alpha \in \mathbb{N}^3$, the length scale r is tempered in the following sense:

$$|\partial^\alpha r(x)| \lesssim r(x)^{-(|\alpha|-1)}. \quad (2.3)$$

In particular, r is uniformly Lipschitz.

In terms of the effective magnetic field b this reads

$$|\partial^\alpha b(x)| \lesssim b(x)^{|\alpha|/2+1}. \quad (2.4)$$

b) For $\varepsilon > 0$ small enough

$$|x - y| \leq \varepsilon r(x) \implies \frac{1}{2} \leq \frac{r(y)}{r(x)} \leq 2. \quad (2.5)$$

Proof of (2.3). Setting $z = (y - x)/r(x)$ we have

$$\frac{\partial z}{\partial x} = -\frac{1}{r(x)}(\mathbf{1} + z \otimes \nabla r) \quad (2.6)$$

and

$$(1 - m(x))\partial_i r(x) = m_i(x) , \quad (2.7)$$

where

$$m(x) = r(x) \int z \cdot \nabla \varphi(z) U(y) d^3 y , \quad m_i(x) = r(x) \int (\partial_i \varphi)(z) U(y) d^3 y .$$

We denote by V_n , ($n \in \mathbb{N}$), the space of finite sums of functions of the form

$$f(x) = r(x)^{-(n-1)} P(\{\partial^\alpha r\}) \int \psi(z) U(y) d^3 y ,$$

where ψ is of the form $D_1 \dots D_k \varphi$ and P is a monomial in the derivatives $\{\partial^\alpha r\}_{|\alpha| \leq n}$ of order 0 in the sense that it contains as many powers of ∂ as of r . One verifies $r^{-1} V_n \subset V_{n+1}$ and, using (2.6), $\partial_i V_n \subset V_{n+1}$. Moreover, $m, m_i \in V_0$. We can now prove (2.3) for $|\alpha| = n + 1$ assuming it for $|\alpha| \leq n$, the latter being true for $n = 0$ because of (2.2). Then $f \in V_n$ satisfies $|f| \lesssim r(x)^{-n}$. By applying ∂^α , ($|\alpha| = n$) to (2.7) we obtain $(1 - m(x))\partial^\alpha \partial_i r(x) \in V_n$ and thus the desired bound since $m \leq 0$ due to (2.1). ■

Proof of (2.5). Setting $C = \sup_x |\nabla r(x)| (\lesssim 1)$, we have the inequality $r(y) \leq r(x) + C|x - y|$ as well as the one with x and y interchanged, so that

$$(1 - C\varepsilon)r(x) \leq r(y) \leq (1 + C\varepsilon)r(x) . \quad \blacksquare$$

The localization of the physical space \mathbb{R}^3 to neighbourhoods of radius $\varepsilon r(x)$ is done by introducing the functions

$$j_y(x) = (\varepsilon r(x))^{-3/2} \chi\left(\frac{x - y}{\varepsilon r(x)}\right) , \quad (y \in \mathbb{R}^3)$$

where $0 < \varepsilon \leq 1$ and $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\text{supp } \chi \subset \{z \mid |z| \leq 1\}$ and $\int \chi(z)^2 dz = 1$. Actually, they constitute a partition of unity with a special additional property:

Lemma II.2.

$$\int j_y(x)^2 d^3 y = 1 , \quad (2.8)$$

$$\int |\partial^\alpha j_y(x) \partial^\beta j_y(x)| d^3 y \lesssim (\varepsilon r(x))^{-(|\alpha|+|\beta|)} \quad (2.9)$$

for any $\alpha, \beta \in \mathbb{N}^3$, where $\partial = \partial/\partial x$.

Proof. The proof of (2.9) is similar to that of (2.3). We set $z = (x - y)/(\varepsilon r(x))$, so that

$$\frac{\partial z}{\partial x} = -\frac{1}{\varepsilon r(x)}(\mathbf{1} - \varepsilon z \otimes \nabla r) . \quad (2.10)$$

Let V_n , ($n \in \mathbb{N}$), be the space of finite sums of functions of the form

$$f_y(x) = (\varepsilon r(x))^{-(n+\frac{3}{2})} P(\{(\varepsilon \partial)^\alpha r\}) \psi(z), \quad (2.11)$$

where ψ is of the form $D_1 \dots D_k \chi$ and P is a monomial in the derivatives $\{(\varepsilon \partial)^\alpha r\}_{|\alpha| \leq n}$ of order 0 in the sense that it contains as many powers of ∂ as of r . From (2.10) we obtain $\partial_i V_n \subset V_{n+1}$ and hence $\partial^\alpha j_y(x) \in V_{|\alpha|}$. For $f_{i,y} \in V_{n_i}$, ($i = 1, 2$) of the form (2.11) we have

$$\int |f_{1,y}(x) f_{2,y}(x)| d^3y \leq \text{const } (\varepsilon r(x))^{-(n_1+n_2)} \int (\varepsilon r(x))^{-3} |\psi_1(z) \psi_2(z)| d^3y,$$

because the $P_i(\{(\varepsilon \partial)^\alpha r\})$, ($i = 1, 2$) are uniformly bounded due to (2.3). The last integral is seen to be uniformly bounded by means of the change of variable $y \rightarrow z$, which also yields (2.8). \blacksquare

We now intend to prove the energy estimate mentioned in Section I.6 and a local variant of it, that show how the quantity $b(x) = r(x)^{-2}$ is controlled by $U(x) = B(x)^2$.

Lemma II.3.

a)

$$\int r(x)^{-4} d^3x \lesssim \int U(x) d^3x. \quad (2.12)$$

b) Let $\Omega_L = \{x \mid \text{dist}(x, \Omega) < L\}$ for $\Omega \subset \mathbb{R}^3$ and $L > 0$. Then, for any $L > 0$ and $\Omega \subset \mathbb{R}^3$ there is a function $\Phi_{\Omega,L} \geq 0$ satisfying $\|\Phi_{\Omega,L}\|_\infty \lesssim 1$ and $\|\Phi_{\Omega,L}\|_1 \lesssim |\Omega_L|$, uniformly in Ω, L , such that

$$\int_{\Omega_L} r(x)^{-4} d^3x \lesssim \int \Phi_{\Omega,L}(x) U(x) d^3x + |\Omega_L| \cdot L^{-4}. \quad (2.13)$$

The following lemma is at the basis of the previous inequalities.

Lemma II.4. Let $s = g_+(t) \geq 1$ and $s = g_-(t) \leq 1$ be the two positive solutions of

$$t^2 = 2(s^{5/2} - 1)(1 - s^{-1/2}). \quad (2.14)$$

for $t \geq 0$. Then

$$r(y) g_- \left(\frac{|y-x|}{r(y)} \right) \leq r(x) \leq r(y) g_+ \left(\frac{|y-x|}{r(y)} \right) \quad (2.15)$$

for all $x, y \in \mathbb{R}^3$.

Proof. Note that the r.h.s. of (2.14) is strictly increasing (resp. decreasing) on $s \in [1, \infty)$ (resp. $s \in (0, 1]$), both having range $[0, \infty)$. Thus g_\pm are well defined. By scaling we may assume $r(y) = 1$ and $y = 0$. We claim that

$$g_+(|x|) \varphi \left(\frac{z-x}{g_+(|x|)} \right) \geq \varphi(z) \quad (2.16)$$

for all $x, z \in \mathbb{R}^3$. Integrating against $U(z)d^3z$ gives

$$g_+ (|x|) \int \varphi \left(\frac{z-x}{g_+ (|x|)} \right) U(z) d^3z \geq 1 ,$$

which, by definition of $r(x)$, implies $r(x) \leq g_+ (|x|)$, as was to be proved. To prove (2.16), we raise both sides to the power $-\frac{1}{2}$ and multiply them by 2 to reduce matters to the estimate

$$g_+^{-1/2} (2 + g_+^{-2} (z-x)^2) \leq 2 + z^2 , \quad (2.17)$$

where $g_+ = g_+ (|x|)$. Pick a coordinate system in which $x = (|x|, 0)$ and $z = (z_1, z_\perp)$ in $\mathbb{R} \times \mathbb{R}^2$. Then (2.17) reads

$$g_+^{-1/2} (2 + g_+^{-2} (z_1 - |x|)^2 + g_+^{-2} z_\perp^2) \leq 2 + z_1^2 + z_\perp^2 .$$

Since $g_+ \geq 1$, this reduces to the case $z_\perp = 0$ and, after some algebra, to the quadratic inequality

$$(g_+^{5/2} - 1) z_1^2 + 2|x| z_1 + [2g_+^{5/2} (1 - g_+^{-1/2}) - |x|^2] \geq 0 .$$

Due to $g_+^{5/2} - 1 \geq 0$, it is enough to check the vanishing of its discriminant, i.e.,

$$4|x|^2 = 4(g_+^{5/2} - 1)[2g_+^{5/2} (1 - g_+^{-1/2}) - |x|^2] .$$

A little manipulation reduces this to the definition of $g_+ (|x|)$. A similar argument proves the other half of (2.15). ■

Proof of (2.12). The inequality

$$r(x)^{-3} \varphi \left(\frac{y-x}{r(x)} \right) \leq r(y)^{-3} \left(g_- (|z|)^{3/2} + \frac{1}{2} g_+ (|z|)^{-1/2} z^2 \right)^{-2} \equiv r(y)^{-3} g(|z|) ,$$

where $z = (y-x)/r(y)$, follows from (2.15) by means of

$$\left(\frac{r(x)^3}{r(y)^3} \frac{1}{\varphi \left(\frac{y-x}{r(x)} \right)} \right)^{1/2} = \left(\frac{r(x)}{r(y)} \right)^{3/2} + \frac{1}{2} \left(\frac{r(x)}{r(y)} \right)^{-1/2} z^2 \geq g_- (|z|)^{3/2} + \frac{1}{2} g_+ (|z|)^{-1/2} z^2 .$$

It implies, using the change of variables $x \rightarrow z$,

$$\int r(x)^{-3} \varphi \left(\frac{y-x}{r(x)} \right) d^3x \leq \int g(|z|) d^3z \quad (2.18)$$

for all $y \in \mathbb{R}^3$. The last integral is finite due to $g(t) \lesssim (1+t)^{-16/5}$. This follows from $g_-(0) = 1$ and from $g_+(t) \lesssim t^{4/5}$ for large t . Integrating (2.18) against $U(y)d^3y$ yields (2.12). ■

Proof of (2.13). We begin with the case of $\Omega = \{y\}$ consisting of a single point, for which we claim

$$\int_{|x-y| < L} r(x)^{-4} d^3x \lesssim \int \varphi\left(\frac{x-y}{L}\right) U(x) d^3x + L^{-1}. \quad (2.19)$$

We assume, without loss, $y = 0$ and distinguish between $\sup_{|x| < L} \varepsilon r(x) > 2L$ and the opposite inequality, where ε is as in (2.5). In the first case, i.e., $\varepsilon r(x_0) > 2L$ for some $|x_0| < L$, we have $|x - x_0| < 2L$ for all $|x| < L$, which implies $r(x) \geq r(x_0)/2 \geq \varepsilon^{-1}L$ and $\int_{|x| < L} r(x)^{-4} d^3x \lesssim L^{-1}$. In the second case, i.e., $\varepsilon r(x) \leq 2L$ for $|x| < L$, we will first show

$$\int_{|x| < L} r(x)^{-3} \varphi\left(\frac{y-x}{r(x)}\right) d^3x \lesssim \varphi(y/L). \quad (2.20)$$

Integrating against $U(y) d^3y$ yields (2.19), even without the L^{-1} term. To prove (2.20), it suffices that the l.h.s. is uniformly bounded in y , due to (2.18), and bounded by a constant times $(L/|y|)^4$ for $|y| \geq 2L$. Indeed, from $\varphi(z) \leq 4z^{-4}$ and from $|y-x| \geq |y|/2$ for $|x| < L$ we have

$$r(x)^{-3} \varphi\left(\frac{y-x}{r(x)}\right) \lesssim \frac{r(x)}{|y-x|^4} \lesssim \frac{L}{|y|^4},$$

and hence the bound just stated. Then the l.h.s. of (2.20) is bounded above by a constant times $\min(1, (L/|y|)^4) \lesssim \varphi(y/L)$. The l.h.s. of (2.13) can now be estimated by using $\chi_{\Omega_L}(x) \lesssim L^{-3} \int_{\Omega_L} \chi(|x-y| < L) d^3y$ and (2.19). The result is (2.13) with

$$\Phi_{\Omega, L}(x) = L^{-3} \int_{\Omega_L} \varphi\left(\frac{x-y}{L}\right) d^3y,$$

which clearly satisfies the claimed bounds. ■

3. Localization of the Birman-Schwinger kernel

It is from now on understood that $\varepsilon > 0$ is small enough.

As one may guess by looking at $K_>(E)$ and $K_<(E)$, equations (1.4) and (1.6), it will turn out that it suffices to localize $(\not{p}^2 + \varepsilon^{-3}b)^2$.

Lemma II.5.

$$(\not{p}^2 + \varepsilon^{-3}b)^2 \geq \int j_y(\not{p}^4 + \frac{1}{2}\varepsilon^{-6}b^2)j_y d^3y. \quad (3.1)$$

For the proof of (3.1) we need some additional tools, that are provided by the following two lemmas.

Lemma II.6. *Let $U \in L^q(\mathbb{R}^3)$ with $q = 3/2$, resp. 1. Then*

$$U \leq \frac{1}{3}(\frac{\pi}{2})^{-4/3}\|U\|_{3/2}p^2, \quad (3.2)$$

$$U \leq (4\pi)^{-1}\|U\|_1(lp^4 + l^{-3}) \quad (3.3)$$

for all $l > 0$.

Remark. *By the diamagnetic inequality ([63] and (I.5.2)), p^2 in (3.2) can be replaced by $D^2 = (p - A)^2$, that is,*

$$U \leq \frac{1}{3}(\frac{\pi}{2})^{-4/3}\|U\|_{3/2}D^2, \quad (3.4)$$

for $U \in L^{3/2}(\mathbb{R}^3)$.

Proof of Lemma II.6. By Hölder's and Sobolev's inequalities we have, for $U \in L^{3/2}(\mathbb{R}^3)$ and $\psi \in H^{2,2}(\mathbb{R}^3)$,

$$(\psi, U\psi) = \|U|\psi|^2\|_1 \leq \|U\|_{3/2}\|\psi\|_6^2 \leq \frac{1}{3}(\frac{\pi}{2})^{-4/3}\|U\|_{3/2}\|\nabla\psi\|_2^2,$$

which proves (3.2). The other inequality is just

$$(\psi, U\psi) \leq \|U\|_1\|\psi\|_\infty^2 \leq (4\pi)^{-1}\|U\|_1(l\|\Delta\psi\|_2^2 + l^{-3}\|\psi\|_2^2),$$

where the estimate for $\|\psi\|_\infty$ is found in [59] ((IX.25) and proof). ■

Lemma II.7.

$$DbD \lesssim \not{p}^2b + b\not{p}^2 + \varepsilon^{-2}b^2. \quad (3.5)$$

Proof. We have

$$\begin{aligned} 2DbD &= D^2b + bD^2 - [D, [D, b]] = \not{p}^2b + b\not{p}^2 + 2bB \cdot \sigma + \Delta b \\ &\leq \not{p}^2b + b\not{p}^2 + 2b|B| + \text{const } b^2, \end{aligned}$$

where we used $\not{p}^2 = D^2 - B \cdot \sigma$ and (2.4). The proof will be completed once we show

$$b|B| \lesssim \varepsilon^{1/2}(DbD + \varepsilon^{-2}b^2).$$

To this end we set $\tilde{\chi}_y(x)$ to be the characteristic function of $\tilde{K}_y = \{x \mid |x - y| < \varepsilon r(x)\}$ and note that $\text{supp } j_y \subset \tilde{K}_y$. We may thus decompose

$$b|B| = \int j_y b|B| \tilde{\chi}_y j_y d^3 y ,$$

where

$$\|b|B|\tilde{\chi}_y\|_{3/2} \leq \|b\tilde{\chi}_y\|_\infty \|B\tilde{\chi}_y\|_2 \|\tilde{\chi}_y\|_6 \lesssim b(y) \cdot r(y)^{-1/2} \cdot (\varepsilon r(y))^{1/2} = \varepsilon^{1/2} b(y) .$$

To obtain this estimate, the first factor has been bounded by using (2.5), which also yields $\tilde{K}_y \subset K_y = \{x \mid |x - y| < 2\varepsilon r(y)\}$. The second factor is bounded in terms of (1.2), and the third one by $|K_y|^{1/6}$. (3.4) now implies

$$b|B| \lesssim \varepsilon^{1/2} \int j_y D b(y) D j_y d^3 y \leq 4\varepsilon^{1/2} \int j_y D b D j_y d^3 y \lesssim \varepsilon^{1/2} (D b D + \varepsilon^{-2} b^2) ,$$

where we used (2.5), the identity $2j_y D b D j_y = j_y^2 D b D + D b D j_y^2 + 2b(\nabla j_y)^2$, as well as the bound (2.9) for $|\alpha| = |\beta| = 1$. \blacksquare

Proof of (3.1). We expand

$$(\not{p}^2 + \varepsilon^{-3} b)^2 = \not{p}^4 + \varepsilon^{-3} (\not{p}^2 b + b \not{p}^2) + \varepsilon^{-6} b^2 ,$$

localize \not{p}^4 ,

$$\not{p}^4 = \int (j_y \not{p}^4 j_y + \tfrac{1}{2} [j_y, [j_y, \not{p}^4]]) d^3 y , \quad (3.6)$$

compute the double commutator

$$\tfrac{1}{2} [j_y, [j_y, \not{p}^4]] = \tfrac{1}{2} \{ [j_y, [j_y, \not{p}^2]], \not{p}^2 \} + [j_y, \not{p}^2]^2 , \quad (3.7)$$

as well as the expressions

$$\begin{aligned} [j_y, \not{p}^2] &= [j_y, D^2] = i(\nabla j_y \cdot D + D \cdot \nabla j_y) , & [j_y, [j_y, \not{p}^2]] &= -2(\nabla j_y)^2 , \\ [j_y, \not{p}^2]^2 &= -4(D \cdot \nabla j_y)(\nabla j_y \cdot D) - (\Delta j_y)^2 + 2\nabla \cdot (\nabla j_y \Delta j_y) . \end{aligned}$$

The localization error for \not{p}^2 ,

$$L(x) = \int (\nabla j_y(x))^2 d^3 y ,$$

satisfies $L \lesssim \varepsilon^{-2} b$ and $(\nabla L)^2 \lesssim \varepsilon^{-6} b^3$ due to (2.9). Moreover, we have

$$\pm(\not{p}^2 f + f \not{p}^2 - 2\not{p} f \not{p}) \lesssim \varepsilon^{-1} \not{p} b \not{p} + \varepsilon^{-5} b^2$$

for $f = L$ or $f = \varepsilon^{-3}b$. Indeed, the l.h.s. is

$$\pm[\not{D}, [\not{D}, f]] = \mp i[\not{D}, \nabla f \cdot \sigma] = -X^* X + \varepsilon^{-1} \not{D} b \not{D} + \varepsilon b^{-1} (\nabla f)^2$$

with $X = (\varepsilon^{-1}b)^{1/2} \not{D} \pm i(\varepsilon b^{-1})^{1/2} \nabla f \cdot \sigma$. The contribution to (3.6) of the first term in (3.7) is thus, up to the sign,

$$\begin{aligned} - \int \frac{1}{2} \{ [j_y, [j_y, \not{D}^2]], \not{D}^2 \} d^3y &= \not{D}^2 L + L \not{D}^2 \leq 2 \not{D} L \not{D} + \text{const} (\varepsilon^{-1} \not{D} b \not{D} + \varepsilon^{-5} b^2) \\ &\leq \text{const} (\varepsilon^{-2} \not{D} b \not{D} + \varepsilon^{-5} b^2) \leq \frac{1}{2} \varepsilon^{-3} (\not{D}^2 b + b \not{D}^2) + \text{const} \varepsilon^{-5} b^2. \end{aligned} \quad (3.8)$$

The contribution from the second term in (3.7) is, again up to the sign,

$$- \int [j_y, \not{D}^2]^2 d^3y \leq \text{const} (\varepsilon^{-2} D b D + \varepsilon^{-4} b^2) \leq \frac{1}{2} \varepsilon^{-3} (\not{D}^2 b + b \not{D}^2) + \varepsilon^{-5} b^2, \quad (3.9)$$

where we used the bound on L and (2.9, 3.5). Together, (3.6-9) show that

$$(\not{D}^2 + \varepsilon^{-3}b)^2 \geq \int j_y \not{D}^4 j_y d^3y + (\varepsilon^{-6} - \text{const} \varepsilon^{-5}) b^2. \quad \blacksquare$$

4. Choice of an appropriate gauge

In the foregoing section we localized $(\not{D}^2 + \varepsilon^{-3}b)^2$, obtaining (3.1):

$$(\not{D}^2 + \varepsilon^{-3}b)^2 \geq \int j_y (\not{D}^4 + \frac{1}{2} \varepsilon^{-6} b^2) j_y d^3y.$$

In this section we are going to locally replace on the r.h.s. the fields A by a gradient and b by a constant. We state the result in a concise and elegant way by means of the direct integral $\widehat{\mathcal{H}} = \int_{\mathbb{R}^3}^{\oplus} \mathcal{H} d^3y = L^2(\mathbb{R}^3; \mathcal{H})$ with elements $\Psi = \{\psi_y\}_{y \in \mathbb{R}^3}$, $\psi_y \in \mathcal{H}$. It is a Hilbert space with respect to the scalar product $(\Psi, \Phi) = \int d^3y (\psi_y, \varphi_y)_{\mathcal{H}}$, where $\Psi, \Phi \in \widehat{\mathcal{H}}$ and $(\cdot, \cdot)_{\mathcal{H}}$ is the scalar product in \mathcal{H} . The localization is effectively carried out with the linear map

$$J : \mathcal{H} \rightarrow \widehat{\mathcal{H}}, \quad J = \int_{\mathbb{R}^3}^{\oplus} j_y d^3y,$$

(i.e., $J\psi = \{j_y \psi\}_{y \in \mathbb{R}^3}$). In these terms, (2.8) simply reads $J^* J = 1$. Remark that $J^* : \widehat{\mathcal{H}} \rightarrow \mathcal{H}$ acts on $\Psi \in \widehat{\mathcal{H}}$ as $J^* \Psi = \int d^3y j_y(x) \psi_y(x)$. As already said, we localize to neighbourhoods where the magnetic vector potential A can be taken to be a gradient (i.e., $B = \nabla \wedge A = 0$) and b a constant. Precisely, we want to find a gauge where the operator $\not{D}^2 + \varepsilon^{-3}b$ then takes the local form

$$H_y = [(p - \nabla f_y) \cdot \sigma]^2 + \varepsilon^{-3}b(y),$$

acting on \mathcal{H} , where $f_y(x)$ is a function to be specified later. In terms of the direct integral representation we have

$$\hat{H} : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}} , \quad \hat{H} = \int_{\mathbb{R}^3}^{\oplus} H_y d^3y ,$$

(i.e., $\hat{H}\{\psi_y\}_{y \in \mathbb{R}^3} = \{H_y\psi_y\}_{y \in \mathbb{R}^3}$: \hat{H} acts on fibers), and our aim is the proof of

Lemma II.8.

$$j_y(\hat{p}^4 + \frac{1}{2}\varepsilon^{-6}b^2)j_y \gtrsim j_y H_y^2 j_y . \quad (4.1)$$

Our main intermediate result in the proof of (1.1) is then obtained combining Lemma II.5 and Lemma II.8:

$$(\hat{p}^2 + \varepsilon^{-3}b)^2 \gtrsim J^* \hat{H}^2 J . \quad (4.2)$$

As a consequence we also have

$$\hat{p}^2 + \varepsilon^{-3}b \gtrsim J^* \hat{H} J . \quad (4.3)$$

To see this, we note that $JJ^* \leq 1$ yields $J^* \hat{H}^2 J \geq (J^* \hat{H} J)^2$. Moreover, $X^2 \geq Y^2$ implies $X \geq Y$ for operators $X, Y \geq 0$ (notice by the way that the converse is not true).

Our strategy relies on the choice of an appropriate gauge (we exploit of course the freedom that is left when we work in the Coulomb gauge $\nabla \cdot A = 0$, without breaking it), and this is done in the next lemma.

Lemma II.9. [25] *Let $K = \{x \mid |x| < 1\}$ be the unit ball, and $K^* = 2K$. Given a vector field $B \in L^2(K^*, \mathbb{R}^3)$ with $\nabla \cdot B = 0$ (as a distribution), there is a vector field A such that*

$$\begin{aligned} \nabla \wedge A &= B , \quad \nabla \cdot A = 0 , \\ \int_K A(x) d^3x &= 0 \end{aligned} \quad (4.4)$$

and

$$\int_K (\nabla \otimes A(x))^2 d^3x \lesssim \int_{K^*} B(x)^2 d^3x . \quad (4.5)$$

Remark. Sobolev's inequality and (4.4) imply

$$\left(\int_K A(x)^6 d^3x \right)^{1/3} \lesssim \int_K (\nabla \otimes A(x))^2 d^3x . \quad (4.6)$$

Proof. The operator Δ on $L^2(K^*, \mathbb{R}^3)$ with boundary conditions

$$n \cdot (n \cdot \nabla) F = 0 , \quad n \wedge F = 0 \quad (4.7)$$

on ∂K^* (with normal n) is self-adjoint. Moreover, $\Delta F = 0$ implies $\int_{K^*} F \Delta F = 0$ and, after an integration by parts, $F = 0$, i.e., $\text{Ker } \Delta = \{0\}$. The equation

$$-\Delta F = B$$

has thus a solution F with $\|F\|_{2,K^*} \lesssim \|B\|_{2,K^*}$: since Δ has discrete spectrum and is one-to-one it has a bounded inverse. We remark that $\nabla \cdot F$ is harmonic on K^* (because of $\nabla \cdot B = 0$) with $\nabla \cdot F = 0$ on ∂K^* by (4.7). Hence $\nabla \cdot F = 0$ on K^* . Setting $A = \nabla \wedge F$ we have $\nabla \cdot A = 0$ and

$$B = -\Delta F = \nabla \wedge (\nabla \wedge F) - \nabla(\nabla \cdot F) = \nabla \wedge A.$$

The estimate (4.5) follows from $(\nabla \otimes A(x))^2 \leq 2(\nabla \otimes \nabla \otimes F(x))^2$ and from the elliptic estimate

$$\|\nabla \otimes \nabla \otimes F\|_{2,K} \lesssim \|F\|_{2,K^*} + \|\Delta F\|_{2,K^*}. \quad (4.8)$$

This, on its turn, can be proved as follows: let $j \in C_0^\infty(\mathbb{R}^3)$ be a radially decreasing and symmetric function satisfying $j(x) = 1$ for $x \in K$ and $j(x) = 0$ for $x \in \mathbb{R}^3 \setminus K^*$. The l.h.s. of (4.8) is smaller than $\|\nabla \otimes \nabla \otimes (jF)\|_2 = \|p \otimes p \widehat{(jF)}\|_2$, by Plancherel. With $p \otimes p \leq 1 + p^2$ and Plancherel again, we find

$$\|\nabla \otimes \nabla \otimes F\|_{2,K} \lesssim \|F\|_{2,K^*} + \|\Delta jF\|_2.$$

Computing the last term explicitly,

$$\Delta jF = (\Delta j)F + 2(\nabla j \cdot \nabla)F + j\Delta F,$$

we easily see that the first and third term are bounded above by a constant times the first resp. second term on the r.h.s. of (4.8). It remains to estimate the middle term:

$$\|(\nabla j \cdot \nabla)F\|_2^2 \lesssim \|\nabla \otimes F\|_{2,K^*}^2 = \sum_{i,n=1}^3 \int_{K^*} (\partial_i F_n)^2 d^3x = \sum_{n=1}^3 \int_{K^*} (\nabla \cdot (F_n \nabla F_n) - F_n \Delta F_n) d^3x.$$

The first term vanishes because of Gauss theorem and (4.7). The second one is bounded above by $\frac{1}{2} F_n^2 + \frac{1}{2} (\Delta F_n)^2$, that is, after summation over n , by a constant times the r.h.s. of (4.8)

By adding a constant vector to $A(x)$ we ensure (4.4) without spoiling the other properties. \blacksquare

With the help of the gauge described in this last lemma we can prove (4.1).

Proof of (4.1). Let $A_y(x)$ be the vector potential on $K_y = \{x \mid |x-y| < 2\epsilon r(y)\}$ which by scaling corresponds to the one given in Lemma II.9 on K . Since $\nabla \wedge A = \nabla \wedge A_y$ on K_y we have $A_y = A - \nabla f_y$ for some function f_y on K_y (which we extend to \mathbb{R}^3 arbitrarily). Upon

making the gauge transformation $\psi \mapsto e^{-if_y}\psi$, $A \mapsto A - \nabla f_y$ we may assume $A_y = A$ on K_y and $H_y = (p \cdot \sigma)^2 + \varepsilon^{-3}b(y)$. The comparison of $p \cdot \sigma$ and \not{D} begins with

$$p^2 = (p \cdot \sigma)^2 = (\not{D} + A \cdot \sigma)^2 = \not{D}^2 + A^2 + \{A \cdot \sigma, \not{D}\} = \not{D}^2 - A^2 + \{A, p\} + B \cdot \sigma ,$$

where we used

$$\{\not{D}, v \cdot \sigma\} = \{D, v\} + (\nabla \wedge v) \cdot \sigma . \quad (4.9)$$

Note that $A \cdot p = p \cdot A$ due to $\nabla \cdot A = 0$. We therefore have $\{A, p\}^2 = 4(p \cdot A)(A \cdot p) \leq 4p(A^2)p$ and

$$p^4 \leq 4(\not{D}^4 + A^4 + 4p(A^2)p + B^2) . \quad (4.10)$$

Upon multiplying this on both sides by j_y we may replace A (resp. B) by $\chi_y A$ (resp. $\chi_y B$), where $\chi_y(x)$ is the characteristic function of K_y , due to $\text{supp } j_y \subset K_y$. To estimate these terms, note that the scale invariant inequalities (4.6, 4.5) followed by (1.2) yield $\|A^2 \chi_y\|_3 \lesssim \|B^2 \chi_y^*\|_1 \lesssim r(y)^{-1}$. This implies, together with $\|\chi_y\|_3 = |K_y|^{1/3} \lesssim \varepsilon r(y)$,

$$\begin{aligned} \|B^2 \chi_y\|_1 &\lesssim r(y)^{-1} , \\ \|A^2 \chi_y\|_{3/2} &\leq \|A^2 \chi_y\|_3 \|\chi_y\|_3 \lesssim \varepsilon , \\ \|A^4 \chi_y\|_1 &\leq \|A^2 \chi_y\|_3^2 \|\chi_y\|_3 \lesssim \varepsilon r(y)^{-1} \leq r(y)^{-1} . \end{aligned}$$

We then get from (3.2, 3.3) by taking $l = \varepsilon r(y)$

$$j_y (A^4 + p(A^2)p + B^2) j_y \lesssim j_y [\varepsilon p^4 + \varepsilon^{-3}b(y)^2] j_y .$$

Finally, we apply this to (4.10) and use (2.5), so as to obtain

$$\begin{aligned} j_y [\not{D}^4 + \frac{1}{2}\varepsilon^{-6}b(x)^2] j_y &\gtrsim j_y [p^4(1 - \text{const } \varepsilon) + \varepsilon^{-6}(1 - \text{const } \varepsilon^3)b(y)^2] j_y \\ &\geq \frac{1}{4} j_y [p^2 + \varepsilon^{-3}b(y)]^2 j_y = \frac{1}{4} j_y H_y^2 j_y . \end{aligned} \quad \blacksquare$$

5. High modes

We direct our attention to high modes first. For these modes we don't need the full power of our main inequality (4.2), but only its weaker version (4.3). We let $\widehat{H}^0 = \int_{\mathbb{R}^3}^{\oplus} H_y^0 d^3y$ with $H_y^0 = [(p - \nabla f_y) \cdot \sigma]^2$ and bound the r.h.s. of (4.3) from below by $J^* \widehat{H}^0 J$. The contribution of $K_>$ to (1.3) is estimated by means of

$$(\mathcal{P}^2 + \varepsilon^{-3} b + E)^{-1} \lesssim (J^* (\widehat{H}^0 + E) J)^{-1} \leq J^* (\widehat{H}^0 + E)^{-1} J. \quad (5.1)$$

The first one of these two inequalities follows from $X \geq Y$ implying $X^{-1} \leq Y^{-1}$ for operators $X, Y \geq 0$. The second one comes from this and

$$J^* X J \geq (J^* X^{-1} J)^{-1} \quad (5.2)$$

for $J^* J = 1$. This inequality [60, 67] is $J^* X J \geq J^* X^{1/2} \Pi X^{1/2} J$ for the orthogonal projection $\Pi = X^{-1/2} J (J^* X^{-1} J)^{-1} J^* X^{-1/2}$. By (5.1) we have

$$n(K_>(E), \frac{1}{2}) \leq n((\widehat{H}^0 + E)^{-1/2} J (V - E)_+^{1/2}, \text{const}),$$

which is further estimated thanks to $n(X, 1) \leq \text{tr}((X^* X)^2)$:

$$\begin{aligned} n(K_>(E), \frac{1}{2}) &\lesssim \text{tr}[(V - E)_+^{1/2} J^* (\widehat{H}^0 + E)^{-1} J (V - E)_+ J^* (\widehat{H}^0 + E)^{-1} J (V - E)_+^{1/2}] \\ &= \int \text{tr}[j_y j_{y'} e^{i(f_y - f_{y'})} (V - E)_+ (p^2 + E)^{-1} j_y j_{y'} e^{-i(f_y - f_{y'})} (V - E)_+ (p^2 + E)^{-1}] d^3y d^3y', \end{aligned} \quad (5.3)$$

where we used $H_y^0 = e^{if_y} p^2 e^{-if_y}$, since $\nabla_x \wedge \nabla_x f_y = 0$. By the proof of the usual Lieb-Thirring inequality (see I.3.6-9), the trace in (5.3) is bounded by $(8\pi)^{-1} E^{-1/2} \int (V(x) - E)_+^2 j_y(x)^2 j_{y'}(x)^2 d^3x$, that is, the contribution of the high modes is, with (2.8),

$$n(K_>(E), \frac{1}{2}) \lesssim E^{-1/2} \int (V(x) - E)_+^2 d^3x. \quad (5.4)$$

After integration over E we obtain

$$\int_0^\infty n(K_>(E), \frac{1}{2}) dE \lesssim \int V(x)^{5/2} d^3x,$$

i.e., the first term on the r.h.s. of (1.1), as expected.

6. Low modes

We turn our attention to the low modes, where spin matters and causes some technical difficulties. It is here that we use (4.2), but before we have to prove

Lemma II.10. *There are two bounded operators $R_1(E)$ and $R_2(E)$ satisfying $\|R_i(E)\| \lesssim 1$, ($i = 1, 2$) uniformly in $E > 0$, such that*

$$K_<(E) = E^{-1/2} R_1(E) (\not{p}^2 + \varepsilon^{-3} b)^{-1} \varepsilon^{-3} b V^{1/2} R_2(E). \quad (6.1)$$

Proof. Using that $X^{-1/2} = \frac{1}{\pi} \int_0^\infty \mu^{-1/2} (X + \mu)^{-1} d\mu$ for $X \geq 0$, together with the second resolvent identity $X^{-1} - (X + Y)^{-1} = X^{-1} Y (X + Y)^{-1}$ we obtain

$$K_<(E) = \frac{1}{\pi} \int_0^\infty \frac{d\mu}{\mu^{1/2}} (\not{p}^2 + E + \mu)^{-1} \varepsilon^{-3} b (\not{p}^2 + \varepsilon^{-3} b + E + \mu)^{-1} (V - E)_+^{1/2}.$$

We will prove

$$b (\not{p}^2 + \varepsilon^{-3} b + E)^{-1} = R(E) (\not{p}^2 + \varepsilon^{-3} b)^{-1} b \quad (6.2)$$

with $\sup_{E > 0} \|R(E)\| \lesssim 1$, such that $K_<(E)$ is of the form (6.1) with

$$\begin{aligned} R_1(E) &= \frac{E^{1/2}}{\pi} \int_0^\infty \frac{d\mu}{\mu^{1/2}} (\not{p}^2 + E + \mu)^{-1} R(E + \mu), \\ R_2(E) &= V^{-1/2} (V - E)_+^{1/2}, \end{aligned}$$

which are indeed bounded as

$$\|R_1(E)\| \lesssim \frac{E^{1/2}}{\pi} \int_0^\infty \frac{d\mu}{\mu^{1/2}} (E + \mu)^{-1} = 1, \quad \|R_2(E)\| \leq 1.$$

The proof of (6.2) runs as follows: we set $f = \log b$ and introduce the boosted Hamiltonian [10]

$$\begin{aligned} H_f &= b \not{p}^2 b^{-1} = e^f \not{p}^2 e^{-f} = [(D + i\nabla f) \cdot \sigma]^2, \\ \operatorname{Re} H_f &= \frac{1}{2} (H_f + H_f^*) = \not{p}^2 - (\nabla f)^2. \end{aligned}$$

With these notations, $R(E) = b (\not{p}^2 + \varepsilon^{-3} b + E)^{-1} b^{-1} (\not{p}^2 + \varepsilon^{-3} b) = (H_f + \varepsilon^{-3} b + E)^{-1} (\not{p}^2 + \varepsilon^{-3} b)$ and the bound (6.2) is equivalent to

$$(\not{p}^2 + \varepsilon^{-3} b)^2 \lesssim (H_f + \varepsilon^{-3} b + E) (H_f + \varepsilon^{-3} b + E)^*, \quad (6.3)$$

uniformly in $E > 0$, because $\|R\|^2 = \|RR^*\|$. Here the r.h.s. is $(H_f + \varepsilon^{-3} b) (H_f + \varepsilon^{-3} b)^* + E^2 + 2E \operatorname{Re}(H_f + \varepsilon^{-3} b)$, with

$$\operatorname{Re}(H_f + \varepsilon^{-3} b) = \not{p}^2 + \varepsilon^{-3} b - (\nabla f)^2 \geq \not{p}^2 + (\varepsilon^{-3} - \text{const}) b \geq 0$$

due to (2.4). It thus suffices to prove (6.3) for $E = 0$. We write $H_f = \not{p}^2 + X$ with

$$X = i\{\not{p}, \sigma \cdot \nabla f\} - (\nabla f)^2 = i\{D, \nabla f\} - (\nabla f)^2 = 2iD \cdot \nabla f - g ,$$

where we used (4.9) and set $g = \Delta f + (\nabla f)^2$. We can now estimate

$$\begin{aligned} (H_f + \varepsilon^{-3}b)(H_f + \varepsilon^{-3}b)^* &= (\not{p}^2 + \varepsilon^{-3}b)^2 + (\not{p}^2 + \varepsilon^{-3}b)X^* + X(\not{p}^2 + \varepsilon^{-3}b) + XX^* \\ &= \frac{1}{2}(\not{p}^2 + \varepsilon^{-3}b)^2 - XX^* + \frac{1}{2}(\not{p}^2 + \varepsilon^{-3}b + 2X)(\not{p}^2 + \varepsilon^{-3}b + 2X)^* \\ &\geq \frac{1}{2}(\not{p}^2 + \varepsilon^{-3}b)^2 - XX^* . \end{aligned}$$

By using (2.4, 3.5), the last term can be estimated as $XX^* \leq 8D(\nabla f)^2D + 2g^2 \lesssim DbD + b^2 \lesssim \not{p}^2b + b\not{p}^2 + \varepsilon^{-2}b^2 \leq \varepsilon^3(\not{p}^2 + \varepsilon^{-3}b)^2$. This concludes the proof of (6.2). \blacksquare

The form of $K_<(E)$ found in Lemma II.10 implies

$$n(K_<(E), \frac{1}{2}) \lesssim n((\not{p}^2 + \varepsilon^{-3}b)^{-1}\varepsilon^{-3}bV^{1/2}, \text{const } E^{1/2}) , \quad (6.4)$$

because of $n(XR, \lambda), n(RX, \lambda) \leq n(X, \|R\|^{-1}\lambda)$. The contribution of the low modes is then estimated by applying to (6.4) the identity $\int_0^\infty n(X, \mu^{1/2})d\mu = \text{tr } X^*X$:

$$\int_0^\infty n(K_<(E), \frac{1}{2})dE \lesssim \varepsilon^{-6} \text{tr}(V^{1/2}b(\not{p}^2 + \varepsilon^{-3}b)^{-2}bV^{1/2}) . \quad (6.5)$$

We see here where inequality (4.2) comes into play: from it and (5.2) namely follows

$$(\not{p}^2 + \varepsilon^{-3}b)^{-2} \lesssim (J^* \widehat{H}^2 J)^{-1} \leq J^* \widehat{H}^{-2} J ,$$

and inserting this inequality into (6.5) we have

$$\int_0^\infty n(K_<(E), \frac{1}{2})dE \lesssim \varepsilon^{-6} \int \text{tr}(j_y V^{1/2}bH_y^{-2}bV^{1/2}j_y) d^3y . \quad (6.6)$$

Upon inserting $H_y = e^{if_y}(p^2 + \varepsilon^{-3}b(y))e^{-if_y}$, the trace under the last integral equals

$$\begin{aligned} \int j_y(x)^2 V(x) b(x)^2 d^3x \cdot (2\pi)^{-3} \int [k^2 + \varepsilon^{-3}b(y)]^{-2} d^3k \\ = (8\pi)^{-1} \varepsilon^{3/2} b(y)^{-1/2} \int j_y(x)^2 V(x) b(x)^2 d^3x , \end{aligned}$$

where $b(y)^{-1/2} \leq 2b(x)^{-1/2}$ for $x \in \text{supp } j_y$, because of (2.5). From this and (2.8) we see that (6.6) is bounded by a constant times $\varepsilon^{-9/2} \int b(x)^{3/2} V(x) d^3x$, i.e., by the second term on the r.h.s. of (1.1). The proof of (1.1) is therefore complete.

7. Higher moments

There is an easy way, due to A. Laptev [65], in order to express moments of order higher than 1 of the negative eigenvalues $0 > -e_i = -e_i(V)$ (we sometimes indicate explicitly, but in this section only, the dependence on the potential) of the Pauli operator $h = \not{p}^2 - V$ as a function of their sum. Let us call $M_\gamma(V) = \sum e_i(V)^\gamma$ the moment $\gamma (\geq 0)$ of the eigenvalues $e_i(V)$. Higher moments $M_\nu(V)$, $\nu > \gamma$, satisfy

$$M_\nu(V) = a_{\nu,\gamma} \int_0^\infty \mu^{\nu-\gamma-1} M_\gamma(V-\mu) d\mu, \quad (7.1)$$

with $a_{\nu,\gamma}^{-1} = \int_0^1 s^{\nu-\gamma-1} (1-s)^\gamma ds = B(\nu-\gamma, \gamma+1)$, $B(x, y)$ being Euler's Beta function.

Proof. Letting $e_i > \mu$ we can easily check that

$$0 < e_i(V) - \mu = e_i(V - \mu). \quad (7.2)$$

On the other hand, the ν -th power of the eigenvalue $e_i(V)$ can be expressed as

$$e_i(V)^\nu = a_{\nu,\gamma} \int_0^\infty \mu^{\nu-\gamma-1} (e_i(V) - \mu)_+^\gamma d\mu,$$

as we can see by making a variable transformation. Summing over i and using (7.2), the claim follows. ■

Theorem II.11. *There exist two positive constants $C_{1,\nu}$ and $C_{2,\nu}$ depending on ν such that the moments of degree $\nu \geq 1$ of the negative eigenvalues $-e_i < 0$ of $h = \not{p}^2 - V$ satisfy*

$$\sum e_i^\nu \leq C_{1,\nu} \int V(x)^{3/2+\nu} d^3x + C_{2,\nu} \int b(x)^{3/2} V(x)^\nu d^3x. \quad (7.3)$$

Remark. We can give the explicit dependence of these constants $C_{1,\nu}$ and $C_{2,\nu}$ on ν , but since they also involve the value of the constants appearing in the estimate (1.1) for the sum of the negative eigenvalues, that are not known, their importance is relative. Nevertheless, we find

$$C_{1,\nu} = C' \frac{15}{8} \sqrt{\pi} \frac{\Gamma(\nu+1)}{\Gamma(\nu+5/2)}, \quad C_{2,\nu} = C'' . \quad (7.4)$$

In these expressions, C' and C'' are the constants appearing in front of the first resp. second term in (1.1), while Γ denotes Euler's Gamma function.

Proof. The case $\nu = 1$ is Theorem I.6. For $\nu > 1$ we insert into the r.h.s. of (7.1) the Lieb-Thirring type estimate (1.1) for the potential $(V - \mu)_+$, since $\not{p}^2 - (V - \mu) \geq \not{p}^2 - (V - \mu)_+$, and obtain

$$\begin{aligned} M_\nu(V) &\leq C' a_{\nu,1} \int_0^\infty d\mu \mu^{\nu-2} \int d^3x (V(x) - \mu)_+^{5/2} \\ &\quad + C'' a_{\nu,1} \int_0^\infty d\mu \mu^{\nu-2} \int d^3x (V(x) - \mu)_+ b(x)^{3/2}. \end{aligned}$$

By a substitution of variables, the first term equals $B(\nu - 1, 7/2) \int V(x)^{3/2+\nu} d^3x$, while the second integral is seen to equal the second term in (7.3), with a constant 1 in front. Using the expansion of the Beta function in terms of the Gamma function, $B(x, y) = \Gamma(x)\Gamma(y)[\Gamma(x+y)]^{-1}$, the computation of $C_{1,\nu}$ in (7.4) reduces to a trivial manipulation. \blacksquare

8. Local stability of matter in external magnetic fields

In this section we are concerned with the proof of local stability of matter (Theorem I.4), where we exploit our Lieb-Thirring inequality (1.1).

Theorem I.4. *Let $\mathcal{R} = \{R_1, \dots, R_K\}$ be the collection of all the nuclei and $L, Z, \Gamma > 0$. Then there is a positive $C(Z, \Gamma)$, an $\varepsilon > 0$ and a function $\Phi_{\mathcal{R}}(x) \geq 0$ with*

$$\|\Phi_{\mathcal{R}}\|_{\infty} \lesssim 1, \quad \|\Phi_{\mathcal{R}}\|_1 \lesssim L^3 K, \quad (8.1)$$

uniformly in \mathcal{R}, Z , such that the Hamiltonian

$$H = \sum_{i=1}^N \not{p}_i^2 + V_C + \Gamma \int \Phi_{\mathcal{R}}(x) B(x)^2 d^3x, \quad (8.2)$$

acting on $\mathcal{H}_{el} = \wedge^N \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, satisfies

$$H \geq -C(Z, \Gamma)(Z+1)L^{-1}(N+K) \quad (8.3)$$

for arbitrary $L \leq (Z+1)^{-1}$, provided $\Gamma^{-1}(Z+1) \leq \varepsilon$, where

$$C(Z, \Gamma) = \text{const} [\Gamma(Z+1)^{-1} + (Z+1)].$$

Proof. The reduction to a one-body problem is as in Theorem I.2, eq. (I.3.11), and the subsequent application. Therefore we partition the physical space \mathbb{R}^3 into Voronoi cells $\Gamma_j = \{x \mid |x-R_j| \leq |x-R_k| \text{ for } k = 1, \dots, K\}$, $(j = 1, \dots, K)$. Let $D_j = \min\{|R_j - R_k| \mid j \neq k\}/2$. Then, for any $\nu > 0$, the Hamiltonian (8.2) is bounded below as

$$H \geq \sum_{i=1}^N h_i - \nu N + \frac{Z^2}{8} \sum_{j=1}^K D_j^{-1} + \Gamma \int \Phi_{\mathcal{R}}(x) B(x)^2 d^3x, \quad (8.4)$$

where $h = \not{p}^2 - (W - \nu)_+$, and $W(x)$ is the one-particle potential introduced in Theorem I.2 satisfying $W(x) \leq Q|x - R_j|^{-1}$ for $x \in \Gamma_j$, where $Q = Z + \sqrt{2Z} + 2.2$ (see (I.3.12)). This implies $\text{supp}(W - \nu)_+ \subset \{x \mid x \in \Gamma_j, |x - R_j| \leq Q/\nu\} \subset \Omega_L = \{x \mid \text{dist}(x, \Omega) < L\}$ for $\Omega = \mathcal{R}$ and $\nu = QL^{-1}$.

Application of (1.1) yields

$$\sum_{i=1}^N h_i \geq -C' \int (W(x) - \nu)_+^{5/2} d^3x - C'' \int b(x)^{3/2} (W(x) - \nu)_+ d^3x,$$

where the first integral is bounded by $\text{const } Q^3 \nu^{-1/2} K$ (see (I.3.13)), i.e., since $L \leq (Z+1)^{-1}$ and $Q \lesssim Z+1$, by $\text{const } Q^{5/2} L^{1/2} K \lesssim QL^{-1} K$. We estimate

$$\int_{\Omega_L} b(x)^{3/2} (W(x) - \nu)_+ d^3x = \sum_{j=1}^K \int_{\Gamma_j \cap \Omega_L} b(x)^{3/2} (W(x) - \nu)_+ d^3x$$

by splitting the integrals over Γ_j into an inner integral over $U_j = \{x \mid |x - R_j| \leq \tilde{D}_j\}$ and an outer integral over $(\Gamma_j \setminus U_j) \cap \Omega_L$, where $\tilde{D}_j = \min(D_j, \delta \cdot r(R_j), L)$ with some small $\delta > 0$. Note that $U_j \subset \Omega_L$ by definition. In view of (2.5), the inner integral is bounded above by a constant times

$$b(R_j)^{3/2} \int_{U_j} W(x) d^3x \leq r(R_j)^{-3} \cdot 2\pi Q \tilde{D}_j^2 \leq 2\pi \delta^3 Q \tilde{D}_j^{-1},$$

whereas the outer integral is bounded by

$$\int_{(\Gamma_j \setminus U_j) \cap \Omega_L} \left(\frac{3}{4} Q \delta^{-1} b(x)^2 + \frac{1}{4} Q^{-3} \delta^3 W(x)^4 \right) d^3x \leq \frac{3}{4} Q \delta^{-1} \int_{(\Gamma_j \setminus U_j) \cap \Omega_L} b(x)^2 d^3x + \pi \delta^3 Q \tilde{D}_j^{-1}.$$

Moreover, we have

$$\tilde{D}_j^{-1} \lesssim \delta^{-4} \int_{U_j} b(x)^2 d^3x + D_j^{-1} + L^{-1}. \quad (8.5)$$

In fact, again by (2.5),

$$\delta^{-4} \int_{U_j} b(x)^2 d^3x \gtrsim \delta^{-4} b(R_j)^2 \cdot \frac{4}{3} \pi \tilde{D}_j^3 \geq \tilde{D}_j^{-1} \left(\frac{\tilde{D}_j}{\delta r(R_j)} \right)^4,$$

so that the r.h.s. of (8.5) is bounded below by a constant times $\tilde{D}_j^{-1} [(\tilde{D}_j / \delta r(R_j))^4 + \tilde{D}_j / D_j + \tilde{D}_j / L] \geq \tilde{D}_j^{-1}$. Collecting these estimates and (2.13) we find

$$\begin{aligned} & C'' \int b(x)^{3/2} (W(x) - \nu)_+ d^3x \\ & \leq C \left(Q \delta^{-1} \int \Phi_{\mathcal{R}}(x) B(x)^2 d^3x + Q \delta^3 \sum_{j=1}^K D_j^{-1} + Q(\delta^{-1} + \delta^3) L^{-1} K \right) \end{aligned} \quad (8.6)$$

for some constant C . We may assume $Z \geq 1$, since for $Z < 1$ the result follows by monotonicity in Z . Since $Q \lesssim Z+1$, we may choose δ so small that $CQ\delta^3 \leq Z^2/8$ (see (8.4)) for all $Z \geq 1$. Then $CQ\delta^{-1} \leq \Gamma$ for sufficiently small $Q\Gamma^{-1}$. The last term in (8.6) is thus bounded above by $\Gamma Q^{-1} + \frac{Z^2}{8} Q^{-1} \leq \Gamma(Z+1)^{-1} + (Z+1)$, in view of $Z+1 \leq Q$. Not forgetting the term proportional to N in (8.4), i.e., $-\text{const} (Z+1)L^{-1}N$, we obtain (8.3) with the corresponding constant $C(Z, \Gamma)$. ■

III. The second Lieb-Thirring estimate

1. Skeleton of the proof

In this chapter, that has the same structure as the previous one, we turn our attention to the second Lieb-Thirring type inequality given in Theorem I.7, that, together with the reduction of the Coulomb potential given in Theorem I.2, is one of the basic tools employed in the proof of the stability result given in Theorem I.5. The combination “reduction of V_C + magnetic Lieb-Thirring estimate” is in fact again the carrying structure of the proof, as in Chapter II. Nevertheless, we make use here of two length scales, r and l (they were both introduced in Chapter I):

$$\frac{1}{r(x)} = \int \varphi\left(\frac{y-x}{r(x)}\right) B(y)^2 d^3y , \quad (1.1)$$

$$\frac{1}{l(x)^3} = \int \varphi\left(\frac{y-x}{l(x)}\right) (\nabla \otimes B(y))^2 d^3y . \quad (1.2)$$

Let $-e_i < 0$ denote again the negative eigenvalues of the one-particle Pauli operator $h = \not{p}^2 - V$. The Lieb-Thirring estimate suitable for the purpose of this chapter is

Theorem I.7. *For sufficiently small $\varepsilon > 0$ there are constants $C', C'' > 0$ such that for any vector potential $A \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$*

$$\sum e_i \leq C' \int V(x)^{3/2} (V(x) + \widehat{B}(x)) d^3x + C'' \int V(x) P(x)^{1/2} (P(x) + \widehat{B}(x)) d^3x , \quad (1.3)$$

where $\widehat{B}(x)$ is the average of $|B(y)|$ over a ball of radius $\varepsilon l(x)$ centered at x .

The proof of (1.3) is close in spirit to that of our first Lieb-Thirring estimate (II.1.1), but is modified in such a way as to estimate the effects of a non-vanishing magnetic field gradient on top of the Lieb-Thirring inequality by Lieb, Solovej, Yngvason (I.6.5), rather than to estimate the effects of a non-vanishing magnetic field on top of the usual Lieb-Thirring inequality (I.3.4). We begin like before by applying the Birman-Schwinger principle in the form (II.1.3),

$$\sum e_i \leq 2 \int_0^\infty n((\not{p}^2 + E)^{-1/2} (V - E)_+^{1/2}, 1) dE , \quad (1.4)$$

but then we “regularize” the operator on the r.h.s. by means of the new effective field $P(x) = l(x)^{-1}(r(x)^{-1} + l(x)^{-1})$ and not of $b(x) = r(x)^{-2}$, as we did in Chapter II. The operator whose singular values bigger than 1 we want to estimate is now split into the sum $K_>(E) + K_<(E)$ of

$$K_>(E) = (\not{p}^2 + \varepsilon^{-3} P + E)^{-1/2} (V - E)_+^{1/2} ,$$

$$K_<(E) = [(\not{p}^2 + E)^{-1/2} - (\not{p}^2 + \varepsilon^{-3} P + E)^{-1/2}] (V - E)_+^{1/2} .$$

This last operator can again be rewritten as

$$K_<(E) = E^{-1/2} R_1(E) (\not{p}^2 + \varepsilon^{-3} P)^{-1} \varepsilon^{-3} P V^{1/2} R_2(E), \quad (1.5)$$

where $\|R_i(E)\| \lesssim 1$, ($i = 1, 2$) uniformly in $E > 0$, and by means of Weyl's inequality (II.1.5) we have to estimate the contributions of $K_>$ and $K_<$ to (1.4) separately. Our first step is once again a localization of the problem, but now over the length scale that "measures the variation of B ", i.e., over $\varepsilon l(x)$, for each point $x \in \mathbb{R}^3$. This procedure produces an error that is of the order of $\varepsilon^{-2} l(x)^{-2} \leq \varepsilon^{-2} P(x)$. The second step is a local comparison of the physical magnetic field $B(x)$ with a constant one (that we denote by \widehat{B}), namely the average of $|B(y)|$ over the neighbourhoods we are considering, that is balls of radius $\varepsilon l(x)$ centered at x . Decomposing $B = \widehat{B} + \tilde{B}$ into a constant part $\widehat{B} = \nabla \wedge \widehat{A}$ and a varying part $\tilde{B} = \nabla \wedge \tilde{A}$, we find a gauge, where $\tilde{A}(x)$ is locally comparable with $|\nabla \otimes B(x)|$, i.e., $|\tilde{A}(x)| \sim |\nabla \otimes B(x)| l(x)^2$, or $\tilde{A}(x)^2 \sim (\nabla \otimes B)(x)^2 l(x)^4 \sim l(x)^{-6} l(x)^4 = l(x)^{-2} \leq P(x)$. In this way the difference between the full Pauli operator \not{p}^2 and $[(p - \widehat{A}) \cdot \sigma]^2$ can be controlled by $\varepsilon^{-3} P(x)$ (for ε small enough), the term that we added to \not{p}^2 in $K_>$ and $K_<$. Thus, we are left with a Pauli operator with constant magnetic field (whose spectrum is described by the Landau bands $E = p_3^2 + 2\nu|\widehat{B}|$, where $\nu = 0, 1, 2, \dots$ and \widehat{B} is chosen to point in the 3-direction), and we can handle this problem like Lieb, Solovej, Yngvason did [51]. In both operators $K_>$ and $K_<$ we split the contributions coming from the lowest Landau band from those coming from higher bands. In the lowest band $[(p - \widehat{A}) \cdot \sigma]^2 = p_3^2$, while in the higher bands

$$[(p - \widehat{A}) \cdot \sigma]^2 = (p - \widehat{A})^2 - \sigma \cdot \widehat{B} \geq \frac{2}{3}(p - \widehat{A})^2, \quad (1.6)$$

because $-\sigma \cdot \widehat{B} \geq -\frac{1}{3}(p - \widehat{A})^2$. This comes from the fact that on the orthogonal complement of the lowest Landau band, $\nu \geq 1$, we have $(p - \widehat{A})^2 - \sigma \cdot \widehat{B} \geq 2\nu|\widehat{B}| \geq 2\nu\sigma \cdot \widehat{B}$. So, in one case we are back to the non-magnetic case, in the other one we have instead the "magnetic Schrödinger operator" $(p - \widehat{A})^2$, for which we use the diamagnetic inequality to relate it to the non-magnetic case as well. Since we don't make use of the diamagnetic inequality in the form given in (I.5.2), we state it in a more general way:

Lemma III.1. *Let $A \in L^2_{loc}(\mathbb{R}^3)$ with $\nabla \cdot A = 0$ (in distributional sense). Then*

$$\left| e^{-t(p-A)^2} \psi \right| \leq e^{-tp^2} |\psi| \quad (1.7)$$

for all $\psi \in L^2(\mathbb{R}^3)$.

Proof. Set $V \equiv 0$ in (15.9) of [63]. ■

Corollary. *Let $E > 0$. Under the same hypotheses of Lemma III.1, the diamagnetic inequality for the resolvent reads*

$$|(p - A)^2 + E|^{-1} \psi \leq (p^2 + E)^{-1} |\psi|, \quad (1.8)$$

or, equivalently, for the resolvent kernel

$$|((p - A)^2 + E)^{-1}(x, x')| \leq (p^2 + E)^{-1}(x - x') . \quad (1.9)$$

Proof. For $\mathbb{R} \ni E > 0$ and an arbitrary operator $X > 0$ on $L^2(\mathbb{R}^3)$, we have

$$(X + E)^{-1}\psi = \int_0^\infty e^{-Et} e^{-tX} \psi dt , \quad \forall \psi \in L^2(\mathbb{R}^3) .$$

Applying this equality to the l.h.s. of (1.8), we immediately obtain the r.h.s., if we recall that $|e^{-Et}| = e^{-Et}$. The second inequality follows easily from the first one. ■

In the next sections we pass to the proof of (1.3), but since it closely follows that of (II.1.1), we will often only indicate the small changes that we must introduce into the proofs of Chapter II, without reproducing them in their entirety.

2. Localization tools

In addition to the length scale r , about which we have been talking thoroughly in Section II.2, we define a new length scale (see (1.2)) as the solution $l = l(x) > 0$ of the equation

$$l^3 \int \varphi \left(\frac{y-x}{l} \right) (\nabla \otimes B(y))^2 d^3y = 1 . \quad (2.1)$$

The function $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$, $\varphi(z) = (1 + \frac{1}{2}z^2)^{-2}$ is the same as in Chapter II and satisfies again

$$z \cdot \nabla \varphi(z) \leq 0 , \quad (2.2)$$

$$|D_1 \dots D_n \varphi| \lesssim \varphi , \quad (n \in \mathbb{N}) \quad (2.3)$$

where $D_j = \partial_j$, ($i = 1, 2, 3$) or $D_j = z \cdot \nabla$.

The solution of (2.1) exists and is unique, except for the case $\nabla \otimes B \equiv 0$ (a.e.), where we set $l \equiv \infty$. l as well (see Section II.2) is a smooth function of $x \in \mathbb{R}^3$.

The nice differentiation property of r (II.2.3) and its commensurability (II.2.5) carry over to $l(x)$, while actually, property (II.2.3) for r can be sharpened.

Lemma III.2.

a) *The length scales l and r are tempered in the following sense:*

$$|\partial^\alpha l(x)| \lesssim l(x)^{-(|\alpha|-1)} , \quad (|\alpha| \geq 0) \quad (2.4)$$

$$|\partial^\alpha r(x)| \lesssim r(x)^{-(|\alpha|-1)} \min \left(1, \left(\frac{r(x)}{l(x)} \right)^{3/2} \right) , \quad (|\alpha| \geq 1) \quad (2.5)$$

where $\alpha \in \mathbb{N}^3$ is a multiindex. For $P(x)$ this (and in particular the last improvement) implies

$$|\nabla P(x)| \lesssim P(x)l(x)^{-1}, \quad |\Delta P(x)| \lesssim P(x)^2. \quad (2.6)$$

b) For ε small enough we have

$$|x - y| \leq \varepsilon l(x) \implies \begin{cases} \frac{1}{2} \leq \frac{l(y)}{l(x)} \leq 2 \end{cases} \quad (2.7)$$

$$\begin{cases} \frac{1}{2} \leq \frac{P(y)}{P(x)} \leq 2 \end{cases} \quad (2.8)$$

Proof of a). We omit the proof of (2.4), since it consists of a minor adaptation of that of (II.2.3). For $r(x) > l(x)$ (2.5) reduces to (II.2.3), so that we may assume $r(x) < l(x)$. We discuss this case using a variant of the argument given in Chapter II. We recall that it was based on the equation

$$(1 - m(x))\partial_i r(x) = m_i(x), \quad (2.9)$$

where

$$m(x) = r(x) \int z \cdot \nabla \varphi(z) B(y)^2 d^3 y, \quad m_i(x) = r(x) \int (\partial_i \varphi)(z) B(y)^2 d^3 y,$$

with $z = (y - x)/r(x)$. Moreover, we denoted by V_n , ($n \in \mathbb{N}$), the space of finite sums of functions of the form

$$f(x) = r(x)^{-(n-1)} P(\{\partial^\alpha r\}) \int \psi(z) B(y)^2 d^3 y,$$

where ψ is of the form $D_1 \dots D_k \varphi$ and P is a monomial in the derivatives $\{\partial^\alpha r\}_{|\alpha| \leq n}$ of order 0 in the sense that it contains as many powers of ∂ as of r . In addition we consider here the subspace $\tilde{V}_n \subset V_n$ obtained by restricting f to satisfy: (i) some $\partial^\alpha r$ with $1 \leq |\alpha| \leq n$ occurs among the factors of P ; or else (ii) $D_1 = \partial_i$, i.e., $\psi = \partial_i \tilde{\psi}$ with $\tilde{\psi}$ of the form previously stated for ψ . One verifies that $\partial_i V_n \subset \tilde{V}_{n+1}$ and $r^{-1} \tilde{V}_n \subset \tilde{V}_{n+1}$.

The induction assumption states that (2.5) holds for $1 \leq |\alpha| \leq n$. (It is empty for $n = 0$). We now prove it for $n + 1$ instead of n . First, we claim that $f \in \tilde{V}_n$ satisfies

$$|f(x)| \lesssim r(x)^{-n} \left(\frac{r(x)}{l(x)} \right)^{3/2}.$$

In case (i) this follows directly from the induction assumption; in case (ii) by integration by parts:

$$\int \partial_i \tilde{\psi}(z) B(y)^2 d^3 y = 2r(x) \int \tilde{\psi}(z) B(y) \cdot \partial_i B(y) d^3 y,$$

which by (2.3) and the Cauchy-Schwarz inequality is bounded in absolute value by

$$2r(x) \left(\int \varphi(z) B(y)^2 d^3 y \right)^{1/2} \left(\int \varphi(z) (\nabla \otimes B(y))^2 d^3 y \right)^{1/2} \lesssim r(x)^{-1} \left(\frac{r(x)}{l(x)} \right)^{3/2}.$$

In the last estimate we used that the first integral equals $r(x)^{-1}$, whereas the second may be estimated by replacing z by $(y - x)/l(x)$, since $r(x)^{-1} > l(x)^{-1}$ and $\varphi(z)$ is radially decreasing. Hence that integral is bounded by $l(x)^{-3}$. We can turn to (2.5): Applying ∂^α , ($|\alpha| = n$) to (2.9) and using $m \in V_0$ we obtain $(1 - m(x))\partial^\alpha \partial_i r(x) \in \partial^\alpha m_i + \tilde{V}_n$. The last set is \tilde{V}_n (even for $|\alpha| = n = 0$), since $m_i \in \tilde{V}_0$. The result follows with $m \leq 0$. ■

Proof of b). The proof of (2.7) is the same as that of (II.2.5). Combining (2.6) with (2.7) we find that for $|x - y| \leq \varepsilon l(x)$ we have $|\log P(y) - \log P(x)| \lesssim \varepsilon$, and hence, by the mean value theorem, (2.8). ■

A partition of unity based on the length scale $l(x)$ (we localize now to neighbourhoods of radius $\varepsilon l(x)$ around each $x \in \mathbb{R}^3$) is

$$j_y(x) = (\varepsilon l(x))^{-3/2} \chi \left(\frac{x - y}{\varepsilon l(x)} \right), \quad (y \in \mathbb{R}^3)$$

where $0 < \varepsilon \leq 1$ and $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\text{supp } \chi \subset \{z \mid |z| \leq 1\}$ and $\int \chi(z)^2 d^3z = 1$. Analogously to Lemma II.2 we have

Lemma III.3.

$$\int j_y(x)^2 d^3y = 1, \quad (2.10)$$

$$\int |\partial^\alpha j_y(x) \partial^\beta j_y(x)| d^3y \lesssim (\varepsilon l(x))^{-(|\alpha|+|\beta|)} \quad (2.11)$$

for any $\alpha, \beta \in \mathbb{N}^3$, where $\partial = \partial/\partial x$.

The length scale $l(x)$ will be the one most frequently used in the following sections. At one point however (in the proof of Lemma III.6), we will use the length scale $\lambda(x)$ defined by $\lambda(x)^{-1} = r(x)^{-1} + l(x)^{-1}$. It also satisfies (2.4) and (2.7) (with l replaced by λ), and Lemma III.3 applies accordingly to the partition based on $\lambda(x)$.

At the end of this section we discuss the equivalent of the magnetic energy estimates (II.2.12) and (II.2.13), that relate l to the physical quantity $\nabla \otimes B$.

Lemma III.4.

a)

$$\int l(x)^{-6} d^3x \lesssim \int (\nabla \otimes B(x))^2 d^3x. \quad (2.12)$$

b) Let $\Omega_L = \{x \mid \text{dist}(x, \Omega) < L\}$ for $\Omega \subset \mathbb{R}^3$ and $L > 0$. Then, for any $L > 0$ and $\Omega \subset \mathbb{R}^3$ there is a function $\Phi_{\Omega, L} \geq 0$ satisfying $\|\Phi_{\Omega, L}\|_\infty \lesssim 1$ and $\|\Phi_{\Omega, L}\|_1 \lesssim |\Omega_L|$, uniformly in Ω, L , such that

$$\int_{\Omega_L} l(x)^{-6} d^3x \lesssim \int \Phi_{\Omega, L}(x) (\nabla \otimes B(x))^2 d^3x + |\Omega_L| L^{-6}. \quad (2.13)$$

Proof. The same proofs as for the corresponding estimates (II.2.12) and (II.2.13) for r are valid once the following remark about the proof of Lemma II.4 has been made: We replace there $r(x)$ by $l(x)$. Because of $g_+ (|x|) \geq 1$, (II.2.15) implies

$$g_+ (|x|)^3 \varphi \left(\frac{z - x}{g_+ (|x|)} \right) \geq \varphi(z),$$

which after integration against $(\nabla \otimes B(z))^2 d^3 z$ implies $l(x) \leq g_+ (|x|)$. Then the proof continues as before. \blacksquare

3. Localization of the Birman-Schwinger kernel

It is from now on understood that $\varepsilon > 0$ is small enough.

Analogously to Chapter II, we must localize the operator $(\not{p}^2 + \varepsilon^{-3} P)^2$.

Lemma III.5.

$$(\not{p}^2 + \varepsilon^{-3} P)^2 \geq \int j_y (\not{p}^4 + \tfrac{1}{2} \varepsilon^{-6} P^2) j_y d^3 y. \quad (3.1)$$

The tools we need are inequality (II.3.4), namely

$$U \leq \tfrac{1}{3} (\tfrac{\pi}{2})^{-4/3} \|U\|_{3/2} D^2. \quad (3.2)$$

for $U \in L^{3/2}(\mathbb{R}^3)$, and the next lemma, that plays the role of Lemma II.7 in Chapter II.

Lemma III.6.

$$Dl^{-2} D \lesssim \not{p}^2 P + P \not{p}^2 + \varepsilon^{-2} P^2. \quad (3.3)$$

Proof. The first step towards (3.3) consists in showing

$$Dl^{-2} D \lesssim \not{p}^2 l^{-2} + l^{-2} \not{p}^2 + \varepsilon^{-2} P^2. \quad (3.4)$$

Similarly to Lemma II.7 its proof reduces to that of

$$l^{-2} |B| \lesssim \varepsilon^{1/2} (Dl^{-2} D + \varepsilon^{-2} P^2). \quad (3.5)$$

This is again proved as in Chapter II, except for the fact that we use here (and only here) a partition of unity based on the length scale $\varepsilon \lambda(x)$ as discussed at the end of Section 2, with $\lambda(x)^{-1} = r(x)^{-1} + l(x)^{-1}$. In particular, we now set $\tilde{K}_y = \{x \mid |x - y| < \varepsilon \lambda(x)\}$ with characteristic function $\tilde{\chi}_y$. It then still holds that

$$\|l^{-2} |B| \tilde{\chi}_y\|_{3/2} \leq \|l^{-2} \tilde{\chi}_y\|_\infty \|B \tilde{\chi}_y\|_2 \|\tilde{\chi}_y\|_6 \lesssim l(y)^{-2} \cdot r(y)^{-1/2} \cdot (\varepsilon r(y))^{1/2} = \varepsilon^{1/2} l(y)^{-2},$$

where: we used $\lambda(x) \leq l(x)$ in estimating the first factor; $\lambda(x) \leq r(x)$ and (1.1) in the second; and again $\lambda(x) \leq r(x)$ in the last one. We hence obtain, just as in Chapter II,

$$l^{-2} |B| \lesssim \varepsilon^{1/2} \left(Dl^{-2} D + l^{-2} \int (\nabla j_y)^2 d^3 y \right)$$

with the integral bounded by $(\varepsilon \lambda(x))^{-2}$ due to (2.11). The proof of (3.5), and hence of (3.4), is completed by noticing that $l^{-2} \lambda^{-2} = P^2$. We now come back to (3.3). We have

$$\pm(\not{P}^2 f + f \not{P}^2 - 2 \not{P} f \not{P}) \lesssim \varepsilon \not{P} P \not{P} + \varepsilon^{-1} P^2$$

for $f = l^{-2}$ or $f = P$. Indeed, the l.h.s. is

$$\pm[\not{P}, [\not{P}, f]] = \mp i[\not{P}, \nabla f \cdot \sigma] = -X^* X + \varepsilon \not{P} P \not{P} + \varepsilon^{-1} P^{-1} (\nabla f)^2$$

with $X = (\varepsilon P)^{1/2} \not{P} \pm i(\varepsilon P)^{-1/2} \nabla f \cdot \sigma$ and $(\nabla f)^2 \lesssim P^3$ due to (2.10) resp. (2.6). Taking $f = l^{-2}$ we first obtain from (3.4)

$$Dl^{-2} D \lesssim \not{P} l^{-2} \not{P} + \varepsilon \not{P} P \not{P} + \varepsilon^{-1} P^2 + \varepsilon^{-2} P^2 \leq 2(\not{P} P \not{P} + \varepsilon^{-2} P^2),$$

and then, with $f = P$, we obtain (3.3). ■

Proof of (3.1). The localization argument begins as that given for (II.3.1), with b replaced by P , i.e., we have

$$\not{P}^4 = \int (j_y \not{P}^4 j_y + \tfrac{1}{2} \{ [j_y, [j_y, \not{P}^2]], \not{P}^2 \} + [j_y, \not{P}^2]^2) d^3 y,$$

with the estimate

$$- \int \tfrac{1}{2} \{ [j_y, [j_y, \not{P}^2]], \not{P}^2 \} d^3 y \leq \tfrac{1}{2} \varepsilon^{-3} (\not{P}^2 P + P \not{P}^2) + \varepsilon^{-5} P^2$$

for the first localization error. The other one is estimated similarly:

$$- \int [j_y, \not{P}^2]^2 d^3 y \leq \text{const} (\varepsilon^{-2} Dl^{-2} D + \varepsilon^{-4} l^{-4}) \leq \tfrac{1}{2} \varepsilon^{-3} (\not{P}^2 P + P \not{P}^2) + \varepsilon^{-5} P^2,$$

by using (3.3). The conclusion then is as in Chapter II. ■

4. Choice of an appropriate gauge

The strategy of this section is to locally replace in the operator on the r.h.s. of (3.1) the magnetic field $B = \nabla \wedge A$ by a constant magnetic field and P by a constant. This step is performed by introducing, once again in the direct integral framework, the same linear map (but the underlying partition of unity is based on the new length scale l)

$$J : \mathcal{H} \rightarrow \widehat{\mathcal{H}} = \int_{\mathbb{R}^3}^{\oplus} \mathcal{H} d^3y, \quad J = \int_{\mathbb{R}^3}^{\oplus} j_y d^3y,$$

(see also Section II.4), and the Hamiltonian

$$\widehat{H} : \widehat{\mathcal{H}} \rightarrow \widehat{\mathcal{H}}, \quad \widehat{H} = \int_{\mathbb{R}^3}^{\oplus} e^{if_y} H_y e^{-if_y} d^3y,$$

where $H_y = H(B_y) + \varepsilon^{-3}P(y)$, $H(B) = [(p - \frac{1}{2}B \wedge x) \cdot \sigma]^2$, $f_y(x)$ is a function to be specified later and $B_y = |K_y|^{-1} \int_{K_y} B(x) d^3x$ is the average magnetic field in the ball $K_y = \{x \mid |x - y| < 2\varepsilon l(y)\}$. In summary, \widehat{H} acts on fibers of $\widehat{\mathcal{H}}$ as a Pauli Hamiltonian with constant magnetic field. The comparison of \widehat{P}^2 with this Hamilton operator is given by

Lemma III.7.

$$j_y(\widehat{P}^4 + \frac{1}{2}\varepsilon^{-6}P^2)j_y \gtrsim j_y e^{if_y} H_y^2 e^{-if_y} j_y. \quad (4.1)$$

The combination of Lemma III.5 and Lemma III.7 yields the important inequality

$$(\widehat{P}^2 + \varepsilon^{-3}P)^2 \gtrsim J^* \widehat{H}^2 J, \quad (4.2)$$

together with its weaker version (see (II.4.3))

$$\widehat{P}^2 + \varepsilon^{-3}P \gtrsim J^* \widehat{H} J. \quad (4.3)$$

The key point is again a “magnetic lemma” stating that there exists a gauge where this comparison is possible.

Lemma III.8. [25] *Let $K = \{x \mid |x| < 1\}$ be the unit ball, and $K^* = 2K$. Let $B \in L^2(K^*, \mathbb{R}^3)$ be a vector field with $\nabla \cdot B = 0$ (as a distribution) and*

$$\int_K B(x) d^3x = 0. \quad (4.4)$$

Then there is a vector field A such that

$$\nabla \wedge A = B, \quad \nabla \cdot A = 0, \quad (4.5)$$

and

$$\|A\|_{\infty, K} \lesssim \|\nabla \otimes B\|_{2, K^*}. \quad (4.6)$$

Proof. A solution A to (4.5) is constructed as in Chapter II, i.e., as $A = \nabla \wedge F$, where F is the solution of $-\Delta F = B$ with boundary conditions (II.4.7). By $\|F\|_{2,K^*} \lesssim \|B\|_{2,K^*}$ and the elliptic estimate

$$\|\nabla^{\otimes 3} F\|_{2,K} \lesssim \|F\|_{2,K^*} + \|\Delta F\|_{2,K^*} + \|\nabla \otimes \Delta F\|_{2,K^*}$$

(which has a similar derivation as (II.4.8)) we have

$$\|\nabla^{\otimes 2} A\|_{2,K} \lesssim \|B\|_{2,K^*} + \|\nabla \otimes B\|_{2,K^*} \lesssim \|\nabla \otimes B\|_{2,K^*}.$$

In establishing the last inequality we used that a Poincaré inequality (see e.g. [69], Theorem 4.4.2) applies to $\|B\|_{2,K^*}$, due to (4.4). Another Poincaré type inequality ([69], Corollary 4.2.3) yields

$$\|A - \alpha - \beta x\|_{\infty,K} \lesssim \|\nabla^{\otimes 2} A\|_{2,K},$$

for $\alpha_i = |K|^{-1} \int_K A_i(x) d^3x$ and $\beta_{ij} = |K|^{-1} \int_K \partial_j A_i(x) d^3x$. This proves (4.6) for $A - \alpha - \beta x$ instead of A . Equation (4.5) is preserved under this replacement, since it implies $\beta_{ij} - \beta_{ji} = 0$ and $\text{tr } \beta = 0$. \blacksquare

Proof of (4.1). Let $B_y = |K_y|^{-1} \int_{K_y} B(x) d^3x$ be the average magnetic field over $K_y = \{x \mid |x - y| < 2\epsilon l(y)\}$. It is generated by the vector potential $A_y(x) = \frac{1}{2} B_y \wedge (x - y)$. On the other hand, let $\tilde{A}_y(x)$ be the vector potential of $\tilde{B}_y(x) = B(x) - B_y$, which by scaling corresponds to the one constructed in the previous lemma. It satisfies

$$|\tilde{A}_y(x)| \lesssim \epsilon^{1/2} l(y)^{-1} \quad (4.7)$$

for $x \in K_y$ because of (1.2, 4.6) (pay attention: (4.6) is not scale invariant). Since $B = \nabla \wedge (A_y + \tilde{A}_y)$, we may assume, upon making a gauge transformation, $A = A_y + \tilde{A}_y$. The Pauli operators corresponding to $\tilde{\psi}_y = (p - A_y) \cdot \sigma$ and $\tilde{\psi}$ are related as

$$\tilde{\psi}_y^2 = (\tilde{\psi} + \tilde{A}_y \cdot \sigma)^2 = \tilde{\psi}^2 + (\tilde{A}_y)^2 + \{\tilde{A}_y \cdot \sigma, \tilde{\psi}\} = \tilde{\psi}^2 + (\tilde{A}_y)^2 + \{\tilde{A}_y, D\} + \tilde{B}_y \cdot \sigma.$$

This and $\nabla \cdot \tilde{A}_y = 0$ yield

$$\tilde{\psi}_y^4 \leq 4(\tilde{\psi}^4 + (\tilde{A}_y)^4 + 4D(\tilde{A}_y)^2 D + (\tilde{B}_y)^2).$$

After multiplying from both sides with j_y we may replace \tilde{A}_y by $\chi_y \tilde{A}_y$ and similarly for \tilde{B}_y , where $\chi_y(x)$ is the characteristic function of K_y . Note that, besides (4.7), we have by (1.2) and $\|\chi_y\|_3 \lesssim \epsilon l(y)$

$$\|\tilde{B}_y^2 \chi_y\|_{3/2} \leq \|\tilde{B}_y^2 \chi_y\|_3 \|\chi_y\|_3 \lesssim \|(\nabla \otimes B)^2 \chi_y\|_1 \|\chi_y\|_3 \lesssim \epsilon l(y)^{-2}.$$

We can thus estimate, using (3.2),

$$j_y \tilde{\psi}_y^4 j_y \lesssim j_y (\tilde{\psi}^4 + \epsilon^2 l(y)^{-4} + \epsilon D l(y)^{-2} D) j_y$$

and finally, using (2.7, 2.8, 3.3),

$$j_y (\tilde{\psi}_y^2 + \epsilon^{-3} P(y))^2 j_y \leq 2j_y (\tilde{\psi}_y^4 + \epsilon^{-6} P(y)^2) j_y$$

$$\lesssim j_y (\tilde{\psi}^4 + \frac{1}{2} \epsilon^{-6} P(x)^2 + \epsilon D l(x)^{-2} D) j_y \leq j_y (\tilde{\psi}^2 + \epsilon^{-3} P)^2 j_y. \quad \blacksquare$$

5. High modes

The contribution of high and low modes results in a combination of our approach of Sections II.5 and II.6 with that of Lieb, Solovej, Yngvason [51]. Let

$$\hat{H}^0 : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}, \quad \hat{H}^0 = \int_{\mathbb{R}^3}^{\oplus} e^{if_y} H(B_y) e^{-if_y} d^3y.$$

Then $\hat{H} \geq \hat{H}^0$ and, as in Chapter II, we obtain from (4.3)

$$n(K_>(E), \frac{1}{2}) \leq n((\hat{H}^0 + E)^{-1/2} J(V - E)_+^{1/2}, \text{const}) . \quad (5.1)$$

From now on the computation closely follows the line given in [51], where the contribution of the lowest Landau band is split from that of the higher bands. We set

$$\hat{\Pi} : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}, \quad \hat{\Pi} = \int_{\mathbb{R}^3}^{\oplus} e^{if_y} \Pi(B_y) e^{-if_y} d^3y,$$

where $\Pi(B)$ is the projection in $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ onto the lowest band of $H(B)$. Its integral kernel is

$$\Pi(B)(x, x') = \frac{|B|}{2\pi} \exp \left[i(x_\perp \wedge x'_\perp) \cdot \frac{B}{2} - (x_\perp - x'_\perp)^2 \frac{|B|}{4} \right] \delta(x_3 - x'_3) \mathcal{P}^\perp, \quad (5.2)$$

in coordinates $x = (x_\perp, x_3)$ where $B = (0, |B|)$, and $\mathcal{P}^\perp = (1 + \sigma_3)/2$ is the projection in \mathbb{C}^2 onto the subspace where $B \cdot \sigma = |B|$. We remark that $\hat{\Pi}$ commutes with \hat{H}^0 . The operator appearing on the r.h.s. of (5.1) is then split as $(\hat{H}^0 + E)^{-1/2} J(V - E)_+^{1/2} = K_0(E) + K_1(E)$, with

$$\begin{aligned} K_0(E) &= (\hat{H}^0 + E)^{-1/2} \hat{\Pi} J(V - E)_+^{1/2}, \\ K_1(E) &= (\hat{H}^0 + E)^{-1/2} (1 - \hat{\Pi}) J(V - E)_+^{1/2}, \end{aligned}$$

so that by Weyl's inequality (II.1.5) it suffices to estimate $n(K_i(E), \text{const})$, ($i = 0, 1$) separately. The first term is bounded from above by a constant times $\text{tr } K_0(E)^* K_0(E)$, that, on its turn, equals

$$\begin{aligned} &\int d^3y \text{tr}[j_y(V - E)_+^{1/2} \Pi(B_y) (H(B_y) + E)^{-1} \Pi(B_y) (V - E)_+^{1/2} j_y] \\ &= \int d^3x d^3y d^3z j_y(x)^2 (V(x) - E)_+ \text{tr}_{\mathbb{C}^2} [\Pi(B_y)(x, z)] (p_3^2 + E)^{-1} (z_3, x_3) \delta(z_\perp - x_\perp), \end{aligned}$$

because of (5.2), of $\Pi(B)(H(B) + E)^{-1} = \Pi(B)(p_3^2 + E)^{-1}$ (in the coordinates used there) and of the cyclicity of the trace, that made the gauge transformation e^{if_y} disappear. Using

$$(p_3^2 + E)^{-1}(x_3, x_3) = (2\pi)^{-1} \int_{-\infty}^{\infty} (p^2 + E)^{-1} dp = \frac{1}{2} E^{-1/2},$$

we end up with

$$n(K_0(E), \text{const}) \lesssim (4\pi E^{1/2})^{-1} \int d^3y d^3x (V(x) - E)_{+} j_y(x)^2 |B_y| . \quad (5.3)$$

For the second term we use inequality (1.6), which states that $\frac{3}{2}H(B_y) \geq D_y^2 \equiv (p - \frac{1}{2}B_y \wedge x)^2$ on the orthogonal complement $\text{Ran}(1 - \Pi(B_y))$ of the lowest Landau band. We hence get

$$\hat{H}^0 \geq \frac{2}{3} \int_{\mathbb{R}^3} e^{if_y} D_y^2 e^{-if_y} d^3y \equiv \hat{H}_S \quad (5.4)$$

on $\text{Ran}(1 - \hat{\Pi})$, as well as $(1 - \hat{\Pi})(\hat{H}^0 + E)^{-1}(1 - \hat{\Pi}) \leq (\hat{H}_S + E)^{-1}$, because $\hat{\Pi}$ and \hat{H}_S commute. Together with $n(X, 1) \leq \text{tr}((X^*X)^2)$ this yields

$$\begin{aligned} n(K_1(E), \text{const}) &\lesssim \text{tr}[(V - E)_{+}^{1/2} J^* (\hat{H}_S + E)^{-1} J(V - E)_{+} J^* (\hat{H}_S + E)^{-1} J(V - E)_{+}^{1/2}] \\ &= \int \text{tr}[j_y j_{y'} e^{i(f_y - f_{y'})} (V - E)_{+} (\frac{2}{3} D_y^2 + E)^{-1} j_y j_{y'} e^{-i(f_y - f_{y'})} (V - E)_{+} (\frac{2}{3} D_{y'}^2 + E)^{-1}] d^3y d^3y'. \end{aligned}$$

Using the pointwise diamagnetic inequality (1.9) for the resolvent kernel

$$|(\frac{2}{3} D_y^2 + E)^{-1}(x, x')| \leq (\frac{2}{3} p^2 + E)^{-1}(x - x') , \quad (5.5)$$

the trace under the integral is bounded as in (II.5.3) by

$$\frac{3}{8\pi} \left(\frac{3}{2E} \right)^{1/2} \int (V(x) - E)_{+}^2 j_y(x)^2 j_{y'}(x)^2 d^3x .$$

This leads to $n(K_1(E), \text{const}) \lesssim E^{-1/2} \int (V(x) - E)_{+}^2 d^3x$ by (2.10) and, together with (5.3), to

$$\int_0^\infty n(K_>(E), \frac{1}{2}) dE \lesssim \int d^3x V(x)^{3/2} \left(V(x) + \int d^3y |B_y| j_y(x)^2 \right) . \quad (5.6)$$

In order to put this result into the form given in Theorem I.7 we estimate

$$|B_y| \leq |K_y|^{-1} \int_{K_y} |B(z)| d^3z = |K_y|^{-1} \int |B(z)| \theta(|z - y| < 2\varepsilon l(y)) d^3z ,$$

where $\theta(A)$ is the characteristic function of the set A , so that

$$\int d^3y |B_y| j_y(x)^2 \leq \int d^3z |B(z)| \int d^3y |K_y|^{-1} \theta(|z - y| < 2\varepsilon l(y)) j_y(x)^2 . \quad (5.7)$$

We recall that $\text{supp } j_y \subset \{x \mid |x - y| \leq \varepsilon l(x)\}$. Using again (2.7) and the triangle inequality $|x - z| \leq |x - y| + |z - y|$ we bound (5.7) by a constant times

$$|K_x|^{-1} \int d^3z |B(z)| \theta(|x - z| < 5\varepsilon l(x)) \int d^3y j_y(x)^2 = |K_x|^{-1} \int_{|x - z| < 5\varepsilon l(x)} d^3z |B(z)| ,$$

i.e., by $\hat{B}(x)$ after a redefinition of ε :

$$\int d^3y |B_y| j_y(x)^2 \lesssim \hat{B}(x) . \quad (5.8)$$

The bound (5.6), after application of (5.8), corresponds then to the first integral term in (1.3).

6. Low modes

The first necessary step is as in Chapter II the proof of (1.5).

Lemma III.9. *There are two bounded operators $R_1(E)$ and $R_2(E)$ satisfying $\|R_i(E)\| \lesssim 1$, ($i = 1, 2$) uniformly in $E > 0$, such that*

$$K_<(E) = E^{-1/2} R_1(E) (\not{P}^2 + \varepsilon^{-3} P)^{-1} \varepsilon^{-3} P V^{1/2} R_2(E). \quad (6.1)$$

Proof. The proof can be taken over literally from that of (II.6.1), after replacing b by P . To be checked however is that $f = \log P$ satisfies $(\nabla f)^2 \lesssim l^{-2} \leq P$ and $|\Delta f| \lesssim P$, as well as $D(\nabla f)^2 D \lesssim \not{P}^2 P + P \not{P}^2 + \varepsilon^{-2} P^2$. This follows from (2.6, 3.3). \blacksquare

The inequality

$$\int_0^\infty n(K_<(E), \tfrac{1}{2}) dE \lesssim \varepsilon^{-6} \operatorname{tr}[V^{1/2} P J^* \widehat{H}^{-2} J P V^{1/2}]$$

follows from (6.1), from

$$n(K_<(E), \tfrac{1}{2}) \lesssim n((\not{P}^2 + \varepsilon^{-3} P)^{-1} \varepsilon^{-3} P V^{1/2}, \operatorname{const} E^{1/2}),$$

from $\int_0^\infty n(X, \mu^{1/2}) d\mu = \operatorname{tr} X^* X$, and from (4.2). We then split $\widehat{H}^{-2} = \widehat{\Pi} \widehat{H}^{-2} \widehat{\Pi} + (1 - \widehat{\Pi}) \widehat{H}^{-2} (1 - \widehat{\Pi})$. The contribution of the first term is

$$\begin{aligned} \int d^3 y \operatorname{tr}[j_y V^{1/2} P \Pi(B_y) (H(B_y) + \varepsilon^{-3} P(y))^{-2} \Pi(B_y) P V^{1/2} j_y] \\ = \frac{1}{8\pi} \int (\varepsilon^{-3} P(y))^{-3/2} |B_y| P(x)^2 V(x) j_y(x)^2 d^3 y d^3 x, \end{aligned}$$

because of (5.2) and of $\Pi(B)(H(B) + E)^{-2} = \Pi(B)(p_3^2 + E)^{-2}$ (see also last section). For the second term we use (see (5.4)) $\widehat{H}^2 \geq (\widehat{H}_S + \widehat{P})^2$ on $\operatorname{Ran}(1 - \widehat{\Pi})$, since \widehat{H} and $\widehat{H}_S + \widehat{P}$ commute, where $\widehat{P} = \varepsilon^{-3} \int_{\mathbb{R}^3} P(y) d^3 y$. This yields a contribution bounded by

$$\begin{aligned} \int \operatorname{tr}[j_y V^{1/2} P (\tfrac{2}{3} D_y^2 + \varepsilon^{-3} P(y))^{-2} P V^{1/2} j_y] d^3 y \\ \leq \frac{3}{8\pi} \int \left(\frac{3}{2\varepsilon^{-3} P(y)} \right)^{1/2} P(x)^2 V(x) j_y(x)^2 d^3 y d^3 x, \end{aligned}$$

where we used again (5.5). Taking into account (2.8) and (2.10) we thus obtain

$$\int_0^\infty n(K_<(E), \tfrac{1}{2}) dE \lesssim \int d^3 x V(x) \left(\varepsilon^{-9/2} P(x)^{3/2} + \varepsilon^{-3/2} P(x)^{1/2} \int d^3 y |B_y| j_y(x)^2 \right).$$

At this point we apply (5.8) to the last term and Theorem I.7 is proved.

7. Higher moments

In the same way as in Section II.7 we can generalize Theorem I.7 for the sum of the negative eigenvalues $-e_i < 0$ of the one-particle Pauli Hamiltonian $h = \not{p}^2 - V$ to moments of these eigenvalues of degree ≥ 1 .

Theorem III.10. *For every $\nu \geq 1$ there exist two positive constants $C_{1,\nu}$ and $C_{2,\nu}$ such that*

$$\sum e_i^\nu \leq C_{1,\nu} \int V(x)^{\frac{1}{2}+\nu} (V(x) + \hat{B}(x)) d^3x + C_{2,\nu} \int V(x)^\nu P(x)^{\frac{1}{2}} (P(x) + \hat{B}(x)) d^3x .$$

Remark. The dependence on ν of these two constants, $C_{1,\nu}$ and $C_{2,\nu}$, is the same as in (II.7.4), where now C' and C'' are the constants appearing in front of the first resp. second term in (1.3).

8. Local stability of matter in external magnetic fields

Theorem I.5. *Let $\mathcal{R} = \{R_1, \dots, R_K\}$ be the collection of all the nuclei and $L, Z, \Gamma, \gamma > 0$. There is a positive $C(Z, \Gamma, \gamma)$ and a function $\Phi_{\mathcal{R}}(x) \geq 0$ with*

$$\|\Phi_{\mathcal{R}}\|_\infty \lesssim 1 , \quad \|\Phi_{\mathcal{R}}\|_1 \lesssim L^3 K , \quad (8.1)$$

uniformly in \mathcal{R} , Z , such that the Hamiltonian

$$H = \sum_{i=1}^N \not{p}_i^2 + V_C + \Gamma \int \Phi_{\mathcal{R}}(x) (B(x)^2 + \gamma L^2 (\nabla \otimes B)(x)^2) d^3x , \quad (8.2)$$

acting on $\mathcal{H}_{\text{el}} = \wedge^N \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, satisfies

$$H \geq -C(Z, \Gamma, \gamma)(Z+1)L^{-1}(N+K) \quad (8.3)$$

for arbitrary $L \leq (Z+1)^{-1}$. For $\Gamma \leq Z+1$ and $1 \leq \gamma \leq z^4$ one can take

$$C(Z, \Gamma, \gamma) = \text{const} [z^3 + z^5 \gamma^{-1/2} \log(z^5 \gamma^{-1/2})] \quad (8.4)$$

with $z = 1 + (Z+1)\Gamma^{-1}$.

Remark. One may modify the definition (1.2) of $l(x)$ by replacing $(\nabla \otimes B)^2$ by $(\nabla \otimes B)^2 + L^{-6}$ for some $L > 0$. Theorem I.7 continues to hold. On the r.h.s. of (2.13) a term L^{-6} should also be added to $(\nabla \otimes B)^2$, but it can be absorbed into the last term. The purpose of this variant is to ensure

$$l(x) \lesssim L , \quad (8.5)$$

for every $x \in \mathbb{R}^3$.

Proof. By monotonicity, it will be enough to prove the theorem for $Z \geq 1$, $\Gamma \leq Q$ and $\gamma \leq z^4$. We partition again \mathbb{R}^3 into Voronoi cells $\Gamma_j = \{x \mid |x - R_j| \leq |x - R_k| \text{ for } k = 1, \dots, K\}$, $(j = 1, \dots, K)$. Let $D_j = \min\{|R_j - R_k| \mid j \neq k\}/2$. For any $\nu > 0$ the reduction to a one-body problem reads (see (I.3.11, II.8.4))

$$H_N \geq \sum_{i=1}^N h_i - \nu N + \frac{Z^2}{8} \sum_{j=1}^K D_j^{-1} + \Gamma \int \Phi_{\mathcal{R}}(x) (B(x)^2 + \gamma L^2 (\nabla \otimes B)(x)^2) d^3 x, \quad (8.6)$$

where $h = \mathcal{P}^2 - (W - \nu)_+$ and W is defined in Theorem I.2 and satisfies $W(x) \leq Q|x - R_j|^{-1}$ for $x \in \Gamma_j$, with $Q = Z + \sqrt{2Z} + 2.2$ (see (I.3.12)).

We choose $\nu = QL^{-1}$ and apply Theorem I.7 (in the variant discussed above) to obtain

$$\sum_{i=1}^N h_i \gtrsim - \int V^{5/2} d^3 x - \int P^{3/2} V d^3 x - \int \widehat{B} V^{3/2} d^3 x - \int \widehat{B} P^{1/2} V d^3 x, \quad (8.7)$$

where $V = (W - QL^{-1})_+$. Comparing with (8.6) it appears to be enough to show that each of the integrals (8.7), which we shall denote by (i-iv) below, is bounded by the bound (8.3) or by a small (universal) constant times

$$\frac{Z^2}{8} \sum_{j=1}^K D_j^{-1} + \Gamma \int \Phi_{\mathcal{R}}(x) (B(x)^2 + \gamma L^2 (\nabla \otimes B)(x)^2) d^3 x. \quad (8.8)$$

i) Note that $\text{supp } V \subset \Omega_L$ for $\Omega = \mathcal{R}$ (see proof of (II.8.3)). This integral is thus bounded by $\text{const } Q^{5/2} L^{1/2} K \lesssim QL^{-1} K$.

ii) We note that for any $\beta_1 > 0$

$$P^{3/2} \leq \sqrt{2} l^{-3/2} (r^{-3/2} + l^{-3/2}) \leq \sqrt{2} \frac{\beta_1}{2} r^{-3} + \sqrt{2} (1 + \frac{\beta_1^{-1}}{2}) l^{-3} \quad (8.9)$$

and we estimate the contributions to (ii) of the two terms separately. For the first one we use that

$$\int_{\Omega_L} r(x)^{-3} V(x) d^3 x \lesssim Q \int \Phi_{\mathcal{R}}(x) B(x)^2 d^3 x + Q \sum_{j=1}^K D_j^{-1} + QL^{-1} K,$$

as was shown in (II.8.6) (take there $\delta = 1$ for instance). We have to ensure, on one hand, that $\text{const } Q \ll \Gamma$ (first term), on the other hand, that $\text{const } \beta_1 Q \ll Z^2/8$ (second term). (By $a \ll b$ we mean $a = \text{const } b$ for some sufficiently small universal constant). Therefore we pick $\beta_1 \ll \min(Q^{-1}\Gamma, 1)$. The term proportional to K is then of the order of $QL^{-1}K$. For the last term in (8.9) we use instead

$$\begin{aligned} \int_{\Omega_L} l(x)^{-3} V(x) d^3 x &\leq \frac{\beta_2}{2} \int_{\Omega_L} l(x)^{-6} d^3 x + \frac{\beta_2^{-1}}{2} \int_{\Omega_L} V(x)^2 d^3 x \\ &\lesssim \beta_2 \int \Phi_{\mathcal{R}}(x) (\nabla \otimes B)(x)^2 d^3 x + (\beta_2 L^{-3} + \beta_2^{-1} Q^2 L) K, \end{aligned}$$

due to (2.13). The desired bound holds provided we pick $z \cdot \beta_2 \ll \Gamma \gamma L^2$, because $1 + \beta_1^{-1}/2 \lesssim z = 1 + (Z + 1)\Gamma^{-1}$ in view of the choice for β_1 . Then

- $(1 + \frac{\beta_1^{-1}}{2})\beta_2 L^{-3}K \lesssim \Gamma \gamma L^{-1}K \leq z^3 \Gamma z L^{-1}K \leq z^3 (Z + 1)L^{-1}K$,

for $\gamma \leq z^4$, since $\Gamma z = \Gamma + (Z + 1) \leq Z + 1$;

- $(1 + \frac{\beta_1^{-1}}{2})\beta_2^{-1} Q^2 LK \lesssim z \cdot (z\Gamma^{-1}\gamma^{-1}L^{-2})Q^2 LK \lesssim z^3 QL^{-1}K$,

because $\Gamma^{-1}Q \lesssim z$.

iii) We split the integral into K inner integrals over $U_j = \{x \mid |x - R_j| \leq \hat{D}_j\}$, $\hat{D}_j = \min(D_j, \varepsilon \cdot l(R_j), L)$ for some small $\varepsilon > 0$; and one outer integral over $\mathbb{R}^3 \setminus \bigcup_{j=1}^K U_j$. The inner integrals can be estimated as

$$\int_{U_j} \hat{B}(x) V(x)^{3/2} d^3x \lesssim \left(\sup_{x \in U_j} \hat{B}(x) \right) \hat{D}_j^{3/2} Q^{3/2} \leq \frac{\beta}{2} \hat{D}_j^3 \left(\sup_{x \in U_j} \hat{B}(x)^2 \right) + \frac{\beta^{-1}}{2} Q^3.$$

Because of (2.7) we have $\frac{1}{2}l(R_j) \leq l(x) \leq 2l(R_j)$ for $x \in U_j$ and thus

$$\begin{aligned} \hat{B}(x)^2 &= |K_x|^{-2} \left(\int_{K_x} |B(y)| d^3y \right)^2 \leq |K_x|^{-1} \int_{K_x} B(y)^2 d^3y \\ &\lesssim (\varepsilon l(R_j))^{-3} \int \theta(|y - R_j| \leq 3\varepsilon l(R_j)) B(y)^2 d^3y. \end{aligned} \quad (8.10)$$

Altogether we find for any $\beta > 0$

$$\begin{aligned} \int_{\bigcup_{j=1}^K U_j} \hat{B}(x) V(x)^{3/2} d^3x &\lesssim \beta \int \Phi(y) B(y)^2 d^3y + \beta^{-1} Q^3 K, \\ \Phi(y) &= \sum_{j=1}^K \hat{D}_j^3 (\varepsilon l(R_j))^{-3} \theta(|y - R_j| \leq 3\varepsilon l(R_j)). \end{aligned}$$

For $\beta \ll \Gamma$ this will be bounded as claimed once we show that $\Phi \lesssim \theta_{\Omega_L}$ (note that $\beta^{-1} Q^3 K \lesssim \Gamma^{-1} Q (Z+1) Q K \leq z^3 (Z+1) L^{-1} K$ since $\Gamma^{-1} Q \lesssim z \leq z^3$ and $Q \lesssim Z+1 \leq L^{-1}$). First, $\text{supp } \Phi \subset \Omega_L$ for small $\varepsilon > 0$ because of (8.5), independently of the magnetic field. It thus suffices to show $\|\Phi\|_\infty \lesssim 1$: from $\hat{D}_j \leq \varepsilon l(R_j)$, the triangle inequality and (2.7) we find

$$\begin{aligned} \|\Phi\|_\infty &\leq \sup_y \sum_{j=1}^K (\varepsilon l(R_j))^{-3} \theta(|y - R_j| \leq 3\varepsilon l(R_j)) \int_{U_j} \theta(|x - R_j| \leq \varepsilon l(R_j)) d^3x \\ &\lesssim \sup_y \sum_{j=1}^K (\varepsilon l(y))^{-3} \int_{U_j} \theta(|x - y| \leq 8\varepsilon l(y)) d^3x \lesssim 1, \end{aligned}$$

since the U_j are disjoint.

The outer integral can be written and estimated as

$$\begin{aligned} & \int_{\Omega_L \setminus (\cup_{j=1}^K U_j)} d^3x V(x)^{3/2} |K_x|^{-1} \int d^3y |B(y)| \theta(|x - y| < \varepsilon l(x)) \\ & \leq \frac{\beta_1}{2} \int_{\Omega_L \times \mathbb{R}^3} d^3x d^3y |B(y)|^2 |K_x|^{-1} \theta(|x - y| < \varepsilon l(x)) \\ & \quad + \frac{\beta_1^{-1}}{2} \int_{\Omega_L \setminus (\cup_{j=1}^K U_j) \times \mathbb{R}^3} d^3x d^3y V(x)^3 |K_x|^{-1} \theta(|x - y| < \varepsilon l(x)). \end{aligned} \quad (8.11)$$

By the usual argument (2.7), the first integral is bounded by a constant times $\int \Phi(y) |B(y)|^2 d^3y$ for

$$\Phi(y) = |K_y|^{-1} \int_{\Omega_L} \theta(|x - y| < 2\varepsilon l(y)) d^3x \lesssim 1.$$

Moreover, $\text{supp } \Phi \subset \Omega_{2L}$ as before. It thus suffices to take $\beta_1 \ll \Gamma$. In the second term on the r.h.s. of (8.11) the integration over y is explicit, and the integral is

$$\int_{\Omega_L \setminus (\cup_{j=1}^K U_j)} V(x)^3 d^3x \lesssim \sum_{j=1}^K Q^3 \log L \hat{D}_j^{-1} \leq \beta_2 Q^3 \sum_{j=1}^K L \hat{D}_j^{-1} + (\log \beta_2^{-1}) Q^3 K, \quad (8.12)$$

where we used that $\log t \leq \beta_2 t + \log \beta_2^{-1}$ for $t, \beta_2 > 0$. We shall take $\Gamma^{-1} \cdot \beta_2 Q^2 L \ll 1$, so that the last term is of the desired form: this means indeed that $\Gamma^{-1} (\log \beta_2^{-1}) Q^3 K \lesssim z^3 L^{-1} K \log \beta_2^{-1}$, while the argument of the logarithm is bounded above by a constant times $\Gamma^{-1} Q^2 L \lesssim z \leq z^5 \gamma^{-1/2}$, since $1 \leq z^4 z^{-2} \leq z^4 \gamma^{-1/2}$ for $1 \leq \gamma \leq z^4$. The first term in (8.12) reduces to an arbitrarily small constant times $Q \sum_{j=1}^K \hat{D}_j^{-1}$. Note that

$$\hat{D}_j^{-1} \lesssim \varepsilon^{-2} \left(\int_{U_j} l(x)^{-6} d^3x \right)^{1/3} + D_j^{-1} + L^{-1}. \quad (8.13)$$

In fact, by (2.7), the integral is bounded below by a constant times $(\varepsilon l(R_j))^{-2} \hat{D}_j$, and thus the whole r.h.s. by

$$\hat{D}_j^{-1} \left[\left(\frac{\hat{D}_j}{\varepsilon l(R_j)} \right)^2 + \frac{\hat{D}_j}{D_j} + \frac{\hat{D}_j}{L} \right] \geq \hat{D}_j^{-1},$$

by definition of \hat{D}_j . The contribution of the last two terms of (8.13) are then controlled by the first term (8.8), resp. by (8.3). For the integral, I , we use $I^{1/3} \leq \frac{2}{3} \beta_3^{-1/2} + \frac{1}{3} \beta_3 I$ and choose β_3 such that $\beta_1^{-1} \beta_2 Q^3 L \cdot \beta_3 \varepsilon^{-2} \ll Q \cdot \beta_3 \varepsilon^{-2} \ll \Gamma z^{-4} L^2$. Note that the U_j are disjoint, allowing for the application of (2.13). In this way, $\Gamma z^{-4} L^2 \leq \Gamma \gamma L^2$, because $z^{-4} \leq \gamma^{-1} \leq 1 \leq \gamma$, and $\beta_1^{-1} \beta_2 Q^3 L \cdot \beta_3^{-1/2} \varepsilon^{-2} K \lesssim Q \cdot \Gamma^{-1/2} Q^{1/2} z^2 L^{-1} \varepsilon^{-3} K \lesssim \varepsilon^{-3} z^3 (Z + 1) L^{-1} K$.

iv) Using

$$P^{1/2} \leq l^{-1/2}(r^{-1/2} + l^{-1/2}) \leq \frac{\beta_1}{2}r^{-1} + (1 + \frac{\beta_1^{-1}}{2})l^{-1}, \quad (8.14)$$

we estimate the contributions to (iv) of the two terms separately. The first integral is

$$\begin{aligned} \int_{\Omega_L} d^3x r(x)^{-1} V(x) |K_x|^{-1} \int d^3y |B(y)| \theta(|x - y| < \varepsilon l(x)) \\ \leq \frac{Q}{2} \int_{\Omega_L \times \mathbb{R}^3} d^3x d^3y |B(y)|^2 |K_x|^{-1} \theta(|x - y| < \varepsilon l(x)) \\ + \frac{Q^{-1}}{2} \int d^3x d^3y r(x)^{-2} V(x)^2 |K_x|^{-1} \theta(|x - y| < \varepsilon l(x)). \end{aligned} \quad (8.15)$$

The first term on the r.h.s. is like the corresponding one in (8.11) and hence acceptable provided $\beta_1 \cdot Q \ll \Gamma$. The second integral, $Q^{-1} \int r(x)^{-2} V(x)^2 d^3x$, is dealt with by splitting it with respect to $\tilde{U}_j = \{x \mid |x - R_j| < \tilde{D}_j\}$, $\tilde{D}_j = \min(D_j, \varepsilon \cdot r(R_j), L)$ (see Section II.8). Then

$$\int_{\tilde{U}_j} r(x)^{-2} V(x)^2 d^3x \lesssim r(R_j)^{-2} \int_{\tilde{U}_j} V(x)^2 d^3x \lesssim \varepsilon^2 Q^2 \tilde{D}_j^{-1},$$

and

$$\int_{\mathbb{R}^3 \setminus (\cup_{j=1}^K \tilde{U}_j)} r(x)^{-2} V(x)^2 d^3x \leq \frac{\varepsilon^2 Q^{-2}}{2} \int_{\mathbb{R}^3 \setminus (\cup_{j=1}^K \tilde{U}_j)} V(x)^4 d^3x + \frac{\varepsilon^{-2} Q^2}{2} \int_{\Omega_L} r(x)^{-4} d^3x.$$

Since the first integral is bounded above by $\text{const } Q^4 \sum_{j=1}^K \tilde{D}_j^{-1}$ we have that

$$\begin{aligned} Q^{-1} \int r(x)^{-2} V(x)^2 d^3x &\lesssim Q \sum_{j=1}^K \tilde{D}_j^{-1} + Q \int_{\Omega_L} r(x)^{-4} d^3x \\ &\lesssim Q \sum_{j=1}^K D_j^{-1} + Q \int \Phi_{\mathcal{R}}(x) B(x)^2 d^3x + Q L^{-1} K \end{aligned}$$

due (II.8.5) and (II.2.13). These terms fit (8.3) for our choice of β_1 since $\beta_1 Q \Gamma \leq Z + 1$ (first and last term).

The integral corresponding to the last term in (8.14) is estimated similarly to (iii) and is split accordingly. The inner integrals can be estimated as

$$\begin{aligned} \int_{U_j} \widehat{B}(x) l(x)^{-1} V(x) d^3x &\lesssim \left(\sup_{x \in U_j} \widehat{B}(x) l(x)^{-1} \right) \widehat{D}_j^2 Q \\ &\leq \frac{2\beta_2^{1/2}}{3} \widehat{D}_j^3 \left(\sup_{x \in U_j} \widehat{B}(x) l(x)^{-1} \right)^{3/2} + \frac{\beta_2^{-1}}{3} Q^3, \end{aligned} \quad (8.16)$$

where

$$(\widehat{B}l^{-1})^{3/2} \leq \frac{1}{4}\gamma^{-1/4}L^{-1/2}(3\widehat{B}^2 + \gamma L^2l^{-6}) . \quad (8.17)$$

The term coming from \widehat{B}^2 will be dealt with by (8.10), the other one by using $\widehat{D}_j^3 \sup_{x \in U_j} l(x)^{-6} \lesssim \int_{U_j} l(x)^{-6} d^3x$. Reminding that $\beta_1^{-1} \lesssim \Gamma^{-1}Q \lesssim z$ and choosing $z \cdot \beta_2^{1/2}\gamma^{-1/4}L^{-1/2} \ll \Gamma$, we ensure that both terms (8.17) are controlled by (8.8) and (8.3) (from the second term in (8.17) we also get through (2.13) a term $\lesssim \gamma\Gamma L^2L^{-3}K \leq z^5(Z+1)L^{-1}K$, for $\gamma \leq z^4 \leq z^5$). The contribution of the last term (8.16) is then of order $z \cdot \beta_2^{-1}Q^3K \lesssim z \cdot \Gamma^{-2}\gamma^{-1/2}L^{-1}L^{-1}z^2Q^3K \lesssim z^5\gamma^{-1/2}QL^{-1}K \lesssim z^5\gamma^{-1/2}QL^{-1}K$. The estimate of the outer integral follows the line of (8.15):

$$\begin{aligned} & \int_{\Omega_L \setminus (\cup_{j=1}^K U_j)} d^3x l(x)^{-1} V(x) |K_x|^{-1} \int d^3y |B(y)| \theta(|x-y| < \varepsilon l(x)) \\ & \leq \frac{\beta_3}{2} \int_{\Omega_L \times \mathbb{R}^3} d^3x d^3y |B(y)|^2 |K_x|^{-1} \theta(|x-y| < \varepsilon l(x)) \\ & \quad + \frac{\beta_3^{-1}}{2} \int_{\Omega_L \setminus (\cup_{j=1}^K U_j) \times \mathbb{R}^3} d^3x d^3y l(x)^{-2} V(x)^2 |K_x|^{-1} \theta(|x-y| < \varepsilon l(x)) . \end{aligned}$$

The first term just requires $z \cdot \beta_3 \ll \Gamma$. The second one is

$$\int_{\mathbb{R}^3 \setminus (\cup_{j=1}^K U_j)} l(x)^{-2} V(x)^2 d^3x \leq \frac{2}{3}\beta_4^{-1/2} \int_{\mathbb{R}^3 \setminus (\cup_{j=1}^K U_j)} V(x)^3 d^3x + \frac{1}{3}\beta_4 \int_{\Omega_L} l(x)^{-6} d^3x .$$

To accomodate the last term, after application of (2.13), we require $\beta_1^{-1}\beta_3^{-1}\cdot\beta_4 \lesssim z^2\Gamma^{-1}$. $\beta_4 \ll \Gamma z^{-4}L^2$ (compare with the end of the discussion of integral (iii)). The first term is dealt as in (8.12), with $\beta_2 \ll z^{-7}$ there, because we have to impose $\beta_1^{-1}\beta_3^{-1}\beta_4^{-1/2}\cdot\beta_2 Q^2L \lesssim z^5\Gamma^{-2}L^{-1}\cdot\beta_2 Q^2L \lesssim z^7\beta_2 \ll 1$ in order for the coefficient in front of the sum on the r.h.s. of (8.12) to be small as compared to $Z^2/8$. The logarithmic term is thus of the order $\beta_1^{-1}\beta_3^{-1}\beta_4^{-1/2}(\log \beta_2^{-1})Q^3K \lesssim z^5\Gamma^{-2}Q^2(\log z)QL^{-1}K \lesssim z^7(\log z z^4 z^{-2})QL^{-1}K \lesssim z^5\gamma^{-1/2}(\log z^5\gamma^{-1/2})QL^{-1}K$. ■

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