A Manual for Conformal Field Theories in 4D

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August 2017

Abstract

This thesis conducts an investigation of four dimensional conformal field theories (CFT). With the application of the bootstrap techniques in mind, we set out to compute the conformal partial waves (CPW) needed to bootstrap a generic four-point function in a 4D CFT. These CPW can correspond to the exchange of a bosonic or fermionic operator, in an irreducible representation $(\ell, \bar{\ell})$ of the Lorentz group. Utilizing the embedding formalism in twistor space, we introduce a basis of differential operators that can relate any CPW to a "seed" CPW with the same exchanged operator. We compute in a closed analytic form the seed CPW. We solve the Casimir equations by using an educated ansatz and reduce the problem to an algebraic linear system. Many of the properties of the ansatz are deduced using the shadow formalism.The seed CPW depends on the representation of the exchanged operator, particularly on the value of $p = |\ell - \bar{\ell}|$ and its complexity grows with p. As an application of our results, we write the bootstrap equations for a four-point function of two scalar and two spin-1/2 fermions. We solve the equations in the light-cone limit and compute the anomalous dimensions of double-twist operators as an expansion in 1/spin in the large spin limit.

Acknowledgement

I am very fortunate to have been an ICTP/ SISSA student, where I've met very friendly, knowledgeable and supportive people. In particular, I am very grateful to my supervisor, Marco Serone. It was a privilege and a pleasure to learn and work with him. For all his advice and support and that he presented to me an example of how to be a great researcher, I cannot thank him enough.

I want also to thank Alejandro Castedo Echeverri, Gabriel Francisco Cuomo and Denis Karteev for the pleasure of working with them and all the useful discussions we had.

I am very grateful to all of my friends, specially those I've met in Trieste who enriched my life and shared with me happiness and difficult times. I owe special thanks to Amna, Nehad and Rafaa for being there for me all the time.

Last but not least, I am forever indebted to my family for their unending support, encouragement and love.

To my parents

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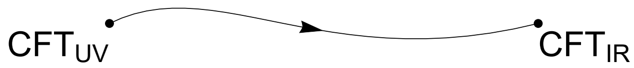

Figure 1.1: One can think of UV complete QFT as a flow between UV fixed point to an IR fixed point.

1. Introduction

1.1 Conformal Field Theory

The study of Quantum field theories (QFT) can be seen as a study of RG (Renormalization Group) flow. Usually we start with a QFT in the UV and compute how it behaves as we change the energy scale. At large distances the theory generally becomes scale invariant, i.e. arrives at an IR fixed point in the space of theories.

Scale invariance implies the invariance under the larger conformal symmetry group[1]. Conformal symmetry group consists of transformations that looks locally as scaling and rotations. This extra symmetry puts strong constraints on the theory and might facilitate the search for its solutions. So RG fixed points are described by conformal field theories (CFT). Studying CFTs will let us map out the possible endpoints of RG flows, and thus understand the space of QFTs.

Many physical systems flow to IR fixed points that are described by a CFT, for example 3d Ising model, water at the critical point of its phase diagram and uni-axial magnets at the critical temperature. Actually these three examples are described by the same CFT, which also arises at the Wilson-Fisher fixed point in the IR of ϕ^4 theory. This phenomenon is called critical universality: that in the continuum limit microscopic details of the Lagrangian don't matter and all theories with the same symmetry look the same (up to identification of couplings).

CFTs play another fundamental role: they can help us by means of the AdS/CFT correspondence, in shedding light on various aspects of quantum gravity and string theory.

The conformal bootstrap program [3, 4] aims to study CFTs without referring to any Lagrangian or microscopic description, by making full use of the symmetry and imposing consistency conditions to solve the theory. This non-perturbative approach was resurrected in [5], and it was since followed by impressive improvements to the current understanding of the space of CFTs.

1.1.1 Why 4D CFT ?

This is hardly a legitimate question, understanding the space of 4D QFT is a long-time dream of theorists. Studying 4D CFTs serve as a starting point. The conformal bootstrap could help explore 4D fixed points, for example in QCD conformal window [6] and as the bootstrap techniques evolve, one might move from a non-perturbative CFT to a non-perturbative QFT.

The 4D conformal bootstrap also has phenomenological importance, relevant to model building beyond the standard model. For example partially composite Higgs model, with a strongly coupled conformal

[1]The statement scale invariance implies conformal invariance is supported by examples, it has been proven in $d = 2$ and $d = 4$ for Lorentz-invariant unitary theories [1, 2]. The required conditions in other dimensions are not known yet.

sector, the bootstrap could be used to check whether a CFT leading to a phenomenologically viable composite Higgs model exist [7].

To achieve these goals, we need to be able to get as much constraints as possible from the bootstrap. This was not always possible, the seminal paper [5] and many after used the 4D conformal bootstrap of only scalars. The reason is that some technical pieces, basically what is called conformal blocks, were missing. Our aim in this is to present these previously missing pieces and provide a manual to use the bootstrap program in 4D CFT. First we will start by reviewing the basics of CFT andwhat the conformal bootstrap entails .

1.2 Conformal Symmetry

1.2.1 The Conformal Algebra

On a Minkowski flat space of dimension d, whose metric is $\eta_{\mu\nu}$, infinitesimal conformal transformations are defined as the coordinate transformations

$$x^\mu \to x^\mu + \epsilon^\mu(x), \tag{1.1}$$

that leaves the metric invariant up to a (local) scale factor

$$\delta g_{\mu\nu} = (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = c(x)\eta_{\mu\nu}, \tag{1.2}$$

where $\mu = 0, \ldots .d-1$. In dimensions $d > 2$ the equation above has four classes of solutions

$$\begin{aligned}
&\epsilon^\mu = a && \text{translation, } c(x) = 0. \\
&\epsilon^\mu = \omega^{[\mu\nu]}\, x_\nu && \text{rotations, } c(x) = 0, \\
&\epsilon^\mu = \lambda\, x^\mu && \text{dilatations, } c(x) = 2\lambda, \\
&\epsilon^\mu = 2(b\cdot x)x^\mu - x^2 b^\mu && \text{special conformal transformation, } c(x) = 4(b\cdot x),
\end{aligned} \tag{1.3}$$

where a and λ are constants, b^μ is a constant vector and $\omega^{[\mu\nu]}$ is a constant antisymmetric tensor. The first two classes $c(x) = 0$ are just Poincaré tansformations generated by

$$P^\mu = i\partial^\mu\,, \quad M^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu), \tag{1.4}$$

while the dilatations are generated by

$$D = ix^\mu\partial_\mu \tag{1.5}$$

and special conformal transformations by

$$K^\mu = i(2x^\mu x\cdot\partial - x^2\partial^\mu). \tag{1.6}$$

The operators generates form an algebra, the conformal algebra

$$\begin{aligned}
&[D, P_\mu] = -i\, P_\mu\,, \quad [P_\mu, M_{\nu\rho}] = i\, (\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu) \\
&[D, K_\mu] = i\, K_\mu\,, \quad [K_\mu, M_{\nu\rho}] = i\, (\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu) \\
&[K_\mu, P_\nu] = -2i\, (\eta_{\mu\nu}D + M^{\mu\nu}) \\
&[M_{\mu\nu}, M_{\rho\sigma}] = i\,(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\rho} - \rho\leftrightarrow\sigma)\,,
\end{aligned} \tag{1.7}$$

and the rest of possible commutations are zero.

The conformal algebra is isomorphic to $SO(d,2)$, the algebra of Lorentz transformations in $\mathbb{R}^{d;2}$ space. This can be seen as follows. Consider in the latter space the coordinates

$$X^0, \dots, X^d, X^{d+1}, \tag{1.8}$$

and the metric $\eta_{AB} = \text{diag}(-1, 1, \dots, 1, -1)$, where $A = \mu, d, \dots d+1$. We will also use the light-cone coordinates

$$X^+ = X^d + X^{d+1}\,, \quad X^- = X^d - X^{d+1}. \tag{1.9}$$

The Lorentz generators are given by

$$L_{AB} = i\left(X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}\right), \tag{1.10}$$

and they satisfy the $SO(d,2)$

$$[L_{AB}, L_{CD}] = i\,(\eta_{BC} L_{AD} - \eta_{AC} L_{BD} - \eta_{BD} L_{AC} + \eta_{AD} L_{BC}) \tag{1.11}$$

The conformal algebra generators will be identified with the $SO(d,2)$ generators as follows

$$M_{\mu\nu} = L_{\mu\nu}\,, \quad P_\mu = L_{\mu+}\,, \quad K_\mu = L_{\mu-}\,, \quad D = L_{+-}\,. \tag{1.12}$$

This identification will be very useful later, for one the conformal quadratic casimir is defined as $L_{AB}L^{AB}$. Also the action of $SO(d,2)$ on $\mathbb{R}^{d;2}$ is linear unlike the complicated action of the conformal group on $\mathbb{R}^{d-1;1}$. This identification inspired the idea to embed the CFT in $d+2$ space (chapter 2).

1.2.2 Conformal Transformations of Fields

Now that we have defined the symmetry generators, we can classify the operators in the CFT into representation (rep) of this symmetry. Let's define the conformal conserved charges as $\hat{P}\mu, \hat{M}_{\mu\nu}, \hat{D}$ and $\hat{K}_\mu$ [2] . Since $[D, M_{\mu\nu}] = 0$, we can classify operators into scale+Lorentz rep. Local operators at the origin transform in the irreducible rep of the Lorentz group

$$[\hat{M}_{\mu\nu}, \mathcal{O}^a(0)] = i\,(\mathcal{S}_{\mu\nu})^a{}_b \mathcal{O}^b(0), \tag{1.13}$$

where $\mathcal{S}_{\mu\nu}$ are matrices that satisfy the same algebra as $M_{\mu\nu}$, and a and b are indices of the Lorentz rep of $\mathcal{O}^a$. We can simultaneously diagonalize the action of $\hat{D}$ at the origin

$$[\hat{D}, \mathcal{O}^a(0)] = -i\,\Delta\,\mathcal{O}^a(0), \tag{1.14}$$

Δ is the scaling dimension of $\mathcal{O}$. Noting that K_μ acts as a lowering operator for the scaling dimension

$$[\hat{D}, [\hat{K}_\mu, \mathcal{O}^a(0)]] = -i\,(\Delta - 1)[\hat{K}_\mu, \mathcal{O}^a(0)]. \tag{1.15}$$

The dimensions are bounded from bellow for any physically sensible theory, there should exist operators such that

[2] Here and in what follows we use a hat to denote an operator in the Hilbert space and to distinguish it from its explicit form in terms of differential operators, where no hat appears.

$$[\hat{K}_\mu, \mathcal{O}^a(0)] = 0. \tag{1.16}$$

Operators that satisfy (1.13) ,(1.14) and (1.16) are called primary operators. We can construct operators with higher dimensions $\Delta + n$, which are called the descendants of $\mathcal{O}^a$, by acting with $\hat{P}_\mu$

$$\left[\hat{P}_{\mu 1}, [\dots, [\hat{P}_{\mu n}, \mathcal{O}^a(0)] \dots \right] \quad \text{Descendants,} \tag{1.17}$$

and primary $\mathcal{O}$ operator and its descendants make up a conformal multiplet.

The action of the conserved charges on a primary local operators away from the origin can be deduced from the definition

$$\mathcal{O}(x) = e^{-ix\cdot\hat{P}}\mathcal{O}(0)e^{ix\cdot\hat{P}}, \quad [\hat{P}, \mathcal{O}(x)] = -i\partial_\mu \mathcal{O}(x), \tag{1.18}$$

combined with (1.13) ,(1.14), (1.16) and the algebra (1.7) to be

$$\begin{aligned}
[\hat{D}, \mathcal{O}^a(x)] &= -i\, (\Delta + x\cdot\partial)\, \mathcal{O}^a(x), \\
[\hat{M}_{\mu\nu}, \mathcal{O}^a(x)] &= -i\, (x_\mu\partial_\nu - x_\nu\partial_\mu)\mathcal{O}^a(x) + i(\mathcal{S}_{\mu\nu})^a{}_b\mathcal{O}^b(x), \\
[\hat{K}_\mu, \mathcal{O}^a(x)] &= -i\, \left(2\Delta x_\mu + 2x_\mu x\cdot\partial - x^2\partial\right)\mathcal{O}^a(x) + 2i\, x^\nu(\mathcal{S}_{\nu\mu})^a{}_b\mathcal{O}^b(x).
\end{aligned} \tag{1.19}$$

1.2.3 Correlation Functions in CFT

A CFT does not admit a particle interpretation, but we can compute correlation functions. The conformal symmetry strongly constrains the correlation function. Two and three-point functions of primary operators are fixed up to constants. We can see this through a scalar primary example. Lets consider two-point correlation functions of scalar primary operators ϕ_1 and ϕ_2 with a conformal invariant vacuum $|0\rangle$.The Lorentz and translation invariance require that the correlator depends on the norm of position difference

$$\langle 0|\phi_1(x_1)\phi_2(x_2)|0\rangle = f(x_{12}^2) \tag{1.20}$$

where here and henceforth we use the notation

$$x_{ij}{}^\mu \equiv x_i{}^\mu - x_j{}^\mu, \quad x^2 \equiv x^\mu x_\mu. \tag{1.21}$$

We can see the implication of scale invariance by requiring that the simultaneous action of $\hat{D}$ on the two operators vanishes

$$\begin{aligned}
0 = \langle 0| \left[\hat{D}, \phi_1\phi_2\right] |0\rangle &= \langle 0| \left[\hat{D}, \phi_1\right] \phi_2 + \phi_1 \left[\hat{D}, \phi_2\right] |0\rangle \\
&= -i\, (x_1\cdot\partial_1 + \Delta_1 + x_2\cdot\partial_2 + \Delta_2)\, \langle 0|\phi_1\phi_2|0\rangle,
\end{aligned} \tag{1.22}$$

which, with a simple algebra, implies that

$$f(x_{12}^2) = \frac{\text{constant}}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}}. \tag{1.23}$$

The implication of special conformal invariance can be determined in a similar fashion

$$0 = \langle 0| \left[\hat{K}_\mu, \phi_1\phi_2\right] |0\rangle\,, \quad \text{requires} \quad \Delta_1 = \Delta_2\,. \tag{1.24}$$

The two point function is fixed up to a constant c and a delta function $\delta_{\Delta_1,\Delta_2}$

$$\langle 0|\phi_1(x_1)\phi_2(x_2)|0\rangle = \frac{c\delta_{\Delta_1,\Delta_2}}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}}. \tag{1.25}$$

It is possible to diagonalize the two point functions so that only two identical operators[3] have a non-zero correlator.

We can find the implication of conformal symmetry on $n-$point function following the same steps, that is, within a correlation function the action of a conformal charge should vanish $\langle 0|[\hat{Q}, \mathcal{O}_1 \ldots \mathcal{O}_n]|0\rangle = 0$. We find that three-point function of scalar primary operators is fixed up to a constant

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{\langle\phi_1\phi_2\phi_3\rangle}}{(x_{12}^2)^{\Delta_{123}}(x_{13}^2)^{\Delta_{312}}(x_{23}^2)^{\Delta_{231}}}, \tag{1.26}$$

where $\lambda_{\langle\phi_1\phi_2\phi_3\rangle}$ is a constant, which we cannot scale away after diagonalizing the 2-point functions, and we define

$$\Delta_{ijk} \equiv \Delta_i + \Delta_j - \Delta_k\,. \tag{1.27}$$

We can keep doing this for higher $n-$point functions and work out the conformal invariant correlators. However starting at $n = 4$, one can define conformal invariant functions of $x_1, x_2, \ldots x_n$. For $n = 4$ there are two such conformal invariant ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \tag{1.28}$$

and thus a 4-point function can be fixed up to an arbitrary function of u and v, for example a correlator of four identical scalar primary operators

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{(x_{12}^2)^{\Delta}(x_{34}^2)^{\Delta}}. \tag{1.29}$$

For $n > 4$ even more conformal invariant ratios can be defined.

Here we have restricted ourselves to scalar operators, for other Lorentz rep applying the same arguments is possible in principle but complicated in practice. We will use a different formalism to find the form of general two and three point functions of other Lorentz reps, they will be also fixed up to a set of constants (usually more than one).

1.3 Operator Product Expansion (OPE)

In any QFT it is possible to expand the product of two local operators $\mathcal{O}_1(x)\mathcal{O}_2(0)$, as $x \to 0$, as a sum of local operators at 0

[3] Identical real scalars and tensors, for complex reps the two operators have to be complex conjugate of each other.

$$\mathcal{O}_1(x) \times \mathcal{O}_2(0) = \sum_i C_{12i} \mathcal{O}_i(0). \tag{1.30}$$

In a general QFT, this sum is asymptotic. However in a CFT the OPE is actually convergent[4] and applies at finite separation. This is the result of state-operator correspondence in CFT, that any state $|\mathcal{O}\rangle$ defines a local operator $\mathcal{O}(0)$ and vice versa. So in a CFT the OPE (1.30) is equivalent to the expansion of a state in the complete basis of states $|\Psi_i\rangle$

$$\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle = \sum_i c_{12i} |\Psi_i\rangle \,. \tag{1.31}$$

In a CFT the sum (1.30) can be taken over primary operators. The coefficients $C_{12i}(x, \partial_x)$, thus encodes the descendants. Furthermore in a CFT, the functions C_{12i} are fixed by the conformal symmetry, as can be seen by using the OPE in a three-point function, for simplicity we consider scalar primary operators ϕ_i

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \sum_k C_{12k}(x_{12}, \partial_{x_{12}})\langle \mathcal{O}_k(x_2)\phi_3(x_3)\rangle = C_{123}(x_{12}, \partial_{x_{12}})\langle \phi_3(x_2)\phi_3(x_3)\rangle, \tag{1.32}$$

and since 2- and 3-point functions are fixed by conformal symmetry up to a constant $\lambda_{\langle\phi_1\phi_2\phi_3\rangle}$, so is the function $C_{123} \propto \lambda_{\langle\phi_1\phi_2\phi_3\rangle}$. That is why the constants $\lambda_{\langle\phi_1\phi_2\phi_3\rangle}$ which appear in three-point functions are called OPE coefficients.

The OPE is an important tool. If used in an $n-$point functions, it reduces the $n-$point function to a sum over $(n-1)-$point functions, after $n-2$ steps we arrive at a 2-point function. So, in principle, all correlation functions in a CFT can be computed in terms of the OPE coefficients. A CFT is completely defined by its spectrum of primary operators and the set of OPE Coefficients, together called "local CFT data".

$$\text{CFT Data} = \{\mathcal{O}_i^{a_i} : \Delta_i, \lambda_{\langle\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\rangle}\} \tag{1.33}$$

1.4 Conformal Bootstrap

Lets consider an arbitrary CFT data, in order for this data to define a good CFT it has to satisfy consistency conditions. The conformal bootstrap is the procedure of applying consistency conditions to a data and solving for a consistent CFT.

The main consistency condition a CFT has to satisfy is the crossing symmetry of all 4-point functions. Namely all possible ways to use OPE to reduce the 4-point function to a 2-point function should be equal. An OPE takes two operators and gives a sum, there are three ways to make pairs out of four operators:

$(12)(34)$ called the $s-$channel, $(14)(23)$ called the $t-$channel, $(13)(24)$ called the $u-$channel.

[4] As long as $\mathcal{O}_1$ is closer to $\mathcal{O}_2$ than any other operator.

The $s-$channel expansion of scalar correlator look as the following

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = \sum_{\mathcal{O}} \lambda_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle}\lambda_{\langle \mathcal{O}\mathcal{O}_3\mathcal{O}_4\rangle} C_{12\mathcal{O}}(x_{12},\partial_{x12}) C_{34\mathcal{O}}(x_{34},\partial_{x34})\, \langle \mathcal{O}(x_2)\mathcal{O}(x_4)\rangle, \tag{1.34}$$

and we define kinematic function

$$W_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle\langle \mathcal{O}\mathcal{O}_3\mathcal{O}_4\rangle} \equiv C_{12\mathcal{O}}(x_{12},\partial_{x12}) C_{34\mathcal{O}}(x_{34},\partial_{x34})\, \langle \mathcal{O}(x_2)\mathcal{O}(x_4)\rangle \tag{1.35}$$

which is fixed by conformal symmetry, we call such function Conformal Partial Waves (CPW). A CPW incorporates the contribution of a conformal multiplet (a primary and its descendants) to the 4-point function. If the CPW's are known, the unknowns are only the CFT data. The the bootstrap equations

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = \langle \mathcal{O}_1\,\mathcal{O}_2\,\mathcal{O}_3\,\mathcal{O}_4\rangle = \langle \mathcal{O}_1\,\mathcal{O}_2\,\mathcal{O}_3\,\mathcal{O}_4\rangle \tag{1.36}$$

will put constraints on the CFT data.

Determining the CPW's is essential for the bootstrap. Even though the idea of the bootstrap was suggested in the '70s by [3, 4], the breakthrough for $d > 2$ CFT came in 2008 [5] only after the CPW relevant for a scalar correlator was computed in a compact form in refs.[8, 9]. Many concrete results from the bootstrap followed, among which is the recent most precise calculation of critical exponents of the 3d Ising Model [10, 11, 12, 13, 14].

Meanwhile, several efficient techniques for determining CPW for correlators with spins have been developed over the last decade, including index-free embedding formalism [15, 16, 17, 18, 19, 20], the shadow formalism in the embedding space [18], "differential bases" for three-point functions [17, 15], and recursion relations [21, 22, 23]. These developments for spinning operators lead,for example, to the universal numerical bounds on wide classes of CFTs [15, 22] and the analytic proofs of the conformal collider bounds [24, 25, 26, 27] and the average null energy condition [28].

One expect that more advanced understanding for operators with spin will lead to new sophisticated bootstrap results.

1.5 Thesis Structure

Our aim from this thesis is to provide the main ingredients needed to bootstrap all possible correlators in 4D CFT. Namely, to calculate the CPW incorporating the contribution of a general conformal multiplet to a general four-point function. We start in chapter 2 by embedding the theory in 6D space, where the conformal symmetry acts linearly and it is easier to impose the symmetry on the correlation functions. In particular we find the complete kinematic basis to write any three-point function. In chapter 3 we construct a completer differential basis that relates any three-point function to a "seed" correlator. Because of the relation between OPE and three-point function (1.32), this basis effectively allows us to relate any CPW to a seed CPW W_p^{seed}, where $p = 0, 1, \ldots$ and there are infinite seed CPW. We proceed to determine W_p^{seed} by solving the Casimr equations in chapter 4, we make use of the shadow formalism to reduce the second order casimir differential equations to algebraic equations and manage to find solutions in analytic closed form.

In the last chapter 5, we use the results of the previous chapters to write the bootstrap equations for a (scalar-spin 1/2) four-point function and we solve the equations in the light-cone limit.

The material presented in this thesis is based on published work [29, 30, 31] and soon to be published work [32].

2. Embedding Formalism in 6D Twistor Space

As we have seen in section 1.2, conformal symmetry puts a lot of constrains on correlation functions. However the symmetry acts non linearly and working out these constrains is not straightforward and it gets harder when considering higher point functions or operators with spin.

However the implication of conformal symmetry are easier to understand if we go from the normal space to the embedding space. The conformal group in d- dimensional space is isomorphic to $SO(2,d)$ which is the Lorentz group in $d+2$ dimensional space. One can embed the d dimensional CFT in $d+2$ space, the non-linear action of the conformal group is induced from the Lorentz linear action. This was first suggested by Dirac [33].

We will start by embedding d dimensional CFT in $d+2$ space in section 2.1 . A primary tensor operator in d space will be embedded in a tensor field on the $d+2$ space. The latter has extra d.o.f, and one needs to use gauge conditions to account for that. The question of writing a conformal invariant correlators will reduce to writing $d+2$ Lorentz invariants. This formalism is good for considering primary symmetric traceless tensor operators.

For our case of interest $d = 4$, the group $SO(2,4)$ is locally isomorphic to the group $SU(2,2)$. The latter group has spinor and dual-spinor reps, these reps are called twistors. It is then possible to embed 4D spinors (and by extension any 4D Lorentz rep) in 6D twisor fields. In section 2.2 we how the 4D fields are embedded in twistor fields and the gauge conditions needed to account for the extra degrees of freedom. We also contract introduce an index free notation, where open indices are contracted by auxiliary twistors. Subsequently, the problem of writing conformal invariant correlators reduces to a problem of identifying $SU(2,2)$ invariants, we identify all possible invariants relevant for 2- , 3- and 4-point function in 2.3, and then we proceed to classify general 3-point functions in 4D CFT in section 2.5. We also study their properties under parity (section 2.7) and their projection to 4D (section 2.6) and formulate the conservation condition in 6D formalism (section 2.8). Extra details about our notation is provided in appendices A and B.

2.1 Embedding Formalism : Vector Notation

What we need is to embed our d space within this $d+2$ space in a way such that it inherits these linear transformations. The Lorentz invariant condition

$$X^2 = 0, \tag{2.1}$$

defines a subspace of d+1 dimensions, null cone .We get a d dimensional space by quotienting the null cone by the rescaling

$$X \sim \lambda X, \quad \lambda \in \mathbb{R} \tag{2.2}$$

Since both conditions respect Lorentz rotations, they define a subspace (projective null cone) that naturally inherits the actions of $SO(2,d)$. The standard $\mathbb{R}^{1,d-1}$ coordinates x^μ should not depend on λ and are defined as

$$x^\mu = \frac{X^\mu}{X^+}\,. \tag{2.3}$$

It can be shown that conformal transformations acting on x^μ are mapped to Lorentz transformations acting on the null-cone.

We identify this projective null cone with $\mathbb{R}^{1,d-1}$ by "gauge fixing" the rescaling (2.2). For example by setting $X^+ = 1$, where this gauge slice is called the Poincaré section. Lorentz transformations acting on X have to be followed by a rescaling to keep it within the Poincaré section. On this section the coordinates takes the form $X = (1, -x^2, x^\mu)$ and the inner product of two null vectors

$$X_{ij} \equiv -2X_i \cdot X_j = x_{ij}^{\;2} \tag{2.4}$$

Primary operators on $\mathbb{R}^{1,d-1}$ can be lifted to homogeneous, conformally-covariant fields on the null-cone. For a primary scalar $\phi(x)$ whose scaling dimension is Δ, we can define a field over all X as

$$\Phi(X) = (X^+)^{-\Delta}\phi\left(X^\mu/X^+\right), \tag{2.5}$$

this field transforms simply under conformal transformations $\Phi(X) \to \Phi(X')$. Its degree of homogeneity depends on the scaling dimension of ϕ,

$$\Phi(\lambda X) = \lambda^{-\Delta}\Phi(X) \tag{2.6}$$

Imposing conformal symmetry on correlation functions in this space boils down to respecting Lorentz invariance, homogeneity of the fields and the null condition $X_i^2 = 0$. For example, the two point function of $\Phi_i(X_i)$ that satisfy these conditions can only be

$$\langle \Phi_1(X_1)\Phi_2(X_2)\rangle = c\,\delta_{\Delta_1,\Delta_2}/X_{12}^{\Delta_1}, \tag{2.7}$$

To get d space correlator, we need to restrict X to the Poincaré section.

$$\phi(x) = \Phi(X)|_{\text{Poincaré}} \tag{2.8}$$

From (2.7) one easily recovers the two point function of two scalar primaries (1.25).

This embedding formalism can be extended to traceless tensor $\phi_{\mu_1\ldots\mu_\ell}(x)$ whose Lorentz representations are specified by some pattern of symmetries in their indices. Such field can be uplifted to $d+2$ homogeneous fields $\Phi_{M_1\ldots M_\ell}(X)$. This tensor embedding was first introduced in [34] and further developed in [16, 35], where it was determined that for the tensor field Φ to be a consistent uplift of the tensor ϕ, it has to satisfy particular properties

1. defined on the null cone $X^2 = 0$,
2. traceless and posses the same index symmetries as $\phi_{\mu_1\ldots\mu_\ell}(x)$,
3. defined module tensors proportional to X,

$$\Phi_{M_1\ldots M_\ell} \sim \Phi_{M_1\ldots M_\ell} + X_{M_i} V_{M_1\ldots \hat{M}_i \ldots M_\ell}, \tag{2.9}$$

   for an arbitrary tensor $V_{M_1\ldots \hat{M}_i \ldots M_\ell}$, the hat on the index M_i means it is missing.

4. transverse,

$$X^{M_i}\Phi_{M_1\ldots M_i \ldots M_\ell} = 0, \tag{2.10}$$

5. homogeneous of degree $-\Delta$ in X.

The extra conditions (3) and (4) are needed to insure that the two fields have the same number of degrees of freedom. One can recover the d field ϕ from the embedding space tensor by projecting it

$$\phi_{\mu_1\dots\mu_\ell} = \frac{\partial X^{M_1}}{\partial x^{\mu_1}} \cdots \frac{\partial X^{M_\ell}}{\partial x^{\mu_\ell}} \Phi_{M_1\dots M_\ell} \tag{2.11}$$

There is further simplification to work with index-free formalism. The indecies are contracted by auxiliary vectors. For example we will consider a symmetric traceless tensor, so that the field $\phi_{\mu_1\dots\mu_\ell}$ transform in a spin ℓ rep. of the Lorentz group. We introduce an auxiliary vectors Z^M,

$$\Phi(X, Z) = \Phi_{M_1\dots M_\ell}(X)\, Z^{M_1} \cdots Z^{M_\ell}, \tag{2.12}$$

which is a homogeneous polynomial of degree ℓ in Z. So that under rescaling

$$\Phi(\lambda X, \mu Z) = \lambda^{-\Delta} \mu^\ell \Phi(X, Z) \tag{2.13}$$

The components of $\Phi_{M_1\dots M_\ell}$ can be recovered by taking derivatives with respect to Z,

$$\Phi_{M_1\dots M_\ell}(X) = \frac{1}{\ell!} \frac{\partial}{\partial Z^{M_1}} \cdots \frac{\partial}{\partial Z^{M_\ell}} \Phi(X, Z) - \text{traces} \tag{2.14}$$

Each property of $\Phi_{M_1\dots M_\ell}$ should be reflected by those of $\Phi(X, Z)$. We can restrict $\Phi(X, Z)$ to the null cone $Z^2 = 0$ (because of the tracelessness) and to the plane $Z \cdot X = 0$ (because of gauge equivalence (2.9)).

The advantage of index-free notation is that complicated conformally-covariant tensors can become simple algebraic expressions in terms of conformal invariants. Correlators of symmetric tensors $\Phi(X_i, Z_i)$ must be gauge- and conformally-invariant functions of X_i and Z_i with the correct homogeneity properties. In two- and three-point correlators, such functions can be constructed as polynomials in the basic invariants

$$H_{ij} \equiv \frac{X_i \cdot X_j\; Z_i \cdot Z_j - X_i \cdot Z_j\; X_j \cdot Z_i}{X_i \cdot X_j}, \qquad V_{i,jk} \equiv \frac{X_j \cdot Z_i}{X_{ij}} - \frac{X_k \cdot Z_i}{X_{ik}} \tag{2.15}$$

This embedding formalism has successfully been used in ordinary space to study correlation functions of traceless symmetric tensors [3, 35, 16, 17].

While this form of embedding applies in any dimension $d > 2$, it does not accommodate fermionic operators. From now on we will consider only $d = 4$ conformal theories. In $d = 4$ however, a little twist to this embedding can achieve a formalism where its simpler to workout the conformal constrains and applies equally to fermions and bosons.

2.2 The 6D Embedding in Twistor Space

There is local isomorphism between the 4D conformal group $SO(4, 2)$ and $SU(2, 2)$. This means we can use rep. of $SU(2, 2)$ group to embed fields of 4D CFT. Conformal transformations are then induced form linear unitary transformations of fundamental and anti-fundamental reps.

An object transforming in the fundamental (anti-fundamental) rep of $SU(2,2)$ is a 4-component object, Y_a (Z^b) and it is called a twistor (dual twistor). These twistors will be used to embed 4D spinors ψ_α ($\overline{\chi}^{\dot{\beta}}$) and consequently any possible rep of 4D Lorentz group. since arbitrary 4D Lorentz representation can be built from products of spinors.

In our notation we reserve Latin letters $a, b, \ldots$ for $SU(2,2)$ indices and Greek letters, dotted and undotted, $\alpha, \dot{\beta}, \ldots$ for 4D spinor indices. For extra details on our notations see please appendix A.

This form of embedding formalism in twistor space has been used sporadically in the literature, mainly in the context of super conformal field theories (see e.g. refs.[36, 37, 38, 39]), and it has been applied in ref.[18, 29] to study correlation functions in 4D CFTs.

2.2.1 Embedding of 1/2 spin Fields

Let us now consider spin 1/2 primary fermions $\psi_\alpha(x)$ and $\bar{\phi}^{\dot{\alpha}}(x)$, with scaling dimension Δ. As shown in ref.[35], such fields are uplifted to 6D homogeneous twistors $\Psi_a(X)$ and $\bar{\Phi}^a(X)$, with degree $n = \Delta - 1/2$. A transversality condition is imposed on the 6D fields, in order to match the number of degrees of freedom:

$$
\begin{aligned}
\overline{\mathbf{X}}^{ab}\Psi_b(X) &= 0\,,\\
\bar{\Phi}^a(X)\mathbf{X}_{ab} &= 0\,,
\end{aligned}
\tag{2.16}
$$

where $\mathbf{X}$ and $\overline{\mathbf{X}}$ are twistor space-time coordinates, defined in terms of the antisymmetric chiral gamma matrices Σ and $\overline{\Sigma}$ as

$$
\mathbf{X}_{ab} \equiv X_M \Sigma^M_{ab}\,, \quad \overline{\mathbf{X}}^{ab} \equiv X_M \overline{\Sigma}^{Mab}\,,
\tag{2.17}
$$

our convention for the gamma matrices is given in Appendix A. By solving eq.(2.16), we get

$$
\begin{aligned}
\Psi_a(X) &= (X^+)^{-\Delta+1/2}\begin{pmatrix} \psi_\alpha(x) \\ -(x_\mu\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\psi_\beta(x)\end{pmatrix}\,,\\
\bar{\Phi}^a(X) &= (X^+)^{-\Delta+1/2}\begin{pmatrix} \bar{\phi}_{\dot{\beta}}(x)(x_\mu\bar{\sigma}^\mu)^{\dot{\beta}\alpha} \\ \bar{\phi}_{\dot{\alpha}}(x)\end{pmatrix}\,.
\end{aligned}
\tag{2.18}
$$

As discussed in ref.[18], it is more convenient to embed $\psi_\alpha(x)$ and $\bar{\phi}^{\dot{\alpha}}(x)$ to twistors $\bar{\Psi}^a(X)$ and $\Phi_a(X)$, respectively, with degree $n = \Delta + 1/2$. In this way, we essentially trade the transversality condition for a gauge redundancy. A generic solution of eq.(2.16) is given by

$$
\Psi_a = \mathbf{X}_{ab}\bar{\Psi}^b \quad \text{and} \quad \bar{\Phi}^a = \Phi_b\overline{\mathbf{X}}^{ba},
\tag{2.19}
$$

for some $\bar{\Psi}$ and Φ, since on the cone

$$
\mathbf{X}_{ab}\overline{\mathbf{X}}^{bc} = \overline{\mathbf{X}}^{ad}\mathbf{X}_{dc} = 0.
\tag{2.20}
$$

We can then equivalently associate $\psi_\alpha(x)$ to a twistor $\bar{\Psi}^a(X)$, and $\bar{\phi}^{\dot{\alpha}}(x)$ to a twistor $\Phi_a(X)$ as follows:

$$
\begin{aligned}
\psi_\alpha(x) &= \mathbf{X}_{\alpha a}\bar{\Psi}^a(X)|_{\text{Poincaré}}\,,\\
\bar{\phi}^{\dot{\alpha}}(x) &= \overline{\mathbf{X}}^{\dot{\alpha}a}\Phi_a(X)|_{\text{Poincaré}}\,,
\end{aligned}
\tag{2.21}
$$

where $\overline{\mathbf{X}}^{\dot\beta b} = \epsilon^{\dot\beta\dot\gamma}\overline{\mathbf{X}}_{\dot\gamma}{}^{b}$. The twistors $\bar\Psi(X)$ and $\Phi(X)$ are subject to an equivalence relation,

$$\begin{aligned}\bar\Psi(X) &\sim \bar\Psi(X) + \overline{\mathbf{X}}V\,,\\ \Phi(X) &\sim \Phi(X) + \mathbf{X}\overline{W}\,,\end{aligned} \tag{2.22}$$

with V and $\overline{W}$ generic twistors.

2.2.2 Embedding of General Fields

We are now ready to consider a 4D primary spinor-tensor in an arbitrary irreducible representation of the Lorentz group, with scaling dimension Δ:

$$f_{\alpha_1\ldots\alpha_l}^{\dot\beta_1\ldots\dot\beta_{\bar l}}(x)\,, \tag{2.23}$$

where dotted and undotted indices are symmetrized. We will denote such a representation as

$$(l,\bar l)\,,$$

namely by the number of undotted and dotted indices that appear. Hence, a spin 1/2 Weyl fermion will be in the $(1,0)$ or $(0,1)$, a vector in the $(1,1)$, an antisymmetric tensor in the $(2,0)\oplus(0,2)$ and so on. Generalizing eq.(2.21), we encode $f_{\alpha_1\ldots\alpha_l}^{\dot\beta_1\ldots\dot\beta_{\bar l}}$ in a 6D multi-twistor field $F^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}$ as follows:

$$f_{\alpha_1\ldots\alpha_l}^{\dot\beta_1\ldots\dot\beta_{\bar l}}(x) = \mathbf{X}_{\alpha_1 a_1}\ldots\mathbf{X}_{\alpha_l a_l}\overline{\mathbf{X}}^{\dot\beta_1 b_1}\ldots\overline{\mathbf{X}}^{\dot\beta_{\bar l} b_{\bar l}}F^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}(X)|_{\text{Poincaré}}\,. \tag{2.24}$$

F is homogeneous function of X with degree $n = -\Delta + (l+\bar l)/2$

$$F^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}(\lambda X) = \lambda^{-(\Delta+(l+\bar l)/2)}F^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}(X) \tag{2.25}$$

Given the gauge redundancy (2.22) in each index, the 4D field f is uplifted to an equivalence class of 6D fields F. Any two fields varying by a term proportional to $\mathbf{X}$ or $\overline{\mathbf{X}}$ are equivalent uplifts of f

$$F \sim F + \overline{\mathbf{X}}V \sim F + \mathbf{X}\overline{W}, \tag{2.26}$$

for some multi twistors V and $\overline{W}$ because of eq.(2.20). There is yet another equivalence class, due again to eq.(2.20). Twistors of the form $F^{a_1a_2\cdots}_{b_1b_2\ldots} = \delta^{a_1}_{b_1}Z^{a_2\cdots}_{b_2\ldots}$ give a vanishing contribution in eq. (2.24). Hence, without loss of generality, we can take as uplift of f a multi-twistor F with vanishing trace, namely:

$$\delta^{b_j}_{a_i}F^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}(X) = 0\,,\quad \forall i=1,\ldots,l, \forall j = 1,\ldots,\bar l\,. \tag{2.27}$$

It is very useful to use an index-free notation by defining

$$f(x,s,\bar s) \equiv f_{\alpha_1\ldots\alpha_l}^{\dot\beta_1\ldots\dot\beta_{\bar l}}(x)s^{\alpha_1}\ldots s^{\alpha_l}\bar s_{\dot\beta_1}\ldots\bar s_{\dot\beta_{\bar l}}\,, \tag{2.28}$$

where s^α and $\bar s_{\dot\beta}$ are auxiliary (commuting and independent) spinors. Similarly, we define

$$F(X,S,\bar{S}) \equiv F^{a_1\dots a_l}_{b_1\dots b_{\bar{l}}}(X)\; S_{a_1}\dots S_{a_l}\bar{S}^{b_1}\dots\bar{S}^{b_{\bar{l}}} \tag{2.29}$$

in terms of auxiliary (again commuting and independent) twistors S_a and $\bar{S}^a$. We can define a projection operation that takes 6D fields to 4D ones

$$f(x,s,\bar{s}) = F(X,S,\bar{S})|_{\text{Poincaré}}, \tag{2.30}$$

where X is constrained to the Poincaré section and

$$S_a|_{\text{Poincaré}} = s^\alpha \mathbf{X}_{\alpha a}|_{X=(1,-x^2,x^\mu)}\,, \quad \bar{S}^a|_{\text{Poincaré}} = \bar{s}_{\dot{\beta}}\overline{\mathbf{X}}^{\dot{\beta}a}|_{X=(1,-x^2,x^\mu)}\,. \tag{2.31}$$

The gauge redundancies (2.27) and (2.26) are easy to see from the projection (2.31), they also permit us to restrict to

$$\overline{\mathbf{X}}^{ab}S_b = \bar{S}^b\mathbf{X}_{ba} = \bar{S}^aS_a = 0\,, \tag{2.32}$$

consistently with the gauge redundancies we have in choosing F. Given a 6D multi-twistor field F, the corresponding 4D field f is explicitly given by

$$f^{\dot{\beta}_1\dots\dot{\beta}_{\bar{l}}}_{\alpha_1\dots\alpha_l}(x) = \frac{1}{l!\bar{l}!}\frac{\partial}{\partial s^{\alpha_1}}\dots\frac{\partial}{\partial s^{\alpha_l}}\frac{\partial}{\partial \bar{s}_{\dot{\beta}_1}}\dots\frac{\partial}{\partial \bar{s}_{\dot{\beta}_{\bar{l}}}}F\left(X, s\mathbf{X}, \bar{s}\overline{\mathbf{X}}\right)|_{X=(1,-x^2,x^\mu)}\,. \tag{2.33}$$

It is useful to compare the index-free notation introduced here with the one introduced in ref.[16] and reviewed in the last section 2.1 for symmetric traceless tensors in terms of polynomials in auxiliary variables z^μ and Z^M. In vector notation, a 4D symmetric traceless tensor $t_{\mu_1\dots\mu_l}$ can be embedded in a 6D tensor $T_{M_1\dots M_l}$. The 4D and 6D fields can be encoded in the polynomials

$$\begin{aligned} t(x,z) &= t_{\mu_1\dots\mu_n}z^{\mu_1}\dots z^{\mu_n}\,, \\ T(X,Z) &= T_{M_1\dots M_n}Z^{M_1}\dots Z^{M_n}\,, \end{aligned} \tag{2.34}$$

where in Minkowski space z_μ is a light-cone vector, $z_\mu z^\mu = 0$. A null vector can always be written as a product of two spinors:

$$z^\mu = \sigma^\mu_{\alpha\dot{\beta}}s^\alpha\bar{s}^{\dot{\beta}}\,. \tag{2.35}$$

Given the relation (B.6) between symmetric traceless tensors written in vector and spinor notation, the spinors s^α and $\bar{s}^{\dot{\alpha}}$ appearing in eq.(2.35) are exactly the ones defined in eq.(2.28). On the contrary, there is not a simple relation between the 6D coordinates Z^A and the 6D twistors S_a and $\bar{S}^a$.

2.3 Ingredients of Correlation Functions

Let us denote by $F_i = F_i(X_i, S_i, \bar{S}_i)$ the index-free 6D multi tensor field corresponding to some $(l_i, \bar{l}_i)$ 4D tensor field f_i. Homogeneity properties of F_i are

$$F_i(\lambda X_i, \mu S_i, \nu\bar{S}_i) = \lambda^{-\kappa_i}\mu^{\ell_i}\nu^{\bar{\ell}_i}F_i(X_i,S_i,\bar{S}_i), \tag{2.36}$$

where we have defined

$$\kappa_i \equiv \Delta_i + \frac{\ell_i + \bar{\ell}_i}{2} \,. \tag{2.37}$$

A correlator involving F_i would be a function of $X_i, S_i, \bar{S}_i$ that is

1. invariant under $SU(2,2)$ transformations,
2. satisfies the homogeneity properties (2.36) for each field and
3. gauge redundancy at every point $X_i^2 = \overline{\mathbf{X}}_i S_i = \bar{S}_i \mathbf{X}_i = \bar{S}_i S_i$

We deduce that an m-point correlator involving F_i fields will have the following form

$$\langle F_1 F_2 \ldots F_m \rangle = \mathcal{K}_m(X_1, \ldots, X_m) \sum_{s=1}^{N_m} \lambda_s \mathcal{T}_s(X_1, S_1, \bar{S}_1, \ldots X_m, S_m, \bar{S}_m). \tag{2.38}$$

Here the tensor structures $\mathcal{T}_s$ are $SU(2,2)$ invariant homogeneous functions with degree 0 $\forall X_i$ and degree ℓ_i $(\bar{\ell}_i)$ in each S_i $(\bar{S}_i)$. The index s runs over all the possible different independent tensor structures compatible with conformal invariance. While $\mathcal{K}_m$ is a kinematic factor which is a homogeneous function with degree $-\kappa_i$ in each X_i and λ_s are either constants (for $m < 4$) or functions of conformal invariant cross-ratios (for $m \geq 4$). For $m = 4$, the conformal invariant cross ratios (1.28) are uplifted to 6D cross-ratios

$$U = \frac{X_{12} X_{34}}{X_{13} X_{24}} \,, \quad V = \frac{X_{14} X_{23}}{X_{13} X_{24}}, \tag{2.39}$$

where we used a 6D short-hand notation

$$X_{ij} \equiv -2 X_i \cdot X_j \,. \tag{2.40}$$

2.3.1 Kinematic Factor

We will start by the kinematic factor, for $(m = 2)$-point functions

$$\mathcal{K}_2 \equiv X_{12}^{-\kappa_1} \delta_{\kappa_1, \kappa_2} \,, \tag{2.41}$$

for $(m = 3)$-point functions

$$\mathcal{K}_3 \equiv X_{12}^{\frac{1}{2}(\kappa_3 - \kappa_1 - \kappa_2)} X_{13}^{\frac{1}{2}(\kappa_2 - \kappa_3 - \kappa_1)} X_{23}^{\frac{1}{2}(\kappa_1 - \kappa_2 - \kappa_3)} \,, \tag{2.42}$$

and for $(m = 4)$-point functions, the homogeneity condition does not fully determine $\mathcal{K}_4$ since we can multiply it by arbitrary powers of U and V, we choose

$$\mathcal{K}_4 \equiv X_{12}^{-\frac{1}{2}(\kappa_1 + \kappa_2)} X_{34}^{-\frac{1}{2}(\kappa_3 + \kappa_4)} \left(\frac{X_{14}}{X_{24}} \right)^{\frac{1}{2}(\kappa_2 - \kappa_1)} \left(\frac{X_{13}}{X_{14}} \right)^{\frac{1}{2}(\kappa_4 - \kappa_3)} \,. \tag{2.43}$$

Finding the tensor structures $\mathcal{T}_s$ is a much less trivial problem. Any $\mathcal{T}_s$ will be a product of some fundamental $SU(2,2)$ invariant building blocks that are to be determined.

2.3.2 Invariant Building Blocks

The fundamental group-theoretical objects carrying $SU(2,2)$ indices, which should eventually be combined with the auxiliary twistors S_a and $\bar{S}^b$ to form $SU(2,2)$ invariants, are obtained as products of

$$\delta_b^a,\ \varepsilon_{abcd},\ \varepsilon^{abcd},\ \mathbf{X}_{ab},\ \overline{\mathbf{X}}^{ab}. \tag{2.44}$$

Let us first focus on the ε tensors. Their contraction with any other object in eq.(2.44) does not give any new structures, because they reduce to a sum of already existing elements in eq.(2.44), for example:

$$\varepsilon^{abcd}\varepsilon_{aefg} = \delta_e^b\delta_f^c\delta_g^d - \delta_e^b\delta_g^c\delta_f^d - \delta_f^b\delta_e^c\delta_g^d + \delta_f^b\delta_g^c\delta_e^d + \delta_g^b\delta_e^c\delta_f^d - \delta_g^b\delta_f^c\delta_e^d, \tag{2.45}$$

$$\varepsilon^{abcd}\mathbf{X}_{ae} = -\delta_e^b\overline{\mathbf{X}}^{cd} + \delta_e^c\overline{\mathbf{X}}^{bd} - \delta_e^d\overline{\mathbf{X}}^{bc}, \tag{2.46}$$

Actually, the ε-symbols drop from the discussion completely. It can be seen using the index-free formalism where ε is encoded into $\varepsilon_{abcd}\bar{S}_i^a\bar{S}_j^b\bar{S}_k^c\bar{S}_l^d$, which vanishes unless $i \neq j \neq k \neq l$, such structure is still related to other elements because of the gauge redundancy condition (see section 2.4).

The fundamental group-theoretical objects can be grouped into three sets

$$\left\{\delta_a^b,\ [\mathbf{X}_i\overline{\mathbf{X}}_j]_a{}^b,\ [\mathbf{X}_i\overline{\mathbf{X}}_j\mathbf{X}_k\overline{\mathbf{X}}_l]_a{}^b,\dots\right\},\left\{[\overline{\mathbf{X}}_i]^{ab},[\overline{\mathbf{X}}_i\mathbf{X}_j\overline{\mathbf{X}}_k]^{ab},\dots\right\},\left\{[\mathbf{X}_i]_{ab},[\mathbf{X}_i\overline{\mathbf{X}}_j\mathbf{X}_k]_{ab},\dots\right\}. \tag{2.47}$$

Multiplying these objects by auxiliary twistors S and $\bar{S}$ will give us the SU(2,2) invariant building blocks needed to characterize the m-point function. They are not all independent, given the relations (2.20), (2.32) and (A.13).

2.3.3 Tensor Structures of 2- and 3-point Functions

Let us first determine the general form of two-point functions $\langle F_1F_2\rangle$. It is clear in this case that the only non-vanishing independent SU(2,2) invariant is obtained by contracting one twistor $\bar{S}_1$ with S_2 or viceversa. The form of the two-point function is uniquely determined:

$$\langle F_1(X_1,S_1,\bar{S}_1)F_2(X_2,S_2,\bar{S}_2)\rangle = cX_{12}^{-\kappa_1}\left(I^{21}\right)^{l_1}\left(I^{12}\right)^{\bar{l}_1}\delta_{l_1,\bar{l}_2}\delta_{l_2,\bar{l}_1}\delta_{\Delta_1,\Delta_2}\,, \tag{2.48}$$

where c is a normalization factor and we have defined the SU(2,2) invariant

$$I^{ij} \equiv \bar{S}_iS_j\,. \tag{2.49}$$

For three-point functions three more invariants arise:

$$\hat{K}_i^{jk} \equiv N_{i,jk}S_j\overline{\mathbf{X}}_iS_k\,, \tag{2.50}$$

$$\hat{\overline{K}}{}_i^{jk} \equiv N_{i,jk}\bar{S}_j\mathbf{X}_i\bar{S}_k\,, \tag{2.51}$$

$$\hat{J}_{jk}^i \equiv N_{j,k}\bar{S}_i\mathbf{X}_j\overline{\mathbf{X}}_kS_i\,. \tag{2.52}$$

The normalization factors

$$N_{jk} \equiv \frac{1}{X_{jk}}\,, \quad N_{i,jk} \equiv \sqrt{\frac{X_{jk}}{X_{ij}X_{ik}}}\,, \tag{2.53}$$

are introduced to make the $SU(2,2)$ invariants in eqs.(2.49)-(2.52) dimensionless and well-defined on the 6D light-cone.[1] Notice that in eqs. (2.50)-(2.52) $i \neq j \neq k$ and indices are not summed. The invariants (2.50)-(2.52) are anti-symmetric in some indices:

$$\hat{K}_i^{jk} = -\hat{K}_i^{kj}, \;\; \hat{\overline{K}}_i^{jk} = -\hat{\overline{K}}_i^{kj}, \;\; \hat{J}^i_{jk} = -\hat{J}^i_{kj}, \tag{2.54}$$

due to the anti-symmetry of $\mathbf{X}$, $\overline{\mathbf{X}}$ and the relations (2.32), (A.13).

Every other $SU(2,2)$ invariant object obtained from eq.(2.47) can be written in terms of different combinations of $\hat{I}^{ij}, \hat{K}_i^{jk}, \hat{\overline{K}}_i^{jk}$ and $\hat{J}^i_{jk}$. Using eqs.(2.49)-(2.52), the most general tensor structure can be written as follows:

$$\mathcal{T}_s = \prod_{i\neq j\neq k=1}^{3} \left(\hat{I}^{ij}\right)^{m_{ij}} \left(\hat{K}_i^{jk}\right)^{k_i} \left(\hat{\overline{K}}_i^{jk}\right)^{\bar{k}_i} \left(\hat{J}^i_{jk}\right)^{j_i} \tag{2.55}$$

where m_{ij}, k_{ij}, $\bar{k}_{ij}$ and j_i are a set of non-negative integers. Demanding that $\mathcal{T}_s$ is homogeneous function of degree ℓ_i in each S_i and degree $\bar{\ell}_i$ in each $\bar{S}_i$ gives us six constraints:

$$l_i = j_i + \sum_{n\neq i}(m_{ni} + k_n)\,, \quad \bar{l}_i = j_i + \sum_{n\neq i}(m_{in} + \bar{k}_n) \quad \forall i, \tag{2.56}$$

It is useful to define

$$\Delta\ell \equiv \ell_1 + \ell_2 + \ell_3 - (\bar{\ell}_1 + \bar{\ell}_2 + \bar{\ell}_3)\,. \tag{2.57}$$

Using the system (2.56), we immediately get

$$\begin{aligned} \Delta\ell &= 2(k_1 + k_2 + k_3 - \bar{k}_1 - \bar{k}_2 - \bar{k}_3)\,, \\ k_1 + k_2 + k_3 &\leq min(\ell_1 + \ell_2, l_1 + \ell_3, \ell_2 + \ell_3)\,, \quad \bar{k}_1 + \bar{k}_2 + \bar{k}_3 \leq min(\bar{\ell}_1 + \bar{\ell}_2, \bar{\ell}_1 + \bar{\ell}_3, \bar{\ell}_2 + \bar{\ell}_3)\,, \end{aligned} \tag{2.58}$$

and hence

$$-2\min(\bar{\ell}_1 + \bar{\ell}_2,\, \bar{\ell}_1 + \bar{\ell}_3,\, \bar{\ell}_2 + \bar{\ell}_3) \leq \Delta\ell \leq 2\min(\ell_1 + \ell_2,\, \ell_1 + \ell_3,\, \ell_2 + \ell_3)\,. \tag{2.59}$$

These are the conditions for the 4D three-point function $\langle f_1 f_2 f_3\rangle$ to be non-vanishing. They exactly match the findings of ref.[41]. Indeed, in that paper it was found that the 3-point function $\langle f_1 f_2 f_3\rangle$, with f_i primary fields in the $(l_i, \bar{l}_i)$ representations of $SL(2,C)$, is non-vanishing if the decomposition of the tensor product $(l_1, \bar{l}_1) \otimes (l_2, \bar{l}_2) \otimes (l_3, \bar{l}_3)$ contains a traceless-symmetric representation (l, l).

This would have completed the classification of the three-point functions if the tensor structures $\mathcal{T}_s$ were all linearly independent, but they are not, and hence a more refined analysis is necessary.

[1] Notice the different normalization and slight different index notation in the definition of the invariants I, K, $\overline{K}$ and J with respect to the ones defined in ref.[18]. The notation here matches the recent paper [40] with the most comprehensive presentation yet of 4D CFT.

2.3.4 Tensor Structures of 4-Point Functions

The tensor structures $\mathcal{T}_s$ are formed from the three-point invariants (2.49) and (2.50)-(2.52) (where i, j, k now run from 1 to 4) and the following new ones:

$$\hat{I}^{ij}_{kl} \equiv N_{kl}\, \bar{S}_i \mathbf{X}_k \overline{\mathbf{X}}_l S_j \,, \tag{2.60}$$
$$\hat{L}^{i}_{jkl} \equiv N_{jkl}\, S_i \overline{\mathbf{X}}_j \mathbf{X}_k \overline{\mathbf{X}}_l S_i \,, \tag{2.61}$$
$$\hat{\bar{L}}^{i}_{jkl} \equiv N_{jkl}\, \bar{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k \mathbf{X}_l \bar{S}_i \,, \tag{2.62}$$

where $i \neq j \neq k \neq l = 1, 2, 3, 4$; $\hat{L}^{i}_{jkl}$ and $\hat{\bar{L}}^{i}_{jkl}$ are totally anti-symmetric in the last three indices and the normalization factor is given by

$$N_{jkl} \equiv \frac{1}{\sqrt{X_{jk} X_{kl} X_{lj}}} . \tag{2.63}$$

The invariants $\hat{I}^{ij}_{kl}$ satisfy the relation

$$\hat{I}^{ij}_{kl} = -\hat{I}^{ij}_{lk} + I^{ij} \tag{2.64}$$

Any four-point function can be expressed as a sum of products of the invariants (2.49)-(2.52) and (2.60)-(2.62). However, not every product is independent, due to many relations between them.

2.4 Relations between Invariants

The dependence of the structures (2.55) has its roots in a set of identities among the twistors S_i and the coordinates $\mathbf{X}_j$, when $i = j$. Combining the guage redundancy condition (2.32) and (2.20) with

$$\epsilon^{abcd} \mathbf{X}_{cd} = -2 \overline{\mathbf{X}}^{ab} , \qquad -2 \mathbf{X}_{ab} = \epsilon_{abcd} \overline{\mathbf{X}}^{cd} , \tag{2.65}$$

lead to the identities

$$S_a \mathbf{X}_{bc} + S_b \mathbf{X}_{ca} + S_c \mathbf{X}_{ab} = 0 \,, \tag{2.66}$$
$$\mathbf{X}_{ab} \mathbf{X}_{cd} + \mathbf{X}_{ca} \mathbf{X}_{bd} + \mathbf{X}_{bc} \mathbf{X}_{ad} = 0 \,. \tag{2.67}$$

Analogous relations apply for the dual twistors $\bar{S}$ and $\overline{\mathbf{X}}$. We have not found identities involving more S's or $\mathbf{X}$'s that do not boil down to eqs.(2.66) and (2.67).

Now we can explain why we dropped the ε symbol when we constructed the invariants, using the identity (2.66) we see

$$-2 X_{ij} \varepsilon_{abcd} \bar{S}^a_i \bar{S}^b_j \bar{S}^c_k \bar{S}^d_l = \bar{S}_i \mathbf{X}_j \bar{S}_l \;\; \bar{S}_j \mathbf{X}_i \bar{S}_k - \bar{S}_j \mathbf{X}_i \bar{S}_l \;\; \bar{S}_i \mathbf{X}_j \bar{S}_k \tag{2.68}$$

2.4.1 Relations between Three-Point Function Invariants

Applying eqs.(2.66) and (2.67) (actually it is enough to use only eq.(2.66)) to bi-products of invariants we get the following relations (no sum over indices):

$$
\begin{aligned}
\hat{K}_j^{ik}\hat{\bar{K}}_i^{jk} &= I^{ki}I^{jk} - I^{ji}\hat{J}^k_{ij}\,, &(2.69)\\
\hat{K}_k^{ij}\hat{\bar{K}}_k^{ij} &= \hat{J}^i_{jk}\hat{J}^j_{ik} - I^{ij}I^{ji}\,, &(2.70)\\
\hat{J}^j_{ik}\hat{K}_j^{ik} &= -I^{ji}K_i^{jk} + I^{jk}\hat{K}_k^{ij}\,, &(2.71)\\
\hat{J}^j_{ik}\hat{\bar{K}}_j^{ik} &= -I^{ij}\hat{\bar{K}}_i^{kj} - I^{kj}\hat{\bar{K}}_k^{ij}\,. &(2.72)
\end{aligned}
$$

We have verified that higher order relations involving more than two invariants always arise as the composition of the relations (2.69)-(2.72). This is expected, since the fundamental identities (2.66) and (2.67) involve only two tensors. A particularly useful third-order relation is

$$
\hat{J}^1_{23}\hat{J}^2_{13}\hat{J}^3_{12} = I^{21}I^{13}I^{32} - I^{12}I^{31}I^{23} + I^{23}I^{32}\hat{J}^1_{23} - I^{13}I^{31}\hat{J}^2_{13} + I^{12}I^{21}\hat{J}^3_{12}, \tag{2.73}
$$

which is obtained by applying, in order, eqs.(2.70), (2.72) and (2.69). The relations (2.69)-(2.73) have been originally obtained in ref.[18], though it was not clear there whether additional relations were possible.

2.4.2 Relations between Four-Point Function Invariants

When we consider the basis of invariants (2.49)-(2.52) and (2.60)-(2.62), the eqs.(2.66) and (2.67) will lead to much more relations. Case in point, we get a linear relation

$$
\hat{J}^i_{jl} = n_{ijkl}\hat{J}^i_{kl} + n_{lijk}\hat{J}^i_{jk}\,, \tag{2.74}
$$

where we have defined

$$
n_{ijkl} \equiv \frac{X_{ij}X_{kl}}{X_{ik}X_{jl}}. \tag{2.75}
$$

And of-course more bi-product relations. For example the product $\hat{K}\hat{\bar{K}}$ have seven relations

$$
\begin{aligned}
\hat{K}_i^{jk}\hat{\bar{K}}_i^{jk} &= \hat{J}^j_{ik}\hat{J}^k_{ij} - I^{jk}I^{kj}\,, &(2.76)\\
\hat{K}_i^{jk}\hat{\bar{K}}_l^{jk} &= \sqrt{n_{ijkl}}\Big(n_{iljk}I^{jk}\hat{I}^{kj}_{li} - n_{ikjl}\hat{J}^j_{ik}\hat{J}^k_{jl} - I^{jk}I^{kj}\Big)\,, &(2.77)\\
\hat{K}_i^{jk}\hat{\bar{K}}_j^{ik} &= I^{ij}\hat{J}^k_{ij} + I^{ik}I^{kj}\,, &(2.78)\\
\hat{K}_i^{jk}\hat{\bar{K}}_j^{lk} &= \sqrt{n_{ijkl}}\Big(I^{kj}\hat{I}^{lk}_{ji} + I^{lj}\hat{J}^k_{ij}\Big)\,, &(2.79)\\
\hat{K}_i^{jk}\hat{\bar{K}}_l^{ij} &= -\sqrt{n_{ilkj}}\Big(I^{ij}\hat{I}^{jk}_{li} + I^{ik}\hat{J}^j_{il}\Big)\,, &(2.80)\\
\hat{K}_i^{jk}\hat{\bar{K}}_j^{li} &= \sqrt{n_{ilkj}}\Big(I^{ij}\hat{I}^{lk}_{ji} - I^{ik}I^{lj}\Big)\,, &(2.81)\\
\hat{K}_i^{jk}\hat{\bar{K}}_i^{jl} &= -\sqrt{n_{ilkj}}\Big(I^{lj}\hat{I}^{jk}_{li} + \hat{J}^j_{il}\hat{I}^{lk}_{ji}\Big)\,, &(2.82)
\end{aligned}
$$

compared to just two relations (2.69) and (2.70) in the three point function case. In both cases the relations will allow us to eliminate the all product of the $\hat{K}\hat{\bar{K}}$. Another relation is

$$
\hat{I}^{ji}_{kl}\,\hat{I}^{lk}_{ij} = 4\Big(I^{li}I^{jk} - n_{ikjl}I^{li}I^{jk} + n_{iljk}I^{ji}I^{lk}\Big) + 2n_{iljk}\Big(I^{li}\hat{I}^{jk}_{li} - I^{jk}\hat{I}^{li}_{kj}\Big). \tag{2.83}
$$

Much more relations can be found and classifying them is a very laborious task. Having a general classification of 4-point tensor structures is crucial to to bootstrap a four-point function with non-zero external spins. When we equate correlators in different channels, we have to identify all the factors in front of the same tensor structure, thus it is important to have a common basis of independent tensor structures. It seems impractical to classify structures of 4-point function in twistor formalism, in [40] they showed that this problem is better addressed in non-covariant way by going to the conformal frame.

2.5 Three-Point Function Classification

The application of therelations in subsection 2.4.1 depend on the value of $\Delta\ell$ defined in (2.57). Lets first consider the case $\Delta\ell = 0$: Combining eqs.(2.69) and (2.70), we see that a product of any $\hat{K}$ and $\hat{\overline{K}}$ can be reduced to a combination of I's and $\hat{J}$'s and thus

$$\Delta\ell = 0 \implies k_i = \bar{k}_i = 0 \quad \forall i \tag{2.84}$$

We can also apply eq.(2.73) successively. At each step the tensor structure splits into five ones, each time with a reduced number of J's. We keep applying eq.(2.73) until the initial tensor structure is written as a sum of tensor structures where all have at least one value of j_1, j_2, or j_3 equal to zero.

$$\Delta\ell = 0 \implies j_1 = 0 \quad \textbf{or} \quad j_2 = 0 \quad \textbf{or} \quad j_3 = 0. \tag{2.85}$$

Next we consider the case $\Delta\ell > 0$: the combined application of eqs.(2.69) and (2.70) remove all the $\hat{\overline{K}}$

$$\Delta\ell > 0 \implies \bar{k}_i = 0 \quad \forall i \tag{2.86}$$

Then we apply eq.(2.72) so that products of the form $\hat{K}_i^{\cdot\cdot}\hat{J}^i_{\cdot\cdot}$ can be rewritten using only $\hat{K}$'s of a different type. It is not difficult to convince oneself that this boils down to the following further constraints on eq.(2.55):

$$\Delta\ell > 0 \implies \begin{cases} k_1 = 0 \quad \textbf{or} \quad j_1 = 0 \\ k_2 = 0 \quad \textbf{or} \quad j_2 = 0 \\ k_3 = 0 \quad \textbf{or} \quad j_3 = 0 \end{cases} \tag{2.87}$$

The last case of $\Delta\ell < 0$ is equivalent to the case $\Delta\ell > 0$ replacing $\hat{K}_i$ with $\hat{\overline{K}}_i$. The result of this analysis is that the most general three-point function $\langle F_1 F_2 F_3\rangle$ can be written as[2]

$$\langle F_1 F_2 F_3\rangle = \sum_{s=1}^{N_3} \lambda^s_{\langle F_1 F_2 F_3\rangle} \langle F_1 F_2 F_3\rangle_s\,, \tag{2.88}$$

where

$$\langle F_1 F_2 F_3\rangle_s = \mathcal{K}_3 \Big(\prod_{i\neq j=1}^{3} (I^{ij})^{m_{ij}} \Big) C^{n_1}_{1,23} C^{n_2}_{2,31} C^{n_3}_{3,12}\,. \tag{2.89}$$

The index s runs over all the independent tensor structures parametrized by the integers m_{ij} and n_i, each multiplied by a constant OPE coefficient λ_s. The invariants $C_{i,jk}$ equal to one of the three-index invariants (2.50)-(2.52), depending on the value of $\Delta\ell$ defined in (2.57) for the external fields.

[2]The points X_1, X_2 and X_3 are assumed to be distinct.

Three-point functions are non-vanishing only when $\Delta\ell$ is an even integer and satisfy the condition (2.59) [41]. We have

- $\Delta l = 0$: $C_{i,jk} = \hat{J}^i_{jk}$ and the power of J's satisfy the constraint (2.85)
- $\Delta l > 0$: $C_{i,jk} = \hat{J}^i_{jk}$ or $\hat{K}^{jk}_i$
- $\Delta l < 0$: $C_{i,jk} = \hat{J}^i_{jk}$ or $\hat{\overline{K}}{}^{jk}_i$.

The number of tensor structures is given by all the possible allowed choices of nonnegative integers m_{ij} and n_i in eq.(2.88) subject to the above constraints and the ones coming from matching the correct powers of S_i and $\bar{S}_i$ for each field (2.56). The latter requirement gives in total six constraints.

2.6 6D to 4D Dictionary

The transition from the 6D index-free form to the 4D one is extremely easy. Given a 6D three-point function, we just need to rewrite the invariants $I, K, \bar{K}, J$ in 4D form. We have

$$\begin{aligned}
\hat{I}^{ij}|_{\text{Poincaré}} &= \bar{s}_{i\dot{\alpha}}(\overline{\mathbf{X}}_i \mathbf{X}_j)^{\dot{\alpha}}_{\ \alpha} s^\alpha_j|_{\text{Poincaré}} = \bar{s}_{i\dot{\alpha}}(x_{ij}\cdot\sigma\epsilon)^{\dot{\alpha}}_{\ \alpha}\, s^\alpha_j\,, && (2.90)\\
\hat{K}^{ij}_k|_{\text{Poincaré}} &= N_{k,ij}\; s^\alpha_i s^\beta_j (\mathbf{X}_i\overline{\mathbf{X}}_k\mathbf{X}_j)_{\alpha\beta}|_{\text{Poincaré}} \\
\hat{\overline{K}}{}^{ij}_k|_{\text{Poincaré}} &= \frac{1}{2}\frac{|x_{ij}|}{|x_{ik}||x_{jk}|}\bar{s}_{i\dot{\alpha}}\bar{s}_{j\dot{\beta}}\left((x^2_{ik}+x^2_{jk}-x^2_{ij})\epsilon^{\dot{\alpha}\dot{\beta}} + 4x^\mu_{ik}x^\nu_{kj}(\bar{\sigma}_{\mu\nu}\epsilon)^{\dot{\alpha}\dot{\beta}}\right)\,, && (2.91)\\
\hat{J}^k_{ij}|_{\text{Poincaré}} &= N_{ij}\;\bar{s}_{k\dot{\alpha}}(\overline{\mathbf{X}}_k\mathbf{X}_i\overline{\mathbf{X}}_j\mathbf{X}_k)^{\dot{\alpha}}_{\ \alpha}s^\alpha_k|_{\text{Poincaré}} = -\frac{x^2_{ik}x^2_{jk}}{x^2_{ij}}s^\alpha_k(Z_{k,ij}\cdot\sigma\epsilon)^{\dot{\alpha}}_{\ \alpha}\bar{s}_{k\dot{\alpha}}\,, && (2.92)
\end{aligned}$$

where

$$Z^\mu_{k,\,ij} \equiv \frac{x^\mu_{ki}}{x^2_{ik}} - \frac{x^\mu_{kj}}{x^2_{jk}},\quad Z^\mu_{k,\,ij} = -Z^\mu_{k,\,ji}\,. \tag{2.93}$$

Explicit 4D correlation functions with indices are obtained by removing the auxiliary spinors s_i and $\bar{s}_i$ through derivatives, as described in eq.(2.33).

2.7 Transformations under 4D Parity

Under the 4D parity transformation $(x^0,\vec{x}) \to (x^0,-\vec{x})$, a 4D field in the $(l,\bar{l})$ representation of the Lorentz group is mapped to a field in the complex conjugate representation $(\bar{l},l)$. We parametrize the transformation as follows:

$$f^{\dot{\beta}_1\ldots\dot{\beta}_{\bar{l}}}_{\alpha_1\ldots\alpha_l}(x) \to \eta(-)^{\frac{l+\bar{l}}{2}} f^{\dot{\alpha}_1\ldots\dot{\alpha}_l}_{\beta_1\ldots\beta_{\bar{l}}}(\tilde{x})\,, \tag{2.94}$$

where η is the intrinsic parity of the field and $\tilde{x}$ is the parity transformed coordinate. Applying parity twice gives

$$\eta\,\eta_c(-)^{l+\bar{l}} = 1\,, \tag{2.95}$$

where η_c is the intrinsic parity of the conjugate field. We then see that $\eta\eta_c = +1$ for bosonic operators and $\eta\eta_c = -1$ for fermionic ones. Under parity, in particular, we have

$$(x\cdot\sigma\epsilon)_\alpha^{\dot\beta} \leftrightarrow -(x\cdot\bar\sigma\epsilon)_{\dot\beta}^{\dot\alpha}\,,\qquad \epsilon_{\alpha\beta}\leftrightarrow -\epsilon^{\dot\alpha\dot\beta}\,,\qquad x^\mu y^\nu(\sigma_{\mu\nu}\epsilon)_{\alpha\beta}\leftrightarrow -x^\mu y^\nu(\bar\sigma_{\mu\nu}\epsilon)^{\dot\alpha\dot\beta}\,. \tag{2.96}$$

We can see how parity acts on the 6D invariants (2.49)-(2.52) by using their 4D expressions (2.90)-(2.92) on the null cone and eqs.(2.96). We get

$$\begin{aligned} I^{ij} &\to\ -I^{ji}\,,\\ \hat K_i^{jk} &\to\ +\hat{\overline{K}}_i^{jk}\,,\\ \hat{\overline{K}}_i^{jk} &\to\ +\hat K_i^{jk}\,,\\ \hat J^i_{jk} &\to\ +\hat J^i_{jk}\,. \end{aligned} \tag{2.97}$$

In general, parity maps correlators of fields into correlators of their complex conjugate fields. Imposing parity in a CFT implies that for each primary field $(l,\bar l)$ there must exist its conjugate one $(\bar l, l)$, and the constants entering in their correlators are related. Of course, we can also have correlators that are mapped to themselves under parity. Since $\Delta l\to-\Delta l$ under parity, where Δl is defined in eq.(2.57), such correlators should have $\Delta l = 0$. Due to eqs.(2.59) and (2.84), the structures K and $\overline{K}$ cannot enter in correlators with $\Delta l = 0$, which depend only on the invariants I^{ij} and $\hat J^i_{jk}$. For correlators that are mapped to themselves under parity, one has to take linear combinations of the tensor structures appearing in eq.(2.88) that are even or odd under parity, according to the transformation rules for I's and J's in eq.(2.97). Depending on the intrinsic parity of the product of the fields entering the correlator, the coefficients multiplying the parity even or parity odd structures should then be set to zero if parity is conserved.

A particular relevant class of correlators that are mapped to themselves under parity are those involving symmetric traceless tensors only. In this case we have verified that eqs.(2.97) lead to the correct number of parity even and parity odd structures as separately computed in ref.[16].

2.8 Conserved Operators

Primary tensor fields whose scaling dimension Δ saturates the unitarity bound [42, 43]

$$\Delta\geq\frac{l+\bar l}{2}+2\,,\quad l\neq0\ \textbf{and}\ \ \bar l\neq 0\,, \tag{2.98}$$

are conserved. Three-point functions with conserved operators are subject to further constraints which will be analyzed in this section. Given a conserved spinor-tensor primary field in the $(l,\bar l)$ representation of the Lorentz group, with scaling dimension Δ, we define

$$(\partial\cdot f)^{\dot\beta_2\ldots\dot\beta_{\bar l}}_{\alpha_2\ldots\alpha_l}(x)\equiv(\epsilon\sigma^\mu)^{\alpha_1}_{\ \dot\beta_1}\partial_\mu f^{\dot\beta_1\ldots\dot\beta_{\bar l}}_{\alpha_1\ldots\alpha_l}(x)=0\,. \tag{2.99}$$

Let us see how the 4D current conservation (2.99) can be uplifted to 6D as a constraint on the field $F^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}$. This will allow us to work directly with the 6D invariants (2.49)-(2.52), providing a great simplification. The analysis that follows is essentially a generalization to arbitrary conserved currents of the one made in ref.[16], where only symmetric traceless currents were considered. From eq. (2.24), we get

$$(\partial\cdot f)^{\dot\beta_2\ldots\dot\beta_{\bar l}}_{\alpha_2\ldots\alpha_l}(x)=(X^+)^{\Delta-(l+\bar l)/2}\partial_\mu\Big((e^\mu)^{\ b_1}_{a_1}\mathbf{X}_{\alpha_2a_2}\ldots\mathbf{X}_{\alpha_la_l}\overline{\mathbf{X}}^{\dot\beta_2b_2}\ldots\overline{\mathbf{X}}^{\dot\beta_{\bar l}b_{\bar l}}F^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}(X)\Big)\,, \tag{2.100}$$

where

$$(e^\mu)_a^{\ b} \equiv -\mathbf{X}_{a\alpha}(\epsilon\sigma^\mu)^\alpha_{\ \dot\beta}\overline{\mathbf{X}}^{\dot\beta b} = (M^{\mu+})_a^{\ b}\,, \tag{2.101}$$

in terms of the tensor

$$M^{MN} = 2\big(X^M\Sigma^N_{\ P}X^P - X^N\Sigma^M_{\ P}X^P\big)\,. \tag{2.102}$$

Applying the derivative to each term gives

$$\begin{aligned}(\partial\cdot f)^{\dot\beta_2\ldots\dot\beta_{\bar l}}_{\alpha_2\ldots\alpha_l} &= (X^+)^{\Delta-(l+\bar l)/2}\mathbf{X}_{\alpha_3 a_3}\ldots\mathbf{X}_{\alpha_l a_l}\overline{\mathbf{X}}^{\dot\beta_3 b_3}\ldots\overline{\mathbf{X}}^{\dot\beta_{\bar l} b_{\bar l}}\bigg((\partial_\mu e^\mu)_{a_1}^{\ b_1}\mathbf{X}_{\alpha_2 a_2}\overline{\mathbf{X}}^{\dot\beta_2 b_2} + \frac{\partial X^M}{\partial x^\nu}(e^\nu)_{a_1}^{\ b_1}\\ &\bigg((l-1)\Big(\frac{\partial\mathbf{X}_{\alpha_2 a_2}}{\partial X^M}\Big)\overline{\mathbf{X}}^{\dot\beta_2 b_2} + (\bar l-1)\Big(\frac{\partial\overline{\mathbf{X}}^{\dot\beta_2 b_2}}{\partial X^M}\Big)\mathbf{X}_{\alpha_2 a_2} + \mathbf{X}_{\alpha_2 a_2}\overline{\mathbf{X}}^{\dot\beta_2 b_2}\frac{\partial}{\partial X^M}\bigg)\bigg)F^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}\,.\end{aligned} \tag{2.103}$$

After some algebraic manipulations, eq.(2.103) can be recast in the form

$$(\partial\cdot f)^{\dot\beta_2\ldots\dot\beta_{\bar l}}_{\alpha_2\ldots\alpha_l} = (X^+)^{\Delta-(l+\bar l)/2+2}\mathbf{X}_{\alpha_2 a_2}\ldots\mathbf{X}_{\alpha_l a_l}\overline{\mathbf{X}}^{\dot\beta_2 b_2}\ldots\overline{\mathbf{X}}^{\dot\beta_{\bar l} b_{\bar l}}R^{a_2\ldots a_l}_{b_2\ldots b_{\bar l}}\,, \tag{2.104}$$

where

$$R^{a_2\ldots a_l}_{b_2\ldots b_{\bar l}} = 2\bigg(-\Big(X_M\Sigma^{MN}\frac{\partial}{\partial X^N}\Big)_{a_1}^{\ b_1} + \frac{1}{X^+}\Big(\Delta - \frac{l+\bar l}{2} - 2\Big)X_M(\Sigma^{M+})_{a_1}^{\ b_1}\bigg)F^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}\,. \tag{2.105}$$

In writing eq.(2.105), we used the fact that F is a homogeneous function of degree $\Delta + (l+\bar l)/2$ and the following two identities hold:

$$\bigg(\Big(X_M\Sigma^{MN}\frac{\partial}{\partial X^N}\Big)_{a_1}^{\ b_1}\mathbf{X}_{\alpha_2 a_2}\bigg)F^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}} = \bigg(\Big(X_M\Sigma^{MN}\frac{\partial}{\partial X^N}\Big)_{a_1}^{\ b_1}\overline{\mathbf{X}}^{\dot\beta_2 b_2}\bigg)F^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}} = 0\,, \tag{2.106}$$

since F is symmetric in its indices and satisfies eq. (2.27).

Analogously to what found in ref.[16] for symmetric traceless operators, we see here what is special about operators that saturate the unitarity bound (2.98). They are the only ones for which the 6D uplifted tensor R is $SO(4,2)$ covariant. In our index-free notation, current conservation in 6D takes an extremely simple form:

$$\partial\cdot f(x,s,\bar s) = (\partial\cdot f(x))^{\dot\beta_2\ldots\dot\beta_{\bar l}}_{\alpha_2\ldots\alpha_l}s^{\alpha_2}\ldots s^{\alpha_l}\bar s_{\dot\beta_2}\ldots\bar s_{\dot\beta_{\bar l}} = D\cdot F(X,S,\bar S) = 0\,, \tag{2.107}$$

where[3]

$$D = \frac{2}{\ell\bar\ell(\ell+\bar\ell+2)}\left(X_M\Sigma^{MN}\partial_N\right)_a^{\ b}\partial_b^{\ a}, \tag{2.108}$$

where

$$\begin{aligned}\partial_b^{\ a} &= \frac{1}{\ell+\bar\ell+1}\partial^a\partial_b\\ &= \left(4 + S\cdot\partial_S + \bar S\cdot\partial_{\bar S}\right)\frac{\partial}{\partial S_a}\frac{\partial}{\partial\bar S^b} - \left(S_b\frac{\partial}{\partial S_a} + \bar S^a\frac{\partial}{\partial\bar S^b}\right)\frac{\partial^2}{\partial S\cdot\partial\bar S}\end{aligned} \tag{2.109}$$

[3] the operator we computed in [29] was wrong and was corrected by the authors of [40]

2.9 Examples

In this section we show some examples on how to use the formalism presented in section 2.5. We will consider here correlation functions where two of the three primary operators are either scalars (0,0) or spin 1/2 Weyl fermions in reps (1,0) and (0,1). We consider these particular correlators because they are very simple and also because they are relevant for later discussions in this thesis.

2.9.1 Scalar - Scalar - Tensor Correlator

Two scalars with scaling dimensions $\Delta_{\phi 1}$ and $\Delta_{\phi 2}$ and tensor $\mathcal{O}$ in rep $(\ell, \bar{\ell})$ and scaling dimension $\Delta_{\mathcal{O}}$. For this correlator $\Delta\ell$ defined in (2.57) should satisfy the condition (2.59)

$$\Delta\ell = \ell - \bar{\ell}, \quad \text{satisfies} \quad 0 \leq \Delta\ell \leq 0 \tag{2.110}$$

So only symmetric traceless operators $\mathcal{O}$ in rep (ℓ, ℓ) have a non-vanishing 3-point functions with two scalars. Using 6D embedding formalism

$$\langle \Phi_1(X_1)\Phi_2(X_2)O(X_3, S_3, \bar{S}_3)\rangle = \mathcal{K}_3(\kappa_{\phi 1}, \kappa_{\phi 2}, \kappa_{\mathcal{O}})\lambda_{\langle\phi_1\phi_2\mathcal{O}\rangle}\left(\hat{J}^3_{12}\right)^{\ell}, \tag{2.111}$$

where $\lambda_{\langle\phi_1\phi_2\mathcal{O}\rangle}$ is the OPE coefficient and $\mathcal{K}_3$ is defined in eq.(4.39). This correlator can be projected to 4D using (2.90)-(2.92) and $X_{ij}|_{\text{Poincaré}} = x^2_{ij}$

$$\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}(x_3, s_3, \bar{s}_3)\rangle = \langle\Phi_1(X_1)\Phi_2(X_2)O(X_3, S, \bar{S})\rangle|_{\text{Poincaré}} \tag{2.112}$$

Furthermore, if needed, we can get open indices by taking derivatives of s_3 and $\bar{s}_3$ as in (2.33)

$$\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}^{\dot\beta_1\ldots\dot\beta_\ell}_{\alpha_1\ldots\alpha_\ell}(x_3)\rangle = \frac{1}{\ell!\bar{\ell}!}\prod_{i=1}^{\ell}\frac{\partial}{\partial s_3^{\alpha_i}}\frac{\partial}{\partial \bar{s}_{3\dot\beta_i}}\left(\langle\Phi_1(X_1)\Phi_2(X_2)O(X_3, S, \bar{S})\rangle|_{\text{Poincaré}}\right) \tag{2.113}$$

If $\mathcal{O}$ is a conserved tensor $\ell > 0$, its dimension fixed to $\Delta_{\mathcal{O}} = 2 + \ell$. Taking the divergence (2.108) of (2.111) and using eqs. (2.32) and (A.13) gives

$$D_3\langle\Phi_1(X_1)\Phi_2(X_2)O(X_3, S_3, \bar{S}_3)\rangle = \mathcal{K}_3(\kappa_{\phi 1}, \kappa_{\phi 2}, \kappa_{\mathcal{O}})\ell(\ell+1)^2(\Delta_{\phi 2} - \Delta_{\phi 1})\lambda_{\langle\phi_1\phi_2\mathcal{O}\rangle}\left(\hat{J}^3_{12}\right)^{\ell-1} = 0, \tag{2.114}$$

where the subscript 3 in D indicates that derivatives are taken with respect to X_3, S_3 and $\bar{S}_3$. Eq. (2.114) has the form of scalar-scalar-spin $(\ell - 1)$ correlator as it should. This implies that the correlator (2.111) with a conserved $\mathcal{O}$ vanishes unless $\Delta_{\phi 1} = \Delta_{\phi 2}$.

2.9.2 Scalar - Fermion - Tensor Correlator

A scalar ϕ and spin-1/2 ψ_α primary operators with scaling dimensions Δ_ϕ and Δ_ψ and a primary operator in rep $(\ell, \bar{\ell})$. For this correlator $\Delta\ell$ defined in (2.57) should be an even number and satisfy the condition (2.59)

$$\Delta\ell = \ell - \bar{\ell} + 1, \quad \text{satisfies} \quad 0 \leq \Delta\ell \leq 2, \tag{2.115}$$

so the third operator has to be in the rep $(1+\ell, \ell)$ or $(\ell, 1+\ell)$, where ℓ is clearly a non-negative integer. Using 6D embedding formalism, $\mathcal{A}$ is a primary operator in rep $(\ell, 1+\ell)$ with scaling dimension $\Delta_{\mathcal{A}}$

$$\langle\Phi(X_1)\Psi(X_2, S_2)A(X_3, S_3, \bar{S}_3)\rangle = \mathcal{K}_3(\kappa_\phi, \kappa_\psi, \kappa_{\mathcal{A}})\lambda_{\langle\phi\psi\mathcal{A}\rangle}I^{32}\left(\hat{J}^3_{12}\right)^{\ell}, \tag{2.116}$$

while $\mathcal{B}$ is a primary operator in rep $(1+\ell,\ell)$ with scaling dimension $\Delta_{\mathcal{B}}$

$$\langle\Phi(X_1)\Psi(X_2,S_2)B(X_3,S_3,\bar{S}_3)\rangle = \mathcal{K}_3(\kappa_\phi,\kappa_\psi,\kappa_{\mathcal{B}})\lambda_{\langle\phi\psi\mathcal{B}\rangle}\hat{K}_1^{32}\left(\hat{J}_{12}^3\right)^\ell\,, \tag{2.117}$$

where $\lambda_{\langle\phi\psi\mathcal{A}\rangle}$ and $\lambda_{\langle\phi\psi\mathcal{B}\rangle}$ are the corresponding OPE coefficients and again $\mathcal{K}_3$ is defined in eq.(4.39).

2.9.3 Fermion - Fermion - Tensor Correlator

We will determine all the three-point functions involving two fermion fields $\psi_{1\alpha}$, $\psi_2^{\dot\beta}$ and a general primary operator $\mathcal{O}$. According to eq.(2.59), the only non-vanishing 3-point function occurs when $\mathcal{O}$ is in one of the following three Lorentz representations: (l,l), $(l+2,l)$ and $(l,l+2)$, with $l\geq 0$. We will determine the form of the correlators in two cases: with non-conserved and conserved operator $\mathcal{O}$.

Non-conserved Tensor

Let us start by considering the (l,l) representations. According to eq. (2.89), for $l=0$ there is only one possible structure to this correlator,

$$\langle O^{(0,0)}(X_1)\Psi_1(X_2,S_2)\Psi_2(X_3,\bar{S}_3)\rangle = \mathcal{K}_3(\kappa_{\mathcal{O}},\kappa_{\psi 1},\kappa_{\psi 2})\;\lambda^1_{\langle\mathcal{O}\psi_1\psi_2\rangle}I^{32}, \tag{2.118}$$

with $\lambda^1_{\langle\mathcal{O}\psi_1\psi_2\rangle}$ a complex parameter. For $l\geq 1$, two independent structures are present,

$$\begin{aligned}&\langle O(X_1,S_1,\bar{S}_1)\Psi_1(X_2,S_2)\Psi_2(X_3,\bar{S}_3)\rangle\\ &=\mathcal{K}_3(\kappa_{\mathcal{O}},\kappa_{\psi 1},\kappa_{\psi 2})\left(\lambda^1_{\langle\mathcal{O}\psi_1\psi_2\rangle}I^{32}\left(\hat{J}^1_{23}\right)^\ell+\lambda^2_{\langle\mathcal{O}\psi_1\psi_2\rangle}I^{12}I^{31}\left(\hat{J}^1_{23}\right)^{\ell-1}\right)\,,\quad l\geq 1\,,\end{aligned} \tag{2.119}$$

where $\lambda^{1,2}_{\langle\mathcal{O}\psi_1\psi_2\rangle}$ are two complex parameters and $\mathcal{K}_3$ is defined in eq.(4.39).

Using eqs.(2.33) and (2.90) we find

$$\begin{aligned}&\langle\mathcal{O}^{\dot\beta_1\ldots\dot\beta_l}_{\alpha_1\ldots\alpha_l}(x_1)\psi_{1\,\alpha}(x_2)\psi_2^{\dot\beta}(x_3)\rangle=\mathcal{K}_3(\kappa_{\mathcal{O}},\kappa_{\psi 1},\kappa_{\psi 2})|_{\text{Poincaré}}\frac{1}{(\ell!)^2}\times\\ &\Big(\tilde\lambda^1_{\langle\mathcal{O}\psi_1\psi_2\rangle}(x_{32}\cdot\sigma\epsilon)_\alpha^{\;\dot\beta}(Z_{1,23}\cdot\sigma\epsilon)_{\alpha_1}^{\;\dot\beta_1}\ldots(Z_{1,23}\cdot\sigma\epsilon)_{\alpha_l}^{\;\dot\beta_l}+\frac{x_{23}^2}{x_{12}^2x_{13}^2}\\ &\tilde\lambda^2_{\langle\mathcal{O}\psi_1\psi_2\rangle}(x_{12}\cdot\sigma\epsilon)_\alpha^{\;\dot\beta_1}(x_{31}\cdot\sigma\epsilon)_{\alpha_1}^{\;\dot\beta}(Z_{1,23}\cdot\sigma\epsilon)_{\alpha_2}^{\;\dot\beta_2}\ldots(Z_{1,23}\cdot\sigma\epsilon)_{\alpha_l}^{\;\dot\beta_l}+\text{perms.}\Big)\,.\end{aligned} \tag{2.120}$$

In eq.(2.120), $\tilde\lambda^1_{\langle\mathcal{O}\psi_1\psi_2\rangle}$ and $\tilde\lambda^2_{\langle\mathcal{O}\psi_1\psi_2\rangle}$ are proportional to $\lambda^1_{\langle\mathcal{O}\psi_1\psi_2\rangle}$ and $\lambda^2_{\langle\mathcal{O}\psi_1\psi_2\rangle}$ in eq.(2.119) respectively with the same proportionality factor, $Z^\mu_{1,23}$ is defined in eq.(2.93) and perms. refer to the $(l!)^2-1$ terms obtained by permuting the α_i and $\dot\beta_i$ indices.

When ψ_2 is the complex conjugate of ψ_1, namely $\psi_2^{\dot\beta}=\bar\psi_1^{\dot\beta}=(\psi_{1\beta})^\dagger$ and the symmetric traceless tensor components are real, the OPE coefficients $\tilde\lambda^{1,2}_{\langle\mathcal{O}\psi_1\psi_2\rangle}$ are either purely real or purely imaginary, depending on l. When $x^\mu_{1,2,3}$ are space-like separated, causality implies that the operators commute between each other [5]. Taking $\beta=\alpha$ and $\beta_i=\alpha_i$, we then have

$$\langle\mathcal{O}^{\dot\alpha_1\ldots\dot\alpha_l}_{\alpha_1\ldots\alpha_l}(x_1)\psi_{1\alpha}(x_2)\bar\psi_1^{\dot\alpha}(x_3)\rangle^*=-\langle\mathcal{O}^{\dot\alpha_1\ldots\dot\alpha_l}_{\alpha_1\ldots\alpha_l}(x_1)\psi_{1\alpha}(x_3)\bar\psi_1^{\dot\alpha}(x_2)\rangle\,. \tag{2.121}$$

Since $Z_{1,23} = -Z_{1,32}$ we get

$$\left(\tilde{\lambda}^1_{\langle\mathcal{O}\psi_1\bar{\psi}_1\rangle}\right)^* = (-1)^\ell \tilde{\lambda}^1_{\langle\mathcal{O}\psi_1\bar{\psi}_1\rangle}\,, \quad \left(\tilde{\lambda}^2_{\langle\mathcal{O}\psi_1\bar{\psi}_1\rangle}\right)^* = (-1)^\ell \tilde{\lambda}^2_{\langle\mathcal{O}\psi_1\bar{\psi}_1\rangle}\,. \tag{2.122}$$

Let us now consider the parity transformations of eq.(2.120). Parity maps the three-point function $\langle\mathcal{O}^{\dot\beta_1\ldots\dot\beta_l}_{\alpha_1\ldots\alpha_l}(x_1)\psi_{1\alpha}(x_2)\psi_2^{\dot\beta}(x_3)\rangle$ to the complex conjugate three-point function $\langle\mathcal{O}^{\dot\alpha_1\ldots\dot\alpha_l}_{\beta_1\ldots\beta_l}(\tilde{x}_1)\bar{\psi}_1^{\dot\alpha}(\tilde{x}_2)\bar{\psi}_{2\beta}(\tilde{x}_3)\rangle$. When $\psi_2 = \bar{\psi}_1$, and $\alpha_i = \beta_i$, the three-point function is mapped to itself, provided the exchange $x_2 \leftrightarrow x_3$ and $\alpha \leftrightarrow \beta$. The two structures appearing in eq.(2.120) have the same parity transformations. If we impose parity conservation in the CFT and we choose a negative intrinsic parity for the traceless symmetric tensor, $\eta_{\mathcal{O}} = -1$, then the three-point function must vanish: $\tilde{\lambda}^1_{\langle\mathcal{O}\psi_1\bar{\psi}_1\rangle} = \tilde{\lambda}^2_{\langle\mathcal{O}\psi_1\bar{\psi}_1\rangle} = 0$. For positive intrinsic parity $\eta_{\mathcal{O}} = 1$, instead, parity invariance does not give any constraint.

Let us next consider the $(\ell+2,\ell)$ representations. According to eq. (2.89), there is only one possible structure to this correlator, for any ℓ:

$$\langle O(X_1,S_1,\bar{S}_1)\Psi_1(X_2,S_2)\Psi_2(X_3,\bar{S}_3)\rangle = \mathcal{K}_3(\kappa_{\mathcal{O}},\kappa_{\psi1},\kappa_{\psi2})\lambda_{\langle\mathcal{O}\psi_1\psi_2\rangle} I^{31}\hat{K}_3^{12}\left(\hat{J}^1_{23}\right)^\ell\,, \tag{2.123}$$

that gives rise to the 4D correlator

$$\begin{aligned}\langle\mathcal{O}^{\dot\beta_1\ldots\dot\beta_l}_{\alpha_1\ldots\alpha_{\ell+2}}(x_1)\psi_{1\alpha}(x_2)\bar{\psi}_2^{\dot\beta}(x_3)\rangle =& \mathcal{K}_3(\kappa_{\mathcal{O}},\kappa_{\psi1},\kappa_{\psi2})|_{\text{Poincaré}}\frac{\tilde{\lambda}_{\langle\mathcal{O}_{2+\ell,\ell}\psi_1\psi_2\rangle}}{(l!)(l+2)!}\Big((x_{31}\cdot\sigma\epsilon)^{\dot\beta}_{\alpha_{l+1}}\times\\ &\Big((x_{23}^2+x_{31}^2-x_{21}^2)\epsilon_{\alpha\alpha_{l+2}}+4x_{23}^\mu x_{31}^\nu(\sigma_{\mu\nu}\epsilon)_{\alpha\alpha_{l+2}}\Big)\times\\ &(Z_{1,23}\cdot\sigma\epsilon)^{\dot\beta_1}_{\alpha_1}\ldots(Z_{1,23}\cdot\sigma\epsilon)^{\dot\beta_l}_{\alpha_l}+\text{perms.}\Big)\,,\end{aligned} \tag{2.124}$$

where $\tilde{\lambda}_{\langle\mathcal{O}\psi_1\psi_2\rangle}$ is proportional to $\lambda_{\langle\mathcal{O}\psi_1\psi_2\rangle}$ in eq.(2.123).

A similar analysis applies to the complex conjugate $(l,l+2)$ representations. The only possible 6D structure is

$$\langle O(X_1,S_1,\bar{S}_1)\Psi_1(X_2,S_2)\Psi_2(X_3,\bar{S}_3)\rangle = \mathcal{K}_3(\kappa_{\mathcal{O}},\kappa_{\psi1},\kappa_{\psi2})\lambda_{\langle\mathcal{O}\psi_1\psi_2\rangle} I^{12}\hat{\overline{K}}{}_2^{13}\left(J^1_{23}\right)^\ell\,, \tag{2.125}$$

and gives

$$\begin{aligned}\langle\mathcal{O}^{\dot\beta_1\ldots\dot\beta_{\ell+2}}_{\alpha_1\ldots\alpha_\ell}(x_1)\psi_{1\alpha}(x_2)\bar{\psi}_2^{\dot\beta}(x_3)\rangle =& \mathcal{K}_3(\kappa_{\mathcal{O}},\kappa_{\psi1},\kappa_{\psi2})|_{\text{Poincaré}}\frac{\tilde{\lambda}_{\langle\mathcal{O}_{\ell,2+\ell}\psi_1\psi_2\rangle}}{(l!)(l+2)!}\Big((x_{12}\cdot\sigma\epsilon)^{\dot\beta_{l+1}}_{\alpha}\times\\ &\Big((x_{23}^2+x_{21}^2-x_{31}^2)\epsilon^{\dot\beta\dot\beta_{\ell+2}}+4x_{32}^\mu x_{21}^\nu(\bar{\sigma}_{\mu\nu}\epsilon)^{\dot\beta\dot\beta_{l+2}}\Big)\times\\ &(Z_{1,23}\cdot\sigma\epsilon)^{\dot\beta_1}_{\alpha_1}\ldots(Z_{1,23}\cdot\sigma\epsilon)^{\dot\beta_l}_{\alpha_l}+\text{perms.}\Big)\,.\end{aligned} \tag{2.126}$$

If $\bar{\psi}_2 = \psi_1$, as expected, eq.(2.126) is mapped to eq.(2.124) under parity transformation. In particular, in a parity invariant CFT, we should have the same number of $(l,l+2)$ and conjugate $(l+2,l)$ fields, with

$$\lambda_{\langle\mathcal{O}_{(2+\ell,\ell)}\psi_1\psi_2\rangle} = \eta_{\mathcal{O}}\lambda_{\langle\mathcal{O}_{(\ell,2+\ell)}\psi_1\psi_2\rangle}\,. \tag{2.127}$$

Conserved Tensor

Let us start by considering (ℓ,ℓ) representations. The scaling dimension of $\mathcal{O}$ is now fixed to be $\Delta_{\mathcal{O}}=\ell+2$. Taking the divergence (2.108) of eq. (2.119) and using eqs. (2.32) and (A.13) gives

$$\begin{aligned}&D_1\langle O(X_1,S_1,\bar{S}_1)\Psi_1(X_2,S_2)\Psi_2(X_3,\bar{S}_3)\rangle=\mathcal{K}_3(\kappa_{\mathcal{O}},\kappa_{\psi1},\kappa_{\psi2})(\ell+1)(\Delta_{\psi1}-\Delta_{\psi2})\\&\left(\lambda^2_{\langle\mathcal{O}\psi_1\psi_2\rangle}(\ell+2)(1-\ell)I^{12}I^{31}\left(\hat{J}^1_{23}\right)^{\ell-2}+(\lambda^2_{\langle\mathcal{O}\psi_1\psi_2\rangle}-\ell(\ell+1)\lambda^1_{\langle\mathcal{O}\psi_1\psi_2\rangle})I^{32}\left(J^1_{23}\right)^{\ell-1}\right)\end{aligned}\tag{2.128}$$

where the subscript 1 in D indicates that derivatives are taken with respect to X_3, S_3 and $\bar{S}_3$. Eq.(2.128) has the correct form for a Fermion-Fermion-spin $(l-1)$ symmetric tensor, as it should, and is automatically satisfied if $\Delta_{\psi1}=\Delta_{\psi2}$. For $\Delta_{\psi1}\neq\Delta_{\psi2}$ we have

$$\lambda^2_{\langle\mathcal{O}\psi_1\psi_2\rangle}(\ell+2)(1-\ell)=0\,,\quad \lambda^1_{\langle\mathcal{O}\psi_1\psi_2\rangle}=\frac{\lambda^2_{\langle\mathcal{O}\psi_1\psi_2\rangle}}{\ell(\ell+1)}\,.\tag{2.129}$$

For $\ell=1$ we get one independent structure in eq.(2.119) with $\lambda^2_{\langle\mathcal{O}\psi_1\psi_2\rangle}=2\lambda^1_{\langle\mathcal{O}\psi_1\psi_2\rangle}$. For $\ell>1$ eq.(2.129) admits only the trivial solution

$$\langle\mathcal{O}^{\dot\beta_1\ldots\dot\beta_l}_{\alpha_1\ldots\alpha_l}(x_1)\psi_{1\alpha}(x_2)\psi_2^{\dot\beta}(x_2)\rangle=0\,,\quad \ell>1\,,\quad \Delta_{\psi1}\neq\Delta_{\psi2}\,.\tag{2.130}$$

Let us next consider the $(\ell+2,\ell)$ representations, where $\mathcal{O}^{(\ell+2,\ell)}$ is a conserved tensor with $\Delta_{\mathcal{O}}=\ell+3$, $\ell>0$. The divergence (2.108) of eq.(2.123) gives now

$$D_1\langle O(X_1,S_1,\bar{S}_1)\Psi_1(X_2,S_2)\Psi_2(X_3,\bar{S}_3)\rangle=\frac{\lambda_{\langle\mathcal{O}\psi_1\psi_2\rangle}\ell(\ell+2)(\ell+3)}{\mathcal{K}_3(\kappa_{\mathcal{O}},\kappa_{\psi1},\kappa_{\psi2})}(\Delta_{\psi2}-\Delta_{\psi1}-1)I^{31}\hat{K}^{12}_3\left(\hat{J}^1_{23}\right)^{\ell-1}\,.\tag{2.131}$$

For $\Delta_{\psi2}=\Delta_{\psi1}+1$ eq.(2.131) is automatically satisfied. When $\Delta_{\psi2}\neq\Delta_{\psi1}+1$, there are no non-trivial solutions of eq.(2.131) for $\ell>0$:

$$\langle\mathcal{O}^{\dot\beta_1\ldots\dot\beta_l}_{\alpha_1\ldots\alpha_{\ell+2}}(x_1)\psi_{1\alpha}(x_2)\psi_2^{\dot\beta}(x_3)\rangle=0\,,\quad \ell>0\,,\quad \Delta_{\psi2}\neq\Delta_{\psi1}+1\,.\tag{2.132}$$

A similar result applies for conserved $\mathcal{O}^{(\ell,\ell+2)}$ operators replacing $\Delta_{\psi1}\leftrightarrow\Delta_{\psi2}$.

3. Deconstructing Conformal Partial Waves

In this chapter we make use of 6D embedding formalism to find relations between CPW. We will see how three-point functions of spinors/tensors can be related to three-point functions of lower spin fields by means of differential operators. We explicitly construct a basis of differential operators that allows one to express any three-point function of two traceless symmetric and an arbitrary bosonic operator $\mathcal{O}^{l,\bar{l}}$ with $l \neq \bar{l}$, in terms of "seed" three-point functions, that admit a unique tensor structure.This would allow to express all the CPW entering a four-point function of traceless symmetric correlators in terms of a few CPW seeds. These seeds will be computed in chapter 4.

We first start by seeing how a relation between three-point functions leads to a relation between CPW.We introduce a basis of differential operators for three point functions. We will construct an explicit basis of differential operators for external symmetric traceless operators, where the exchanged operator is traceless symmetric and then pass to the more involved case of mixed tensor exchange. We also propose a set of seed CPW needed to get CPW associated with the exchange of a bosonic operator $\mathcal{O}^{l,\bar{l}}$. As an example a four correlation function of two scalars two fermions is deconstructed.

3.1 Relation between CPW

Let us consider for instance the 4-point function of four primary tensor operators:

$$\langle \mathcal{O}_1^{I_1}(x_1)\mathcal{O}_2^{I_2}(x_2)\mathcal{O}_3^{I_3}(x_3)\mathcal{O}_4^{I_4}(x_4)\rangle = \mathcal{K}_4 \sum_{n=1}^{N_4} g_n(u,v)\mathcal{T}_n^{I_1I_2I_3I_4}(x_i)\,. \tag{3.1}$$

In eq.(3.1) we have schematically denoted by I_i the Lorentz indices of the operators $\mathcal{O}_i(x_i)$,

$$\mathcal{K}_4 = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\kappa_1-\kappa_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\kappa_3-\kappa_4}{2}} (x_{12}^2)^{-\frac{\kappa_1+\kappa_2}{2}} (x_{34}^2)^{-\frac{\kappa_3+\kappa_4}{2}} \tag{3.2}$$

is a kinematical factor, the 4D equivalent of (2.43), κ_i are defined in (2.37), u and v are the conformally invariant cross ratios (1.28).

$\mathcal{T}_n^{I_1I_2I_3I_4}(x_i)$ are tensor structures. These are functions of the x_i's and can be kinematically determined.

Their total number N_4 depends on the Lorentz properties of the external primaries. For correlators involving scalars only, one has $N_4 = 1$, but in general $N_4 > 1$ and rapidly grows with the spin of the external fields.

All the non-trivial dynamical information of the 4-point function is encoded in the N_4 functions $g_n(u,v)$. As we mentioned, a bootstrap analysis requires to rewrite the 4-point function (3.1) in terms of the operators exchanged in any channel. In the s-channel (12-34), for instance, we have

$$\langle \mathcal{O}_1^{I_1}(x_1)\mathcal{O}_2^{I_2}(x_2)\mathcal{O}_3^{I_3}(x_3)\mathcal{O}_4^{I_4}(x_4)\rangle = \sum_{i,j}\sum_{\mathcal{O}_r} \lambda^i_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_r}\lambda^j_{\bar{\mathcal{O}}_{\bar{r}}\mathcal{O}_3\mathcal{O}_4} W^{(i,j)I_1I_2I_3I_4}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4,\mathcal{O}_r}(x_i)\,, \tag{3.3}$$

where i and j run over the possible independent tensor structures associated to the three point functions $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_r\rangle$ and $\langle \bar{\mathcal{O}}_{\bar{r}}\mathcal{O}_3\mathcal{O}_4\rangle$, whose total number is N_{3r}^{12} and $N_{3\bar{r}}^{34}$ respectively,[1] the λ's being their corresponding structure constants, and finally $W^{(p,q)I_1I_2I_3I_4}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4}(u,v)$ are the associated CPWs.

[1]Strictly speaking these numbers depend also on $\mathcal{O}_r$, particularly on its spin. When the latter is large enough, however, N_{3r}^{12} and $N_{3\bar{r}}^{34}$ are only functions of the external operators.

The CPWs depend on the external as well as the exchanged operator scaling dimension and spin, dependence we omitted in order not to clutter further the notation.[2]

The sum over the exchanged primary operators $\mathcal{O}_r$ includes a sum over all possible representations $(\ell,\bar\ell)$ that can appear in the 4-point function and, for each representation, a sum over all the possible primaries, i.e. a sum over all possible scaling dimensions $\Delta_{\mathcal{O}_r}$. It is useful to define $\delta = |\bar\ell-\ell|$ and rearrange the sum over $(\ell,\bar\ell)$ in a sum over, say, ℓ and δ. There is an important difference between these two sums. For any 4-point function, the sum over l extends up to infinity, while the sum over δ is always finite. More precisely, we have

$$\begin{aligned} &\delta = 0,\, 2\,,\, \ldots\,,\, p-2,\, p, \qquad && \mathcal{O}_r \quad \text{bosonic} \\ &\delta = 1,\, 3\,,\, \ldots\,,\, p-2,\, p, \qquad && \mathcal{O}_r \quad \text{fermionic.} \end{aligned} \tag{3.4}$$

In both cases, the integer p is defined to be

$$p = \min(\ell_1+\bar\ell_1+\ell_2+\bar\ell_2, \ell_3+\bar\ell_3+\ell_4+\bar\ell_4)\,, \tag{3.5}$$

and is automatically an even or odd integer when $\mathcal{O}_r$ is a boson or a fermion operator.

By comparing eqs.(3.1) and (3.3) one can infer that the number of allowed tensor structures in three and four-point functions is related:[3]

$$N_4 = \sum_r N_{3r}^{12} N_{3\bar r}^{34}\,. \tag{3.6}$$

There are several CPW for each exchanged primary operator $\mathcal{O}_r$, depending on the number of allowed 3-point function structures. They encode the contribution of all the descendant operators associated to the primary $\mathcal{O}_r$. Contrary to the functions $g_n(u,v)$ in eq.(3.1), the CPW do not carry dynamical information, being determined by conformal symmetry alone. They admit a parametrization like the 4-point function itself,

$$W^{(i,j)I_1I_2I_3I_4}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4,\mathcal{O}_r}(x_i) = \mathcal{K}_4 \sum_{n=1}^{N_4} g^{(i,j)}_{\mathcal{O}_r,n}(u,v)\mathcal{T}_n^{I_1I_2I_3I_4}(x_i)\,, \tag{3.7}$$

where $g^{(i,j)}_{\mathcal{O}_r,n}(u,v)$ are the CBs, scalar functions of u and v that depend on the dimensions and spins of the external and exchanged operators.

Once the CPW are determined, by comparing eqs.(3.1) and (3.3) we can express $g_n(u,v)$ in terms of the OPE coefficients of the exchanged operators. This procedure can be done in other channels as well, $(13-24)$ and $(14-23)$. Imposing crossing symmetry, requiring the equality of different channels, gives us the bootstrap equations.

The computation of CPW of tensor correlators is possible, but technically is not easy. In particular it is desirable to have a relation between different CPW, so that it is enough to compute a small subset of them, which determines all the others. In order to understand how this reduction process works, it is very use the embedding formalism in the 6D twistor space with index-free notation explained in chapter 2.

[2]For further simplicity, in what follows we will often omit the subscript indicating the external operators associated to the CPW.

[3]We do not have a formal proof of eq.(3.6), although the agreement found in ref.[29] using eq.(3.6) in different channels is a strong indication that it should be correct.

It is useful to consider the parametrization of CPW in the shadow formalism [44, 45, 46, 47]. It has been shown in ref.[18] that a generic CPW can be written in 6D as

$$W^{(p,q)}_{O_1O_2O_3O_4,O_r}(X_i) \propto \int d^4X d^4Y \langle O_1(X_1)O_2(X_2)O_r(X,S,\bar{S})\rangle_p G \langle \bar{O}_{\bar{r}}(Y,T,\bar{T})O_3(X_3)O_4(X_4)\rangle_q\,. \tag{3.8}$$

In eq.(3.8)

$$O_i(X_i) \equiv O_i(X_i,S_i,\bar{S}_i),$$

are the index-free 6D fields associated to the 4D fields $\mathcal{O}_i(x_i)$, $O_r(X,S,\bar{S})$ and $\bar{O}_{\bar{r}}(Y,T,\bar{T})$ are the exchanged operator and its conjugate, G is a sort of "propagator", function of X,Y and of the twistor derivatives $\partial/\partial S$, $\partial/\partial T$, $\partial/\partial\bar{S}$ and $\partial/\partial\bar{T}$, and the subscripts p and q label the three-point function tensor structures. Finally, in order to remove unwanted contributions, the transformation $X_{12}\to e^{4\pi i}X_{12}$ should be performed and the integral should be projected to the suitable eigenvector under the above monodromy.

Suppose one is able to find a relation between three-point functions of this form:

$$\langle O_1(X_1)O_2(X_2)O_r(X,S,\bar{S})\rangle_p = D_{pp'}(X_{12},S_{1,2},\bar{S}_{1,2})\langle O'_1(X_1)O'_2(X_2)O_r(X,S,\bar{S})\rangle_{p'}\,, \tag{3.9}$$

where $D_{pp'}$ is some operator that depends on $X_{12},S_{1,2},\bar{S}_{1,2}$ and their derivatives, but is crucially independent of X, S, and $\bar{S}$, and $O'_i(X_i)$ are some other, possibly simpler, tensor operators. As long as the operator $D_{pp'}(X_{12},S_{1,2},\bar{S}_{1,2})$ does not change the monodromy properties of the integral, one can use eq.(3.9) in both three-point functions entering eq.(3.8) and move the operator $D_{pp'}$ outside the integral. In this way we get, with obvious notation,

$$W^{(p,q)}_{O_1O_2O_3O_4,O_r}(X_i) = D^{12}_{pp'}D^{34}_{qq'}W^{(p',q')}_{O'_1O'_2O'_3O'_4,O_r}(X_i)\,. \tag{3.10}$$

Using the embedding formalism in vector notation, ref.[17] has shown how to reduce, in any space-time dimension, CPW associated to a correlator of traceless symmetric operators which exchange a traceless symmetric operator to the known CPW of scalar correlators [8, 9].

Focusing on 4D CFTs and using the embedding formalism in twistor space, we will see how the reduction of CPW can be generalized for arbitrary external and exchanged operators.

3.2 Differential Representation of Three-Point Functions

We look for an explicit expression of the operator $D_{pp'}$ defined in eq.(3.9) as a linear combination of products of simpler operators. They must raise (or more generically change) the degree in $S_{1,2}$ and have to respect the gauge redundancy we have in the choice of O. As we explained in section 2.5, multitwistors of the form

$$O \sim O + \mathcal{O}(\bar{S}X)G + \mathcal{O}(\overline{X}S)G'\,, \qquad O\sim O+\mathcal{O}(X^2)G\,, \tag{3.11}$$

where G and G' are some other multi-twistors fields, are equivalent uplifts of the same 4D tensor field. Eq.(3.9) is gauge invariant with respect to the equivalence classes (3.11) only if we demand

$$D_{pp'}\mathcal{O}(\overline{\mathbf{X}}_i\mathbf{X}_i,\overline{\mathbf{X}}_iS_i,\overline{S}_i\mathbf{X}_i,X_i^2,\overline{S}_iS_i) = \mathcal{O}(\overline{\mathbf{X}}_i\mathbf{X}_i,\overline{\mathbf{X}}_iS_i,\overline{S}_i\mathbf{X}_i,X_i^2,\overline{S}_iS_i)\,,\quad i=1,2\,. \tag{3.12}$$

It is useful to classify the building block operators according to their value of Δl, as defined in eq.(2.57).

At zero order in derivatives, we have three possible operators, with $\Delta l = 0$:

$$\sqrt{X_{12}}, I^{12}, I^{21}. \tag{3.13}$$

At first order in derivatives (in X and S), four operators are possible with $\Delta l = 0$:

$$\begin{aligned}
D_1 &\equiv \frac{1}{2}\overline{S}_1\Sigma^M\overline{\Sigma}^N S_1\Big(X_{2M}\frac{\partial}{\partial X_1^N} - X_{2N}\frac{\partial}{\partial X_1^M}\Big)\,,\\
D_2 &\equiv \frac{1}{2}\overline{S}_2\Sigma^M\overline{\Sigma}^N S_2\Big(X_{1M}\frac{\partial}{\partial X_2^N} - X_{1N}\frac{\partial}{\partial X_2^M}\Big)\,,\\
\widetilde{D}_1 &\equiv \overline{S}_1\mathbf{X}_2\overline{\Sigma}^N S_1\frac{\partial}{\partial X_2^N} + 2I^{12}\,S_{1a}\frac{\partial}{\partial S_{2a}} - 2I^{21}\,\overline{S}_1^a\frac{\partial}{\partial \overline{S}_2^a}\,,\\
\widetilde{D}_2 &\equiv \overline{S}_2\mathbf{X}_1\overline{\Sigma}^N S_2\frac{\partial}{\partial X_1^N} + 2I^{21}\,S_{2a}\frac{\partial}{\partial S_{1a}} - 2I^{12}\,\overline{S}_2^a\frac{\partial}{\partial \overline{S}_1^a}\,.
\end{aligned} \tag{3.14}$$

The extra two terms in the last two lines of eq.(3.14) are needed to satisfy the condition (3.12). The $SU(2,2)$ symmetry forbids any operator at first order in derivatives with $\Delta l = \pm 1$.

When $\Delta l = 2$, we have the two operators

$$d_1 \equiv S_2\overline{X}_1\frac{\partial}{\partial \overline{S}_1}\,, \qquad d_2 \equiv S_1\overline{X}_2\frac{\partial}{\partial \overline{S}_2}\,, \tag{3.15}$$

and their conjugates with $\Delta l = -2$:

$$\overline{d}_1 \equiv \overline{S}_2 X_1\frac{\partial}{\partial S_1}\,, \qquad \overline{d}_2 \equiv \overline{S}_1 X_2\frac{\partial}{\partial S_2}\,. \tag{3.16}$$

The operator $\sqrt{X_{12}}$ just decreases the dimensions at both points 1 and 2 by one half. The operator I^{12} increases by one the spin $\bar{l}_1$ and by one l_2. The operator D_1 increases by one the spin l_1 and by one $\bar{l}_1$, increases by one the dimension at point 1 and decreases by one the dimension at point 2. The operator $\widetilde{D}_1$ increases by one the spin l_1 and by one the spin $\bar{l}_1$ and it does not change the dimension of both points 1 and 2. The operator d_1 increases by one the spin l_2 and decreases by one $\bar{l}_1$, decreases by one the dimension at point 1 and does not change the dimension at point 2. The action of the remaining operators is trivially obtained by $1 \leftrightarrow 2$ exchange or by conjugation.

Two more operators with $\Delta l = 2$ are possible:

$$\begin{aligned}
\widetilde{d}_1 &\equiv X_{12}S_1\overline{\Sigma}^M S_2\frac{\partial}{\partial X_1^N} - I^{12}S_{1a}\overline{\mathbf{X}}_2^{ab}\frac{\partial}{\partial \overline{S}_1^b}\,,\\
\widetilde{d}_2 &\equiv X_{12}S_2\overline{\Sigma}^M S_1\frac{\partial}{\partial X_2^N} - I^{21}S_{2a}\overline{\mathbf{X}}_1^{ab}\frac{\partial}{\partial \overline{S}_2^b}\,,
\end{aligned} \tag{3.17}$$

together with their conjugates with $\Delta l = -2$. We will shortly see that the operators (3.17) are redundant and can be neglected.

The above operators satisfy the commutation relations

$$
\begin{aligned}
&[D_i, \widetilde{D}_j] = [d_i, d_j] = [\bar{d}_i, \bar{d}_j] = [d_i, \widetilde{d}_j] = [\bar{d}_i, \bar{\widetilde{d}}_j] = [\widetilde{d}_i, \widetilde{d}_j] = [\bar{\widetilde{d}}_i, \bar{\widetilde{d}}_j] = 0\,, \qquad i,j = 1,2\,, \\
&[D_1, D_2] = 4I^{12}I^{21}\Big(- X_1^M \frac{\partial}{\partial X_1^M} + X_2^M \frac{\partial}{\partial X_2^M}\Big)\,, \\
&[\widetilde{D}_1, \widetilde{D}_2] = 4I^{12}I^{21}\Big(X_1^M \frac{\partial}{\partial X_1^M} - X_2^M \frac{\partial}{\partial X_2^M} + S_1 \frac{\partial}{\partial S_1} + \bar{S}_1 \frac{\partial}{\partial \bar{S}_1} - S_2 \frac{\partial}{\partial S_2} - \bar{S}_2 \frac{\partial}{\partial \bar{S}_2}\Big)\,, \\
&[\widetilde{d}_1, \bar{\widetilde{d}}_2] = 2X_{12}I^{12}I^{21}\Big(- X_1^M \frac{\partial}{\partial X_1^M} + X_2^M \frac{\partial}{\partial X_2^M} - \bar{S}_1 \frac{\partial}{\partial \bar{S}_1} + S_2 \frac{\partial}{\partial S_2}\Big)\,, \\
&[d_i, \bar{d}_j] = 2X_{12}\Big(S_j \frac{\partial}{\partial S_j} - \bar{S}_i \frac{\partial}{\partial \bar{S}_i}\Big)(1 - \delta_{i,j})\,, \qquad i,j = 1,2\,, \\
&[d_i, D_j] = -2\delta_{i,j}\widetilde{d}_i\,, \qquad i,j = 1,2\,, \\
&[d_1, \widetilde{D}_1] = 2\widetilde{d}_2\,, \qquad\qquad [d_2, \widetilde{D}_1] = 0\,, \\
&[\widetilde{d}_1, D_1] = 0\,, \qquad\qquad [\widetilde{d}_2, D_1] = -2I^{12}I^{21}d_2\,, \\
&[\widetilde{d}_1, \widetilde{D}_1] = 2I^{12}I^{21}d_2\,, \qquad [\widetilde{d}_2, \widetilde{D}_1] = 0\,, \\
&[d_1, \bar{\widetilde{d}}_1] = -X_{12}\widetilde{D}_2\,, \qquad [d_1, \bar{\widetilde{d}}_2] = X_{12}D_2\,.
\end{aligned}
\tag{3.18}
$$

Some other commutators are trivially obtained by exchanging 1 and 2 and by the parity transformation (3.24). The operators $\sqrt{X_{12}}$, I^{12} and I^{21} commute with all the differential operators. Acting on the whole correlator, we have

$$
S_i \frac{\partial}{\partial S_i} \to l_i\,, \qquad \bar{S}_i \frac{\partial}{\partial \bar{S}_i} \to \bar{l}_i\,, \qquad X_i^M \frac{\partial}{\partial X_i^M} \to -\kappa_i\,,
\tag{3.19}
$$

and hence the above differential operators, together with X_{12} and $I^{12}I^{21}$, form a closed algebra when acting on three-point correlators. Useful information on conformal blocks can already be obtained by considering the rather trivial operator $\sqrt{X_{12}}$. For any three point function tensor structure, we have

$$
\langle O_1 O_2 O_3 \rangle_s = (\sqrt{X_{12}})^a \langle O_1^{\frac{a}{2}} O_2^{\frac{a}{2}} O_3 \rangle_s\,,
\tag{3.20}
$$

where a is an integer and the superscript indicates a shift in dimension. If $\Delta(\mathcal{O}) = \Delta_{\mathcal{O}}$, then $\Delta(\mathcal{O}^a) = \Delta_{\mathcal{O}} + a$. Using eqs.(3.20) and (3.10), we get for any 4D CPW and pair of integers a and b:

$$
W^{(p,q)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4,\mathcal{O}_r} = x_{12}^a x_{34}^b W^{(p,q)}_{\mathcal{O}_1^a\mathcal{O}_2^a\mathcal{O}_3^b\mathcal{O}_4^b,\mathcal{O}_r}\,.
\tag{3.21}
$$

In terms of the conformal blocks defined in eq.(3.7) one has

$$
\mathcal{G}^{(p,q)}_{\mathcal{O}_r,n}(u,v) = \mathcal{G}^{(p,q)a,a,b,b}_{\mathcal{O}_r,n}(u,v)\,,
\tag{3.22}
$$

where the superscripts indicate the shifts in dimension in the four external operators. Equation (3.22) significantly constrains the dependence of $\mathcal{G}^{(p,q)}_{\mathcal{O}_r,n}$ on the external operator dimensions Δ_i. The conformal blocks can be periodic functions of Δ_1, Δ_2 and Δ_3, Δ_4, but can arbitrarily depend on $\Delta_1 - \Delta_2$, $\Delta_3 - \Delta_4$. This is in agreement with the known form of scalar conformal blocks. Since we are mostly concerned in deconstructing tensor structures, we will neglect in the following the operator $\sqrt{X_{12}}$.

The set of differential operators is redundant, namely there is generally more than 1 combination of products of operators that lead from one three-point function structure to another one. In particular, without any loss of generality we can forget about the operators (3.17), since their action is equivalent to commutators of d_i and D_j.

On the other hand, it is not difficult to argue that the above operators do not allow to connect any three-point function structure to any other one. For instance, it is straightforward to verify that there is no way to connect a three-point correlator with one $(l,\bar{l})$ field to another correlator with a $(l\pm 1,\bar{l}\mp 1)$ field, with the other fields left unchanged. This is not an academic observation because, as we will see, connections of this kind will turn out to be useful in order to simplify the structure of the CPW seeds. The problem is solved by adding to the above list of operators the following second-order operator with $\Delta l = 0$:

$$\nabla_{12} \equiv \frac{(\overline{\mathbf{X}}_1 \mathbf{X}_2)^a_{\;b}}{X_{12}} \frac{\partial^2}{\partial \overline{S}_1^a \partial S_{2,b}} \tag{3.23}$$

and its conjugate ∇_{21}. The above operators transform as follows under 4D parity:

$$D_i \to D_i\,, \qquad \widetilde{D}_i \to \widetilde{D}_i\,, \quad d_i \leftrightarrow -\bar{d}_i\,, \qquad \widetilde{d}_i \leftrightarrow \widetilde{\bar{d}}_i\,, \quad (i=1,2)\,, \quad \nabla_{12} \leftrightarrow -\nabla_{21}\,. \tag{3.24}$$

It is clear that all the operators above are invariant under the monodromy $X_{12} \to e^{4\pi i} X_{12}$. The addition of ∇_{12} and ∇_{21} makes the operator basis even more redundant. It is clear that the paths connecting two different three-point correlators that make use of the least number of these operators are preferred, in particular those that also avoid (if possible) the action of the second order operators ∇_{12} and ∇_{21}. We will not attempt here to explicitly construct a minimal differential basis connecting two arbitrary three-point correlators. Such an analysis is in general complicated and perhaps not really necessary, since in most applications we are interested in CPW involving external fields with spin up to two. Given their particular relevance, we will instead focus in the next section on three-point correlators of two traceless symmetric operators with an arbitrary field $O^{(l,\bar{l})}$.

3.3 Differential Basis for Traceless Symmetric Operators

In this section we show how three-point correlators of two traceless symmetric operators with an arbitrary field $O^{(l_3,\bar{l}_3)}$ can be reduced to seed correlators, with one tensor structure only. We first consider the case $l_3 = \bar{l}_3$, and then go on with $l_3 \neq \bar{l}_3$.

3.3.1 Traceless Symmetric Exchanged Operators

The reduction of traceless symmetric correlators to lower spin traceless symmetric correlators has been successfully addressed in ref.[17]. In this subsection we essentially reformulate the results of ref.[17] in our formalism. This will turn out to be crucial to address the more complicated case of antisymmetric operator exchange. Whenever possible, we will use a notation as close as possible to that of ref.[17], in order to make any comparison more transparent to the reader.

Three-point correlators of traceless symmetric operators can be expressed only in terms of the $SU(2,2)$ invariants I^{ij} and $\hat{J}^i_{jk}$ defined in eqs.(2.49)-(2.50), since Δl defined in eq.(2.57) vanishes. It is useful to consider separately parity even and parity odd tensor structures. Given the action of parity, eq.(2.97), the most general parity even tensor structure is given by products of the following invariants:

$$(I^{21}I^{13}I^{32} - I^{12}I^{31}I^{23}), (I^{12}I^{21}), (I^{13}I^{31}), (I^{23}I^{32}), \hat{J}^1_{23}, \hat{J}^2_{31}, \hat{J}^3_{12}\,. \tag{3.25}$$

These structures are not all independent, because of the identity (2.73).

While in chapter 2, eq.(2.73) has been used to define an independent basis where no tensor structure contains the three $SU(2,2)$ invariants $\hat{J}^1_{23}, \hat{J}^2_{31}, \hat{J}^3_{12}$ at the same time. A more symmetric and convenient basis is obtained by using eq.(2.73) to get rid of the first factor in eq.(3.25).

We define the most general parity even tensor structure of traceless symmetric tensor correlator as

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix} \equiv \mathcal{K}_3 (I^{12}I^{21})^{m_{12}} (I^{13}I^{31})^{m_{13}} (I^{23}I^{32})^{m_{23}} (\hat{J}^1_{23})^{j_1} (\hat{J}^2_{31})^{j_2} (\hat{J}^3_{12})^{j_3} \,, \tag{3.26}$$

where l_i and Δ_i are the spins and scaling dimensions of the fields, the kinematical factor $\mathcal{K}_3$ is defined in eq.(2.42) and

$$\begin{aligned} j_1 &= l_1 - m_{12} - m_{13} \geq 0 \,, \\ j_2 &= l_2 - m_{12} - m_{23} \geq 0 \,, \\ j_3 &= l_3 - m_{13} - m_{23} \geq 0 \,. \end{aligned} \tag{3.27}$$

Notice the similarity of eq.(3.26) with eq.(3.15) of ref.[17], with $(I^{ij}I^{ji}) \to H_{ij}$ and $\hat{J}^i_{jk} \to V_{i,jk}$. The structures (3.26) can be related to a seed scalar-scalar-tensor correlator. Schematically

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix} = \mathcal{D} \begin{bmatrix} \Delta_1' & \Delta_2' & \Delta_3 \\ 0 & 0 & l_3 \\ 0 & 0 & 0 \end{bmatrix} \,, \tag{3.28}$$

where $\mathcal{D}$ is a sum of products of the operators introduced in section 3.2. Since symmetric traceless correlators have $\Delta l = 0$, it is natural to expect that only the operators with $\Delta l = 0$ defined in eqs.(3.13) and (3.14) will enter in $\mathcal{D}$.

Starting from the seed, we now show how one can iteratively construct all tensor structures by means of recursion relations. The analysis will be very similar to the one presented in ref.[17] in vector notation. We first construct tensor structures with $m_{13} = m_{32} = 0$ for any l_1 and l_2 by iteratively using the relation (analogue of eq.(3.27) in ref.[17], with $D_1 \to D_{12}$ and $\widetilde{D}_1 \to D_{11}$)

$$\begin{aligned} & D_1 \begin{bmatrix} \Delta_1 & \Delta_2 + 1 & \Delta_3 \\ l_1 - 1 & l_2 & l_3 \\ 0 & 0 & m_{12} \end{bmatrix} + \tilde{D}_1 \begin{bmatrix} \Delta_1 + 1 & \Delta_2 & \Delta_3 \\ l_1 - 1 & l_2 & l_3 \\ 0 & 0 & m_{12} \end{bmatrix} = \\ & (2 + 2m_{12} - l_1 - l_2 - \Delta_3) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ 0 & 0 & m_{12} \end{bmatrix} - 8(l_2 - m_{12}) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ 0 & 0 & m_{12} + 1 \end{bmatrix} . \end{aligned} \tag{3.29}$$

The analogous equation with D_2 and $\widetilde{D}_2$ is obtained from eq.(3.29) by exchanging $1 \leftrightarrow 2$ and changing sign of the coefficients in the right hand side of the equation. The sign change arises from the fact that $\hat{J}^1_{23} \leftrightarrow -\hat{J}^2_{31}$ and $\hat{J}^3_{12} \to -\hat{J}^3_{12}$ under $1 \leftrightarrow 2$. Hence structures that differ by one spin get a sign change. This observation applies also to eq.(3.31) below. Structures with $m_{12} > 0$ are deduced using (analogue of eq(3.28) in ref.[17])

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix} = (I^{12}I^{21}) \begin{bmatrix} \Delta_1 + 1 & \Delta_2 + 1 & \Delta_3 \\ l_1 - 1 & l_2 - 1 & l_3 \\ m_{23} & m_{13} & m_{12} - 1 \end{bmatrix} . \tag{3.30}$$

Structures with non-vanishing m_{13} (m_{23}) are obtained by acting with the operator D_1 (D_2):

$$\begin{aligned}4(l_3-m_{13}-m_{23})&\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ l_1 & l_2 & l_3\\ m_{23} & m_{13}+1 & m_{12}\end{bmatrix} = D_1\begin{bmatrix}\Delta_1 & \Delta_2+1 & \Delta_3\\ l_1-1 & l_2 & l_3\\ m_{23} & m_{13} & m_{12}\end{bmatrix}\\ +4(l_2-m_{12}-m_{23})&\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ l_1 & l_2 & l_3\\ m_{23} & m_{13} & m_{12}+1\end{bmatrix} -\\ \tfrac{1}{2}(2+2m_{12}-2m_{13}+\Delta_2-\Delta_1-\Delta_3-l_1-l_2+l_3)&\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ l_1 & l_2 & l_3\\ m_{23} & m_{13} & m_{12}\end{bmatrix},\end{aligned} \tag{3.31}$$

and is the analogue of eq (3.29) in ref.[17]. In this way all parity even tensor structures can be constructed starting from the seed correlator.

Let us now turn to parity odd structures. The most general parity odd structure is given by

$$\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ l_1 & l_2 & l_3\\ m_{23} & m_{13} & m_{12}\end{bmatrix}_{odd} \equiv (I^{12}I^{23}I^{31}+I^{21}I^{32}I^{13})\begin{bmatrix}\Delta_1+1 & \Delta_2+1 & \Delta_3+1\\ l_1-1 & l_2-1 & l_3-1\\ m_{23} & m_{13} & m_{12}\end{bmatrix}. \tag{3.32}$$

Since the parity odd combination $(I^{12}I^{23}I^{31}+I^{21}I^{32}I^{13})$ commutes with $D_{1,2}$ and $\widetilde{D}_{1,2}$, the recursion relations found for parity even structures straightforwardly apply to the parity odd ones. One could define a "parity odd seed"

$$16l_3(\Delta_3-1)\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ 1 & 1 & l_3\\ 0 & 0 & 0\end{bmatrix}_{odd} = (d_2\bar{d}_1-\bar{d}_2 d_1)D_1D_2\begin{bmatrix}\Delta_1+2 & \Delta_2+2 & \Delta_3\\ 0 & 0 & l_3\\ 0 & 0 & 0\end{bmatrix} \tag{3.33}$$

and from here construct all the parity odd structures. Notice that the parity odd seed cannot be obtained by applying only combinations of $D_{1,2}$, $\widetilde{D}_{1,2}$ and $(I^{12}I^{21})$, because these operators are all invariant under parity, see eq.(3.24). This explains the appearance of the operators d_i and $\bar{d}_i$ in eq.(3.33). The counting of parity even and odd structures manifestly agrees with that performed in ref.[16].

Once proved that all tensor structures can be reached by acting with operators on the seed correlator, one might define a differential basis which is essentially identical to that defined in eq.(3.31) of ref. [17]:

$$\left\{\begin{matrix}\Delta_1 & \Delta_2 & \Delta_3\\ l_1 & l_2 & l_3\\ m_{23} & m_{13} & m_{12}\end{matrix}\right\}_0 = (I^{12}I^{21})^{m_{12}}D_1^{m_{13}}D_2^{m_{23}}\widetilde{D}_1^{j_1}\widetilde{D}_2^{j_2}\begin{bmatrix}\Delta_1' & \Delta_2' & \Delta_3\\ 0 & 0 & l_3\\ 0 & 0 & 0\end{bmatrix}, \tag{3.34}$$

where $\Delta_1' = \Delta_1+l_1+m_{23}-m_{13}$, $\Delta_2' = \Delta_2+l_2+m_{13}-m_{23}$. The recursion relations found above have shown that the differential basis (3.34) is complete: all parity even tensor structures can be written as linear combinations of eq.(3.34). The dimensionality of the differential basis matches the one of the ordinary basis for any spin l_1, l_2 and l_3. Since both bases are complete, the transformation matrix relating them is ensured to have maximal rank. Its determinant, however, is a function of the scaling dimensions Δ_i and the spins l_i of the fields and one should check that it does not vanish for some specific values of Δ_i and l_i. We have explicitly checked up to $l_1=l_2=2$ that for $l_3\geq l_1+l_2$ the rank of the transformation matrix depends only on Δ_3 and l_3 and never vanishes, for any value of Δ_3 allowed by the unitarity bound [42]. On the other hand, a problem can arise when $l_3<l_1+l_2$, because

in this case a dependence on the values of Δ_1 and Δ_2 arises and the determinant vanishes for specific values (depending on the l_i's) of $\Delta_1 - \Delta_2$ and Δ_3, even when they are within the unitarity bounds.[4]

This issue is easily solved by replacing $\widetilde{D}_{1,2} \to (\widetilde{D}_{1,2} + D_{1,2})$ in eq.(3.34), as suggested by the recursion relation (3.29), and by defining an improved differential basis

$$\left\{ \begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{array} \right\} = (I^{12}I^{21})^{m_{12}} D_1^{m_{13}} D_2^{m_{23}} \sum_{n_1=0}^{j_1} \binom{j_1}{n_1} D_1^{n_1} \widetilde{D}_1^{j_1-n_1} \sum_{n_2=0}^{j_2} \binom{j_2}{n_2} D_2^{n_2} \widetilde{D}_2^{j_2-n_2} \left[\begin{array}{ccc} \Delta_1' & \Delta_2' & \Delta_3 \\ 0 & 0 & l_3 \\ 0 & 0 & 0 \end{array} \right] \tag{3.35}$$

where $\Delta_1' = \Delta_1 + l_1 + m_{23} - m_{13} + n_2 - n_1$, $\Delta_2' = \Delta_2 + l_2 + m_{13} - m_{23} + n_1 - n_2$. A similar basis for parity odd structures is given by

$$\left\{ \begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{array} \right\}_{odd} = (d_2\bar{d}_1 - \bar{d}_2 d_1) D_1 D_2 \left\{ \begin{array}{ccc} \Delta_1+2 & \Delta_2+2 & \Delta_3 \\ l_1-1 & l_2-1 & l_3 \\ m_{23} & m_{13} & m_{12} \end{array} \right\}. \tag{3.36}$$

In practical computations it is more convenient to use the differential basis rather than the recursion relations and, if necessary, use the transformation matrix to rotate the results back to the ordinary basis. We have explicitly constructed the improved differential basis (3.35) and (3.36) up to $l_1 = l_2 = 2$. The rank of the transformation matrix depends on Δ_3 and l_3 for any value of l_3, and never vanishes, for any value of Δ_3 allowed by the unitary bound.[5]

3.3.2 Antisymmetric Exchanged Operators

In this subsection we consider correlators with two traceless symmetric and one antisymmetric operator $O^{(l_3,\bar{l}_3)}$, with $l_3 - \bar{l}_3 = 2\delta$, with δ an integer. A correlator of this form has $\Delta l = 2\delta$ and according to the analysis of section 2.5, any of its tensor structures can be expressed in a form containing an overall number δ of $\hat{K}_i^{jk}$'s if $\delta > 0$, or $\hat{\overline{K}}_i^{jk}$'s if $\delta < 0$. We consider in the following $\delta > 0$, the case $\delta < 0$ being easily deduced from $\delta > 0$ by means of a parity transformation. The analysis will proceed along the same lines of subsection 3.3.1. We first show a convenient parametrization for the tensor structures of the correlator, then we prove by deriving recursion relations how all tensor structures can be reached starting from a single seed, to be determined, and finally present a differential basis.

We first consider the situation where $l_3 \geq l_1 + l_2 - \delta$ and then the slightly more involved case with unconstrained l_3.

Recursion Relations for $l_3 \geq l_1 + l_2 - \delta$

It is convenient to look for a parametrization of the tensor structures which is as close as possible to the one (3.26) valid for $\delta = 0$. When $l_3 \geq l_1 + l_2 - \delta$, any tensor structure of the correlator contains enough $\hat{J}^3_{12}$'s invariants to remove all possible $\hat{K}_3^{12}$'s invariants using the identity

$$\hat{J}^3_{12} \hat{K}_3^{12} = I^{31} \hat{K}_1^{23} - I^{32} \hat{K}_2^{31} \,. \tag{3.37}$$

[4] A similar problem seems also to occur for the basis (3.31) of ref. [17] in vector notation.

[5] The transformation matrix is actually not of maximal rank when $l_3 = 0$ and $\Delta_3 = 1$. However, this case is quite trivial. The exchanged scalar is free and hence the CFT is the direct sum of at least two CFTs, the interacting one and the free theory associated to this scalar. So, either the two external l_1 and l_2 tensors are part of the free CFT, in which case the whole correlator is determined, or the OPE coefficients entering the correlation function must vanish.

There are four possible combinations in which the remaining $\hat{K}_1^{23}$ and $\hat{K}_2^{31}$ invariants can enter in the correlator: $\hat{K}_1^{23}I^{23}$, $\hat{K}_1^{23}I^{21}I^{13}$ and $\hat{K}_2^{31}I^{13}$, $\hat{K}_2^{31}I^{12}I^{23}$. These structures are not all independent. In addition to eq.(3.37), using the two identities

$$\begin{aligned} I^{12}\hat{K}_2^{31} &= \hat{J}_{23}^1\hat{K}_1^{23} + I^{13}\hat{K}_3^{12}\,, \\ I^{21}\hat{K}_1^{23} &= -\hat{J}_{31}^2\hat{K}_2^{31} + I^{23}\hat{K}_3^{12}\,, \end{aligned} \tag{3.38}$$

we can remove half of them and keep only, say, $\hat{K}_1^{23}I^{23}$ and $\hat{K}_2^{31}I^{13}$. The most general tensor structure can be written in terms of the parity even structures of traceless symmetric correlators as

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}_p \equiv \left(\frac{\hat{K}_1^{23}I^{23}}{X_{23}}\right)^{\delta-p}\left(\frac{\hat{K}_2^{31}I^{13}}{X_{13}}\right)^{p} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1-p & l_2-\delta+p & l_3 \\ m_{23} & m_{13} & \widetilde{m}_{12} \end{bmatrix}\,, \quad p=0,\ldots,\delta\,, \tag{3.39}$$

where the subscript p in the lhs differentiate this tensor structure from the traceless-symmetric case[6]. On the rhs we have parity even structures (3.26) of traceless symmetric correlators, where

$$\begin{aligned} j_1 &= l_1 - p - \widetilde{m}_{12} - m_{13} \geq 0\,, \\ j_2 &= l_2 - \delta + p - \widetilde{m}_{12} - m_{23} \geq 0\,, \\ j_3 &= l_3 - m_{13} - m_{23} \geq 0 \end{aligned} \qquad \widetilde{m}_{12} = \begin{cases} m_{12} & if \quad p=0 \;\; or \;\; p=\delta \\ 0 & otherwise \end{cases}\,. \tag{3.40}$$

The condition in m_{12} derives from the fact that, using eqs.(3.38), one can set m_{12} to zero in the tensor structures with $p \neq 0, \delta$, see below. Attention should be paid to the subscript p. Structures with no subscript refer to purely traceless symmetric correlators, while those with the subscript p refer to three-point functions with two traceless symmetric and one antisymmetric field. All tensor structures are classified in terms of $\delta+1$ classes, parametrized by the index p in eq.(3.39). The parity odd structures of traceless symmetric correlators do not enter, since they can be reduced in the form (3.39) by means of the identities (3.38). The class p exists only when $l_1 \geq p$ and $l_2 \geq \delta - p$. If $l_1 + l_2 < \delta$, the entire correlator vanishes.

Contrary to the symmetric traceless exchange, there is no obvious choice of seed that stands out. The allowed correlator with the lowest possible spins in each class, $l_1 = p$, $l_2 = \delta - p$, $m_{ij} = 0$, can all be seen as possible seeds with a unique tensor structure. Let us see how all the structures (3.39) can be iteratively constructed using the operators defined in section 3.2 in terms of the $\delta+1$ seeds. It is convenient to first construct a redundant basis where $m_{12} \neq 0$ for any p and then impose the relation that leads to the independent basis (3.39). The procedure is similar to that followed for the traceless symmetric exchange. We first construct all the tensor structures with $m_{13} = m_{32} = 0$ for any spin l_1 and l_2, and any class p, using the following relations:

$$\begin{aligned} &D_1 \begin{bmatrix} \Delta_1 & \Delta_2+1 & \Delta_3 \\ l_1-1 & l_2 & l_3 \\ 0 & 0 & m_{12} \end{bmatrix}_p + \widetilde{D}_1 \begin{bmatrix} \Delta_1+1 & \Delta_2 & \Delta_3 \\ l_1-1 & l_2 & l_3 \\ 0 & 0 & m_{12} \end{bmatrix}_p = (\delta-p) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ 0 & 0 & m_{12} \end{bmatrix}_{p+1} \\ &-8(l_2-\delta+p-m_{12}) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ 0 & 0 & m_{12}+1 \end{bmatrix}_p + (2m_{12}-l_1-l_2-\Delta_3+2+\delta-p) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ 0 & 0 & m_{12} \end{bmatrix}_p\,, \end{aligned} \tag{3.41}$$

together with the relation

$$\begin{bmatrix} \Delta_1-1 & \Delta_2-1 & \Delta_3 \\ l_1+1 & l_2+1 & l_3 \\ 0 & 0 & m_{12}+1 \end{bmatrix}_p = (I^{12}I^{21}) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ 0 & 0 & m_{12} \end{bmatrix}_p\,. \tag{3.42}$$

[6] The tensor structure defined here clearly depends on the value of δ, but we drop this dependence from the notation we use hereafter.

Notice that the operators $D_{1,2}$ and $\widetilde{D}_{1,2}$ relate nearest neighbour classes and the iteration eventually involves all classes at the same time. The action of the D_2 and $\widetilde{D}_2$ derivatives can be obtained by replacing $1 \leftrightarrow 2$, $p \leftrightarrow (\delta - p)$ in the coefficients multiplying the structures and $p+1 \to p-1$ in the subscripts, and by changing sign on one side of the equation. Structures with non-vanishing m_{13} and m_{23} are obtained using

$$\begin{aligned} 4(l_3 - m_{13} - m_{23} + \delta - p) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13}+1 & m_{12} \end{bmatrix}_p - 4(\delta - p) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23}+1 & m_{13} & m_{12} \end{bmatrix}_{p+1} &= \\ 4(l_2 - \delta + p - m_{23} - m_{12}) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12}+1 \end{bmatrix}_p + D_1 \begin{bmatrix} \Delta_1 & \Delta_2+1 & \Delta_3 \\ l_1-1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}_p & \qquad (3.43) \\ -\frac{1}{2}(2m_{12} - 2m_{13} + \Delta_2 - \Delta_1 - \Delta_3 - l_1 - l_2 + l_3 + 2\delta - 2p + 2) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}_p & \end{aligned}$$

together with the corresponding relation with $1 \leftrightarrow 2$ and $p \to p+1$. All the structures (3.39) are hence derivable from $\delta + 1$ seeds by acting with the operators $D_{1,2}$, $\widetilde{D}_{1,2}$ and $(I^{12}I^{21})$. The seeds, on the other hand, are all related by means of the following relation:

$$(\delta - p)^2 \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ p+1 & \delta-p-1 & l_3 \\ 0 & 0 & 0 \end{bmatrix}_{p+1} = R \begin{bmatrix} \Delta_1+1 & \Delta_2+1 & \Delta_3 \\ p & \delta-p & l_3 \\ 0 & 0 & 0 \end{bmatrix}_p , \qquad (3.44)$$

where

$$R \equiv -\frac{1}{2}\bar{d}_2 d_2 \,. \qquad (3.45)$$

We conclude that, starting from the single seed correlator with $p = 0$,

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 0 & \delta & l_3 \\ 0 & 0 & 0 \end{bmatrix}_0 \equiv \left(\frac{\hat{K}_1^{23} I^{23}}{X_{23}} \right)^\delta \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 0 & 0 & l_3 \\ 0 & 0 & 0 \end{bmatrix} , \qquad (3.46)$$

namely the three-point function of a scalar, a spin δ traceless symmetric operator and the antisymmetric operator with spin $(l_3 + 2\delta, l_3)$, we can obtain all tensor structures of higher spin correlators.

Let us now see how the constraint on m_{12} in eq.(3.40) arises. When $p \neq 0, \delta$, namely when both K_1 and K_2 structures appear at the same time, combining eqs.(3.38), the following relation is shown to hold:

$$\begin{aligned} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12}+1 \end{bmatrix}_p = -\frac{1}{4} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}_p - \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13}+1 & m_{12} \end{bmatrix}_p - \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23}+1 & m_{13} & m_{12} \end{bmatrix}_p \\ -8 \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23}+1 & m_{13}+1 & m_{12} \end{bmatrix}_p + \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13}+1 & m_{12} \end{bmatrix}_{p-1} + 4 \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13}+2 & m_{12} \end{bmatrix}_{p-1} \\ + \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23}+1 & m_{13} & m_{12} \end{bmatrix}_{p+1} + 4 \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23}+2 & m_{13} & m_{12} \end{bmatrix}_{p+1} . \qquad (3.47) \end{aligned}$$

Using it iteratively, we can reduce all structures with $p \neq 0, \delta$ to those with $m_{12} = 0$ and with $p = 0, \delta$, any m_{12}.[7] This proves the validity of eq.(3.39). As a further check, we have verified that the number of tensor structures obtained from eq.(3.39) agrees with those found from eq.(3.38) of ref.[29].

[7] One has to recall the range of the parameters (3.40), otherwise it might seem that non-existant structures can be obtained from eq.(3.47).

Recursion Relations for general l_3

The tensor structures of correlators with $l_3 < l_1 + l_2 - \delta$ cannot all be reduced in the form (3.39), because we are no longer ensured to have enough $\hat{J}^3_{12}$ invariants to remove all the $\hat{K}^{12}_3$'s by means of eq.(3.37). In this case the most general tensor structure reads

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}_{p,q} \equiv \eta \Big(\frac{\hat{K}^{23}_1 I^{23}}{X_{23}}\Big)^{\delta-p} \Big(\frac{\hat{K}^{31}_2 I^{13}}{X_{13}}\Big)^{q} \Big(\frac{\hat{K}^{12}_3 I^{13} I^{23}}{\sqrt{X_{12}X_{13}X_{23}}}\Big)^{p-q} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 - p & l_2 - \delta + q & l_3 \\ m_{23} & m_{13} & \widetilde{m}_{12} \end{bmatrix}, \quad (3.48)$$

with $p = 0, \ldots, \delta$, $q = 0, \ldots, \delta$, $p - q \geq 0$ and

$$\begin{aligned} j_1 &= l_1 - p - \widetilde{m}_{12} - m_{13} \geq 0\,, \\ j_2 &= l_2 - \delta + q - \widetilde{m}_{12} - m_{23} \geq 0\,, \\ j_3 &= l_3 - m_{13} - m_{23} \geq 0\,, \end{aligned} \qquad \begin{aligned} \widetilde{m}_{12} &= \begin{cases} m_{12} & if \quad q = 0 \ \ or \ \ p = \delta \\ 0 & otherwise \end{cases} \\ \eta &= \begin{cases} 0 & if \quad j_3 > 0 \ \ and \ \ p \neq q \\ 1 & otherwise \end{cases} \, . \end{aligned} \quad (3.49)$$

The parameter η in eq.(3.49) is necessary because the tensor structures involving $\hat{K}^{12}_3$ (i.e. those with $p \neq q$) are independent only when $j_3 = 0$, namely when the traceless symmetric structure does not contain any $\hat{J}^3_{12}$ invariant. All the tensor structures (3.48) can be reached starting from the single seed with $p = 0$, $q = 0$, $l_1 = 0$, $l_2 = \delta$ and $m_{ij} = 0$. The analysis follows quite closely the one made for $l_3 \geq l_1 + l_2 - \delta$, although it is slightly more involved. As before, it is convenient to first construct a redundant basis where $m_{12} \neq 0$ for any p, q and we neglect the factor η above, and impose only later the relations that leads to the independent basis (3.48). We start from the structures with $p = q$, which are the same as those in eq.(3.39): first construct the structures with $m_{13} = m_{23} = 0$ by applying iteratively the operators $D_{1,2} + \widetilde{D}_{1,2}$, and then apply D_1 and D_2 to get the structures with non-vanishing m_{13} and m_{23}. Structures with $p \neq q$ appear when acting with D_1 and D_2. We have:

$$\begin{aligned} D_1 \begin{bmatrix} \Delta_1 & \Delta_2 + 1 & \Delta_3 \\ l_1 - 1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}_{p,p} &= 2(\delta - p) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}_{p+1,p} \\ -4(l_2 + p - \delta - m_{12} - m_{23}) &\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} + 1 \end{bmatrix}_{p,p} + 4(l_3 - m_{13} - m_{23}) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} + 1 & m_{12} \end{bmatrix}_{p,p} \\ + \frac{1}{2}\Big(2m_{12} - 2m_{13} + \Delta_2 - \Delta_1 - \Delta_3 - l_1 - l_2 + l_3 + 2(\delta - p + 1)\Big) &\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}_{p,p} \, . \end{aligned} \quad (3.50)$$

The action of D_2 is obtained by exchanging $1 \leftrightarrow 2$ and $\delta - p \leftrightarrow q$ in the coefficients multiplying the structures and replacing the subscript $(p+1, p)$ with $(p, p-1)$. For $m_{13} + m_{23} < l_3$ the first term in eq.(3.50) is redundant and can be expressed in terms of the known structures with $p = q$. An irreducible structure is produced only when we reach the maximum allowed value $m_{13} + m_{23} = l_3$, in which case the third term in eq.(3.50) vanishes and we can use the equation to get the irreducible structures with $p \neq q$. Summarizing, all tensor structures can be obtained starting from a single seed upon the action of the operators $D_{1,2}$, $(D_{1,2} + \widetilde{D}_{1,2})$, $I^{12}I^{21}$ and R.

Differential Basis

A differential basis that is well defined for any value of l_1, l_2, l_3 and δ is

$$\left\{\begin{array}{ccc}\Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12}\end{array}\right\}_{p,q} = \eta\,(I^{12}I^{21})^{\widetilde{m}_{12}} D_1^{m_{13}+p-q} D_2^{m_{23}} \sum_{n_1=0}^{j_1} \binom{j_1}{n_1} D_1^{n_1} \widetilde{D}_1^{j_1-n_1} \sum_{n_2=0}^{j_2} \binom{j_2}{n_2} D_2^{n_2} \widetilde{D}_2^{j_2-n_2} R^q \left[\begin{array}{ccc}\Delta_1' & \Delta_2' & \Delta_3 \\ 0 & \delta & l_3 \\ 0 & 0 & 0\end{array}\right]_0 , \tag{3.51}$$

where $\Delta_1' = \Delta_1 + l_1 + m_{23} - m_{13} + n_2 - n_1 - p + q$, $\Delta_2' = \Delta_2 + l_2 + m_{13} - m_{23} + n_1 - n_2 + 2q - \delta$, and all parameters are defined as in eq.(3.49). The recursion relations found above have shown that the differential basis (3.51) is complete. One can also check that its dimensionality matches the one of the ordinary basis for any l_1, l_2, l_3 and δ. Like in the purely traceless symmetric case, the specific choice of operators made in eq.(3.51) seems to be enough to ensure that the determinant of the transformation matrix is non-vanishing regardless of the choice of Δ_1 and Δ_2. We have explicitly checked this result up to $l_1 = l_2 = 2$, for any l_3. The transformation matrix is always of maximal rank, except for the case $l_3 = 0$ and $\Delta_3 = 2$, which saturates the unitarity bound for $\delta = 1$. Luckily enough, this case is quite trivial, being associated to the exchange of a free $(2,0)$ self-dual tensor [48] (see footnote 5). The specific ordering of the differential operators is a choice motivated by the form of the recursion relations, as before, and different orderings can be trivially related by using the commutators defined in eq.(3.18).

3.4 Computation of Four-Point Functions

We have shown in section 3.1 how relations between three-point functions lead to relations between CPW. The latter are parametrized by 4-point, rather than 3-point, function tensor structures, so in order to make further progress it is important to classify four-point functions. It should be clear that even when acting on scalar quantities, tensor structures belonging to the class of 4-point functions are generated. For example $\widetilde{D}_1 U = -U J_{1,24}$.

A general classification of 4-point tensor structure is very complicated in twistor language. The reason is that while the formalism enable us to write all possible tensor structures, the problem lies with determining a linearly independent bases. Third forth order relations between structures arise. Finding all possible relations then working out their consequences is not very practical. An easier way to do this is to work in a non-covariant method, by using conformal symmetry to fix (x_1, x_2, x_3, x_4) to special values, on what is called conformal frame. In this frame the correlator has only to satisfy the remaining symmetry requirements. This has been work out nicely in [40]. In this thesis however, we will work with low spin 4-point functions that don't require such classification.

3.4.1 Relation between "Seed" Conformal Partial Waves

Using the results of the last section, we can compute the CPW associated to the exchange of arbitrary operators with external traceless symmetric fields, in terms of a set of seed CPW, schematically denoted by $W^{(p,q)}_{\mathcal{O}^{l+2\delta,l}}(l_1, l_2, l_3, l_4)$. We have

$$W^{(p,q)}_{\mathcal{O}^{l+2\delta,l}}(l_1, l_2, l_3, l_4) = D^{(p)}_{(12)} D^{(q)}_{(34)} W_{\mathcal{O}^{l+2\delta,l}}(0, \delta, 0, \delta)\,, \tag{3.52}$$

where $D_{12}^{(p)}$ schematically denotes the action of the differential operators reported in the last section, and $D_{34}^{(q)}$ are the same operators for the fields at X_3 and X_4, obtained by replacing $1 \to 3$, $2 \to 4$ everywhere in eqs.(3.14)-(3.17) and (3.23). For simplicity we do not report the dependence of W on U, V, and on the scaling dimensions of the external and exchanged operators.

The seed CPW are the simplest among the ones appearing in correlators of traceless symmetric tensors, but they are *not* the simplest in general. These will be the CPW arising from the four-point functions with the *lowest* number of tensor structures with a non-vanishing contribution of the field $O^{l+2\delta,l}$ in some of the OPE channels. Such minimal four-point functions are[8]

$$\langle O^{(0,0)}(X_1)O^{(2\delta,0)}(X_2)O^{(0,0)}(X_3)O^{(0,2\delta)}(X_4)\rangle = \mathcal{K}_4 \sum_{n=0}^{2\delta} g_n(U,V) I_{42}^n (\hat{I}_{31}^{42})^{2\delta-n}\,, \tag{3.53}$$

with just

$$N_4^{seed}(\delta) = 2\delta + 1 \tag{3.54}$$

tensor structures. In the s-channel (12-34) operators $O^{l+n,l}$, with $-2\delta \leq n \leq 2\delta$, are exchanged. We denote by $W_{seed}(\delta)$ and $\overline{W}_{seed}(\delta)$ the single CPW associated to the exchange of the fields $O^{l+2\delta,l}$ and $O^{l,l+2\delta}$ in the four-point function (3.53). They are parametrized in terms of $2\delta+1$ conformal blocks as follows ($\mathcal{G}_0^{(0)} = \overline{\mathcal{G}}_0^{(0)}$):

$$\begin{aligned} W_{seed}(\delta) &= \mathcal{K}_4 \sum_{n=0}^{2\delta} G_n^{(\delta)}(U,V) I_{42}^n (\hat{I}_{31}^{42})^{2\delta-n}\,, \\ \overline{W}_{seed}(\delta) &= \mathcal{K}_4 \sum_{n=0}^{2\delta} \overline{G}_n^{(\delta)}(U,V) I_{42}^n (\hat{I}_{31}^{42})^{2\delta-n}\,. \end{aligned} \tag{3.55}$$

In contrast, the number of tensor structures in $\langle O^{(0,0)}(X_1)O^{(\delta,\delta)}(X_2)O^{(0,0)}(X_3)O^{(\delta,\delta)}(X_4)\rangle$ grows rapidly with δ. Denoting it by $\widetilde{N}_4(\delta)$ we have, using eq.(6.6) of ref.[29]:

$$\widetilde{N}_4(\delta) = \frac{1}{3}\Big(2\delta^3 + 6\delta^2 + 7\delta + 3\Big)\,. \tag{3.56}$$

It is important to stress that a significant simplification occurs in using seed CPW even when there is no need to reduce their number, i.e. $p = q = 1$. For instance, consider the correlator of four traceless symmetric spin 2 tensors. The CPW $W_{O^{l+8,l}}(2,2,2,2)$ is unique, yet it contains 1107 conformal blocks (one for each tensor structure allowed in this correlator), to be contrasted to the 85 present in $W_{O^{l+8,l}}(0,4,0,4)$ and the 9 in $W_{seed}(4)$! We need to relate $\langle O^{(0,0)}(X_1)O^{(2\delta,0)}(X_2)O^{(l+2\delta,l)}(X_3)\rangle$ and $\langle O^{(0,0)}(X_1)O^{(\delta,\delta)}(X_2)O^{(l+2\delta,l)}(X_3)\rangle$ in order to be able to use the results of section 3.3 together with $W_{seed}(\delta)$. As explained at the end of Section 3.2, there is no combination of first-order operators which can do this job and one is forced to use the operator (3.23):

$$\langle O_{\Delta_1}^{(0,0)}(X_1)O_{\Delta_2}^{(\delta,\delta)}(X_2)O_{\Delta}^{(l,l+2\delta)}(X)\rangle_1 = \Big(\prod_{n=1}^{\delta} c_n\Big)(\bar{d}_1\nabla_{12}\widetilde{D}_1)^{\delta}\langle O_{\Delta_1+\delta}^{(0,0)}(X_1)O_{\Delta_2}^{(2\delta,0)}(X_2)O_{\Delta}^{(l,l+2\delta)}(X)\rangle_1\,, \tag{3.57}$$

[8] Instead of eq.(3.53) one could also use 4-point functions with two scalars and two $O^{(0,2\delta)}$ fields or two scalars and two $O^{(2\delta,0)}$ fields. Both have the same number $2\delta+1$ of tensor structures as the correlator (3.53).

where[9]

$$c_n^{-1} = 2(1-n+2\delta)\Big(2(n+1)+\delta+l+\Delta_1-\Delta_2+\Delta\Big)\,. \tag{3.58}$$

Equation (3.57) implies the following relation between the two CPW:

$$W_{O^{l+2\delta,l}}(0,\delta,0,\delta) = \Big(\prod_{n=1}^{\delta} c_n^{12}c_n^{34}\Big)\Big(\nabla_{43}d_3\widetilde{D}_3\Big)^{\delta}\Big(\nabla_{12}\bar{d}_1\widetilde{D}_1\Big)^{\delta}W_{seed}(\delta)\,, \tag{3.59}$$

where $c_n^{12}=c_n$ in eq.(3.58), c_n^{34} is obtained from c_n by exchanging $1\to 3, 2\to 4$ and the scaling dimensions of the corresponding external operators are related as indicated in eq.(3.57).

Summarizing, the whole highly non-trivial problem of computing $W^{(p,q)}_{O^{l+2\delta,l}}(l_1,l_2,l_3,l_4)$ has been reduced to the computation of the $2\times(2\delta+1)$ conformal blocks $\mathcal{G}_n^{(\delta)}(U,V)$ and $\overline{\mathcal{G}}_n^{(\delta)}(U,V)$ entering eq.(3.55). Once they are known, one can use eqs.(3.59) and (3.52) to finally reconstruct $W^{(p,q)}_{O^{l+2\delta,l}}(l_1,l_2,l_3,l_4)$.

3.5 Example

In this section we would like to elucidate various aspects of our construction. In the subsection 3.5.1 we give an example in which we deconstruct a correlation function of four fermions. We leave the domain of traceless symmetric external operators to show the generality of our formalism. It might also have some relevance in phenomenological applications beyond the Standard Model [7].

3.5.1 Four Fermions Correlator

Our goal here is to deconstruct the CPW in the s-channel associated to the four fermion correlator

$$\langle\bar{\psi}^{\dot\alpha}(x_1)\psi_\beta(x_2)\chi_\gamma(x_3)\bar{\chi}^{\dot\delta}(x_4)\rangle\,. \tag{3.60}$$

For simplicity, we take $\bar\psi$ and $\bar\chi$ to be conjugate fields of ψ and χ, respectively, so that we have only two different scaling dimensions, Δ_ψ and Δ_χ. Parity invariance is however not imposed in the underlying CFT. The correlator (3.60) admits six different tensor structures. An independent basis of tensor structures for the 6D uplift of eq. (3.60) can be found using the relation (2.83). A possible choice is

$$\langle\Psi(X_1,\bar S_1)\,\bar\Psi(X_2,S_2)\,\bar{\mathcal{X}}(X_3,S_3)\,\mathcal{X}(X_4,\bar S_4)\rangle = \frac{1}{X_{12}^{\Delta_\psi+\frac12}X_{34}^{\Delta_\chi+\frac12}}\Big(g_1(U,V)I^{12}I^{43}+ \tag{3.61}$$
$$g_2(U,V)I^{42}I^{13}+g_3(U,V)I^{12}J_{43,21}(\hat I^{43}_{21})+g_4(U,V)I^{42}(\hat I^{13}_{24})+g_5(U,V)I^{43}(\hat I^{12}_{34})+g_6(U,V)I^{13}(\hat I^{42}_{31})\Big).$$

For $l\geq 1$, four CPW $W^{(p,q)}_{O^{l,l}}$ $(p,q=1,2)$ are associated to the exchange of traceless symmetric fields, and one for each antisymmetric field, $W_{O^{l+2,l}}$ and $W_{O^{l,l+2}}$. Let us start with $W^{(p,q)}_{O^{l,l}}$. The traceless symmetric CPW are obtained as usual by relating the three point function of two fermions and one $O^{l,l}$

[9] Notice that the scalings dimension Δ_1 and Δ_2 in eq.(3.58) do not exactly correspond in general to those of the external operators, but should be identified with Δ_1' and Δ_2' in eq.(3.51). It might happen that the coefficient c_n vanishes for some values of Δ_1 and Δ_2. As we already pointed out, there is some redundancy that allows us to choose a different set of operators. Whenever this coefficient vanishes, we can choose a different operator, e.g. $\widetilde{D}_1\to D_1$.

to that of two scalars and one $O^{l,l}$. This relation requires to use the operator (3.23). There are two tensor structures for $l \geq 1$:

$$\begin{aligned} \langle \Psi(\bar{S}_1)\bar{\Psi}(S_2)O^{l,l}\rangle_1 &= \mathcal{K} I^{12} J^l_{0,12} = I^{12}\langle \Phi^{\frac{1}{2}}\Phi^{\frac{1}{2}} O^{l,l}\rangle_1 , \\ \langle \Psi(\bar{S}_1)\bar{\Psi}(S_2)O^{l,l}\rangle_2 &= \mathcal{K} I_{10} I_{02} J^{l-1}_{0,12} = \frac{1}{16 l(\Delta-1)}\nabla_{21}\Big(\widetilde{D}_2\widetilde{D}_1 + \kappa I^{12}\Big)\langle \Phi^{\frac{1}{2}}\Phi^{\frac{1}{2}} O^{l,l}\rangle_1 , \end{aligned} \tag{3.62}$$

where $\kappa = 2\big(4\Delta - (\Delta+l)^2\big)$, the superscript n in Φ indicates the shift in the scaling dimensions of the field and the operator $O^{l,l}$ is taken at X_0. Plugging eq.(3.62) (and the analogous one for $\mathcal{X}$ and $\bar{\mathcal{X}}$) in eq.(3.10) gives the relation between CPW. In order to simplify the equations, we report below the CPW in the differential basis, the relation with the ordinary basis being easily determined from eq.(3.62):

$$\begin{aligned} W^{(1,1)}_{O^{l,l}} &= I^{12} I^{12} I_{43} W^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{seed}(0) , \\ W^{(1,2)}_{O^{l,l}} &= I^{12}\nabla_{34}\widetilde{D}_4\widetilde{D}_3 W^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{seed}(0) , \\ W^{(2,1)}_{O^{l,l}} &= I_{43}\nabla_{21}\widetilde{D}_2\widetilde{D}_1 W^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{seed}(0) , \\ W^{(2,2)}_{O^{l,l}} &= \nabla_{21}\widetilde{D}_2\widetilde{D}_1\nabla_{34}\widetilde{D}_4\widetilde{D}_3 W^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{seed}(0) , \end{aligned} \tag{3.63}$$

where $\widetilde{D}_3$ and $\widetilde{D}_4$ are obtained from $\widetilde{D}_1$ and $\widetilde{D}_2$ in eq.(3.14) by replacing $1 \to 3$ and $2 \to 4$ respectively. The superscripts indicate again the shift in the scaling dimensions of the external operators. As in ref.[17] the CPW associated to the exchange of traceless symmetric fields is entirely determined in terms of the single known CPW of four scalars $W_{seed}(0)$. For illustrative purposes, we report here the explicit expressions of $W^{(1,2)}_{O^{l,l}}$:

$$\begin{aligned} \mathcal{K}_4^{-1} W^{(1,2)}_{O^{l,l}} = 8 I^{12} I_{43} \Big(& U(V-U-2)\partial_U + U^2(V-U)\partial_U^2 + \big(V^2-(2+U)V+1\big)\partial_V + \\ & V\big(V^2-(2+U)V+1\big)\partial_V^2 + 2UV(V-U-1)\partial_U\partial_V\Big)\mathcal{G}_0^{(0)} \\ + 4 U I^{12} J_{43,21} \Big(& U\partial_U + U^2\partial_U^2 + (V-1)\partial_V + V(V-1)\partial_V^2 + 2UV\partial_U\partial_V\Big)\mathcal{G}_0^{(0)} , \end{aligned} \tag{3.64}$$

where $\mathcal{G}_0^{(0)}$ are the known scalar conformal blocks [8, 9]. It is worth noting that the relations (2.76)-(2.83) have to be used to remove redundant structures and write the above result (3.64) in the chosen basis (3.61).

The analysis for the antisymmetric CPW $W_{O^{l+2,l}}$ and $W_{O^{l,l+2}}$ is simpler. The three point function of two fermions and one $O^{l,l+2}$ field has a unique tensor structure, like the one of a scalar and a $(2,0)$ field F. One has

$$\begin{aligned} \langle \Psi(\bar{S}_1)\bar{\Psi}(S_2)O^{l+2,l}\rangle_1 &= \mathcal{K} I_{10} K_{1,20} J^l_{0,12} = \frac{1}{4}\bar{d}_2\langle \Phi^{\frac{1}{2}} F^{\frac{1}{2}} O^{l+2,l}\rangle_1 , \\ \langle \Psi(\bar{S}_1)\bar{\Psi}(S_2)O^{l,l+2}\rangle_1 &= \mathcal{K} I_{02} \overline{K}_{2,10} J^l_{0,12} = \frac{1}{2}\bar{d}_2\langle \Phi^{\frac{1}{2}} F^{\frac{1}{2}} O^{l,l+2}\rangle_1 \end{aligned} \tag{3.65}$$

and similarly for the conjugate $(0,2)$ field $\bar{F}$. Using the above relation, modulo an irrelevant constant factor, we get

$$\begin{aligned} W_{O^{l+2,l}} &= \bar{d}_2 d_4 W^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{seed}(1) , \\ W_{O^{l,l+2}} &= \bar{d}_2 d_4 \overline{W}^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{seed}(1) , \end{aligned} \tag{3.66}$$

where $W_{seed}(1)$ and $\overline{W}_{seed}(1)$ are defined in eq.(3.55). Explicitly, one gets

$$\begin{aligned}\frac{\sqrt{U}}{4}\mathcal{K}_4^{-1}W_{O^{l+2,l}} =& I^{12}I_{43}\Big(\mathcal{G}_2^{(1)}+(V-U-1)\mathcal{G}_1^{(1)}+4U\mathcal{G}_0^{(1)}\Big)-4UI_{42}I^{13}\mathcal{G}_1^{(1)}+UI^{12}J_{43,21}\mathcal{G}_1^{(1)}\\ &-UI_{42}J_{13,24}\mathcal{G}_2^{(1)}+UI_{43}J_{12,34}\mathcal{G}_1^{(1)}-4UI^{13}(\hat{I}^{42}_{31})\mathcal{G}_0^{(1)}\,.\end{aligned} \tag{3.67}$$

The same applies for $W_{O^{l,l+2}}$ with $\mathcal{G}_n^{(1)} \to \overline{\mathcal{G}}_n^{(1)}$. The expression (3.67) shows clearly how the six conformal blocks entering $W_{O^{l,l+2}}$ are completely determined in terms of the three $\mathcal{G}_n^{(1)}$.

4. Computing Seed Conformal Blocks

Now we are at the last step of obtaining general CBs in 4D CFT. In the last chapter 3, we have seen how to relate, by means of differential operators, mixed tensor CBs appearing in an arbitrary spinor/tensor 4-point correlator to a basis of minimal mixed tensor CBs. These "seed" blocks arise from 4-point functions involving two scalars and two tensor fields in the $(0,p)$ and $(p,0)$ representations of the Lorentz group, with p an arbitrary integer. Such 4-point functions are the simplest ones (i.e. with the least number of tensor structures) where $(\ell+p,\ell)$ or $(\ell,\ell+p)$ mixed symmetry (bosonic or fermionic) tensors can be exchanged in some OPE limit, for any ℓ.

The aim of this chapter is to compute the those "seed" CBs. We set out to solve the Casimir system of differential equations ($p+1$ coupled equations), but we first use the shadow formalism to get an educated ansatz. Using this ansatz, we manage to reduce the Casimir second-order differential equations to a system of algebraic equations and get the blocks in a closed analytic form.

We start in section 4.1 where we summarize the results of the last chapter, generalizing to the case where p can be odd as well as even integer. In section 4.2 and derive the system of $p+1$ Casimir equations satisfied by the CBs $G_e^{(p)}$, for any p. Next, in section 4.3, we use the shadow formalism to deduce the asymptotic behaviour of the CBs from which we write an ansatz for the CBs in section 4.4. Finally, we insert the ansatz into the Casimir system of equations and solve it for any p and ℓ, using generalizations of the methods introduced in ref.[9] (and further refined in ref.[49]) to compute 6D symmetric CBs for scalar correlators. Like scalar blocks in higher even dimensions, the mixed tensor CBs are found using an ansatz given by a sum of hyper-geometric functions with unknown coefficients $c^e_{m,n}$. In this way a system of $p+1$ linear coupled differential equations of second order in two variables is reduced to an algebraic linear system for $c^e_{m,n}$. The set of non-trivial coefficients $c^e_{m,n}$, determined by solving the linear system, admits a useful geometric interpretation. They span a two-dimensional lattice in the (m,n) plane. For large p, the total number of coefficients $c^e_{m,n}$ grows like p^3 and their explicit form becomes more and more complicated as p increases. We point out that a similar geometric interpretation applies also to the set of non-trivial coefficients $x_{m,n}$ entering the solution for the symmetric scalar blocks in even number of dimensions.

4.1 Deconstructing Conformal Partial Waves

It has been shown in chapter 3 that the CPWs associated to an operator $\mathcal{O}^{(\ell,\ell+p)}$ (and similarly for its conjugate $\overline{\mathcal{O}}^{(\ell+p,\ell)}$) exchanged in the OPE channel $(12)(34)$ of a 4-point function $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle$, can be obtained from a single CPW $W^{seed}_{\mathcal{O}^{(\ell,\ell+p)}}$ as follows:

$$W^{(i,j)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4,\mathcal{O}^{(\ell,\ell+p)}} = \mathcal{D}^i_{12}\mathcal{D}^j_{34}W^{seed}_{\mathcal{O}^{(\ell,\ell+p)}}\,, \tag{4.1}$$

where $\mathcal{D}^i_{12}$ and $\mathcal{D}^i_{34}$ are differential operators that depend on $\mathcal{O}_{1,2}$ and $\mathcal{O}_{3,4}$, respectively. For even integer $p=2n$, the seed CPWs are those associated to 4-point functions of two scalar fields with one $(2n,0)$ and one $(0,2n)$ bosonic operators, while for odd integer $p=2n+1$, they consist of 4-point functions of two scalar fields with one $(2n+1,0)$ and one $(0,2n+1)$ fermionic operators:

$$\langle\phi_1(x_1)F_{2,\alpha_1\alpha_2\ldots\alpha_{2n}}(x_2)\phi_3(x_3)\overline{F}_4^{\dot\beta_1\dot\beta_2\ldots\dot\beta_{2n}}(x_4)\rangle\,,\qquad p=2n\,, \tag{4.2}$$

$$\langle\phi_1(x_1)\psi_{2,\alpha_1\alpha_2\ldots\alpha_{2n+1}}(x_2)\phi_3(x_3)\overline{\psi}_4^{\dot\beta_1\dot\beta_2\ldots\dot\beta_{2n+1}}(x_4)\rangle\,,\quad p=2n+1\,. \tag{4.3}$$

In the above correlators, in the OPE channel $\langle(12)(34)\rangle$ primary operators $\mathcal{O}^{(\ell,\ell+\delta)}$ and their conjugates $\overline{\mathcal{O}}^{(\ell+\delta,\ell)}$ can be exchanged only with the values of δ indicated in eq. (3.4) and any ℓ. There are several 4-point functions in which the operators $\mathcal{O}^{(\ell,\ell+p)}$ and $\overline{\mathcal{O}}^{(\ell+p,\ell)}$ are exchanged and in which the corresponding CPWs have a unique structure. Among these, the correlators (4.2) and (4.3) are the ones with the minimum number of tensor structures and hence the simplest. This is understood by noticing that for *any* value of δ (and not only for $\delta = p$) the operators $\mathcal{O}^{(\ell,\ell+\delta)}$ and their conjugates $\overline{\mathcal{O}}^{(\ell+\delta,\ell)}$ appear in both the (12) and (34) OPE's with one tensor structure only, since there is only one tensor structure in the corresponding three-point functions:

$$\langle\phi(x_1)F_{\alpha_1\ldots\alpha_{2n}}(x_2)\mathcal{O}^{\dot\beta_1\ldots\dot\beta_{\ell+\delta}}_{\alpha_1\ldots\alpha_\ell}(x_0)\rangle\,,\qquad \langle\overline{\mathcal{O}}^{\dot\beta_1\ldots\dot\beta_\ell}_{\alpha_1\ldots\alpha_{\ell+\delta}}(x_0)\phi(x_3)\overline{F}^{\dot\beta_1\ldots\dot\beta_{2n}}(x_4)\rangle\,, \tag{4.4}$$

$$\langle\phi(x_1)\psi_{\alpha_1\ldots\alpha_{2n+1}}(x_2)\mathcal{O}^{\dot\beta_1\ldots\dot\beta_{\ell+\delta}}_{\alpha_1\ldots\alpha_\ell}(x_0)\rangle\,,\quad \langle\overline{\mathcal{O}}^{\dot\beta_1\ldots\dot\beta_\ell}_{\alpha_1\ldots\alpha_{\ell+\delta}}(x_0)\phi(x_3)\overline{\psi}^{\dot\beta_1\ldots\dot\beta_{2n+1}}(x_4)\rangle\,. \tag{4.5}$$

This implies then that the number of 4-point tensor structures appearing in eqs.(4.2) and (4.3) is the minimum possible and equals to $N_4 = p+1$.

Summarizing, the problem of computing CPWs and CBs associated to the exchange of mixed symmetry operators $\mathcal{O}^{(\ell,\ell+p)}$ and $\overline{\mathcal{O}}^{(\ell+p,\ell)}$ in any 4-point function is reduced to the computation of the $p+1$ CBs appearing in the decomposition of $W^{seed}_{\mathcal{O}^{(\ell,\ell+p)}}$ and $\overline{W}^{seed}_{\mathcal{O}^{(\ell+p,\ell)}}$.

Despite this simplification, the above computation is still technically challenging. A further great simplification occurs by using the 6D embedding formalism of chapter 2.

An independent basis for the $p+1$ tensor structures appearing in the 6D uplift of the correlators (4.2) and (4.3) can be given as:

$$\langle\Phi_1(X_1)F_2^{(p,0)}(X_2,S_2)\Phi_3(X_3)\overline{F}_4^{(0,p)}(X_4,\overline{S}_4)\rangle = \mathcal{K}_4\sum_{n=0}^{p} g_n(U,V)(I^{42})^n(\hat I^{42}_{31})^{p-n}\,, \tag{4.6}$$

where I^{42} , $\hat I^{42}_{31}$, $\mathcal{K}_4$, U and V are defined in section 2.3.

We denote the 6D seed CPW associated to the exchange of the fields $O^{(\ell,\ell+p)}$ and $\overline{O}^{(\ell+p,\ell)}$ in the 4-point function (4.6) by $W^{seed}(p)$ and $\overline{W}^{seed}(p)$, respectively. They are parametrized in terms of $p+1$ CBs as follows:

$$\begin{aligned} W^{seed}(p) &= \mathcal{K}_4\sum_{e=0}^{p} G_e^{(p)}(U,V)(I^{42})^e(\hat I^{42}_{31})^{p-e},\\ \overline{W}^{seed}(p) &= \mathcal{K}_4\sum_{e=0}^{p} \overline{G}_e^{(p)}(U,V)(I^{42})^e(\hat I^{42}_{31})^{p-e}. \end{aligned} \tag{4.7}$$

For simplicity, we have dropped in eq.(4.7) the dependence of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ on Δ and ℓ. The CBs depend also on the external operator dimensions, more precisely on a and b, defined as

$$a \equiv \frac{\tau_2-\tau_1}{2} = \frac{\Delta_2-\Delta_1}{2}+\frac{p}{4}\,,\qquad b \equiv \frac{\tau_3-\tau_4}{2} = \frac{\Delta_3-\Delta_4}{2}-\frac{p}{4}\,. \tag{4.8}$$

For simplicity of notation, we no longer distinguish between even and odd values of p, since we can consider both cases simultaneously. It is then understood that in the corrrelator (4.6) $F_2^{(p,0)}$ and $\overline{F}_4^{(0,p)}$ are 6D uplifts of 4D fermion fields for p odd.

It is possible to get $W^{seed}(p)$ from $\overline{W}^{seed}(p)$, or vice versa, using the results of chapter 3 and a parity transformation $\mathcal{P}$. We have

$$\overline{W}^{seed}(p) = \mathcal{P}\; W_{\Phi_1 \overline{F}_2 \Phi_3 F_4, O^{(\ell,\ell+p)}} \,, \tag{4.9}$$

where

$$W_{\Phi_1 \overline{F}_2 \Phi_3 F_4, O^{(\ell,\ell+p)}} = \frac{1}{2^{2p}\,(p!)^2}\Big(\prod_{n=1}^{p} c_n\Big)\; (\nabla_{12}\bar{d}_1\widetilde{D}_1)^p(\nabla_{43}d_3\widetilde{D}_3)^p W^{seed}(p)\Big|_{a\to a-\frac{p}{2},\, b\to b+\frac{p}{2}} \tag{4.10}$$

is the CPW associated to the parity dual 4-point function $\langle \Phi_1 \overline{F}_2^{(0,p)} \Phi_3 F_4^{(p,0)} \rangle$, and

$$(c_n)^{-1} = (4+3p-2a-\kappa-2n)(4+3p+2b-\kappa-2n)\,, \quad \kappa = \Delta+\ell+\frac{p}{2}\,. \tag{4.11}$$

In fact, we will not use eq.(4.9) to compute $\overline{W}^{seed}(p)$, because we will find an easier way to directly compute both $W^{seed}(p)$ and $\overline{W}^{seed}(p)$.

Instead of eq.(4.6), we could have considered the alternative 4-point function

$$\langle \Phi_1(X_1) F_2^{(p,0)}(X_2) \overline{F}_3^{(0,p)}(X_3) \Phi_4(X_4) \rangle \tag{4.12}$$

to calculate an analogue seed CPW $\widetilde{W}^{seed}(p)$. Since eq.(4.12) is equal to eq.(4.6) under the permutation $3 \leftrightarrow 4$, the CBs appearing in the decomposition of $W^{seed}(p)$ and $\widetilde{W}^{seed}(p)$ are related as follows:

$$\widetilde{G}_e^{(p)}(U,V;a,b) = V^a G_e^{(p)}\Big(\frac{U}{V},\frac{1}{V};a,-b\Big)\,, \quad e=0,\ldots,p\,. \tag{4.13}$$

The 4D CPWs $W^{seed}_{\mathcal{O}(\ell,\ell+p)}$ and $\overline{W}^{seed}_{\mathcal{O}(\ell+p,\ell)}$ are obtained by projecting to 4D their 6D counterparts $W^{seed}(p)$ and $\overline{W}^{seed}(p)$. There is no need to explicitly perform such projection, because the 4D CBs are directly identified with their 6D counterparts. One has simply

$$G_e^{(p)}(U,V) = G_e^{(p)}(u,v)\,, \quad \overline{G}_e^{(p)}(U,V) = \overline{G}_e^{(p)}(u,v)\,, \tag{4.14}$$

where $G_e^{(p)}(u,v)$ and $\overline{G}_e^{(p)}(u,v)$ are the 4D CBs entering the r.h.s. of eq.(3.7) when expanding the 4D CPWs $W^{seed}_{\mathcal{O}(\ell,\ell+p)}$ and $\overline{W}^{seed}_{\mathcal{O}(\ell+p,\ell)}$.

4.2 The System of Casimir Equations

In this section we derive the system of second order Casimir equations for the seed conformal blocks defined in eq. (4.7). Before addressing the more complicated case of interest, let us recall how the Casimir equation for scalar correlators is derived. One starts by considering the 4-point function

$$\langle [\hat{C}, \phi_1(x_1)\phi_2(x_2)]\phi_3(x_3)\phi_4(x_4)\rangle\,, \tag{4.15}$$

where $\hat{C}$ is the quadratic Casimir operator.[1] Recasting the generators of the 4D conformal group in a 6D form as $\hat{L}_{MN}$, with M,N 6D indices, we have

$$\hat{C} = \frac{1}{2}\hat{L}_{MN}\hat{L}^{MN}\,. \tag{4.16}$$

[1] CBs satisfy also higher order equations obtained by means of higher Casimir invariants. We will not consider them here, since the quadratic Casimir will be enough for us to find the CB's.

The Casimir equation is derived by expressing eq.(4.15) in two different ways. On one hand, we can replace in eq.(4.16) the operator $\hat{L}_{MN}$ with its explicit action in terms of differential operators acting on the scalar fields inserted at the points x_1 and x_2: $[\hat{L}_{MN}, \phi(x)] = L_{MN}(x,\partial)\phi(x)$. On the other hand, we might consider the (12) OPE. Scalar operators can only exchange symmetric traceless operators, so $p=0$ in this case, and one has

$$\phi_1(x_1)\phi(x_2) = \sum_{\mathcal{O}^{(\ell,\ell)}} \lambda_{\phi_1\phi_2\mathcal{O}} \mathcal{T}^{\mu_1\ldots\mu_\ell} \mathcal{O}^{(\ell,\ell)}_{\mu_1\ldots\mu_\ell}(x_2) + \text{descendants}\,, \tag{4.17}$$

where $\mathcal{T}$ is a tensor structure factor whose explicit form will not be needed. In the latter view, we end up having the commutator of $\hat{C}$ with $\mathcal{O}^{(\ell,\ell)}$ which gives the Casimir eigenvalue

$$[\hat{C}, \mathcal{O}^{(\ell,\ell)}(x)] = E^0_\ell \mathcal{O}^{(\ell,\ell)}(x) \tag{4.18}$$

where

$$E^p_\ell = \Delta\,(\Delta-4) + \ell^2 + (2+p)(\ell+\frac{p}{2}) \tag{4.19}$$

is the value associated to an operator in the $(\ell+p,\ell)$ or $(\ell,\ell+p)$ Lorentz representations. Using then eq.(3.3) one derives a differential equation for each CPW, for any fixed Δ and ℓ.

The explicit form of the second order differential operator acting on the CPW or directly on the CB is best derived in the $4+2$-dimensional embedding space. The CPW of scalar correlators is parametrized by a single conformal block $G_0^{(0)}(z,\bar{z})$. When acting on scalar operators at x_1 and x_2, the Lorentz generator can be written as $L_{MN} = L_{1,MN} + L_{2,MN}$, where

$$L_{i\,MN} = i\Big(X_{i\,M}\frac{\partial}{\partial X_i^N} - X_{i\,N}\frac{\partial}{\partial X_i^M}\Big)\,. \tag{4.20}$$

Plugging eq.(4.20) in eq.(4.16), one finds after a bit of algebra the Casimir equation [9]

$$\Delta_2^{(a,b;0)} G_0^{(0)}(z,\bar{z}) = \frac{1}{2} E^0_\ell G_0^{(0)}(z,\bar{z})\,, \tag{4.21}$$

where a and b are defined in eq.(4.8), $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$. The second-order differential operator Δ is defined as

$$\Delta_\epsilon^{(a,b;c)} = D_z^{(a,b;c)} + D_{\bar{z}}^{(a,b;c)} + \epsilon\,\frac{z\bar{z}}{z-\bar{z}}\Big((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\Big)\,, \tag{4.22}$$

in terms of the second-order holomorphic operator

$$D_z^{(a,b;c)} \equiv z^2(1-z)\partial_z^2 - \big((a+b+1)z^2 - cz\big)\partial_z - abz\,. \tag{4.23}$$

The above derivation can be generalized for CPWs entering 4-point correlators of tensor fields. As we have seen in section 4.1, in the most general case the exchange of a given field $\mathcal{O}^{(\ell,\bar{\ell})}$ is not parametrized by a single CPW, but by a set of CPWs $W^{(i,j)}$, whose number depends on the number of tensor structures defining the three-point functions $(12\mathcal{O})$ and $(34\overline{\mathcal{O}})$. In order to derive the second order differential equation satisfied by $W^{(i,j)}$ one has to properly identify the OPE coefficients λ^i appearing in the generalization of eq.(4.17) with those in eq.(3.3). This is not needed for the seed correlators (4.6) since the CPW is unique, like in the scalar correlator. For each p, we have

$$C\,W^{seed}(p) = E^p_\ell\,W^{seed}(p), \tag{4.24}$$

where C is the explicit differential form of the Casimir operator to be determined and E^p_ℓ is as in eq.(4.19). An identical equation is satisfied by $\overline{W}^{seed}(p)$. Contrary to the scalar case, the single differential equation (4.24) for $W^{seed}(p)$ turns into a system of equations for the $p+1$ CBs $G_e^{(p)}$. Let us see how this system of equations can be derived for any p.

The action of the Lorentz generators $L_{i,MN}$ on tensor fields should include, in addition to the orbital contribution (4.20), the spin part. one can label the 6D spin representations by two integers $(s,\,\bar{s})$ which count the number of twistor indices in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations respectively (see Appendix A for details and our conventions). The Lorentz generators acting on generic 6D fields in the $(s,\,\bar{s})$ representation are then given by

$$\begin{aligned}
[L_{i\,MN}]_{a_1..\,a_{\bar{s}};\,d_1..\,d_s}^{b_1..\,b_s;\,c_1..\,c_{\bar{s}}} &= i(X_{i\,M}\partial_{i\,N}-X_{i\,N}\partial_{i\,M})(\delta_{a_1}^{c_1}..\,\delta_{a_{\bar{s}}}^{c_{\bar{s}}})(\delta_{d_1}^{b_1}..\,\delta_{d_s}^{b_s}) \\
&+ i\Big(\,[\Sigma_{MN}]_{a_1}^{c_1}\delta_{a_2}^{c_2}..\delta_{a_{\bar{s}}}^{c_{\bar{s}}}+[\Sigma_{MN}]_{a_2}^{c_2}\delta_{a_1}^{c_1}..\delta_{a_{\bar{s}}}^{c_{\bar{s}}}+..\Big)\delta_{d_1}^{b_1}..\delta_{d_s}^{b_s} \\
&+ i\Big(\,[\overline{\Sigma}_{MN}]_{d_1}^{b_1}\delta_{d_2}^{b_2}..\delta_{d_s}^{b_s}+[\overline{\Sigma}_{MN}]_{d_2}^{b_2}\delta_{d_1}^{b_1}..\delta_{d_s}^{b_s}+..\Big)\delta_{a_1}^{c_1}..\delta_{a_{\bar{s}}}^{c_{\bar{s}}}\,.
\end{aligned} \tag{4.25}$$

We can get rid of all the twistor indices by defining the index-free Lorentz generators

$$L_{iMN}=i(X_{iM}\partial_{iN}-X_{iN}\partial_{iM})+i(S_i\overline{\Sigma}_{MN}\partial_{S_i})+i(\bar{S}_i\Sigma_{MN}\partial_{\bar{S}_i}). \tag{4.26}$$

Given any 6D tensor $O(X,S,\bar{S})$, we have

$$[\hat{L}_{MN},O_i(X_i,S_i,\bar{S}_i)]=L_{iMN}O_i(X_i,S_i,\bar{S}_i)\,, \tag{4.27}$$

where $\hat{L}_{MN}$ satisfy the Lorentz algebra

$$[\hat{L}_{MN},\hat{L}_{RS}]=i\Big(\eta_{MS}\hat{L}_{NR}+\eta_{NR}\hat{L}_{MS}-\eta_{MR}\hat{L}_{NS}-\eta_{NS}\hat{L}_{MR}\Big). \tag{4.28}$$

The explicit form of the Casimir differential operator entering eq.(4.24) is obtained by plugging eq.(4.26) in eq.(4.16). The single equation (4.24) for the CPW turns into a system of second-order coupled differential equations for the $p+1$ conformal blocks $G_e^{(p)}$, $e=0,\dots,p$, since the coefficients multiplying the $p+1$ tensor structures in eq.(4.7) should vanish independently. Schematically

$$(C-E^p_\ell)\Big(\mathcal{K}_4\sum_{e=0}^{p}G_e^{(p)}(U,V)(I^{42})^e(\hat{I}^{42}_{31})^{p-e}\Big)=\mathcal{K}_4\sum_{e=0}^{p}Cas_e^{(p)}(G)(I^{42})^e(\hat{I}^{42}_{31})^{p-e}=0\ \Rightarrow Cas_e^{(p)}(G)=0\,, \tag{4.29}$$

where $Cas_e^{(p)}(G)$ are the $p+1$ Casimir equations, in general each one involving all conformal blocks $G_e^{(p)}$. Determining the Casimir system $Cas_e^{(p)}(G)$ is conceptually straightforward but technically involved. The main complication arises from the spin part of the Lorentz generator (4.26) that generates products of $SU(2,2)$ invariants not present in eq.(4.7). The new invariants are linearly dependent and must be eliminated using relations among them. See section 2.4.2 for a list of such relations. This is a lengthy step, that however can be automatized in a computer. When redundant structures have been eliminated, one is finally able to read from eq.(4.29) the Casimir system $Cas_e^{(p)}(G)$. Despite the complicacy of the computation, the final system of $p+1$ equations can be written into the following remarkably compact form:

$$Cas_e^{(p)}(G)=\Big(\Delta_{2+p}^{(a_e,b_e;c_e)}-\frac{1}{2}\left(E^p_\ell-\varepsilon^p_e\right)\Big)\,G_e^{(p)}+A^p_e\,z\bar{z}\,L(a_{e-1})\,G_{e-1}^{(p)}+B_e\;L(b_{e+1})\,G_{e+1}^{(p)}=0\,, \tag{4.30}$$

where $e = 0, \ldots, p$,

$$\varepsilon_e^p \equiv \tfrac{3}{4}\,p^2 - (1+2e)\,p + 2e\,(2+e), \quad A_e^p \equiv 2(p-e+1), \quad B_e \equiv \frac{e+1}{2}\,, \tag{4.31}$$

and the coefficients E_ℓ^p are given in eq.(4.19). In eq.(4.30) it is understood that $G_{-1}^{(p)} = G_{p+1}^{(p)} = 0$. An identical system of equations is satisfied by the conjugate CBs $\overline{G}_e^{(p)}$. Interestingly enough, only two differential operators enter into the Casimir system: the second-order operator (4.22) that already features $p = 0$, with coefficients a_e, b_e and c_e given by

$$a_e \equiv a, \;\; b_e \equiv b + (p-e), \;\; c_e \equiv p - e\,, \tag{4.32}$$

and the new linear operator $L(\mu)$ given by

$$L(\mu) \equiv -\frac{1}{z-\bar z}\,\Big(z(1-z)\partial_z - \bar z(1-\bar z)\partial_{\bar z} \Big) + \mu. \tag{4.33}$$

Another remarkable property of the Casimir system (4.30) is that, for each given e and p, at most three conformal blocks mix with each other in a sort of "nearest-neighbour interaction": G_e mixes only with G_{e+1} and G_{e-1}. The Casimir equations at the "boundaries" $Cas_0^{(p)}$ and $Cas_p^{(p)}$ involve just two blocks. For $p = 0$, the second and third terms in eq.(4.30) vanish and the system trivially reduces to the single equation (4.21).

Finding the solution of the system (4.30) is a complicated task, that we address in the next sections.

4.3 Shadow Formalism

Another method to obtain CBs in closed analytical form uses the so called shadow formalism. It was first introduced by Ferrara, Gatto, Grillo, and Parisi [44, 45, 46, 47] and used in ref.[8] to get closed form expressions for the scalar CBs. In this section we apply the shadow formalism, using the recent formulation given in ref.[18], to get compact expressions for $W^{seed}(p)$ and $\overline{W}^{seed}(p)$ in an integral form for any p and ℓ.[2] Using these expressions, we compute the CBs $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $\ell = 0$ and generic p. We then provide a practical way to obtain $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for any ℓ in a compact form. We finally use this method to compute $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $p = 1$ and $G_e^{(p)}$ for $p = 2$ explicitly.

Despite the power of the above technique, it is computationally challenging to go beyond the $p = 2$ case. Moreover, as we will see, we do not have any control on the final analytic form of CBs. In light of this, we will provide the full analytic solution for $G_e^{(p)}$ and $\overline{G}_e^{(p)}$, for any p, only in section 4.4, where we solve directly the set of Casimir differential equations by using an educated ansatz for the solution. The results obtained in this section are however of essential help to argue the proper ansatz. They will also allow us to get the correct physical asymptotic behaviour of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ that will be used as boundary conditions to solve the Casimir system of equations (4.30). Finally, the explicit computation of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $p = 1$ and $G_e^{(p)}$ for $p = 2$ using the shadow formalism provides an important consistency check for the validity of the full general solution (4.112) to be found in section 4.4.

[2] The shadow formalism given in an index-free 6D embedding twistor space has also been used in refs.[50, 51] to compute CBs in supersymmetric CFTs.

4.3.1 CPW in Shadow Formalism

We start by briefly reviewing the shadow formalism along the lines of ref.[18], where the reader can find more details. The CPW associated to the exchange of a given operator O_r with spin $(\ell,\bar{\ell})$ in a correlator of four operators $O_n(X_n)$, $n=1,2,3,4$ (in embedding space and twistor language) is given by

$$W^{(i,j)}_{O^{(\ell,\bar{\ell})}}(X_i) = \nu \int d^4X_0 \langle O_1(X_1)O_2(X_2)O_r(X_0,S,\bar{S})\rangle_i \overleftrightarrow{\Pi}_{\ell,\bar{\ell}} \langle \widetilde{O}_r(X_0,T,\bar{T})O_3(X_3)O_4(X_4)\rangle_j \Big|_M , \tag{4.34}$$

where ν is a normalization factor, the projector gluing two 3-point functions is given by

$$\overleftrightarrow{\Pi}_{\ell,\bar{\ell}} = (\overleftarrow{\partial}_S X_0 \overrightarrow{\partial}_T)^{\ell} (\overleftarrow{\partial}_{\bar{S}} \overline{X}_0 \overrightarrow{\partial}_{\bar{T}})^{\bar{\ell}} , \tag{4.35}$$

and $\widetilde{O}_r$ is the shadow operator

$$\widetilde{O}_r(X,S,\bar{S}) \equiv \int d^4Y \frac{1}{(-2X\cdot Y)^{4-\Delta+\ell+\bar{\ell}}} O_{\bar{r}}(Y, Y\bar{S}, \bar{Y}S) . \tag{4.36}$$

In eq.(4.34) we have omitted for simplicity the dependence of O_n on their auxiliary twistors S_n, $\bar{S}_n$, and the subscripts i and j in $\langle O_1 O_2 O_r\rangle$ and $\langle \widetilde{O}_r O_3 O_4\rangle$ denote the three point functions stripped of their OPE coefficients:

$$\langle O_1 O_2 O_3\rangle \equiv \sum_i \lambda^i_{O_1O_2O_3} \langle O_1 O_2 O_3\rangle_i . \tag{4.37}$$

The integral in eq.(4.34) would actually determine the CPW associated to the operator $O_r(X,S,\bar{S})$ plus its unwanted shadow counterpart, that corresponds to the exchange of a similar operator but with the scaling dimension $\Delta \to 4-\Delta$. The two contributions can be distinguished by their different behaviour under the monodromy transformation $X_{12} \to e^{4\pi i}X_{12}$. In particular, the physical CPW should transform with the phase $e^{2i\pi(\Delta-\Delta_1-\Delta_2)}$, independently of the Lorentz quantum numbers of the external and exchanged operators. This projection on the correct monodromy component explains the subscript M in the bar at the end of eq.(4.34).

We use eq.(4.34) to get an integral form of $W^{seed}(p)$ and $\overline{W}^{seed}(p)$ in eq.(4.7). The explicit expressions of the needed 3-point functions are given by

$$\begin{aligned} \langle \Phi_1(X_1)F_2(X_2)O^{(\ell,\ell+p)}(X_0)\rangle &= \mathcal{K}_3(\tau_1,\tau_2,\tau) I^p_{02} J^{\ell}_{0,12} , \\ \langle \Phi_1(X_1)F_2(X_2)\overline{O}^{(\ell+p,\ell)}(X_0)\rangle &= \mathcal{K}_3(\tau_1,\tau_2,\tau) K^p_{1,02} J^{\ell}_{0,12} , \end{aligned} \tag{4.38}$$

where

$$\mathcal{K}_3(\tau_1,\tau_2,\tau_3) = X_{12}^{\frac{\tau_3-\tau_1-\tau_2}{2}} X_{13}^{\frac{\tau_2-\tau_1-\tau_3}{2}} X_{23}^{\frac{\tau_1-\tau_2-\tau_3}{2}} , \tag{4.39}$$

is a kinematic factor and

$$K_{i,jk} \equiv \sqrt{\frac{X_{jk}}{X_{ij}X_{ik}}} S_j \overline{\mathbf{X}}_i S_k , \quad \overline{K}_{i,jk} \equiv \sqrt{\frac{X_{jk}}{X_{ij}X_{ik}}} \bar{S}_j \mathbf{X}_i \bar{S}_k , \quad J_{i,jk} \equiv \frac{1}{X_{jk}} \bar{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k S_i \tag{4.40}$$

are $SU(2,2)$ invariants for three-point functions. The "shadow" 3-point function counterparts are given by

$$\begin{aligned} \langle \widetilde{O}^{(\ell,\ell+p)}(X_0)\Phi_3(X_3)\bar{F}_4(X_4)\rangle &\propto \langle O^{(\ell,\ell+p)}(X_0)\Phi_3(X_3)\bar{F}_4(X_4)\rangle\Big|_{\Delta\to 4-\Delta} = \mathcal{K}_3\Big|_{\Delta\to 4-\Delta} \overline{K}^p_{3,04} J^{\ell}_{0,34} , \\ \langle \widetilde{\overline{O}}^{(\ell+p,\ell)}(X_0)\Phi_3(X_3)\bar{F}_4(X_4)\rangle &\propto \langle \overline{O}^{(\ell+p,\ell)}(X_0)\Phi_3(X_3)\bar{F}_4(X_4)\rangle\Big|_{\Delta\to 4-\Delta} = \mathcal{K}_3\Big|_{\Delta\to 4-\Delta} I^p_{40} J^{\ell}_{0,34} . \end{aligned}$$

Using the above relations, after a bit of algebra, one can write

$$W^{seed}(p) = \frac{\nu}{X_{12}^{a_{12}+\frac{\ell}{2}} X_{34}^{a_{34}+\frac{\ell+p}{2}}} \int D^4X_0 \frac{\mathcal{N}_\ell(p)}{X_{01}^{a_{01}+\frac{\ell}{2}} X_{02}^{a_{02}+\frac{\ell+p}{2}} X_{03}^{a_{03}+\frac{\ell+p}{2}} X_{04}^{a_{04}+\frac{\ell}{2}}}\Big|_{M=1}, \tag{4.41}$$

$$\overline{W}^{seed}(p) = \frac{\overline{\nu}}{X_{12}^{a_{12}+\frac{\ell+p}{2}} X_{34}^{a_{34}+\frac{\ell}{2}}} \int D^4X_0 \frac{\overline{\mathcal{N}}_\ell(p)}{X_{01}^{a_{01}+\frac{\ell+p}{2}} X_{02}^{a_{02}+\frac{\ell}{2}} X_{03}^{a_{03}+\frac{\ell}{2}} X_{04}^{a_{04}+\frac{\ell+p}{2}}}\Big|_{M=1}, \tag{4.42}$$

where

$$\begin{aligned} a_{01} &= \frac{\Delta}{2} + \frac{p}{4} - a, & a_{02} &= \frac{\Delta}{2} - \frac{p}{4} + a, & a_{12} &= \frac{\Delta_1+\Delta_2}{2} - \frac{\Delta}{2}, \\ a_{03} &= \frac{4-\Delta}{2} + \frac{p}{4} + b, & a_{04} &= \frac{4-\Delta}{2} - \frac{p}{4} - b, & a_{34} &= \frac{\Delta_3+\Delta_4}{2} - \frac{4-\Delta}{2}, \end{aligned} \tag{4.43}$$

and

$$\mathcal{N}_\ell(p) \equiv (\bar{S}S_2)^p(\bar{S}X_2\bar{X}_1S)^\ell \overleftrightarrow{\Pi}_{\ell,\ell+p}(\bar{S}_4X_3\bar{T})^p(\bar{T}X_4\bar{X}_3T)^\ell, \tag{4.44}$$

$$\overline{\mathcal{N}}_\ell(p) \equiv (\bar{S}_4S)^p(\bar{S}X_3\bar{X}_4S)^\ell \overleftrightarrow{\Pi}_{\ell+p,\ell}(S_2\overline{X}_1T)^p(\bar{T}X_1\bar{X}_2T)^\ell. \tag{4.45}$$

We will not need to determine the normalization factors ν and $\bar{\nu}$ in eqs.(4.41) and (4.42). Notice that the correct behaviour of the seed CPWs under $X_{12} \to e^{4\pi i}X_{12}$ is saturated by the factor X_{12} multiplying the integrals in eqs.(4.41) and (4.42). Hence the latter should be projected to their trivial monodromy components $M = 1$, as indicated. Notice that eqs.(4.44) and (4.45) are related by a simple transformation:

$$\overline{\mathcal{N}}_\ell(p) = \mathcal{P}\mathcal{N}_\ell(p)\Big|_{1\leftrightarrow3,\ 2\leftrightarrow4}, \tag{4.46}$$

where $\mathcal{P}$ is the parity operator.

We can recast the expression (4.44) in a compact and convenient form using some manipulations. We first define 3 variables

$$s \equiv X_{12}X_{34}\prod_{n=1}^{4}X_{0n},\ t \equiv \frac{1}{2\sqrt{s}}\Big(X_{02}X_{03}X_{14} - X_{01}X_{03}X_{24} - (3\leftrightarrow4)\Big),\ u \equiv \frac{X_{02}X_{03}X_{34}}{\sqrt{s}}. \tag{4.47}$$

Then we look for a relation expressing the generic $\mathcal{N}_\ell(p)$ in terms of the known $\mathcal{N}_\ell(0)$:

$$\mathcal{N}_\ell(0) = (-1)^\ell(\ell!)^4\ s^{\ell/2}C^1_\ell(t)\,, \tag{4.48}$$

where C^p_ℓ are Gegenbauer polynomials of rank p. Starting from eq.(4.44), after acting with the S and T derivatives, one gets

$$\mathcal{N}_\ell(p) = (\ell!)^2(\overrightarrow{\partial}_{\bar{S}}\overline{X}_0\overrightarrow{\partial}_{\bar{T}})^{\ell+p}\left((\bar{S}S_2)^p(\bar{S}_4X_3\bar{T})^p(\bar{S}\Omega\bar{T})^\ell\right), \tag{4.49}$$

where we have defined $\Omega_{ab} = (X_2\bar{X}_1X_0\bar{X}_3X_4)_{ab}$. In order to relate $\mathcal{N}_\ell(p)$ above to $\mathcal{N}_{\ell+p}(0)$ in eq.(4.48), we look for an operator $\widetilde{\mathcal{D}}$ satisfying

$$\widetilde{\mathcal{D}}^p\,(\overrightarrow{\partial}_{\bar{S}}\overline{X}_0\overrightarrow{\partial}_{\bar{T}})^{\ell+p}(\bar{S}\Omega\bar{T})^{\ell+p} = (\overrightarrow{\partial}_{\bar{S}}\overline{X}_0\overrightarrow{\partial}_{\bar{T}})^{\ell+p}\left((\bar{S}S_2)^p(\bar{S}_4X_3\bar{T})^p(\bar{S}\Omega\bar{T})^\ell\right). \tag{4.50}$$

We deduce that $\widetilde{\mathcal{D}}$ should be bilinear in $\bar{S}_4$ and S_2 and should commute with $(\overrightarrow{\partial}_{\bar{S}}\overline{X}_0\overrightarrow{\partial}_{\bar{T}})$. In addition to that, it should have the correct scaling in X's and should be gauge invariant, namely it should be well defined on the light-cone $X^2 = 0$ and preserve the conditions (2.32). It is not difficult to see that the choice $\widetilde{\mathcal{D}} = \mathcal{D}/(8X_{01}X_{04})$, where

$$\mathcal{D} = (\bar{S}_4 X_0 \bar{\Sigma}^N S_2)\frac{\partial}{\partial X_2^N} \tag{4.51}$$

fulfills all the requirements. One has $\widetilde{\mathcal{D}}(\bar{S}\Omega\bar{T}) = (\bar{S}S_2)(\bar{S}_4 X_3 \bar{T})$. Iterating it p times gives the desired relation:

$$\mathcal{N}_\ell(p) \propto \widetilde{\mathcal{D}}^p \mathcal{N}_{\ell+p}(0)\,. \tag{4.52}$$

The operator $\mathcal{D}$ annihilates all the scalar products with the exception of X_{12}, in which case we have $\mathcal{D}X_{12} = I_2$, and we define

$$I_1 \equiv X_{03}\, \hat{I}^{42}_{30}, \qquad I_2 \equiv X_{01}\, \hat{I}^{42}_{01}\,. \tag{4.53}$$

The action on the s, t, and u variables is

$$\mathcal{D}\, s = X_{12}^{-1}\, s\, I_2, \;\; \mathcal{D}\, t = -\frac{1}{2}\, X_{12}^{-1}\, (u^{-1}\, I_1 + t\, I_2), \;\; \mathcal{D}\, u^{-1} = \frac{1}{2}\, X_{12}^{-1}\, u^{-1}\, I_2\,, \tag{4.54}$$

on Gegenbauer polynomials is

$$\mathcal{D}\, C_n^\lambda(t) = 2\lambda\, C_{n-1}^{\lambda+1}(t)\, \mathcal{D}\, t\,, \tag{4.55}$$

and vanishes on $J_{42,01}$ and $J_{42,30}$. Using recursively the identity for Gegenbauer polynomials

$$\frac{n}{2\lambda}\, C_n^\lambda(t) - t\, C_{n-1}^{\lambda+1}(t) = -C_{n-2}^{\lambda+1}(t)\,, \tag{4.56}$$

we can write the following expression for $\mathcal{N}_\ell(p)$:

$$\mathcal{N}_\ell(p) \propto s^{\frac{\ell}{2}} \sum_{w=0}^{p} \binom{p}{w} u^w\, C_{\ell-w}^{p+1}(t)\; I_1^{p-w} I_2^w, \tag{4.57}$$

where $\binom{p}{w}$ is the binomial coefficient and for compactness we have defined the dimensionful tensor structures Combining together eqs.(4.41), (4.42), (4.46), (4.47) and (4.57) we can finally write

$$\begin{aligned} W^{seed}(p) &= \nu' \sum_{w=0}^{p} \binom{p}{w} \frac{1}{X_{12}^{a_{12}+\frac{w}{2}} X_{34}^{a_{34}+\frac{p-w}{2}}} \int D^4 X_0 \frac{C_{\ell-w}^{p+1}(t)\; I_1^{p-w} I_2^w}{X_{01}^{a_{01}+\frac{w}{2}} X_{02}^{a_{02}+\frac{p-w}{2}} X_{03}^{a_{03}+\frac{p-w}{2}} X_{04}^{a_{04}+\frac{w}{2}}}\Bigg|_{M=1}, \\ \overline{W}^{seed}(p) &= \bar{\nu}' \sum_{w=0}^{p} \binom{p}{w} \frac{1}{X_{12}^{a_{12}+\frac{p-w}{2}} X_{34}^{a_{34}+\frac{w}{2}}} \int D^4 X_0 \frac{C_{\ell-w}^{p+1}(t)\; I_1^{w} I_2^{p-w}}{X_{01}^{a_{01}+\frac{p-w}{2}} X_{02}^{a_{02}+\frac{w}{2}} X_{03}^{a_{03}+\frac{w}{2}} X_{04}^{a_{04}+\frac{p-w}{2}}}\Bigg|_{M=1} \end{aligned} \tag{4.58}$$

where ν' and $\bar{\nu}'$ are undetermined normalization factors.

4.3.2 Seed Conformal Blocks and Their Explicit Form for $\ell = 0$

The computation of the CBs $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ starting form eq.(4.58) is a non-trivial task for generic ℓ and p, since we are not aware of a general formula for an integral that involves $C_{\ell-w}^{p+1}(t)$ for $p \neq 0$. For any given ℓ, one can however expand the Gegenbauer polynomial, in which case the CBs $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ can be computed. In this subsection we discuss the structure of CBs for generic ℓ and compute $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $\ell = 0$ and generic p.

Recalling the definition of t in eq.(4.47), one realizes that the Gegenbauer polynomials in eq.(4.58), when expanded, do not give rise to intrinsically new integrals but just amounts to shifting the exponents in the denominator. The tensor structures in the numerators bring p open indices in the form $X_0^{N_1}\ldots X_0^{N_p}$, which can be removed by using eq.(3.21) in ref. [18]. In this way the problem is reduced to the computation of scalar integrals in $2h = 2(2+p)$ effective dimensions, of the form:

$$I^{(h)}_{A_{02},\,A_{03},\,A_{04}} \equiv \int D^{2h}X_0 \frac{1}{X_{01}^{A_{01}}X_{02}^{A_{02}}X_{03}^{A_{03}}X_{04}^{A_{04}}}\bigg|_{M=1}, \tag{4.59}$$

where $A_{01}+A_{02}+A_{03}+A_{04} = 2h$. The capital A_{0i} are used for the exponents in the denomentaor with all possible shifts introduced by the Gegenbaur polynomials. This integral is given by

$$I^{(h)}_{A_{02},\,A_{03},\,A_{04}} \propto X_{13}^{A_{04}-h}X_{14}^{A_{02}+A_{03}-h}X_{24}^{-A_{02}}X_{34}^{h-A_{03}-A_{04}} \times R^{(h)}(z,\bar z;\, A_{02},A_{03},A_{04}), \tag{4.60}$$

where

$$R^{(h)}(z,\bar z;\, A_{02},A_{03},A_{04}) \equiv \Big(-\frac{\partial}{\partial v}\Big)^{h-1} f(z;\, A_{02},A_{03},A_{04})f(\bar z;\, A_{02},A_{03},A_{04}), \tag{4.61}$$

$$f(z;\, A_{02},A_{03},A_{04}) \equiv {}_2F_1(A_{02}-h+1,\, -A_{04}+1;\, -A_{03}-A_{04}+h+1;\, z). \tag{4.62}$$

The derivative $-\partial/\partial v$ in $(z,\bar z)$ coordinates equals

$$-\frac{\partial}{\partial v} = \frac{1}{z-\bar z}\Big(z\frac{\partial}{\partial z} - \bar z\frac{\partial}{\partial \bar z}\Big). \tag{4.63}$$

In the case of $\ell = 0$, all the above manipulations simplify drastically. The Gegenbauer polynomials $C^{p+1}_{\ell-w}(t)$ vanishe for all the values w except for $w=0$, leaving only one type of tensor structure: I_1^p for $W^{seed}(p)$ and I_2^p for $\overline{W}^{seed}(p)$. This leads to a one-to-one correspondence between CBs and integrals:

$$G_e^{(p)} \propto X_{13}^{p-e}X_{34}^{e}\,\mathcal{K}_4^{-1}\, I^{(2+p)}_{a_{02}+\frac{p}{2},\,a_{03}+\frac{p}{2},\,a_{04}+e} \propto (z\bar z)^{\frac{\Delta+\frac{p}{2}}{2}} R^{(2+p)}(z,\bar z;\, a_{02}+\frac{p}{2}, a_{03}+\frac{p}{2}, a_{04}+e), \tag{4.64}$$

$$\overline{G}_e^{(p)} \propto X_{12}^{e}X_{13}^{p-e}\,\mathcal{K}_4^{-1}\, I^{(2+p)}_{a_{02}+e,\,a_{03}+p-e,\,a_{04}+\frac{p}{2}} \propto (z\bar z)^{\frac{\Delta-\frac{p}{2}}{2}+e} R^{(2+p)}(z,\bar z;\, a_{02}+e, a_{03}+p-e, a_{04}+\frac{p}{2}).$$

We have omitted here the relative factors between different CBs. They must be restored if one wants to check that $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ in eq.(4.64) satisfy the Casimir system (4.30). For generic ℓ the CBs are a sum of expressions like eq.(4.64) with different shifts of the parameters A_{0i}, weighted by the relative constants and powers of v (coming from the Gegenbauer polynomial). Since all these terms have $p+1$ derivatives with respect to v, the highest power in $1/(z-\bar z)$ appearing in $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ is

$$\Big(\frac{1}{z-\bar z}\Big)^{1+2p}. \tag{4.65}$$

The asymptotic behaviour of the CBs when $z,\bar z\to 0$ ($u\to 0$, $v\to 1$) for $\ell=0$ is easily obtained from eq.(4.64) by noticing that $R^{(h)}(z,\bar z;\, A_{02},A_{03},A_{04})$ is constant in this limit. Then we have

$$\lim_{z\to 0,\,\bar z\to 0} G_e^{(p)} \propto (z\bar z)^{\frac{\Delta}{2}+\frac{p}{4}}, \qquad \lim_{z\to 0,\,\bar z\to 0} \overline{G}_e^{(p)} \propto (z\bar z)^{\frac{\Delta}{2}-\frac{p}{4}+e}. \tag{4.66}$$

By knowing that the CBs should be proportional to the factor in eq.(4.65), we can refine eq.(4.66) and write

$$\lim_{z\to 0,\,\bar z\to 0} G_e^{(p)} \propto \frac{(z\bar z)^{\frac{\Delta}{2}+\frac{p}{4}}}{(z-\bar z)^{1+2p}}(z^{1+2p}-\bar z^{1+2p}), \tag{4.67}$$

$$\lim_{z\to 0,\,\bar z\to 0} \overline{G}_e^{(p)} \propto \frac{(z\bar z)^{\frac{\Delta}{2}-\frac{p}{4}+e}}{(z-\bar z)^{1+2p}}(z^{1+2p}-\bar z^{1+2p}). \tag{4.68}$$

Notice that the behavior (4.67) and (4.68) of the CBs for $z, \bar{z} \to 0$ when $\ell = 0$ is not guaranteed to be straightforwardly extended for any $\ell \neq 0$. Indeed, we see from eq.(4.58) that for a given p, the generic CPW is obtained when $\ell \geq p$, in which case all terms in the sum over w are present. All the values of $\ell < p$ should be treated separately.

4.3.3 Computing the Conformal Blocks for $\ell \neq 0$

A useful expression of the CBs for generic values of ℓ can be obtained using eq.(4.52) and the known closed form of $W^{seed}(0)$. Recall that

$$W^{seed}(0) = \left(\frac{X_{14}}{X_{13}}\right)^b \left(\frac{X_{24}}{X_{14}}\right)^{-a} \frac{G_0^{(0)}(Z,\bar{Z})}{X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}}}\,, \tag{4.69}$$

where a and b are as in eq.(4.8) for $p=0$ and $G^{(0)}(z,\bar{z})$ are the known scalar CBs [8, 9]

$$G_0^{(0)}(z,\bar{z}) = G_0^{(0)}(z,\bar{z};\Delta,l,a,b) = (-1)^\ell \frac{z\bar{z}}{z-\bar{z}} \left(k^{(a,b;0)}_{\frac{\Delta+\ell}{2}}(z) k^{(a,b;0)}_{\frac{\Delta-\ell-2}{2}}(\bar{z}) - (z \leftrightarrow \bar{z}) \right)\,, \tag{4.70}$$

expressed in terms of the function[3]

$$k_\rho^{(a,b;c)}(z) \equiv z^\rho\, {}_2F_1(a+\rho,\, b+\rho;\, c+2\rho;\, z)\,. \tag{4.71}$$

Comparing eq.(4.69) with eq.(4.58) for $p=0$, one can extract the value of the shadow integral in closed form for generic spin ℓ [18]:

$$I_\ell \equiv \int D^4 X_0 \frac{C^1_\ell(t)}{X_{01}^{a_{01}} X_{02}^{a_{02}} X_{03}^{a_{03}} X_{04}^{a_{04}}}\Big|_{M=1} \propto \left(\frac{X_{14}}{X_{13}}\right)^b \left(\frac{X_{24}}{X_{14}}\right)^{-a} \frac{G_0^{(0)}(Z,\bar{Z};\Delta,\ell,a,b)}{X_{12}^{\frac{\Delta}{2}} X_{34}^{\frac{4-\Delta}{2}}}\,. \tag{4.72}$$

Using the relations (4.48) and (4.52) one can recast $W^{seed}(p)$ and $\overline{W}^{seed}(p)$ in the form

$$\begin{aligned} W^{seed}(p) &\propto \frac{\mathcal{D}_{N_1}\ldots\mathcal{D}_{N_p}}{X_{12}^{a_{12}+\frac{\ell}{2}} X_{34}^{a_{34}}} X_{12}^{\frac{\ell+p}{2}} \int D^4 X_0 \frac{C^1_{\ell+p}(t) X_0^{N_1}\ldots X_0^{N_p}}{X_{01}^{a_{01}+\frac{p}{2}} X_{02}^{a_{02}} X_{03}^{a_{03}} X_{04}^{a_{04}+\frac{p}{2}}}\Big|_{M=1}\,, \\ \overline{W}^{seed}(p) &\propto \frac{\overline{\mathcal{D}}_{N_1}\ldots\overline{\mathcal{D}}_{N_p}}{X_{12}^{a_{12}} X_{34}^{a_{34}+\frac{\ell}{2}}} X_{34}^{\frac{\ell+p}{2}} \int D^4 X_0 \frac{C^1_{\ell+p}(t) X_0^{N_1}\ldots X_0^{N_p}}{X_{01}^{a_{01}} X_{02}^{a_{02}+\frac{p}{2}} X_{03}^{a_{03}+\frac{p}{2}} X_{04}^{a_{04}}}\Big|_{M=1}\,, \end{aligned} \tag{4.73}$$

where $\overline{\mathcal{D}} = \mathcal{P}\mathcal{D}|_{1\leftrightarrow 3, 2\leftrightarrow 4}$, as follows from eq.(4.46), $\mathcal{D} = \mathcal{D}_M X_0^M$, $\overline{\mathcal{D}} = \overline{\mathcal{D}}_M X_0^M$. The tensor integral is evaluated using $SO(4,2)$ Lorentz symmetry. One writes

$$\int D^4 X_0 \frac{C^1_{\ell+p}(t) X_0^{M_1}\ldots X_0^{M_p}}{X_{01}^{a_{01}+\frac{p}{2}} X_{02}^{a_{02}} X_{03}^{a_{03}} X_{04}^{a_{04}+\frac{p}{2}}} = \sum_n A_n(X_i)\, \tau_n^{M_1\ldots M_p}(X_i)\,, \tag{4.74}$$

where n runs over all possible rank p traceless symmetric tensors τ_n which can be constructed from X_1, X_2, X_3, X_4 and η_{MN}'s, with arbitrary scalar coefficients A_n to be determined. Performing all possible contractions, which do not change the monodromy of the integrals, the A_n coefficients can be

[3] We adopt here the notation first used in ref.[5] for this function, but notice the slight difference in the definition: $k_\rho^{there} = k_{\rho/2}^{here}$.

solved as linear combinations of the scalar block integrals I_ℓ defined in eq.(4.72), with shifted external dimensions.

In this way, we have computed the CBs $G_e^{(p)}$ with $p=1,2$ and $\overline{G}_e^{(p)}$ with $p=1$ for general Δ,ℓ,a,b. We have also verified that the CBs $\overline{G}_e^{(1)}$ obtained from $G_e^{(1)}$ using eqs.(4.10) and (4.9) agree with those arising from the direct shadow computation. There is a close connection among the CBs $G_e^{(p)}$ and $\overline{G}_{p-e}^{(p)}$, for any p. More on this point in section 4.4. In all cases the CBs satisfy the Casimir system (4.30).

As mentioned at the end of subsection 4.3.2, the asymptotic behaviour of the CBs for $z,\bar{z}\to 0$ depends on whether $\ell \geq p$ or not. For $p=1$ we can expand the obtained solutions, which for $\ell\geq 1$ read as

$$\lim_{z\to 0,\,\bar{z}\to 0} G_e^{(1)} \propto \frac{(z\bar{z})^{\frac{\Delta-\ell}{2}+\frac{1}{4}}}{(z-\bar{z})^3}\left(\bar{z}^{\ell+e+2}-(z\leftrightarrow\bar{z})\right), \qquad \ell\geq 1 \tag{4.75}$$

$$\lim_{z\to 0,\,\bar{z}\to 0} \overline{G}_e^{(1)} \propto \frac{(z\bar{z})^{\frac{\Delta-\ell}{2}-\frac{1}{4}}}{(z-\bar{z})^3}\left(z^e\bar{z}^{\ell+3}-(z\leftrightarrow\bar{z})\right), \qquad \ell\geq 1\,, \tag{4.76}$$

while for $\ell=0$ they match eqs.(4.67) and (4.68). The above relations, together with eqs.(4.65), (4.67) and (4.68), will allow us to settle the problem of the boundary values of the CBs for any value of p and ℓ, that will be reported in eqs.(4.85) and (4.89). The explicit form of $G_e^{(p)}$ found for $p=2$ using the shadow formalism provides a further check of the whole derivation.

4.4 Solving the System of Casimir Equations

The goal of this section is to find the explicit form of the conformal blocks $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ appearing in eq.(4.7) by solving the Casimir system (4.30). In doing it we adopt and expand the methods introduced by Dolan and Osborn in refs. [9, 49] to obtain 6D scalar conformal blocks. We will mostly focus on the blocks $G_e^{(p)}$, since the same analysis will apply to $\overline{G}_e^{(p)}$ with a few modifications that we will point out.

Before jumping into details let us outline the main logical steps of our derivation. We first find, with the guidance of the results obtained in section 4.3, the behaviour of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ in the limit $z,\bar{z}\to 0$ in which the Casimir system (4.30) can be easily solved. Using this information and eq.(4.65), we then write an educated ansatz for the form of the CBs. Using this ansatz, we reduce the problem of solving a system of linear partial differential equations of second order in two variables to a system of linear algebraic equations for the unknown coefficients entering the ansatz. Then we show that the non-zero coefficients in the ansatz admit a geometric interpretation. They form a two-dimensional lattice with an octagon shape structure. This interpretation allows us to precisely predict which coefficients enter in our ansatz for any value of p. Finally, we show that the linear algebraic system admits a recursive solution and we discuss the complexity of deriving full solutions for higher values of p.

4.4.1 Asymptotic Behaviour

Not all solutions of the Casimir system (4.30) give rise to sensible CBs. The physical CBs are obtained by demanding the correct boundary values for $G_e^{(p)}$ and $\overline{G}_e^{(p)}$. Possible boundary values are given by considering the OPE limit $z,\bar{z}\to 0$ of $W^{seed}(p)$ and $\overline{W}^{seed}(p)$. The limits of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $z,\bar{z}\to 0$ could be computed by a careful analysis of tensor structures. This analysis has been partly done in section 4.3, where we have obtained the boundary values of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $z,\bar{z}\to 0$ for special values of p and/or ℓ. Luckily enough, there will be no need to extend such analysis because the form of the system (4.30) in the OPE limit, together with eqs.(4.67), (4.75) and (4.76), will clearly indicate the general form of the boundary values of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$.

Let us then consider the form of the conformal blocks $G_e^{(p)}$ in the limit $z, \bar{z} \to 0$, with $z \to 0$ *taken first*. In this limit

$$G_e^{(p)} \to N_e z^{\lambda^{(e)}} \bar{z}^{\bar{\lambda}^{(e)}}\,, \tag{4.77}$$

where N_e, $\lambda^{(e)}$ and $\bar{\lambda}^{(e)}$ are parameters to be determined. For simplicity of notation we have omitted their p-dependence. The differential operators (4.22) and (4.33), when acting on eq.(4.77) give, at leading order in z and $\bar{z}$,

$$\Delta_\epsilon^{(a_e,b_e;c_e)} \to \lambda^{(e)}(\lambda^{(e)}-1) + c_e(\lambda^{(e)}+\bar{\lambda}^{(e)}) + \bar{\lambda}^{(e)}(\bar{\lambda}^{(e)}-1) - \epsilon\lambda^{(e)}\,, \tag{4.78}$$

$$L(\mu) \to \frac{1}{\bar{z}}(\lambda^{(e)}-\bar{\lambda}^{(e)})\,. \tag{4.79}$$

Let us now focus on the specific equation $Cas_e^{(p)}$ with $e=p$. In the limit $z,\bar{z}\to 0$ it reads

$$\begin{aligned} Cas_p^{(p)}(G) \to\ & N_p\Big(\lambda^{(p)}(\lambda^{(p)}-1) + \bar{\lambda}^{(p)}(\bar{\lambda}^{(p)}-1) - (p+2)\lambda^{(p)} - \frac{1}{2}(E_{\ell,p}-\epsilon_p^p)\Big) z^{\lambda^{(p)}}\bar{z}^{\lambda^{(p)}} \\ & +2N_{p-1}(\lambda^{(p-1)}-\bar{\lambda}^{(p-1)}) z^{\lambda^{(p-1)}+1}\bar{z}^{\bar{\lambda}^{(p-1)}} = 0\,. \end{aligned} \tag{4.80}$$

For generic values of ℓ, we have $\lambda^{(e)} \neq \bar{\lambda}^{(e)}$. Hence we cannot have $\lambda^{(p-1)}+1 < \lambda^{(p)}$ in eq.(4.80), since this would imply that the last term dominates in the limit and N_{p-1} vanishes, in contradiction with the initial hypothesis (4.77).

Let us first consider the case in which $\lambda^{(p-1)}+1 > \lambda^{(p)}$, so that the terms in the second row of eq.(4.80), coming from $G_{p-1}^{(p)}$, vanish. It is immediate to see that the only sensible solution for $\lambda^{(p)}$ and $\bar{\lambda}^{(p)}$ that reproduce the known OPE limit for the $p=0$ case is

$$\lambda^{(p)} = \frac{\Delta-\ell}{2}+\frac{p}{4}, \qquad \bar{\lambda}^{(p)} = \frac{\Delta+\ell}{2}+\frac{p}{4}\,. \tag{4.81}$$

Notice that eq.(4.81) agrees with the asymptotic behaviour for the CBs $G_e^{(p)}$ found in eq.(4.75) for $e=p=1$ and $\ell\geq 1$. Consider now the equation $Cas_{p-1}^{(p)}$. For $z,\bar{z}\to 0$ we have

$$\begin{aligned} Cas_{p-1}^{(p)}(G) \to\ & N_{p-1}\Big(\lambda^{(p-1)}(\lambda^{(p-1)}-1) + \bar{\lambda}^{(p-1)}(\bar{\lambda}^{(p-1)}-1) + (\lambda^{(p-1)}+\bar{\lambda}^{(p-1)}) - (p+2)\lambda^{(p-1)} \\ & -\frac{1}{2}(E_{\ell,p}-\epsilon_{p-1}^p)\Big) z^{\lambda^{(p-1)}}\bar{z}^{\bar{\lambda}^{(p-1)}} + \frac{p}{2}N_p(\lambda^{(p)}-\bar{\lambda}^{(p)}) z^{\lambda^{(p)}}\bar{z}^{\bar{\lambda}^{(p)}-1} \\ & +4N_{p-2}(\lambda^{(p-2)}-\bar{\lambda}^{(p-2)}) z^{\lambda^{(p-2)}+1}\bar{z}^{\bar{\lambda}^{(p-2)}} = 0\,. \end{aligned} \tag{4.82}$$

According to eq.(4.75), we expect $\lambda^{(p-2)} = \lambda^{(p-1)} = \lambda^{(p)}$, $\bar{\lambda}^{p-1} = \bar{\lambda}^{(p)}-1$, $\bar{\lambda}^{p-2} = \bar{\lambda}^{(p)}-2$ in eq.(4.82). In this case the last term is higher order in z and eq.(4.82) is satisfied by simply taking

$$\frac{N_{p-1}}{N_p} = -\frac{\ell p}{2(\ell+p)}\,. \tag{4.83}$$

Notice that we have tacitly assumed above that $\lambda^{(p)}-\bar{\lambda}^{(p)} = -\ell$ does not vanish, i.e. $\ell\neq 0$. For $\ell=0$, more care is required and one should consider the first subleading term in $\bar{z}$ in the expansion (4.77).

The above analysis can be iteratively repeated until the last equation $Cas_0^{(p)}$ is reached and all the coefficients N_e, $\lambda^{(e)}$ and $\bar{\lambda}^{(e)}$ are determined. Analogously to the $\ell = 0$ case in eq.(4.82), all the low spin cases up to $\ell = p$ should be treated separately at some step in the iteration, as already pointed out in subsection 4.3.2. Skipping the detailed derivation, the final values of $\lambda^{(e)}$ and $\bar{\lambda}^{(e)}$ are given by

$$\begin{aligned} \lambda^{(e)} &= \lambda^{(p)}\,, && \forall \ell = 0,1,2,\ldots \\ \bar{\lambda}^{(e)} &= \bar{\lambda}^{(p)} - (p-e)\,, && \forall \ell = p-e, p-e+1,\ldots \\ \bar{\lambda}^{(e)} &= \bar{\lambda}^{(p)}\,, && \forall \ell = 0,1,\ldots,p-e-1\,, \end{aligned} \tag{4.84}$$

where $\lambda^{(p)}$ and $\bar{\lambda}^{(p)}$ are as in eq.(4.81) and $e = 0,\ldots,p-1$. The asymptotic behaviour of the CBs in the OPE limit is given for any ℓ and p by

$$\lim_{z\to 0,\,\bar{z}\to 0} G_e^{(p)} \propto \frac{(z\bar{z})^{\lambda^{(p)}}}{(z-\bar{z})^{1+2p}} \Big(\bar{z}^{\bar{\lambda}^{(e)}-\lambda^{(p)}+1+2p} - (z \leftrightarrow \bar{z}) \Big)\,. \tag{4.85}$$

We do not report the explicit form of the normalization factors N_e, since they will be of no use in what follows.

We still have to consider the case in which $\lambda^{(p-1)} + 1 = \lambda^{(p)}$ in eq.(4.80). By looking at eq.(4.76), it is clear that this case corresponds to the asymptotic behaviour of the conjugate CBs $\overline{G}_e^{(p)}$. We do not report here the similar derivation of the Casimir equations for $\overline{G}_e^{(p)}$ in the OPE limit. It suffices to say that the analysis closely follows the ones made for $G_e^{(p)}$ starting now from the equation with $e = 0$. If we denote by

$$\overline{G}_e^{(p)} \to \bar{N}_e z^{\omega^{(e)}} \bar{z}^{\bar{\omega}^{(e)}} \tag{4.86}$$

the boundary behaviour of $\overline{G}_e^{(p)}$ when $z, \bar{z} \to 0$ ($z \to 0$ taken first), one finds

$$\begin{aligned} \omega^{(e)} &= \omega^{(0)} + e\,, && \forall \ell = 0,1,2,\ldots \\ \bar{\omega}^{(e)} &= \bar{\omega}^{(0)}\,, && \forall \ell = p-e, p-e+1,\ldots \\ \bar{\omega}^{(e)} &= \bar{\omega}^{(0)} + e\,, && \forall \ell = 0,1,\ldots,p-e-1 \end{aligned} \tag{4.87}$$

where

$$\omega^{(0)} = \frac{\Delta-\ell}{2} - \frac{p}{4}, \qquad \bar{\omega}^{(0)} = \frac{\Delta+\ell}{2} - \frac{p}{4}\,. \tag{4.88}$$

The asymptotic behaviour of the conjugate CBs are given for any ℓ and p by

$$\lim_{z\to 0,\,\bar{z}\to 0} \overline{G}_e^{(p)} \propto \frac{(z\bar{z})^{\omega^{(e)}}}{(z-\bar{z})^{1+2p}} \Big(\bar{z}^{\bar{\omega}^{(e)}-\omega^{(e)}+1+2p} - (z \leftrightarrow \bar{z}) \Big)\,. \tag{4.89}$$

4.4.2 The Ansatz

The key ingredient of the ansatz is the function $k_\rho^{(a,b;c)}(z)$ defined in eq.(4.71), which is an eigenfunction of the hyper-geometric like operator $D_z^{(a,b;c)}$:

$$D_z^{(a,b;c)}\, k_\rho^{(a,b;c)}(z) = \rho\,(\rho + c - 1)\, k_\rho^{(a,b;c)}(z). \tag{4.90}$$

Using eq.(4.90) one can define an eigenfunction of the operator $\Delta_0^{(a,b;c)}$ as the product of two k's:

$$\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)}(z,\bar{z}) \equiv k_{\rho_1}^{(a,b;c)}(z) k_{\rho_2}^{(a,b;c)}(\bar{z}), \tag{4.91}$$

$$\mathcal{F}_{\rho_1,\rho_2}^{\pm\,(a,b;c)}(z,\bar{z}) \equiv \mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)}(z,\bar{z}) \pm \mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)}(\bar{z},z). \tag{4.92}$$

These functions played an important role in ref.[9] for the derivation of an analytic closed expression of the scalar CBs in even space-time dimensions. In our case, the situation is much more complicated, because we have different blocks appearing in the Casimir equations. We notice, however, that the second order operator Δ in each equation $Cas_e^{(p)}$ acts only on the block $G_e^{(p)}$, while the blocks $G_{e-1}^{(p)}$ and $G_{e+1}^{(p)}$ are multiplied by first order operators only. Since, as we will shortly see, first order derivatives and factors of z and $\bar{z}$ acting on the functions $\mathcal{F}$ can always be expressed in terms of functions $\mathcal{F}$ with shifted parameters, a reasonable ansatz for the CBs is to take each G_e proportional to a sum of functions of the kind $\mathcal{F}_{\rho_1,\rho_2}^{(a_e,b_e;c_e)}(z,\bar{z})$ for some ρ_1 and ρ_2. Taking also into account eq.(4.65), found using the shadow formalism, the form of the ansatz for the blocks $G_e^{(p)}$ should be[4]

$$G_e^{(p)}(z,\bar{z}) = \Big(\frac{z\bar{z}}{z-\bar{z}}\Big)^{2p+1} g_e^{(p)}(z,\bar{z}), \qquad g_e^{(p)}(z,\bar{z}) \equiv \sum_{m,n} c_{m,n}^e \mathcal{F}_{\rho_1+m,\,\rho_2+n}^{-\,(a_e,b_e;c_e)}(z,\bar{z}), \tag{4.93}$$

where $c_{m,n}^e$ are coefficients to be determined and the sum over the two integers m and n in eq.(4.93) is so far unspecified and possibly infinite. Notice that all the functions $\mathcal{F}$ entering the sum over m and n have the same values of a_e, b_e and c_e. Matching eq.(4.93) in the limit $z,\bar{z}\to 0$ with eq.(4.85) allows us to determine ρ_1 and ρ_2, modulo a shift by an integer. We take

$$\rho_1 = \bar{\lambda}^{(p)}\,, \qquad \rho_2 = \lambda^{(p)} - p - 1\,, \tag{4.94}$$

in which case the sum over n is bounded from below by $n_{min} = -p$. At this value of n, we have $m(n_{min}) = e - p$. There is no need to discuss separately the behaviour of the blocks with $\ell \leq p$. Their form is still included in the ansatz (4.93) with the additional requirement that some coefficients $c_{m,n_{min}}^e$ should vanish. This condition is automatically satisfied in the final solution. In the next subsections we will discuss the precise range of the sum over m and n and explain how the coefficients $c_{m,n}^e$ can be determined.

4.4.3 Reduction to a Linear System

The eigenfunctions $\mathcal{F}_{\rho_1,\rho_2}^{\pm\,(a,b;c)}(z,\bar{z})$ have several properties that would allow us to find a solution to the system (4.30). In order to exploit such properties, we first have to express the system (4.30) for $G_e^{(p)}$ in terms of the functions $g_e^{(p)}(z,\bar{z})$ defined in eq.(4.93). We plug the ansatz (4.93) in eq.(4.30) and use the following relations

$$\begin{aligned}
\Delta_\epsilon^{(a,b;c)}\Big(\frac{z\bar{z}}{z-\bar{z}}\Big)^k &= \Big(\frac{z\bar{z}}{z-\bar{z}}\Big)^k\Big(\Delta_{\epsilon-2k}^{(a,b;c)} + k\,(k-\epsilon+c-1) - k\,(k-\epsilon+1)\,\frac{z\bar{z}(z+\bar{z})-2z\bar{z}}{(z-\bar{z})^2}\Big)\,, \\
L(\mu)\Big(\frac{z\bar{z}}{z-\bar{z}}\Big)^k &= \Big(\frac{z\bar{z}}{z-\bar{z}}\Big)^k\Big(L(\mu) + k\,\frac{z+\bar{z}-2z\bar{z}}{(z-\bar{z})^2}\Big)\,,
\end{aligned} \tag{4.95}$$

to obtain the system of Casimir equations for $g_e^{(p)}$:

$$\widetilde{Cas}_e^{(p)}(g) \equiv Cas^0\, g_e^{(p)} + Cas^+\, g_{e+1}^{(p)} + Cas^-\, g_{e-1}^{(p)} = 0\,. \tag{4.96}$$

We have split each Casimir equation in terms of three differential operators Cas^0, Cas^+, Cas^-, that act on $g_e^{(p)}$, $g_{e+1}^{(p)}$ and $g_{e-1}^{(p)}$, respectively. In order to avoid cluttering, we have omitted the obvious e

[4]Recall that the conformal blocks are even under $z\leftrightarrow\bar{z}$ exchange, that leaves u and v unchanged.

and p dependences of such operators. Their explicit form is as follows:

$$\begin{aligned} Cas^0 &= \left(\frac{z-\bar{z}}{z\bar{z}}\right)^2 \left(\Delta_0^{(a_e,b_e;c_e)} + (1+2p)(2p-2-e) - \frac{1}{2}\left(E_\ell^p - \varepsilon_e^p\right)\right) \\ &\quad -3p\,\frac{z-\bar{z}}{z\bar{z}} \times \Big((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\Big) - p\,(1+2p)\,\frac{z+\bar{z}-2}{z\bar{z}}, \qquad (4.97) \\ Cas^+ &= B_e\,\frac{z-\bar{z}}{z\bar{z}} \times \frac{z-\bar{z}}{z\bar{z}} L(b_{e+1}) + (1+2p)\,B_e\,\frac{z+\bar{z}-2z\bar{z}}{z\bar{z}}\,\frac{1}{z\bar{z}}, \qquad (4.98) \\ Cas^- &= A_e^p\,\frac{z-\bar{z}}{z\bar{z}} \times (z-\bar{z}) L(a_{e-1}) + (1+2p)\,A_e^p\,\frac{z+\bar{z}-2z\bar{z}}{z\bar{z}}. \qquad (4.99) \end{aligned}$$

Notice that the action of $\Delta_0^{(a_e,b_e;c_e)}$ in eq.(4.97) on $g_e^{(p)}$ is trivial and gives just the sum of the eigenvalues of the $\mathcal{F}^{-\,(a,b;c)}_{\rho_1,\,\rho_2}(z,\bar{z})$ entering $g_e^{(p)}$. It is clear from the form of the ansatz (4.93) that the system (4.96) involves three different kinds of functions $\mathcal{F}^-$, with different values of a, b and c (actually only b and c differ, recall eq.(4.32)).

Using properties of hypergeometric functions, however, we can bring the Casimir system (4.96) into an algebraic system involving functions $\mathcal{F}^{-\,(a_e,b_e;c_e)}_{\rho_1+r,\,\rho_2+t}(z,\bar{z})$ only, with different values of r and t, but crucially with the same values of a_e, b_e and c_e. In order to do that, it is useful to interpret each of the terms entering the definitions of Cas^0, Cas^+ and Cas^- as an operator acting on the functions $\mathcal{F}^-$ shifting their parameters. Their action can be reconstructed from the more fundamental operators provided in the appendix C. For each function $\mathcal{F}^-$ appearing in the ansatz (4.93), we have

$$Cas^0\,\mathcal{F}^{-(a,b;c)}_{\rho_1+m,\,\rho_2+n}(z,\bar{z}) = \sum_{(r,t)\in\mathcal{R}_0} A^0_{r,t}(m,n)\,\mathcal{F}^{-\,(a,b;c)}_{\rho_1+m+r,\,\rho_2+n+t}(z,\bar{z})\,, \qquad (4.100)$$

$$Cas^+\,\mathcal{F}^{-\,(a,b;c)}_{\rho_1+m,\,\rho_2+n}(z,\bar{z}) = \sum_{(r,t)\in\mathcal{R}_+} A^+_{r,t}(m,n)\,\mathcal{F}^{-\,(a,b+1;c+1)}_{\rho_1+m+r,\,\rho_2+n+t}(z,\bar{z})\,, \qquad (4.101)$$

$$Cas^-\,\mathcal{F}^{-\,(a,b;c)}_{\rho_1+m,\,\rho_2+n}(z,\bar{z}) = \sum_{(r,t)\in\mathcal{R}_-} A^-_{r,t}(m,n)\,\mathcal{F}^{-\,(a,b-1;c-1)}_{\rho_1+m+r,\,\rho_2+n+t}(z,\bar{z})\,, \qquad (4.102)$$

where A^0, A^- and A^+ are coefficients that in general depend on all the parameters involved: a, b, Δ, ℓ, e and p but not on z and $\bar{z}$, namely they are just constants. For future purposes, in eqs.(4.100)-(4.102) we have only made explicit the dependence of A^0, A^- and A^+ on the integers m and n. The sum over (r,t) in each of the above terms runs over a given set of pairs of integers. We report in fig. 4.1 the values of (r,t) spanned in each of the three regions $\mathcal{R}_0$, $\mathcal{R}_+$ and $\mathcal{R}_-$. We do not report the explicit and quite lengthy expression of the coefficients $A^0_{r,t}$, $A^+_{r,t}$ and $A^-_{r,t}$, but we refer the reader again to appendix C where we provide all the necessary relations needed to derive them. Using eqs.(4.93) and (4.100)-(4.102), the Casimir system (4.96) can be rewritten in terms of the functions $\mathcal{F}^-$ only, with the same set of coefficients a_e, b_e and c_e:[5]

$$\sum_{m,n}\Big(\sum_{(r,t)\in\mathcal{R}_0} A^0_{r,t}(m,n)\,c^e_{m,n} + \sum_{(r,t)\in\mathcal{R}_+} A^+_{r,t}(m,n)\,c^{e+1}_{m,n} + \sum_{(r,t)\in\mathcal{R}_-} A^-_{r,t}(m,n)\,c^{e-1}_{m,n}\Big)\mathcal{F}^{-\,(a_e,b_e;c_e)}_{\rho_1+m+r,\,\rho_2+n+t} = 0\,. \qquad (4.103)$$

The functions $\mathcal{F}^-$ appearing in eq.(4.103) are linearly independent among each other, since they all have a different asymptotic behaviour as $z,\bar{z}\to 0$. Hence the only way to satisfy eq.(4.103) is to demand

[5] It is understood that $c^{-1}_{m,n} = c^{p+1}_{m,n} = 0$ in eq.(4.103).

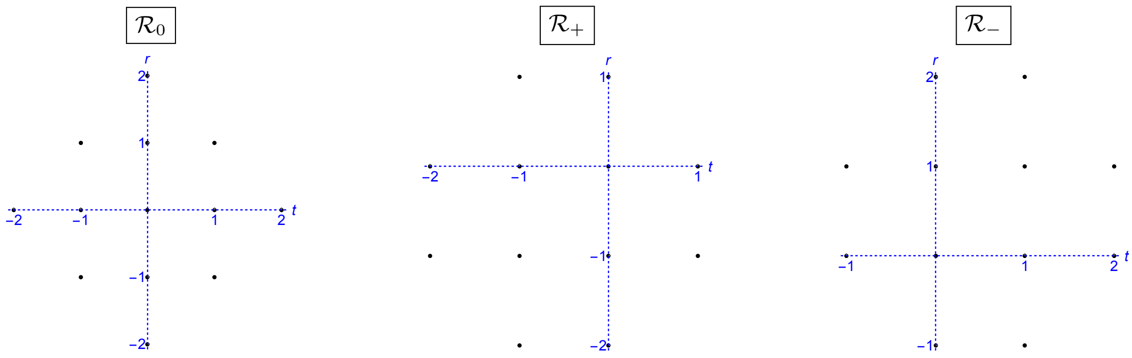

Figure 4.1: Set of points in the (r,t) plane forming the regions $\mathcal{R}_0$ (13 points), $\mathcal{R}_+$ (12 points) and $\mathcal{R}_-$ (12 points) defined in eqs.(4.100)-(4.102).

that terms multiplying different $\mathcal{F}^-$ vanish on their own:

$$\sum_{(r,t)\in\mathcal{R}_0} A^0_{r,t}(m'-r,n'-t)c^e_{m'-r,n'-t} + \sum_{(r,t)\in\mathcal{R}_+} A^+_{r,t}(m'-r,n'-t)c^{e+1}_{m'-r,n'-t}$$
$$+ \sum_{(r,t)\in\mathcal{R}_-} A^-_{r,t}(m'-r,n'-t)c^{e-1}_{m'-r,n'-t} = 0\,, \quad \forall m',n', \quad e=0,\dots p\,, \tag{4.104}$$

where $m' = m+r$, $n' = n+t$. The Casimir system is then reduced to the over-determined *linear algebraic system of equations* (4.104).

4.4.4 Solution of the System

In order to solve the system (4.104), we have to determine the range of values of (m,n) entering the ansatz (4.93), that also determines the size of the linear system. Because of the results of the shadow formalism we expect that the size of the linear system, and the range of (m,n), is finite. By rewriting the known $p=1$ and $p=2$ CBs found using the shadow formalism in the form of eq.(4.93), we have deduced the range in (m,n) of the coefficients $c^e_{m,n}$ for any p (a posteriori proved using the results below). For each value of e, the non-trivial coefficients $c^e_{m,n}$ span a two-dimensional lattice in the (m,n) plane. For each e, the shape of the lattice is an octagon, with p and e dependent edges. The position and shape of the generic octagon in the (m,n) plane is depicted in fig. 4.2. One has

$$n_{min} = -p\,, \quad n_{max} = e+p\,, \quad m_{min} = e-2\,p\,, \quad m_{max} = p\,. \tag{4.105}$$

For $e=0$ and $e=p$, the octagons collapse to hexagons. The number N^e_p of points inside a generic octagon is

$$N^e_p = 2p\,(2p-e) + (1+e)\,(3p+1-e) \tag{4.106}$$

and correspond to the number of non-trivial coefficients $c^e_{m,n}$ entering the ansatz (4.93). The total number N_p of coefficients to be determined at level p is then

$$N_p \equiv \sum_{e=0}^{p} N^e_p = (1+p)\left(1+\frac{17}{6}\,p+\frac{25}{6}\,p^2\right). \tag{4.107}$$

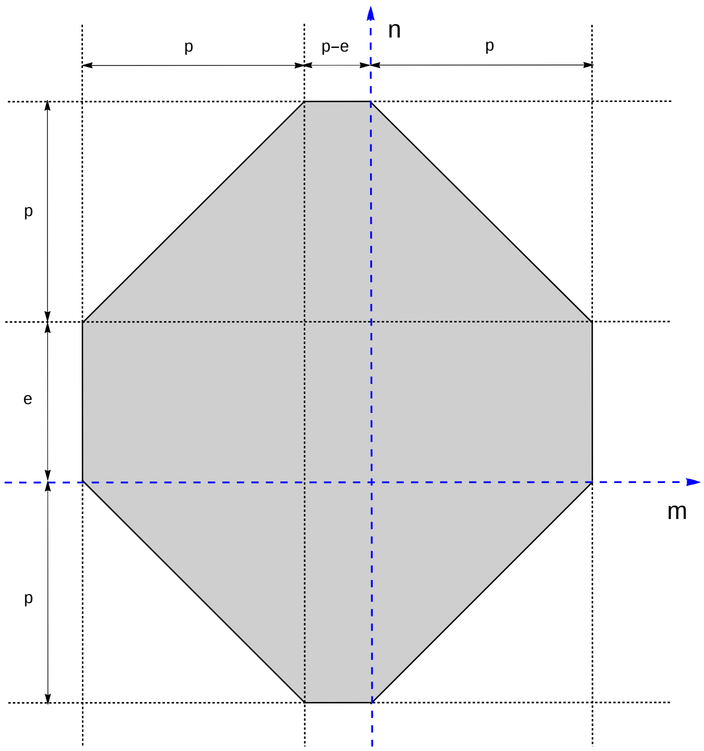

Figure 4.2: The dimensions of the generic octagon enclosing the lattice of non-vanishing coefficients $c^e_{m,n}$ entering the ansatz for mixed tensor CBs in eq.(4.112).

The size of the linear system grows as p^3. The first values are $N_1 = 16$, $N_2 = 70$, $N_3 = 188$, $N_4 = 395$. For illustration, we report in fig. 4.3 the explicit lattice of non-trivial coefficients $c^e_{m,n}$ for $p = 3$.

The system (4.104) is always over-determined, since it is spanned by the values (m', n') whose range is bigger than the range of $(m, n) \in Oct_e^{(p)}$ (spanning all the coefficients to be determined) due to the presence of $(r, t) \in [-2, 2]$. There are only $N_p - 1$ linearly independent equations, because the system of Casimir equations can only determine conformal blocks up to an overall factor. The most important property of the system (4.104) is the following: while the number of equations grows with p, the total number of coefficients $c^e_{m,n}$ entering any given equation in the system (4.104) does not. This is due to the "local nearest-neighbour" nature of the interaction between the blocks, for which at most three conformal blocks can enter the Casimir system (4.30), independently of the value of p. More precisely, all the equations (4.104) involve from a minimum of one coefficient $c^e_{m,n}$ up to a maximum of 37 ones. Thirty seven corresponds to the total number of coefficients A^0, A^+ and A^- entering eqs.(4.100)-(4.102), see fig.4.1. The only coefficients that enter alone in some equations are the ones corresponding to the furthermost vertices of the hexagons, namely

$$c^p_{0,-p}, c^p_{0,2p}, c^0_{p,0}, c^0_{-2p,0} \,. \tag{4.108}$$

For instance, let us take $n' = -2 - p$ and $e = p$ in eq.(4.104), with m' generic. Since $n_{min} = -p$, a non-vanishing term can be obtained only by taking $t = -2$. Considering that $c^{p+1} = 0$ and $\mathcal{R}_-$ does not include $t = -2$ (see fig.4.1), this equation reduces to

$$A^0_{0,-2}(m, -p)|_{e=p}\, c^p_{m,-p} = 0\,, \quad \forall m\,, \tag{4.109}$$

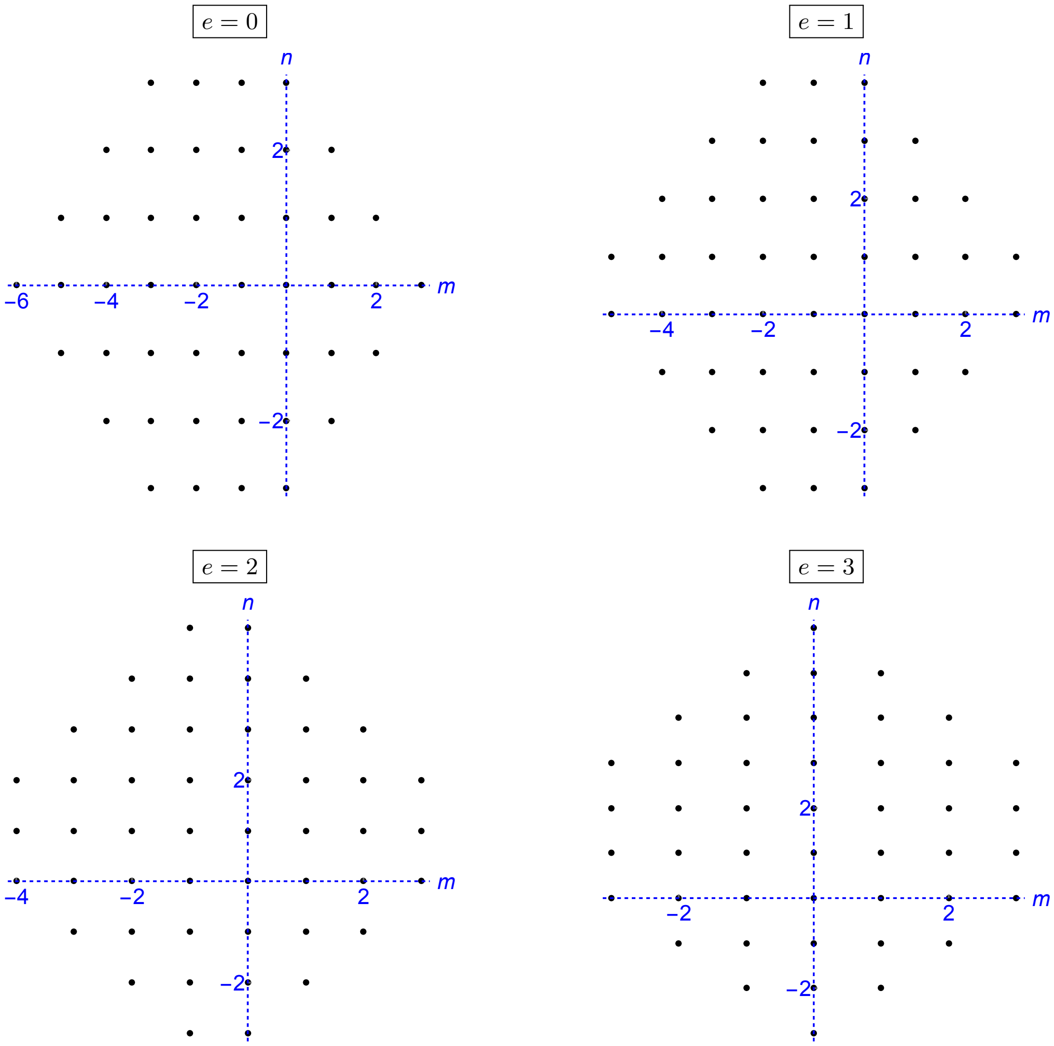

Figure 4.3: Set of non-vanishing coefficients $c^e_{m,n}$ (represented as black dots) entering the ansatz for mixed tensor CBs in eq.(4.112) for $p = 3$ and $e = 0, 1, 2, 3$. For $e = 0$ and $e = p$ the octagons collapse to hexagons.

where $m' = m$, since the point in $\mathcal{R}_0$ with $t = -2$ has $r = 0$. This equation forces all the coefficients $c^p_{m,-p}$ to vanish, unless the factor $A^0_{0,-2}(m, -p)$ vanishes on its own. One has

$$A^0_{0,-2}(m, n)|_{e=p} \propto (m + n + p)\Delta + (m - n - p)\ell + m^2 + \frac{1}{2}m(p - 2) + (n + p)(n + \frac{3}{2}p - 2)\,.$$

This factor is generally non-vanishing, unless $m = 0$ and $n = -p$, in which case it vanishes for any Δ, ℓ and p. In this way eq.(4.109) selects $c^p_{0,-p}$ as the only non-vanishing coefficient at level $n = -p$ for $e = p$. Notice that it is crucial that $A^0_{0,-2}(m, n)|_{e=p}$ vanishes automatically for a given pair (m, n), otherwise either the whole set of equations would only admit the trivial solution $c^e_{m,n} = 0$, or the system would be infinite dimensional. A similar reasoning applies for the other three coefficients. One has in particular

$$
\begin{aligned}
A^0_{0,2}(0,2p)|_{e=p} \quad c^p_{0,2p} &= 0\,, \\
A^0_{2,0}(p,0)|_{e=0} \quad c^0_{p,0} &= 0\,, \\
A^0_{-2,0}(-2p,0)|_{e=0} \quad c^0_{-2p,0} &= 0\,,
\end{aligned}
\tag{4.110}
$$

that are automatically satisfied because the three coefficients $A^0_{0,2}$, $A^0_{2,0}$ and $A^0_{-2,0}$ vanish when evaluated for the specific values reported in eq.(4.110) for any Δ, ℓ and p.

The system (4.104) is efficiently solved by extracting a subset of $N_p - 1$ linearly independent equations. This can be done by fixing the values $(r,t) = (r^*, t^*)$ entering the definitions of (m', n'). There are 4 very special subsets of the $N_p - 1$ equations (corresponding to very specific values (r^*, t^*)) which allows us to determine the solution iteratively starting from eq.(4.104). They correspond to a solution where one of the four coefficients (4.108) is left undetermined, in other words (r^*, t^*) can be set to be $(0,-2)$, $(0,2)$, $(2,0)$ or $(-2,0)$. For instance, if we choose $c_0 \equiv c^p_{0,-p}$ as the undetermined coefficient, a recursion relation is found from eq.(4.104) by just singling out the term with $t=-2$ in A^0 and setting $(r^*, t^*) = (0,-2)$. Such a choice leads to $m' = m$, $n' = n-2$, and one finally gets

$$
\begin{aligned}
-A^0_{0,-2}(m,n) c^e_{m,n} = & \sum_{\substack{(r,t)\in\mathcal{R}_0 \\ (r,t)\neq(0,-2)}} A^0_{r,t}(m-r, n-2-t) c^e_{m-r,n-2-t} \\
& + \sum_{(r,t)\in\mathcal{R}_+} A^+_{r,t}(m-r, n-2-t) c^{e+1}_{m-r,n-2-t} \\
& + \sum_{(r,t)\in\mathcal{R}_-} A^-_{r,t}(m-r, n-2-t) c^{e-1}_{m-r,n-2-t}\,.
\end{aligned}
\tag{4.111}
$$

It is understood in eq.(4.111) that $c^e_{m,n} = 0$ if the set (m,n) lies outside the e-octagon of coefficients. The recursion (4.111) allows us to determine all the coefficients $c^e_{m,n}$ at a given $e = e_0$ and $n = n_0$ in terms of the ones $c^e_{m,n}$ with $n < n_0$ and c^e_{m,n_0} with $e > e_0$. Hence, starting from c_0, one can determine all $c^e_{m,n}$ as a function of c_0 for any p. The overall normalization of the CBs is clearly irrelevant and can be reabsorbed in a redefinition of the OPE coefficients. However, some care should be taken in the choice of c_0 if one wants to avoid the appearance of spurious divergencies in the CBs for specific values of ℓ and Δ. These divergencies are removed by a proper Δ and ℓ dependent rescaling of c_0. From eq.(4.104) one can easily write the three other relations similar to eq.(4.111) to determine recursively $c^e_{m,n}$ starting from $c^p_{0,2p}$, $c^0_{p,0}$ or $c^0_{-2p,0}$.

We can finally write down the full analytic solution for the CBs $G^{(p)}_e$:

$$
G^{(p)}_e(z,\bar z) = \left(\frac{z\bar z}{z-\bar z}\right)^{2p+1} \sum_{(m,n)\in Oct^{(p)}_e} c^e_{m,n} \mathcal{F}^{-\,(a_e,b_e;c_e)}_{\frac{\Delta+\ell+\frac{p}{2}}{2}+m,\ \frac{\Delta-\ell+\frac{p}{2}}{2}-(p+1)+n}(z,\bar z),
\tag{4.112}
$$

where $c^e_{m,n}$ satisfy the recursion relation (4.111) (or any other among the four possible ones) and (m,n) runs over the points within the e-octagon depicted in fig.4.2.

A similar analysis can be performed for the conjugate blocks $\overline{G}^{(p)}_e$. We do not report here the detailed derivation that is logically identical to the one above, but just the final solution:

$$
\overline{G}^{(p)}_e(z,\bar z) = \left(\frac{z\bar z}{z-\bar z}\right)^{2p+1} \sum_{(m,n)\in Oct^{(p)}_{p-e}} \bar c^e_{m,n} \mathcal{F}^{-\,(a_e,b_e;c_e)}_{\frac{\Delta+\ell-\frac{p}{2}}{2}+e+m,\ \frac{\Delta-\ell-\frac{p}{2}}{2}+e-(p+1)+n}(z,\bar z).
\tag{4.113}
$$

where

$$\bar{c}^{e}_{m,n}(a,b,\Delta,l,p) = 4^{e} c^{p-e}_{m,n}\Big(-a+\frac{p}{2}, -b-\frac{p}{2}, \Delta, l, p\Big)\,. \tag{4.114}$$

Generating the full explicit solution from eq.(4.111) can be computationally quite demanding for large values of p. For concreteness, we only report in appendix D the explicit form of the 16 coefficients $c^{e}_{m,n}$ for $p=1$ and $a=-b=1/2$. The blocks $G_e^{(p)}$ for $p=1,2$ and $\overline{G}_e^{(p)}$ for $p=1$ are in complete agreement with those computed using the shadow formalism. By choosing specific values for the parameters a and b, we also have determined the coefficients $c^{e}_{m,n}$ up to $p=8$, i.e. the value of p that is obtained in the 4-point function of four energy momentum tensors, see eq.(3.5).

It is important to remind the reader that the CBs $G_e^{(p)}$ computed here are supposed to be the seed blocks for possibly other 4-point correlation functions, whose CBs are determined by acting with given operators on $G_e^{(p)}$ [30]. The complexity of the form of the blocks $G_e^{(p)}$ at high p is somehow compensated by the fact that the operators one has to act with become simpler and simpler, the higher is p. An example should clarify the point. Let us consider a 4-point function of spin two operators. In this case, one has to determine conformal blocks associated to the exchange of operators $\mathcal{O}^{(\ell,\ell+p)}$ (and $\overline{\mathcal{O}}^{(\ell+p,\ell)}$) for $p=0,2,4,6,8$ (and any ℓ). The conformal blocks associated to the traceless symmetric operators are obtained by applying up to 8 derivative operators in several different combinations to the scalar CB $G_0^{(0)}$. Despite the seed block is very simple, the final blocks are given by (many) complicated sum of derivatives of $G_0^{(0)}$. The $p=8$ CBs, instead, are essentially determined by the very complicated $G_e^{(8)}$ (and $\overline{G}_e^{(8)}$) blocks, but no significant extra complications come from the external operators. An example of such phenomenon in a four fermion correlator is shown (though in a less significative way) in section 7.1 of ref.[30]. For any given 4-point function, after the use of the differential operators introduced in ref. [30], there is no need to compute the coefficients $c^{e}_{m,n}$ for any a and b but only for the values of interest. This considerably simplifies the expression of $c^{e}_{m,n}$.

4.4.5 Analogy with Scalar Conformal Blocks in Even Dimensions

It is worth pointing out in more detail some similarities between the CBs $G_e^{(p)}$ for mixed symmetry tensors computed above and the scalar conformal blocks G_d in $d>2$ even space-time dimensions ($G_4 = G_0^{(0)}$ in our previous notation). The quadratic Casimir equation for scalar CBs in any number of dimensions is

$$\Delta^{(a,b;0)}_{d-2} G_d(z,\bar{z}) = \frac{1}{2} E_\ell(d) G_d(z,\bar{z})\,, \tag{4.115}$$

where

$$E_\ell(d) = \Delta\,(\Delta-d) + \ell(\ell+d-2) \tag{4.116}$$

is the quadratic Casimir eigenvalue for traceless symmetric tensors. The explicit analytical form of scalar blocks in $d=2,4,6$ dimensions has been found in refs.[8, 9]. The same authors also found a relation between scalar blocks in any even space-time dimensionality, eq.(5.4) of ref.[9] (see also the more elegant eq.(4.36) of ref.[49]), that allows us to iteratively determine G_d for any d, starting from G_2. The $d=4$ and $d=6$ solutions found in ref.[9] have the form

$$G_d(z,\bar{z}) = \Big(\frac{z\bar{z}}{z-\bar{z}}\Big)^{d-3} g_d(z,\bar{z})\,, \qquad g_d(z,\bar{z}) = \sum_{m,n} x_{m,n} \mathcal{F}^{-\,(a,b;0)}_{\frac{\Delta+\ell}{2}+m,\ \frac{\Delta-\ell+2-d}{2}+n}(z,\bar{z}), \tag{4.117}$$

where a and b are as in eq.(4.8) with $p=0$ and $x_{m,n}$ are coefficients that in general depend on Δ, l, a and b. In $d=4$ there is only one non-vanishing coefficient centered at $(m,n)=(0,0)$, while in $d=6$

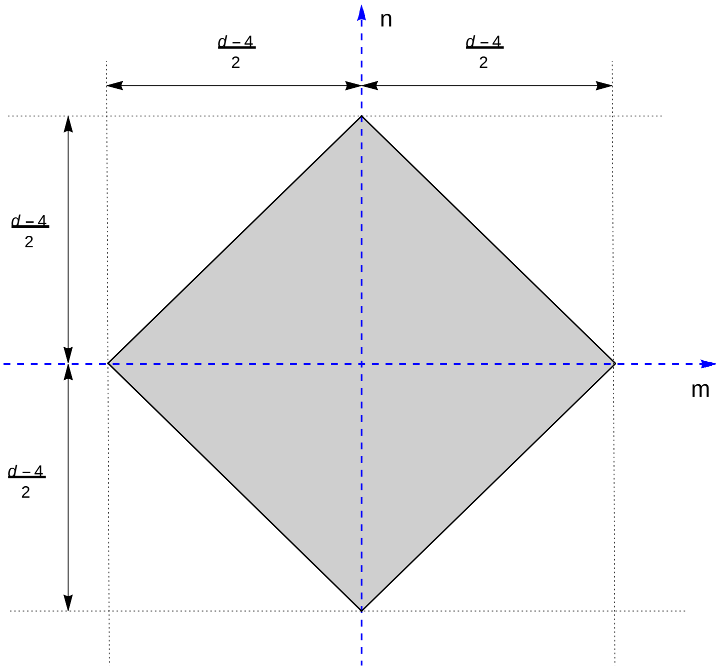

Figure 4.4: The dimensions of the generic slanted square enclosing the lattice of non-vanishing coefficients $x_{m,n}$ entering the ansatz for scalar symmetric CBs in eq.(4.117).

there are five of them. They are at $(m, n) = (0, -1)$, $(-1, 0)$, $(0, 0)$, $(1, 0)$ and $(0, 1)$. These five points form a slanted square in the (m, n) plane, centered at the origin. The explicit form of the coefficients $x_{m,n}$ is known, but it will not be needed in what follows.[6] It is natural to expect that eq.(4.117) should apply for any even $d \geq 4$, with a number of non-vanishing coefficients that increases with d.[7] This is not difficult to prove. From the first relation in eq.(4.95) we can get the form of the Casimir equation for the function $g_d(z, \bar{z})$ defined in eq.(4.117), that can be written as

$$\left(\frac{1}{\bar{z}} - \frac{1}{z}\right)\left(\Delta_0^{(a,b;0)} + 6 - 2d - \frac{1}{2}E_\ell(d)\right)g_d = (d-4)\Big((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\Big)g_d\,. \tag{4.118}$$

Using the techniques explained in subsection 4.4.3 and the results of appendix C, it is now straightforward to identify which is the range of (m, n) of the non-vanishing coefficients $x_{m,n}$ for any d (see fig.4.4).[8] In d dimensions, the minimum and maximum values of m and n are given by

$$n_{min} = \frac{4-d}{2}\,, \quad n_{max} = \frac{d-4}{2}\,, \quad m_{min} = \frac{4-d}{2}\,, \quad m_{max} = \frac{d-4}{2}\,. \tag{4.119}$$

The number $\widetilde{N}_d$ of coefficients $x_{m,n}$ entering the ansatz (4.117) for scalar blocks in d even space-time dimensions is easily computed by counting the number of lattice points enclosed in the slanted square. We have

$$\widetilde{N}_d = \frac{d^2}{2} - 3d + 5\,. \tag{4.120}$$

[6] En passant, notice that there is a typo in eq.(2.20) of ref.[9] where the block G_6 is reported. In the denominator appearing in the last row of that equation, one should replace $(\Delta+\ell-4)(\Delta+\ell-6) \to (\Delta-\ell-4)(\Delta-\ell-6)$.

[7] See also ref.[52], where similar considerations were conjectured.

[8] Alternatively, one might use eq.(4.36) of ref.[49] to compute G_d and then recast it in the form (4.117).

For large d, $\widetilde{N}_d \propto d^2$ and matches the behavior of $N_e^p \propto p^2$ for large p in eq.(4.106).

In light of the above analogy between scalar CBs G_d in even d dimensions and mixed tensor CBs $G_e^{(p)}$ in four dimensions, it would be interesting to investigate whether there exist a set of differential operators that links the blocks $G_e^{(p+1)}$ (or $G_e^{(p+2)}$) to the blocks $G_e^{(p)}$, in analogy to the operator (4.35) of ref.[49] relating G_{d+2} to G_d. It would be very useful to find, in this or some other way, a more compact expression for the blocks $G_e^{(p)}$.

Let us finally emphasize a technical, but relevant, point where the analogy between G_d in d dimensions and $G_e^{(p)}$ in 4 dimensions does not hold. A careful reader might have noticed that in the Casimir equation for g_d the term proportional to $(z+\bar{z})-2$, namely the third term in the r.h.s. of the first equation in eq.(4.95), automatically vanishes. Indeed, if we did not know the power $d-3$ in the ansatz (4.117), we could have guessed it by demanding that term to vanish. On the contrary, no such simple guess seems to be possible for the power $2p+1$ entering $G_e^{(p)}$, given also the appearance of the operator L defined in eq.(4.95). As discussed, we have fixed the power $2p+1$ by means of the shadow formalism.

5. Application: Analytic Bootstrap

The analytic exploration of the bootstrap equations in one of the most notable directions of developments of the bootstrap program. The underlying idea is to take a kinematic limit in which the crossing equations simplify enough to be solved analytically, because they admit a perturbative expansion in small parameter.

It was shown in [53, 54] that crossing equations in the light-cone limit admit a large-spin expansion. The light-cone limit mentioned here amounts to taking $u \to 0$ while v is fixed. From the definition of crossing ratios, this limit is equivalent to $x_{12}^2 \to 0$ in Lorentzian signature, i.e x_2 approaches the the light-cone of x_1 (and not necessarily $x_2 \to x_1$, see fig. 5.1). In [53, 54] it was proved that if a CFT spectrum contains two scalar primary operators Φ_1 and Φ_2 with scaling dimensions Δ_1 and Δ_2, crossing symmetry requires the existence of infinite towers of operator with increasing spin ℓ whose twist $\tau \equiv \Delta - \ell$ goes as $\Delta_1 + \Delta_2 + 2n + \mathcal{O}(1/\ell)$, for each non-negative integer n. These operators can be written schematically as

$$\Phi_1 \partial_{\mu 1} \dots \partial_{\mu \ell} (\partial^2)^n \Phi_2. \tag{5.1}$$

These operators are usually called double-twist operators. Of course operators like (5.1) do not make sense in a strongly coupled theory, however the results of [53, 54] say that they do in large spin limit. We will follow [55] and denote the family of large-spin operators whose twist $\sim \Delta_1 + \Delta_2 + 2n$ as

$$[\Phi_1 \Phi_2]_n \,.$$

The dimensions and OPE coefficients of $[\Phi_1 \Phi_2]_n$ are computable as an asymptotic expansion in $1/spin$. Analytically, [53, 54, 56, 57, 58, 59, 60, 61, 62] computed anomalous dimensions and OPE coefficients of high-spin operators starting in terms of low-twist primaries exchanged in a scalar four point correlator. The results matched AdS observables such as binding energies [53, 54, 63, 25] and Eikonal phases [64, 65, 66].

In this work instead, we will analytically explore a scalar-fermion correlator using the knowledge of spinning conformal blocks in 4D to write the bootstrap equations and solve them in the light-cone limit; we uncover a picture similar to the scalar case [53, 54], i.e. double-twist operators whose dimensions and OPE coefficients we compute as an asymptotic expansion in $1/\text{spin}$. In $d = 4$ we define the twist τ of an operator in the rep $(\ell, \bar{\ell})$ as

$$\tau \equiv \Delta - \frac{\ell + \bar{\ell}}{2}, \quad \text{in unitary CFT} \quad \begin{cases} \tau \geq 1 & \ell = 0 \text{ or } \bar{\ell} = 0 \\ \tau \geq 2 & \text{otherwise} \end{cases} \tag{5.2}$$

To a leading order, double-twist operators resemble a generalized free theory (GFT), indicating that interactions in the corresponding AdS$_5$ falls for large impact parameter [53].

The picture here also differs from the scalar simple case, operators exchanged in different OPE channels can belong to one of three classes of representations of the Lorentz group $SO(1,3)$. These are bosonic operators of the rep (ℓ, ℓ), fermionic operators in the reps $(1 + \ell, \ell)$ and $(\ell, 1 + \ell)$. Each of these classes will contribute in the light-cone limit. The main results of this chapter are the corrections to OPE coefficients (5.51) (and (5.54)) and corrections to the twist, referred to as anomalous dimensions, in

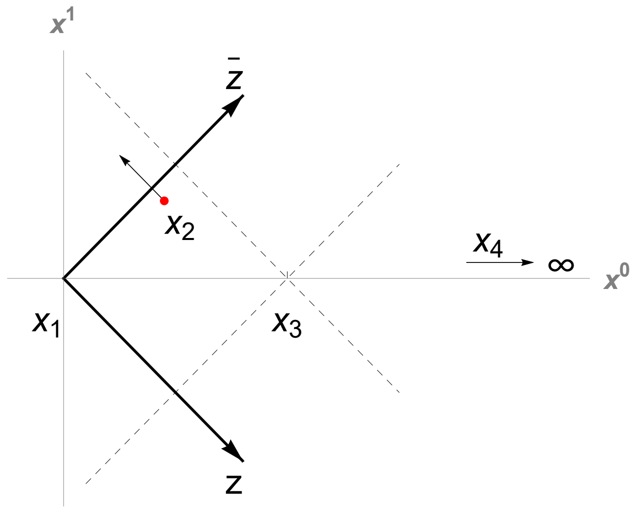

Figure 5.1: Lght-Cone Limit I: On $z, \bar{z}$ plane we use the conformal symmetry to fix the points to the configuration x_1 at $(0,0)$, x_2 at $(z, \bar{z})$, x_3 at $(1,1)$ and send x_4 to ∞ . In the light-cone limit $z \to 0$ x_2 is approaching the light cone of x_1.

(5.50) (and (5.53)) corresponding to the existence of a bosonic (and fermionic) operator with low twist in the spectrum of CFT.

Some tensor correlators have been explored in the light-cone limit, correlators involving conserved spin-1 current and spin-2 stress-energy tensor were used to prove the conformal collider bounds, first proposed in [67], from CFT first principles [25, 26, 27].

The layout of the chapter is as follows: we will start by reviewing the scalar case in section 5.1 to show how the computation works in a simpler instance, following arguments of [53]. In section 5.2 we write the two different sets of bootstrap equations for the correlator $\langle\phi\psi\overline{\psi}\phi\rangle$. We solve the equations in two light-cone limits , we first prove the existence of double-twist operators $[\phi\psi]_n$ and then compute the bosonic and fermionic corrections to their OPE coefficients and scaling dimensions. We calculate in appendix E the OPE coefficients of double-twist operators in a GFT.

5.1 Review of Scalar Correlator Case

We will consider four point function of identical scalars Φ with scaling dimension Δ_Φ in 4 dimensions for simplicity

$$\langle\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle = \frac{g(u,v)}{(x_{12}^2 x_{34}^2)^{\Delta_\Phi}}, \tag{5.3}$$

where g is an arbitrary function for now, but using OPE it can be written as a sum of conformal blocks. The crossing equation

$$(z\bar{z})^{-\Delta_\Phi}\sum_{\mathcal{O}}|\lambda_{\langle\Phi\Phi\mathcal{O}\rangle}|^2 f_{\Delta,\ell}(z,\bar{z}) = ((1-z)(1-\bar{z}))^{-\Delta_\Phi}\sum_{\mathcal{O}}|\lambda_{\langle\Phi\Phi\mathcal{O}\rangle}|^2 f_{\Delta,\ell}(1-z,1-\bar{z}), \tag{5.4}$$

where $\mathcal{O}$ runs over primary operators in the $\Phi\times\Phi$ OPE and Δ,ℓ are the dimension and spin of $\mathcal{O}$. Note that in this case $\mathcal{O}$ are even spin primary tensors. The functions $f_{\Delta,\ell}(z,\bar{z})$ are the $p=0$ conformal blocks defined in eq. (4.70)

$$f_{\Delta,\ell}(z,\bar{z}) = G_0^{(0)}(z,\bar{z})|_{a=b=0}\,. \tag{5.5}$$

The light cone limit is given by $z \ll 1-\bar{z} \equiv \epsilon \ll 1$ (equivalent to $u \to 0$ with $u \ll v$) . In this limit, the functions in the left-hand side of (5.4) behaves as follows, taking into account that ℓ is even:

$$f_{\Delta,\ell}(z,\bar{z}) = z^{\tau/2}\left(k^{0,0;0}_{\frac{\tau}{2}+\ell}(1-\epsilon) + \mathcal{O}(z)\right), \tag{5.6}$$

where the $k^{a,b;c}_{\rho}(x)$ function (4.71) is basically a hypergeometric function. It is clear that the left-hand side of (5.4) is dominated by operators of minimum twist, these are lead by the unit operator $\tau_{\text{unit}}=0$. Next leading terms will come operators which satisfy unitarity bounds

$$\tau \equiv \Delta - \ell, \quad \text{in unitary CFT} \quad \begin{cases} \tau \geq 1 & \ell = 0. \\ \tau \geq 2 & \text{otherwise} \end{cases} \tag{5.7}$$

Meanwhile a conformal block on the right-hand side of (5.4) can be replaced by it expansion for small $\epsilon = 1-\bar{z}$

$$\begin{aligned} f_{\Delta,\ell}(1-z,1-\bar{z}) &= \epsilon^{\frac{\tau}{2}-\Delta_\Phi}\,\tilde{f}_\tau(\epsilon)\,k^{0,0;0}_{\tau/2+\ell}(1-z) + \mathcal{O}\left(\epsilon^{\tau/2+\ell}\right), \\ \tilde{f}_\tau(\epsilon) &\equiv \frac{1}{1-\epsilon}\,{}_2F_1(\frac{\tau}{2}-1,\frac{\tau}{2}-1,\tau-2,\epsilon). \end{aligned} \tag{5.8}$$

Hence the bootstrap eq. (5.4) in the light-cone limit

$$z^{-\Delta_\Phi}(1-\epsilon)^{-\Delta_\Phi} + \cdots = \epsilon^{-\Delta_\Phi}\sum_{\mathcal{O}} |\lambda_{\langle\Phi\Phi\mathcal{O}\rangle}|^2 \epsilon^{\tau/2}\,\tilde{f}_\tau(\epsilon)\,k^{0,0;0}_{\tau/2+\ell}(1-z), \tag{5.9}$$

where we have isolated the contribution of the unit operator in the left-hand side, which is a power-law divergence in z. However, individual terms on the right-hand side are analytic in z up to a logarithm $\ln z$

$$\begin{aligned} k^{a,b;c}_{\rho}(1-z) = \frac{\Gamma(a+b+2\rho)}{\Gamma(a+\rho)\Gamma(b+\rho)} &\sum_{m,n=0}^{\infty} \frac{(-\rho)_m(a+\rho)_n(b+\rho)_n}{m!(n!)^2}\, z^{m+n} \\ &\Big(2\psi(n+1) - \psi(a+n+\rho) - \psi(b+n+\rho) - \ln(z)\Big), \end{aligned} \tag{5.10}$$

the notation used here $(x)_n = \Gamma\left(x+n\right)/\Gamma\left(x\right)$ is the Pocchamer symbol and $\psi(y) = \Gamma'(y)/\Gamma\left(y\right)$ is the digamma function.

The resolution of this dilemma is that the sum over $\mathcal{O}$, which is a double sum over τ and ℓ, does not converge uniformly around $z=0$. The sum over τ is actually convergent, but the sum over ℓ diverge for $z<0$. To understand the convergence of the blocks we only need to see how they behave for large τ or large ℓ. In large τ limit with $|z|,|\epsilon|<1$, the blocks are suppressed by powers $\epsilon^{\tau/2}$. This means that the sum over τ will converge for small z and ϵ.

Instead, in the large ℓ limit[1]

$$k^{a,b;c}_{\frac{\tau}{2}+\ell}(1-z) = \frac{2^{2\ell+\tau+c}\ell^{\frac{1}{2}}}{\sqrt{\pi}z^{\frac{1}{2}(a+b-c)}}\,K_{a+b-c}(2\ell\sqrt{z}) + \mathcal{O}\left(\ell^{-\frac{1}{2}}\right) \approx \frac{2^{c+2\ell+\tau-1}}{z^{\frac{1}{2}(a+b-c+\frac{1}{2})}}e^{-2\ell\sqrt{z}} \tag{5.11}$$

[1] One can get this approximation by writing the hypergeometric function in the integral form, expanding in powers of $1/\ell$ while $z\ell^2 \sim 1$. It was proved in [53] that the region $z\ell^2 \sim 1$ gives the greater contribution.

where $K_\alpha(x)$ is the modified Bessel function. Note that for Re$(\sqrt{z}) > 0$ there is an exponential suppression, but for Re$(\sqrt{z}) < 0$ the block diverge as $\ell \to \infty$. Hence the sum over ℓ in the right-hand side of (5.9) converges at large ℓ for positive real $\sqrt{z}$, and so we will define it by analytic continuation elsewhere in the complex z plane. Crucially, the analytic continuation of the sum contains the power law divergence in z that is not exhibited by any of the individual terms in the sum. The terms in the sum over large ℓ which reproduce $z^{-\Delta_\Phi}$ should also have $\tau \to \Delta_\Phi$ since the leading term on the left-hand side in independent of ϵ. These are the double-twist operators $[\Phi\Phi]_0$.

5.1.1 Existence of $[\Phi\Phi]_n$

The sum over ℓ of families $[\Phi\Phi]_n$ reproduce the $z^{-\Delta_\Phi}$ divergence. To see that, first we assume the dependence of the OPE coefficients on τ and ℓ factorizes for $\ell \gg 1$

$$|\lambda_{\langle\Phi\Phi\,[\Phi\Phi]_n\rangle}|^2 = q(\tau)2^{-2\ell}\ell^X + \mathcal{O}(\ell^{X-1}), \tag{5.12}$$

for some unknown power X and function q. The the sum in the right-hand side of (5.9) will contain the sum over spin of the $[\Phi\Phi]_0$ family of operators

$$\sum_{\mathcal{O}\in[\Phi\Phi]_0} |\lambda_{\langle\Phi\Phi\mathcal{O}\rangle}|^2\, \tilde{f}_{2\Delta_\Phi}(\epsilon)\, k^{0,0;0}_{\Delta_\Phi+\ell}(1-z) \approx \frac{p(2\Delta_\Phi)\, \tilde{f}_{2\Delta_\Phi}(\epsilon)}{\sqrt{\pi}} \sum_{\ell\gg 1} 2^{2\Delta_\Phi}\ell^{X+\frac{1}{2}} K_0(2\ell\sqrt{z}), \tag{5.13}$$

where we used the approximation (5.11). Next we approximate the sum over ℓ as an integral and make use of the formula

$$\int_0^\infty \mathrm{d}x\, x^\alpha K_\beta(x) = 2^{\alpha-1}\Gamma\left(\frac{1+\alpha+\beta}{2}\right)\Gamma\left(\frac{1+\alpha-\beta}{2}\right), \tag{5.14}$$

Since the sum admit only even spin, we have to divide the integral by 2 ,so that $\sum_\ell \to \frac{1}{2}\int \mathrm{d}\ell$. We get the right divergence $z^{-\Delta_\Phi}$ provided that the OPE coefficients behave at large ℓ as:

$$|\lambda_{\langle\Phi\Phi\,[\Phi\Phi]_0\rangle}|^2 = \frac{\sqrt{\pi}\,\ell^{2\Delta_\Phi-\frac{3}{2}}}{2^{2\Delta_\Phi+2\ell-3}\Gamma(\Delta_\Phi)^2} + \mathcal{O}(\ell^{2\Delta_\Phi-\frac{5}{2}}) \tag{5.15}$$

Considering higher orders in ϵ one needs the contribution of operators families $[\Phi\Phi]_n$, for all positive integers n, to match the identity contribution to in the bootstrap equation (5.9). The OPE coefficients can be computed to be

$$|\lambda_{\langle\Phi\Phi\,[\Phi\Phi]_n\rangle}|^2 = \lim_{\ell\to\infty} P^{\mathsf{GFT}}_{n,\ell} + \mathcal{O}(\ell^{2\Delta_\Phi-\frac{5}{2}}), \tag{5.16}$$

where $P^{\mathsf{GFT}}_{n,\ell}$ are the OPE coefficients of double-twist operators $[\Phi\Phi]_n$ in a generalized free theory (GFT). A GFT (also called mean field theory) is dual to a free theory on AdS. It is a theory in which any correlator is given as a sum over the 2-pt function contractions:

$$\langle\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle\Big|_{\mathsf{GFT}} = \frac{1}{(x_{12}^2x_{34}^2)^{\Delta_\Phi}} + \frac{1}{(x_{14}^2x_{23}^2)^{\Delta_\Phi}} + \frac{1}{(x_{13}^2x_{24}^2)^{\Delta_\Phi}} \tag{5.17}$$

and double-twist operators $[\Phi\Phi]_n$ are the only primary operators that appear in the OPE $\Phi \times \Phi$.

5.1.2 Corrections to Dimensions and OPE Coefficients

We can determine the anomalous dimensions of double-twist operators by matching additional terms on the left-hand side of (5.4). Let $\mathcal{O}_m$ be be the smallest-twist operator in the $\Phi\times\Phi$ OPE that is not the unit operator. It can be a low-dimension scalar or the conserved stress energy tensor $\mathcal{O}_m = T_{\mu\nu}$. $\mathcal{O}_m$ contributes a power law divergence in z to the crossing equation

$$\begin{aligned}
&z^{-\Delta_\Phi}(1-\epsilon)^{-\Delta_\Phi} + |\lambda_{\langle\Phi\Phi\mathcal{O}_m\rangle}|^2 \frac{z^{\frac{\tau_m}{2}-\Delta_\Phi}}{(1-\epsilon)^{\Delta_\Phi}}\, k^{0,0;0}_{\tau_m/2+\ell_m}(1-\epsilon) + \dots \\
&= z^{-\Delta_\Phi}(1-\epsilon)^{-\Delta_\Phi} + |\lambda_{\langle\Phi\Phi\mathcal{O}_m\rangle}|^2 \frac{z^{\frac{\tau_m}{2}-\Delta_\Phi}}{(1-\epsilon)^{\Delta_\Phi}} \frac{\Gamma(\tau_m+2\ell_m)}{\Gamma(\tau_m/2+\ell)^2}\left(2\psi(1) - 2\psi\left(\frac{\tau_m}{2}+\ell_m\right) - \ln(\epsilon)\right) + \dots \\
&= \epsilon^{-\Delta_\Phi}\sum_{\mathcal{O}} |\lambda_{\langle\Phi\Phi\mathcal{O}\rangle}|^2 \epsilon^{\tau/2}\, \tilde{f}_\tau(\epsilon)\, k^{0,0;0}_{\tau/2+\ell}(1-z),
\end{aligned} \tag{5.18}$$

where we have used the first term of (5.10). The divergence $z^{\frac{\tau_m}{2}-\Delta_\Phi}$ is reproduced on the right-hand side by the sum over $[\Phi\Phi]_0$ with corrections to scaling dimensions and OPE coefficients that goes as $1/\ell^{\tau_m}$

$$|\lambda_{\langle\Phi\Phi\,[\Phi\Phi]_0\rangle}|^2 = \left(1+\frac{\delta P_0}{\ell^{\tau_m}}\right)\lim_{\ell\to\infty} P^{\mathsf{GFT}}_{0,\ell}. \tag{5.19}$$

To match the $\ln(\epsilon)$ term in (5.18), we can take

$$\tau_{[\Phi\Phi]_0} = 2\Delta_\Phi + \gamma_0(\ell). \tag{5.20}$$

and calculate

$$\gamma_0(\ell) = -\frac{2}{\ell^{\tau_m}}\frac{|\lambda_{\langle\Phi\Phi\mathcal{O}_m\rangle}|^2\Gamma(\Delta_\Phi)^2\Gamma(2\ell_m+\tau_m)}{\Gamma\left(\Delta_\Phi-\frac{\tau_m}{2}\right)^2\Gamma\left(\ell_m+\frac{\tau_m}{2}\right)^2},\quad \delta P_0(\ell) = \gamma_0(\ell)\left(\psi\left(\ell_m+\frac{\tau_m}{2}\right)+\gamma+\ln(2)\right) \tag{5.21}$$

Again comparing higher order in ϵ will allow us to compute similar corrections for $[\Phi\Phi]_n$ for $n>0$.

Note that $|\lambda_{\langle\Phi\Phi\,[\Phi\Phi]_n\rangle}|^2$ calculated here is an asymptotic density of OPE coefficients at large spin. From the argument used here it's not clear how this density is distributed, it could be distributed evenly with one operator at each spin, or one operator every other spin or another way.

Assuming that there is only one operator of the family $[\Phi\Phi]_n$ at each spin, crossing symmetry had allowed us to the compute their scaling dimensions and OPE coefficients as an asymptotic expansion $1/$ spin [56] This assumption has been confirmed in explicit examples, for instance in the three-dimensional Ising model, where the resulting expansion appears to remain accurate all the way down to spin two [59, 60, 55]. Recently, [68, 69, 70, 71] have developed an inversion formula to extract CFT data from 4-point correlators, in the light-cone limit this formula gives an exact analytic result confirming the previous assumptions and providing a new promising method to solve bootstrap equations.

5.2 The Scalar-Fermion Bootstrap Equations

Let ϕ be a primary scalar with scaling dimension Δ_ϕ, ψ a spin $1/2$ Weyl fermion in the rep $(1,0)$ with scaling dimension Δ_ψ and $\overline{\psi} = \psi^\dagger$ transform in the rep $(0,1)$. We are going to study the 4-point correlator

$$\langle\phi(x_1)\psi(x_2,s_2)\phi(x_3)\overline{\psi}(x_4,\bar{s}_4)\rangle \tag{5.22}$$

in the light-cone limits, where one operator approaches the light-cone of another. Let's first set the notation we are going to use in this section.

5.2.1 Notations

There are three ways to take OPE in the correlator (5.22): $s-$channel $\phi(x_1)\times\psi(x_2,s_2)$, $t-$channel $\phi(x_1)\times\overline{\psi}(x_4,\bar{s}_4)$ and $u-$channel $\phi(x_1)\times\phi(x_3)$.

The operators that appears in these OPE's belong to one of three Lorentz reps: bosonic symmetric traceless tensors (ℓ,ℓ) and fermionic rep $(\ell+1,\ell)$ and $(\ell,1+\ell)$, where ℓ is a non-negative integer. To make it easier to differentiate between these different families of operators in the following discussion we will refer to them with different letters

$$\begin{aligned}
&\mathcal{O}^{(\ell)}_{\Delta} \quad \text{in the rep} \quad (\ell,\ell),\\
&\mathcal{A}^{(\ell)}_{\Delta} \quad \text{in the rep} \quad (\ell,\ell+1),\\
&\mathcal{B}^{(\ell)}_{\Delta} \quad \text{in the rep} \quad (\ell+1,\ell),
\end{aligned} \tag{5.23}$$

the subscript Δ is the scaling dimension of the respective operator. Note that the reps $(\ell,\ell+1)$ is the complex conjugate of $(\ell+1,\ell)$ and vice versa. So $\overline{\mathcal{A}}\equiv\mathcal{A}^{\dagger}$ transform as a $\mathcal{B}$ operator, while $\overline{\mathcal{B}}\equiv\mathcal{B}^{\dagger}$ transform as an $\mathcal{A}$ operator.

In the absence of global symmetries (charges) all the bosonic operators are hermitian

$$\overline{\mathcal{O}}^{(\ell)}_{\Delta}\equiv\left(\mathcal{O}^{(\ell)}_{\Delta}\right)^{\dagger}=\mathcal{O}^{(\ell)}_{\Delta}. \tag{5.24}$$

The Correlation Function 5.22 can be written in terms of two tensor structures

$$\langle\phi(x_1)\psi(x_2,s_2)\phi(x_3)\overline{\psi}(x_4,\bar{s}_4)\rangle=\sum_{I=0}^{1}g^{I}(z,\bar{z})\,\mathbf{T}^{I}, \tag{5.25}$$

where the tensor structures are defined as

$$\mathbf{T}^0\equiv\mathcal{K}_4\,\hat{I}^{42}_{31}\Big|_{\text{Poincaré}},\quad \mathbf{T}^1\equiv\mathcal{K}_4\,\hat{I}^{42}\Big|_{\text{Poincaré}},\quad \mathcal{K}_4\equiv\left(x_{12}^2\,x_{34}^2\right)^{-\frac{\Delta_\phi+\Delta_\psi}{2}-\frac{1}{4}}\left(\frac{x_{13}^2}{x_{24}^2}\right)^{\frac{\Delta_\psi-\Delta_\phi}{2}+\frac{1}{4}}. \tag{5.26}$$

$\hat{I}^{42}_{31}$, $\hat{I}^{42}_{31}$ are 4 point tensor structures where defined in section 2.3 and $\mathcal{K}_4$ is the kinematic factor.

5.2.2 The OPE Decompositions

Note that the correlator (5.25) is exactly the $p=1$ seed correlator in section 4. 4.1, therefor the CPW that appear in the $s-$channel (and in $t-$channel) are the seed CPW $W^{\text{seed}}(1)$ and $\overline{W}^{\text{seed}}(1)$ computed there. In the $u-$ channel expansion the operators exchanged are $\mathcal{O}^{(\ell)}_{\Delta}$, so the CPW are related by differential operators to $W^{\text{seed}}(0)$. Here we will change the notation slightly compared to chapter 4 .

The $s-$channel decomposition is given by

$$\langle\phi(x_1)\,\psi(x_2,s_2)\,\phi(x_3)\,\overline{\psi}(x_4,\bar{s}_4)\rangle = \sum_{\mathcal{A},\mathcal{B}}\Big(P_{\mathcal{A}}\,\overline{W}^{\text{seed}}_{\langle\phi\psi\mathcal{A}\rangle\langle\overline{\mathcal{A}}\phi\overline{\psi}\rangle} + P_{\mathcal{B}}\,W^{\text{seed}}_{\langle\phi\psi\mathcal{B}\rangle\langle\overline{\mathcal{B}}\phi\overline{\psi}\rangle}\Big), \tag{5.27}$$

$W^{\text{seed}}_{\langle\phi\psi\mathcal{B}\rangle\langle\overline{\mathcal{B}}\phi\overline{\psi}\rangle}$ and $\overline{W}^{\text{seed}}_{\langle\phi\psi\mathcal{A}\rangle\langle\overline{\mathcal{A}}\phi\overline{\psi}\rangle}$ are $W^{\text{seed}}(p=1)$ and $\overline{W}^{\text{seed}}(p=1)$ computed in chapter 4 , with the right replacement of parameters that depend on external operators. We have also introduced the subscript to clearly indicate what operator is exchanged. We have also defined the OPE coefficients

$$P_{\mathcal{A}} \equiv \lambda_{\langle\phi\psi\mathcal{A}\rangle}\lambda_{\langle\overline{\mathcal{A}}\phi\overline{\psi}\rangle}\,, \qquad P_{\mathcal{B}} \equiv \lambda_{\langle\phi\psi\mathcal{B}\rangle}\lambda_{\langle\overline{\mathcal{B}}\phi\overline{\psi}\rangle}, \tag{5.28}$$

where $\lambda_{\langle f_1 f_2 f_3\rangle}$ is the OPE coefficient that appears in the 3-point function $\langle f_1 f_2 f_3\rangle$. The 3-point functions involving the coefficients (5.28) are defined in the section 2.9 and they have the following relation

$$\lambda^*_{\langle\phi\psi\mathcal{A}\rangle} = \lambda_{\langle\overline{\mathcal{A}}\phi\overline{\psi}\rangle}, \qquad \lambda^*_{\langle\phi\psi\mathcal{B}\rangle} = -\lambda_{\langle\overline{\mathcal{B}}\phi\overline{\psi}\rangle}. \tag{5.29}$$

or equivalently

$$P_{\mathcal{A}} = \left|\lambda_{\langle\phi\psi\mathcal{A}\rangle}\right|^2, \qquad P_{\mathcal{B}} = -\left|\lambda_{\langle\phi\psi\mathcal{B}\rangle}\right|^2, \tag{5.30}$$

and $P_{\mathcal{A}}$ is a positive real number while $P_{\mathcal{B}}$ is negative real number.

The $t-$ channel expansion is similar to the $s-$channel up to exchanging $x_1 \leftrightarrow x_3$

$$\langle\phi(x_1)\,\psi(x_2,s_2)\,\phi(x_3)\,\overline{\psi}(x_4,\bar{s}_4)\rangle = \sum_{\mathcal{A},\mathcal{B}}\Big(P_{\mathcal{A}}\,\overline{W}^{\text{seed}}_{\langle\phi\psi\mathcal{A}\rangle\langle\overline{\mathcal{A}}\phi\overline{\psi}\rangle} + P_{\mathcal{B}}\,W^{\text{seed}}_{\langle\phi\psi\mathcal{B}\rangle\langle\overline{\mathcal{B}}\phi\overline{\psi}\rangle}\Big)\bigg|_{1\leftrightarrow 3}. \tag{5.31}$$

In the $u-$channel expansion symmetric traceless operators $\mathcal{O}$ appear. Their CPWs are related to $W^{\text{seed}}(p=0)$ by differential operators as explained in chapter 3

$$\langle\phi(x_1)\,\psi(x_2,s_2)\,\phi(x_3)\,\overline{\psi}(x_4,\bar{s}_4)\rangle = \sum_{\mathcal{O}}\sum_{a=1}^{2}\lambda_{\langle\phi\phi\mathcal{O}\rangle}\lambda^a_{\langle\mathcal{O}\psi\overline{\psi}\rangle}\mathcal{D}^a_{34}W^{\text{seed}}_{\langle\phi\phi\mathcal{O}\rangle\langle\mathcal{O}\phi'\phi'\rangle}\bigg|_{2\leftrightarrow 3}. \tag{5.32}$$

Let's explain how to arrive to this form (5.32): $W^{\text{seed}}_{\langle\phi\phi\mathcal{O}\rangle\langle\mathcal{O}\phi'\phi'\rangle}$ is $W^{\text{seed}}(p=0)$ with subscript that to external and exchanged operator. ϕ' is a scalar. The CPW $W^{\text{seed}}(p=0)$ is computed for $\mathcal{O}$ appearing in the correlator $s-$channel, because of this we have to take $2\leftrightarrow 3$. The differential operators $\mathcal{D}^a_{34}$ act on the points 3 and 4, they relate the correlator $\langle\mathcal{O}\psi\overline{\psi}\rangle$ to $\langle\mathcal{O}\phi'\phi'\rangle$. The former correlator has two tensor structures, hence the index $a=1,2$. Using the techniques of chapter 3, we find that $[\phi'] = \Delta_\psi + \frac{1}{2}$ and the differential operators to be

$$\begin{aligned}
\mathcal{D}^1_{34} &= I^{43}\,,\\
\mathcal{D}^2_{34} &= \frac{4\Delta-(\ell+\Delta)^2}{4\ell(\Delta-1)}\,I^{43} + \frac{1}{4\ell(1-\Delta)}\nabla_{43}\tilde{D}_{34}\tilde{D}_{43}X^{-1}_{34},
\end{aligned} \tag{5.33}$$

where ℓ, Δ are the spin and dimension of the exchanged operator $\mathcal{O}$.

We will define

$$P_{\mathcal{O}}{}^{a} \equiv \lambda_{\langle\phi\phi\mathcal{O}\rangle}\lambda^{a}_{\langle\mathcal{O}\psi\overline{\psi}\rangle} \tag{5.34}$$

One of the operators $\mathcal{O}$ exchanged in this channel is the unit operator, its contribution is just a product of 2-point correlators

$$\langle\phi(x_1)\,\psi(x_2,s_2)\,\phi(x_3)\,\overline{\psi}(x_4,\bar{s}_4)\rangle = \frac{-i\,I^{42}|_{\text{Poincaré}}}{(x_{13}^2)^{\Delta_\phi}(x_{24}^2)^{\Delta_\psi+\frac{1}{2}}} + \cdots = \frac{-i\,\mathbf{T}^1}{(z\bar{z})^{-\frac{1}{2}(\Delta_\phi+\Delta_\psi)-\frac{1}{4}}} + \ldots \tag{5.35}$$

Conformal Partial Waves (CPW): Each one of the CPW in $s-$, $t-$ and $u-$channels can be expanded in the basis $(\mathbf{T}^0, \mathbf{T}^1)$ of the correlation function 5.25. So when we equate any two of the three crossing channels we get two equations, one for each of the linearly independent tensor structure $\mathbf{T}^I$.

$$W^{\text{seed}}_{\langle\phi\psi\mathcal{B}\rangle\langle\overline{\mathcal{B}}\phi\overline{\psi}\rangle} = \sum_{e=0}^{1} G_e^{(1)}(z,\bar{z})\mathbf{T}^e \tag{5.36}$$

$$W^{\text{seed}}_{\langle\phi\psi\mathcal{B}\rangle\langle\overline{\mathcal{B}}\phi\overline{\psi}\rangle}\Big|_{1\leftrightarrow3} = \sum_{e=0}^{1} G_e^{(1)}(1-z,1-\bar{z})(\mathbf{T}^e|_{1\leftrightarrow3}) \tag{5.37}$$

where $G_e^{(p)}$ are the seed conformal blocks which were computed in chapter 4 , under permutation $z|_{1\leftrightarrow3} = 1-z$ and $\bar{z}|_{1\leftrightarrow3} = 1-\bar{z}$. The CPW $\overline{W}^{\text{seed}}_{\langle\phi\psi\mathcal{A}\rangle\langle\overline{\mathcal{A}}\phi\overline{\psi}\rangle}$ have similar expansion in terms of $\overline{G}_e^{(p)}$ seed blocks. The tensor structures defined in (5.26) have the following properties under permutation

$$\mathbf{T}^0|_{1\leftrightarrow3} = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\frac{\Delta_\phi+\Delta_\psi+\frac{1}{2}}{2}}(\mathbf{T}^1-\mathbf{T}^0)\,, \quad \mathbf{T}^1|_{1\leftrightarrow3} = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\frac{\Delta_\phi+\Delta_\psi+\frac{1}{2}}{2}}\mathbf{T}^1\,. \tag{5.38}$$

Meanwhile $u-$channel CPW can be written

$$W^{a}_{\langle\phi\phi\mathcal{O}\rangle\langle\mathcal{O}\psi\overline{\psi}\rangle} = (\omega\bar{\omega})^{\frac{1}{2}(\Delta_\phi+\Delta_\psi+\frac{1}{2})}\sum_{e=0}^{1}\tilde{\mathcal{D}}_e^a(\partial_\omega,\partial_{\bar{\omega}})\,f_{\Delta,\ell}(\omega,\bar{\omega})\,\mathbf{T}^e, \tag{5.39}$$

where

$$\omega \equiv \frac{1}{z} = z|_{2\leftrightarrow3}\,, \quad \bar{\omega} \equiv \frac{1}{\bar{z}} = \bar{z}|_{2\leftrightarrow3}\,, \tag{5.40}$$

and $f_{\Delta,\ell}$ is the seed scalar block defined in (5.5).

5.2.3 The Bootstrap Equations

Here we investigate two sets of bootstrap equations :

the $u-t$ equations

$$\langle \phi(x_1)\,\psi(x_2,s_2)\,\phi(x_3)\,\overline{\psi}(x_4,\bar{s}_4)\rangle = \langle \phi(x_1)\,\psi(x_2,s_2)\,\phi(x_3)\,\overline{\psi}(x_4,\bar{s}_4)\rangle \tag{5.41}$$

and the $s-t$ equations

$$\langle \phi(x_1)\,\psi(x_2,s_2)\,\phi(x_3)\,\overline{\psi}(x_4,\bar{s}_4)\rangle = \langle \phi(x_1)\,\psi(x_2,s_2)\,\phi(x_3)\,\overline{\psi}(x_4,\bar{s}_4)\rangle \tag{5.42}$$

5.3 Analytic Analysis of Bootstrap Equations

In this section we will analyze the bootstrap equations (5.41) and (5.42) in the respective light-cone limit. These limits will be $\omega \ll 1-\bar{\omega} \ll 1$ for $u-t$ bootstrap eq. (5.41) and $z \ll 1-\bar{z} \ll 1$ for the $s-t$ bootstrap eq. (5.42). If the scalar case is an indication, we would expect to prove the existence of double-twist operators $[\phi\overline{\psi}]_n$ in the large spin tail of the spectrum of the CFT. Our analysis will prove this picture, double-twist operators are required to produce the contribution of low-twist operators in the crossed channel. It then possible to compute the anomalous dimensions and OPE coefficients of the double-twist operators as asymptotic expansions in $1/$spin.

In this case double-twist operators belong to two families: $\mathcal{A}$ in rep $(\ell,\ell+1)$ and $\mathcal{B}$ in the rep $(\ell+1,\ell)$. Because of the light-cone limits we study these operators dominate the $t-$channel expansion. At leading order of $1/\ell$ they resemble GFT operators and reproduce the unit operator in the $u-$channel. At subleading order of $1/\ell$ they also reproduce low-twist bosonic operators in the $u-$channel and low-twist fermionic oprators in the $s-$channel.

$$\begin{aligned}
\tau_{\mathcal{A}} &= \Delta_\phi + \Delta_\psi - \frac{1}{2} + 2n + \gamma_{\mathcal{A}}(\ell,n), \qquad \gamma_{\mathcal{A}}(\ell,n) = \gamma^b_{\mathcal{A}}(\ell,n) + \gamma^f_{\mathcal{A}}(\ell,n) \\
\tau_{\mathcal{B}} &= \Delta_\phi + \Delta_\psi + \frac{1}{2} + 2n + \gamma_{\mathcal{B}}(\ell,n), \qquad \gamma_{\mathcal{B}}(\ell,n) = \gamma^b_{\mathcal{B}}(\ell,n) + \gamma^f_{\mathcal{B}}(\ell,n),
\end{aligned} \tag{5.43}$$

where we differentiate the anomalous dimensions γ according to their source, they can result from low-twist bosonic operator γ_b (sec 5.3.1) or a low-twist fermionic operator γ_f (sec 5.3.2). We make a similar distinction for the corrections of OPE coefficients

$$P_{\mathcal{A}} = \lim_{\ell\to\infty} P^{GFT}_{\mathcal{A}}(n,\ell) + \delta P_{\mathcal{A}}, \quad \delta P_{\mathcal{A}} = \delta P^b_{\mathcal{A}} + \delta P^f_{\mathcal{A}}, \tag{5.44}$$

$$P_{\mathcal{B}} = \lim_{\ell\to\infty} P^{GFT}_{\mathcal{B}}(n,\ell) + \delta P_{\mathcal{B}}, \quad \delta P_{\mathcal{B}} = \delta P^b_{\mathcal{B}} + \delta P^f_{\mathcal{B}}. \tag{5.45}$$

In the next two sections we go into details of these computations, which largely follows that of [53] we reviewed in section 5.1.

5.3.1 Equations: u-t-channel

We start by analyzing the equations (5.41). The light-cone limit we will take here is $\omega \ll 1-\overline{\omega} \ll 1$ which in equivalent to x_2 approaching the light-cone of x_4 (see fig. 5.2).

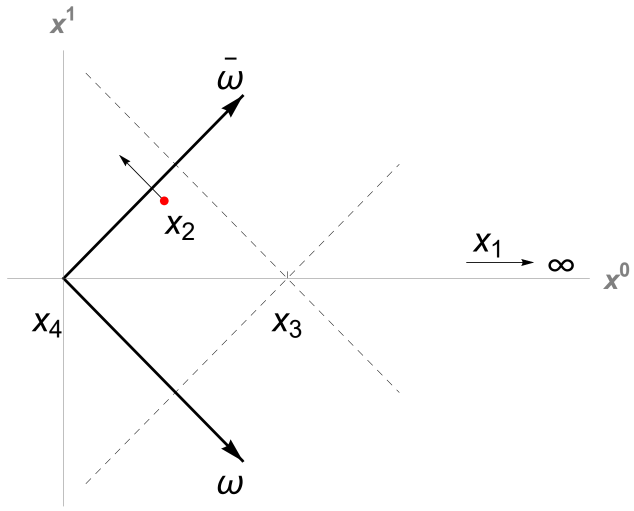

Figure 5.2: Lght-Cone Limit II: On $\omega, \overline{\omega}$ plane we use the conformal symmetry to fix the points to the configuration x_4 at $(0,0)$, x_2 at $(\omega, \overline{\omega})$, x_3 at $(1,1)$ and send x_1 to ∞ . In the light-cone limit $\omega \to 0$ x_2 is approaching the light cone of x_4.

Within this limit the, individual terms on the sides of bootstrap equation (5.41) behaves differently, schematically as[2]

$$R^I + \sum_{\mathcal{O}_m} P^a_{\mathcal{O}_m} {G'_a}^I(\omega, \overline{\omega}) = \sum_{\mathcal{A}} P_{\mathcal{A}} {G''_{\mathcal{A}}}^I(1-\omega, 1-\overline{\omega}) + \sum_{\mathcal{B}} P_{\mathcal{B}} {G''_{\mathcal{B}}}^I(1-\omega, 1-\overline{\omega}), \tag{5.47}$$

where G' and G'' is a leading term in the light cone limit, R^I is the identity contribution, where $R^1 = 1$ and $R^0 = 0$ and $\mathcal{O}_m$ are low-twist symmetric traceless operators exchanged in the $u-$channel. Here we just indicate the leading behaviour

$$G' \sim \omega^{\tau_{\mathcal{O}_m}/2}, \quad G'' \sim \omega^{\Delta_\psi + 1/2} \ln \omega. \tag{5.48}$$

The LHS is dominated by power-law divergence in ω coming from low-twist operators, leading by the unit operator ($\tau_1 = 0$). On RHS however each term made up of hypergeometric function of $1-\omega$ and has at most a logarithmic divergence.

Only an infinite sum of terms in the RHS can possibly reproduce the power low divergence on the LHS. Specifically, the sum of high spin operators with fixed twist will give a power low divergence on the RHS. Matching also the powers of $\overline{\omega}$, we conclude that these large-spin operators should have twists, in leading powers of $1/\ell$, that match those of double-twist operators in GFT (5.43).

The sum over ℓ will give rise to the leading singularity in ω (unit operator contribution the $u-$ch)

[2]To express $t-$channnel blocks in $\omega, \overline{\omega}$ variables, we make use of the $\mathcal{F}$ function property

$$\mathcal{F}^{-(a,b,c)}_{\rho_1,\rho_2}\left(1-\frac{1}{\omega}, 1-\frac{1}{\overline{\omega}}\right) = (-1)^{\rho_1-\rho_2}(\omega\overline{\omega})^a \; \mathcal{F}^{-(a,c-b,c)}_{\rho_1,\rho_2}(1-\omega, 1-\overline{\omega}) \tag{5.46}$$

provided that the OPE coefficients behave asymptotically as:

$$\begin{aligned}
P_{\mathcal{A}}(n,\ell) &\approx -\frac{2^{-2n-2\ell}\sqrt{2\pi}\,(\Delta_\phi-1)_n\left(\Delta_\psi-\frac{3}{2}\right)_n}{n!\,\Gamma(\Delta_\phi)\,\Gamma\left(\Delta_\psi+\frac{1}{2}\right)\left(\Delta_\phi+\Delta_\psi+n-\frac{7}{2}\right)_n}\left(\frac{\ell}{2}\right)^{\Delta_\phi+\Delta_\psi-1} &&= \lim_{\ell\to\infty} P_{\mathcal{A}}^{\mathsf{GFT}},\\
P_{\mathcal{B}}(n,\ell) &\approx \frac{2^{-2n-2\ell-2}\sqrt{2\pi}\,(\Delta_\phi-1)_n\left(\Delta_\psi-\frac{3}{2}\right)_{n+1}}{n!\,\Gamma(\Delta_\phi)\,\Gamma\left(\Delta_\psi+\frac{1}{2}\right)\left(\Delta_\phi+\Delta_\psi+n-\frac{5}{2}\right)_n}\left(\frac{\ell}{2}\right)^{\Delta_\phi+\Delta_\psi-2} &&= \lim_{\ell\to\infty} P_{\mathcal{B}}^{\mathsf{GFT}},
\end{aligned} \tag{5.49}$$

matching GFT coefficients in the limit $\ell\to\infty$, while n takes all non-negative integer values.

Actually the asymptotic behavior of $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ at large ℓ match the leading and subleading terms in GFT coefficients expansion around $\ell\to\infty$. We can state these results by saying the theory spectrum contains operators that looks as general free theory at high spin.

Now we solve the bootstrap equation to sub-leading order in ω going beyond the identity contribution in the $u-$ channel to low-twist operators $\mathcal{O}_m$ in the OPE $\phi\times\phi$ (and $\psi\times\overline{\psi}$). Assuming there is a stress-tensor, then $\tau_{\mathcal{O}_m}\leq 2$, in this case the $\mathcal{O}_m$ contribute a power low divergence in ω in the LHS of (5.47). This divergence is again reproduced by a sum over spin of double-twist operators, with corrections to their twists (γ_X^b, $\gamma_{\overline{Y}}^b$) and OPE coefficients ($\delta P_{\mathcal{A}}^b$, $\delta P_{\mathcal{B}}^b$). The corrections fall with increasing spin with a rate controlled by the $\tau_{\mathcal{O}_m}$, they go as $\ell^{-\tau_{\mathcal{O}_m}}$

$$\begin{aligned}
\gamma_{\mathcal{A}}(\ell,n) &= -i\,\ell^{-\tau_{\mathcal{O}_m}}\frac{2(-1)^{\ell_{\mathcal{O}_m}}\Gamma\left(\tau_{\mathcal{O}_m}+2\ell_{\mathcal{O}_m}\right)P^1_{\mathcal{O}_m}}{(\Delta_\phi)_{-\frac{\tau_{\mathcal{O}_m}}{2}}\left(\Delta_\psi+\frac{1}{2}\right)_{-\frac{\tau_{\mathcal{O}_m}}{2}}\Gamma\left(\frac{\tau_{\mathcal{O}_m}}{2}+\ell_{\mathcal{O}_m}\right)^2}\\
&\quad\times\sum_{i=0}^{n}\frac{(n)_i\left(\Delta_\phi+\Delta_\psi+n-\frac{7}{2}\right)_i\left(\frac{\tau_{\mathcal{O}_m}}{2}+\ell_{\mathcal{O}_m}-i\right)_{2i}}{i!(\Delta_\phi-1)_i\left(\Delta_\psi-\frac{3}{2}\right)_i},\\
\gamma_{\mathcal{B}}(\ell,n) &= -i\,\ell^{-\tau_{\mathcal{O}_m}}\frac{2(-1)^{\ell_{\mathcal{O}_m}}\Gamma\left(\tau_{\mathcal{O}_m}+2\ell_{\mathcal{O}_m}\right)}{(\Delta_\phi)_{-\frac{\tau_{\mathcal{O}_m}}{2}}\left(\Delta_\psi+\frac{1}{2}\right)_{-\frac{\tau_{\mathcal{O}_m}}{2}}\Gamma\left(\frac{\tau_{\mathcal{O}_m}}{2}+\ell_{\mathcal{O}_m}\right)^2}\\
&\quad\sum_{i=0}^{n}\frac{(n)_i\left(\Delta_\phi+\Delta_\psi+n-\frac{5}{2}\right)_i\left(\frac{\tau_{\mathcal{O}_m}}{2}+\ell_{\mathcal{O}_m}-i\right)_{2i}}{i!(\Delta_\phi-1)_i(\Delta_\psi-\frac{1}{2})_i}\times\\
&\quad\times\left(\frac{2\Delta_\psi-\tau_{\mathcal{O}_m}-3}{(2\Delta_\psi-3)}P^1_{\mathcal{O}_m}-\frac{2\Delta_\psi-\tau_{\mathcal{O}_m}-1}{(2\Delta_\psi-3)}P^2_{\mathcal{O}_m}\right),
\end{aligned} \tag{5.50}$$

We also solved for $\delta P(n)$ and $\delta\overline{P}(n)$ for $n=0,1,2$

$$\begin{aligned}
\delta P_{\mathcal{A}}(\ell,0) &= \gamma_{\mathcal{A}}(\ell,0)\left(\psi\left(\tfrac{\tau_{\mathcal{O}_m}}{2}+\ell_{\mathcal{O}_m}\right)-\psi(1)-\ln 2\right),\\
\delta P_{\mathcal{B}}(\ell,0) &= \gamma_{\mathcal{B}}(\ell,0)\left(\psi\left(\tfrac{\tau_{\mathcal{O}_m}}{2}+\ell_{\mathcal{O}_m}\right)-\psi(1)-\ln 2\right),
\end{aligned} \tag{5.51}$$

where $\tau_{\mathcal{O}_m}$ and $\ell_{\mathcal{O}_m}$ are the twist and spin of O_m the minimum-twist operator exchanged in the $u-$ channel, $(x)_n=\Gamma(x+n)/\Gamma(x)$ is the Pocchamer symbol and $\psi(y)=\Gamma'(y)/\Gamma(y)$ is the digamma function.

When $\mathcal{O}_m$ is the stress-energy tensor. The Ward identities imply

$$\lambda_{\langle\phi\phi T\rangle}=-\frac{2\Delta_\phi}{3\pi^2},\quad \lambda^1_{\langle T\psi\overline{\psi}\rangle}=\frac{-i\,(2\Delta_\psi-3)}{12\pi^2},\quad \lambda^2_{\langle T\psi\overline{\psi}\rangle}=\frac{i}{2\pi^2}, \tag{5.52}$$

see appendix F for details. The anomalous dimensions (5.50) are negative $\forall n$.
Negativity of the anomalous dimensions caused by the exchange of the stress tensor in 4 dimensional CFT has been related to the gravity being attractive in 5 dimensional Ads [53, 54].

5.3.2 Equations: s-t-channel

We have already proved from the $t-u$ bootstrap equations that at high spin, the spectrum of a CFT contains a GFT-like part. We also computed the correction due to the exchange of a bosonic minimum-twist operator. In this section, we will instead compute the correction due a fermionic minimum-twist operator, since in both $s-$ and $t-$ channel fermionic operators are exchanged.

The limit we will take here is $z \ll 1-\bar{z} \ll 1$. In this limit the $s-$ channel expansion get the leading contributions from minimum-twist fermions. Suppose a low-twist operator Ψ_m, which transform in the rep $(\ell_{\Psi_m}+1, \ell_{\Psi_m})$ and has a twist τ_{Ψ_m} , is exchanged in the $s-$channel with OPE coefficient P_{Ψ_m}. The complex conjugate operator $\overline{\Psi}_m$, which transform in the rep $(\ell_{\Psi_m}, \ell_{\Psi_m}+1)$, is exchanged in the $s-$channel with OPE coefficient $P_{\overline{\Psi}_m}$ [3]. In a similar reasoning to the previous sections we compute the $n=0$ fermionic contribution to the anomalous dimensions

$$\begin{aligned}
\gamma^f_{\mathcal{A}}(\ell,0) &= 2\,(-1)^{\ell}\ell^{-\tau_{\Psi_m}}\,P_{\overline{\Psi}_m}\frac{\Gamma(\Delta_\phi)\Gamma(\Delta_\psi+\frac{1}{2})\Gamma(1+2\ell_{\Psi_m}+\tau_{\Psi_m})}{\Gamma\big(\frac{1}{4}(2\Delta_\phi+2\Delta_\psi-2\tau_{\Psi_m}+1)\big)^2}\times \\
&\frac{(-1)^{\ell_{\Psi_m}}}{\Gamma\big(\frac{1}{4}(2\Delta_\phi-2\Delta_\psi+2\tau_{\Psi_m}+4\ell_{\Psi_m}+3)\big)\Gamma\big(\frac{1}{4}(-2\Delta_\phi+2\Delta_\psi+2\tau_{\Psi_m}+4\ell_{\Psi_m}+1)\big)}, \\
\gamma^f_{\mathcal{B}}(\ell,0) &= 2\,(-1)^{1+\ell}\ell^{-\tau_{\Psi_m}}\,P_{\Psi_m}\frac{\Gamma(\Delta_\phi)\Gamma(\Delta_\psi+\frac{1}{2})\Gamma(1+2\ell_{\Psi_m}+\tau_{\Psi_m})}{(\Delta_\psi-\frac{3}{2})\,\Gamma\big(\frac{1}{4}(2\Delta_\phi+2\Delta_\psi-2\tau_{\Psi_m}-1)\big)^2}\times \\
&\frac{(-1)^{\ell_{\Psi_m}}}{\Gamma\big(\frac{1}{4}(2\Delta_\phi-2\Delta_\psi+2\tau_{\Psi_m}+4\ell_{\Psi_m}+1)\big)\Gamma\big(\frac{1}{4}(-2\Delta_\phi+2\Delta_\psi+2\tau_{\Psi_m}+4\ell_{\Psi_m}+3)\big)},
\end{aligned} \tag{5.53}$$

and the correction to the OPE coefficients

$$\begin{aligned}
\delta P^f_{\mathcal{A}}(\ell,0) &= \frac{1}{2}\,\gamma^f_{\mathcal{A}}(\ell,0)\,\Big(-2\psi(1)-\ln 4+\psi\left(\frac{\Delta_\phi-\Delta_\psi+\tau_{\Psi_m}}{2}+\ell_{\Psi_m}+\frac{3}{4}\right) \\
&\quad+\psi\big(\frac{1}{4}(-2\Delta_\phi+2\Delta_\psi+2\tau_{\Psi_m}+4\ell_{\Psi_m}+1)\big)\Big), \\
\delta P^f_{\mathcal{B}}(\ell,0) &= \frac{1}{2}\,\gamma^f_{\mathcal{B}}(\ell,0)\,\Big(-2\psi(1)-\ln 4+\psi\big(\frac{1}{4}(2\Delta_\phi-2\Delta_\psi+2\tau_{\Psi_m}+4\ell_{\Psi_m}+1)\big) \\
&\quad+\psi\big(\frac{1}{4}(-2\Delta_\phi+2\Delta_\psi+2\tau_{\Psi_m}+4\ell_{\Psi_m}+3)\big)\Big).
\end{aligned} \tag{5.54}$$

[3] It is not necessary that both the operator Ψ_m and its complex conjugate $\overline{\Psi}_m$ appear in the OPE expansion, their OPE coefficients P_{Ψ_m} and $P_{\overline{\Psi}_m}$ are independent and either can be set to zero.

6. Conclusions

In this thesis we explore 4D CFT. For the larger part of (chapter 2 - 4) our gaol is to provide the tools needed to bootstrap a correlators with operators in general Lorentz rep.

We computed in chapter 2 the most general three point function occurring in a 4D CFT between bosonic and fermionic primary fields in arbitrary representations of the Lorentz group. We have used the 6D embedding formalism in twistor space with an index free notation, as introduced in ref.[18], to efficiently recast the result in terms of 6D SU(2,2) invariants. The main result of the chapter is the compact 6D formula (2.88), from which any 4D correlator can easily be extracted. The constraints arising from conserved operators take a very simple form, see eqs.(2.107) and (2.108), and can be solved within the formalism.

Understanding three-point functions is the first step. Next, in chapter 3, we use our knowledge of three-point functions and introduce a set of differential operators, eqs.(3.14), (3.15) (3.16) and (3.23), that enables us to relate different three-point functions. In particular, three-point tensor correlators with different tensor structures can always be related to a three-point function with a single tensor structure.

Particular attention has been devoted to the three point functions of two traceless symmetric and one mixed tensor operator, where explicit independent bases have been provided, eqs.(3.48) and (3.51). These results allow us to deconstruct four point tensor correlators, since we can express the CPW in terms of a few CPW seeds. We argue that the simplest CPW seeds are those associated to the four point functions of two scalars, one $\mathcal{O}^{2\delta,0}$ and one $\mathcal{O}^{0,2\delta}$ field, that have only $2\delta+1$ independent tensor structures. The argument extends beyond the traceless symmetric operators and $\delta = p/2$ can be an integer or half integer.

In chapter 4 we do the computation of the seed conformal blocks $G_e^{(p)}$ (and $\overline{G}_e^{(p)}$) associated to the exchange of mixed symmetry bosonic and fermionic primary operators $\mathcal{O}^{(\ell,\ell+p)}$ (and $\overline{\mathcal{O}}^{(\ell+p,\ell)}$) in the four point functions (4.6). We have found a totally general expression for $G_e^{(p)}$ for any e, p, Δ, ℓ and external scaling dimensions, by solving the Casimir set of differential equations, that can be written in the compact form (4.30). The shadow formalism has been of fundamental assistance to deduce it and also as a useful cross check for the validity of the results. The final expression for the conformal blocks is given in eq.(4.112).

The conformal blocks are expressed in terms of coefficients $c^e_{m,n}$, that can be determined recursively, e.g. by means of eq.(4.111). For each CB, the coefficients $c^e_{m,n}$ span a 2D octagon-shape lattice in the (m,n) plane, with sizes that depend on p and e and increase as p increases. We have reported in Appendix D the explicit form of $c^e_{m,n}$ for the simplest case $p=1$. We have not reported the $c^e_{m,n}$ for higher values of p, since their number and complexity grows with p.

In chapter 5 we use the technology provided in chapter 2 - 4 to write the bootstrap equations for $\langle\phi(x_1)\psi(x_2,s_2)\phi(x_3)\overline{\psi}(x_4,\bar{s}_4)\rangle$ and solve it in the light-cone limit. We prove the existence of two families of fermionic double-twist operators, resembling the generalized free theory (GFT) . The main results of the chapter is anomalous dimensions (5.50) and corrections to OPE coefficients (5.51) related to low-twist boson in the spectrum, and the corresponding (5.53) and (5.54) related to low-twist fermions in the spectrum. In appendix E we compute the OPE coefficients of double-twist operators in a GFT.

Appendix A. Conventions of 6D Twistor Space

We follow the conventions of Wess and Bagger [72] for the two-component spinor algebra in 4D (See in particular Appendix A). Six dimensional vector indices are denoted by $M, N, \ldots$, with $M = \{\mu, +, -\}$; four dimensional vector indices are denoted by $\mu, \nu, \ldots$; four-dimensional spinor indices are denoted by dotted and undotted Greek letters, $\alpha, \beta, \ldots$, $\dot{\alpha}, \dot{\beta}, \ldots$; six-dimensional spinor (twistor) indices are denoted by $a, b, \ldots$, with $a = \{\alpha, \dot{\alpha}\}$. We use capital and small letters for 6D and 4D tensors; in particular, 6D and 4D coordinates are denoted as X^M and x^μ, where $x^\mu = X^\mu / X^+$.

The conformal group $SO(4,2)$ is locally isomorphic to $SU(2,2)$. The spinorial representations $\mathbf{4}_\pm$ of $SO(4,2)$ are mapped to the fundamental and anti-fundamental representations of $SU(2,2)$. Roughly speaking, $SO(4,2)$ spinor indices turn into $SU(2,2)$ twistor indices. We denote by V_a and $\overline{W}^a \equiv W^{\dagger b} \rho_b^a$, where ρ is the SU(2,2) metric, twistors transforming in the fundamental and anti-fundamental of $SU(2,2)$, respectively:

$$V \to UV\,, \quad \overline{W} \to \overline{W}\,\overline{U}\,. \tag{A.1}$$

In eq.(A.1), U and $\overline{U} \equiv \rho\, U^\dagger \rho$ satisfy the condition $\overline{U}U = U\overline{U} = 1$.

The non-vanishing components of the 6D metric η_{MN} and its inverse η^{MN} in light-cone coordinates are

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)\,, \quad \eta_{+-} = \eta_{-+} = \frac{1}{2}\,, \quad \eta^{+-} = \eta^{-+} = 2\,. \tag{A.2}$$

Six dimensional Gamma matrices Γ^M are constructed by means of the 6D matrices Σ^M and $\overline{\Sigma}^M$, analogues of σ^μ and $\bar{\sigma}^\mu$ in 4D:

$$\Gamma^M = \begin{pmatrix} 0 & \Sigma^M \\ \overline{\Sigma}^M & 0 \end{pmatrix}\,, \tag{A.3}$$

obeying the commutation relation

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}\,. \tag{A.4}$$

It is very useful to choose a basis for the Σ and $\bar{\Sigma}$ matrices where they are antisymmetric. This is explicitly given by

$$\begin{aligned} \Sigma^M_{ab} &= \left\{ \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\gamma}} \\ -\bar{\sigma}^{\mu\dot{\alpha}\gamma}\epsilon_{\beta\gamma} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \begin{pmatrix} -2\epsilon_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \right\}, \\ \overline{\Sigma}^{Mac} &= \left\{ \begin{pmatrix} 0 & -\epsilon^{\alpha\gamma}\sigma^\mu_{\gamma\dot{\beta}} \\ \epsilon_{\dot{\alpha}\dot{\gamma}}\bar{\sigma}^{\mu\dot{\gamma}\beta} & 0 \end{pmatrix}, \begin{pmatrix} -2\epsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2\epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \right\}, \end{aligned} \tag{A.5}$$

where, in order, $M = \{\mu, +, -\}$ in eq.(A.5). The 6D spinor Lorentz generators are defined as

$$\begin{aligned} \left(\Sigma^{MN}\right)_a^{\ b} &= \frac{1}{4}(\Sigma^M_{ac}\overline{\Sigma}^{N\,cb} - \Sigma^N_{ac}\overline{\Sigma}^{M\,cb})\,, \\ \left(\overline{\Sigma}^{MN}\right)^a_{\ b} &= \frac{1}{4}(\overline{\Sigma}^{M\,ac}\Sigma^N_{cb} - \overline{\Sigma}^{N\,ac}\Sigma^M_{cb})\,. \end{aligned} \tag{A.6}$$

Useful relations among the Σ^M and $\overline{\Sigma}^M$ matrices, used repeatedly in this thesis, are the following:

$$\begin{aligned} \overline{\Sigma}^{Mab} &= -\frac{1}{2}\epsilon^{abcd}\Sigma^M_{cd}, \qquad \Sigma^M_{ab} = -\frac{1}{2}\epsilon_{abcd}\overline{\Sigma}^{Mcd}, \\ \Sigma^M_{ab}\Sigma_{Mcd} &= 2\epsilon_{abcd}, \qquad \overline{\Sigma}^{Mab}\overline{\Sigma}^{cd}_M = 2\epsilon^{abcd}, \\ \Sigma^M_{ab}\overline{\Sigma}^{cd}_M &= -2(\delta^c_a\delta^d_b - \delta^d_a\delta^c_b)\,, \end{aligned} \tag{A.7}$$

where $\epsilon_{1234} = \epsilon^{1234} = +1$.

The 6D null cone is defined by

$$X^2 = X^M X^N \eta_{MN} = 0 \Longrightarrow X^- = -\frac{X_\mu X^\mu}{X^+} \,. \tag{A.8}$$

On the null cone we have

$$X_1 \cdot X_2 = X_1^M X_2^N \eta_{MN} = -\frac{1}{2} X_1^+ X_2^+ (x_1 - x_2)^\mu (x_1 - x_2)_\mu \,, \tag{A.9}$$

where $x^\mu = X^\mu / X^+$ are the standard 4D coordinates. We define

$$x_{ij}^\mu \equiv x_i^\mu - x_j^\mu \,, \qquad x_{ij}^2 \equiv x_{ij}^\mu x_{\mu,ij} \,. \tag{A.10}$$

Twistor space-coordinates are defined as

$$\mathbf{X}_{ab} \equiv X_M \Sigma^M_{ab} = -\mathbf{X}_{ba} \,, \quad \overline{\mathbf{X}}^{ab} \equiv X_M \overline{\Sigma}^{Mab} = -\overline{\mathbf{X}}^{ba} \,. \tag{A.11}$$

A very useful relation is

$$\mathbf{X}_{ac} \overline{\mathbf{X}}^{cb} = X_M X_N \Sigma^M_{ac} \overline{\Sigma}^{N\,cb} = \frac{1}{2} X_M X_N (\Sigma^M_{ac} \overline{\Sigma}^{N\,cb} + \Sigma^N_{ac} \overline{\Sigma}^{M\,cb}) = X_M X^M \, \delta_a^b = X^2 \, \delta_a^b, \tag{A.12}$$

and similarly, suppressing twistorial indices, $\overline{\mathbf{X}}\mathbf{X} = X^2$. One also has

$$\mathbf{X}_{1\,ac} \overline{\mathbf{X}}_2^{cb} + \mathbf{X}_{2\,ac} \overline{\mathbf{X}}_1^{cb} = \overline{\mathbf{X}}_1^{bd} \mathbf{X}_{2\,da} + \overline{\mathbf{X}}_2^{bd} \mathbf{X}_{1\,da} = 2 X_1 \cdot X_2 \, \delta_a^b \,. \tag{A.13}$$

In the basis defined by eq.(A.5), we have

$$\begin{cases} \mathbf{X}_{\alpha\gamma} = -X^+ \epsilon_{\alpha\gamma} \\ \mathbf{X}_\alpha{}^{\dot\gamma} = -X_\mu \sigma^\mu_{\alpha\dot\beta} \epsilon^{\dot\beta\dot\gamma} \\ \mathbf{X}^{\dot\alpha}{}_\gamma = X_\mu \overline{\sigma}^{\mu\dot\alpha\beta} \epsilon_{\beta\gamma} \\ \mathbf{X}^{\dot\alpha\dot\gamma} = X^- \epsilon^{\dot\alpha\dot\gamma} \end{cases} \qquad \begin{cases} \overline{\mathbf{X}}^{\alpha\gamma} = -X^- \epsilon^{\alpha\gamma} \\ \overline{\mathbf{X}}^\alpha{}_{\dot\gamma} = -X_\mu \epsilon^{\alpha\beta} \sigma^\mu_{\beta\dot\gamma} \\ \overline{\mathbf{X}}_{\dot\alpha}{}^\gamma = X_\mu \epsilon_{\dot\alpha\dot\beta} \overline{\sigma}^{\mu\dot\beta\gamma} \\ \overline{\mathbf{X}}_{\dot\alpha\dot\gamma} = X^+ \epsilon_{\dot\alpha\dot\gamma} \end{cases} \tag{A.14}$$

The 4D spinors are embedded as follows in the 6D chiral spinors (twistors):

$$\Psi_a = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} , \qquad \bar\Phi^a = \begin{pmatrix} \phi^\alpha \\ \bar\xi_{\dot\alpha} \end{pmatrix} . \tag{A.15}$$

In order to avoid a proliferation of spinor indices, we define

$$(\sigma^\mu \epsilon)_\alpha^{\dot\gamma} \equiv \sigma^\mu_{\alpha\dot\beta} \epsilon^{\dot\beta\dot\gamma} \,. \tag{A.16}$$

Notice that in writing eq.(A.16) we have used the usual convention of matrix multiplication. A similar comment applies for other similar expressions involving $\bar\sigma^\mu$, $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$.

Appendix B. Spinor and Vector Notation for 4D Tensor Fields

We usually write bosonic fields transforming in the lowest representations of the Lorentz group in vector notation: A_μ, $T_{\mu\nu}$, etc. With the notable exception of symmetric traceless tensors of the form $T_{(\mu_1\ldots\mu_l)}$, the vector notation becomes awkward for higher spin. On the contrary, by using the isomorphism between $SO(3,1)$ and $SL(2,C)$, a generic irreducible representation of the Lorentz group is defined by two integers $(l,\bar{l})$. The matrix σ^μ provides the link between the vector and spinor representations of fields. Given a reducible bosonic tensor field $t_{\mu_1\ldots\mu_n}$ or fermionic spinor-tensor fields $\psi_{\alpha,\mu_1\ldots\mu_n}$, $\bar\psi^{\dot\alpha}_{\mu_1\ldots\mu_n}$, we have

$$
\begin{aligned}
(\sigma^{\mu_1}\epsilon)_{\alpha_1}^{\dot\beta_1}\ldots(\sigma^{\mu_n}\epsilon)_{\alpha_n}^{\dot\beta_n}t_{\mu_1\ldots\mu_n} &= \sum_{l,\bar l}^{n} t_{\alpha_1\ldots\alpha_l}^{\dot\beta_1\ldots\dot\beta_{\bar l}}\epsilon_{\alpha_{l+1}\alpha_{l+2}}\ldots\epsilon_{\alpha_{n-1}\alpha_n}\epsilon^{\dot\beta_{\bar l+1}\dot\beta_{\bar l+2}}\ldots\epsilon^{\dot\beta_{n-1}\dot\beta_n}\,,\\
(\sigma^{\mu_1}\epsilon)_{\alpha_1}^{\dot\beta_1}\ldots(\sigma^{\mu_n}\epsilon)_{\alpha_n}^{\dot\beta_n}\psi_{\gamma\mu_1\ldots\mu_n} &= \sum_{l,\bar l}^{n} \psi_{\gamma\alpha_1\ldots\alpha_l}^{\dot\beta_1\ldots\dot\beta_{\bar l}}\epsilon_{\alpha_{l+1}\alpha_{l+2}}\ldots\epsilon_{\alpha_{n-1}\alpha_n}\epsilon^{\dot\beta_{\bar l+1}\dot\beta_{\bar l+2}}\ldots\epsilon^{\dot\beta_{n-1}\dot\beta_n}\,,\\
(\sigma^{\mu_1}\epsilon)_{\alpha_1}^{\dot\beta_1}\ldots(\sigma^{\mu_n}\epsilon)_{\alpha_n}^{\dot\beta_n}\bar\psi^{\dot\gamma}_{\mu_1\ldots\mu_n} &= \sum_{l,\bar l}^{n} \bar\psi_{\alpha_1\ldots\alpha_l}^{\dot\gamma\dot\beta_1\ldots\dot\beta_{\bar l}}\epsilon_{\alpha_{l+1}\alpha_{l+2}}\ldots\epsilon_{\alpha_{n-1}\alpha_n}\epsilon^{\dot\beta_{\bar l+1}\dot\beta_{\bar l+2}}\ldots\epsilon^{\dot\beta_{n-1}\dot\beta_n}\,,
\end{aligned}
\tag{B.1}
$$

where the sum over l, $\bar l$ runs over even or odd integers, for even or odd n, respectively. Taking symmetric and antisymmetric combinations in the undotted and dotted indices of the r.h.s. of eq.(B.1) allows us to find the explicit relations between the different field components in vector and spinor notations. Inverse relations are obtained by multiplying eq.(B.1) by powers of $(\epsilon\sigma^\mu)$:

$$
\begin{aligned}
t_{\mu_1\ldots\mu_n} &= 2^{-n}\sum_{l,\bar l}^{n}(\epsilon\sigma_{\mu_1})^{\alpha_1}_{\dot\beta_1}\ldots(\epsilon\sigma_{\mu_n})^{\alpha_n}_{\dot\beta_n} t_{\alpha_1\ldots\alpha_l}^{\dot\beta_1\ldots\dot\beta_{\bar l}}\epsilon_{\alpha_{l+1}\alpha_{l+2}}\ldots\epsilon_{\alpha_{n-1}\alpha_n}\epsilon^{\dot\beta_{\bar l+1}\dot\beta_{\bar l+2}}\ldots\epsilon^{\dot\beta_{n-1}\dot\beta_n}\,,\\
\psi_{\gamma\mu_1\ldots\mu_n} &= 2^{-n}\sum_{l,\bar l}^{n}(\epsilon\sigma_{\mu_1})^{\alpha_1}_{\dot\beta_1}\ldots(\epsilon\sigma_{\mu_n})^{\alpha_n}_{\dot\beta_n} \psi_{\gamma\alpha_1\ldots\alpha_l}^{\dot\beta_1\ldots\dot\beta_{\bar l}}\epsilon_{\alpha_{l+1}\alpha_{l+2}}\ldots\epsilon_{\alpha_{n-1}\alpha_n}\epsilon^{\dot\beta_{\bar l+1}\dot\beta_{\bar l+2}}\ldots\epsilon^{\dot\beta_{n-1}\dot\beta_n}\,,\\
\bar\psi^{\dot\gamma}_{\mu_1\ldots\mu_n} &= 2^{-n}\sum_{l,\bar l}^{n}(\epsilon\sigma_{\mu_1})^{\alpha_1}_{\dot\beta_1}\ldots(\epsilon\sigma_{\mu_n})^{\alpha_n}_{\dot\beta_n} \bar\psi_{\alpha_1\ldots\alpha_l}^{\dot\gamma\dot\beta_1\ldots\dot\beta_{\bar l}}\epsilon_{\alpha_{l+1}\alpha_{l+2}}\ldots\epsilon_{\alpha_{n-1}\alpha_n}\epsilon^{\dot\beta_{\bar l+1}\dot\beta_{\bar l+2}}\ldots\epsilon^{\dot\beta_{n-1}\dot\beta_n}\,.
\end{aligned}
\tag{B.2}
$$

It may be useful to work out in detail the case for, say, a bosonic rank-two tensor $t_{\mu\nu}$. We have

$$
(\sigma^\mu\epsilon)_{\alpha_1}^{\dot\beta_1}(\sigma^\nu\epsilon)_{\alpha_2}^{\dot\beta_2}t_{\mu\nu} = t\epsilon_{\alpha_1\alpha_2}\epsilon^{\dot\beta_1\dot\beta_2} + t_{\alpha_1\alpha_2}\epsilon^{\dot\beta_1\dot\beta_2} + t^{\dot\beta_1\dot\beta_2}\epsilon_{\alpha_1\alpha_2} + t_{\alpha_1\alpha_2}^{\dot\beta_1\dot\beta_2}\,,
\tag{B.3}
$$

which corresponds to the decomposition $(0,0)\oplus(1,0)\oplus(0,1)\oplus(1,1)$, scalar, self-dual antisymmetric tensor, anti self-dual antisymmetric tensor, symmetric tensor. From eq.(B.3) we get

$$
\begin{aligned}
t &= \frac{1}{2}\eta^{\mu\nu}t_{\mu\nu}\,,\\
t_{\alpha_1\alpha_2} &= t_{\mu\nu}(\sigma^{\mu\nu}\epsilon)_{\alpha_1\alpha_2}\,,\\
t^{\dot\beta_1\dot\beta_2} &= t_{\mu\nu}(\epsilon\bar\sigma^{\mu\nu})^{\dot\beta_1\dot\beta_2}\,,\\
t_{\alpha_1\alpha_2}^{\dot\beta_1\dot\beta_2} &= \frac{1}{4}t_{\mu\nu}\Big((\sigma^\mu\epsilon)_{\alpha_1}^{\dot\beta_1}(\sigma^\nu\epsilon)_{\alpha_2}^{\dot\beta_2} + (\sigma^\mu\epsilon)_{\alpha_2}^{\dot\beta_1}(\sigma^\nu\epsilon)_{\alpha_1}^{\dot\beta_2} + (\mu\leftrightarrow\nu)\Big)\,.
\end{aligned}
\tag{B.4}
$$

Notice that in the last relation in eq.(B.4) the trace part of $t_{\mu\nu}$ automatically gives a vanishing contribution. We get the inverse relations by means of eq.(B.2). Decomposing $t_{\mu\nu} = \eta_{\mu\nu} t/2 + t_{[\mu\nu]} + t_{(\mu\nu)}$, where $t_{(\mu\nu)} = 1/2(t_{\mu\nu} + t_{\nu\mu}) - \eta_{\mu\nu} t/2$ and $t_{[\mu\nu]} = 1/2(t_{\mu\nu} - t_{\nu\mu})$, one has

$$\begin{aligned} t_{[\mu\nu]} &= \frac{1}{2}(\epsilon\sigma_{\mu\nu})^{\alpha_1\alpha_2} t_{\alpha_1\alpha_2} + \frac{1}{2}(\bar{\sigma}_{\mu\nu}\epsilon)_{\dot{\beta}_1\dot{\beta}_1} t^{\dot{\beta}_1\dot{\beta}_2}\,, \\ t_{(\mu\nu)} &= (\epsilon\sigma_\mu)^{\alpha_1}_{\ \dot{\beta}_1} (\epsilon\sigma_\nu)^{\alpha_2}_{\ \dot{\beta}_2} t^{\dot{\beta}_1\dot{\beta}_2}_{\alpha_1\alpha_2}\,. \end{aligned} \tag{B.5}$$

For arbitrary symmetric traceless fields $t_{(\mu_1\ldots\mu_l)}$, in particular, we have

$$\begin{aligned} t^{\dot{\beta}_1\ldots\dot{\beta}_l}_{\alpha_1\ldots\alpha_l} =& \frac{1}{l!} t_{(\mu_1\ldots\mu_l)} \Big((\sigma^{\mu_1}\epsilon)^{\ \dot{\beta}_1}_{\alpha_1} \ldots (\sigma^{\mu_l}\epsilon)^{\ \dot{\beta}_l}_{\alpha_l} + \mathrm{perms.} \Big)\,, \\ t_{(\mu_1\ldots\mu_l)} =& (\epsilon\sigma_{\mu_1})^{\alpha_1}_{\ \dot{\beta}_1} \ldots (\epsilon\sigma_{\mu_l})^{\alpha_l}_{\ \dot{\beta}_l} t^{\dot{\beta}_1\ldots\dot{\beta}_l}_{\alpha_1\ldots\alpha_l}\,. \end{aligned} \tag{B.6}$$

Appendix C. Properties of the $\mathcal{F}$ Functions

In this Appendix we provide all the properties of the functions $\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)}$ needed for the system of Casimir equations and more specifically to derive eqs.(4.100)-(4.102). We will not consider the functions $\mathcal{F}_{\rho_1,\rho_2}^{\pm\,(a,b;c)}$ here, since their properties can trivially be deduced from the ones below by demanding both sides to be symmetric/anti-symmetric under the exchange $z \leftrightarrow \bar{z}$.

The fundamental identities to be considered can be divided in two sets, depending on whether the values (a,b,c) of the functions $\mathcal{F}$ are left invariant or not. The former identities read

$$
\begin{aligned}
\Big(\frac{1}{z}-\frac{1}{2}\Big)\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \mathcal{F}_{\rho_1-1,\rho_2}^{(a,b;c)} - D_{\rho_1}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} + B_{\rho_1}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2}^{(a,b;c)} && \text{(C.1)}\\
\Big(\frac{1}{\bar{z}}-\frac{1}{2}\Big)\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \mathcal{F}_{\rho_1,\rho_2-1}^{(a,b;c)} - D_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} + B_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2+1}^{(a,b;c)} && \text{(C.2)}\\
L_0\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \rho_2\mathcal{F}_{\rho_1,\rho_2-1}^{(a,b;c)} - \rho_1\mathcal{F}_{\rho_1-1,\rho_2}^{(a,b;c)} - (\rho_2+c-1)B_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2+1}^{(a,b;c)} + && \text{(C.3)}\\
&\quad (\rho_1+c-1)B_{\rho_1}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2}^{(a,b;c)} + \frac{1}{2}(2-c)(D_{\rho_1}^{(a,b,c)} - D_{\rho_2}^{(a,b,c)})\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)},
\end{aligned}
$$

where $L_0 = \Big((1-\bar{z})\partial_{\bar{z}} - (1-z)\partial_z\Big)$ and we have defined

$$
\begin{aligned}
C_\rho^{(a,b,c)} &= \frac{(a+\rho)(b-c-\rho)}{(c+2\rho)(c+2\rho-1)}, && \text{(C.4)}\\
B_\rho^{(a,b,c)} &= C_\rho^{(a,b,c)}\,C_{\rho+1}^{(b-1,a,c-1)} = \frac{(\rho+a)(\rho+b)(\rho+c-b)(\rho+c-a)}{(2\rho+c)^2(c+2\rho+1)(c+2\rho-1)},\\
D_\rho^{(a,b,c)} &= \frac{(2a-c)(2b-c)}{2(c+2\rho)(c+2\rho-2)}. && \text{(C.5)}
\end{aligned}
$$

The latter identities read

$$
\begin{aligned}
\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \mathcal{F}_{\rho_1,\rho_2}^{(a,b-1;c-1)} - C_{\rho_1}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2}^{(a,b-1;c-1)} - && \text{(C.6)}\\
&\quad C_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2+1}^{(a,b-1;c-1)} + C_{\rho_1}^{(a,b,c)}C_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2+1}^{(a,b-1;c-1)},\\
\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \mathcal{F}_{\rho_1,\rho_2}^{(a-1,b;c-1)} - C_{\rho_1}^{(b,a,c)}\mathcal{F}_{\rho_1+1,\rho_2}^{(a-1,b;c-1)} - && \text{(C.7)}\\
&\quad C_{\rho_2}^{(b,a,c)}\mathcal{F}_{\rho_1,\rho_2+1}^{(a-1,b;c-1)} + C_{\rho_1}^{(b,a,c)}C_{\rho_2}^{(b,a,c)}\mathcal{F}_{\rho_1+1,\rho_2+1}^{(a-1,b;c-1)},\\
\frac{1}{z\bar{z}}\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \mathcal{F}_{\rho_1-1,\rho_2-1}^{(a+1,b+1;c+2)}, && \text{(C.8)}\\
(z-\bar{z})L(a)\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= (\rho_2-\rho_1)\mathcal{F}_{\rho_1,\rho_2}^{(a,b-1;c-1)} - (\rho_1+\rho_2+c-1)C_{\rho_1}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2}^{(a,b-1;c-1)} + && \text{(C.9)}\\
&\quad (\rho_1+\rho_2+c-1)C_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2+1}^{(a,b-1;c-1)} - (\rho_2-\rho_1)C_{\rho_1}^{(a,b,c)}C_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2+1}^{(a,b-1;c-1)},\\
\frac{z-\bar{z}}{z\bar{z}}L(b)\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= (\rho_2-\rho_1)\mathcal{F}_{\rho_1-1,\rho_2-1}^{(a,b+1;c+1)} - (\rho_1+\rho_2+c-1)C_{\rho_1}^{(b,a,c)}\mathcal{F}_{\rho_1,\rho_2-1}^{(a,b+1;c+1)} + && \text{(C.10)}\\
&\quad (\rho_1+\rho_2+c-1)C_{\rho_2}^{(b,a,c)}\mathcal{F}_{\rho_1-1,\rho_2}^{(a,b+1;c+1)} - (\rho_2-\rho_1)C_{\rho_1}^{(b,a,c)}C_{\rho_2}^{(b,a,c)}\mathcal{F}_{\rho_1,\rho_2}^{(a,b+1;c+1)}.
\end{aligned}
$$

The relations (C.1)-(C.3) were first derived in ref.[9] (see also ref.[49]), while the relations (C.9) and (C.10) were derived in [31]. It is straightforward to see that eqs.(4.100)-(4.102) can be derived using proper combinations of eqs.(C.1)-(C.10). For instance, the action of the first term appearing in the r.h.s. of eq.(4.99) is reproduced (modulo a trivial constant factor) by taking the combined action given by ((C.2)−(C.1))× (C.9)× (C.6). All other terms in eqs.(4.97)-(4.99) are similarly deconstructed.

Appendix D. The Conformal Blocks for $p = 1$

We report in this appendix the full explicit solution for the two conformal blocks $G_0^{(1)}$ and $G_1^{(1)}$ associated to the exchange of fermion operators of the kind $\mathcal{O}^{(\ell,\ell+1)}$ for the specific values

$$a = \frac{1}{2}\,, \qquad b = -\frac{1}{2}\,. \tag{D.1}$$

We choose as undetermined coefficient $c^1_{0,-1}$ and report below the values of the coefficients normalized to $c^1_{0,-1}$. We have

$$c^0_{-2,0} = \frac{(2+\ell)}{2\,(1+\ell)}, \;\; c^0_{-1,-1} = -\frac{\ell}{2\,(1+\ell)}, \;\; c^1_{-1,0} = -\frac{(3+\ell)}{1+\ell}. \tag{D.2}$$

$$\begin{aligned}
c^0_{-1,0} &= \frac{(3+\ell)(-1+2\Delta)(-1+2\ell+2\Delta)}{8(1+\ell)(-3+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{-1,1} &= -\frac{(2+\ell)(5+2\ell-2\Delta)^2(-7+2\Delta)}{32(1+\ell)(3+2\ell-2\Delta)(7+2\ell-2\Delta)(-3+2\Delta)},\\
c^0_{0,-1} &= -\frac{(-1+2\Delta)(-1+2\ell+2\Delta)}{8(-3+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{0,0} &= \frac{\ell(-7+2\Delta)(-1+2\ell+2\Delta)^2}{32(1+\ell)(-3+2\Delta)(-3+2\ell+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{0,1} &= -\frac{(3+\ell)(5+2\ell-2\Delta)^2(-5+2\Delta)(-1+2\ell+2\Delta)}{128(1+\ell)(3+2\ell-2\Delta)(7+2\ell-2\Delta)(-3+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{1,0} &= \frac{(-5+2\Delta)(-1+2\ell+2\Delta)(3+2\ell+2\Delta)^2}{128(-3+2\Delta)(1+2\ell+2\Delta)^2(5+2\ell+2\Delta)},\\
c^1_{-1,1} &= -\frac{(2+\ell)(5+2\ell-2\Delta)(-1+2\Delta)}{4(1+\ell)(7+2\ell-2\Delta)(-3+2\Delta)},\\
c^1_{0,2} &= \frac{(2+\ell)(1+2\ell-2\Delta)(5+2\ell-2\Delta)^2(-5+2\Delta)}{64(1+\ell)(3+2\ell-2\Delta)^2(7+2\ell-2\Delta)(-3+2\Delta)},\\
c^1_{1,0} &= -\frac{(-7+2\Delta)(-1+2\ell+2\Delta)(3+2\ell+2\Delta)}{16(-3+2\Delta)(1+2\ell+2\Delta)^2},\\
c^1_{1,1} &= -\frac{\ell(5+2\ell-2\Delta)(-5+2\Delta)(-1+2\ell+2\Delta)(3+2\ell+2\Delta)}{64(1+\ell)(7+2\ell-2\Delta)(-3+2\Delta)(1+2\ell+2\Delta)^2}
\end{aligned}$$

$$
\begin{aligned}
c^1_{0,0} &= \frac{1}{4(1+\ell)(11+2\ell-2\Delta)(-3+2\Delta)(-3+2\ell+2\Delta)(1+2\ell+2\Delta)} \times \\
&\Big(576-384\Delta+\ell\Big(627-2\ell(-29+2\ell(7+2\ell))-472\Delta+4\ell(-47+4\ell(3+\ell))\Delta \\
&+8(-9+\ell(19+2\ell))\Delta^2-16(-6+\ell)\Delta^3-16\Delta^4\Big)\Big)\,, \\
c^1_{0,1} &= \frac{(5+2\ell-2\Delta)}{16(1+\ell)(3+2\ell-2\Delta)(7+2\ell-2\Delta)(-3+2\Delta)(-3+2\ell+2\Delta)(1+2\ell+2\Delta)} \times \\
&\Big(\ell(643-14\ell(-3+2\ell(9+2\ell)))+4\ell(-232+\ell(-115+4\ell(1+\ell)))\Delta+8(3+\ell) \\
&(-24+\ell(17+2\ell))\Delta^2-16(-7+\ell)(3+\ell)\Delta^3-16(3+\ell)\Delta^4+27(9+4\Delta)\Big)\,.
\end{aligned}
$$

The asymptotic behaviour of the CBs for $z,\bar{z}\to 0$ ($z\to 0$ first) is dominated by the coefficients with $n=-1$ and the lowest value of m, i.e. $c^0_{-1,-1}$ and $c^1_{0,-1}$. For $\ell=0$, the asymptotic behaviour of $G_0^{(1)}$ is given by the next term $c^0_{0,-1}$, since $c^0_{-1,-1}$ in eq.(D.2) vanishes. This in agreement with the asymptotic behaviour of the CBs found in subsection 4.4.1. Notice how the complexity of the $c^e_{m,n}$ varies from coefficient to coefficient. In general the most complicated ones are those in the "interior" of the octagons (hexagons only for $p=1$).

Appendix E. Generalized Free Field Theory (GFT)

Generalized free theories, also called mean field theories, are defined as the theories where all the correlators are computed by the sum over all possible 2-point function Wick-contractions.

Here we consider a GFT with a scalar and a fermion of scaling dimensions Δ_ϕ and Δ_ψ. All correlators in the theory will be sums of the 2-point functions

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{(x_{12}^2)^{\Delta_\phi}}\,,\quad \langle\psi(x_1,s_1)\overline{\psi}(x_2,\bar{s}_2)\rangle = \frac{-i\,\mathbb{I}^{21}}{(x_{12}^2)^{\Delta_\psi+\frac{1}{2}}} \tag{E.1}$$

The 3-point function

$$\langle\phi(x_1)\psi(x_2,s_2)\mathcal{O}^{(\ell,\bar\ell)}(x_3,s_3,\bar{s}_3)\rangle \neq 0, \tag{E.2}$$

then the primary $\mathcal{O}^{(\ell,\bar\ell)}$ has to be some fermionic operator with $|\ell-\bar\ell|=1$, it can be one of the reps

$$\mathcal{O}^{(\ell,1+\ell)} = \mathcal{A}^{(\ell)} \quad \text{or} \quad \mathcal{O}^{(\ell+1,\ell)} = \mathcal{B}^{(\ell)}, \tag{E.3}$$

These operators clearly have to be double-twist operators $[\phi\overline{\psi}]_n$, schematically

$$\begin{aligned}\mathcal{A}_n^{(\ell)}(x_3,s_3,\bar{s}_3) &= \Big(s_3\sigma\partial\bar{s}_3\Big)^\ell[\partial^2]^n\,\phi(x_3)\Big(\bar{s}_3^{\dot\alpha}\bar\psi_{\dot\alpha}(x_3)\Big),\\ \mathcal{B}_n^{(\ell)}(x_3,s_3,\bar{s}_3) &= \Big(s_3\sigma\partial\bar{s}_3\Big)^\ell[\partial^2]^n\,\phi(x_3)\Big(s_3^\alpha\sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar\psi^{\dot\beta}(x_3)\Big).\end{aligned} \tag{E.4}$$

One can then read off the scaling dimensions

$$\Delta_\mathcal{A} = \Delta_\phi+\Delta_\psi+\ell+2n,\quad \Delta_\mathcal{B} = \Delta_\phi+\Delta_\psi+\ell+2n+1, \tag{E.5}$$

or the twists

$$\tau_\mathcal{A} = \Delta_\phi+\Delta_\psi+2n-\frac{1}{2},\quad \tau_\mathcal{B} = \Delta_\phi+\Delta_\psi+\ell+2n+\frac{1}{2}. \tag{E.6}$$

GFT Solution for the OPE Coefficients We want to calculate the OPE coefficients of the double-twist operators (E.4). Since we can calculate the 4-point function $\langle\phi\psi\phi\overline{\psi}\rangle$ as a sum of Wick contractions, we use its expansion in conformal blocks to calculate the OPE coefficients. The 4-point function is given by the expression

$$\langle\phi(x_1)\psi(x_2,s_2)\phi(x_3)\bar\psi(x_4,\bar{s}_4)\rangle = \langle\phi(x_1)\phi(x_2)\rangle\langle\psi(x_2,s_2)\overline{\psi}(x_4,\bar{s}_4)\rangle = \frac{-i\,\mathbb{I}^{42}}{(x_{13}^2)^{\Delta_\phi}(x_{24}^2)^{\Delta_\psi+\frac{1}{2}}}, \tag{E.7}$$

Using the $s-$ channel expansion given in (5.27) with the twists (E.6)

$$\langle\phi(x_1)\psi(x_2,s_2)\phi(x_3)\bar\psi(x_4,\bar{s}_4)\rangle = \sum_{n,\ell=0}^{\infty}\Big(P_\mathcal{A}^{\text{GFT}}\,\overline{W}^{\text{seed}}_{\langle\phi\psi\mathcal{A}\rangle\langle\overline{\mathcal{A}}\phi\overline{\psi}\rangle} + P_\mathcal{B}^{\text{GFT}}\,W^{\text{seed}}_{\langle\phi\psi\mathcal{B}\rangle\langle\overline{\mathcal{B}}\phi\overline{\psi}\rangle}\Big). \tag{E.8}$$

Now we can equate the equations (E.7) and (E.8), the only unknowns are $P_\mathcal{A}^{\text{GFT}}$'s and $P_\mathcal{B}^{\text{GFT}}$'s . Expanding around $z=\bar{z}=0$, the OPE coefficients can be computed order by order to be

$$\begin{aligned}
P_{\mathcal{A}}^{\mathsf{GFT}}(n,\ell) =&\frac{(\Delta_\phi-1)_n(\Delta_\psi-\frac{3}{2})_{n+1}}{n!(\Delta_\phi+\Delta_\psi+n-\frac{5}{2})_n}\times\\
&\frac{(\ell+1)(\Delta_\phi)_{\ell+n+1}(\Delta_\psi+\frac{1}{2})_{\ell+n}}{(l+n+2)!(\Delta_\phi+\Delta_\psi+n+\ell-\frac{3}{2})_{1+n}(\Delta_\phi+\Delta_\psi+2n+\ell+\frac{1}{2})_\ell}\\
P_{\mathcal{B}}^{\mathsf{GFT}}(n,\ell) =&-\frac{(\Delta_\phi-1)_n(\Delta_\psi-\frac{3}{2})_n}{n!(\Delta_\phi+\Delta_\psi+n-\frac{7}{2})_n}\times\\
&\frac{(1+\ell)(\Delta_\phi)_{\ell+n}(\Delta_\psi+\frac{1}{2})_{\ell+n}}{(\ell+n+1)!(\Delta_\phi+\Delta_\psi+\ell+n-\frac{3}{2})_n(\Delta_\phi+\Delta_\psi+\ell+2n-\frac{1}{2})_\ell}\,.
\end{aligned} \tag{E.9}$$

It is nice to note that $P_{\mathcal{A}}^{\mathsf{GFT}}\geq 0$ and $P_{\mathcal{B}}^{\mathsf{GFT}}\leq 0$ as required by (5.30).

Appendix F. Ward Identities

Here we want to work out the constraints on the OPE coefficients $\lambda^i_{\langle T\psi\overline{\psi}\rangle}$ resulting form Conformal Ward identities. The stress-tensor operator $T_{\mu\nu}$ is conserved and traceless, the operator equations

$$\partial_\mu T^\mu{}_\nu = 0 \text{ and } T^\mu{}_\mu = 0$$

should be satisfied within any correlator, up to contact terms.
The OPE coefficients $\lambda^i_{\langle T\psi\overline{\psi}\rangle}$ appear in the 3-point function (we will drop the subscript in the calculations $\lambda^i_{\langle T\psi\overline{\psi}\rangle} \to \lambda^i$)

$$\langle \Psi(x_1, S_1)\overline{\Psi}(x_2, \bar{S}_2)T(x_3, S_3, \bar{S}_3)\rangle = \Big(X_{12}^{\Delta_\psi - 5/2} X_{13}^3 X_{23}^3\Big)^{-1}\Big(\lambda^1 I^{21}(J^3_{12})^2 + \lambda^2 I^{23} I^{31} J^3_{12}\Big), \quad \text{(F.1)}$$

projecting to 4D, and restoring the tensor indices using (B.6)

$$\begin{aligned}
&\langle \psi_\alpha(x_1)\bar{\psi}^{\dot{\alpha}}(x_2)T^{\sigma\omega}(x_3)\rangle = \\
&\Big(-\frac{\lambda^1}{2}\big(x_{12}^{2\Delta_\psi - 1} x_{13}^2 x_{23}^2\big)^{-1} x_{12}^\mu Z^\nu_{3,12} Z^\rho_{3,12}\,{}_\alpha(\sigma_\mu\epsilon)^{\dot{\alpha}}\big(8\eta^\sigma_\nu\eta^\omega_\rho - 2\eta_{\rho\nu}\eta^{\sigma\omega}\big) \\
&+\frac{\lambda^2}{4}\big(x_{12}^{2\Delta_\psi - 3} x_{13}^4 x_{23}^4\big)^{-1}\, x_{23}^\mu x_{13}^\nu Z^\rho_{3,12}\big(4\eta^\omega_\rho\,{}_\alpha(\sigma_\mu\bar{\sigma}^\sigma\sigma_\nu\epsilon)^{\dot{\alpha}} + 4\eta^\sigma_\rho\,{}_\alpha(\sigma_\mu\bar{\sigma}^\omega\sigma_\nu\epsilon)^{\dot{\alpha}} \\
&-2\eta^{\sigma\omega}\,{}_\alpha(\sigma_\mu\bar{\sigma}_\rho\sigma_\nu\epsilon)^{\dot{\alpha}}\big)\Big),
\end{aligned} \quad \text{(F.2)}$$

where $Z^\mu_{3,12} = x^\mu_{23}/x^2_{23} - x^\mu_{13}/x^2_{13}$. This 3-point function has to satisfy the ward identity:

$$\frac{\partial}{\partial x_3^\sigma}\langle \psi_\alpha(x_1)\bar{\psi}^{\dot{\alpha}}(x_2)T^{\sigma\omega}(x_3)\rangle = -\delta^4(x_1 - x_3)\partial^\omega\langle\psi_\alpha\bar{\psi}^{\dot{\alpha}}\rangle - \delta^4(x_2 - x_3)\partial^\omega\langle\psi_\alpha\bar{\psi}^{\dot{\alpha}}\rangle, \quad \text{(F.3)}$$

We take the limit $x_1 \to x_3$, $s = x_1 - x_3$ and $x_{23} = s - x_{12}$. In this limit

$$Z^\mu_{3,12} \to -\frac{s^\mu}{s^2} - \frac{x^\mu_{12}}{x^2_{12}}. \quad \text{(F.4)}$$

we can write the 3-point function in this limit as

$$\begin{aligned}
&\langle \psi_\alpha(x_1)\bar{\psi}^{\dot{\alpha}}(x_2)T^{\sigma\omega}(x_3)\rangle = \\
&-\frac{\lambda^1}{2x_{12}^{2\Delta_\psi+1}}\Big((\partial^\sigma\partial^\omega s^{-2}) + \frac{8s^\omega x^\sigma_{12} + 8s^\sigma x^\omega_{12}}{x^2_{12}s^4} + 16\frac{s\cdot x_{12}s^\sigma s^\omega}{x^2_{12}s^6} - 8\eta^{\sigma\omega}\frac{s\cdot x_{12}}{x^2_{12}s^4}\Big)x^\mu_{12}\,{}_\alpha(\sigma_\mu\epsilon)^{\dot{\alpha}} \\
&+\frac{\lambda^2}{4x_{12}^{2\Delta_\psi+1}}\, x^\mu_{12}\frac{1}{8}(\partial^\nu\partial^\rho s^{-2})\big(4\eta^\omega_\rho\,{}_\alpha(\sigma_\mu\bar{\sigma}^\sigma\sigma_\nu\epsilon)^{\dot{\alpha}} + 4\eta^\sigma_\rho\,{}_\alpha(\sigma_\mu\bar{\sigma}^\omega\sigma_\nu\epsilon)^{\dot{\alpha}}\big) \\
&+\frac{\lambda^2}{4x_{12}^{2\Delta_\psi+1}}\Big(\frac{s^\nu x^\rho_{12}x^\mu_{12}}{x^2_{12}s^4} - \frac{s^\nu s^\rho s^\mu}{s^6} + 4\frac{s\cdot x_{12}\, x^\mu_{12}s^\nu s^\rho}{x^2_{12}\, s^6}\Big)\big(4\eta^\omega_\rho\,{}_\alpha(\sigma_\mu\bar{\sigma}^\sigma\sigma_\nu\epsilon)^{\dot{\alpha}} \\
&+4\eta^\sigma_\rho\,{}_\alpha(\sigma_\mu\bar{\sigma}^\omega\sigma_\nu\epsilon)^{\dot{\alpha}} - 2\eta^{\sigma\omega}\,{}_\alpha(\sigma_\mu\bar{\sigma}_\rho\sigma_\nu\epsilon)^{\dot{\alpha}}\big).
\end{aligned} \quad \text{(F.5)}$$

Here we Adopt the methods of differential regularisation [73]. Once we apply ∂_σ on the correlator (F.5), we will have a term $\partial^2 s^{-2} = -4\pi^2\delta^4(s)$, and this can be checked taking the integration $\int \mathrm{d}s^4$.

Other terms in (F.5), once taking the divergence, are zero for any $s \neq 0$ and so equivalent to a $\#\delta^4(s)$

$$\int \mathsf{ds}^4 \partial_\sigma \Big(\frac{8 s^\omega x_{12}^\sigma + 8 s^\sigma x_{12}^\omega}{x_{12}^2 s^4} - 8\eta^{\sigma\omega} \frac{s \cdot x_{12}}{x_{12}^2 s^4} + \frac{16\ s \cdot x_{12} s^\sigma s^\omega}{x_{12}^2 s^6} \Big) = 24\pi^2 \frac{x_{12}^\omega}{x_{12}^2},$$
$$\int \mathsf{ds}^4 \partial_\sigma \Big(\frac{s^\nu x_{12}^\rho x_{12}^\mu}{x_{12}^2 s^4} - \frac{s^\nu s^\rho s^\mu}{s^6} + 4 \frac{s \cdot x_{12}\ x_{12}^\mu s^\nu s^\rho}{x_{12}^2\ s^6} \Big) \tag{F.6}$$
$$= \frac{\pi^2}{2} \frac{\eta_\sigma^\nu x_{12}^\rho x_{12}^\mu}{x_{12}^2} - \frac{\pi^2}{12} (\eta_\sigma^\nu \eta^{\rho\mu} + \eta_\sigma^\mu \eta^{\rho\nu} + \eta_\sigma^\rho \eta^{\nu\mu}) + \frac{\pi^2}{3} \frac{x_{12}^\lambda\ x_{12}^\mu}{x_{12}^2} (\eta_{\lambda\sigma} \eta^{\nu\rho} + \eta_\lambda^\nu \eta_\sigma^\rho + \eta_\lambda^\rho \eta_\sigma^\nu)$$

The integrands in the last two equations (when contracted with sigma matrices as in (F.5)) are zero for any $s \neq 0$. So ,as implied by (F.6), the integrands are equivalent to a $\#\delta^4(s)$. So taking divergence of (F.5):

$$\frac{\partial}{\partial x_3^\sigma} \langle \psi_\alpha(x_1) \bar{\psi}^{\dot\alpha}(x_2) T^{\sigma\omega}(x_3) \rangle$$
$$= \frac{2\pi^2 \lambda^1}{x_{12}^{2\Delta_\psi + 1}} (\partial^\omega \delta^4(s)) x_{12}^\mu\ {}_\alpha(\sigma_\mu \epsilon)^{\dot\alpha} - \frac{\pi^2 \lambda^2\ x_{12}^\mu}{2 x_{12}^{2\Delta_\psi + 1}} \big(\partial^\omega \delta^4(s)\ {}_\alpha(\sigma_\mu \bar\epsilon)^{\dot\alpha} + \partial^\nu \delta^4(s)\ {}_\alpha(\sigma_\mu \bar\sigma^\omega \sigma_\nu \epsilon)^{\dot\alpha} \big) \tag{F.7}$$
$$+ \Big(- 2\pi^2 \lambda^2\ \frac{{}_\alpha(\sigma^\omega \epsilon)^{\dot\alpha}}{x_{12}^{2\Delta_\psi + 1}} - 4\pi^2 (3\lambda^1 - 2\lambda^2) \frac{x_{12}^\omega\ x_{12}^\nu\ {}_\alpha(\sigma_\nu \epsilon)^{\dot\alpha}}{x_{12}^{2\Delta_\psi + 3}} \Big) \delta^4(s).$$

Meanwhile the identity (F.3) tells us

$$\partial_\sigma \langle \psi_\alpha(x_1) \bar\psi^{\dot\alpha}(x_2) T^{\sigma\omega}(x_3) \rangle = -i \Big(\frac{{}_\alpha(\sigma^\omega \epsilon)^{\dot\alpha}}{x_{12}^{2\Delta_\psi + 1}} - (2\Delta_\psi + 1) \frac{x_{12}^\omega x_{12}^\nu\ {}_\alpha(\sigma_\nu \epsilon)^{\dot\alpha}}{x_{12}^{2\Delta_\psi + 3}} \Big) \delta^4(s)\,, \tag{F.8}$$

using the normalization $\langle \Psi(x_1, S_1) \overline{\Psi}(x_2, \bar S_2) \rangle = -i\, I^{21} X_{12}^{-\Delta_\psi - 1/2}$. The equations (F.7) and (F.8) imply:

$$\lambda^2_{\langle T\psi\bar\psi \rangle} = \frac{i}{2\pi^2} \quad \text{and} \quad \lambda^1_{\langle T\psi\bar\psi \rangle} = \frac{-i\,(2\Delta_\psi - 3)}{12\pi^2} \tag{F.9}$$

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