Dispersion in a bent-solenoid channel with symmetric focusing ⋆

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Abstract

Longitudinal ionization cooling of a muon beam is essential for muon colliders and will be useful for neutrino factories. Bent-solenoid channels with symmetric focusing has been considered for beam focusing and for generating the required dispersion in the "emittance exchange" scheme of longitudinal cooling. In this paper, we derive the Hamiltonian that governs the linear beam dynamics of a bent-solenoid channel, solve the single-particle dynamics, and give equations for determining the lattice functions, in particular, the dispersion functions.

Key words: dispersion, bent-solenoid, emittance exchange, ionization cooling, muon collider
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1 Introduction

To build muon colliders with luminosity interesting to high-energy physics experiments, the phase-space distribution of a muon beam collected from decay of pions created on a proton target, needs to be reduced by 10^6. Both transverse and longitudinal cooling are required [1,2]. For neutrino factories [3], cooling in the longitudinal direction will help to increase the intensity and reduce the cost. To obtain the desired cooling, ionization cooling channels are being developed. When passing through absorbers, muons' momentum vectors are reduced due to ionization energy loss. By accelerating muons only longitudinally in the rf cavities, transverse cooling can be achieved. However, the

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ionization process itself does not effectively cool the longitudinal momentum spread because the energy-loss rate is not sensitive to beam momentum except for very low-energy muons. To achieve longitudinal cooling, a promising option is the "emittance exchange" scheme: introducing dispersion to spatially separate muons of different energies and then using wedge absorbers to discriminatively cool them [2].

Since solenoid channels are well-suited for focusing low-energy muon beams with large acceptance, they are the primary candidate for a transverse cooling channel [1,3]. Thus, a natural choice of transport channel for emittance exchange is to add dispersion in a solenoid channel. The straightforward approach is to superimpose a dipole field with the solenoid field and make the solenoids bend along the curved reference orbit determined by the dipole field. Since the main solenoid field continuously rotates the beam and tends to make the beam rotationally symmetric, it is advantageous to have symmetric focusing in a bent-solenoid channel. To achieve this, gradient dipoles (with field index n=1/2) could be used. [1]

In a bent-solenoid cooling channel, neither the solenoid nor the dipole field can usually be treated as piecewise constant elements. Thus the lattice consists of rather complicated combined function magnets. In this paper, we study the single-particle linear dynamics in such a bent-solenoid channel with no absorbers. The result, especially the equations for the dispersion functions, can be useful for designing a bent-solenoid focusing channel with the desired dispersion. Our result is also useful as a basis for a comprehensive ionization cooling theory or for studying the nonlinear dynamics in a bent-solenoid channel.

2 Hamiltonian

The magnetic field guiding muons in a bent-solenoid channel consists of a longitudinal solenoidal field for focusing, a vertical dipole field for dispersion, and a quadrupole field from gradient dipoles for symmetric focusing. The magnetic field and vector potential in the usual Frenet-Serret coordinate system $\{x, y, s\}$ can be written, up to the linear order, as [4]

$$B_x(x,y,s) = -\frac{1}{2}b_s'\,x + b_1\,y$$

[1] R. Palmer first suggested the possibility of replacing some of the weak solenoids of a transverse cooling channel with gradient dipoles to keep the required focusing while introducing the dipole field.

$$\begin{aligned} B_y(x,y,s) &= b_0 + b_1 x - \frac{1}{2} b_s' y \\ B_s(x,y,s) &= b_s - \kappa b_s\, x + b_0'\, y \end{aligned} \tag{1}$$

and

$$\begin{aligned} A_x(x,y,s) &= -\frac{1}{2} b_s\, y + \frac{1}{3}\kappa b_s\, xy - \frac{1}{3} b_0'\, y^2 \\ A_y(x,y,s) &= \ \frac{1}{2} b_s\, x - \frac{1}{3}\kappa b_s\, x^2 + \frac{1}{3} b_0'\, xy \\ A_s(x,y,s) &= -b_0\, x - \frac{1}{2}(b_1 - \kappa b_0) x^2 + \frac{1}{2} b_1\, y^2, \end{aligned} \tag{2}$$

where b_s, b_0, b_1 are the solenoidal, dipolar, quadrupolar components. They are all s-dependent to account for the fringe field. A prime denotes differentiation with respect to s. $\kappa(s)$ is the curvature of the reference orbit and is normally chosen to be $qb_0(s)/p_0$ for a reference particle of charge q and nominal momentum p_0. Here A_s is the normal, instead of the canonical, s-component of A.

The Hamiltonian for a bent-solenoid channel, with s as the independent variable, can be derived from the standard procedure. We start from the basic expression [5,6]

$$H = -\frac{qA_s}{p_0}\left(1+\frac{x}{\rho}\right) - \left\{ \left(1+\frac{x}{\rho}\right)\sqrt{1 - \left(\frac{p_x - qA_x/p_0}{1+\delta}\right)^2 - \left(\frac{p_y - qA_y/p_0}{1+\delta}\right)^2} - 1 \right\}(1+\delta). \tag{3}$$

Here p_x, p_y are the normalized momenta with respect to p_0, $\delta = (p-p_0)/p_0$ is the relative momentum deviation, and z is the longitudinal position relative to the reference particle. Inserting the vector potential in Eq. (2) and expanding the Hamiltonian up to the second order yields the linear Hamiltonian as

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\hat{b}_s^2\left(x^2+y^2\right) - \hat{b}_s L_z \tag{4}$$

$$-\frac{x\delta}{\rho(s)} + \frac{x^2}{2\rho(s)^2} + \frac{1}{2}\hat{b}_1\left(x^2 - y^2\right), \tag{5}$$

where $\hat{b}_s = \frac{q}{2p_0} b_s = \frac{q}{2p_0} B_s(0,0,s)$, $\hat{b}_1 = \frac{q}{p_0} b_1 = \frac{q}{p_0}\frac{\partial B_y}{\partial x}\Big|_{x=y=0}$, $\frac{1}{\rho} = \kappa = \frac{q}{p_0} b_0 = \frac{q}{p_0} B_y(0,0,s)$. The canonical angular momentum $L_z = xp_y - yp_x$. Note that, at linear order, the fringe field terms (b_s', b_0') and curvature terms (κb_s, κb_0) do not appear in the Hamiltonian, and the potential part of the Hamiltonian is additive for superimposed magnets.

For symmetric focusing channels, the quadrupole components of the gradient dipoles must be tied to the bending radius as $\hat{b}_1(s) = -1/2\rho(s)^2$. Thus the total focusing strength becomes $K(s) = \hat{b}_s(s)^2 + 1/2\rho(s)^2$. The Hamiltonian reduces to

$$
\begin{aligned}
H = & \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}K(s)\left(x^2 + y^2\right) - \hat{b}_s(s)L_z \\
& - \frac{x\delta}{\rho(s)} + \frac{1}{2}\left[\delta^2 + \mathrm{V}(s)z^2\right].
\end{aligned} \tag{6}
$$

Here, for simplicity, we added a simple oscillator with focusing strength $V(s)$ as the longitudinal Hamiltonian and assumed no acceleration of the reference particle.

3 Dispersion and beta functions

To solve the Hamiltonian of Eq. (6), we transform it to the Larmor frame (a rotating frame that rotates at half of the cyclotron frequency) so that the x-y coupling term L_z is removed. Using the $\tilde{}$ over a symbol to indicate that it is in the Larmor frame, the transformation reads

$$
x = \tilde{x}\cos\theta + \tilde{y}\sin\theta\,, \quad y = \tilde{y}\cos\theta - \tilde{x}\sin\theta\,, \quad \text{etc.,} \tag{7}
$$

where $\theta(s) = \int_0^s \hat{b}_s(\bar{s})d\bar{s}$ is the rotating angle of the Larmor frame. The generating function is

$$
F_2 = x\left[\tilde{p}_x\cos\theta + \tilde{p}_y\sin\theta\right] + y\left[\tilde{p}_y\cos\theta - \tilde{p}_x\sin\theta\right]. \tag{8}
$$

In the Larmor frame, the Hamiltonian becomes

$$
\begin{aligned}
\tilde{H} = & \frac{1}{2}\left(\tilde{p}_x^2 + \tilde{p}_y^2\right) + \frac{1}{2}K(s)\left(\tilde{x}^2 + \tilde{y}^2\right) \\
& - \frac{\tilde{x}\delta\cos[\theta(s)]}{\rho(s)} - \frac{\tilde{y}\delta\sin[\theta(s)]}{\rho(s)} + \frac{1}{2}\left[\delta^2 + \mathrm{V}(s)z^2\right].
\end{aligned} \tag{9}
$$

To further decouple the transverse and longitudinal degrees of freedom, we introduce the dispersions $\tilde{D}_x$, $\tilde{D}_y$ and a corresponding canonical transformation from $(\tilde{x}, \tilde{p}_x, \tilde{y}, \tilde{p}_y, z, \delta)$ to $(\tilde{x}_\beta, \tilde{p}_{x\beta}, \tilde{y}_\beta, \tilde{p}_{y\beta}, \hat{z}, \hat{\delta})$ as

$$
\tilde{x} = \tilde{x}_\beta + \tilde{D}_x\delta\ , \quad \tilde{p}_x = \tilde{p}_{x\beta} + \tilde{D}_x'\delta\ ,
$$

$$\tilde{y} = \tilde{y}_\beta + \tilde{D}_y\delta \ , \quad \tilde{p}_y = \tilde{p}_{y_\beta} + \tilde{D}'_y\delta \ , \tag{10}$$
$$z = \hat{z} - \tilde{D}'_x\tilde{x} + \tilde{D}_x\tilde{p}_x - \tilde{D}'_y\tilde{y} + \tilde{D}_y\tilde{p}_y \ , \quad \delta = \hat{\delta},$$

which can be generated by the generating function

$$F = (\tilde{x} - \tilde{D}_x\delta)\tilde{p}_{x_\beta} + D'_x\tilde{x}\delta + (\tilde{y} - \tilde{D}_y\delta)\tilde{p}_{y_\beta} + D'_y\tilde{y}\delta - \hat{z}\delta - \frac{1}{2}(D_xD'_x + D_yD'_y)\delta^2. \tag{11}$$

The transformed Hamiltonian $\tilde{H}_\beta$ is complicated. However, it can be dramatically simplified and decoupled by requiring 1) the cavity regions are dispersion free and 2) the dispersion functions satisfy the differential equations

$$\tilde{D}''_x + K(s)\tilde{D}_x = \frac{\cos[\theta(s)]}{\rho(s)}, \ \tilde{D}''_y + K(s)\tilde{D}_y = \frac{\sin[\theta(s)]}{\rho(s)}. \tag{12}$$

These are similar to the well-known dispersion equation in quadrupole channels. However, because of the solenoidal field, a vertical dipole will generate dispersion in both horizontal and vertical planes with driving terms depending on the Larmor rotation angle through the $\cos\theta$ and $\sin\theta$. It is more difficult to find a dispersion solution for bent-solenoid channels because there are two equations to satisfy and furthermore the right-hand-side driving terms depend on the focusing strength $K(s)$ due to the focusing from the gradient dipole.

Under the above two conditions, the new Hamiltonian $\tilde{H}_\beta$ is reduced to a simple form

$$\tilde{H}_\beta = \frac{1}{2}\left(\tilde{p}^2_{x_\beta} + \tilde{p}^2_{y_\beta}\right) + \frac{1}{2}K(s)\left(\tilde{x}^2_\beta + \tilde{y}^2_\beta\right) + \frac{1}{2}\left[I(s)\delta^2 + \mathrm{V}(s)\hat{z}^2\right]. \tag{13}$$

Here $I(s) = 1 - \frac{\tilde{D}_x\cos[\theta(s)]}{\rho(s)} - \frac{\tilde{D}_y\sin[\theta(s)]}{\rho(s)}$ reflects the momentum compaction effect.

Now that all three degrees of freedom are decoupled in $\tilde{H}_\beta$, we can introduce lattice functions for them in analog to the Courant-Snyder theory [5]. There is one set of lattice functions (β_T, α_T, γ_T) for both transverse degrees of freedom and one set (β_L, α_L, γ_L) for the longitudinal motion, which satisfy the familiar equations

$$\beta'_T = -2\alpha_T \, , \ \alpha'_T = K(s)\beta_T - \gamma_T \, , \ \gamma_T = \frac{1+\alpha_T^2}{\beta_T} \tag{14}$$

and

$$\beta'_L = -2I(s)\alpha_T \, , \ \alpha'_L = V(s)\beta_T - I(s)\gamma_T \, , \ \gamma_L = \frac{1+\alpha_L^2}{\beta_L}. \tag{15}$$

With appropriate boundary conditions, these two sets of lattice functions define the transverse and longitudinal machine ellipses that characterize the betatron and synchrotron oscillations. Using the lattice functions determined by Eqs. (12, 14, 15) and the transformations in Eqs. (7, 10) it is straightforward to write down the complete solution for single-particle motion in a bent-solenoid channel.

4 Design of dispersion functions

As in storage rings, optimizing the dispersion is an important part of a longitudinal cooling lattice design. Dispersion functions need to fulfill many conditions for a lattice to function well. Equation (12) provides a useful tool for dispersion design in a symmetrically focused bent-solenoid channel. Successful design of a longitudinal ionization cooling channel is still under investigation. In this section, we provide an example to illustrate some of the issues and compare the result of Eq. (12) with simulation.

The thoughts behind this example are as follows. Since longitudinal cooling is in fact achieved via transverse cooling, a natural starting point for a bent-solenoid cooling channel is to add dispersion in suitable regions of a successful transverse (straight) solenoid cooling channel while maintaining the original transverse focusing properties, especially the periodicity and beta function that are critical to transverse cooling. This can be achieved by reducing the solenoid focusing to balance the focusing from added gradient dipoles, i.e., adjust $\rho(s)$ as desired but keep the total focusing strength $K(s) = \hat{b}_s^2 + 1/2\rho^2$ unchanged. Obviously there is a limit on the dipole strengths and locations. Also, the dispersion design is complicated because reducing the solenoid strength will affect the Larmor rotation angle and thus the dispersion driving terms in Eq. (12). Other conceivable conditions are: 1) Cavity regions should be dispersion free in order to decouple the transverse and longitudinal degrees of freedom. Thus the dispersion has to be localized between the cavities, which cannot be too far apart due to longitudinal focusing requirement. 2) Maximum dispersion better occurs in a minimum beta region due to the large beam size and limited aperture. 3) The dispersion insertion is better if it is a first-order achromat.

In the example shown in the figure, we start with a "superFoFo" lattice used in the transverse cooling channel of the neutrino factory feasibility study-II [3]. Two periods of the periodic solenoid field ($\hat{b}_s$) and beta function (β) of the superFoFo lattice are shown as a thin dashed line and a thick solid line, respectively. To generate a closed dispersion bump that fulfills the above conditions and the more stringent requirement that $D_x = D'_x = D_y = D'_y = 0$ at both ends for three different muon energies (to control chromatic effect), twenty short piecewise-constant vertical dipoles with fringe field are added. By adjusting the dipole strengths and the solenoid focusing as discussed above, we found a solution of Eq. (12) and plot the D_x (D_y) as thick (thin) dotted line. The added dipole field ($1/\rho$) is plotted as a thin dash-dot line and the adjusted solenoid strength ($\hat{b}_s$) as a thin solid line. This bent-solenoid channel was tracked with the ICOOL simulation code [7] and the tracked dispersion D_x (D_y) is also shown as thick (thin) gray line. We see that the tracking result and Eq. (12) agree very well. This example demonstrates that Eq. (12) can be used to design sophisticated dispersion functions.

Since the dispersions begin and end with zeros and the beta function is unchanged, such a dispersion section could be used to implement longitudinal ionization cooling by replacing certain periods of the original straight solenoid channel. We emphasize that this is just an example solution. Much simpler dispersion solutions exist if we require the zero boundary condition for only one muon energy. Furthermore, if the solenoid and dipole fields do not overlap, the channels discussed in this paper reduce to simpler separated function lattices as proposed in Ref. [8].

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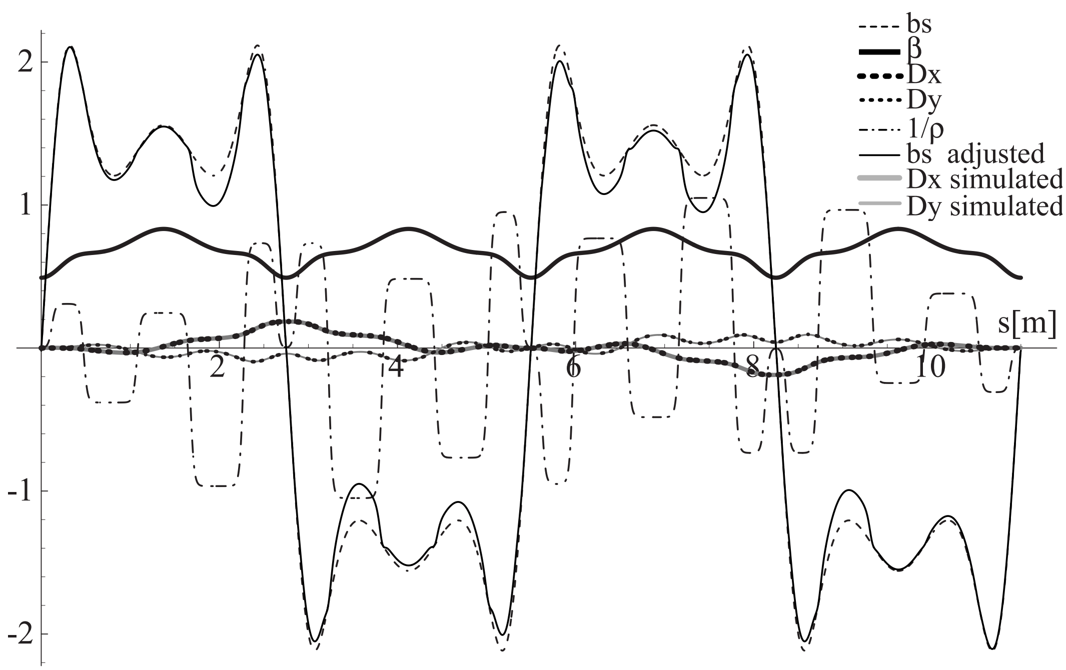

Fig. 1. Dispersion example of a bent-solenoid channel. Lattice functions β, D_x, and D_y are in [m]. Field strength functions $\hat{b}_s$ and $1/\rho$ are in [1/m]. The original and adjusted solenoid strengths are in dashed and solid (adjusted) lines. The dispersions calculated with Eq. (12) and ICOOL tracking are in dotted and gray (simulated) lines.