

CATEGORICAL ASPECTS OF BPS STATES

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ABSTRACT OF THE DISSERTATION

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This dissertation is devoted to the study of categorical aspects of BPS states in two-dimensional quantum field theories with $\mathcal{N} = (2, 2)$ supersymmetry. The basic aim of a categorical discussion is to study spaces of BPS states, which carry much more refined information than their traditionally studied characters. In a two-dimensional theory, whereas BPS states are supersymmetric states defined on a one-dimensional spatial slice, carrying out the discussion at a categorical level requires one to incorporate two-dimensional supersymmetric instantons. We motivate these instantons and the differential equations they obey in a broader physical context. We show how these two-dimensional instanton effects can be incorporated to result in a categorification of the Cecotti-Vafa wall-crossing formula. We generalize the discussion to incorporate two-dimensional theories with non-trivial twisted mass terms. The presence of twisted masses require us to incorporate Fock spaces of periodic solitons into the discussion, and we show how these Fock spaces affect the categorical wall-crossing formalism. We sketch two important future directions. The

first involves the application of the ideas of this thesis to the study of three-manifolds and homological knot invariants. The second has us graduate from two-dimensional theories and enter the world of four-dimensional $\mathcal{N} = 2$ theories and their BPS states.

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Table of Contents

Abstract	ii
Acknowledgements	iv
1. Introduction: Instantons, Gradient Flow Equations, Supersymmetry, and Categorification	1
1.1. Instantons and Gradient Flow Equations	1
1.1.1. Holomorphic Liouville Functional	11
1.1.2. Symplectic Bosons	13
1.1.3. Complex Chern-Simons	14
1.1.4. 4d Chern-Simons	16
1.2. Instantons and Supersymmetry	18
1.3. Instantons and Categorification	21
2. Categorification of Cecotti-Vafa Wall-Crossing	26
2.1. Introduction and Outline	26
2.1.1. A Failure of Naive Categorification	28
2.1.2. Missing Instantons	31
2.1.3. Wall-Crossing Invariants	33
2.1.4. Outline of the Chapter	35
2.2. Wall-Crossing of BPS Indices	36

2.3.	BPS Chain Complexes and ζ -instantons	45
2.3.1.	BPS Chain Complexes	45
2.3.2.	ζ -instantons and Interior Amplitudes	50
2.3.3.	Homotopy Equivalence of BPS Data	56
2.4.	Statement of Categorical Wall-Crossing	57
2.5.	ζ -instantons and Brane Categories	65
2.5.1.	Bare Thimble Category	65
2.5.2.	Interior Amplitudes and Deformations of \widehat{R}	67
2.6.	Homotopy Equivalence of Brane Categories	69
2.6.1.	Left Configuration	70
2.6.2.	Right Configuration	71
2.6.3.	Canonical Representatives Satisfy Wall-Crossing Constraint	73
2.6.4.	Homotopy Equivalence \implies Categorical WCF	76
2.7.	The Fermion Degree of a ζ -instanton	80
2.7.1.	Gradient Polygons in \mathbb{C}	83
2.7.2.	Paths in \mathbb{C}^*	85
2.7.3.	Fermion Degrees for \mathbb{Z}_N -symmetric Models	86
	Deformed A_{N-1} -Model	86
	Trigonometric Models	88
2.8.	Examples	89
2.8.1.	Quartic LG Model	89
2.8.2.	Trigonometric LG Model	91
2.8.3.	Elliptic LG Model	93
2.9.	Conclusions and Future Directions	98

3. Algebra of the Infrared and Categorical Wall-Crossing with Twisted	
Masses	101
3.1. Introduction	101
3.2. Basics of Twisted Masses in $\mathcal{N} = (2, 2)$ Theories	111
3.2.1. Abstract Vacuum and BPS data	111
3.2.2. Realization of BPS data in Landau-Ginzburg Models	120
3.2.3. Realization in Sigma Models with Isometries	123
3.2.4. Examples	127
The Free Chiral	128
The \mathbb{CP}^1 Model	129
Weierstrass Model	133
Class S theories with trivial DSZ pairing	136
3.3. Boundary Amplitudes and Framed Wall-Crossing	136
3.3.1. S-Walls	136
Example: Cubic LG Model	140
S-Wall Interface	143
3.3.2. K-Walls	149
Example: Mirror to the Free Chiral	152
K-Wall Interface	158
3.4. Bulk Amplitudes and Wall-Crossing of Vanilla BPS States	161
3.4.1. Interior Amplitudes in Cecotti-Vafa Wall-Crossing	163
3.4.2. L_∞ -Algebra Governing Interior Amplitudes	166
Example: The \mathbb{CP}^1 Model at Weak Coupling	174
3.4.3. The A_∞ Category of Thimbles	177
Polygon Viewpoint	183

3.4.4. Equivalence of A_∞ -Categories and Categorical Wall-Crossing	185
Categorical Wall-Crossing in the \mathbb{CP}^1 Model	187
3.5. Algebra of the Infrared	199
3.5.1. No Periodic Solitons	204
R_c for the Weierstrass Model	205
3.5.2. Periodic Solitons Only	209
R_c for Free Chiral	213
3.5.3. Mixed Case	214
3.5.4. A_∞ -Category of Branes	215
3.5.5. Remarks on the Polygon Viewpoint	218
4. Future Directions: Algebraic Knot Invariants, $4d \mathcal{N} = 2$ BPS States . .	221
4.1. Introduction	221
4.2. Algebraic Knot Invariants	221
4.3. Holomorphic-Topological Twists and BPS States in $4d \mathcal{N} = 2$ Theories . . .	234
Appendix A. Some Basic Homological Algebra	245
A.1. Some Basic Homological Algebra	245
Appendix B. A_∞ and L_∞ Algebras and Their Morphisms	249
B.1. A_∞ Algebras and Morphisms	249
Appendix C. BPS Degeneracies in One-Variable Polynomial Superpoten-	
tials	255
C.1. $N_{ij} \in \{0, 1\}$ for $W \in \mathbb{C}[X]$	255
References	256

Chapter 1

Introduction: Instantons, Gradient Flow Equations, Supersymmetry, and Categorification

Much of this thesis is devoted to the study of instanton effects in two-dimensional $\mathcal{N} = 2$ supersymmetric Landau-Ginzburg models. In this introductory chapter we explain the role of instantons in modern-day physics from a few different perspectives. We begin by motivating instanton equations in the broader context of physics, and explain how these equations often arise as the gradient flow equations for an appropriate action functional. In particular, we use the gradient flow viewpoint to derive the instanton equation that appears in Landau-Ginzburg models. Next we explain how instanton equations come up as the supersymmetric equations in supersymmetric theories. Finally we explain the role of instantons in categorification.

1.1 Instantons and Gradient Flow Equations

The equations of mathematical physics, whether they describe the motion of a particle through Newton's equation

$$\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x}, \tag{1.1}$$

the shapes of minimal area surfaces, via the harmonic map equation

$$\Delta\phi = 0, \tag{1.2}$$

or the properties of the nuclear force via the Yang-Mills equations

$$D_A F = 0, \tag{1.3}$$

$$D_A * F = 0, \tag{1.4}$$

very often tend to be second order, non-linear partial differential equations. Intimately tied with these properties is that these differential equations follow from an action principle: there is a functional S of the relevant fields such that the equations of interest are equivalent to the stationarity of S . Such partial differential equations and the action principles they follow from have been the cornerstone of physics for many centuries.

These differential equations and the action principles that govern them are concise and elegant summaries of the laws of nature. Furthermore, the physical predictions that follow from them are compatible with empirical observation. Indeed it is very satisfying when a vast array of physical phenomenon is captured by a few equations that can be written on the back of an envelope. On the other hand, the world that we observe around us is full of rich and complex phenomenon, many of which we still do not understand. How is the apparent simplicity of physical laws and the richness and complexity of the world around us compatible? The resolution lies in the non-linear nature of these equations¹. While it is simple to derive these equations and analyze their symmetry properties, their non-linear nature prevents us from solving them beyond the simplest of cases.

¹The skeptical reader might ask if this is really true. Indeed, many important physics equations are in fact linear! For instance, consider the Dirac equation, which predicts the existence of anti-matter. Similarly Maxwell's equations are also linear. However, the observed phenomenon always involve some type of interactions. For instance, the Dirac and Maxwell equations are separately linear theories and explain anti-matter and light as an electromagnetic wave respectively, but if we want to describe how electrons and light interact, we need Quantum Electrodynamics, a non-linear theory.

In what situations can one say something substantive about the solutions of these non-linear equations of mathematical physics? The first is when one takes a special case when the non-linearity drops out and the equations reduce to linear ones. For instance, with Newton's equations one could take $V = 0$, where one is describing the motion of a free particle, or V to be a quadratic polynomial so that the system reduces to a combination of free motion and harmonic oscillations. Similarly, one could take the harmonic map equation for maps $\phi : \Sigma \rightarrow M$ on flat manifolds, say $\Sigma = \mathbb{R}^2$ and $M = \mathbb{R}^3$ where the equation reduces to the linear Laplace equation which can be analyzed by several well-developed methods. Finally, in the Yang-Mills equations, one could take the gauge group G to be Abelian so that one is describing electromagnetic waves propagating in a vacuum and we are once again in a situation where an explicit analysis is possible. Another situation where one might describe things explicitly, is a highly symmetric situation, such as describing the motion of a symmetric top, or studying the harmonic map equation propagating in a symmetric space. In these situations the equations are non-linear yet solvable due to the enhanced symmetry of the problem.

Is there a class of solutions one can be explicit about, but does not involve making specific choices, such as choosing favorable potentials V , manifolds (Σ, M) or Yang-Mills gauge group G ? It turns out that there is a class of solutions known as instantons where this can be so. An instanton is a field configuration that not only minimizes the action locally, but also globally (in a given connected component of field space). Equivalently, it is a solution of a first-order partial differential equation (usually obtained by applying the “Bogomolnyi trick”) that also satisfies the second-order equation. The first-order PDEs obeyed by instantons are known as instanton equations. Let us discuss one-by-one in each of our three examples what the corresponding instanton equation is. For the case of particle

mechanics, we take V to be of the form $V = -\frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2$, and the instanton equation reads

$$\frac{dx}{dt} = \pm \frac{\partial h}{\partial x}. \quad (1.5)$$

Similarly, assuming (Σ, j) is a Riemann surface, and M admits an almost complex structure J , the instanton equation for harmonic maps is the Cauchy-Riemann (or pseudo-holomorphic map) equation

$$d\phi \circ j = \pm J \circ d\phi. \quad (1.6)$$

For Yang-Mills theory the instanton equation is the (anti) self-duality equation

$$F = \pm * F. \quad (1.7)$$

Each equation is now first order in the fields x , ϕ , and A respectively. It is also easy to directly verify that if a field configuration satisfies the respective instanton equations, they also satisfy the Newton, harmonic map and Yang-Mills equations.

What makes instanton equations more amenable to study when compared to their second-order parent equations? That they satisfy first-order equations certainly leads to more simplicity. For instance, the Yang-Mills equations were considered intractable for non-Abelian gauge groups and it was a breakthrough when Belavin, Polyakov, Schwartz and Tyupkin [BPST] were able to write down an explicit solution for $G = SU(2)$ by reducing first to the self-dual instanton equation. More generally, instanton equations obey a property known as ellipticity², so that their linearization leads to first-order elliptic differential operators. This is a favorable situation where mathematicians have developed tools for the analysis of such PDEs. In particular, the Atiyah-Singer index theorem, applicable

²A partial differential equation is said to be elliptic if its linearization leads to an invertible principle symbol.

to elliptic differential operators, can be used to extract a great deal of information about instantons and their moduli spaces [ADHM, DK].

What do these instantons teach us? Naively, due to the Wick rotation involved, a classical physicist might think that the solutions of instanton equations are unphysical and therefore uninteresting. However this could not be further from the truth. All one has to do is to step into the quantum world, where Wick rotation is necessary to get a well-behaved path integral. Instantons are then indispensable. Indeed, in his classic lectures “The Uses of Instantons,” [Col] Sidney Coleman opens with

In the last two years there have been astonishing developments in quantum field theory. We have obtained control over problems previously believed to be of insuperable difficulty and we have obtained deep and surprising insights into the structure of the leading candidate for the field theory of strong interactions, quantum chromodynamics. These goodies have come from a family of computational methods that are the subject of these lectures.

Instanton equations have also lead to deep new discoveries in pure mathematics. The study of the holomorphic map equation lead to the field of Gromov-Witten theory/Floer theory and advances in symplectic topology [F2]. The (anti) self-dual instanton equation introduced in the study of four-manifolds by Donaldson, has revolutionized the field of low-dimensional topology [Don].

Given the usefulness of instanton equations, is there a method to derive them systematically? For instance, one might wonder if there is an action principle which directly gives us these first order non-linear PDEs? The answer to this question turns out to be negative. The gradient flow equation, the pseudo-holomorphic map equation and the self-duality equations all have the common property that there is no known method to derive them

directly from an action principle. The equations cannot be identified as the critical set of a functional S . There is however a very useful point of view which is the closest one can get to having an action principle. Instanton equations are usually the **gradient flow** equations for an action functional in one-dimension less than the original equations.

Suppose (M, g) is a Riemannian manifold and $h : M \rightarrow \mathbb{R}$ is a function on M . The gradient flow equation is the ordinary differential equation

$$\frac{d\phi^a}{d\tau} = g^{ab} \frac{\partial h}{\partial \phi^b}. \quad (1.8)$$

It has the property that the value of h always increases along a solution as τ increases.

Consider the one-dimensional instanton equation

$$\frac{dx}{d\tau} = \frac{\partial h}{\partial x}. \quad (1.9)$$

Consider the zero-dimensional action h . Then tautologically, indeed, the gradient flow of h coincides with the instanton equation. Similarly, consider the Cauchy-Riemann equation with $\Sigma = \mathbb{R} \times I$, where I is a one-manifold. Letting Σ be parametrized by the complex coordinate $z = x + i\tau$, the Cauchy-Riemann equations in real coordinates read

$$\frac{\partial \phi^a}{\partial \tau} = J^a_b \frac{\partial \phi^b}{\partial x}. \quad (1.10)$$

Let $\omega = d\lambda$ be the symplectic form on M and consider the one-dimensional action functional

$$h = \int_I \varphi^*(\lambda) \quad (1.11)$$

on the space of maps from I to M . Then the Cauchy-Riemann equations can indeed be identified as the flow equation

$$\frac{\partial \phi^a}{\partial \tau} = g^{ab} \frac{\delta h}{\delta \phi^b}. \quad (1.12)$$

Finally consider the self dual equations on a four-manifold of the type $M_4 = \mathbb{R} \times M_3$, and consider the action functional

$$h = \frac{1}{4\pi} \int_{M_3} \text{Tr} \left(A \, dA + \frac{2}{3} A^3 \right) \quad (1.13)$$

to be the Chern-Simons functional for gauge fields on M_3 . Then one can show that the gradient flow equations for the Chern-Simons functional give a (gauged-fixed) version of the self-duality equations.

Thus as we have just demonstrated, d -dimensional instanton equations, rather than following from a d -dimensional action principle, tend to be gradient flow equations for $(d - 1)$ -dimensional action functionals. Does this mean we can write down a usual sort of d -dimensional action and study flow equations to get desirable instanton equations in $(d + 1)$ -dimensions? Things are not so easy. The main reason for the “niceness” of the instanton equations we have described so far is their ellipticity. Without the equations being elliptic they lose much of their power. On the other hand, the gradient flow equation of the usual sort of action does not lead to elliptic equations. Indeed, gradient flow of the standard particle mechanics action $S = \int dx \, \frac{1}{2} \left(\frac{d\phi}{dx} \right)^2$ leads to the diffusion equation $\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$ which is parabolic rather than elliptic.

The action functionals which *do* lead to elliptic equations are special. Stressing this point at a seminar³, Edward Witten states

“Gradient flow will always be a differential equation in one dimension more, but generically it won’t be elliptic. Try it at home. Write down a functional, and write down the

³The seminar “The Jones Polynomial and Gauge Theory” by Edward Witten, was given in May 2021, virtually via Zoom at MATRIX (Mathematical Research Institute), The University of Melbourne.

gradient flow equation in one dimension more. You'll more often get a parabolic equation than an elliptic one. You can get lots of parabolic equations that way, but good luck finding an elliptic equation. If you find one, I recommend to publish it. I think they're all significant."

Let us go back to the Cauchy-Riemann equations. For simplicity we work on $M = \mathbb{R}^2$ with coordinates (p, q) and symplectic form $\omega = dp \wedge dq$. The Cauchy-Riemann equations for maps from the $x + i\tau$ -plane to the $q + ip$ -plane are

$$\frac{\partial p}{\partial \tau} = \frac{\partial q}{\partial x}, \quad \frac{\partial q}{\partial \tau} = -\frac{\partial p}{\partial x}, \quad (1.14)$$

and indeed follow from the gradient flow (in the variable τ) of the action

$$S = \int dx p \frac{dq}{dx}. \quad (1.15)$$

By way of generalizing, a natural guess would be to add a Hamiltonian $H(p, q)$ to the symplectic action S , so that we take

$$S \rightarrow S = \int dx \left(p \frac{dq}{dx} - H(p, q) \right). \quad (1.16)$$

The gradient flow equation of this symplectic action leads to a deformed version of the Cauchy-Riemann equations that read

$$\frac{\partial p}{\partial \tau} = \frac{\partial q}{\partial x} - \frac{\partial H}{\partial p}, \quad (1.17)$$

$$\frac{\partial q}{\partial \tau} = \frac{\partial p}{\partial x} + \frac{\partial H}{\partial q}. \quad (1.18)$$

More generally, the equations on an arbitrary symplectic manifold X with almost complex structure J are

$$\frac{\partial \phi^a}{\partial \tau} = J^a_b \frac{\partial \phi^b}{\partial x} - g^{ab} \frac{\partial H}{\partial \phi^b}. \quad (1.19)$$

These equations are not particularly nice. For arbitrary H , it is not difficult to see that these equations have no ellipticity (for instance, take H to be a quadratic polynomial $p^2 + q^2$). We also break the two-dimensional conformal symmetry of the Cauchy-Riemann equations. So it seems like adding a Hamiltonian to the Morse function for Cauchy-Riemann equations is not a desirable deformation. However, a miracle occurs when the almost complex structure on M is integrable, and H is taken to be the real (or imaginary) part of a J -holomorphic function on M ,

$$H = \operatorname{Re} W. \quad (1.20)$$

When this is the case our deformed Cauchy-Riemann equations can indeed be shown to be elliptic⁴!

Going back to the more general case, the summary is that on a Kähler manifold with $\omega = d\lambda$ and a holomorphic function W on X , the symplectic action

$$S = \int \phi^*(\lambda) - \operatorname{Im} W \, dx \quad (1.21)$$

via gradient flow, leads to an elliptic instanton equation

$$\frac{\partial \phi^a}{\partial x} + J^a_b \frac{\partial \phi^b}{\partial \tau} = g^{ab} \frac{\partial}{\partial \phi^b} \operatorname{Re} W. \quad (1.22)$$

Thus we have succeeded in finding an elliptic deformation of the holomorphic map equation. The action (1.21) meets Witten's standard of being an action functional whose gradient flow equations are elliptic. The equation we have arrived at was first studied in [Wit3], and is known as Witten's $\bar{\partial}$ -equation. It is also known as the ζ -instanton⁵ equation in [GMW].

⁴One can show ellipticity by differentiating the equation to arrive at second order equations that coincide with the harmonic map equation deformed by the potential $V = |dW|^2$. This is indeed elliptic

⁵Here ζ is a phase, and the ζ -instanton equation comes from replacing $\operatorname{Re} W$ in equation 1.22 with $\operatorname{Re} \zeta^{-1} W$.

The ζ -instanton equation, motivated from gradient flow above, will be responsible for important instanton effects in two-dimensional $\mathcal{N} = 2$ supersymmetric Landau-Ginzburg models that we will extensively study in this thesis.

For completeness with regards to later use, it also useful to record an equivariant version of the story above. Suppose the Kähler manifold X admits a G -action acting by holomorphic isometries with moment map μ , such that the holomorphic function W is invariant under the G -action. Then we can consider the G -equivariant version of the symplectic action

$$S = \int \phi^*(\lambda) + \langle A, \mu \rangle - \text{Im}W dx \quad (1.23)$$

where A is a Lie algebra valued connection on \mathbb{R} . The instanton equations are now

$$D_x \phi^a + J^a_b D_\tau \phi^b = g^{ab} \frac{\partial}{\partial \phi^b} \text{Re}W, \quad (1.24)$$

$$F_A = *\mu \quad (1.25)$$

where

$$D_\mu \phi^a = \partial_\mu \phi^a + A_\mu^i K^{ia}, \quad (1.26)$$

where i is a $\mathfrak{g} = \text{Lie}(G)$ index and K^{ia} are vector fields on X that generate the G -action. The second equation is the gradient flow equation for A which gives

$$F_{\tau x} = \mu. \quad (1.27)$$

One can further motivate Witten's $\bar{\partial}$ -equation by considering some infinite-dimensional examples. At first it might seem like the deformation of the Cauchy-Riemann equation as in (1.21) only gives us a new elliptic instanton equation in two dimensions. However, one can exploit the freedom we have. W is an arbitrary holomorphic function, and by taking W

to be a holomorphic functional, namely a holomorphic function on an infinite-dimensional Kähler manifold one might get elliptic instanton equations in higher dimensions. We discuss four separate cases where this works. One is quickly lead to examples of first-order, elliptic partial differential equations which are either completely new, or are known, but at the forefront of modern research.

1.1.1 Holomorphic Liouville Functional

Let (Y, Ω_{AB}) be a complex symplectic manifold. Let $\Omega = d\Lambda$, and let $\omega = d\lambda$ be a real symplectic form on Y . We also assume that Y is equipped with a Kähler metric for ω . Consider the infinite-dimensional Kähler manifold

$$X = \{\varphi : \mathcal{I} \rightarrow Y\}, \quad (1.28)$$

where \mathcal{I} is a one-manifold parametrized by y , and choose the symplectic form and Kähler metric on X to be the ones inherited from the real symplectic form ω on Y . Consider the holomorphic function

$$W = \int_{\mathcal{I}} \varphi^*(\Lambda). \quad (1.29)$$

The symplectic action functional now reads

$$S = \int dx dy \left(\lambda_A \frac{\partial \phi^A}{\partial x} - \text{Im}(\zeta^{-1} \Lambda_A \frac{\partial \phi^A}{\partial y}) \right). \quad (1.30)$$

The critical points of S satisfy the two-dimensional PDE

$$\frac{\partial \phi^A}{\partial x} = \omega^{AB} \text{Im}(\zeta^{-1} \Omega_{BC}) \frac{\partial \phi^C}{\partial y}, \quad (1.31)$$

which may be rewritten as

$$\frac{\partial \phi^A}{\partial x} = g^{AB} \operatorname{Re}(\zeta^{-1} \Omega_{BC}) \frac{\partial \phi^C}{\partial y}. \quad (1.32)$$

By simple linear algebra, it is possible to choose a metric g^{AB} on Y so that

$$I_C^A(\zeta) = -g^{AB} \operatorname{Im}(\zeta^{-1} \Omega_{BC}) \quad (1.33)$$

is an almost complex structure. This almost complex structure for ζ an arbitrary phase is distinct from the original one for which Y is a Kähler manifold. The stationarity of S therefore gives the pseudo-holomorphic map equation

$$\frac{\partial \phi^A}{\partial x} + I_B^A(\zeta) \frac{\partial \phi^B}{\partial y} = 0 \quad (1.34)$$

in the almost complex structure $I(\zeta)$.

Set $\zeta = 1$. Witten's $\bar{\partial}$ -equation on the other hand for this superpotential reads

$$\frac{\partial \phi^A}{\partial \tau} + J_B^A \left(\frac{\partial \phi^B}{\partial x} + I_C^B \frac{\partial \phi^C}{\partial y} \right) = 0. \quad (1.35)$$

This is a non-linear, three-dimensional elliptic PDE. The ellipticity follows from applying for instance $\frac{\partial}{\partial \tau}$ to the equation, and using the quaternion relation $IJ = -JI$ along with $I^2 = J^2 = -1$ to show that the equation implies the three-dimensional Laplace equation.

To the author's knowledge this equation and its properties have not been discussed in the literature. We expect it to be important for several applications. The paper [Wit4] considers precisely the holomorphic Liouville functional, so it is tempting to speculate that the three-dimensional instanton equation might play a role in A-model quantization. Another place it is expected to play a role, as we will briefly discuss in Chapter 4, is the BPS states in four-dimensional $\mathcal{N} = 2$ theories. Finally it might give an explicit construction of the 2-category of A-type boundary conditions associated to a complex symplectic manifold,

a sort of mirror construction to the 2-category of B-brane boundary conditions studied in [KRS].

To conclude this example, we also record a G -equivariant version of the formalism. Suppose G acts on Y by Ω -preserving symplectomorphisms, and let μ_c be the associated complex moment map. We now let

$$W = \int_{\mathcal{I}} \phi^*(\Lambda) + \text{Tr}(A\mu_c) \quad (1.36)$$

where $A = A_y dy$ is a G -connection on \mathcal{I} and Tr refers to the inner product on the Lie algebra \mathfrak{g} of G . It is straightforward to derive the symplectic action, its critical points and the gradient flow equation in the G -equivariant setting. We omit the details.

1.1.2 Symplectic Bosons

There is another natural generalization of the Liouville action on a complex symplectic manifold (Y, Ω) . This comes from considering the action of two-dimensional “symplectic bosons” rather than the one-dimensional Liouville action. We take the target space X to be the space of maps

$$X = \{\phi : \mathbb{C} \rightarrow Y\} \quad (1.37)$$

with real symplectic form

$$\omega_X = \int d^2z \omega_{AB} \delta\phi^A \wedge \delta\phi^B. \quad (1.38)$$

Consider the holomorphic functional

$$W = \int dz \wedge \phi^*(\Lambda), \quad (1.39)$$

which in local coordinates reads

$$W = \int d^2z \Lambda_A \partial_{\bar{z}} \phi^A. \quad (1.40)$$

The symplectic action is

$$S = \int dx d^2z \left(\lambda_A \frac{\partial \phi^A}{\partial x} - \text{Im}(\zeta^{-1} \Lambda_A \partial_{\bar{z}} \phi^A) \right) \quad (1.41)$$

and its stationary points, in complex coordinates are

$$\frac{\partial \phi^i}{\partial x} = \zeta I^i_{\bar{j}} \partial_z \bar{\phi}^{\bar{j}}, \quad (1.42)$$

where $I^i_{\bar{j}} = g^{i\bar{k}} \Omega_{\bar{k}\bar{j}}$. Letting $s = x + i\tau$, the gradient flow equation on the other hand reads

$$\partial_{\bar{s}} \phi^i = I^i_{\bar{j}} \partial_z \bar{\phi}^{\bar{j}}. \quad (1.43)$$

This is a non-linear elliptic PDE in four dimensions. The ellipticity again follows from showing that this equation implies the Laplace equation

$$(\partial_s \partial_{\bar{s}} + \partial_z \partial_{\bar{z}}) \phi^i = 0 \quad (1.44)$$

on $\mathbb{C}_s \times \mathbb{C}_z$. Again we are not aware of any discussion of this instanton equation and its applications in the literature.

It is also useful to consider a G -equivariant version of the symplectic boson superpotential. Again suppose G acts on Y by Ω -symplectomorphisms and suppose that μ_c is the corresponding complex moment map. The G -equivariant holomorphic functional is

$$W = \int d^2z (\Lambda_A \partial_{\bar{z}} \phi^A + A_{\bar{z}} \mu_c). \quad (1.45)$$

This will be used in 1.1.4 when discussing 4d Chern-Simons.

1.1.3 Complex Chern-Simons

In this subsection and the next we work out some gauge theoretic examples. Let M_3 be a three-manifold, G be a real compact gauge group and $G_{\mathbb{C}}$ its complexification. Let the target space be

$$X = \{G_{\mathbb{C}}\text{-connections on } M_3\}. \quad (1.46)$$

with symplectic form

$$\omega_X = \frac{i}{2} \int_{M_3} d^3x \sqrt{g} g^{ij} \text{Tr}(\delta \mathcal{A}_i \wedge \delta \bar{\mathcal{A}}_j). \quad (1.47)$$

Let the superpotential be the complex Chern-Simons functional

$$W = \int_{M_3} \text{Tr}(\mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3). \quad (1.48)$$

We consider the \mathcal{G} -equivariant version of the story where $\mathcal{G} = \text{Map}(M_3, G)$ acts on X by gauge transformations:

$$\delta_\varepsilon A_i = -D_i \varepsilon, \quad (1.49)$$

$$\delta_\varepsilon \phi_i = [\varepsilon, \phi_i]. \quad (1.50)$$

The corresponding moment map for this action is

$$\mu = \int_{M_3} d^3x \sqrt{g} g^{ij} \text{Tr}(\varepsilon D_i \phi_j). \quad (1.51)$$

The symplectic action is then

$$S = - \int dy d^3x \text{Tr} \left(\sqrt{g} g^{ij} \phi_i F_{yj} + \frac{1}{2} \text{Im}(\zeta^{-1} \epsilon^{ijk} (\mathcal{A}_i \partial_j \mathcal{A}_k + \frac{2}{3} [\mathcal{A}_i, \mathcal{A}_j])) \right). \quad (1.52)$$

The stationary condition for S is equivalent to the Kapustin-Witten equations [KW]

$$\mathcal{F} = \zeta * \bar{\mathcal{F}}, \quad (1.53)$$

$$d_A * \phi = 0 \quad (1.54)$$

specialized to $M_4 = \mathbb{R} \times M_3$.⁶ When $\zeta = -i$ taking the real and imaginary parts of these equations gives us the standard Kapustin-Witten equations

$$F - \phi \wedge \phi + * d_A \phi = 0, \quad (1.55)$$

$$d_A * \phi = 0. \quad (1.56)$$

⁶We also have to set $\phi_y = 0$ in order to precisely match the stationary condition of S with the four-dimensional equation $\mathcal{F} = \zeta * \bar{\mathcal{F}}$.

The gradient flow equations for this action functional on the other hand are the Haydys-Witten equations on $M_5 = \mathbb{R}^2 \times M_3$

$$ds \wedge \mathcal{F} = \zeta * \overline{\mathcal{F}}, \quad (1.57)$$

$$F_{\tau x} + \sqrt{g} g^{ij} D_i \phi_j = 0. \quad (1.58)$$

Once again, when $\zeta = -i$ and assuming that the connection of M_5 is of the form

$$A = A_\tau d\tau + A_x dx + \mathcal{A}_i dx^i \quad (1.59)$$

where A_τ, A_x are real and \mathcal{A}_i are complex, we can write the standard form of the Haydys-Witten equations

$$F_{y\mu} + D^\nu B_{\nu\mu} = 0, \quad (1.60)$$

$$F^+ - \frac{1}{4} B \times B - \frac{1}{2} D_y B = 0. \quad (1.61)$$

The Kapustin-Witten and Haydys-Witten equations provide a gauge theoretic formulation of Khovanov homology [Wit8]. We will revisit this topic briefly in Chapter 4 of this thesis.

1.1.4 4d Chern-Simons

As our final example, we consider the 2d symplectic bosons of Section 1.1.2 with an infinite-dimensional target space. Consider the complex symplectic manifold (Y, Ω) with

$$Y = \{G_{\mathbb{C}}\text{-connections on } \mathbb{R}_{u,v}^2\}, \quad \Omega = \int du dv \operatorname{Tr}(\delta \mathcal{A}_u \wedge \delta \mathcal{A}_v). \quad (1.62)$$

Take the \mathcal{G} -equivariant version where $\mathcal{G} = \operatorname{Map}(\mathbb{R}^2, G)$. Letting ε be the gauge transformation parameter, the complex moment map is

$$\mu_c = \int dx dy \operatorname{Tr}(\varepsilon \mathcal{F}_{uv}). \quad (1.63)$$

Plugging these equations into the \mathcal{G} -equivariant holomorphic symplectic boson functional, the resulting functional is the 4d Chern-Simons theory of Costello [C, CWY1, CWY2, CY]:

$$W = \int_{\mathbb{R}^2 \times \mathbb{C}} dz \wedge \text{CS}(A), \quad (1.64)$$

where

$$A = \mathcal{A}_u du + \mathcal{A}_v dv + A_{\bar{z}} d\bar{z} \quad (1.65)$$

is a partial connection on $\mathbb{R}^2 \times \mathbb{C}$. Note that along \mathbb{R}^2 we have $\mathcal{A}_{u,v} = A_{u,v} + i\phi_{u,v}$ so that the real part of \mathcal{A}_u and \mathcal{A}_v are real G -connections, whereas the complex part is a one-form on \mathbb{R}^2 .

The stationary equations are five-dimensional PDEs for a partial connection

$$A = A_x dx + \mathcal{A}_u du + \mathcal{A}_v dv + A_{\bar{z}} d\bar{z} \quad (1.66)$$

on $\mathbb{R}_s \times \mathbb{R}^2 \times \mathbb{C}$ which read

$$\mathcal{F} = \zeta * (d\bar{z} \wedge \overline{\mathcal{F}}). \quad (1.67)$$

In order to match the above equations with the stationary condition of S we must assume A_x is real. These are supplemented by the zero-moment map condition

$$D_u \phi_u + D_v \phi_v + \partial_z A_{\bar{z}} = 0. \quad (1.68)$$

The gradient flow equation is a six-dimensional PDE for a connection

$$A = A_\tau d\tau + A_x dx + \mathcal{A}_u du + \mathcal{A}_v dv + A_{\bar{z}} d\bar{z} \quad (1.69)$$

on the six-manifold $\mathbb{R}_{s,\tau}^2 \times \mathbb{R}^2 \times \mathbb{C}$ taking the form

$$ds \wedge \mathcal{F} = *(\zeta d\bar{z} \wedge \overline{\mathcal{F}}). \quad (1.70)$$

Once again these are supplemented with the equations

$$F_{s\bar{s}} + D_u\phi_u + D_v\phi_v + \partial_z A_{\bar{z}} = 0. \quad (1.71)$$

Again, the author is not aware of a systematic discussion of these PDEs in the literature. See [ATZ] for some indication of their applications to the categorification of integrable systems.

1.2 Instantons and Supersymmetry

So far we have discussed instantons as solutions to first order non-linear PDEs that give global minima of second-order non-linear Euclidean action functionals. We observed that such field configurations provide non-perturbative insight into the field theory, and that the equations they satisfy often come about as gradient flow equations for an action functional in one dimension less. These remarks are applicable to general field theories. We now explain how instantons play a role of distinguished importance in supersymmetric field theories.

To illustrate this point, we go back to the gradient flow equation

$$\frac{d\phi}{d\tau} = \frac{\partial h}{\partial \phi}. \quad (1.72)$$

One of the simplest examples of a supersymmetric system consists of the super-particle moving under a potential $V = \frac{1}{2} \left(\frac{\partial h}{\partial \phi} \right)^2$. In Euclidean signature the action reads

$$S_E = \int d\tau \left(\frac{1}{2} \left(\frac{d\phi}{d\tau} \right)^2 + \frac{1}{2} \left(\frac{\partial h}{\partial \phi} \right)^2 + \bar{\psi} \frac{d}{d\tau} \psi + \frac{\partial^2 h}{\partial \phi^2} \bar{\psi} \psi \right). \quad (1.73)$$

It is invariant under the odd symmetry transformation

$$Q\phi = \bar{\psi}, \quad (1.74)$$

$$Q\psi = -\frac{d\phi}{d\tau} + \frac{\partial h}{\partial \phi}, \quad (1.75)$$

$$Q\bar{\psi} = 0 \quad (1.76)$$

and

$$\overline{Q}\phi = -\psi, \quad (1.77)$$

$$\overline{Q}\psi = 0, \quad (1.78)$$

$$\overline{Q}\overline{\psi} = \frac{d\phi}{d\tau} + \frac{\partial h}{\partial \phi}. \quad (1.79)$$

Therefore the supersymmetry equations, obtained from setting the action of the supercharges on the fermions to vanish, for Q , are equivalent to the gradient flow equation, whereas for \overline{Q} are equivalent to the anti-gradient flow equation.

As we remarked previously, one of the main uses of instantons is to provide information about the quantum spectrum of a theory which is not accessible through standard perturbative methods. This can be made precise in a beautiful mathematical fashion in supersymmetric field theories [Wit2]. For supersymmetric quantum mechanics, we're interested in diagonalizing the Hamiltonian

$$H = -\frac{1}{2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{2}\left(\frac{\partial h}{\partial \phi}\right)^2 + \frac{1}{2}\frac{\partial^2 h}{\partial \phi^2}[\overline{\psi}, \psi] \quad (1.80)$$

acting on the super-quantum mechanics Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^2$. By using the equation

$$2H = \{Q, Q^\dagger\} \quad (1.81)$$

one can show that the energy eigenvalues satisfy $E \geq 0$ with $E = 0$ if and only if a state is annihilated by both Q and Q^\dagger . The zero energy states, since they preserve both supercharges are known as supersymmetric states and are of particular interest in supersymmetric field theory. The virtue of instantons is that they can be used to obtain the exact, non-perturbative space of supersymmetric states. The basic idea is as follows. One begins by studying the classical vacua, which are given by the critical points of h . One can then construct perturbative ground states $|\psi_\alpha\rangle$ for each critical point α of h . This gives us

the space of perturbative ground states. The tunneling effect however tells us that some of these states might be lifted. This lifting is captured precisely by instantons. It tells us that there are matrix elements of the form

$$\langle \psi_\alpha | Q | \psi_\beta \rangle = n_{\beta\alpha} e^{h_\alpha - h_\beta} \quad (1.82)$$

where $n_{\beta\alpha}$ is the number of instantons from β to α . Since we know the exact matrix element between perturbative ground states, one can determine precisely which states are lifted and determine the space of exact ground states. The virtue of supersymmetry is that it allows us to obtain an exact expression for the non-perturbative correction to the spectrum. This then gives us a method to obtain non-perturbative information about the quantum spectrum.

The above is just the simplest example of a more general phenomenon. The supersymmetric or BPS equations in supersymmetric theories typically are equivalent to instanton equations, and these instantons usually give us non-perturbative insight into our quantum theory. We can verify this remark for the other basic examples we discussed: The Cauchy-Riemann equation is the supersymmetric equation for a two-dimensional field theory known as the A-model with target space (X, ω) . Since we are now in a two-dimensional quantum field theory, we need to choose a spatial slice to define a Hilbert space. Taking the spatial slice to be an interval, and studying the pseudo-holomorphic map equation with various boundary conditions on the strip $\mathbb{R} \times I$, one is lead to the beautiful and rich mathematical structure of the Fukaya A_∞ -category [FOOO1, FOOO2]. Finally, our third basic example, the self-dual instanton equation is the supersymmetric equation for the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory with the Donaldson-Witten twist. This time we consider spatial slices determined by a compact three-manifold M_3 . The space of perturbative ground states are given by flat G -connections on M_3 , whereas the space of quantum exact

ground states are given by counting solutions of the self-dual instanton equation on $\mathbb{R} \times M_3$. The resulting structure is known as instanton Floer cohomology [F1].

Our main interest in this thesis however, as remarked before is the deformed Cauchy-Riemann equation (1.22). The deformed Cauchy-Riemann equation, or Witten's $\bar{\partial}$ -equation arises as the supersymmetric equation for a deformation of the A-model known as the A-model with superpotential. It is also known as a Landau-Ginzburg model with target space X , and superpotential W . The A-model with superpotential and the role of the instanton equation will play a central role in this thesis.

1.3 Instantons and Categorification

We now explain the role of instantons in categorification; an important notion used throughout this thesis. Categorification at its most primitive level aims to upgrade an integer, typically some sort of geometric or physical invariant, such as the Euler characteristic of a manifold, or the Jones polynomial of a knot, to a more refined structure: a vector space, or a chain complex, from which the original integer can be recovered by taking a character. In physical terms, very often the new ingredient in categorification is the incorporation of instanton effects.

To illustrate this, let us go back to the example of supersymmetric quantum mechanics. Given an SQM system with two nilpotent supercharges Q, Q^\dagger , and a \mathbb{Z}_2 -valued fermion number operator $(-1)^F$, one can define the Witten index

$$I = \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H}. \quad (1.83)$$

The virtue of I is that unlike the standard partition function $Z(\beta) = \text{Tr}_{\mathcal{H}} e^{-\beta H}$, the Witten index is a protected quantity; it is insensitive to small deformations of the physical system.

Moreover, one can show that supposing the quantum mechanical system has a discrete spectrum, I is independent of β and is moreover an integer, $I \in \mathbb{Z}$ that only depends on the supersymmetric ground states of H . For the SQM system defined by the pair (M, h) one can compute I exactly in terms of the critical points of h and their Morse indices⁷. It is given by

$$I = \sum_{\{p|\partial h=0\}} (-1)^{\mu_p}. \quad (1.84)$$

In categorification one is interested in vector spaces. The natural vector space \mathcal{R} that categorifies the Witten index I is the Hilbert subspace of supersymmetric states: $|\psi\rangle \in \mathcal{H}$ such that $Q|\psi\rangle = Q^\dagger|\psi\rangle = 0$. The construction of the exact space of quantum ground states, starting from the perturbative ground states labeled by the critical points of h is where instantons come in. In order to construct \mathcal{R} we begin by forming the space of perturbative ground states

$$R = \bigoplus_{\{p|\partial h_p=0\}} \mathbb{C}\langle\psi_p\rangle. \quad (1.85)$$

The all-important instanton effects are then incorporated in an operator

$$d : R \rightarrow R \quad (1.86)$$

of fermion degree +1 such that $d^2 = 0$. Together (R, d) form what is known as a chain complex, such that the d -cohomology gives us a space isomorphic to the space of ground states

$$\mathcal{R} \cong H_d^*(R). \quad (1.87)$$

⁷The Morse index μ_p of a critical point p of h is given by the number of negative eigenvalues of the Hessian of h at p .

The Witten index I is recovered by taking the Euler character of \mathcal{R} (and also the Euler character of R). The chain complex (R, d) is known as the Morse-Smale-Witten complex [Wit2].

This thesis aims to apply the categorification procedure to the study of a natural collection of integers known as BPS indices in the A-model with superpotential. In the A-model with superpotential, we will be interested in the Hilbert space on the real line \mathbb{R} . In order to define a Hilbert space, we must choose the field configurations at the infinite ends of \mathbb{R} , which are constrained to be a critical point of the superpotential. Thus the Hilbert space on \mathbb{R} decomposes into superselection sectors \mathcal{H}_{ij} labeled by ordered pairs of critical points $\{\phi_i, \phi_j\}$. Traditionally one studies the supersymmetric ground states in these sectors by studying the individual Witten indices μ_{ij} of \mathcal{H}_{ij} . The Witten index μ_{ij} is known as a BPS index. BPS indices, similar to the Witten index in SQM, can again be computed by a classical analysis; one studies the ζ -soliton equation

$$\frac{d\phi^a}{dx} = g^{ab} \partial_b \text{Re} \zeta^{-1} W \quad (1.88)$$

with ij boundary conditions, and its solutions in order to compute μ_{ij} .

The study of the BPS indices $\{\mu_{ij}\}$ goes back to work of Cecotti and Vafa [CV1], and Cecotti, Fendley, Intriligator and Vafa [CFIV] from the early 1990s, who observed many interesting properties. One particularly striking property eventually came to be known as “wall-crossing.” Whereas the Witten index I in SQM is independent of small deformations of the potential h , the collection of integers $\{\mu_{ij}\}$ in the A-model with superpotential can jump as we deform the superpotential W . Moreover, the quantitative form of the jump takes the form

$$\mu_{ij} \rightarrow \mu_{ij} \pm \mu_{ik} \mu_{kj}, \quad (1.89)$$

for some critical point ϕ_k . Jumps in the BPS indices of this form are known as Cecotti-Vafa wall-crossing. See Chapter 2 for more details and examples of the Cecotti-Vafa wall-crossing formula.

Our basic aim in this thesis is to categorify BPS indices and their universal properties such as wall-crossing. The main new ingredient involved in this, as mentioned before is the incorporation of instanton effects which can be ignored when working at the level of indices. In Chapter 2 we explain how one can incorporate Witten's $\bar{\partial}$ -equation and construct chain complexes (R_{ij}, d_{ij}) that categorify the BPS index μ_{ij} . The main result of this chapter is then an explicit description of how instanton effects are incorporated to provide a categorification of the Cecotti-Vafa wall-crossing formula.

In Chapter 3, we generalize the discussion of categorical wall-crossing to include theories with twisted masses. To motivate twisted masses, we recall that one reason for studying the A-model with superpotential is that when applied to infinite dimensions and taking W to be a holomorphic functional, we are lead to some of the most interesting equations of mathematical physics. However when discussing this we glossed over some subtleties. For instance, if we look carefully we will note that in general the Chern-Simons superpotential is not single-valued, only its derivative is. Similarly, in the setup of Section 1.1.1, if the complex symplectic form Ω defines a non-trivial class in $H^{(2,0)}(Y)$, then the holomorphic Liouville functional is only locally defined, and therefore not single-valued. This multivaluedness of superpotentials is in fact allowed by Landau-Ginzburg theory, since the superpotential enters the Lagrangian only through its derivatives. Equivalently, one can say the the derivative of the superpotential can have non-trivial periods around 1-cycles of the target space. These non-trivial periods are known as “twisted masses.” Twisted masses can lead to various novel phenomenon in the A-model with superpotential. Chapter 3 is

therefore devoted to the study of the A-model with non-trivial twisted masses, and how the presence of twisted masses affects the discussion of BPS states, instanton equations and categorical wall-crossing.

Finally in Chapter 4 we conclude by discussing some future research directions. We revisit the Chern-Simons superpotential of Section 1.1.3, and conjecture how one can be lead to novel *algebraic* knot invariants. We also remark on how some of our results are expected to generalize to four-dimensional $\mathcal{N} = 2$ theories.

Chapter 2

Categorification of Cecotti-Vafa Wall-Crossing

The contents of this chapter appeared in the preprint [KM1], written jointly with G. W. Moore.

2.1 Introduction and Outline

BPS states have played an important role in many aspects of physical mathematics. As is very well-known, the spaces of BPS states can jump discontinuously as physical parameters are varied, a phenomenon known as wall-crossing. Investigations of BPS wall-crossing have led to a wide variety of very interesting developments. For some reviews of BPS wall-crossing see [Cec, KoSo2, KoSo3, M1, M3, N, Pio].

BPS wall-crossing appears in two-dimensional quantum field theories with $\mathcal{N} = (2, 2)$ supersymmetry, where it was first discovered [CFIV, CV1] as well as in four-dimensional supergravity and field theory with $\mathcal{N} = 2$ supersymmetry [DM, Dor, GMN1, LY, SW]. It also appears in a more elaborate form in coupled 2d-4d systems [GMN4].

Indeed, there are quantitative formulae expressing how BPS indices change across walls of marginal stability. It is natural to ask if one can obtain more refined information about the spaces of BPS states. For example, if BPS states are identified with the cohomology of some chain complexes one would like to know how the chain complexes themselves jump across walls of marginal stability. One cannot expect an answer at the level of chain complexes

per se, since homotopy equivalent chain complexes are also physically equivalent, but it is meaningful to ask how the equivalence class of the chain complexes (up to homotopy) changes¹. In particular, relating the homotopy equivalence class of chain complexes across a marginal stability wall allows us, by taking cohomology, to answer *How do the BPS Hilbert spaces jump across a wall of marginal stability?* This is the question a categorified wall-crossing formula is meant to answer.

This chapter addresses the categorification of the renowned Cecotti-Vafa wall-crossing formula for BPS indices in two-dimensional $\mathcal{N} = (2, 2)$ quantum field theory. We have made use of a formalism developed in [GMW, GMWSh], specifically for the purpose of carrying out the program of categorification of wall-crossing formulae. Indeed, in [GMW, GMWSh] it was explained how to categorify the so-called “framed wall-crossing” or “S-wall-crossing” formulae in the two-dimensional models. The present chapter adds to the story with an improved understanding of how to phrase the categorification of the Cecotti-Vafa wall-crossing formula.

¹Note that the homotopy class of a chain complex contains more information than the index. As a simple example, consider

$$C = (\mathbb{Z} \oplus \mathbb{Z}[1], d = 0) \tag{2.1}$$

and

$$C' = (\mathbb{Z} \oplus \mathbb{Z}[1], d') \tag{2.2}$$

where d' maps a generator of \mathbb{Z} to a generator of $\mathbb{Z}[1]$. Both have vanishing Euler characteristics

$$\chi(C) = \chi(C') = 0, \tag{2.3}$$

but their cohomology is different so they are not homotopy equivalent.

Much remains to be done in the program of the categorification of wall-crossing formulae. In particular, the categorification of the four-dimensional wall-crossing formula of Kontsevich-Soibelman is not known.² We believe an important step forward is to include twisted masses in two-dimensional Landau-Ginzburg models. This will be achieved in Chapter 3.

In the remainder of this introduction we outline in more detail the difficulties which must be overcome to categorify the Cecotti-Vafa wall-crossing formula, and how we will achieve this.

2.1.1 A Failure of Naive Categorification

Supposing that i, j, k denote distinct massive vacua of a two-dimensional $\mathcal{N} = (2, 2)$ theory, recall that the Cecotti-Vafa wall-crossing formula states that across a wall of marginal stability of type ijk , the BPS indices μ and μ' on either side of the wall are related by

$$\mu'_{ij} = \mu_{ij}, \tag{2.4}$$

$$\mu'_{jk} = \mu_{jk}, \tag{2.5}$$

$$\mu'_{ik} = \mu_{ik} \pm \mu_{ij}\mu_{jk}, \tag{2.6}$$

the sign accounting for which way the wall-crossing occurred. As a first step in categorification, it's indeed encouraging, as we recall in section 2.3, that for Landau-Ginzburg models one can formulate finite-dimensional chain complexes (R_{ij}, d_{ij}) such that the BPS index

²The change in the 4d BPS state spaces is nicely understood using the halo formalism of [ADJM, DM, GMN3]. In some sense, this answers the question of the categorification of wall-crossing formulae, but the categorification program is more ambitious, and seeks to describe the full set of BPS states on either side of the wall in homotopical algebra terms.

μ_{ij} is given by a graded trace

$$\mu_{ij} = \text{Tr}_{R_{ij}}(-1)^F. \quad (2.7)$$

The BPS Hilbert space ³ of type ij is isomorphic to the d_{ij} -cohomology,

$$\mathcal{H}_{ij}^{\text{BPS}} = H^\bullet(R_{ij}, d_{ij}). \quad (2.9)$$

A categorified wall-crossing formula should then relate the BPS chain complexes (R'_{ij}, d'_{ij}) upon crossing a wall of marginal stability to the original chain complexes (R_{ij}, d_{ij}) . The simplest guess consistent with (2.6) is to say that the underlying vector spaces of the chain complexes are related by

$$R'_{ij} = R_{ij}, \quad (2.10)$$

$$R'_{jk} = R_{jk}, \quad (2.11)$$

$$R'_{ik} = R_{ik} \oplus (R_{ij} \otimes R_{jk}), \quad (2.12)$$

accompanied possibly with a degree shift on the $(R_{ij} \otimes R_{jk})$ summand to account for which way the wall-crossing occurred. The simplest differentials that one can guess on the primed spaces are

$$d'_{ij} = d_{ij}, \quad (2.13)$$

$$d'_{jk} = d_{jk}, \quad (2.14)$$

$$d'_{ik} = d_{ik} \oplus (d_{ij} \otimes 1 + 1 \otimes d_{jk}). \quad (2.15)$$

³Throughout this chapter, we have factored out the (super)translational mode of the soliton. With it included the chain complex will be

$$\tilde{R}_{ij} = R_{ij} \otimes (\mathbb{Z}[-1] \oplus \mathbb{Z}), \quad (2.8)$$

and the BPS index would be the “new index” $\text{Tr}_{\tilde{R}_{ij}}(F(-1)^F)$ of [CFIV]. The spectrum of F on R_{ij} lies in a \mathbb{Z} -torsor, so after a suitable phase redefinition, the μ_{ij} will be integers.

Indeed, the Cecotti-Vafa statement (2.6) would follow as a corollary from this guess, simply by taking graded traces. Under this formula for the differentials, the primed BPS Hilbert spaces are simply

$$(\mathcal{H}_{ij}^{\text{BPS}})' \cong \mathcal{H}_{ij}^{\text{BPS}}, \quad (2.16)$$

$$(\mathcal{H}_{jk}^{\text{BPS}})' \cong \mathcal{H}_{jk}^{\text{BPS}}, \quad (2.17)$$

$$(\mathcal{H}_{ik}^{\text{BPS}})' \cong \mathcal{H}_{ik}^{\text{BPS}} \oplus (\mathcal{H}_{ij}^{\text{BPS}} \otimes \mathcal{H}_{jk}^{\text{BPS}}). \quad (2.18)$$

Things are not so simple: it is very easy to construct counter-examples to this naive prediction of how BPS Hilbert spaces jump across a wall of marginal stability. Here is a simple one.

Consider the quartic Landau-Ginzburg model, namely the theory of a chiral superfield Φ with superpotential

$$W = \frac{1}{4}\Phi^4 - \Phi. \quad (2.19)$$

Denote the three vacua $\Phi_1 = e^{-2\pi i/3}$, $\Phi_2 = 1$, $\Phi_3 = e^{2\pi i/3}$ with corresponding critical values W_1, W_2, W_3 . One can show that the absolute number of solitons is 1 between each pair of distinct vacua. By taking into account the fermion degree we have that

$$R_{12} = \mathbb{Z}, \quad (2.20)$$

$$R_{23} = \mathbb{Z}, \quad (2.21)$$

$$R_{13} = \mathbb{Z}, \quad (2.22)$$

with all differentials identically zero. We can vary the lower order terms of the superpotential (for instance we can turn on a quadratic term) so that W_2 passes through the line connecting W_1 and W_3 . The naive guess implies that upon this wall-crossing the chain

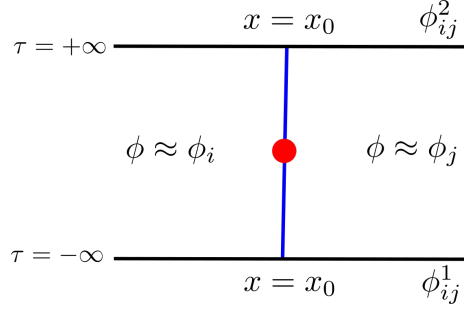


Figure 2.1: An instanton interpolating between two-different ij -solitons

complex R'_{13} is

$$R'_{13} = R_{13} \oplus (R_{12} \otimes R_{23})[1], \quad (2.23)$$

$$= \mathbb{Z} \oplus \mathbb{Z}[1]. \quad (2.24)$$

Because every differential in sight acts trivially, we conclude that $(\mathcal{H}_{13}^{\text{BPS}})'$ is two-dimensional. On the other hand, every Landau-Ginzburg model with target space \mathbb{C} and a polynomial superpotential has an absolute number of solitons between each pair of critical points given by either 0 or 1⁴. Thus the cohomology in such a model is either trivial or one-dimensional and we have found a contradiction. Our naive attempt at categorification has failed.

2.1.2 Missing Instantons

The reason for the failure of the differential d'_{ik} (2.15) is simple, but also interesting: We have missed instantons.

The spaces R_{ij} are made of perturbative BPS states $|\phi_{ij}\rangle$ coming from quantizing around a classical soliton ϕ_{ij} . The differentials d_{ij} on R_{ij} are meant to encode matrix elements

$$\langle \phi_{ij}^b | \mathcal{Q}_{ij} | \phi_{ij}^a \rangle, \quad (2.25)$$

⁴For a proof see Appendix C.1.

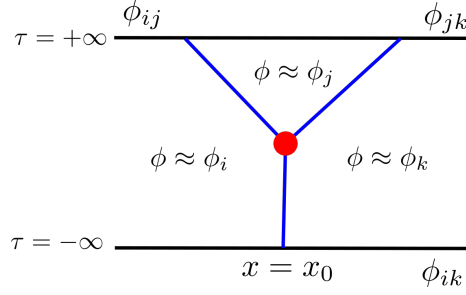


Figure 2.2: An instanton which contributes to an off-diagonal element of d'_{ik} .

where the superscripts a, b label different classical solitons of type ij . When these are non-zero there is a difference between the exact ground states and the perturbative ones. We know from the relation between Morse theory and supersymmetry [Wit2], that the former are computed by considering suitable instantons between these perturbative ground states. Now within a fixed sector, say the ij -sector, solutions of such an instanton on the plane look as in Figure 2.1: The soliton ϕ_{ij}^a is stationary, sitting at a fixed point x_0 , whereas at an instant τ_0 , we transition from the ϕ_{ij}^a to ϕ_{ij}^b . Such a process will contribute to the matrix element if the fermion numbers of ϕ_{ij}^a and ϕ_{ij}^b differ by 1.

Close to a wall of marginal stability, it is reasonable to postulate that bound states of ij and jk -solitons give rise to an approximate ik -soliton, post wall-crossing, thus giving our guess (2.12). Instantons of the sort depicted in Figure 2.1, contribute to matrix elements of the type

$$\langle \phi_{ik}^b | \mathcal{Q}_{ik} | \phi_{ik}^a \rangle \quad (2.26)$$

and

$$\langle \phi_{ij}^a, \phi_{jk}^b | \mathcal{Q}_{ik} | \phi_{ij}^{a'}, \phi_{jk}^{b'} \rangle. \quad (2.27)$$

Such contributions are indeed reflected in our guess for the differential (2.15). Our formula for d'_{ik} has made an implicit assumption that the off-diagonal matrix element

$$\langle \phi_{ij}^a, \phi_{jk}^b | \mathcal{Q}_{ik} | \phi_{ik}^c \rangle \quad (2.28)$$

vanishes. However, it turns out, as we will explain in section 2.3.2 that in addition to the familiar instanton of Figure 2.1, there can be a more interesting object, where a stationary ik -soliton can split into ij and jk solitons traveling at just the correct angles to preserve \mathcal{Q}_{ik} -supersymmetry. Such an instanton is depicted in Figure 2.2. Counting instantons of this type allows one to write down a corrected differential on R'_{ik} . This is the main new ingredient that enters the categorified wall-crossing formula.

2.1.3 Wall-Crossing Invariants

In order to derive wall-crossing formulas such as (2.6) it is extremely useful to introduce certain *wall-crossing invariants*. For Cecotti-Vafa wall-crossing an example of such a wall-crossing invariant is the spectrum generator ⁵

$$S = \prod_{Z_{ij} \in \mathbb{H}}^{\curvearrowright} (\mathbf{1} + \mu_{ij} e_{ij}) \in SL(|\mathbb{V}|, \mathbb{Z}) \quad (2.29)$$

which must be invariant under crossing marginal stability walls [KoSo1], so long as no BPS rays enter or exit the half-plane \mathbb{H} . The wall-crossing invariant S has a simple conceptual meaning. One can show that S_{ij} is the Witten index of the space of boundary local operators at a junction of thimbles of type i and j [GMW] (a related interpretation appeared in [?]), see Figure 2.3. Such a space is insensitive to marginal stability walls. Nonetheless

⁵Notation: \mathbb{V} is the vacuum set, assumed to be finite in this chapter. \mathbb{H} is the upper-half plane, Z_{ij} are central charges and e_{ij} is the ij elementary matrix. \curvearrowright is meant to indicate a clockwise ordered product with respect to the central charges. Implicit in the notation is that an ordering on \mathbb{V} has been chosen.

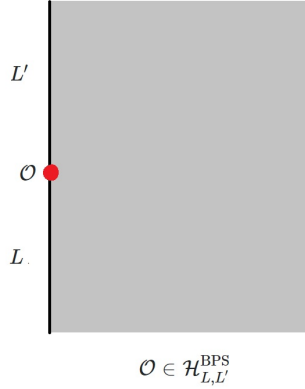


Figure 2.3: A boundary local operator \mathcal{O} between two branes L and L'

the BPS indices S at a given point in parameter space allow the computation of the boundary Witten indices S . Comparing S on different sides of the wall of marginal stability leads to (2.6).

It is natural then to expect that a categorical wall-crossing invariant can also be constructed. The invariance of S is categorically enhanced as follows. The BPS chain complexes (R_{ij}, d_{ij}) , along with counts of ζ -instantons of the type depicted in Figure 2.2, allow for the construction of an A_∞ -category $\widehat{R}[X, W]$ whose objects can be thought of thimble branes⁶ and morphisms are vector spaces of boundary local operators at brane junctions [GMW]⁷. The categorical wall-crossing constraint is then formulated as follows.

⁶Note that considering a category with only thimble objects is not restrictive. $\widehat{R}[X, W]$ can be enlarged to a triangulated A_∞ category for which the thimble objects provide a semi-orthogonal decomposition.

⁷The A_∞ category of [GMW] can be viewed as an infrared construction of the category of A-branes in a Landau-Ginzburg model, which to mathematicians is known as the Fukaya-Seidel category [Seid] of (X, W) , and is denoted by $FS[X, W]$. It is expected that $FS[X, W]$ and $\widehat{R}[X, W]$ are quasi-isomorphic as A_∞ -categories. An outline of a proof of this expectation was given in [GMW].

The homotopy class of $\widehat{R}[X, W]$ is a wall-crossing invariant.

In the above statement *homotopy class* refers to the homotopy equivalence of A_∞ -categories which is defined in Appendix B.1. We show how our categorical wall-crossing formula can be derived from this wall-crossing constraint in section 2.6.

Remark Note that instead of $\widehat{R}[X, W]$, there are other wall-crossing invariants one could have used as a starting point. For instance instead of imposing A_∞ -equivalence of the “open string algebra” $\widehat{R}[X, W]$ across a marginal stability wall like we do in this chapter, one could have imposed L_∞ -equivalence of the closed string algebra R_c , defined in [GMW]. Another way of describing the categorical wall-crossing formula makes use of half-BPS interfaces. These can be used to construct a categorical notion of a flat parallel transport on a bundle of categories of boundary conditions over the space of Morse superpotentials [GMW]. The absence of monodromy around contractible cycles that intersect walls of marginal stability implies a categorified version of the invariance of S defined in equation (2.29). This categorical equation can in turn be reduced to categorified braid relations. For details see [GMW, M2]. These superficially distinct starting points are all expected to lead to the same eventual result.

2.1.4 Outline of the Chapter

The outline of this chapter is as follows. In section 2.2, we recall the standard discussion of wall-crossing at the level of BPS indices. This is followed in section 2.3 by a discussion of how to formulate chain complexes that categorify the BPS indices. The crucial concept of a ζ -instanton with fan boundary conditions is discussed and we formulate the statement of categorical wall-crossing by using counts of certain trivalent instantons in section 2.4. After reviewing the construction of the A_∞ category of half-BPS branes associated to a

Landau-Ginzburg model in section 2.5, we show the equivalence of the categorical wall-crossing formula to the homotopy equivalence of A_∞ categories constructed on either side of a marginal stability wall in section 2.6. After a brief digression on fermion degrees of a ζ -instanton in section 2.7, we turn our attention to some examples that illustrate our formulas in section 2.8. We conclude with some speculations in section 2.9.

2.2 Wall-Crossing of BPS Indices

While our formulas are expected to hold for arbitrary massive two-dimensional $\mathcal{N} = (2, 2)$ theories (with a non-anomalous $U(1)_R$ -symmetry), it is simplest to work in the setting of Landau-Ginzburg models. A Landau-Ginzburg model is a supersymmetric sigma model with a Kähler manifold target X and a potential of the form

$$V = |dW|^2, \quad (2.30)$$

where $W : X \rightarrow \mathbb{C}$ is a holomorphic function known as the superpotential. More precisely, working in two-dimensional $\mathcal{N} = 2$ -superspace, we can use the Kähler structure on X to write D-terms

$$L_D = \int d^4\theta K(\Phi, \bar{\Phi}), \quad (2.31)$$

and the holomorphicity of W to write F-terms

$$L_F = \int d^2\theta W(\Phi) + \int d^2\bar{\theta} \bar{W}(\bar{\Phi}), \quad (2.32)$$

to get a Lagrangian

$$L = L_D + L_F, \quad (2.33)$$

invariant under two-dimensional $\mathcal{N} = (2, 2)$ Poincaré supersymmetry. The reader is encouraged to consult [MS1], whose notation we adopt, for more details. Various non-renormalization theorems [Seib] of W tell us that one can get great mileage simply by

studying the superpotential and its various properties. One use of the superpotential W is that it is sufficient to study many aspects of BPS states.

Supposing that W only has a finite number of isolated singularities, a familiar argument shows that the classical energy in such a theory obeys the BPS bound,

$$E \geq |Z_{ij}| \quad (2.34)$$

where

$$Z_{ij} = W_i - W_j \quad (2.35)$$

and W_i denotes the critical value $W(\phi_i)$ of the critical point ϕ_i . Denoting the bosonic fields of the LG model as ϕ , the standard Bogomolny trick leads to the BPS equation

$$\frac{d\phi}{dx} = \nabla \text{Re}(\zeta^{-1}W), \quad (2.36)$$

known as the ζ -soliton equation, ζ being an arbitrary phase. Solutions on \mathbb{R} with prescribed vacua ϕ_i and ϕ_j at the ends of \mathbb{R} can only exist if

$$\zeta = \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}. \quad (2.37)$$

Using intersection theory of vanishing cycles, it is possible to get a well-defined *signed* count of the number of BPS solitons in the ij -sector. Let

$$L_i(\zeta) = \{p \in X | \lim_{x \rightarrow -\infty} f_x^\zeta(p) = \phi_i\}, \quad (2.38)$$

$$R_i(\zeta) = \{p \in X | \lim_{x \rightarrow +\infty} f_x^\zeta(p) = \phi_i\} \quad (2.39)$$

be the ascending and descending manifolds respectively, emanating from the critical point ϕ_i of the Morse function $\text{Re}(\zeta^{-1}W)$. f_x^ζ denotes the one-parameter map $f_x^\zeta : X \rightarrow X$ defined by the gradient vector field of $\text{Re}(\zeta^{-1}W)$. We then set

$$\mu_{ij} = L_i^- \circ R_j^+ \quad (2.40)$$

where $L_i^- = L_i(\zeta_{ji}e^{-i\epsilon})$ and $R_j^+ = R_j(\zeta_{ji}e^{+i\epsilon})$ and ϵ is a small positive number. The infinitesimal rotation ensures that the intersection is transversal.

The significance of μ_{ij} from the perspective of the $\mathcal{N} = 2$ field theory defined by (X, W) is that one can show [CFIV, CV1] that

$$\mu_{ij} = \text{Tr}_{\mathcal{H}_{ij}^{\text{BPS}}}(-1)^F F \quad (2.41)$$

where F is the fermion number and

$$\mathcal{H}_{ij}^{\text{BPS}} = \ker(\mathcal{Q}_{ij}) \cap \ker(\overline{\mathcal{Q}}_{ij}), \quad (2.42)$$

where

$$\mathcal{Q}_{ij} = Q_- - \zeta_{ij}^{-1} \overline{Q}_+. \quad (2.43)$$

μ_{ij} is thus a supersymmetry protected index that counts the degeneracy of BPS states of type ij . Some of its elementary properties are as follows.

Metric Independence While the BPS soliton equation does depend on the Kähler metric on X , the BPS index μ_{ij} is metric-independent.

CPT Reversing $x \rightarrow -x$ takes $F \rightarrow -F$ so that $\mu_{ij} = -\mu_{ji}$.

It is familiar that supersymmetric indices such as the Witten index are quantities that are piecewise constant in parameter space. For instance, we can consider the one-dimensional system given by the real superpotential

$$h = x^4 + \alpha x^2 + \beta x. \quad (2.44)$$

While the conventional partition function $Z = \text{Tr}(e^{-\beta H})$ of the system will be a very non-trivial function of α and β , the Witten index $I = \text{Tr}(-1)^F e^{-\beta H}$ is simply equal to +1,

$$I = 1, \quad (2.45)$$

irrespective of α and β . In contrast the behavior of the BPS index is more subtle.

Historically⁸ wall-crossing was first noticed by considering points in the parameter space of the Landau-Ginzburg model with

$$W = X^4 + t_1 X^2 + t_2 X \quad (2.46)$$

with distinct symmetry groups. Supposing we start out at $(t_1, t_2) = (0, 1)$, where the model is \mathbb{Z}_3 -symmetric, the latter permuting the three vacua. We can show that there is indeed a single soliton between each pair of distinct critical points,

$$\mu_{12} = 1, \quad (2.47)$$

$$\mu_{23} = 1, \quad (2.48)$$

$$\mu_{13} = 1, \quad (2.49)$$

a spectrum consistent with the \mathbb{Z}_3 symmetry. If we move slightly away from this point, the collection of numbers doesn't change. On the other hand at $(t_1, t_2) = (1, 0)$, the superpotential has \mathbb{Z}_2 -symmetry. Requiring a \mathbb{Z}_2 -symmetric spectrum requires that one of the solitons disappears and the BPS indices are

$$\mu'_{12} = 1, \quad (2.50)$$

$$\mu'_{23} = 1, \quad (2.51)$$

$$\mu'_{13} = 0. \quad (2.52)$$

Thus BPS indices are examples of indices that are not constant but only piecewise constant.

⁸We thank S. Cecotti for narrating this story.

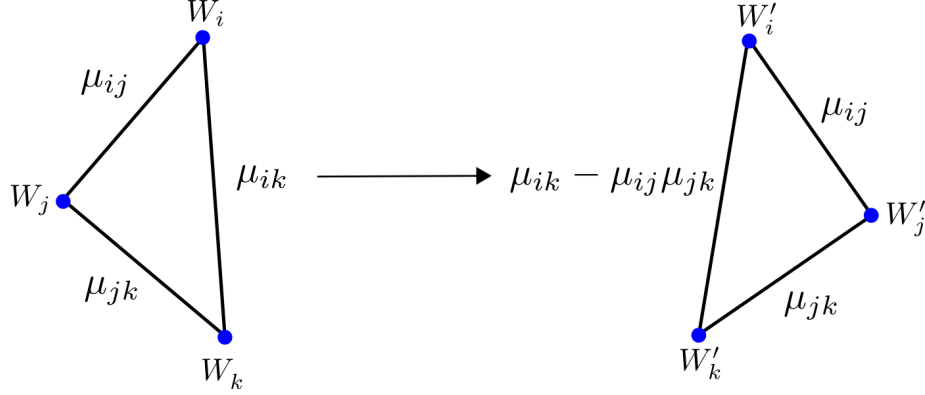


Figure 2.4: Wall-crossing summarized in the W -plane.

The content of the Cecotti-Vafa formula is as follows. It first states that potential discontinuous jumps in the BPS spectrum can occur when three critical values W_i, W_j, W_k become co-linear as we vary parameters. This is the locus where $\text{Im}(Z_{ij}\overline{Z}_{jk}) = 0$. Next it gives an explicit formula for the quantitative nature of this jump: If μ and μ' denote BPS degeneracies on different sides of the wall of marginal stability, they must be related by

$$\mu'_{ij} = \mu_{ij}, \quad (2.53)$$

$$\mu'_{jk} = \mu_{jk}, \quad (2.54)$$

$$\mu'_{ik} = \mu_{ik} \pm \mu_{ij}\mu_{jk}, \quad (2.55)$$

where the sign $-$ is picked in going from the negative side, where $\text{Im}(Z_{ij}\overline{Z}_{jk}) < 0$ to the positive side, where $\text{Im}(Z_{ij}\overline{Z}_{jk}) > 0$ and the $+$ is picked in the reverse move. We summarize the formula from the perspective of the W -plane in Figure 2.4.

The trick in arguing for this is to consider not just BPS states, but rather to look at

$$\mathcal{Q}(\zeta) = Q_- - \zeta^{-1}\overline{Q}_+ \quad (2.56)$$

preserving boundary conditions of our Landau-Ginzburg model when the latter is formulated on a half-space such as $(-\infty, 0] \times \mathbb{R}_t$. Such branes have been analyzed in great detail

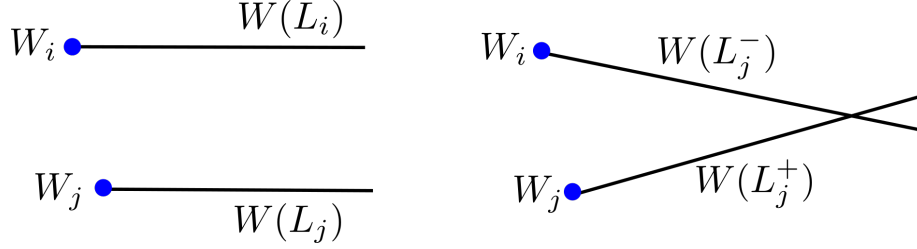


Figure 2.5: The topological intersection numbers $\widehat{\mu}_{ij}$ obtained by looking at intersection numbers of slightly rotated thimbles.

in references, [GMW, ?]. One finds that the homology class of the support of these branes lives in the finite rank \mathbb{Z} -module

$$B(\zeta) := H_{\frac{1}{2}\dim(X)}(X, \text{Re}(\zeta^{-1}W) \rightarrow \infty; \mathbb{Z}). \quad (2.57)$$

We can equip $B(\zeta)$ with a natural bilinear form

$$\widehat{\mu}^\zeta : B(\zeta) \times B(\zeta) \rightarrow \mathbb{Z}, \quad (2.58)$$

defined as follows. When W is Morse, there is a natural \mathbb{Z} -module basis for $B(\zeta)$ given by the homology class of Lefschetz thimbles $\{[L_i(\zeta)]\}_{i \in \mathbb{V}}$. The thimble $L_i(\zeta)$ projects to half-infinite rays emanating from the critical value W_i in the ζ -direction. We then define

$$\widehat{\mu}_{ij}^\zeta := \widehat{\mu}(L_i, L_j) = L_i^- \circ L_j^+, \quad (2.59)$$

where L^\pm denote thimbles with phases slightly rotated by a small positive or negative angle respectively, as in Figure 2.5.

Some basic properties of $\widehat{\mu}^\zeta$ are as follows. First: if i and j are distinct vacua, $\widehat{\mu}_{ij}^\zeta$ and $\widehat{\mu}_{ji}^\zeta$ cannot both be non-zero. In the case they are equal,

$$\widehat{\mu}_{ii} = 1. \quad (2.60)$$

Finally, if the vacuum weights are ζ -generic⁹, we can order the thimble basis in decreasing order of $\text{Im}(\zeta^{-1}W_i)$. Making this choice of ordering, we find that $\widehat{\mu}^\zeta$ is an upper-triangular $|\mathbb{V}| \times |\mathbb{V}|$ matrix with +1 on the diagonal.

For definiteness and to avoid notational clutter we set $\zeta = 1$ and set $\widehat{\mu} = \widehat{\mu}^{\zeta=1}$. This is equivalent to choosing the half-plane in which we take phase ordered products to be the upper-half plane, as was done in (2.29).

The matrix representation $\widehat{\mu}_{ij}$ for the bilinear form can be calculated from the BPS indices μ_{ij} by a nice rule expressed in terms of convex geometry.

Definition: A half-plane fan F of phase ζ is a collection of vacua $F = \{i_1, \dots, i_n\}$ such that $W(F) = \{W_{i_1}, \dots, W_{i_n}\}$ are the clockwise-ordered vertices of a semi-infinite convex polygon going off to infinity in the $-\zeta$ -direction. See Figure 2.6 for an example with $n = 4$. The dual graph looks like a half-plane fan (and indeed has a space-time interpretation), hence the terminology.

To a given half-plane fan $F = \{i_1, i_2, i_3, \dots, i_n\}$ assign the number

$$\mu_F = \mu_{i_1 i_2} \mu_{i_2 i_3} \dots \mu_{i_{n-1} i_n}. \quad (2.61)$$

We then make the

Claim

$$\widehat{\mu}_{ij} = \sum_{\substack{F_{ij} = \{i, i_1, \dots, i_k, j\} \\ F_{ij} \text{ half-plane fan}}} \mu_{i i_1} \dots \mu_{i_k j}. \quad (2.62)$$

⁹A set of critical values is called ζ -generic, following the terminology in [KKS], if none of the relative phases ζ_{ij} are equal to ζ .

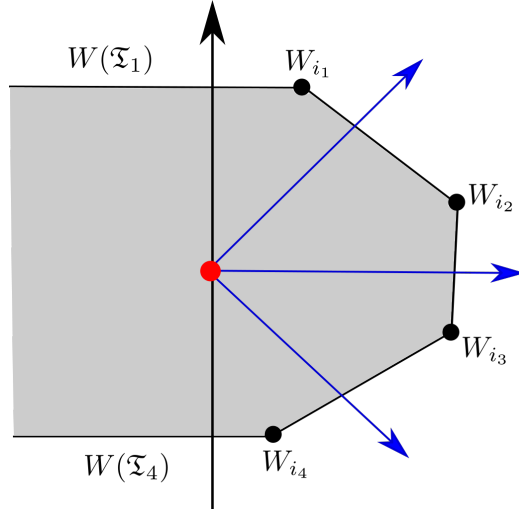


Figure 2.6: A half-plane fan $F_{i_1 i_4} = \{i_1, i_2, i_3, i_4\}$ for $\zeta = 1$ and the semi-infinite polygon it forms in the W -plane.

Proof The proof is a straightforward inductive argument, where we induct on distance between i and j . To show the base case, for two neighboring vacua $i < j$, one has $\widehat{\mu}_{ij} = \mu_{ij}$ due to (2.40)¹⁰. On the other hand there's only one polygon between two neighboring vacua, whose finite segment is given by the segment connecting them, to which we also assign μ_{ij} . For the inductive step, assume that the polygon rule (2.62) holds for vacua that are up to n units apart and consider a pair of vacua $\{i, j\}$ that are $n + 1$ units apart. We know that

$$L_i \circ \widetilde{L}_j = \mu_{ij} \quad (2.63)$$

where $\widetilde{L}_j := L_j(\zeta_{ij} e^{-i\epsilon})$. We thus want to compare \widetilde{L}_j with L_j^+ namely we must rotate this thimble in a clockwise direction by the phase of ζ_{ij} . In doing this rotation we pick up Picard-Lefschetz discontinuities: For each critical value W_k such that $\{i, k, j\}$ forms

¹⁰Note that for ζ being the phase of an ij -soliton left-right intersection number of (2.40) agrees with the left-left intersection number of (2.59)

half-plane fan, we pick up a contribution of $L_k^+ \mu_{ki}$. Summing these up we get

$$L_j^+ = \tilde{L}_j + \sum_{\substack{k \\ \{i,k,j\} \text{ is a fan}}} L_k^+ \mu_{ki}. \quad (2.64)$$

Thus we can compute that

$$\hat{\mu}_{ij} = \mu_{ij} + \sum_{\substack{k \text{ s.t.} \\ \{i,k,j\} \text{ is a fan}}} \hat{\mu}_{ik} \mu_{kj}. \quad (2.65)$$

The polygon rule applies to $\hat{\mu}_{ik}$ so that

$$\hat{\mu}_{ik} \mu_{kj} = \sum_{F_{ik}} \mu_{F_{ik}} \mu_{kj}. \quad (2.66)$$

On the other hand if $\{i, k, j\}$ is a fan, we can form an ij half-plane fan by taking the fan $F_{ik} = \{i, \dots, k\}$ and putting j at the end $\{i, \dots, k, j\}$. To this one precisely assigns $\mu_{F_{ik}} \mu_{kj}$. Conversely, every ij fan can be obtained in this way. \square

To see that this implies the wall-crossing formula, consider $\hat{\mu}$ restricted to the three-dimensional $\{i, j, k\}$ space and note that if we are on the left side of Figure 2.4 then there is only one half-plane of type ij ik jk respectively, so that

$$\hat{\mu} = \begin{pmatrix} 1 & \mu_{ij} & \mu_{ik} \\ 0 & 1 & \mu_{jk} \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.67)$$

On the other side of the wall we have two half-plane fans of type ik , depicted in Figure 2.10, leading us to write

$$\hat{\mu} = \begin{pmatrix} 1 & \mu'_{ij} & \mu'_{ik} + \mu'_{ij} \mu'_{jk} \\ 0 & 1 & \mu'_{jk} \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.68)$$

The two expressions for $\hat{\mu}$ are equal if and only if the wall-crossing formula holds.

More generally suppose that $\{l, m\}$ is any pair of vacua such that there is a fan

$$F_{lm} = \{l, i_1, \dots, i, k, \dots, i_n, m\} \quad (2.69)$$

in which $\{i, k\}$ appears as a subset of consecutive vacua. Then on the other side of the wall, for every such fan, the set of lm -fans gains an additional fan obtained by taking F_{lm} and inserting j in between i and k . Moreover these are the only additional fans we gain, assuming we cross no other marginal stability walls in the move. Thus we compare

$$\mu_{li_1} \dots \mu_{ik} \dots \mu_{i_n m} \quad (2.70)$$

with

$$\mu_{li_1} \dots \mu'_{ik} \dots \mu_{i_n m} + \mu_{li_1} \dots \mu'_{ij} \mu'_{jk} \dots \mu_{i_n m} \quad (2.71)$$

and the two are equal if and only if the wall-crossing formula holds. Therefore we conclude that the wall-crossing formula is equivalent to the invariance of the bilinear form $\hat{\mu}$ across a wall of marginal stability.

2.3 BPS Chain Complexes and ζ -instantons

2.3.1 BPS Chain Complexes

The chain complexes R_{ij} that categorify μ_{ij} can be formulated by using an infinite-dimensional version of Morse theory. Suppose that the symplectic form ω on X is exact and choose a Liouville form λ so that $\omega = d\lambda$. We consider the (family of) “Morse” functions

$$h_\zeta[\phi] = \int_{\mathbb{R}} \phi^*(\lambda) + \text{Im}(\zeta^{-1} W(\phi)) dx \quad (2.72)$$

acting on the space

$$\mathcal{X}_{ij} = \{\phi : \mathbb{R} \rightarrow X \mid \lim_{x \rightarrow -\infty} \phi(x) = \phi_i \quad \lim_{x \rightarrow \infty} \phi(x) = \phi_j\}. \quad (2.73)$$

Generators The critical points are the points where $\delta h_\zeta = 0$ which are solutions of the ζ -soliton equation

$$\frac{d\phi^I}{dx} = \frac{\zeta}{2} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}, \quad (2.74)$$

and so the critical point set is non-empty only for $\zeta = \zeta_{ji}$. The Morse function is actually not Morse because of the translational invariance of the soliton equation but we can mod out the solution space by this \mathbb{R} -action to obtain a (generically) finite set of critical points, in one-to-one correspondence with intersection points

$$L_i(\zeta_{ji} e^{-i\epsilon}) \cap R_j(\zeta_{ji} e^{i\epsilon}). \quad (2.75)$$

Thus we look to the pair

$$(\mathcal{X}_{ij}, h_{-\zeta_{ij}}) \quad (2.76)$$

and assign a \mathbb{Z} -module R_{ij} with one generator for each solution of the ζ_{ji} -soliton equation

$$R_{ij} = \bigoplus_{p \in L_i^- \cap R_j^+} \mathbb{Z} \langle \phi_{ij}^p \rangle. \quad (2.77)$$

Gradations Next we come to the subtle business of defining gradations on R_{ij} . The Fermion number, or homological degree of a generator in the Morse complex for a Morse function f as reviewed in [GMW, MS1] is given by

$$-\frac{1}{2} \sum_{\lambda \in \text{Spec Hess}(f(p))} \text{sign}(\lambda), \quad (2.78)$$

where p is the critical point of f whose degree we're computing. To assign a degree to a ζ -soliton we must therefore compute the second derivative $\delta^2 h_\zeta$. Equivalently we may linearize the ζ -soliton equation (2.74) which leads to

$$D_x^{(1,0)} \delta \phi^I = \frac{\zeta}{2} g^{I\bar{J}} D_{\bar{J}} \partial_{\bar{K}} \bar{W} \delta \bar{\phi}^{\bar{K}} \quad (2.79)$$

where

$$D_x^{(1,0)} \delta \phi^I = \frac{\partial}{\partial x} \delta \phi^I + \Gamma_{JK}^I \frac{\partial \phi^J}{\partial x} \delta \phi^K \quad (2.80)$$

is the pullback connection on $\phi^*(T^{(1,0)}X)$. By considering also the complex conjugate of (2.79), we can write the linearized soliton equation as

$$D_\zeta \delta \phi = 0 \quad (2.81)$$

where D_ζ is a Dirac type operator

$$D_\zeta : \Gamma(\phi^*(T^{(1,0)}X) \oplus \phi^*(T^{(0,1)}X)) \rightarrow \Gamma(\phi^*(T^{(1,0)}X) \oplus \phi^*(T^{(0,1)}X)). \quad (2.82)$$

Writing

$$\delta \phi \in \Gamma(\phi^*(T^{(1,0)}X) \oplus \phi^*(T^{(0,1)}X)) \quad (2.83)$$

as a column vector

$$\delta \phi = \begin{pmatrix} \delta \phi^I \\ \delta \bar{\phi}^{\bar{I}} \end{pmatrix} \quad (2.84)$$

the operator D_ζ reads¹¹

$$D_\zeta = \begin{pmatrix} \delta_J^I \partial_x + \Gamma_{JK}^I \partial_x \phi^K & 0 \\ 0 & \delta_{\bar{J}}^{\bar{I}} \partial_x + \Gamma_{\bar{J}\bar{K}}^{\bar{I}} \partial_x \bar{\phi}^{\bar{K}} \end{pmatrix} - \begin{pmatrix} 0 & \frac{\zeta}{2} g^{I\bar{K}} D_{\bar{K}} \partial_{\bar{J}} \bar{W} \\ \frac{\zeta^{-1}}{2} g^{\bar{I}K} D_K \partial_J W & 0 \end{pmatrix}. \quad (2.85)$$

The operator D_ζ is expressed a little more compactly by identifying

$$\phi^*(T^{(1,0)}X) \oplus \phi^*(T^{(0,1)}X) \cong \phi^*(TX), \quad (2.86)$$

¹¹Note that the operator (2.85) differs from that given in equation 12.6 of [GMW], v1. The authors of [GMW] forgot to include covariant derivatives.

where TX denotes the complexified tangent bundle. Choosing real coordinates indexed by $a = 1, \dots, \dim_{\mathbb{R}}(X)$, we can write

$$D_{\zeta} = \delta_b^a D_x - g^{ac} D_b \partial_c \operatorname{Re}(\zeta^{-1} W), \quad (2.87)$$

where

$$D_x \delta \phi^a = \partial_x \delta \phi^a + \Gamma_{bc}^a \partial_x \phi^b \delta \phi^c \quad (2.88)$$

is now the pullback connection on $\phi^*(TX)$. The Fermion number of an ij -soliton ϕ should thus be given by a regularized version of (2.78):

$$F(\phi) = -\lim_{\epsilon \rightarrow 0} \frac{1}{2} \sum_{\lambda \in \text{Eigenvalues}(D_{\zeta_{ji}}(\phi))} \operatorname{sign}(\lambda) e^{-\epsilon|\lambda|} \quad (2.89)$$

$$= -\frac{1}{2} \eta(D_{\zeta_{ij}}(\phi)). \quad (2.90)$$

One wants chain complexes $R_{ij}^{(1)}, R_{ij}^{(2)}$ constructed from two different choices of Kähler metrics $g^{(1)}, g^{(2)}$ (namely by a different choice of D-terms) to be homotopy equivalent

$$R_{ij}^{(1)} \simeq R_{ij}^{(2)}. \quad (2.91)$$

A necessary condition for is this that if we continuously interpolate between the metrics $g^{(1)}$ and $g^{(2)}$ and evolve the soliton $\phi^{(1)}$ solving the ζ -soliton equation for $g^{(1)}$ to $\phi^{(2)}$ a soliton for $g^{(2)}$ then their Fermion degrees must match. However the variational formula for the η -invariant says that

$$\frac{1}{2} \eta(D(\phi^{(1)}, g^{(1)})) - \frac{1}{2} \eta(D(\phi^{(2)}, g^{(2)})) = 2 \int_{\mathbb{R} \times [0,1]} \tilde{\phi}^* \left(\frac{1}{2\pi} \operatorname{Tr} \mathcal{R} \right), \quad (2.92)$$

where

$$\tilde{\phi} : \mathbb{R} \times [0, 1] \rightarrow X \quad (2.93)$$

is a path in \mathcal{X}_{ij} interpolating between $\phi^{(1)}$ and $\phi^{(2)}$, and

$$\frac{1}{2\pi} \text{Tr } \mathcal{R} \quad (2.94)$$

is the Chern-Weil representative of $c_1(TX)$. This is nothing but a reminder that the LG model has an axial anomaly for arbitrary Kähler target. The axial anomaly is traditionally expressed as the statement that the right hand side of (2.92) measures the net violation of Fermion number. The factor of two comes from taking into account the individual violations of both left and right moving fermions. Thus gradations are unchanged under metric variations only if X is Calabi-Yau. Otherwise to ensure this property we must grade R_{ij} by a cyclic group \mathbb{Z}_N such that the image of $2c_1(X)$ in $H^2(X, \mathbb{Z}_N)$ vanishes.

Differential The differential d_{ij} is provided by counting (with signs) solutions of the ζ_{ji} -instanton equation

$$\partial_{\bar{s}} \phi^I = \frac{\zeta_{ji}}{2} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}} \quad (2.95)$$

interpolating between solitons of fermion number differing by a unit. Here $s = x + i\tau$, where τ is the Euclidean time. Thus we get well-defined chain complexes (R_{ij}, d_{ij}) from which we can construct $\mathcal{H}_{ij}^{\text{BPS}}$ by taking cohomology

$$\mathcal{H}_{ij}^{\text{BPS}} \cong H^\bullet(R_{ij}, d_{ij}). \quad (2.96)$$

A ζ -instanton which contributes to the differential d_{ij} in spacetime looks like Figure 2.1. Physically we expect the following properties.

Metric Dependence BPS chain complexes constructed from two different choices of Kähler metrics should be homotopy equivalent.

CPT Reversing the spatial coordinate, i.e the path $\phi_{ij}^p(-x)$ says that for every basis element ϕ_{ij}^p of R_{ij} we get an element ϕ_{ji}^p such that

$$\deg(\phi_{ji}^p) = 1 - \deg(\phi_{ij}^p). \quad (2.97)$$

The shift in degree by $+1$ is a technical consequence of factoring out the translational mode of the soliton. For more details on this point see the discussion in section 12.3 in [GMW]. In basis independent terms, CPT says that we have a degree -1 non-degenerate pairing

$$K_{ij} : R_{ij} \otimes R_{ji} \rightarrow \mathbb{Z}. \quad (2.98)$$

2.3.2 ζ -instantons and Interior Amplitudes

As alluded to in the introduction, a categorified wall-crossing formula will involve certain “off-diagonal” maps

$$M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik} \quad (2.99)$$

which allow construction of the correct differential. The construction of this map involves counting ζ -instantons with fan boundary conditions, which we now discuss.

We consider solutions of the ζ -instanton equation

$$\bar{\partial}_{\bar{s}} \phi^I = \frac{\zeta}{4} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}} \quad (2.100)$$

which look like a collection of “boosted solitons” at infinity. See [GMW] sections 14.1-14.2 and Appendix E for more details on such boundary conditions. Let

$$I = \{i_1, \dots, i_n\} \quad (2.101)$$

be a cyclic fan of vacua and

$$\phi = \{\phi_{i_1 i_2}, \dots, \phi_{i_n i_1}\} \quad (2.102)$$

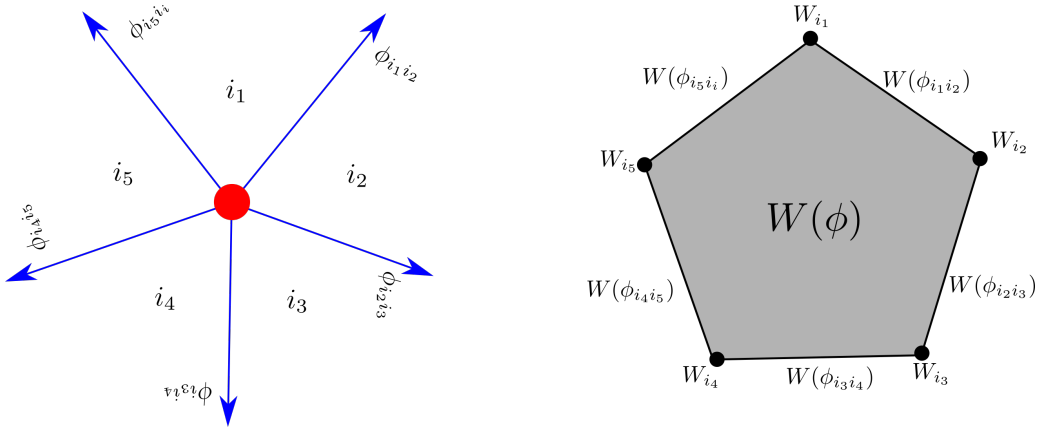


Figure 2.7: Boundary conditions for a ζ -instanton in the (x, τ) -plane on the left, and the image of such a ζ -instanton in the W -plane on the right.

be a fan of solitons. We want to consider ζ -instantons which support these particular solitons on the edges. I is a fan if and only if the critical values

$$W_I = \{W_{i_1}, \dots, W_{i_n}\} \quad (2.103)$$

are the clockwise ordered vertices of a convex polygon in the W -plane. Solutions of the ζ -instanton equation with fan boundary conditions are known as a domain-wall junctions and have been studied in [CHT, GT, INOS], and elsewhere. In particular, it was noted in [CHT], that just the way a ζ_{ij} -soliton maps to a line connecting W_i and W_j in the W -plane, a ζ -instanton maps to the interior of the convex polygon with W_I as vertices. See Figure 2.7 for an example with $n = 5$. This fact motivates the terminology BPS or gradient polygon for ϕ , as was introduced in [KKS].

Solutions of the ζ -instanton equation modulo translations with a fixed fan and fixed soliton collection ϕ supported on edges form a moduli space $\mathcal{M}_\zeta(\phi)$. Its dimension is given by forming the vector

$$e_\phi := \phi_{i_1 i_2} \otimes \cdots \otimes \phi_{i_n i_1} \quad (2.104)$$

in the cyclic tensor product

$$R_I = R_{i_1 i_2} \otimes \cdots \otimes R_{i_n i_1} \quad (2.105)$$

and considering its degree

$$F(\phi) := \deg(e_\phi). \quad (2.106)$$

The (virtual) dimension of these moduli spaces is [GMW]

$$\dim(\mathcal{M}_\zeta(\phi)) = F(\phi) - 2. \quad (2.107)$$

Moreover $\mathcal{M}_\zeta(\phi)$ can be oriented. In particular if $F(\phi) = 2$, we learn that the moduli space $\mathcal{M}_\zeta(\phi)$ is a collection of oriented points and thus we can get a well-defined signed count of ζ -instantons

$$N_\zeta(\phi) := \# \mathcal{M}_\zeta(\phi). \quad (2.108)$$

The integers $N(\phi)$ ¹² satisfy some miraculous identities . There is an identity corresponding to each cyclic fan.

For a cyclic fan of length two, $\{i, j\}$ we have

$$\sum_{\substack{\chi_{ij} \in L_i^- \cap R_j^+ \\ F(\phi_{ij}, \chi_{ji})=2 \\ F(\chi_{ij}, \psi_{ji})=2}} N(\phi_{ij}, \chi_{ji}) N(\chi_{ij}, \psi_{ji}) = 0. \quad (2.109)$$

This is nothing but the identity that the differential d_{ij} counting ζ -instantons between ij -solitons is nilpotent, which is a familiar fact from Morse theory. It involves the fact that the moduli space $\mathcal{M}(\phi_{ij}, \psi_{ji})$ such that $d(\phi_{ij}, \psi_{ji}) = 3$ has ends corresponding to broken flow lines gluing intermediate instantons.

¹²We can safely drop the ζ -subscript from the notation because the integers $N_\zeta(\phi)$ are ζ -independent

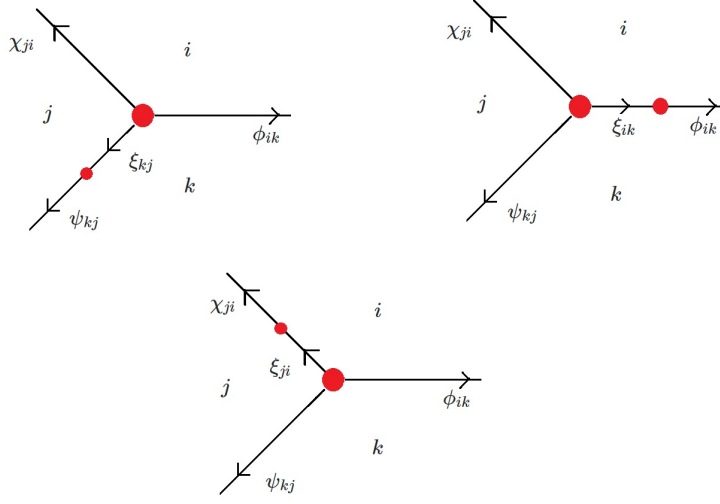


Figure 2.8: The various ends of $\mathcal{M}(\phi_{ik}, \psi_{kj}, \chi_{ji})$ where $F(\phi_{ik}, \psi_{kj}, \chi_{ji}) = 3$.

For $\{i, k, j\}$ a cyclic fan of vacua of length three, we have the identity

$$\begin{aligned}
& \sum_{\substack{\xi_{ij} \in L_i^- \cap R_j^+ \\ F(\phi_{ik}, \psi_{kj}, \xi_{ji})=2 \\ F(\xi_{ij}, \chi_{ji})=2}} N(\phi_{ik}, \psi_{kj}, \xi_{ji}) N(\xi_{ij}, \chi_{ji}) \\
& + \sum_{\substack{\xi_{jk} \in L_j^- \cap R_k^+ \\ F(\chi_{ji}, \phi_{ik}, \xi_{kj})=2 \\ F(\xi_{jk}, \psi_{kj})=2}} N(\chi_{ji}, \phi_{ik}, \xi_{kj}) N(\xi_{jk}, \psi_{kj}) \\
& + \sum_{\substack{\xi_{ik} \in L_i^- \cap R_k^+ \\ F(\xi_{ik}, \psi_{kj}, \chi_{ji})=2 \\ F(\phi_{ik}, \xi_{ki})=2}} N(\phi_{ik}, \xi_{ki}) N(\xi_{ik}, \psi_{kj}, \chi_{ji}) = 0.
\end{aligned} \tag{2.110}$$

The argument for this involves looking at the ends of the moduli space

$$\mathcal{M}(\phi_{ik}, \psi_{kj}, \chi_{ji}) \tag{2.111}$$

of a fan of solitons such that $F(\phi_{ik}, \psi_{kj}, \chi_{ji}) = 3$. There are three types of ends, where a rigid instanton of type $\{i, k\}$ is glued to a rigid instanton of type $\{i, k, j\}$, similarly for

$\{i, j\}$ and $\{j, k\}$. See Figure 2.8. Such “broken flows” give

$$\begin{aligned}
\partial \mathcal{M}(\phi_{ik}, \psi_{kj}, \chi_{ji}) &= \bigsqcup_{\substack{\xi_{ij} \in L_i^- \cap R_j^+ \\ F(\phi_{ik}, \psi_{kj}, \xi_{ji})=2 \\ F(\xi_{ij}, \chi_{ji})=2}} \mathcal{M}(\phi_{ik}, \psi_{kj}, \xi_{ji}) \times \mathcal{M}(\xi_{ij}, \chi_{ji}) \\
&\sqcup \bigsqcup_{\substack{\xi_{jk} \in L_j^- \cap R_k^+ \\ F(\chi_{ji}, \phi_{ik}, \xi_{kj})=2 \\ F(\xi_{jk}, \psi_{kj})=2}} \mathcal{M}(\chi_{ji}, \phi_{ik}, \xi_{kj}) \times \mathcal{M}(\xi_{jk}, \psi_{kj}) \\
&\sqcup \bigsqcup_{\substack{\xi_{ik} \in L_i^- \cap R_k^+ \\ F(\xi_{ik}, \psi_{kj}, \chi_{ji})=2 \\ F(\phi_{ik}, \xi_{ki})=2}} \mathcal{M}(\phi_{ik}, \xi_{ki}) \times \mathcal{M}(\xi_{ik}, \psi_{kj}, \chi_{ji}).
\end{aligned} \tag{2.112}$$

(2.110) then follows from

$$\# \partial \mathcal{M}(\phi_{ik}, \psi_{kj}, \chi_{ji}) = 0. \tag{2.113}$$

More generally, one expects that the moduli spaces $\mathcal{M}_\zeta(\phi)$ can be compactified, such that the compactified moduli space $\overline{\mathcal{M}}_\zeta(\phi)$ has strata labeled by web diagrams of the type in Figure 2.8.

Although the identities (2.109) and (2.110) are all we need for categorical wall-crossing, we should mention for completeness that there are more complicated identities involving fans of longer length which can be deduced from the web combinatorics of [GMW]. The summary is that all identities follow from a single L_∞ -Maurer-Cartan equation. Form the vector space¹³

$$R_c = \oplus_I R_I \tag{2.114}$$

$$= \oplus_{i \in \mathbb{V}} R_i \oplus_{i \neq j} (R_{ij} \otimes R_{ji}) \oplus \dots \tag{2.115}$$

¹³ $R_i \cong \mathbb{Z}$

corresponding to taking all possible cyclic tensor products. R_c has the structure of an L_∞ -algebra. Namely there are maps

$$\rho(\mathbf{t}) : S_+ R_c \rightarrow R_c, \quad (2.116)$$

where $S_+ R_c$ denotes (the positive part of) the symmetric algebra, satisfying L_∞ -axioms. $\rho(\mathbf{t})$ is defined through **taut webs** as in [GMW]. Define

$$\beta_I := \sum_{\substack{\phi \text{ gradient polygons for } I \\ F(\phi)=2}} N(\phi) e_\phi, \quad (2.117)$$

and let

$$\beta := \sum_I \beta_I \in R_c. \quad (2.118)$$

One of the main results of [GMW] is that analysis of various moduli spaces leads one to conclude that β is a Maurer-Cartan element for the L_∞ -structure. Namely it satisfies the L_∞ Maurer-Cartan equation

$$\rho(\mathbf{t})(e^\beta) = 0. \quad (2.119)$$

β was called the **interior amplitude** in [GMW]. The identities (2.109), (2.110) are some simple equations that come from unpacking the L_∞ Maurer-Cartan equation.

Remark In general interior amplitudes will have components associated to arbitrary fans

$$\beta_{i_1 i_2 \dots i_n} \in R_{i_1 i_2} \otimes R_{i_2 i_3} \otimes \dots \otimes R_{i_n i_1}. \quad (2.120)$$

However, only the trivalent components associated to the “wall-crossing triangle”; β_{ikj} on one side and β'_{ijk} on the other, enter the discussion in categorical wall-crossing.

2.3.3 Homotopy Equivalence of BPS Data

We have discussed the construction of the BPS chain complexes

$$\{(R_{ij}, d_{ij})\}, \quad (2.121)$$

the contraction maps

$$\{K_{ij}\} \quad (2.122)$$

and the important vector encoding counts of rigid ζ -instantons

$$\beta \in R_c. \quad (2.123)$$

We have noted however that the BPS complexes by themselves are not physical observables, only their homotopy equivalence class is. It is natural to try to extend the notion of homotopy equivalence from the BPS complexes, to the full categorical BPS data, namely to introduce a natural notion of homotopy equivalence for the contraction pairings and interior amplitudes. We briefly formulate such a notion in this sub-section.

Suppose we are given another collection of BPS data $(\{S_{ij}\}, \{L_{ij}\}, \gamma)$ where S_{ij} denote complexes L_{ij} contraction maps, and γ is now a Maurer-Cartan element of the L_∞ -algebra S_c , constructed from S_{ij} and L_{ij} . We say that the BPS data

$$(\{R_{ij}\}, \{K_{ij}\}, \beta) \text{ and } (\{S_{ij}\}, \{L_{ij}\}, \gamma) \quad (2.124)$$

are **homotopy equivalent** if there are homotopy equivalences of chain complexes

$$f_{ij} : R_{ij} \rightarrow S_{ij} \quad (2.125)$$

that fit into a collection of maps

$$f_n : R_c^{\otimes n} \rightarrow S_c \quad (2.126)$$

with f_1 being induced canonically from the collection $\{f_{ij}\}$ that together define an L_∞ -equivalence from R_c to S_c . The maps $\{f_{ij}\}$ and the L_∞ -morphism $\{f_n\}$ must be such that the diagram

$$\begin{array}{ccc}
 R_{ij} \otimes R_{ji} & \xrightarrow{f_{ij} \otimes f_{ji}} & S_{ij} \otimes S_{ji} \\
 & \searrow K_{ij} & \downarrow L_{ij} \\
 & & \mathbb{Z}
 \end{array} \tag{2.127}$$

commutes up to homotopy, and the Maurer-Cartan element transports naturally:

$$f(e^\beta) \sim \gamma, \tag{2.128}$$

where \sim denotes gauge equivalence of Maurer-Cartan elements, defined in Appendix B.1.

The general philosophy of this thesis is that we should only consider homotopy equivalence classes of the categorical BPS data. For example a D-term variation will only result in homotopy equivalent BPS data. The equivalence in this section can be viewed as a relaxation of the notion of strict isomorphism of categorical BPS data as defined in [GMW] section 4.1.1.

2.4 Statement of Categorical Wall-Crossing

Notation Given an element $r_{ik} \otimes r_{kj} \otimes r_{ji} \in R_{ik} \otimes R_{kj} \otimes R_{ji}$ we can define

$$M[r_{ik} \otimes r_{kj} \otimes r_{ji}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik} \tag{2.129}$$

by using the contraction maps

$$M[r_{ik} \otimes r_{kj} \otimes r_{ji}](r'_{ij} \otimes r'_{jk}) = K_{ji}(r_{ji}, r'_{ij}) K_{kj}(r_{kj}, r'_{jk}) r_{ik}. \tag{2.130}$$

Similarly we define

$$M'[r_{ik} \otimes r_{kj} \otimes r_{ji}] : R_{ki} \rightarrow R_{kj} \otimes R_{ji} \quad (2.131)$$

by contracting the ik factor using K_{ik} , and using the Koszul sign rule. Finally the natural product rule differential on a tensor product chain complex of the form as $R_{ij} \otimes R_{jk}$ is denoted as d_{ijk} :

$$d_{ijk} = d_{ij} \otimes 1 \pm 1 \otimes d_{jk}. \quad (2.132)$$

When we write \pm it means we are not being precise about the exact sign.

Marginal Stability Wall Recall an ijk wall of marginal stability is the locus where

$$\text{Im}(Z_{ij}\overline{Z}_{jk}) = 0. \quad (2.133)$$

See Figure 2.9.

Main Statement Let

$$(R_{ij}, R_{jk}, R_{ik}, \beta_{ikj}) \quad (2.134)$$

be the chain complexes and interior amplitude component in a region where

$$\text{Im}(Z_{ij}\overline{Z}_{jk}) < 0, \quad (2.135)$$

and

$$(R'_{ij}, R'_{jk}, R'_{ik}, \beta'_{ikj}) \quad (2.136)$$

be the chain complexes and interior amplitude component in a region where

$$\text{Im}(Z'_{ij}\overline{Z}'_{jk}) > 0. \quad (2.137)$$

Note that β_{ikj} defines a chain¹⁴ map

$$M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}, \quad (2.138)$$

and β'_{ijk} defines a chain map

$$M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk}. \quad (2.139)$$

The categorical wall-crossing formula states that

$$R'_{ij} \simeq R_{ij}, \quad (2.140)$$

$$R'_{jk} \simeq R_{jk}, \quad (2.141)$$

$$R'_{ik} \simeq \text{Cone}(M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}). \quad (2.142)$$

Furthermore, letting (P, Q) be the chain maps that implement the homotopy equivalence between the primed and unprimed sides, it states that the diagrams

$$\begin{array}{ccc} R'_{ik}[1] & \xrightarrow{M'[\beta'_{ijk}]} & R'_{ij} \otimes R'_{jk} \\ P \downarrow & & \downarrow P \otimes P \\ \text{Cone}(M[\beta_{ikj}])[1] & \xrightarrow{\pi} & R_{ij} \otimes R_{jk} \end{array} \quad (2.143)$$

and

$$\begin{array}{ccc} R'_{ik}[1] & \xrightarrow{M'[\beta'_{ijk}]} & R'_{ij} \otimes R'_{jk} \\ Q \uparrow & & \uparrow Q \otimes Q \\ \text{Cone}(M[\beta_{ikj}])[1] & \xrightarrow{\pi} & R_{ij} \otimes R_{jk} \end{array} \quad (2.144)$$

commute up to homotopy.

¹⁴This follows from β being an interior amplitude, or equivalently, identity (2.110). The taut webs involved in this identity are the ones in Figure 2.8.

Equivalently,

$$R_{ij} \simeq R'_{ij}, \quad (2.145)$$

$$R_{jk} \simeq R'_{jk}, \quad (2.146)$$

$$R_{ik} \simeq \text{Cone}(M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk}), \quad (2.147)$$

and letting (S, T) be the chain maps implementing homotopy equivalence between the two sides, the diagrams

$$\begin{array}{ccc} R_{ij} \otimes R_{jk} & \xrightarrow{M[\beta_{ikj}]} & R_{ik}, \\ T \otimes T \downarrow & & \downarrow T \\ R'_{ij} \otimes R'_{jk} & \xrightarrow{i} & \text{Cone}(M'[\beta'_{ijk}]) \end{array} \quad (2.148)$$

and

$$\begin{array}{ccc} R_{ij} \otimes R_{jk} & \xrightarrow{M[\beta_{ikj}]} & R_{ik}, \\ S \otimes S \uparrow & & \uparrow S \\ R'_{ij} \otimes R'_{jk} & \xrightarrow{i} & \text{Cone}(M'[\beta'_{ijk}]) \end{array} \quad (2.149)$$

commute up to homotopy.

These formulas are also sufficient to relate the contraction maps. Given chain complexes

$$(R_{ij}, R_{jk}, R_{ik}, \beta_{ikj}) \quad (2.150)$$

such that

$$\text{Im}(Z_{ij} \overline{Z}_{jk}) < 0, \quad (2.151)$$

the dual complexes (R_{kj}, R_{ji}, R_{ki}) will be a triple such that

$$\text{Im}(Z_{kj} \overline{Z}_{ji}) > 0. \quad (2.152)$$

Therefore the formulas for going from $\text{Im}(\cdots) > 0$ to $\text{Im}(\cdots) < 0$ imply that

$$R'_{kj} \simeq R_{kj}, \quad (2.153)$$

$$R'_{ji} \simeq R_{ji}, \quad (2.154)$$

$$R'_{ki} \simeq \text{Cone}(M'[\beta_{ikj}] : R_{ki}[1] \rightarrow R_{kj} \otimes R_{ji}). \quad (2.155)$$

Note that there is a canonical degree -1 map

$$L : \text{Cone}(M[\beta_{ikj}]) \otimes \text{Cone}(M'[\beta_{ijk}]) \rightarrow \mathbb{Z} \quad (2.156)$$

given by

$$L = \begin{pmatrix} 0 & K_{ik} \\ K_{ij} \otimes K_{jk} & 0 \end{pmatrix}. \quad (2.157)$$

Denote the chain maps implementing the homotopy equivalence as \tilde{P}, \tilde{Q} . With this, the final part of categorical wall-crossing also determines the homotopy class of the contraction maps, by stating that the diagrams

$$\begin{array}{ccc} R_{ij} \otimes R_{ji} & \xrightarrow{K_{ij}} & \mathbb{Z} \\ \uparrow P \otimes \tilde{P} & \nearrow K'_{ij} & \\ R'_{ij} \otimes R'_{ji} & & \end{array} \quad (2.158)$$

$$\begin{array}{ccc} R_{jk} \otimes R_{kj} & \xrightarrow{K_{jk}} & \mathbb{Z} \\ \uparrow P \otimes \tilde{P} & \nearrow K'_{jk} & \\ R'_{jk} \otimes R'_{kj} & & \end{array} \quad (2.159)$$

$$\begin{array}{ccc}
C(M) \otimes C(M') & \xrightarrow{L} & \mathbb{Z} \\
\uparrow P \otimes \tilde{P} & \nearrow K'_{ik} & \\
R'_{ik} \otimes R'_{ki} & &
\end{array} \tag{2.160}$$

commute up to homotopy. In the above we have abbreviated $\text{Cone}(M[\beta_{ikj}])$ and $\text{Cone}(M'[\beta'_{ijk}])$ as $C(M)$ and $C(M')$ respectively. There will be similar diagrams with (Q, \tilde{Q}) .

Canonical Representatives In practice given the chain complexes on one side, one wants to work with specific representatives within the homotopy equivalence class of chain complexes (and chain maps) for the other. There is a canonical choice for this. Suppose we treat the primed side as unknown. Then the canonical representatives for the primed complexes are

$$R'_{ij} = R_{ij}, \tag{2.161}$$

$$R'_{jk} = R_{jk}, \tag{2.162}$$

$$R'_{ik} = \text{Cone}(M[\beta_{ikj}]). \tag{2.163}$$

By letting P, Q to be identity maps, we can then make the diagrams (2.143), (2.144), strictly commute by letting

$$M'[\beta'_{ijk}] = \pi, \tag{2.164}$$

which is equivalent to saying that

$$\beta'_{ijk} = K_{ij}^{-1} K_{jk}^{-1}. \tag{2.165}$$

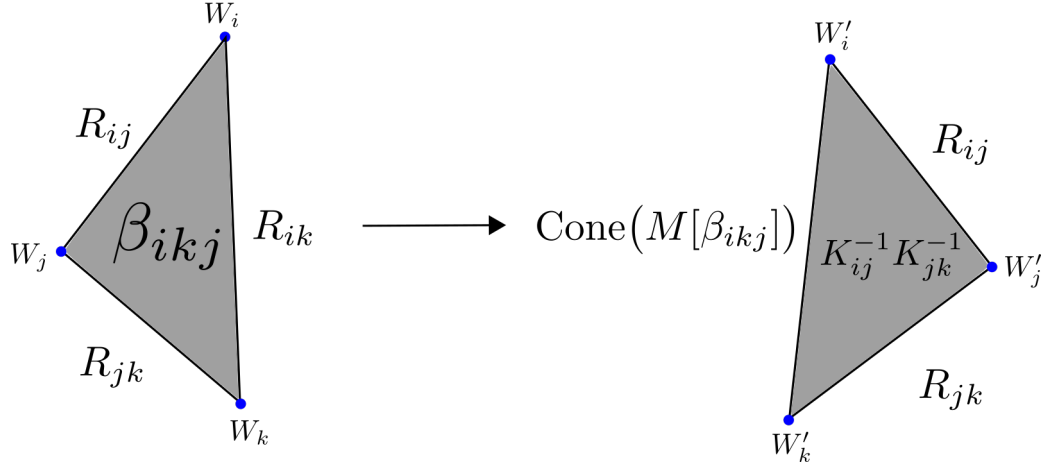


Figure 2.9: Categorical wall-crossing summarized in the W -plane.

The canonical representatives for the dual complexes are

$$R'_{kj} = R_{kj}, \quad (2.166)$$

$$R'_{ji} = R_{ji}, \quad (2.167)$$

$$R'_{ki} = \text{Cone}(M'[\beta_{ikj}]) \quad (2.168)$$

and one can then set the contraction maps to be

$$K'_{ij} = K_{ij}, \quad (2.169)$$

$$K'_{jk} = K_{jk}, \quad (2.170)$$

$$K'_{ik} = \begin{pmatrix} 0 & K_{ik} \\ K_{ij} \otimes K_{jk} & 0 \end{pmatrix}. \quad (2.171)$$

Figure 2.9 summarizes the categorical wall-crossing formula for going from a point in parameter space with $\text{Im}(Z_{ij}\overline{Z}_{jk}) < 0$ to a point where $\text{Im}(Z_{ij}\overline{Z}_{jk}) > 0$ from the perspective of the W -plane. The formulas and the figure summarizing the specific representatives in the inverse move would look similar. These straightforward details are left for the reader.

Remark: Consistency Check A consistency check our formulas must pass is whether jumping from the negative side of the wall of marginal stability where $\text{Im}(Z_{ij}\overline{Z}_{jk}) < 0$ to the positive side where $\text{Im}(Z_{ij}\overline{Z}_{jk}) > 0$ and then jumping back to the negative side is equivalent to doing nothing. We work with the canonical representatives. Starting from the complex R_{ik} the wall-crossing formula says that

$$R'_{ik} = \text{Cone}(M[\beta_{ijk}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}), \quad (2.172)$$

and

$$\beta'_{ijk} = K_{ij}^{-1} K_{jk}^{-1}. \quad (2.173)$$

Jumping back to the right side, gives us

$$R''_{ik} = \text{Cone}(M'[K_{ij}^{-1} K_{jk}^{-1}] : \text{Cone}(M[\beta_{ikj}])[1] \rightarrow R_{ij} \otimes R_{jk}). \quad (2.174)$$

But

$$M'[\beta'_{ijk}] = \pi \quad (2.175)$$

and therefore we have

$$R''_{ik} = \text{Cone}(\pi : \text{Cone}(M[\beta] : R_{ij} \otimes R_{jk} \rightarrow R_{ik})[1] \rightarrow R_{ij} \otimes R_{jk}) \quad (2.176)$$

$$= \text{Cyl}(M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}) \quad (2.177)$$

$$\simeq R_{ik}. \quad (2.178)$$

The cylinder construction of homological algebra, used above is described in Appendix A.1. Therefore we end up with a complex canonically homotopy equivalent to the original complex. A similar check can be performed for β''_{ikj} . One shows that the diagram

$$\begin{array}{ccc} R_{ij} \otimes R_{jk} & \xrightarrow{M[\beta_{ikj}]} & R_{ik} \\ & \searrow M[\beta''_{ikj}] & \downarrow i \\ & & \text{Cyl}(M[\beta_{ikj}]) \end{array} \quad (2.179)$$

commutes up to homotopy. This shows clearly the need to work at the level of homotopy equivalence.

In the next two sections we show how these conditions word-for-word are the homotopy equivalence of A_∞ categories constructed at a point where $\text{Im}(Z_{ij}\overline{Z}_{jk}) > 0$ compared to a point where $\text{Im}(Z_{ij}\overline{Z}_{jk}) < 0$.

2.5 ζ -instantons and Brane Categories

2.5.1 Bare Thimble Category

While the chain complex R_{ij} categorifies μ_{ij} , categorification of $\hat{\mu}_{ij}$ leads to more interesting structure. The correct viewpoint will be that B must be upgraded to a category, and $\hat{\mu}_{ij}$ will be categorified to vector spaces of morphisms.

The construction of the “bare” thimble category \hat{R}^{bare} proceeds as follows.

Objects The objects are an ordered collection of thimbles

$$\mathfrak{T}_1, \dots, \mathfrak{T}_n, \tag{2.180}$$

one for each critical point $i \in \text{Crit}(W)$. They are ordered by $\text{Im}(-W)$ so that $i > j$ if $\text{Im}(W_i) < \text{Im}(W_j)$.

Morphisms The morphisms are given as follows. In order to define ¹⁵

$$\hat{R}_{ij} := \text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j) \tag{2.181}$$

¹⁵ $\text{Hop}(A, B) := \text{Hom}^{\text{opp}}(A, B) = \text{Hom}(B, A)$

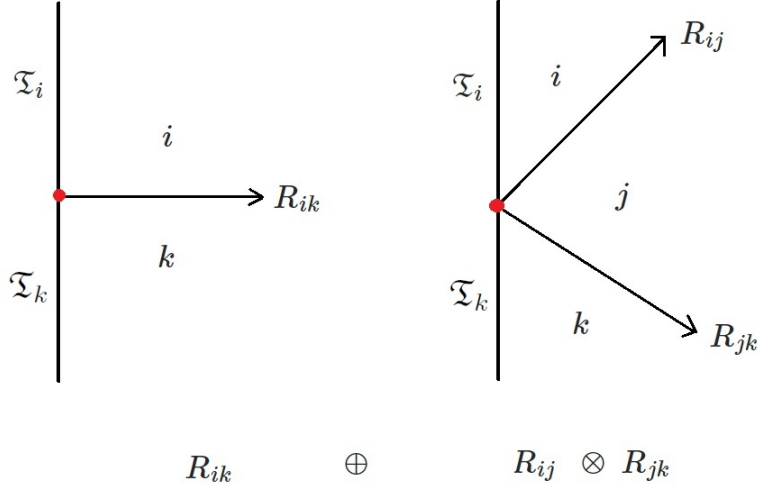


Figure 2.10: Contribution of some half-plane fans to $\text{Hop}(\Sigma_i, \Sigma_k)$

we look at all half-plane fans with “top” vacuum i and “bottom” vacuum j . To an edge separating i and j assign the vector space R_{ij} and take the (ordered) tensor product along each edge. Thus to each half-plane fan F_{ij} of this type we assign a vector space $R_{F_{ij}}$. The morphism space is then defined by taking direct sums over all F_{ij} half-plane fans

$$\widehat{R}_{ij} = \bigoplus_{F_{ij}} R_{F_{ij}}. \quad (2.182)$$

See Figure 2.10 for an example of a morphism space where two fans contribute. Note that

$$\widehat{R}_{ii} = \text{Hop}(\Sigma_i, \Sigma_i) = \mathbb{Z}. \quad (2.183)$$

If there are no half-plane fans then

$$\widehat{R}_{ij} = 0, \quad (2.184)$$

so that the objects $\{\Sigma_1, \dots, \Sigma_n\}$ are an exceptional collection; the matrix of morphism spaces \widehat{R}_{ij} is an upper-triangular matrix with \mathbb{Z} on the diagonal.

Compositions An associative composition law

$$m_{ijk} : \widehat{R}_{ij} \otimes \widehat{R}_{jk} \rightarrow \widehat{R}_{ik} \quad (2.185)$$

is given simply by looking at whether F_{jk} can be placed below F_{ij} to form a fan F_{ik} . If so, we take the tensor product of the vectors in $R_{F_{ij}}$ and $R_{F_{jk}}$ to get a vector in $R_{F_{ik}}$. If not, we set it equal to zero.

Differentials Finally the differential \widehat{d}_{ij} on

$$\widehat{R}_{ij} = R_{ij} \oplus (R_{ik} \otimes R_{kj}) \oplus \dots \quad (2.186)$$

will be inherited from the differentials on the complexes in the obvious way

$$\widehat{d}_{ij} = d_{ij} \oplus (d_{ik} \otimes 1 + 1 \otimes d_{kj}) \oplus \dots \quad (2.187)$$

Remark The differential-graded algebra

$$\text{End}(\oplus_i \mathfrak{T}_i) = \bigoplus_{i,j} \widehat{R}_{ij} \quad (2.188)$$

in which the algebra multiplication is specified by the morphisms as defined above, as explained in Appendix B.1, carries the same information as the category \widehat{R} and so we often use the terms algebra and category interchangeably in what follows.

2.5.2 Interior Amplitudes and Deformations of \widehat{R}

While $\widehat{R}^{\text{bare}}$ indeed gives $\widehat{\mu}$ as its matrix of Euler characters, the cohomology space $H^\bullet(\widehat{R}_{ij}, \widehat{d}_{ij})$ is not very physically meaningful. In particular, it is not isomorphic to the space of boundary BPS local operators at a \mathfrak{T}_i - \mathfrak{T}_j brane junction, like we would want it to be. The reason for this is similar to the failure of our naive categorification: we have not taken into account all ζ -instantons. In particular these ζ -instantons will correct the differential (2.187) and the composition law (2.185) described in the previous section.

The precise way to take ζ -instantons into account again uses the interior amplitude β . Similar to how one can use taut webs with n vertices to define L_∞ -maps,

$$\rho(\mathbf{t}^{(n)}) : S^n R_c \rightarrow R_c, \quad (2.189)$$

we can use taut half-plane webs with p boundary vertices and q bulk vertices to define maps

$$\rho(\mathbf{t}_{\mathcal{H}}^{(p,q)}) : (\widehat{R})^{\otimes p} \otimes (R_c)^{\otimes q} \rightarrow \widehat{R} \quad (2.190)$$

which satisfy the LA_∞ -axioms [GMW] (these are also known as the axioms of an open-closed homotopy algebra, see [KS]). We now make use of the

Theorem Suppose (A, L) is an open-closed homotopy algebra with structure maps

$$m_{k,l} : A^{\otimes k} \otimes L^{\otimes l} \rightarrow A, \quad k \geq 1, l \geq 0 \quad (2.191)$$

and suppose $\gamma \in L$ is a Maurer-Cartan element for the L_∞ algebra L . Then the collection of maps

$$m_k[\gamma] : A^{\otimes k} \rightarrow A, \quad (2.192)$$

defined by

$$m_k[\gamma](-, \dots, -) := \sum_{l \geq 0} \frac{1}{l!} m_{k,l}(-, \dots, -, \gamma^{\otimes l}) \quad (2.193)$$

give a (new) A_∞ -structure on A .

Thus we use the ζ -instanton counting element β to deform the dg-category $\widehat{R}^{\text{bare}}$ to an A_∞ -category denoted by $\widehat{R}[X, W]$. The deformed category $\widehat{R}[X, W]$ is proposed as the physical brane category of the Landau-Ginzburg model associated to the pair (X, W) . In

particular, we correct the differential \widehat{d}_{ij} to $\widehat{d}_{ij}[\beta]$ via (2.193) with $k = 1$, so that the cohomology

$$H^\bullet(\widehat{R}_{ij}, \widehat{d}_{ij}[\beta]) \quad (2.194)$$

is isomorphic to the space of $\frac{1}{2}$ -boundary BPS local operators at a $(\mathfrak{T}_i, \mathfrak{T}_j)$ -brane junction. In addition $k = 2$ of (2.193) also modifies the bilinear composition (2.185). As a result of (2.193) higher operations

$$\{m_k[\beta]\}_{k>2} \quad (2.195)$$

are also introduced. Together these operations turn $\widehat{R}[X, W]$ into a genuine A_∞ -category.

2.6 Homotopy Equivalence of Brane Categories

The categorical wall-crossing constraint is formulated as follows.

Categorical Wall-Crossing Constraint Suppose W and W' are superpotentials on different sides of a wall of marginal stability. Then the β -deformed thimble categories on either side of the wall are homotopy equivalent

$$\widehat{R}[X, W] \simeq \widehat{R}'[X, W'] \quad (2.196)$$

as A_∞ -categories.

We now relate our categorical wall-crossing formulas with the categorical wall-crossing constraint. First we construct the left and right $\{i, j, k\}$ -subcategories. As an instructive first check, we verify that the canonical representatives indeed give homotopy equivalent categories. Finally we unpack the axioms for A_∞ equivalence and show how the general statement follows.

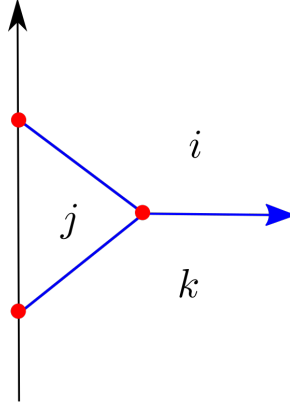


Figure 2.11: A taut half-plane web which contributes to a non-trivial bilinear product, by inserting the interior amplitude β in the bulk vertex.

2.6.1 Left Configuration

Let us first construct the $\{i, j, k\}$ sub-algebra of \widehat{R} for the configuration on the left of Figure 2.4. The soliton complexes are

$$(R_{ij}, d_{ij}), (R_{ik}, d_{ik}), (R_{jk}, d_{jk}). \quad (2.197)$$

Because there are no half-plane fans with more than one edge emanating from the boundary, the morphism spaces are simply

$$\widehat{R}_{ij} = R_{ij}, \quad (2.198)$$

$$\widehat{R}_{jk} = R_{jk}, \quad (2.199)$$

$$\widehat{R}_{ik} = R_{ik}. \quad (2.200)$$

In the undeformed algebra, there are no non-trivial multiplications.

Now consider the interior amplitude component

$$\beta_{ikj} \in R_{ik} \otimes R_{kj} \otimes R_{ji}, \quad (2.201)$$

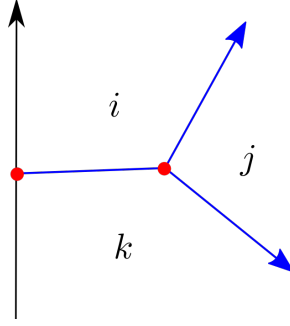


Figure 2.12: A taut half-plane web which contributes to an off-diagonal element in the differential by inserting the interior amplitude β' in the bulk vertex.

and consider the β -deformed algebra $\widehat{R}(X, W)$. In $\widehat{R}(X, W)$ we see that the taut half-plane web shown in Figure 2.11 now gives rise to a non-trivial morphism

$$M[\beta_{ikj}] : \widehat{R}_{ij} \otimes \widehat{R}_{jk} \rightarrow \widehat{R}_{ik} \quad (2.202)$$

given precisely by (2.130) applied to β_{ikj} . The differential \widehat{d} remains uncorrected.

The only A_∞ axiom to check is that

$$d_{ik}(M[\beta_{ikj}](r_{ij}, r_{jk})) = M[\beta_{ikj}](dr_{ij}, r_{jk}) \pm M[\beta_{ikj}](r_{ij}, dr_{jk}) \quad (2.203)$$

which follows from β_{ikj} being an interior amplitude component.

2.6.2 Right Configuration

Suppose the BPS chain complexes on the right configuration are

$$(R'_{ij}, d'_{ij}), (R'_{jk}, d'_{jk}), (R'_{ik}, d'_{ik}). \quad (2.204)$$

There are now two half-plane fans of type ik , shown in Figure 2.10 with one and two edges emanating from the boundary vertex respectively. This gives that the morphism spaces

are

$$\widehat{R}'_{ij} = R'_{ij}, \quad (2.205)$$

$$\widehat{R}'_{jk} = R'_{jk}, \quad (2.206)$$

$$\widehat{R}'_{ik} = R'_{ik} \oplus (R'_{ij} \otimes R'_{jk}). \quad (2.207)$$

Denote the interior amplitude on the right configuration to be

$$\beta'_{ijk} \in R'_{ij} \otimes R'_{jk} \otimes R'_{ki}. \quad (2.208)$$

Writing an element of \widehat{R}'_{ik} as a column vector $\begin{pmatrix} r'_{ik} \\ r'_{ij} r'_{jk} \end{pmatrix}$ the differential on \widehat{R}'_{ik} is of the form

$$\widehat{d}'_{ik}[\beta'] = \begin{pmatrix} d'_{ik} & 0 \\ M'[\beta'_{ijk}] & d'_{ijk} \end{pmatrix} \quad (2.209)$$

where

$$M'[\beta'_{ijk}] : R'_{ik} \rightarrow R'_{ij} \otimes R'_{jk} \quad (2.210)$$

is a degree +1 map defined by Figure 2.12. Nilpotence of $\widehat{d}'_{ik}[\beta']$ holds if

$$d'_{ijk} M'[\beta'_{ijk}] + M'[\beta'_{ijk}] d'_{ik} = 0, \quad (2.211)$$

therefore we may equivalently view $M'[\beta'_{ijk}]$ as a chain map

$$M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk} \quad (2.212)$$

and we can rewrite

$$\widehat{R}'_{ik} = \text{Cone}(M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk}). \quad (2.213)$$

The only non-trivial multiplication map is

$$i : \widehat{R}'_{ij} \otimes \widehat{R}'_{jk} \rightarrow \widehat{R}'_{ik} \quad (2.214)$$

given by inclusion. The A_∞ axiom says that i is a chain map with respect to d'_{ijk} , the product rule differential on $R'_{ij} \otimes R'_{jk}$ and $\widehat{d}'_{ik}[\beta'] = d_{M'}$ the mapping cone differential on \widehat{R}'_{ik} .

2.6.3 Canonical Representatives Satisfy Wall-Crossing Constraint

In this section we show that the canonical representatives (2.161), (2.162), (2.163), (2.165) satisfy the categorical wall-crossing constraint.

Claim: Suppose the primed complexes

$$(R'_{ij}, d'_{ij}), (R'_{jk}, d'_{jk}), (R'_{ik}, d'_{ik}) \quad (2.215)$$

and interior amplitude

$$\beta'_{ijk} \in R'_{ij} \otimes R'_{jk} \otimes R'_{kj} \quad (2.216)$$

are given as in (2.161), (2.162), (2.163) and (2.165). Then there is a functor

$$T : \widehat{R} \rightarrow \widehat{R}' \quad (2.217)$$

which defines a quasi-isomorphism of A_∞ -categories. We call T the wall-crossing functor.

Proof: By virtue of the categorical wall-crossing statement, we have the primed morphism spaces

$$\widehat{R}'_{ij} = R_{ij}, \quad (2.218)$$

$$\widehat{R}'_{jk} = R_{jk}, \quad (2.219)$$

$$\widehat{R}'_{ik} = R_{ik} \oplus (R_{ij} \otimes R_{jk})[-1] \oplus (R_{ij} \otimes R_{jk}). \quad (2.220)$$

The differentials deformed by the interior amplitude component β'_{ijk} are of the form

$$\hat{d}'_{ik}[\beta'] = \hat{d}_{ik}, \quad (2.221)$$

$$\hat{d}'_{jk}[\beta'] = \hat{d}_{jk}, \quad (2.222)$$

$$\hat{d}'_{ik}[\beta'] = \begin{pmatrix} d_{ik} & M[\beta_{ikj}] & 0 \\ 0 & d_{ijk}^{[-1]} & 0 \\ M'_1[\beta'_{ijk}] & M'_2[\beta'_{ijk}] & d_{ijk} \end{pmatrix}, \quad (2.223)$$

where $M[\beta_{ikj}]$ was defined as before and

$$M'_1[\beta'_{ijk}] : R_{ik} \rightarrow R_{ij} \otimes R_{jk}, \quad (2.224)$$

$$M'_2[\beta'_{ijk}] : (R_{ij} \otimes R_{jk})[-1] \rightarrow R_{ij} \otimes R_{jk} \quad (2.225)$$

are the different components of the maps defined by Figure 2.12 by inserting β' in the bulk vertex. The functor T can then be defined as follows. On objects we simply have the identity map. On morphism spaces we define

$$T_1 : \hat{R}_{ij} \rightarrow \hat{R}'_{ij}, \quad (2.226)$$

$$T_1 : \hat{R}_{kj} \rightarrow \hat{R}'_{kj} \quad (2.227)$$

as identity maps, whereas

$$T_1 : \hat{R}_{ik} \rightarrow \hat{R}'_{ik} \quad (2.228)$$

is defined as inclusion,

$$T_1(r_{ik}) = \begin{pmatrix} r_{ik} \\ 0 \\ 0 \end{pmatrix} \quad (2.229)$$

Furthermore

$$T_2 : \hat{R}_{ij} \otimes \hat{R}_{jk} \rightarrow \hat{R}'_{ik} \quad (2.230)$$

is again defined to be inclusion, but into the summand with shifted degree,

$$T_2(r_{ij}r_{jk}) = \begin{pmatrix} 0 \\ (r_{ij}r_{jk})^{[-1]} \\ 0 \end{pmatrix}. \quad (2.231)$$

Indeed (T_1, T_2) have degrees $(0, -1)$ respectively. The higher maps T_n are set to be zero for $n \geq 3$.

First we have to show the axioms of an A_∞ -morphism are satisfied. Here there are just two axioms to check. At $n = 1$ we have to check if T_1 is a chain map. The only non-trivial check is on the ik -component of T_1 and it follows that we have a chain map from the form of the differential (2.223). At $n = 2$ we must check

$$\begin{aligned} & T_1(m_2(r_{ij}, r_{jk})) - m'_2(T_1(r_{ij}), T_1(r_{jk})) \\ &= T_2(r_{ij}, dr_{jk}) \pm T_2(dr_{ij}, r_{jk}) \pm d'(T_2(r_{ij}, r_{jk})). \end{aligned} \quad (2.232)$$

This follows from the following simplification for the expression of $\widehat{d}'_{ik}[\beta']$. The explicit form of β'

$$\beta'_{ijk} = K_{ij}^{-1} K_{jk}^{-1}, \quad (2.233)$$

from (2.165), implies that the off-diagonal maps $M'_{1,2}[\beta'_{ijk}]$ are

$$M'_1[\beta'_{ijk}] = 0 \quad (2.234)$$

$$M'_2[\beta'_{ijk}] = \text{id}. \quad (2.235)$$

Note that the identity map M'_2 has degree $+1$ due to the degree shift on the domain. Thus we can rewrite the differential as

$$\widehat{d}'_{ik}[\beta'] = \begin{pmatrix} d_{ik} & M[\beta_{ikj}] & 0 \\ 0 & d_{ijk}^{[-1]} & 0 \\ 0 & \text{id} & d_{ijk} \end{pmatrix}. \quad (2.236)$$

Using this expression for d' on the right hand side, the axiom easily follows. Thus T defines an A_∞ -functor.

Finally, we must show that the wall-crossing functor T is a quasi-isomorphism. Again this is non-trivial only on the ik -component. The simplification of $\widehat{d}'[\beta']$ in fact allows us to relate this to the mapping cylinder construction: similar to (2.163) one can recognize \widehat{R}'_{ik} as the mapping cone of the projection map

$$\pi : R'_{ik}[1] = \text{Cone}(M[\beta_{ikj}])[1] \rightarrow R_{ij} \otimes R_{jk}. \quad (2.237)$$

In other words we can rewrite

$$\widehat{R}'_{ik} = \text{Cyl}(M[\beta_{ikj}]). \quad (2.238)$$

Applying the **Proposition** about mapping cylinders from Appendix A.1 to $f = M[\beta_{ikj}]$ yields that T is a quasi-isomorphism. \square

Remark Two A_∞ -algebras are homotopy equivalent if and only if they are quasi-isomorphic (this is a theorem of Prouté, [Pro]). We can thus say

$$\widehat{R}[X, W] \simeq \widehat{R}'[X, W'] \quad (2.239)$$

where \simeq is meant to be understood as homotopy equivalence.

2.6.4 Homotopy Equivalence \implies Categorical WCF

Finally we come to the main claim.

Claim The categorical wall-crossing constraint, namely the homotopy equivalence of A_∞ -categories

$$\widehat{R}[X, W] \simeq \widehat{R}[X, W'] \quad (2.240)$$

implies the categorical wall-crossing formula

$$R'_{ij} \simeq R_{ij}, \quad (2.241)$$

$$R'_{jk} \simeq R_{jk}, \quad (2.242)$$

$$R'_{ik} \simeq \text{Cone}(M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}). \quad (2.243)$$

Consider first the A_∞ morphism

$$T : \widehat{R}[X, W] \rightarrow \widehat{R}'[X, W']. \quad (2.244)$$

This in particular means that there are chain maps

$$T_1 : \widehat{R}_{ij} \rightarrow \widehat{R}'_{ij}, \quad (2.245)$$

$$T_1 : \widehat{R}_{jk} \rightarrow \widehat{R}'_{jk}, \quad (2.246)$$

$$T_1 : \widehat{R}_{ik} \rightarrow \widehat{R}'_{ik}. \quad (2.247)$$

We showed in 2.6.1, 2.6.2 that the hatted and un-hatted spaces coincide as chain complexes except for \widehat{R}'_{ik} which is of the form

$$\widehat{R}'_{ik} = \text{Cone}(M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R_{jk}). \quad (2.248)$$

Therefore we have chain maps

$$T_1 : R_{ij} \rightarrow R'_{ij}, \quad (2.249)$$

$$T_1 : R_{jk} \rightarrow R'_{jk}, \quad (2.250)$$

$$T_1 : R_{ik} \rightarrow \text{Cone}(M'[\beta'_{ikj}]). \quad (2.251)$$

In addition the A_∞ -morphism T provides a degree -1 map

$$T_2 : \widehat{R}_{ij} \otimes \widehat{R}_{jk} \rightarrow \widehat{R}'_{ik} = \text{Cone}(M'[\beta'_{ikj}]) \quad (2.252)$$

such that the second A_∞ -morphism axiom, (B.13), which in the present case reads

$$\begin{aligned} T_1(M[\beta_{ikj}](r_{ij}, r_{jk})) &\pm M'_2(T_1(r_{ij}), T_1(r_{jk})) \\ &= \widehat{d}_{ik}[\beta'] T_2(r_{ij}, r_{jk}) \pm T_2(dr_{ij}, r_{jk}) \pm T_2(r_{ij}, dr_{jk}), \end{aligned} \quad (2.253)$$

holds. We showed that M'_2 the bilinear multiplication

$$M'_2 : \widehat{R}'_{ij} \otimes \widehat{R}'_{jk} \rightarrow \text{Cone}(M'[\beta'_{ijk}]) \quad (2.254)$$

is given simply by the inclusion map i in 2.6.2. We therefore see that the conceptual way to interpret this axiom is that it is saying that the square

$$\begin{array}{ccc} R_{ij} \otimes R_{jk} & \xrightarrow{M[\beta_{ikj}]} & R_{ik} \\ T_1 \otimes T_1 \downarrow & & \downarrow T_1 \\ R'_{ij} \otimes R'_{jk} & \xrightarrow{i} & \text{Cone}(M'[\beta'_{ijk}]) \end{array} \quad (2.255)$$

commutes up to homotopy ¹⁶

$$i(T_1 \otimes T_1) \simeq T_1(M[\beta_{ikj}]) \quad (2.257)$$

with T_2 providing the chain homotopy. This condition is precisely (2.148).

Let the morphism in the other direction be

$$S : \widehat{R}'[X, W'] \rightarrow \widehat{R}[X, W] \quad (2.258)$$

which in particular says that we have chain maps

$$S_1 : R'_{ij} \rightarrow R_{ij}, \quad (2.259)$$

$$S_1 : R'_{jk} \rightarrow R_{jk}, \quad (2.260)$$

$$S_1 : \text{Cone}(M'[\beta'_{ijk}]) \rightarrow R_{ik} \quad (2.261)$$

¹⁶Note that the the compositions are chain maps

$$R_{ik} \otimes R_{jk} \xrightarrow[T_1 \circ M[\beta]]{i \circ T_1 \otimes T_1} \text{Cone}(M'[\beta'_{ijk}]), \quad (2.256)$$

that provide homotopy inverses to the T_1 's. S also provides us with a degree -1 map

$$S_2 : R'_{ij} \otimes R'_{ik} \rightarrow R_{ik} \quad (2.262)$$

that satisfies the second A_∞ axiom which in this case says that the square

$$\begin{array}{ccc} R'_{ij} \otimes R'_{jk} & \xrightarrow{i} & \text{Cone}(M'[\beta'_{ijk}]) \\ S_1 \otimes S_1 \downarrow & & \downarrow S_1 \\ R_{ij} \otimes R_{jk} & \xrightarrow{M[\beta_{ijk}]} & R_{ik} \end{array} \quad (2.263)$$

commutes up to homotopy, with S_2 providing the chain homotopy.

$$R'_{ij} \otimes R'_{jk} \xrightarrow{M[\beta] \circ S_1 \otimes S_1} R_{ik}. \quad (2.264)$$

In particular the existence of (S_1, T_1) implies that

$$R_{ik} \simeq R'_{ik}, \quad (2.265)$$

$$R_{jk} \simeq R'_{jk}, \quad (2.266)$$

$$R_{ik} \simeq \text{Cone}(M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk}), \quad (2.267)$$

which are precisely the homotopy equivalences (2.145), (2.146), (2.147) asserted in the categorical wall-crossing statement. The statement that these are homotopy equivalences follows from the definition of homotopy equivalence of A_∞ -algebras. Similarly the commutative square above is precisely (2.149).

Finally we use the Triangularity Lemma from Appendix A.1.

We found above that

$$R_{ik} \simeq \text{Cone}(M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk}) \quad (2.268)$$

so an application of the Triangularity Lemma implies that

$$R'_{ik} \simeq \text{Cone}(S_1 \circ i : R'_{ij} \otimes R'_{jk} \rightarrow R_{ik}). \quad (2.269)$$

Next we recall that the A_∞ -axiom for S_2 implies that

$$S_1 \circ i \simeq M[\beta_{ikj}] \circ (S_1 \otimes S_1) \quad (2.270)$$

and so their cones are homotopy equivalent. This gives

$$R'_{ik} \simeq \text{Cone}(M[\beta_{ikj}] \circ (S_1 \otimes S_1) : R'_{ik} \otimes R'_{jk} \rightarrow R_{ik}). \quad (2.271)$$

Finally since

$$S_1 : R'_{ij} \rightarrow R_{ij}, \quad (2.272)$$

$$S_1 : R'_{jk} \rightarrow R_{jk} \quad (2.273)$$

are individually homotopy equivalences, so is

$$S_1 \otimes S_1 : R'_{ij} \otimes R'_{jk} \rightarrow R_{ij} \otimes R_{jk}. \quad (2.274)$$

Therefore the latter part has a trivial mapping cone and can be “factored out” to conclude that

$$R'_{ik} \simeq \text{Cone}(M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}), \quad (2.275)$$

the result to be shown.

2.7 The Fermion Degree of a ζ -instanton

Recall that a ζ -instanton with boundary conditions labeled by the triple of solitons

$$\boldsymbol{\phi} = (\phi_{ik}, \phi_{kj}, \phi_{ji}) \quad (2.276)$$

that occupy the edges of an ikj wall-crossing triangle contributes to the differential in a categorical wall-crossing process if and only if

$$F(\phi_{ik} \otimes \phi_{kj} \otimes \phi_{ji}) = 2. \quad (2.277)$$

Therefore it is quite important to determine the degree of a given gradient polygon.

By definition the Fermion number is the index of the Dirac operator

$$\mathcal{D}_\zeta : \Gamma(\phi^*(TX)) \rightarrow \Gamma(\phi^*(TX)) \quad (2.278)$$

given by

$$\mathcal{D}_\zeta = \begin{pmatrix} \delta_J^I D_s^{(1,0)} & 0 \\ 0 & \delta_{\bar{J}}^{\bar{I}} D_s^{(0,1)} \end{pmatrix} - \begin{pmatrix} 0 & \frac{\zeta}{2} g^{I\bar{K}} D_{\bar{K}} \partial_{\bar{J}} \bar{W} \\ \frac{\zeta^{-1}}{2} g^{\bar{I}K} D_K \partial_J W & 0 \end{pmatrix}. \quad (2.279)$$

in the background of a ζ -instanton ϕ with ϕ boundary conditions. Clearly such an index will be difficult to compute if we work directly with \mathcal{D} ¹⁷. However a Maslov index type construction, described in [KKS], gives a more geometric prescription to obtain a well-defined integer $d(\phi)$ which is expected to agree with the index of \mathcal{D} up to an overall shift. It would be interesting to prove the equality of $d(\phi)$ with the index of \mathcal{D} , but this would take us too far afield in the present chapter. We proceed assuming the equality holds and use the geometric prescription in what follows. The Maslov index construction also assumes that X is equipped with a nowhere vanishing holomorphic volume form Ω .

Starting from a convex gradient polygon

$$\phi = (\phi_{i_0 i_1}, \dots, \phi_{i_n i_0}) \quad (2.280)$$

the Maslov index prescription gives us $d(\phi) \in \mathbb{Z}$ as follows. The main step consists of assigning to the gradient polygon ϕ a (homotopy class of a) loop in the Lagrangian Grassmannian of X ,

$$\text{Lag}(TX) = \{(p, E) | p \in X, E \text{ Lagrangian subspace of } T_p X\}, \quad (2.281)$$

constructed as follows.

¹⁷Moreover the question of whether \mathcal{D} is even Fredholm is a very delicate one, [CGGLPFZ]

First to each soliton ϕ_{ij} we associate an open path γ in $\text{Lag}(TX)$ simply by taking a point p along the soliton trajectory and assigning to it the Lagrangian subspace

$$T_p L_i(\zeta_{ij}) \subset T_p X \quad (2.282)$$

as the fiber. Let γ_k denote the open path assigned to $\phi_{i_{k-1}i_k}$ in this way. One notices that the endpoint of γ_k and the starting point of γ_{k+1} have the same base point, the k th critical point i_k , but the Lagrangians fibers differ. The endpoint of γ_k has fiber

$$\ell_k := T_{i_k} L_{i_{k-1}}(\zeta_{i_{k-1}i_k}) \quad (2.283)$$

whereas the starting point of γ_{k+1} has the fiber

$$\ell_{k+1} := T_{i_k} L_{i_k}(\zeta_{i_k i_{k+1}}). \quad (2.284)$$

ℓ_k, ℓ_{k+1} are Lagrangians living in the same ambient space $T_{i_k} X$. Between any two Lagrangian subspaces L_1, L_2 in a symplectic vector space V , there is a canonical homotopy class of paths κ_{L_1, L_2} in $\text{Lag}(V)$ that connects these points, known as the **symplectic bridge**¹⁸ connecting L_1 and L_2 . For instance if $\dim(V) = 2$, the Lagrangians are specified by points θ_1, θ_2 in $\mathbb{RP}^1 \cong S^1$ and $\kappa_{\theta_1, \theta_2}$ is the circular arc going in the counter-clockwise direction between these two angles. Therefore there is a well-defined way to connect the open path γ_k to γ_{k+1} . Going around the gradient polygon by gluing adjacent open paths via symplectic bridges, one obtains a loop in $\text{Lag}(TX)$.

Next we need to define a winding number of the loop γ . Let $\bar{\gamma}$ be the loop in X obtained by projecting γ to X . Thus, if $\bar{\gamma}(t) = p \in X$ then $\gamma(t) \subset T_p X$ is a maximal Lagrangian subspace. Let $2n$ denote the rank of TX considered as a real vector bundle over X . Then $\gamma(t)$ is a real vector space of dimension n . The n^{th} exterior product of this space is a real

¹⁸This is also known as the **canonical short path**, see for instance [Aur].

line associated to the point p . Now, recall that TX can also be considered to be a complex vector bundle of rank n . Therefore, the n^{th} exterior power of TX as a complex vector bundle is a complex line associated to p . Indeed, this is the fiber of the canonical bundle at p , denoted \mathcal{K}_p . Note that $\Lambda^n \gamma(t) \subset \mathcal{K}_p$ is a real line inside a complex line. Finally we use Ω to trivialize the canonical bundle and therefore get a real line $\ell_p \subset \mathbb{C}$. That is, to the loop $\gamma : S^1 \rightarrow \text{Lag}(TX)$ we associate a loop in $\text{Lag}(\mathbb{C}) = \mathbb{RP}^1 \cong S^1$. All-in-all we get a map

$$\psi(\phi) : S^1 \rightarrow \text{Lag}(\mathbb{C}) \cong S^1. \quad (2.285)$$

The integer $d(\phi)$ is defined to be the winding number of $\psi(\phi)$. The fermion number is then

$$F(\phi) = d(\phi) + 1. \quad (2.286)$$

We illustrate the computation of $d(\phi)$ in some examples.

2.7.1 Gradient Polygons in \mathbb{C}

Suppose our target space is the complex plane, and say for simplicity that the solitons trace out straight lines so that the gradient polygon $\phi = (\phi_{i_0 i_1}, \dots, \phi_{i_n i_0})$ traces out the boundary of an $(n+1)$ -gon. This boundary can be clockwise or counter-clockwise oriented and we analyze each case.

For the case of clockwise oriented boundaries, the tangent Lagrangian does not vary along the soliton. The symplectic bridge between $\phi_{i_{k-1} i_k}$ and $\phi_{i_k i_{k+1}}$ chooses to take the route that takes θ_k radians where θ_k is an internal angle of the polygon. Adding up these

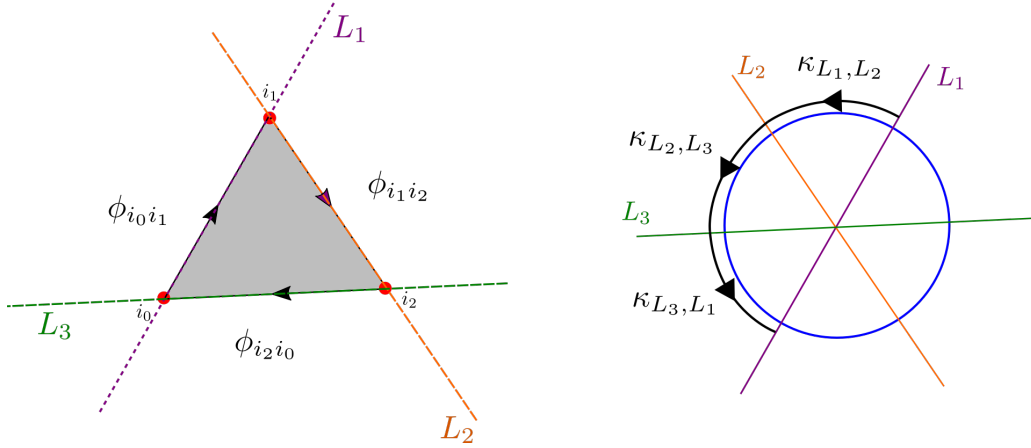


Figure 2.13: The left shows the gradient polygon $\phi = (\phi_{i_0 i_1}, \phi_{i_1 i_2}, \phi_{i_2 i_0})$ assumed to trace out straight lines on the complex plane. The dashed lines depict the Lagrangians tangent to these solitons. On the right we show the symplectic bridges $\kappa_{L_i, L_{i+1}}$ connecting these Lagrangians. The winding number of the total path in $\text{Lag}(\mathbb{C}) = S^1/\mathbb{Z}_2$ is $+1$, therefore $d(\phi) = 1$.

angles gives one a total winding number in S^1/\mathbb{Z}_2 of

$$d(\phi) = \frac{((n+1) - 2)\pi}{\pi} \quad (2.287)$$

$$= n - 1 \quad (2.288)$$

where in the first equality we divide by π (not 2π) because of the \mathbb{Z}_2 quotient. See Figure 2.13 for the case of $n = 2$.

For counter-clockwise oriented (convex) polygons¹⁹, the symplectic bridge chooses to connect adjacent Lagrangians via the route that takes $\pi - \theta_k$ radians. This gives one

$$d(\phi) = 2, \quad (2.289)$$

an index independent of n .

¹⁹We don't know any examples of $W(\phi)$ where this happens, although we don't see a reason why it cannot happen in principle.

That clockwise versus counterclockwise give such different answers might be a bit puzzling first, but its origin is clarified if one thinks about the analogous situation in Morse theory. Suppose that $\mathcal{M}(x_a, x_b)$ denotes the reduced moduli space of solutions of the gradient flow equation

$$\frac{d\phi^I}{dx} = g^{IJ} \frac{\partial h}{\partial \phi^J}, \quad (2.290)$$

between two critical points x_a, x_b of h with Morse indices μ_a, μ_b ²⁰. Then supposing $\mu_b > \mu_a$ we have

$$\dim \mathcal{M}(x_a, x_b) = \mu_b - \mu_a - 1. \quad (2.291)$$

On the other hand,

$$\dim \mathcal{M}(x_b, x_a) = 0. \quad (2.292)$$

$\mathcal{M}(x_b, x_a)$ is in fact empty, as a consequence of the ascending property of the gradient flow. Thus it should not be very surprising that the moduli space of ζ -instantons is not very well-behaved under orientation reversal of a cyclic fan.

2.7.2 Paths in \mathbb{C}^*

Let's now consider a gradient polygon of solitons in the punctured complex plane \mathbb{C}^* so that the total path winds around the origin. We choose the holomorphic volume form that trivializes $T\mathbb{C}^*$ to be

$$\Omega = \frac{dX}{X}. \quad (2.293)$$

One can show that a loop that winds around the origin, by virtue of this trivialization satisfies

$$d(\phi) = 0. \quad (2.294)$$

²⁰Not to be confused with the BPS index μ_{ij}

This will be useful for the trigonometric Landau-Ginzburg models.

2.7.3 Fermion Degrees for \mathbb{Z}_N -symmetric Models

We can use the observations above to determine (integral part of) the fermion degrees of solitons in at least two interesting \mathbb{Z}_N -symmetric family of models. These are

1. $W = \frac{1}{N+1}\phi^{N+1} - t\phi$, the deformed A_{N-1} model.
2. $W = \phi + \frac{1}{N-1}\phi^{-(N-1)}$, the \mathbb{Z}_N invariant “trigonometric” LG model.

Let’s analyze each one.

Deformed A_{N-1} -Model

The model of a single chiral superfield ϕ with superpotential

$$W = \frac{1}{N+1}\phi^{N+1} - t\phi \quad (2.295)$$

is a well-studied one. The critical points are

$$\phi_k = t^{\frac{1}{N}} e^{\frac{2\pi i k}{N}} \quad (2.296)$$

for $k = 0, 1, \dots, N-1$, with critical values

$$W_k = -\frac{N}{N+1} t^{\frac{N+1}{N}} e^{\frac{2\pi i k}{N}}. \quad (2.297)$$

It is well-known that there is a unique soliton ϕ_{ij} interpolating between each pair (ϕ_i, ϕ_j) of distinct critical points. Therefore

$$R_{ij} = \mathbb{Z}\langle\phi_{ij}\rangle. \quad (2.298)$$

The degree F_{ij} of ϕ_{ij} is of the form

$$F_{ij} = n_{ij} + f_{ij} \quad (2.299)$$

where n_{ij} is the integral part and f_{ij} is the fractional part (for which we have a universal formula). It remains to determine n_{ij} .

For this we use the constraint coming from the Maslov index: Let $\phi = (\phi_{i_0 i_1}, \dots, \phi_{i_k i_0})$ be a convex gradient polygon. Then

$$n_{i_0 i_1} + n_{i_1 i_2} + \dots + n_{i_k i_0} = d(\phi) + 1. \quad (2.300)$$

For the present model, we have that

$$(\phi_{i_0 i_1}, \phi_{i_1 i_2}, \dots, \phi_{i_k i_0}) \quad (2.301)$$

is a gradient polygon if and only if $i_0 > i_1 > i_2 \dots > i_n$ up to cyclic reordering. In the complex plane the gradient polygon traces out a clockwise oriented closed path with k -segments, and thus the computation in 2.7.1 implies

$$d(\phi_{i_0 i_1}, \phi_{i_1 i_2}, \dots, \phi_{i_k i_0}) = k - 2. \quad (2.302)$$

We thus get the constraint

$$n_{i_0 i_1} + n_{i_1 i_2} + \dots + n_{i_k i_0} = k - 1, \quad (2.303)$$

which is satisfied by a particularly simple solution:

$$n_{ij} = 1 \text{ for } i > j, \quad (2.304)$$

$$n_{ij} = 0 \text{ for } i < j. \quad (2.305)$$

By induction on k we see the solution is unique up to shifts

$$n_{ij} \rightarrow n_{ij} + n_i - n_j. \quad (2.306)$$

Therefore we conclude that

$$R_{ij} = \mathbb{Z}[1] \text{ for } i > j, \quad (2.307)$$

$$R_{ij} = \mathbb{Z} \text{ for } i < j. \quad (2.308)$$

Trigonometric Models

We can do a similar analysis for the \mathbb{Z}_N -symmetric trigonometric Landau-Ginzburg models.

These have target space \mathbb{C}^* and superpotential

$$W = \phi + \frac{1}{N-1} \phi^{-(N-1)}. \quad (2.309)$$

The critical points are again located at the roots of unity

$$\phi_k = e^{2\pi i k/N} \quad (2.310)$$

for $k = 0, 1, \dots, N-1$ and the critical values are

$$W_k = \frac{N}{N-1} e^{\frac{2\pi i k}{N}}. \quad (2.311)$$

The soliton spectrum of this model is also known (this model is example 3 in section 8.1 of [CV1]): There is a unique soliton between each nearest neighbor pair $(\phi_i, \phi_{i+1}), (\phi_i, \phi_{i-1})$ and none between the other pairs. Therefore the only gradient polygon ϕ with more than 2 solitons consists of the full N -gon

$$\phi = (\phi_{N-1, N-2}, \phi_{N-2, N-3}, \dots, \phi_{0, N-1}). \quad (2.312)$$

The paths these solitons trace out in \mathbb{C}^* consists of round arcs that together wind around the origin once in the clockwise direction. The computation of the Maslov index for paths in \mathbb{C}^* allows us to conclude that $d(\phi) = 0$ and therefore

$$n_{N-1, N-2} + n_{N-2, N-3} + \dots + n_{0, N-1} = 1. \quad (2.313)$$

We choose the solution

$$n_{i, i-1} = 0, \quad (2.314)$$

$$n_{i, i+1} = 1. \quad (2.315)$$

Thus the non-zero BPS chain complexes with this solution read

$$R_{i,i+1} = \mathbb{Z}\langle\phi_{i,i+1}\rangle \cong \mathbb{Z}, \quad (2.316)$$

$$R_{i,i-1} = \mathbb{Z}\langle\phi_{i,i-1}\rangle \cong \mathbb{Z}[1]. \quad (2.317)$$

2.8 Examples

Finally let's illustrate categorical wall-crossing in a few examples.

2.8.1 Quartic LG Model

Let's return to the quartic Landau-Ginzburg model that was alluded to in the introduction.

The target space is the complex plane \mathbb{C} and the superpotential is

$$W = \frac{1}{4}\phi^4 - \frac{t_1}{2}\phi^2 - t_2\phi. \quad (2.318)$$

Consider the point $(t_1, t_2) = (0, 1)$ where the critical points are

$$\phi_1 = e^{-\frac{2\pi i}{3}}, \quad \phi_2 = 1, \quad \phi_3 = e^{\frac{2\pi i}{3}} \quad (2.319)$$

with critical values

$$W_1 = -\frac{3}{4}e^{-\frac{2\pi i}{3}}, \quad W_2 = -\frac{3}{4}, \quad W_3 = -\frac{3}{4}e^{\frac{2\pi i}{3}}. \quad (2.320)$$

The BPS chain complexes consist of

$$R_{12} = \mathbb{Z}\langle\phi_{12}\rangle, \quad (2.321)$$

$$R_{13} = \mathbb{Z}\langle\phi_{13}\rangle, \quad (2.322)$$

$$R_{23} = \mathbb{Z}\langle\phi_{23}\rangle, \quad (2.323)$$

where ϕ_{ij} is the unique soliton interpolating between ϕ_i and ϕ_j . As discussed in 2.7.3, an assignment of degrees consistent with the Maslov index is that all three spaces are concentrated in degree zero.

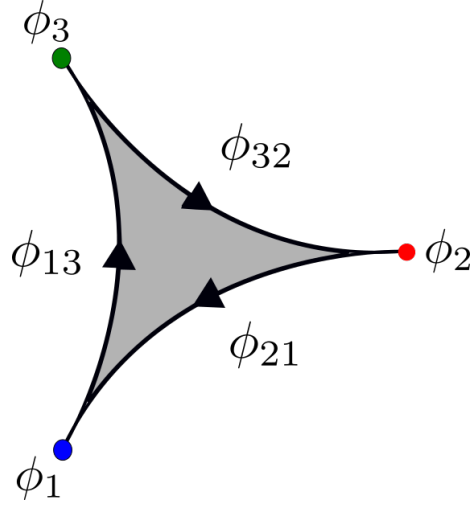


Figure 2.14: Image of the ζ -instanton with fan boundary conditions $\{1, 3, 2\}$ in the X -plane. It sweeps out a region bounded by the soliton paths.

Now we must count ζ -instantons. Consider the cyclic fan $\{1, 3, 2\}$ which has degree $+2$. It is argued in papers on domain wall junctions [GT] that there is indeed a solution with no reduced moduli with these trivalent fan boundary conditions. Therefore we have

$$N(\phi_{13}, \phi_{32}, \phi_{21}) = 1. \quad (2.324)$$

The image swept out by this instanton $\phi(\mathbb{C})$ is depicted in Figure 2.14.

Crossing the wall of marginal stability we consider $(t_1, t_2) = (1, \epsilon)$ where ϵ is some small number. Categorical wall-crossing says that the chain complex is

$$R'_{13} = \mathbb{Z}\langle(\phi_{12}\phi_{23})^{[-1]}\rangle \oplus \mathbb{Z}\langle\phi_{13}\rangle. \quad (2.325)$$

The differential reads

$$d'_{13}((\phi_{12}\phi_{23})^{[-1]}) = \phi_{13}, \quad (2.326)$$

$$d'_{13}(\phi_{13}) = 0, \quad (2.327)$$

by virtue of the ζ -instanton of Figure 2.14. Therefore the cohomology is trivial

$$H^\bullet(R'_{13}, d'_{13}) = 0. \quad (2.328)$$

Indeed this is the correct BPS Hilbert space on the other side of the wall.

2.8.2 Trigonometric LG Model

Next we consider the model with target space the complex cylinder \mathbb{C}^* with coordinate ϕ . The family of superpotentials we consider is

$$W = \phi + \lambda\phi^{-1} + \frac{1}{2}\phi^{-2}. \quad (2.329)$$

The model at $\lambda = 0$ is known in [CV1] as the Bullough-Dodd model and that's where we begin our analysis. Here we have the critical points

$$\phi_1 = e^{\frac{2\pi i}{3}}, \quad \phi_2 = 1, \quad \phi_3 = e^{-\frac{2\pi i}{3}} \quad (2.330)$$

with critical values $W_i = \frac{3}{2}X_i$. As discussed in 2.7.3, there is a single soliton between each pair of vacua and so the BPS chain complexes read

$$R_{12} = \mathbb{Z}\langle\phi_{12}\rangle, \quad (2.331)$$

$$R_{23} = \mathbb{Z}\langle\phi_{23}\rangle, \quad (2.332)$$

$$R_{13} = \mathbb{Z}\langle\phi_{12}\rangle. \quad (2.333)$$

As discussed in 2.7.2, consistent with the Maslov index is to choose these spaces to be concentrated in degree zero (the vacua have been relabeled compared to that section). Note that there's a crucial difference with the quartic Landau-Ginzburg model. The vector space associated to the cyclic fan $\{1, 2, 3\}$ is one-dimensional but now concentrated in degree +1. The interior amplitude must therefore be trivial

$$\beta = 0. \quad (2.334)$$

Therefore, there are no ζ -instantons.

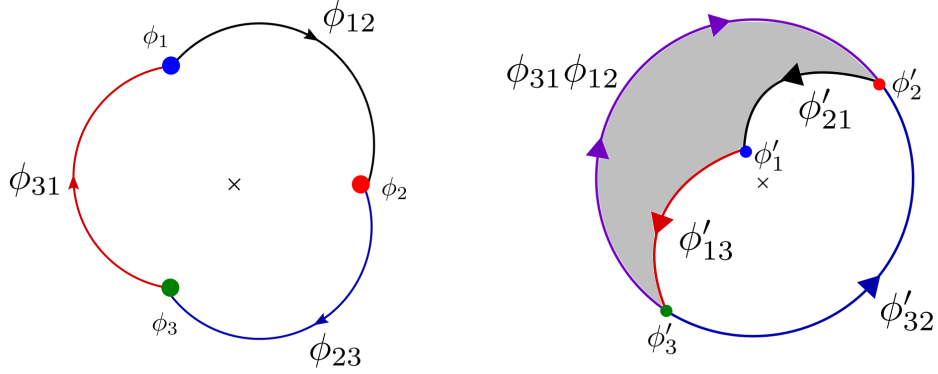


Figure 2.15: Left: the solitons in the $\lambda = 0$ model. There is one between each pair of vacua. On the right we cross the wall of marginal stability and go to $\lambda = 2i$. There are now two solitons in the 32 sector. We also gain a non-trivial ζ -instanton contributing to the interior amplitude.

The absence of ζ -instantons with trivalent boundary conditions may also be geometrically argued as follows. The cyclic fan of solitons sweep out a path that winds around the origin. Were a ζ -instanton to exist, its image would be a region bounded by this path. However, the latter region contains the singular point $\phi = 0$, which means that the ζ -instanton blows up at finite (x, τ) .

We now vary λ by taking it to be purely imaginary and increasing the magnitude from the \mathbb{Z}_3 symmetric point $\lambda = 0$. The wall of marginal stability is crossed at $\lambda \sim 1.5i$. W_1 passes through the line between W_2 and W_3 . Therefore R_{23} jumps. We have

$$R'_{23} = (R_{21} \otimes R_{13})[-1] \otimes R_{23}, \quad (2.335)$$

$$= \mathbb{Z}\langle(\phi_{21}\phi_{13})^{[-1]}\rangle \oplus \mathbb{Z}\langle\phi_{23}\rangle, \quad (2.336)$$

$$\cong \mathbb{Z}^2. \quad (2.337)$$

Trivial β implies that this is also the cohomology. We see that the 23 sector has gained a bound state of the 21 and 13 sectors.

These two states post wall-crossing have a simple interpretation. When λ is large the theory consists of the \mathbb{CP}^1 mirror along with a vacuum W_1 running away to infinity. The solitons between 2 and 3 are the solitons of this model.

Categorical wall-crossing also predicts the interior amplitude after wall-crossing. Formula (2.165) says that the interior amplitude should be

$$\beta'_{132} = (\phi_{31}\phi_{12}) \otimes \phi_{21} \otimes \phi_{13}. \quad (2.338)$$

Indeed the geometry of solitons allows the region between the new soliton that appears, $\phi_{31}\phi_{12}$, between 3 and 2 and the old solitons ϕ_{21} and ϕ_{13} to be filled up by a ζ -instanton. See Figure 2.15.

2.8.3 Elliptic LG Model

Let the target space be $T_\tau^2 \setminus \{0\}$ and

$$W = \wp(\phi, \tau). \quad (2.339)$$

We study the wall-crossing properties as we vary τ , the complex structure parameter of the torus²¹. The critical points are the familiar half-periods

$$\left\{ \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \right\} \bmod (\mathbb{Z} \oplus \mathbb{Z}\tau) \quad (2.340)$$

with critical values being the elliptic constants

$$\{e_1(\tau), e_2(\tau), e_3(\tau)\}. \quad (2.341)$$

²¹The moduli space of models is the stack $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$ where \mathbb{H} is the upper-half plane. The moduli space of models with marked vacua is $\mathbb{H}/\Gamma(2)$ where $\Gamma(2)$ is the level 2 principal congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$. See [BC] for further examples of this type.

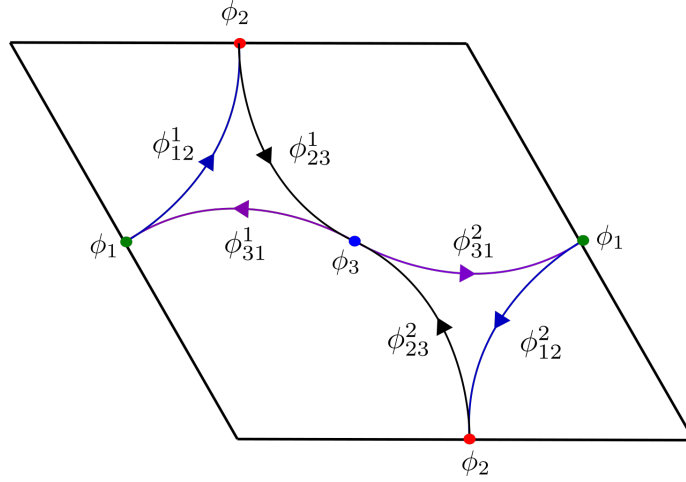


Figure 2.16: BPS solitons in the $W = \wp(\phi, \tau)$ model with $\tau = e^{\frac{2\pi i}{3}}$. There are two solitons between each pair of vacua, the paths they trace out are depicted.

It is well-known [CV1, ?] that this model has precisely two solitons between each pair of critical points, independent of the value of τ . On the other hand, there are still marginal stability walls. For example when τ is pure imaginary the $e_i(\tau)$ are all real and hence co-linear, so the imaginary axis and its $\text{PSL}(2, \mathbb{Z})$ -images are marginal stability walls in the upper-half plane. The fact that there are two solitons in any chamber, is explained at the level of BPS indices by the equations

$$2 = -2 + 2 \cdot 2, \text{ or } , \quad (2.342)$$

$$-2 = 2 - 2 \cdot 2. \quad (2.343)$$

We will now see what happens at the level of chain complexes.

First work at the \mathbb{Z}_3 symmetric point $\tau_0 = e^{\frac{2\pi i}{3}}$. We set

$$\phi_1 = \frac{\tau_0}{2}, \quad \phi_2 = \frac{1}{2}, \quad \phi_3 = \frac{1 + \tau_0}{2}. \quad (2.344)$$

The homogeneity property of $\wp(\phi, \tau)$ at the \mathbb{Z}_3 symmetric point implies that the critical values are proportional to the cubic roots of unity

$$W_1 = W_0 e^{\frac{2\pi i}{3}}, \quad W_2 = W_0, \quad W_3 = W_0 e^{-\frac{2\pi i}{3}}, \quad (2.345)$$

where the proportionality constant is, according to [DLMF]:

$$W_0 = \left(\frac{\Gamma^3(\frac{1}{3})}{2^{\frac{1}{3}} 2\pi} \right)^2. \quad (2.346)$$

The chain complexes are

$$R_{12} = \mathbb{Z}\langle\phi_{12}^1\rangle \oplus \mathbb{Z}\langle\phi_{12}^2\rangle \cong \mathbb{Z}^2[1], \quad (2.347)$$

$$R_{13} = \mathbb{Z}\langle\phi_{13}^1\rangle \oplus \mathbb{Z}\langle\phi_{13}^2\rangle \cong \mathbb{Z}^2[1], \quad (2.348)$$

$$R_{23} = \mathbb{Z}\langle\phi_{23}^1\rangle \oplus \mathbb{Z}\langle\phi_{23}^2\rangle \cong \mathbb{Z}^2[1]. \quad (2.349)$$

A computation similar to the deformed A_{N-1} -minimal models can be performed to conclude that these chain complexes are all concentrated in degree +1 and so all the individual differentials d_{ij} vanish. The trajectories these solitons trace out on $T_{\tau_0}^2$ are depicted in Figure 2.16.

The anti-particles are associated to the BPS complexes

$$R_{21} = \mathbb{Z}\langle\phi_{21}^1\rangle \oplus \mathbb{Z}\langle\phi_{21}^2\rangle \cong \mathbb{Z}^2, \quad (2.350)$$

$$R_{31} = \mathbb{Z}\langle\phi_{31}^1\rangle \oplus \mathbb{Z}\langle\phi_{31}^2\rangle \cong \mathbb{Z}^2, \quad (2.351)$$

$$R_{32} = \mathbb{Z}\langle\phi_{32}^1\rangle \oplus \mathbb{Z}\langle\phi_{32}^2\rangle \cong \mathbb{Z}^2. \quad (2.352)$$

The pairings K_{12}, K_{13}, K_{23} are diagonal in this basis of solitons.

Let's now consider ζ -instantons. The vector space corresponding to the cyclic fan

$$\{1, 2, 3\}, \quad (2.353)$$

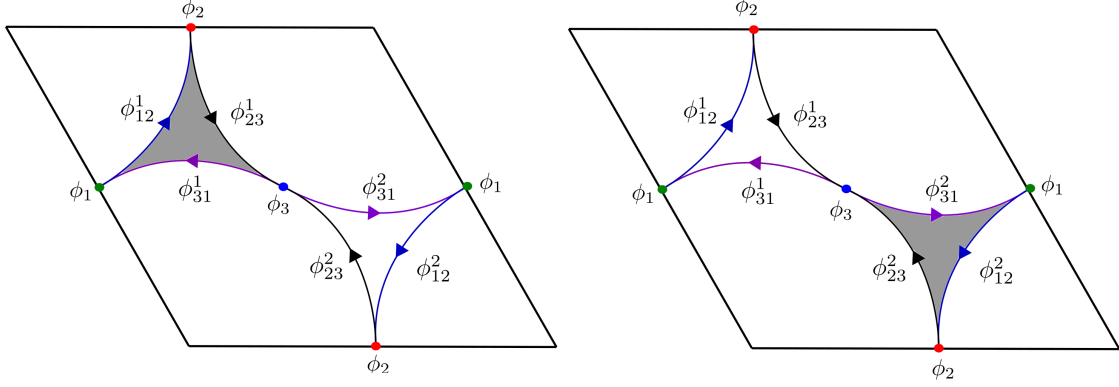


Figure 2.17: ζ -instantons in the $W = \rho(\phi, \tau)$ model with $\tau = e^{\frac{2\pi i}{3}}$.

$R_{12} \otimes R_{23} \otimes R_{31}$ is concentrated in degree +2 and so this model allows rigid instantons. There are eight possible gradient polygons $\phi^{a,b,c} = (\phi_{12}^a, \phi_{23}^b, \phi_{31}^c)$ for $a, b, c = 1, 2$ which could a-priori be occupied. However, the model has additional flavor symmetries whose charges are associated with the winding numbers around the torus²². These symmetries reduce the number of possibilities as follows. Denoting q_1, q_2 the fugacities for the cycles that (half)-wind around the horizontal and τ -direction respectively, the solitons have the following (exponentiated) winding numbers: States in R_{12} have winding numbers $q_1 q_2$ and $(q_1 q_2)^{-1}$, in R_{23} they have q_2, q_2^{-1} , and in R_{13} they have q_1, q_1^{-1} . On the other hand β must have zero winding charge. This cuts down the allowed gradient polygons that can be occupied to

$$\phi^1 = (\phi_{12}^1, \phi_{23}^1, \phi_{31}^1), \quad (2.354)$$

$$\phi^2 = (\phi_{12}^2, \phi_{23}^2, \phi_{31}^2). \quad (2.355)$$

²²More precisely this symmetry doesn't come from translational invariance, since the pole in the superpotential distinguishes a point in the torus (there is a puncture at $X = 0$). Nevertheless we can form a conserved current for each harmonic one-form α given by $j = *\phi^*(\alpha)$.

The simplest non-trivial guess is to posit that these polygons indeed support ζ -instantons with degeneracies

$$N(\phi^1) = 1, \quad (2.356)$$

$$N(\phi^2) = 1. \quad (2.357)$$

Thus we predict the interior amplitude for this model is

$$\beta = \phi_{12}^1 \otimes \phi_{23}^1 \otimes \phi_{31}^1 + \phi_{12}^2 \otimes \phi_{23}^2 \otimes \phi_{31}^2. \quad (2.358)$$

Assuming this is indeed the case, we now evolve from $\tau_0 = e^{2\pi i/3}$ to, a point of the form $\tau_1 = ie^{-i\epsilon}$ with $\epsilon > 0$. In doing so we must cross the wall at $\text{Re}(\tau) = 0$. In such a move, one can check (numerically for instance) that the point W_3 passes through the line connecting W_1 and W_2 . Therefore the chain complexes R_{13}, R_{32} remain the same as before

$$R'_{13} = \mathbb{Z}^2[1], \quad (2.359)$$

$$R'_{32} = \mathbb{Z}^2, \quad (2.360)$$

but R_{12} can jump:

$$R'_{12} = (R_{13} \otimes R_{32})[-1] \oplus R_{12} \quad (2.361)$$

$$= (\mathbb{Z}\langle\phi_{13}^1, \phi_{13}^2\rangle \otimes \mathbb{Z}\langle\phi_{32}^1, \phi_{32}^2\rangle)[-1] \oplus \mathbb{Z}\langle\phi_{12}^1, \phi_{12}^2\rangle. \quad (2.362)$$

The first summand is concentrated in degree zero whereas the second factor is in degree one. The ζ -instanton count imply that the differentials act as follows.

$$d'_{12}((\phi_{13}^1 \phi_{32}^1)^{[-1]}) = \phi_{12}^1, \quad (2.363)$$

$$d'_{12}((\phi_{13}^1 \phi_{32}^2)^{[-1]}) = 0, \quad (2.364)$$

$$d'_{12}((\phi_{13}^2 \phi_{32}^1)^{[-1]}) = 0, \quad (2.365)$$

$$d'_{12}((\phi_{13}^2 \phi_{32}^2)^{[-1]}) = \phi_{12}^2. \quad (2.366)$$

Thus the cohomology is

$$H^\bullet(R'_{12}, d'_{12}) = \mathbb{Z}\langle (\phi_{13}^1 \phi_{32}^2)^{[-1]}, (\phi_{13}^2 \phi_{32}^1)^{[-1]} \rangle, \quad (2.367)$$

which is two-dimensional as expected. Categorical wall-crossing has allowed us to see that there has been a non-trivial reorganization of the BPS states in the 12-sector: in particular their winding numbers jump. This was not visible at the level of ordinary BPS indices²³.

2.9 Conclusions and Future Directions

There are various future directions that might be worth pursuing. While staying in the two-dimensional world, it is desirable to categorify more general wall-crossing statements. In particular the presence of twisted masses leads to interesting new phenomena. These new phenomena and how they affect the discussion of categorical wall-crossing will be the subject of Chapter 3. Similarly, another interesting direction would be to categorify the beautiful formula of Kontsevich and Soibelman, perhaps by constructing the category of infrared line defects in four-dimensional $\mathcal{N} = 2$ theories as a first step.

In a more speculative direction one might wonder about the following. We were studying two-dimensional theories, both in spacetime and from the perspective of the W -plane. Edges between vacua in the latter were initially supported by BPS indices, which are integers, and in particular we can use these edges to form a wall-crossing triangle. Categorifying upgraded these integers to chain complexes, but a lesson we learned is that information about these chain complexes by themselves is not sufficient to describe categorical wall-crossing: they must be accompanied by integers associated to the interior of

²³Of course a *refined* index could have still detected this. In particular upgrading μ_{ij} to a character valued index $\mu_{ij}(q_1, q_2)$ and applying Cecotti-Vafa does the job in this example. In general such a refinement might not always be available.

the wall-crossing triangle. In a higher-dimensional generalization of the formalism, let's say three dimensions, we can imagine having a tetrahedron, whose edges carry categories, faces carry chain complexes and whose interior carries the data of integers. See Figure 2.18. Wall-crossing would occur when the vertices of the tetrahedron lie on a common plane followed by the apex switching sides as viewed from the base. It would be interesting to spell out the wall-crossing structure of this hierarchy of categories, vector spaces and integers that lie on the various faces of the tetrahedron. Even more compelling would be to find a quantum field theoretic realization of such a higher-dimensional “wall-crossing simplex.”

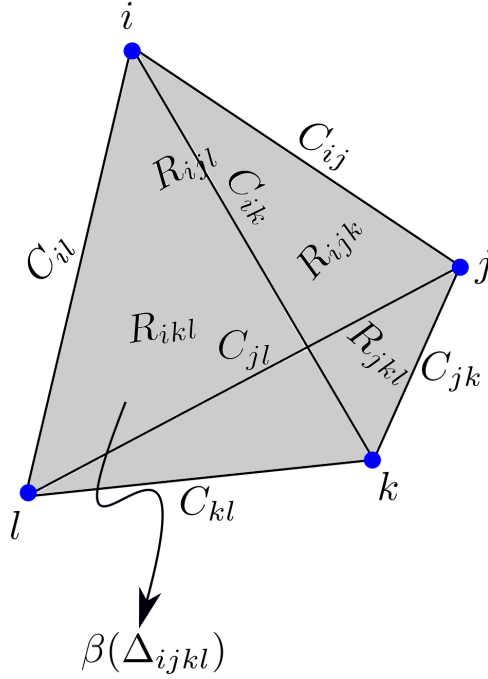


Figure 2.18: A speculative wall-crossing simplex. C_{ij} etc denote categories associated to edges, R_{ijk} etc denote chain complexes associated to faces and $\beta(\Delta_{ijkl})$ denotes a collection of numbers associated to the interior. Wall-crossing would occur when i passes through the base jkl triangle and moves over to the other side.

In the process of categorifying the simplest wall-crossing formula, we have been lead to an interesting blend of mathematics and physics. The physics of domain wall junctions and their moduli spaces allows one to construct canonical objects in homological algebra: the mapping cone and mapping cylinder. These mathematical objects allow us to compactly express the answer to the question we had initially asked. This is the very essence of physical mathematics.

Chapter 3

Algebra of the Infrared and Categorical Wall-Crossing with Twisted Masses

The contents of this paper will appear in a forthcoming preprint [KM2], being prepared jointly with G. W. Moore.

3.1 Introduction

We begin this chapter by recalling how twisted masses can arise in two-dimensional $\mathcal{N} = (2, 2)$ theories, thus generalizing the setup of the previous chapter. Consider a two-dimensional quantum field theory with $\mathcal{N} = (2, 2)$ supersymmetry. To help orient our discussion, we begin by recalling the basic symmetry algebra of our field theory. By definition having an $\mathcal{N} = (2, 2)$ theory means that our theory possesses a symmetry algebra containing the two-dimensional Poincare algebra, along with odd generators

$$Q_+^1, Q_+^2, Q_-^1, Q_-^2 \tag{3.1}$$

with $Q_+^{1,2}$ being the two left movers and $Q_-^{1,2}$ being the two right movers under the Lorentz group. The two left and two right movers are assumed to be distinguished by a $U(1)$ R-symmetry generator F , so that we have Q_+^1, Q_-^1 of positive eigenvalue under F and Q_+^2, Q_-^2 of negative eigenvalue. The symmetry algebra of these odd charges (in Lorentz signature)

is constrained to be

$$\{Q_+^1, Q_+^2\} = P_0 + P_1, \quad (3.2)$$

$$\{Q_-^1, Q_-^2\} = P_0 - P_1 \quad (3.3)$$

where P_μ is the generator of translations in the x^μ direction, and also has central charges

$$\{Q_+^1, Q_-^2\} = Z, \quad (3.4)$$

$$\{Q_-^1, Q_+^2\} = \bar{Z}, \quad (3.5)$$

with all other brackets vanishing. In Lorentz signature we require the reality condition:

$$(Q_+^1)^\dagger = Q_+^2, \quad (Q_-^1)^\dagger = Q_-^2. \quad (3.6)$$

What type of questions might we be interested in? One of the most basic questions one might answer is what are the supersymmetric vacua of our theory on \mathbb{R} , and what properties do they possess? By a supersymmetric vacuum, we mean a state which annihilates all four supercharges. In this paper we are primarily concerned with theories with finitely many discrete vacua, each of which is assumed to be massive. We use the notation introduced in [GMW] and denote the finite set of vacua as \mathbb{V} with a typical element labelled by lowercase Latin letters such as i or j . For a massive theory with finitely many vacua then the next most basic question one might ask is what is the space of half-BPS states? More precisely, if we let

$$Q_{\text{BPS}}(\zeta) = Q_+^1 + \zeta Q_-^1, \quad (3.7)$$

with ζ being an arbitrary phase, then from the elementary equation

$$\{Q_{\text{BPS}}, (Q_{\text{BPS}})^\dagger\} = 2(H + \text{Re}(\zeta^{-1}Z)) \quad (3.8)$$

one may derive the BPS bound: it says the energy spectrum of the Hilbert space on \mathbb{R} , in the ij -sector, satisfies

$$E_{ij} \geq |Z_{ij}|, \quad (3.9)$$

with the space of half-BPS states being states that precisely saturate the bound. The quantum Hilbert space of ij BPS states is denoted as \mathcal{R}_{ij} . If a classical description of the theory is available then the quantum BPS states \mathcal{R}_{ij} can be further described as the space classical BPS states R_{ij} , corrected by instantons (as in Morse theory). Finally another interesting quantity of study involves studying the spectrum of our theory on the spatial half-line \mathbb{R}_+ with a half-BPS boundary condition (namely a boundary condition preserving $Q_{\text{BPS}}(\zeta)$ and its adjoint) \mathfrak{B} placed at the boundary $x = 0$. The basic quantity of study now is the space of (\mathfrak{B}, i) BPS states, namely BPS states that satisfy the \mathfrak{B} -boundary condition at $x = 0$ and settle into a specific vacuum i as $x \rightarrow \infty$. Denote the space of such BPS states $\mathcal{R}_i(\mathfrak{B})$. These are known as “framed BPS states” (whereas spaces of BPS states on \mathbb{R} are sometimes referred to as “vanilla BPS states”). If we are in a situation where a classical description is available, one can again describe $\mathcal{R}_i(\mathfrak{B})$ as the space of classical framed BPS states $\mathcal{E}_i(\mathfrak{B})$ corrected by instantons. In practice we will often work with Landau-Ginzburg models where classical descriptions of vanilla and framed BPS states are indeed available. Therefore for the rest of the paper we will work with R_{ij} and $\mathcal{E}_i(\mathfrak{B})$, with instanton corrections incorporated through differentials d_{ij} and $d_{\mathcal{E}_i(\mathfrak{B})}$, and state our results in terms of these “BPS chain complexes.”¹

To recap, in a given gapped $\mathcal{N} = 2$ theory, the basic objects of interest are the space of BPS states on the line, \mathcal{R}_{ij} , and the space of BPS states on the half-line $\mathcal{R}_i(\mathfrak{B})$. One of

¹Note that using BPS complexes is more general, since spaces of quantum BPS states can always be considered to be a chain complex with vanishing differential.

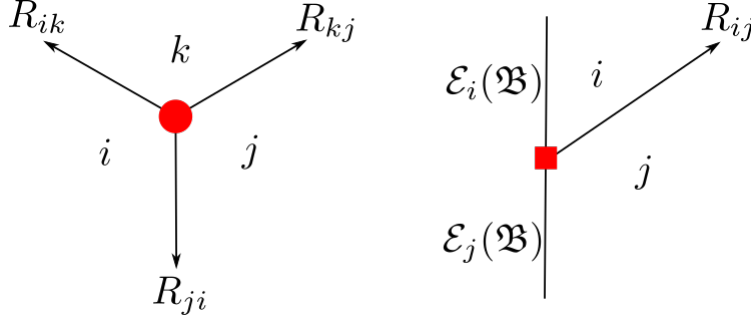


Figure 3.1: Figure of bulk amplitude β_{ikj} , and boundary amplitude \mathcal{B}_{ij} required to describe wall-crossing of vanilla and framed BPS Hilbert spaces respectively. Euclidean time runs vertically.

the most basic properties of these spaces of BPS states is that they undergo wall-crossing. Indeed taking Euler characters, one recovers the BPS indices μ_{ij} and $\overline{\Omega}_i(\mathfrak{B})$, respectively, which are well-known to undergo wall-crossing, with the most basic wall-crossing formulas being of the type

$$\mu_{ij} \rightarrow \mu_{ij} + \mu_{ik}\mu_{ki}, \quad (3.10)$$

for the vanilla BPS indices and

$$\overline{\Omega}_j(\mathfrak{B}) \rightarrow \overline{\Omega}_j(\mathfrak{B}) + \overline{\Omega}_i(\mathfrak{B})\mu_{ij}, \quad (3.11)$$

for the framed BPS indices. Can we describe how the spaces $\mathcal{R}_{ij}, \mathcal{R}_i(\mathfrak{B})$ of BPS states jump rather than just the indices? Surprisingly and interestingly, this is an instance where a naive categorification of the formulas for the indices fails and gives incorrect results. There is additional physical input required to describe the correct way in which the spaces of BPS states jump.

The additional physical data we need to describe the wall-crossing of vanilla and framed BPS states are certain instanton corrections to the differentials that are encoded in certain

linear maps between the underlying chain complexes. To describe the jumping of the vanilla BPS states there is a canonical linear map of vanishing fermion number

$$M[\beta_{ikj}] : R_{ij}^{[1]} \rightarrow R_{ik} \otimes R_{kj}. \quad (3.12)$$

Similarly, for the framed BPS states, there is a linear map

$$T[\mathcal{B}_{ij}] : \mathcal{E}_j^{[1]}(\mathfrak{B}) \rightarrow \mathcal{E}_i(\mathfrak{B}) \otimes R_{ij} \quad (3.13)$$

of vanishing fermion number. The categorical wall-crossing formulas can then be elegantly described in terms of the mapping cone construction of homological algebra applied to these linear transformations: the homotopy class of R_{ij} jumps to

$$R'_{ij} \simeq \text{Cone}(M[\beta_{ikj}] : R_{ij}^{[1]} \rightarrow R_{ik} \otimes R_{kj}) \quad (3.14)$$

and the homotopy class of $\mathcal{E}_j(\mathfrak{B})$ jumps to

$$\mathcal{E}'_j(\mathfrak{B}) \simeq \text{Cone}(T[\mathcal{B}_{ij}] : \mathcal{E}_j^{[1]}(\mathfrak{B}) \rightarrow \mathcal{E}_i(\mathfrak{B}) \otimes R_{ij}). \quad (3.15)$$

See Figure 3.1 for an illustration of these instantons in spacetime. To define a Cone of two complexes, one must use a chain map. One might ask then, why the linear transformations in (3.12) and (3.13) are chain maps? The requirement that they are chain maps imposes certain algebraic constraints on the instanton amplitudes β_{ijk} and \mathcal{B}_{ij} .

The bulk and boundary amplitudes required to describe categorical wall-crossing of vanilla and framed BPS states are in fact the simplest examples of more general bulk and boundary amplitudes. The trivalent amplitude β_{ijk} describing the jump in vanilla BPS states can be easily generalized to n -valent amplitudes $\beta_{i_1 \dots i_n}$, and similarly the brane amplitude \mathcal{B}_{ij} with a single outgoing soliton can be generalized to have a fan of outgoing solitons. The algebraic relations obeyed by β_{ijk} and \mathcal{B}_{ij} are also the simplest examples of

general algebraic constraints obeyed by bulk and boundary amplitudes. The bulk amplitudes are governed by an L_∞ Maurer-Cartan equation whereas the boundary amplitudes satisfy an A_∞ Maurer-Cartan equation. Thus in writing down a categorical wall-crossing formula one is naturally lead to discover a novel class of BPS objects which obey rich algebraic structures. These more general amplitudes enter into the calculations of physically natural objects such as the space of supersymmetric states on a segment with $(\mathfrak{B}_L, \mathfrak{B}_R)$ boundary conditions, or in the computation of spaces of bulk and boundary local operators.

The part of the story that we have described so far was worked out in [GMW] and expanded further upon in Chapter 2 of this dissertation. A different viewpoint, with potential applications to higher dimensional theories appears in [KKS, KSS]. The formalism of all these papers makes an important simplifying assumption: the central charge in the ij sector is assumed to be of the form

$$Z_{ij} = W_i - W_j \tag{3.16}$$

for a well-defined set of complex numbers (defined up to an overall simultaneous shift) W_i associated to each vacuum $i \in \mathbb{V}$. In other words, it is assumed that the central charge gets a purely topological contribution coming from the “boundary” of space, the two ends of \mathbb{R} . In particular, on a compact spatial slice such as S^1 , the central charge vanishes. However this is not always the case. It is easy to imagine a situation where the central charge can be non-zero even on S^1 . Suppose that our theory has a global symmetry group G with charges valued in a lattice Γ . By definition since G is a global symmetry, its charges commute with the rest of the $\mathcal{N} = (2, 2)$ superPoincare generators. Therefore a central charge of the form

$$Z(\gamma) = M \cdot \gamma \tag{3.17}$$

where $\gamma \in \Gamma$ and $M \in \text{Hom}(\Gamma, \mathbb{C})$ is allowed and can be non-zero. Going back to the Hilbert space on \mathbb{R} with ij boundary conditions, the central charge becomes a sum of the topological contributions and the contributions from the global charges. It takes the form

$$Z_{ij}(\gamma) = W_i - W_j + M \cdot \gamma. \quad (3.18)$$

M is commonly referred to as a twisted mass [HH]. We note that the split of the central charge into contributions coming from topological and Noether charges is not fully canonical. A more precise version of (3.18) will appear in Section 2.

The presence of a non-trivial twisted mass in the central charge gives rise to various novel and interesting physical phenomenon:

1. An initial, rather trivial observation is that in the presence of a non-trivial G , the BPS states are refined to carry extra charges

$$R_{ij} = \oplus_{\gamma \in \Gamma} R_{ij}(\gamma), \quad \mathcal{E}_i(\mathfrak{B}) = \oplus_{\gamma \in \Gamma} \mathcal{E}_i(\mathfrak{B}, \gamma), \quad (3.19)$$

and such a refinement can lead to useful selection rules. Moreover if $M \neq 0$, the assumption that each ordered pair of vacua carries a unique central charge, important for the formalism of [GMW], no longer holds. Instead each pair of vacua is now associated with a Γ -torsor.

2. Next, a second slightly less obvious observation is the presence of BPS states on S^1 [GMW, Park]. These compact or periodic BPS states can contribute to the Hilbert space on \mathbb{R} as well, simply by choosing the same vacuum i at both ends on \mathbb{R} . Therefore, with a non-trivial twisted mass, there can be non-trivial BPS states present in the ii -sector. Furthermore, these BPS states typically come in Fock spaces, since by the linear nature of M , multiparticle states can also saturate the BPS bound.

So the BPS particle spectrum of an $\mathcal{N} = (2, 2)$ theory with twisted masses can have Fock spaces of periodic BPS particles.

3. Finally, wall-crossing phenomenon are much more rich and interesting if $M \neq 0$. For vanilla BPS states, the only generic wall with $M = 0$ is the alignment of Z_{ij} and Z_{jk} with Z_{ik} . With twisted masses, $Z_{ij}(\gamma)$ may align with $Z_{ii}(\gamma)$ and because of the linear nature of Z_{ii} , there is simultaneous alignment with $Z_{ii}(k\gamma)$ for $k \geq 1$. Therefore a whole host of wall-crossing decays may occur. Similar remarks apply to wall-crossing for framed BPS states. If a brane \mathfrak{B} preserves the supercharge associated with a phase $\zeta_{\mathfrak{B}}$, then $Z_{k\gamma}$ for each $k \geq 1$ can align with $\zeta_{\mathfrak{B}}$, and at such a wall (known as a K-wall) the spaces of framed BPS states can jump in interesting ways.

The main aim of this paper is to address the third point in a categorical framework. Whereas the wall-crossing formula for the framed and vanilla BPS indices, being a specialization of the 2d-4d wall-crossing formula, is well-understood, a categorical discussion will have to incorporate bulk and boundary amplitudes. The wall-crossing formula for the framed indices says that the framed BPS index

$$\overline{\Omega}_i(\mathfrak{B}) = \sum_{\gamma \in \Gamma} \overline{\Omega}_i(\mathfrak{B}, \gamma) X_{\gamma} \quad (3.20)$$

jumps to

$$\overline{\Omega}_i(\mathfrak{B}) \rightarrow \overline{\Omega}_i(\mathfrak{B}) \prod_{n \geq 1} (1 - X_{n\gamma'})^{-\mu_i(n\gamma')}, \quad (3.21)$$

where X_{γ} generate the group algebra of Γ . To see for instance what type of amplitude is involved in the categorification of this formula, we note that there can be an amplitude for a brane \mathfrak{B} to emit entire Fock spaces of periodic solitons. More precisely, the amplitude relevant for describing wall-crossing when Z_{γ} aligns with $\zeta_{\mathfrak{B}}$ will be encoded in a linear

map of the form

$$T[\mathcal{B}_\gamma] : \mathcal{E}_i(\mathfrak{B}) \rightarrow \mathcal{E}_i(\mathfrak{B}) \otimes \mathcal{F}[V_{\gamma_i}] \quad (3.22)$$

where $\mathcal{F}[V_{\gamma_i}]$ is the graded Fock space of

$$V_{\gamma_i} = \oplus_{n \geq 1} R_{n\gamma_i}. \quad (3.23)$$

Incorporating these amplitudes to describe the wall-crossing of framed Hilbert spaces $\mathcal{R}_i(\mathfrak{B})$ will be one of the main results of this paper.

To conclude the introductory remarks, we explain some of our motivations for carrying out a categorical discussion of BPS states in the presence of twisted masses.

1. First, as we have already discussed, it is very natural to do this from a purely two-dimensional perspective, since a generic two-dimensional $\mathcal{N} = (2, 2)$ model will have a non-trivial global symmetry group, which generically does affect the central charge Z . A canonical example of a useful model where a twisted mass term can be present is the \mathbb{CP}^1 sigma model which has an $SU(2)$ global symmetry group coming from target space isometries.
2. Second, there is a particular Landau-Ginzburg model of interest in the study of three-manifolds and knot homology: this is twisted five-dimensional supersymmetric Yang-Mills theory, which can indeed be formulated as a Landau-Ginzburg model with the target being the space of $G_{\mathbb{C}}$ -connections on a three-manifold M and the superpotential W being the Chern-Simons functional. Since the CS superpotential is multivalued, this is an example of an LG model with twisted masses. Here one of the main interesting points discovered in [Wit8] was that to each knot $K \subset M_3$, one can assign a brane $\mathfrak{B}(K)$ of this model, and then the framed BPS Hilbert spaces

$\mathcal{E}_{\sigma_i}(\mathfrak{B}(K))$ are natural candidates for homological knot invariants that generalize Khovanov’s complex. These “knot invariants” however will be subject to wall-crossing phenomenon. The formulas in this paper are expected to apply to the complexes $\mathcal{E}_{\sigma_i}(\mathfrak{B}(K))$ for $K \subset M_3$ at least for isolated flat connections. Other LG models that appear in the study of knot homology, some of which are directly motivated from the CSLG model [GW, GM], and from other perspectives [A] always involve multivalued superpotentials. All the new phenomenon associated with non-trivial twisted masses are present in these models.

3. Third, from a mathematical perspective, we are outlining novel mutations that are expected to occur in the Fukaya-Seidel category of a multivalued superpotential.
4. Finally, the behavior of BPS indices under wall-crossing in two-dimensional theories with twisted masses, is a special case of the more general 2d-4d wall-crossing formula [GMN4], and the discussion presented here is expected to be directly relevant for discussing categorical wall-crossing of BPS states in four-dimensional $\mathcal{N} = 2$ theories.

The outline of this paper is as follows. In Section 3.2, we recall some basics of twisted mass terms in $\mathcal{N} = (2, 2)$ theories. We discuss spaces of BPS states and the important categorified spectrum generator in the presence of twisted masses, and illustrate these notions with some examples. Next we begin our discussion of categorical wall-crossing formulas with twisted masses. We discuss boundary amplitudes and wall-crossing of framed BPS states in Section 3.3 and illustrate our formulas in the important example of the theory of a free chiral superfield (we work with the mirror Landau-Ginzburg formulation). In Section 3.4 we discuss interior amplitudes and wall-crossing of vanilla BPS states, and demonstrate these notions in the example of the \mathbb{CP}^1 sigma model with twisted masses. Finally, we sketch generalizations of the “Algebra of the Infrared” [GMW] - an algebraic

framework to discuss bulk and boundary amplitudes - to incorporate twisted masses.

3.2 Basics of Twisted Masses in $\mathcal{N} = (2, 2)$ Theories

3.2.1 Abstract Vacuum and BPS data

We begin by recalling the formal setup of [GMW]. One is given a finite set of massive vacua \mathbb{V} with a typical element denoted by a Latin letter $i \in \mathbb{V}$, along with a generic set of vacuum weights $z : \mathbb{V} \rightarrow \mathbb{C}$. Writing $z_i := z(i)$ the central charge in the ij -sector is written as

$$Z_{ij} = z_i - z_j. \quad (3.24)$$

For each (ordered) pair of distinct vacua $i \neq j$, we are given \mathbb{Z} -graded \mathbb{Z} -modules² R_{ij} equipped with a perfect degree -1 -pairing

$$K_{ij} : R_{ij} \otimes R_{ji} \rightarrow \mathbb{Z}. \quad (3.25)$$

Given these concepts one can define the categorified spectrum generator associated to a half-plane $\mathbb{H} \subset \mathbb{C}$. It is given by

$$\hat{R}_{\mathbb{H}} := \bigotimes_{Z_{ij} \in \mathbb{H}}^{\curvearrowright} (\mathbb{Z}\mathbf{1} \oplus R_{ij}e_{ij}), \quad (3.26)$$

where \curvearrowright denotes an ordering on the product by the clockwise order of the central charges $\{Z_{ij}\}$, e_{ij} denotes the ij elementary matrix in $\text{gl}(|\mathbb{V}|, \mathbb{Z})$, and $\mathbf{1}$ denotes the $|\mathbb{V}| \times |\mathbb{V}|$ unit

²In the introduction we indicated that $\{R_{ij}\}$ is a collection of chain complexes rather than just ordinary \mathbb{Z} -modules. We will show how to incorporate the differentials $\{d_{ij}\}$ on this collection of \mathbb{Z} -modules when we discuss interior amplitudes in a later section. Also, in physical theories one should work with chain complexes over \mathbb{C} , but the abstract formalism makes sense for more refined modules over \mathbb{Z} . We work at this more refined level.

matrix. We denote the coefficient of e_{ij} in \widehat{R} as \widehat{R}_{ij} :

$$\bigoplus_{i,j \in \mathbb{V}} \widehat{R}_{ij} e_{ij} = \bigotimes_{Z_{ij} \in \mathbb{H}}^{\curvearrowright} (\mathbb{Z}\mathbf{1} \oplus R_{ij} e_{ij}). \quad (3.27)$$

Most often we will work with \mathbb{H} being the right half-plane, $\{z \mid \operatorname{Re} z > 0\}$ and \widehat{R} without any half-plane in the subscript is understood to mean the spectrum generator associated to the right half plane.

The \widehat{R}_{ij} -spaces give us a formalism to discuss branes. A given brane \mathfrak{B} carries Chan-Paton spaces which by definition are \mathbb{Z} -graded \mathbb{Z} -modules $\mathcal{E}_i(\mathfrak{B})$ for each $i \in \mathbb{V}$. The morphism space of two branes $\operatorname{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$ is then given by

$$\operatorname{Hop}(\mathfrak{B}_1, \mathfrak{B}_2) = \bigoplus_{i,j \in \mathbb{V}} \mathcal{E}_i(\mathfrak{B}_1) \otimes \widehat{R}_{ij} \otimes \mathcal{E}_j^\vee(\mathfrak{B}_2) \quad (3.28)$$

where V^\vee denotes the dual of a \mathbb{Z} -module V .

Finally, it is also useful to define the trace of the categorical BPS monodromy

$$R_c = \operatorname{Tr}(\widehat{R}^{\text{opp}} \otimes \widehat{R}) \quad (3.29)$$

where \widehat{R}^{opp} denotes the spectrum generator associated to the opposite half-plane $\operatorname{Re} z < 0$, and Tr is defined by

$$\operatorname{Tr}(\oplus_{i,j \in \mathbb{V}} V_{ij} e_{ij}) = \oplus_{i \in \mathbb{V}} V_{ii}. \quad (3.30)$$

R_c is a categorification of the wall-crossing invariant quantity $\operatorname{Tr}(S^{-t}S)$ studied in [CV1, CV2].

We now generalize these notions to include twisted masses. As in the introduction, the new piece of data that enters the discussion is a finitely generated free abelian group Γ . The

vacuum set \mathbb{V} is now upgraded to the *vacuum groupoid* as follows. The vacuum groupoid \mathbb{V} consists of a finite collection of objects $i, j, \dots \in \text{Ob}(\mathbb{V})$. Abusing notation, we often write objects as $i \in \mathbb{V}$. The morphism space for each pair of vacua $\Gamma_{ij} := \text{Hom}(i, j)$ is required to be Γ -torsor, with a typical element denoted as $\gamma_{ij} \in \Gamma_{ij}$. The composition map $\Gamma_{ij} \times \Gamma_{jk} \rightarrow \Gamma_{ik}$ is denoted as $\gamma_{ij} + \gamma_{jk}$. For each i we have $\text{Hom}(i, i) = \Gamma_{ii}$ a Γ -torsor canonically isomorphic to Γ and we write u_i as the additive identity. A typical element $\gamma_{ii} \in \Gamma_{ii}$ is often abbreviated as γ_i , and canonically maps to an element $\gamma \in \Gamma$. The data of a central charge is then captured by a groupoid homomorphism

$$Z : \sqcup_{i,j} \Gamma_{ij} \rightarrow \mathbb{C} \quad (3.31)$$

written as $Z_{\gamma_{ij}}$ for $\gamma_{ij} \in \Gamma_{ij}$. On the diagonal components Γ_{ii} , the homomorphism is required to be independent of i simply being determined by a homomorphism $Z : \Gamma \rightarrow \mathbb{C}$ and is denoted as Z_γ . Then $Z_{\gamma_{ii}}$ is determined under the canonical identification of Γ_{ii} with Γ .

The BPS \mathbb{Z} -modules $\{R_{\gamma_{ij}}\}$, is a collection of \mathbb{Z} -graded \mathbb{Z} -modules, one for each ordered pair of vacua i, j and an element $\gamma_{ij} \in \Gamma_{ij}$ along with perfect pairings of degree -1

$$K_{\gamma_{ij}} : R_{\gamma_{ij}} \otimes R_{\bar{\gamma}_{ji}} \rightarrow \mathbb{Z}, \quad (3.32)$$

where $\bar{\gamma}_{ji}$ denotes the inverse of $\gamma_{ij} \in \Gamma_{ij}$: the element of Γ_{ji} such that $\gamma_{ij} + \bar{\gamma}_{ji} = u_i$ and $\bar{\gamma}_{ji} + \gamma_{ij} = u_j$. Unlike before i and j are no longer required to be distinct. The collection of BPS \mathbb{Z} -modules can include the spaces of *periodic* BPS states R_{γ_i} for each $\gamma_i \in \Gamma_{ii}$.

Remark on Notation In this paper γ_{ij} is used to denote a generic element of Γ_{ij} . Similarly, γ_{ji} also denotes a generic element of Γ_{ji} . However, given $\gamma_{ij} \in \Gamma_{ij}$, the element $\bar{\gamma}_{ji} \in \Gamma_{ji}$ denotes the specific additive inverse of γ_{ij} .

We now introduce the categorified spectrum generator with twisted masses. Before doing this, we introduce the groupoid algebra $\mathbb{Z}[\mathbb{V}]$ associated to the vacuum groupoid \mathbb{V} . It is generated by elements $x_{\gamma_{ij}}$ for each $\gamma_{ij} \in \Gamma_{ij}$ and satisfies the relation

$$x_{\gamma_{ij}} x_{\gamma_{kl}} = \delta_{jk} x_{\gamma_{ij} + \gamma_{kl}}. \quad (3.33)$$

The categorified spectrum generator with twisted masses is then given by

$$\widehat{R}_{\mathbb{H}} = \bigotimes_{\substack{Z_{\gamma_{ij}} \in \mathbb{H}, i \neq j \\ Z_{\gamma} \in \mathbb{H}}}^{\curvearrowright} : \mathcal{S}_{\gamma_{ij}} \mathcal{K}_{\gamma} :, \quad (3.34)$$

where

$$\mathcal{S}_{\gamma_{ij}} = \mathbb{Z}\mathbf{1} \oplus R_{\gamma_{ij}} x_{\gamma_{ij}} \quad (3.35)$$

is the categorified $\mathcal{S}_{\gamma_{ij}}$ -factor, and

$$\mathcal{K}_{\gamma} = \oplus_{i \in \mathbb{V}} \mathcal{F}^*[R_{\gamma_i} x_{\gamma_i}] \quad (3.36)$$

is the categorified \mathcal{K}_{γ} -factor. In the above we clarify that

$$\mathbf{1} = \sum_{i \in \mathbb{V}} x_{u_i}, \quad (3.37)$$

and $\mathcal{F}^*[Vx]$ for a graded \mathbb{Z} -module V and a formal variable x denotes the series

$$\mathcal{F}^*[Vx] = \oplus_{n \geq 0} \mathcal{F}^n[V] x^n. \quad (3.38)$$

As before \widehat{R} allows us to define $\widehat{R}_{\gamma_{ij}}$ as the coefficients of $x_{\gamma_{ij}}$

$$\widehat{R} = \bigoplus_{\substack{i, j \in \mathbb{V} \\ \gamma_{ij} \in \Gamma_{ij}}} \widehat{R}_{\gamma_{ij}} x_{\gamma_{ij}}. \quad (3.39)$$

There is also new data involved in defining branes. A brane \mathfrak{B} now carries a Γ -torsor $\Gamma_{\mathfrak{B}, i}$ for each $i \in \mathbb{V}$, equipped with composition maps

$$\circ : \Gamma_{\mathfrak{B}, i} \times \Gamma_{ij} \rightarrow \Gamma_{\mathfrak{B}, j} \quad (3.40)$$

subject to natural coherence conditions, so that the Chan-Paton spaces of \mathfrak{B} are now graded by this collection of Γ -torsors

$$\mathcal{E}_i(\mathfrak{B}) = \bigoplus_{\gamma_{\mathfrak{B},i} \in \Gamma_{\mathfrak{B},i}} \mathcal{E}_{\gamma_{\mathfrak{B},i}}(\mathfrak{B}). \quad (3.41)$$

The morphism space between two branes $\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$ is then graded by a Γ -torsor $\Gamma_{\mathfrak{B}_1, \mathfrak{B}_2}$ so that

$$\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2) = \bigoplus_{\substack{i,j \in \mathbb{V} \\ \gamma_{\mathfrak{B}_1,i} \in \Gamma_{\mathfrak{B}_1,i}, \gamma_{ij} \in \Gamma_{ij}, \gamma_{\mathfrak{B}_2,j} \in \Gamma_{\mathfrak{B}_2,j}}} \mathcal{E}_{\gamma_{\mathfrak{B}_1,i}}(\mathfrak{B}_1) \otimes \hat{R}_{\gamma_{ij}} \otimes \mathcal{E}_{\gamma_{\mathfrak{B}_2,j}}^\vee(\mathfrak{B}_2). \quad (3.42)$$

Finally as before, we can introduce the trace of the categorical BPS monodromy

$$R_c = \text{Tr}(\hat{R}^{\text{opp}} \otimes \hat{R}), \quad (3.43)$$

where Tr denotes the direct sum of all diagonal coefficients x_{γ_i} :

$$\text{Tr} \left(\bigoplus_{\substack{i,j \in \mathbb{V} \\ \gamma_{ij} \in \Gamma_{ij}}} V_{\gamma_{ij}} x_{\gamma_{ij}} \right) = \bigoplus_{\substack{i \in \mathbb{V}, \\ \gamma_i \in \Gamma_{ii}}} V_{\gamma_i}. \quad (3.44)$$

We note that because of the canonical identification of Γ_{ii} with Γ , the \mathbb{Z} -module R_c is graded $\mathbb{Z} \times \Gamma$. We call the grading by \mathbb{Z} the cohomological or fermion degree, and the grading by Γ the equivariant or flavor degree.

In practice, it is useful to express the categorical spectrum generator with twisted masses as a $|\text{Ob}(\mathbb{V})| \times |\text{Ob}(\mathbb{V})|$ matrix, similar to the spectrum generator without twisted masses. In order to do this one has to make some non-canonical choices. One has to choose a reference element $\tau_{ij} \in \Gamma_{ij}$ for each $i \neq j$. Since Γ_{ij} is a Γ -torsor, we can write any other element γ_{ij} as $\gamma_{ij} = \tau_{ij} + \gamma$ for some $\gamma \in \Gamma$. We then identify

$$x_{\gamma_{ij}} \rightarrow x_{\gamma} e_{ij}, \quad (3.45)$$

$$x_{\gamma_i} \rightarrow x_{\gamma} e_{ii} \quad (3.46)$$

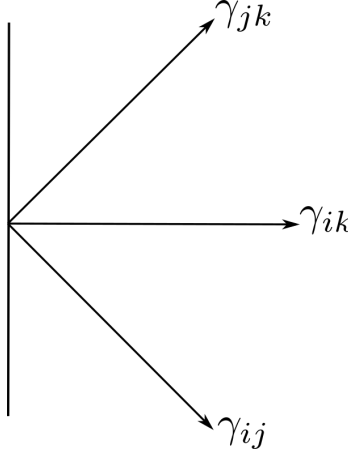


Figure 3.2: A configuration of central charge rays that lie in the right half-plane when $\text{Ob}(\mathbb{V}) = \{i, j, k\}$. This is not to be confused with a half-plane fan, introduced later.

with e_{ij} being elementary matrices as before. In other words the choices of the τ_{ij} 's allow us to define an isomorphism

$$\mathbb{Z}[\mathbb{V}] \rightarrow \text{gl}(|\mathbb{V}|, \mathbb{Z}[\Gamma]) \quad (3.47)$$

from the groupoid algebra to the algebra of $|\mathbb{V}| \times |\mathbb{V}|$ matrices valued in the group ring $\mathbb{Z}[\Gamma]$. Applying this to the spectrum generator, we can write \widehat{R} as a matrix with entries being \mathbb{Z} -module valued polynomials in the x_γ variables, with \mathbb{Z} -modules as coefficients.

We end this subsection by recording the form of \widehat{R} and R_c for two important classes of vacuum data that will be of use later in this paper.

The first case is when $\text{Ob}(\mathbb{V}) = \{i, j, k\}$ consists of three distinct vacua, and the only non-trivial BPS \mathbb{Z} -modules are $R_{\gamma_{ij}}, R_{\gamma_{jk}}, R_{\gamma_{ik}}$ and therefore $R_{\bar{\gamma}_{ji}}, R_{\bar{\gamma}_{kj}}, R_{\bar{\gamma}_{ki}}$ for some $\gamma_{ij} \in \Gamma_{ij}, \gamma_{jk} \in \Gamma_{jk}$ and $\gamma_{ik} \in \Gamma_{ik}$. In particular, there are no non-trivial spaces of periodic solitons, and therefore the categorified \mathcal{K}_γ -factors are trivial. We suppose that

$$\gamma_{ik} = \gamma_{ij} + \gamma_{jk}, \quad (3.48)$$

and that the central charges corresponding to

$$(\gamma_{ik}, \bar{\gamma}_{kj}, \bar{\gamma}_{ji}) \quad (3.49)$$

are in clockwise order in the complex plane. We can choose γ_{ik} to lie in the right half plane and therefore so do γ_{ij} and γ_{jk} . The spectrum generator is therefore

$$\hat{R} = (\mathbb{Z}\mathbf{1} \oplus R_{\gamma_{jk}} e_{jk}) \otimes (\mathbb{Z}\mathbf{1} \oplus R_{\gamma_{ik}} e_{ik}) \otimes (\mathbb{Z}\mathbf{1} \oplus R_{\gamma_{ij}} e_{ij}). \quad (3.50)$$

Choosing the ordering $i < j < k$ on the vacua, the matrix form of \hat{R} therefore reads

$$\hat{R} = \begin{pmatrix} \mathbb{Z} & R_{\gamma_{ij}} & R_{\gamma_{jk}} \\ 0 & \mathbb{Z} & R_{\gamma_{ik}} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}. \quad (3.51)$$

On the other hand for the opposite half-plane we have

$$\hat{R}^{\text{opp}} = (\mathbb{Z}\mathbf{1} \oplus R_{\bar{\gamma}_{kj}} e_{kj}) \otimes (\mathbb{Z}\mathbf{1} \oplus R_{\bar{\gamma}_{ki}} e_{ki}) \otimes (\mathbb{Z}\mathbf{1} \oplus R_{\bar{\gamma}_{ji}} e_{ji}), \quad (3.52)$$

which can be expanded to

$$\hat{R}^{\text{opp}} = \begin{pmatrix} \mathbb{Z} & 0 & 0 \\ R_{\bar{\gamma}_{ji}} & \mathbb{Z} & 0 \\ R_{\bar{\gamma}_{ki}} \oplus (R_{\bar{\gamma}_{kj}} \otimes R_{\bar{\gamma}_{ki}}) & R_{\bar{\gamma}_{kj}} & \mathbb{Z} \end{pmatrix}. \quad (3.53)$$

The categorical trace R_c can then be computed to be

$$\begin{aligned} R_c &= R_i \oplus R_j \oplus R_k \\ &\quad \oplus R_{(\gamma_{ij}, \bar{\gamma}_{ji})} \oplus R_{(\gamma_{ik}, \bar{\gamma}_{ki})} \oplus R_{(\gamma_{jk}, \bar{\gamma}_{kj})} \\ &\quad \oplus R_{(\gamma_{ik}, \bar{\gamma}_{kj}, \bar{\gamma}_{ji})}, \end{aligned} \quad (3.54)$$

where $R_i \cong R_j \cong R_k \cong \mathbb{Z}$ are generated by each respective vacuum state ϕ_i, ϕ_j, ϕ_k ,

$$R_{(\gamma_{ij}, \bar{\gamma}_{ji})} = R_{\gamma_{ij}} \otimes R_{\bar{\gamma}_{ji}}, \quad (3.55)$$

with similar definitions for $R_{(\gamma_{ik}, \bar{\gamma}_{ki})}$ and $R_{(\gamma_{jk}, \bar{\gamma}_{kj})}$, and

$$R_{(\gamma_{ik}, \bar{\gamma}_{kj}, \bar{\gamma}_{ji})} = R_{\gamma_{ik}} \otimes R_{\bar{\gamma}_{kj}} \otimes R_{\bar{\gamma}_{ji}}. \quad (3.56)$$

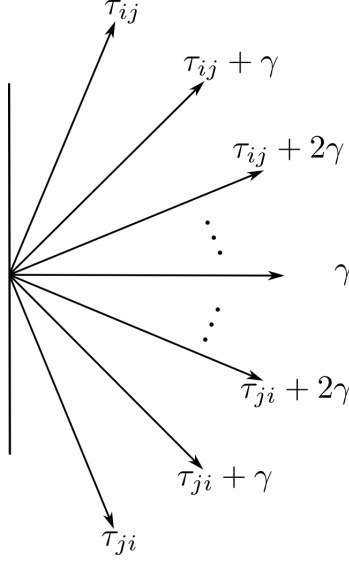


Figure 3.3: A configuration of central charge rays that lie in the right half-plane when $\text{Ob}(\mathbb{V}) = \{i, j\}$ and $\Gamma = \mathbb{Z}\langle\gamma\rangle$. $\tau_{ij} + \tau_{ji} = \gamma_i \cong \gamma$ in the above figure.

The second case we will analyze in detail is when $\text{Ob}(\mathbb{V}) = \{i, j\}$ consists of two vacua, and $\Gamma = \mathbb{Z}\langle\gamma\rangle$ is a rank one lattice with generator γ . We suppose τ_{ij}, τ_{ji} are elements of Γ_{ij}, Γ_{ji} respectively such that $\tau_{ij} + \tau_{ji} = \gamma_i$. We suppose that $(\tau_{ij}, \gamma_j, \tau_{ji}, -\gamma_i)$ are in clockwise order and that τ_{ij} is an element such that $Z_{\tau_{ij}}$ is in the right-half plane, but $-\gamma_i + \tau_{ij}$ is not. Then the clockwise order of central charges that lie in the right half-plane is

$$(\tau_{ij}, \tau_{ij} + \gamma, \tau_{ij} + 2\gamma, \dots, \gamma, \dots, \tau_{ji} + 2\gamma, \tau_{ji} + \gamma, \tau_{ji}), \quad (3.57)$$

and therefore

$$\hat{R} = \mathcal{S}_{\tau_{ij}} \mathcal{S}_{\tau_{ij} + \gamma} \mathcal{S}_{\tau_{ij} + 2\gamma} \cdots \prod_{n=1}^{\infty} \mathcal{K}_{n\gamma} \cdots \mathcal{S}_{\tau_{ji} + 2\gamma} \mathcal{S}_{\tau_{ji} + \gamma} \mathcal{S}_{\tau_{ji}}. \quad (3.58)$$

The configuration of central charges in the right half-plane is depicted in Figure 3.3. In [GGM] it is referred to as a “peacock pattern.” By choosing τ_{ij} as the representative element

in Γ_{ij} , we can write \widehat{R} as a 2×2 matrix as follows

$$\widehat{R} = \begin{pmatrix} \mathbb{Z} & R_{\tau_{ij}} \\ 0 & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & R_{\tau_{ij}+\gamma} \\ 0 & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & R_{\tau_{ij}+2\gamma} \\ 0 & \mathbb{Z} \end{pmatrix} \cdots \begin{pmatrix} \otimes_{n=1}^{\infty} \mathcal{F}^*[R_{n\gamma_i}] & 0 \\ 0 & \otimes_{n=1}^{\infty} \mathcal{F}^*[R_{n\gamma_j}] \end{pmatrix} \quad (3.59)$$

$$\cdots \begin{pmatrix} \mathbb{Z} & 0 \\ R_{\tau_{ji}+2\gamma} & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & 0 \\ R_{\tau_{ji}+\gamma} & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & 0 \\ R_{\tau_{ji}} & \mathbb{Z} \end{pmatrix}.$$

This product can be computed out to be

$$\widehat{R} = \begin{pmatrix} \widehat{R}_{ii} & \widehat{R}_{ij} \\ \widehat{R}_{ji} & \widehat{R}_{jj} \end{pmatrix} \quad (3.60)$$

where

$$\widehat{R}_{ii} = \mathcal{F}[V_{\gamma_i}] \oplus_{n,m \geq 0} R_{\tau_{ij}+n\gamma} \otimes \mathcal{F}[V_{\gamma_j}] \otimes R_{\tau_{ji}+m\gamma}, \quad (3.61)$$

$$\widehat{R}_{ij} = \oplus_{n \geq 0} R_{\tau_{ij}+n\gamma} \otimes \mathcal{F}[V_{\gamma_j}], \quad (3.62)$$

$$\widehat{R}_{ji} = \oplus_{n \geq 0} \mathcal{F}[V_{\gamma_j}] \otimes R_{\tau_{ji}+n\gamma}, \quad (3.63)$$

$$\widehat{R}_{jj} = \mathcal{F}[V_{\gamma_j}]. \quad (3.64)$$

As before we denote $V_{\gamma_a} = \oplus_{n \geq 1} R_{n\gamma_a}$ for $a \in \{i, j\}$. The opposite spectrum generator is

$$\widehat{R}^{\text{opp}} = \mathcal{S}_{\bar{\tau}_{ji}} \mathcal{S}_{\bar{\tau}_{ji}-\gamma} \mathcal{S}_{\bar{\tau}_{ji}-2\gamma} \cdots \prod_{n=1}^{\infty} \mathcal{K}_{-n\gamma} \cdots \mathcal{S}_{\bar{\tau}_{ij}-2\gamma} \mathcal{S}_{\bar{\tau}_{ij}-\gamma} \mathcal{S}_{\bar{\tau}_{ij}}. \quad (3.65)$$

Its matrix elements can be computed out to be

$$\widehat{R}_{ii}^{\text{opp}} = \mathcal{F}^*[V_{-\gamma_i}] \quad (3.66)$$

$$\widehat{R}_{ij}^{\text{opp}} = \oplus_{n \geq 0} \mathcal{F}[V_{-\gamma_i}] \otimes R_{\bar{\tau}_{ij}-n\gamma} \quad (3.67)$$

$$\widehat{R}_{ji}^{\text{opp}} = \oplus_{n \geq 0} R_{\bar{\tau}_{ji}-n\gamma} \otimes \mathcal{F}[V_{-\gamma_i}] \quad (3.68)$$

$$\widehat{R}_{jj}^{\text{opp}} = \oplus_{n,m \geq 0} R_{\bar{\tau}_{ji}-n\gamma} \otimes \mathcal{F}[V_{-\gamma_i}] \otimes R_{\bar{\tau}_{ij}-m\gamma} \oplus \mathcal{F}[V_{-\gamma_i}]. \quad (3.69)$$

The categorical trace

$$R_c = (\widehat{R}_{ii}^{\text{opp}} \otimes \widehat{R}_{ii}) \oplus (\widehat{R}_{ij}^{\text{opp}} \otimes \widehat{R}_{ji}) \oplus (\widehat{R}_{ji}^{\text{opp}} \otimes \widehat{R}_{ij}) \oplus (\widehat{R}_{jj}^{\text{opp}} \otimes \widehat{R}_{jj}), \quad (3.70)$$

can be simplified to

$$R_c = R_i \oplus R_j \oplus R_{(ij)} \quad (3.71)$$

where

$$R_i := \mathcal{F}^*[V_{\gamma_i}] \otimes \mathcal{F}^*[V_{-\gamma_i}], \quad (3.72)$$

$$R_j := \mathcal{F}^*[V_{\gamma_j}] \otimes \mathcal{F}^*[V_{-\gamma_j}], \quad (3.73)$$

and

$$R_{(ij)} = \bigoplus_{(\gamma_{ij}, \gamma_{ji}) \in \Gamma_{ij} \times \Gamma_{ji}} R_{\gamma_{ij}} \otimes \mathcal{F}^*[V_{\gamma_j}] \otimes R_{\gamma_{ji}} \otimes \mathcal{F}^*[V_{-\gamma_i}]. \quad (3.74)$$

3.2.2 Realization of BPS data in Landau-Ginzburg Models

We now explain how one can concretely realize the vacuum and BPS data in Landau-Ginzburg models. Suppose (X, g, J) is a Kähler manifold with Riemannian metric g and complex structure J , and suppose α is a closed holomorphic one-form on X . The pair (X, α) allows one to write down a two-dimensional theory with $\mathcal{N} = (2, 2)$ supersymmetry which we call a Landau-Ginzburg model with twisted masses. The Lagrangian in terms of component fields (ϕ, ψ_{\pm}) is written as

$$L = L_D + L_F \quad (3.75)$$

where

$$\begin{aligned} L_D = g_{I\bar{J}}(\partial_t \phi^I \partial_t \phi^{\bar{J}} - \partial_x \phi^I \partial_x \phi^{\bar{J}}) + i g_{I\bar{J}} \bar{\psi}_+^{\bar{J}} D_- \psi_+^I + i g_{I\bar{J}} \bar{\psi}_-^{\bar{J}} D_+ \psi_-^I \\ + R_{I\bar{J}K\bar{L}} \psi_+^I \bar{\psi}_+^{\bar{J}} \psi_-^K \bar{\psi}_-^{\bar{L}} \end{aligned} \quad (3.76)$$

and

$$L_F = \frac{1}{2} g^{I\bar{J}} \alpha_I \alpha_{\bar{J}} + D_I \alpha_J \psi_+^I \psi_-^J + D_{\bar{I}} \alpha_{\bar{J}} \bar{\psi}_-^{\bar{J}} \bar{\psi}_+^{\bar{I}}. \quad (3.77)$$

Since the potential of the theory is $V = \frac{1}{2}g^{I\bar{J}}\alpha_I\alpha_{\bar{J}}$ we see that the physical vacuum set coincides with the zeroes of the one-form α , which we assume to be isolated and non-degenerate. The latter condition means that the symmetric matrix $D_I\alpha_J$ evaluated at a zero is non-degenerate for each zero.

The vacuum data introduced in the previous subsection is concretely realized in terms of (X, α) as follows. First, the deck group Γ is simply the first homology group of X

$$\Gamma = H_1(X, \mathbb{Z}). \quad (3.78)$$

Since the zeroes of α coincide with the physical vacua of the LG theory, the vacuum groupoid \mathbb{V} has objects given by

$$\text{Ob}(\mathbb{V}) = \text{Zero}(\alpha). \quad (3.79)$$

The morphisms of the vacuum groupoid are given by the relative homology

$$\Gamma_{ij} \subset H_1(X, \{\phi_i, \phi_j\}; \mathbb{Z}), \quad (3.80)$$

of oriented 1-chains c such that $\partial c = \phi_j - \phi_i$, and the composition maps $\Gamma_{ij} \times \Gamma_{jk} \rightarrow \Gamma_{ik}$ are given simply by concatenating an ij -chain with a jk -chain. The central charge homomorphism is given by

$$Z_{\gamma_{ij}} = \int_{\gamma_{ij}} \alpha. \quad (3.81)$$

One can indeed show that this coincides with the definition coming from the supersymmetry algebra

$$Z = \{Q_+^1, Q_-^2\}. \quad (3.82)$$

Finally, if \mathfrak{B} is a brane of the LG model supported on a Lagrangian submanifold $L_{\mathfrak{B}} \subset X$, we can also realize the torsor $\Gamma_{\mathfrak{B},i}$ as

$$\Gamma_{\mathfrak{B},i} \subset H_1(X, \{L_{\mathfrak{B}}, \phi_i\}; \mathbb{Z}), \quad (3.83)$$

the subset of 1-chains homologous to paths that start at a point in $L\mathfrak{B}$, and end on ϕ_i .

The BPS data can also be realized explicitly. Explain how to construct $R_{\gamma_{ij}}$ and $\mathcal{E}_{\gamma_{\mathfrak{B},i}}(\mathfrak{B})$ are constructed explicitly.

Remark on terminology We have been referring to a model defined by a closed holomorphic one-form α as “Landau-Ginzburg model with twisted masses.” On the other hand, in the introduction twisted masses referred to a non-zero contribution of a global symmetry charge to the central charge Z of the theory. The reason why these two uses of terminology is compatible is as follows. Suppose we start out with an LG model defined by (X, W) where W is a single-valued holomorphic function. The Lagrangian of the LG model has the same kinetic term (3.76), and the potential term, determined by W , is

$$L_F = \frac{1}{2} g^{I\bar{J}} \partial_I W \partial_{\bar{J}} \bar{W} + D_I \partial_J W \psi_+^I \psi_-^J + D_{\bar{I}} \partial_{\bar{J}} W \bar{\psi}_-^{\bar{J}} \bar{\psi}_+^{\bar{I}}. \quad (3.84)$$

Given a representative η of a non-trivial cohomology class in $H^1(X, \mathbb{C})$ we can define a conserved current by letting

$$j_\eta^\mu = \epsilon^{\mu\nu} \eta_I \partial_\nu \phi^I. \quad (3.85)$$

The corresponding conserved charge is $J = \int dx j^0$ as usual. Supposing $\{\beta_a\}_{a=1}^{h^{1,0}(X)}$ is a basis of holomorphic differentials, we may couple the corresponding topological currents to a background twisted vector multiplet and freeze the values of the corresponding scalar to $2\pi i(m_1, \dots, m_{h^{1,0}}) \in \mathbb{C}^n$ while setting all other fields to vanish. The resulting deformation of the action is quite simple and amounts to taking

$$\partial_I W \rightarrow \alpha_I := \partial_I W + 2\pi i \sum_{n=1}^k m_n (\beta_n)_I \quad (3.86)$$

in the expression (3.84) for L_F , thus giving a potential term of the form (3.77).

3.2.3 Realization in Sigma Models with Isometries

We now discuss twisted masses and the realization of vacuum data in supersymmetric sigma models with target space isometries. Consider a $\mathcal{N} = (2, 2)$ sigma model defined by a choice of a target Kähler manifold (X, g, J) . The standard Lagrangian in terms of component fields is given by $L = L_D$ where L_D is given as in (3.76). Supposing that X has a holomorphic Killing field K^I , meaning it satisfies

$$D_I K_{\bar{J}} + D_{\bar{J}} K_I = 0, \quad (3.87)$$

we can perform the twisted mass deformation as follows. The Killing field K leads to a $U(1)$ symmetry of the sigma model with corresponding conserved current

$$j^\mu = h^{\mu\nu} g_{I\bar{J}} (\partial_\nu \phi^I K^{\bar{J}} + \partial_\nu \phi^{\bar{J}} K^I) + i D_I K_{\bar{J}} \bar{\psi}^{\bar{J}} \gamma^\mu \psi^I. \quad (3.88)$$

It is useful to write out the individual components for later use

$$j^t = g_{I\bar{J}} (\partial_t \phi^I K^{\bar{J}} + \partial_t \phi^{\bar{J}} K^I) + i D_I K_{\bar{J}} (\bar{\psi}_+^{\bar{J}} \psi_+^I + \bar{\psi}_-^{\bar{J}} \psi_-^I), \quad (3.89)$$

$$j^x = -g_{I\bar{J}} (\partial_x \phi^I K^{\bar{J}} + \partial_x \phi^{\bar{J}} K^I) + i D_I K_{\bar{J}} (-\bar{\psi}_+^{\bar{J}} \psi_+^I + \bar{\psi}_-^{\bar{J}} \psi_-^I). \quad (3.90)$$

We can couple this $U(1)$ global symmetry to a background twisted vector multiplet with component fields

$$(A_t, A_x, \lambda_\pm, \sigma, \bar{\sigma}, D). \quad (3.91)$$

One can consistently set these fields to values

$$(0, 0, 0, m, \bar{m}, 0) \quad (3.92)$$

where m is an arbitrary complex number, while preserving $\mathcal{N} = (2, 2)$ supersymmetry. The resulting deformation to the Lagrangian is

$$\Delta L_K = -|m|^2 g_{I\bar{J}} K^I K^{\bar{J}} + i D_I K_{\bar{J}} (m \bar{\psi}_+^{\bar{J}} \psi_-^I + \bar{m} \bar{\psi}_-^{\bar{J}} \psi_+^I). \quad (3.93)$$

The combined Lagrangian

$$L_K = L + \Delta L_K \quad (3.94)$$

is invariant under the supersymmetry transformations

$$\delta\phi^I = \epsilon_+\psi_-^I - \epsilon_-\psi_+^I, \quad (3.95)$$

$$\delta\psi_+^I = i\bar{\epsilon}_-\partial_+\phi^I + \epsilon_+\Gamma_{JK}^I\psi_+^J\psi_-^K + i\bar{\epsilon}_+\bar{m}K^I, \quad (3.96)$$

$$\delta\psi_-^I = -i\bar{\epsilon}_+\partial_-\phi^I + \epsilon_-\Gamma_{JK}^I\psi_+^J\psi_-^K - i\bar{\epsilon}_-mK^I, \quad (3.97)$$

along with their complex conjugates. Under these the combined action $S := S_0 + S_K$ varies as

$$\delta S = \int d^2x (-\partial_\mu\epsilon_-G_+^\mu + \partial_\mu\epsilon_+G_-^\mu + \partial_\mu\bar{\epsilon}_-\bar{G}_+^\mu - \partial_\mu\bar{\epsilon}_+\bar{G}_-^\mu) \quad (3.98)$$

where

$$G_+^t = g_{I\bar{J}}\partial_+\phi^{\bar{J}}\psi_+^I - \bar{m}g_{I\bar{J}}K^{\bar{J}}\psi_-^I, \quad (3.99)$$

$$G_+^x = -g_{I\bar{J}}\partial_+\phi^{\bar{J}}\psi_+^I - \bar{m}g_{I\bar{J}}K^{\bar{J}}\psi_-^I, \quad (3.100)$$

$$G_-^t = g_{I\bar{J}}\partial_-\phi^{\bar{J}}\psi_-^I - mg_{I\bar{J}}K^{\bar{J}}\psi_+^I, \quad (3.101)$$

$$G_-^x = g_{I\bar{J}}\partial_-\phi^{\bar{J}}\psi_-^I + mg_{I\bar{J}}K^{\bar{J}}\psi_+^I. \quad (3.102)$$

Note that in addition to the $U(1)$ global isometry, there is also a topological current reading

$$T^\mu = i\epsilon^{\mu\nu}(\partial_\nu\phi^IK_I - \partial_\nu\phi^{\bar{I}}K_{\bar{I}}), \quad (3.103)$$

which is conserved, $\partial_\mu T^\mu = 0$, as a consequence of the holomorphic Killing equation. Again it is useful to write out the individual components

$$T^t = i(\partial_x\phi^IK_I - \partial_x\phi^{\bar{I}}K_{\bar{I}}), \quad (3.104)$$

$$T^x = i(-\partial_t\phi^IK_I + \partial_t\phi^{\bar{I}}K_{\bar{I}}). \quad (3.105)$$

When $m \neq 0$, the supersymmetry algebra obtains a non-zero twisted central charge term

$$\tilde{Z} = \{\bar{Q}_+, Q_-\}. \quad (3.106)$$

By using the explicit form of the supercharges, it is straightforward to compute that

$$\tilde{Z} = im \int dx T^t - m \int dx j^t. \quad (3.107)$$

In the case when there is a moment map h , meaning we can write

$$\partial_I h = ig_{I\bar{J}} K^{\bar{J}}, \quad (3.108)$$

we can convert the integrand to a total derivative and write

$$\tilde{Z}_{ij} = im(h(\phi_j) - h(\phi_i)) - mJ \quad (3.109)$$

where $\{\phi_i\}$ are the critical points of the moment map (assumed to be isolated). Thus we find that the twisted central charge is a sum of a contribution from the boundary of space, and a contribution proportional to the conserved charge of the global symmetry. This is of the form anticipated in (3.18).

Unlike the case of Landau-Ginzburg models where non-renormalization theorems tell us that the classical vacua and form of the central charges continue to be valid quantum mechanically, the classical vacuum set given by the critical points of the moment map h , and the expression for Z given by (3.109) are subject to strong quantum corrections. We can anticipate the form of the quantum corrections as follows.

In order to summarize the expected result we have to recall some aspects of the A-model with target (X, ω) . With the twisted mass deformation the A-model observables are equivariant deRham cohomology classes $H_{U(1)}^*(X, m)$ where the equivariant differential is given by

$$d_m = d - im \iota_K. \quad (3.110)$$

There is a distinguished element given by the cohomology class of the equivariant symplectic form

$$\omega_h = \omega - \text{im } h. \quad (3.111)$$

ω_h is special because deforming the action by second descendant of ω_h

$$(\omega_h)^{(2)} = dzd\bar{z} \{ \bar{Q}_+, [Q_-, \omega_h] \} \quad (3.112)$$

via

$$S \rightarrow S + t \int (\omega_h)^{(2)} \quad (3.113)$$

corresponds to rescaling the metric $g_{I\bar{J}} \rightarrow t g_{I\bar{J}}$. In other words, in the correspondence between observables and deformation parameters, ω_h corresponds to the Kähler parameter t . Let C_ω be the corresponding action of ω_h on $H_{U(1)}^*(X, m)$ via the quantum cup product. Let $C = \mathbb{C}_t^*$ be the parameter space. By using C_ω we can define the spectral curve $\Sigma \subset T^*C$ via the spectral equation

$$\det(C_\omega - s) = 0. \quad (3.114)$$

Once we have a natural spectral curve Σ , we can propose the standard formulas familiar from coupled 2d-4d systems [GMN4]. At a point $t \in C$ we propose that

$$\text{Ob } \mathbb{V} = \pi^{-1}(t) \quad (3.115)$$

is the set of quantum vacua,

$$\Gamma = H_1(\Sigma, \mathbb{Z}) \quad (3.116)$$

is the deck group, and

$$\Gamma_{ij} \subset H_1(\Sigma, \{t^{(i)}, t^{(j)}\}; \mathbb{Z}) \quad (3.117)$$

the collection of Γ -torsors is given by oriented 1-chains homologous to paths that begin at $t^{(i)}$ and end at $t^{(j)}$. Finally we propose that the quantum central charge is

$$Z_{\gamma_{ij}} = \int_{\gamma_{ij}} \lambda, \quad (3.118)$$

where $\lambda = s dt$ is the Liouville form on T^*C .

It is interesting to continue and formulate the BPS \mathbb{Z} -modules $R_{\gamma_{ij}}$ directly in terms of the spectral curve Σ . This would involve a categorification of the procedure via spectral networks used in [GMN4, GMN5] to determine the Euler characters $\mu_{\gamma_{ij}}$ of $R_{\gamma_{ij}}$. It is also interesting to express the data of a B-brane \mathfrak{B} and its Chan-Paton spaces $\mathcal{E}_i(\mathfrak{B})$ in terms of the spectral curve Σ . We leave this to future work.

3.2.4 Examples

We now discuss examples of vacuum and BPS data in concrete $\mathcal{N} = (2, 2)$ theories with twisted masses.

We remark that in order to determine the BPS \mathbb{Z} -modules of classical solitons in Landau-Ginzburg models, we directly consider intersection points of left and right thimbles, infinitesimally rotated from the phase of the classical soliton. It is possible to analyze the intersection points explicitly for Landau-Ginzburg models with one-dimensional target spaces, which is exclusively the case in our examples. The fermion degrees mod 2 of the solitons corresponding to the intersection points of thimbles are also determined once the thimbles have been oriented. We postulate that the fermion degrees in $\mathbb{Z} \bmod 2$, being either 0 or 1, continue to hold in \mathbb{Z} for this simple class of examples. That this can be done consistently requires one to either directly compute η -invariants of a Dirac operator coming from linearizing the soliton equation, or an argument based on the Maslov index.

The Free Chiral

The simplest theory one can write down with a non-trivial twisted mass term is the theory of a free chiral superfield Φ with a twisted mass m turned on for the standard $U(1)$ symmetry $\Phi \rightarrow e^{i\varphi}\Phi$. It is instructive to work in the mirror formulation of the model, which is a Landau-Ginzburg model with target space \mathbb{C}^* with coordinate ϕ , and LG one form

$$\alpha = \left(\frac{m}{\phi} - 1\right)d\phi. \quad (3.119)$$

The model has a unique vacuum $\mathbb{V} = \{1\}$ located at $\phi = m$ and the deck group $\Gamma = H_1(\mathbb{C}^*, \mathbb{Z})$ has rank one, generated by the cycle γ that goes counter-clockwise around the origin once. The central charge is given by

$$Z_\gamma = 2\pi i m. \quad (3.120)$$

The only non-trivial BPS \mathbb{Z} -modules consist of³

$$R_\gamma = \mathbb{Z}, \quad (3.121)$$

generated by ϕ_γ , along with its CPT conjugate

$$R_{-\gamma} = \mathbb{Z}^{[1]}, \quad (3.122)$$

where the superscript means that the generator has cohomological degree $+1$. Letting $m = ia$ with $a < 0$, Z_γ lies in the right-half plane, therefore the categorical spectrum generator

$$\widehat{R}_{11} = \mathcal{F}^*[R_\gamma], \quad (3.123)$$

³One can determine these spaces in a standard way by working out the intersections of left and right thimbles explicitly. The fermion degrees mod 2 are determined by working out the orientation of thimble intersections. These will be discussed in Section 3.3.2.

is simply a bosonic Fock space in the ϕ_γ variable. Equivalently,

$$\widehat{R}_{n\gamma} \cong \mathbb{Z} \quad (3.124)$$

for each $n \geq 0$ and trivial for $n < 0$.

We will discuss various examples of branes and their Chan-Paton spaces in Section 3.3.2.

The \mathbb{CP}^1 Model

Another canonical example of an $\mathcal{N} = (2, 2)$ theory with twisted masses consists of the \mathbb{CP}^1 model with a twisted mass m turned on for the diagonal $U(1)$ subgroup of the $SU(2)$ isometry group. One can determine the BPS spectrum of this model in a variety of ways [Dor, GMN4]. We find it useful to illustrate this example once again in the mirror Landau-Ginzburg formulation [HV, MS1]. The mirror theory once again has target the punctured complex plane \mathbb{C}^* with coordinate ϕ , and LG one-form given by

$$\alpha = \left(1 - \frac{m}{\phi} - \frac{\Lambda^2}{\phi^2}\right) d\phi. \quad (3.125)$$

The model has two vacua $\mathbb{V} = \{1, 2\}$ which are located at the two zeroes of ϕ :

$$\phi_1 = \frac{1}{2}(m - \sqrt{m^2 + 4\Lambda^2}), \quad (3.126)$$

$$\phi_2 = \frac{1}{2}(m + \sqrt{m^2 + 4\Lambda^2}). \quad (3.127)$$

We note that the theory has a \mathbb{Z}_2 -symmetry acting on the field ϕ via

$$\phi \rightarrow -\phi^{-1}. \quad (3.128)$$

Let $\tau_{12} \in \Gamma_{12}$ denote the cycle going from ϕ_1 to ϕ_2 and γ denote the cycle going around the origin in a counter-clockwise direction. We can compute

$$Z_{\gamma_{12}} = 2\sqrt{m^2 + 4\Lambda^2} + m \log \left(\frac{m - \sqrt{m^2 + 4\Lambda^2}}{m + \sqrt{m^2 + 4\Lambda^2}} \right), \quad (3.129)$$

$$Z_\gamma = -2\pi i m. \quad (3.130)$$

The strong coupling spectrum is

$$R_{\tau_{12}} = \mathbb{Z}, \quad (3.131)$$

$$R_{\tau_{12}-\gamma} = \mathbb{Z} \quad (3.132)$$

generated by the corresponding solitons $\phi_{\tau_{12}}$ and $\phi_{\tau_{12}-\gamma}$ respectively, along with the CPT conjugates

$$R_{\bar{\tau}_{21}} = \mathbb{Z}^{[1]}, \quad (3.133)$$

$$R_{\bar{\tau}_{21}+\gamma} = \mathbb{Z}^{[1]}. \quad (3.134)$$

Therefore at strong coupling the spectrum generator is

$$\hat{R}_s = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}\langle\phi_{\bar{\tau}_{21}+\gamma}\rangle & \mathbb{Z} \end{pmatrix} \otimes \begin{pmatrix} \mathbb{Z} & \mathbb{Z}\langle\phi_{\tau_{12}}\rangle \\ 0 & \mathbb{Z} \end{pmatrix}. \quad (3.135)$$

The weak coupling spectrum is

$$R_{\gamma_{12}} \cong \mathbb{Z} \quad \text{for any } \gamma_{12} \in \Gamma_{12}, \quad (3.136)$$

$$R_{\gamma_{21}} \cong \mathbb{Z}^{[1]} \quad \text{for any } \gamma_{21} \in \Gamma_{21} \quad (3.137)$$

generated by the corresponding solitons $\phi_{\gamma_{12}}$ and $\phi_{\gamma_{21}}$ respectively, along with

$$R_{\gamma_1} \cong \mathbb{Z}, \quad R_{\gamma_2} \cong \mathbb{Z}^{[1]}, \quad (3.138)$$

$$R_{-\gamma_1} \cong \mathbb{Z}^{[1]}, \quad R_{-\gamma_2} \cong \mathbb{Z}, \quad (3.139)$$

generated by the corresponding periodic solitons $a_{\gamma_1}, \psi_{-\gamma_1}, \psi_{\gamma_2}, a_{-\gamma_2}$ respectively. The spectrum generator is

$$\hat{R}_w = \prod_{n=0}^{\infty} \begin{pmatrix} \mathbb{Z} & \mathbb{Z}\langle\phi_{\tau_{12}+n\gamma}\rangle \\ 0 & \mathbb{Z} \end{pmatrix} \begin{pmatrix} S^*[a_{\gamma_1}] & 0 \\ 0 & \Lambda^*[\psi_{\gamma_2}] \end{pmatrix} \prod_{n=1}^{\infty} \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}\langle\phi_{\tau_{21}+n\gamma}\rangle & \mathbb{Z} \end{pmatrix}. \quad (3.140)$$

The relationship between the strong and weak coupling BPS spectra is usually explained via the invariance of the (deategorified) spectrum generator. Indeed, if one takes graded characters of \widehat{R}_s and \widehat{R}_w we find that the strong and weak coupling spectrum generator agree, namely the wall-crossing identity

$$\begin{pmatrix} 1 & 0 \\ -x_\gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \prod_{n=0}^{\infty} \begin{pmatrix} 1 & x_\gamma^n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1-x_\gamma)^{-1} & 0 \\ 0 & 1-x_\gamma \end{pmatrix} \prod_{n=1}^{\infty} \begin{pmatrix} 1 & 0 \\ -x_\gamma^n & 1 \end{pmatrix} \quad (3.141)$$

holds. However, one finds that such an identity does not hold on the nose as an equality for the categorified spectrum generators $\widehat{R}_w, \widehat{R}_s$; the matrix elements of the spectrum generator at weak coupling consists of infinite-dimensional spaces, whereas at strong coupling they are finite-dimensional. This is rectified by incorporating instanton effects. We will show in Section 3.4.4 how the use of ζ -instantons of the \mathbb{CP}^1 model allow us to construct differentials on the matrix elements of the categorical spectrum generators so that we have a homotopy equivalence between the weak and strong coupling spectrum generators

$$\widehat{R}_s \simeq \widehat{R}_w, \quad (3.142)$$

thus categorifying the wall-crossing formula (3.141).

It is also instructive to discuss branes and their framed \mathbb{Z} -modules in the LG formulation of the \mathbb{CP}^1 model. The superpotential on the covering space \mathbb{C} with coordinate Y (so that $\phi = e^Y$) is given by

$$W = mY - \Lambda^2(e^Y + e^{-Y}). \quad (3.143)$$

Suppose we work at a point in parameter space with $m, \Lambda > 0$ and study branes preserving $\zeta = 1$. A brane \mathfrak{B} is supported on a curve in \mathbb{C} along which $\text{Re}W \rightarrow \infty$. These regions for $\Lambda > 0$ are given by

$$O_k = \left\{ \frac{\pi}{2} + 2\pi k < \text{Im}Y < \frac{3\pi}{2} + 2\pi k \right\}. \quad (3.144)$$

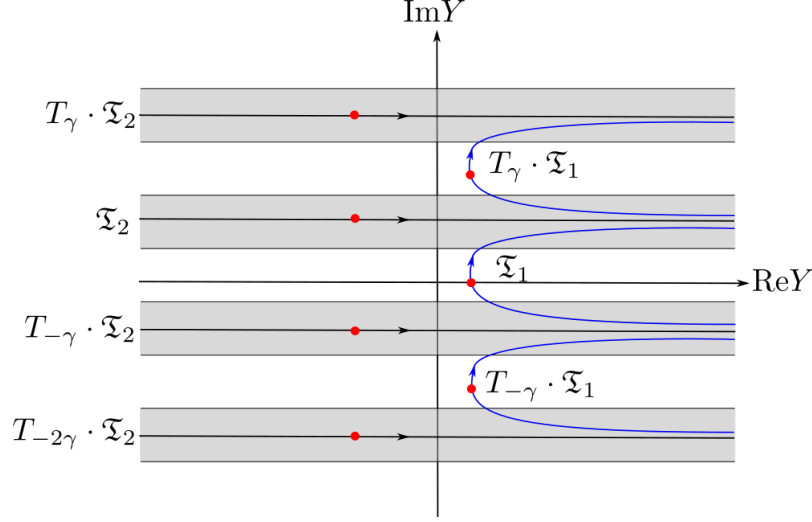


Figure 3.4: Left thimbles for the \mathbb{CP}^1 model in the covering space \mathbb{C} .

The left Lefschetz thimbles $\mathfrak{Z}_1, \mathfrak{Z}_2$ at $\zeta = 1$ along with their deck translates are depicted in Figure 3.4. The thimbles $\mathfrak{Z}_2^{(k)} := T_{k\gamma} \cdot \mathfrak{Z}_2$ is a curve lying entirely in the region O_k whereas \mathfrak{Z}_1^k connects the region O_k with O_{k+1} as depicted in the Figure. A standard example of a brane in the \mathbb{CP}^1 model then is the brane \mathfrak{B}_k whose support consists of a curve that connects O_0 with O_k for any $k \in \mathbb{Z}$. The case $k = 3$ is shown in Figure 3.5. \mathfrak{B}_k intersects non-trivially at a single point with the right thimbles $\mathfrak{R}_2^{(l)}$ only for $l = 0$, whereas for it has one intersection point with $\mathfrak{R}_1^{(l)}$ for each $l = 0, \dots, k-1$. Moreover, all of these intersections have positive orientation. Therefore the brane \mathfrak{B}_k has framed BPS Hilbert spaces given by

$$\mathcal{E}_{1,n\gamma}(\mathfrak{B}_k) = \mathbb{Z} \text{ for } n = 0, 1, \dots, k-1, \quad (3.145)$$

$$\mathcal{E}_{2,n\gamma}(\mathfrak{B}_k) = \delta_{n,0} \mathbb{Z}. \quad (3.146)$$

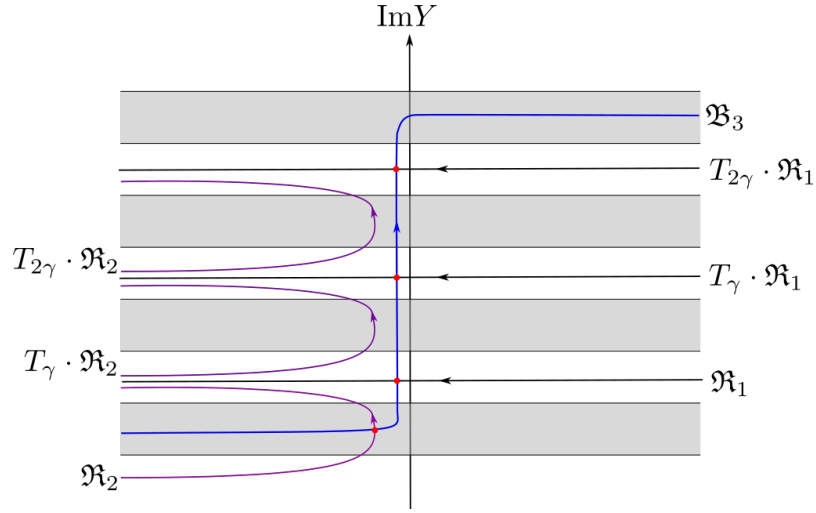


Figure 3.5: The brane \mathfrak{B}_k with $k = 3$ and its intersection points with the right thimbles are depicted.

Weierstrass Model

Finally we also include an example of a model⁴ where the rank of Γ is bigger than one. Consider the model with target being a punctured elliptic curve $X = T_\tau^2 \setminus \{0\}$ and

$$\alpha = \wp(\phi, \tau) d\phi. \quad (3.147)$$

Note the contrast with the “Elliptic LG model” discussed in section 8.3 of [KM1], which had $W = \wp(\phi, \tau)$. In this case we can write

$$W = -\zeta(\phi, \tau), \quad (3.148)$$

the Weierstrass ζ -function, necessarily multivalued. Since $\wp(\phi, \tau)$ has a second-order pole at the additive zero of the elliptic curve X , it must have two zeroes. Thus, the model has two vacua, but in general it is difficult to write down explicit expressions for the zeros of

⁴This model was originally discussed by the author in unpublished work with S. Cecotti, whom we thank.

$\wp(\phi, \tau)$. However one can work at the enhanced symmetry point $\tau_0 = e^{2\pi i/3}$ where the model has \mathbb{Z}_6 symmetry. Making use of this symmetry, it is possible to locate the exact zeroes. On the one hand the \mathbb{Z}_6 symmetry implies that if ϕ_0 is a zero, then so is $\omega^k \phi_0$ where $\omega = e^{\frac{2\pi i}{3}}$ and $k = 0, 1, \dots, 5$. On the other hand there are only two zeroes in the fundamental domain. Therefore some of the six complex numbers above must coincide with a lattice translate of ϕ_0 . In particular we have

$$\omega^2 \phi_0 = \phi_0 - 1 \quad (3.149)$$

which means that $\phi_0 = \frac{1}{\sqrt{3}}e^{\frac{i\pi}{6}}$. The other zero in the fundamental domain is given by $\omega\phi_0$. Thus the two zeroes are located at

$$\phi_1 = \frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right), \quad (3.150)$$

$$\phi_2 = \frac{1}{\sqrt{3}}i. \quad (3.151)$$

The punctured torus has

$$H_1(T_\tau^2 \setminus \{0\}, \mathbb{Z}) \cong \mathbb{Z}\langle\gamma\rangle \oplus \mathbb{Z}\langle\gamma'\rangle \quad (3.152)$$

where γ, γ' are the a and b -cycles respectively. Therefore we have $\mathbb{V} = \{1, 2\}$ and $\Gamma = \mathbb{Z}\langle\gamma\rangle \oplus \mathbb{Z}\langle\gamma'\rangle$. The one-form α has non-zero periods on either cycle which can be computed to be

$$Z_\gamma = -\frac{2\pi}{\sqrt{3}}, \quad (3.153)$$

$$Z_{\gamma'} = \frac{2\pi}{\sqrt{3}}\exp\left(\frac{i\pi}{3}\right). \quad (3.154)$$

Letting γ_{12} be the open cycle going from ϕ_1 to ϕ_2 in a straight line, we can also compute

$$Z_{\gamma_{12}} = \frac{2\pi}{3}i\tau_0. \quad (3.155)$$

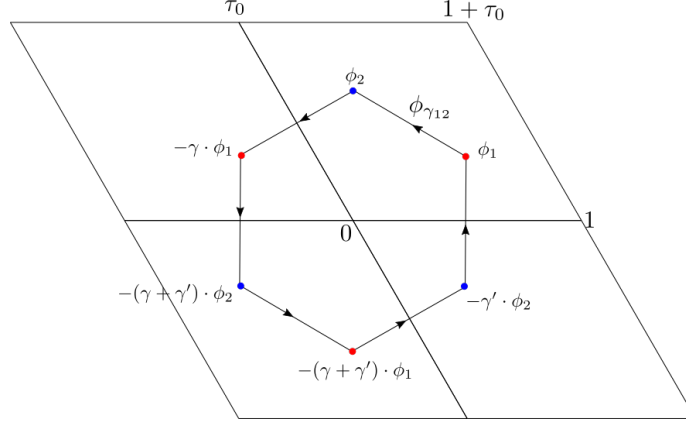


Figure 3.6: The critical points and soliton paths of the $\alpha = \wp(\phi, \tau)d\phi$ model as depicted on the cover.

The BPS spectrum consists of three 12 solitons, all with distinct charges in Γ_{12} :

$$R_{\gamma_{12}} = \mathbb{Z}^{[1]}, \quad (3.156)$$

$$R_{\gamma_{12}-\gamma} = \mathbb{Z}, \quad (3.157)$$

$$R_{\gamma_{12}+\gamma'} = \mathbb{Z}^{[1]}, \quad (3.158)$$

where the solitons are depicted in Figure 3.6, along with the usual spaces of anti-solitons:

$$R_{\bar{\gamma}_{21}} = \mathbb{Z}, \quad (3.159)$$

$$R_{\bar{\gamma}_{21}+\gamma} = \mathbb{Z}^{[1]}, \quad (3.160)$$

$$R_{\bar{\gamma}_{21}-\gamma'} = \mathbb{Z}. \quad (3.161)$$

All other soliton spaces, including spaces of periodic solitons are trivial. The clockwise order of the central charges in the upper-half plane is

$$(Z_{\bar{\gamma}_{21}+\gamma}, Z_{\gamma_{12}+\gamma'}, Z_{\bar{\gamma}_{21}}) \quad (3.162)$$

and so the categorical spectrum generator reads

$$\widehat{R} = \mathcal{S}_{\bar{\gamma}_{21}+\gamma} \mathcal{S}_{\gamma_{12}+\gamma'} \mathcal{S}_{\bar{\gamma}_{21}} \quad (3.163)$$

$$= \begin{pmatrix} \mathbb{Z} & 0 \\ R_{\bar{\gamma}_{21}+\gamma} & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & R_{\gamma_{12}+\gamma'} \\ 0 & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & 0 \\ R_{\bar{\gamma}_{21}} & \mathbb{Z} \end{pmatrix} \quad (3.164)$$

$$= \begin{pmatrix} \mathbb{Z} \oplus (R_{\gamma_{12}+\gamma} \otimes R_{\bar{\gamma}_{21}}) & R_{\gamma_{12}+\gamma'} \\ R_{\bar{\gamma}_{21}} \oplus R_{\bar{\gamma}_{21}+\gamma} \oplus (R_{\bar{\gamma}_{21}+\gamma'} \otimes R_{\gamma_{12}+\gamma} \otimes R_{\bar{\gamma}_{21}}) & \mathbb{Z} \oplus (R_{\bar{\gamma}_{21}+\gamma} \otimes R_{\gamma_{12}+\gamma}) \end{pmatrix} \quad (3.165)$$

It is a good exercise to draw the Lefschetz thimbles, and to work out the framed BPS complexes of the brane $\mathfrak{B}_{n,m}$ whose support winds around γ n -times and γ' m -times.

Class S theories with trivial DSZ pairing

Any four-dimensional, $\mathcal{N} = 2$ theory of class S such that the charge lattice of electromagnetic + flavor charges Γ has a trivial DSZ pairing $\langle , \rangle = 0$, coupled to its canonical surface defect is a theory to which our framework applies. The R_{γ_i} are then spaces of 4d BPS particles, and the $R_{\gamma_{ij}}$ consist of spaces of 2d-4d solitons. The simplest example of such a theory is the free hypermultiplet.

3.3 Boundary Amplitudes and Framed Wall-Crossing

3.3.1 S-Walls

In this section we review framed wall-crossing in the absence of twisted masses. As in the introduction, fix a phase $\zeta_{\mathfrak{B}}$ and let \mathfrak{B} be a supersymmetric boundary condition (placed at the boundary of the right half plane ⁵ i.e at $x = 0$ where $x \geq 0$) that preserves $Q_{\text{BPS}}(\zeta_{\mathfrak{B}})$.

⁵Might need to switch conventions. Either replace $\text{Im}(\zeta^{-1} Z_{ij})$ to $\text{Re}(\zeta^{-1} Z_{ij})$ or work with the upper-half plane instead.

Let $\mathcal{E}_i(\mathfrak{B})$ be the \mathbb{Z} -graded module of classical (\mathfrak{B}, i) framed BPS states, and let R_{ij} be the \mathbb{Z} -graded module of classical ij vanilla BPS states.

As stressed in the introduction, a categorical discussion of framed wall-crossing involves not only the framed and vanilla BPS spaces $\mathcal{E}_i(\mathfrak{B}), R_{ij}$, but forces us to also include a boundary amplitude \mathcal{B} . The essential ingredients of S -wall crossing can be captured in a theory with just two vacua, $\mathbb{V} = \{i, j\}$. Suppose we are at a region of parameter space with $\text{Im}(\zeta_{\mathfrak{B}}^{-1} Z_{ij}) > 0$ (the unprimed side). In order to discuss the boundary amplitude \mathcal{B} , one has to consider the space of self-morphisms of the brane \mathfrak{B} which read

$$\text{Hop}(\mathfrak{B}, \mathfrak{B}) = \oplus_{a \in \mathbb{V}} \mathcal{E}_a(\mathfrak{B}) \otimes \widehat{R}_{ab} \otimes \mathcal{E}_b^\vee(\mathfrak{B}). \quad (3.166)$$

When there are only two vacua with $\zeta_{\mathfrak{B}}^{-1} Z_{ij}$ lying in our chosen half-plane (so we have $\text{Re}(\zeta_{\mathfrak{B}}^{-1} Z_{ij}) > 0$), the different direct sum components of $\text{Hop}(\mathfrak{B}, \mathfrak{B})$ can be organized into a 2×2 matrix

$$\text{Hop}(\mathfrak{B}, \mathfrak{B}) = \begin{pmatrix} \text{End}(\mathcal{E}_i(\mathfrak{B})) & \mathcal{E}_i(\mathfrak{B}) \otimes R_{ij} \otimes (\mathcal{E}_j(\mathfrak{B}))^\vee \\ 0 & \text{End}(\mathcal{E}_j(\mathfrak{B})). \end{pmatrix} \quad (3.167)$$

Recall from Appendix C.1 of [GMW], that the interior amplitude $\beta_{ij} \in R_{ij} \otimes R_{ji}$ in a theory with two vacua is equivalent, by the use of the non-degenerate pairing $K_{ij} : R_{ij} \otimes R_{ji} \rightarrow \mathbb{Z}$, to a differential $d_{ij} : R_{ij} \rightarrow R_{ij}$ so that $d_{ij}^2 = 0$. This turns the morphism space $\text{Hop}(\mathfrak{B}, \mathfrak{B})$ to a differential-graded algebra, where the differential only acts on the 12 component via d_{ij} while acting trivially on the framed BPS spaces, and the multiplication comes from matrix multiplication combined with compositions of linear maps. \mathcal{B} is then a degree one element of $\text{Hop}(\mathfrak{B}, \mathfrak{B})$

$$\mathcal{B} = \begin{pmatrix} d_{\mathcal{E}_i} & \mathcal{B}_{ij} \\ 0 & d_{\mathcal{E}_j} \end{pmatrix} \quad (3.168)$$

which satisfies the Maurer-Cartan equation

$$d_{ij}\mathcal{B} + \mathcal{B}^2 = 0. \quad (3.169)$$

The Maurer-Cartan equation in turn can be translated into saying that $d_{\mathcal{E}_i}$ and $d_{\mathcal{E}_j}$ are differentials on $\mathcal{E}_i(\mathfrak{B})$ and $\mathcal{E}_j(\mathfrak{B})$ respectively, thus turning the framed BPS \mathbb{Z} -modules into chain complexes, and that the degree shifted map

$$\mathcal{B}_{ij} : \mathcal{E}_j^{[1]}(\mathfrak{B}) \rightarrow \mathcal{E}_i(\mathfrak{B}) \otimes R_{ij} \quad (3.170)$$

is a chain map between the complexes $(\mathcal{E}_j^{[1]}(\mathfrak{B}), -d_{\mathcal{E}_j})$ and $(\mathcal{E}_i(\mathfrak{B}) \otimes R_{ij}, d_{\mathcal{E}_i} \otimes 1 + 1 \otimes d_{ij})$.

We are now ready to state the framed wall-crossing formula. Define an S_{ij} wall to be the locus where

$$\frac{Z_{ij}}{\zeta_{\mathfrak{B}}} \in i\mathbb{R}_+. \quad (3.171)$$

Let us cross an S_{ij} wall to go to a region of parameter space with $\text{Re}(\zeta_{\mathfrak{B}}^{-1}Z_{ij}) < 0$ (the primed side). The categorical wall-crossing formula for framed BPS states says that in crossing the wall from the positive side to the negative side, the homotopy class of the BPS complexes jumps to

$$R'_{ij} \simeq R_{ij}, \quad (3.172)$$

$$\mathcal{E}'_i(\mathfrak{B}) \simeq \mathcal{E}_i(\mathfrak{B}), \quad (3.173)$$

$$\mathcal{E}'_j(\mathfrak{B}) \simeq \text{Cone}(\mathcal{B}_{ij} : \mathcal{E}_j^{[1]}(\mathfrak{B}) \rightarrow \mathcal{E}_i(\mathfrak{B}) \otimes R_{ij}). \quad (3.174)$$

Moreover, the primed boundary amplitude, which is now valued in⁶

$$\mathcal{B}'_{ji} \in \mathcal{E}'_j(\mathfrak{B}) \otimes R'_{ji} \otimes (\mathcal{E}'_i(\mathfrak{B}))^\vee \quad (3.175)$$

⁶We remind the reader that the perfect pairing $K_{ij} : R_{ij} \otimes R_{ji} \rightarrow \mathbb{Z}$, provides an isomorphism $R_{ji} \cong (R_{ij}^\vee)^{[1]}$.

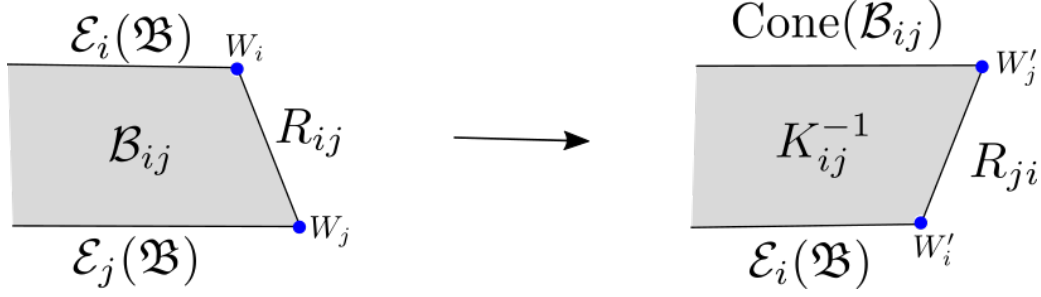


Figure 3.7: Summary of framed S -wall crossing in the W -plane. The horizontal lines represent images of framed BPS solitons in the W -plane, whereas the lines connecting W_i and W_j are images of the vanilla solitons in the W -plane. The shaded region represents the image of a framed ζ -instanton.

i.e valued in

$$(\mathcal{E}_j(\mathfrak{B}) \otimes R_{ji} \otimes \mathcal{E}_i^\vee(\mathfrak{B})) \oplus (\text{End}(\mathcal{E}_i(\mathfrak{B})) \otimes (R_{ij} \otimes R_{ji})) \quad (3.176)$$

is given by

$$\mathcal{B}'_{ij} = \begin{pmatrix} 0 \\ \text{id} \otimes K_{ij}^{-1} \end{pmatrix}. \quad (3.177)$$

This wall-crossing formula is summarized in the Figure 3.7.

Conversely suppose that we begin with a region where $\text{Re}(\zeta^{-1}Z_{ij}) < 0$ (now, the unprimed side) and cross an S_{ij} -wall to go over to the side with $\text{Re}(\zeta^{-1}Z_{ij}) > 0$ (now, the primed side). The boundary amplitude is now valued in

$$\mathcal{B}_{ji} \in \mathcal{E}_j(\mathfrak{B}) \otimes R_{ji} \otimes \mathcal{E}_i^\vee(\mathfrak{B}), \quad (3.178)$$

and defines a chain map

$$\mathcal{B}_{ij} : (R_{ji}^\vee \otimes \mathcal{E}_i(\mathfrak{B}))^{[1]} \rightarrow \mathcal{E}_j(\mathfrak{B}). \quad (3.179)$$

In doing this move, the BPS complexes jump to

$$R'_{ji} \simeq R_{ji}, \quad (3.180)$$

$$\mathcal{E}'_i(\mathfrak{B}) \simeq \mathcal{E}_i(\mathfrak{B}), \quad (3.181)$$

$$\mathcal{E}'_j(\mathfrak{B}) \simeq \text{Cone}(\mathcal{B}_{ji} : (R'_{ji} \otimes \mathcal{E}_i(\mathfrak{B}))^{[1]} \rightarrow \mathcal{E}_j(\mathfrak{B})). \quad (3.182)$$

The primed boundary amplitude, which is valued in

$$\mathcal{B}'_{ij} \in \mathcal{E}'_i(\mathfrak{B}) \otimes R'_{ij} \otimes (\mathcal{E}'_j(\mathfrak{B}))^\vee \quad (3.183)$$

is given by

$$\mathcal{B}'_{ij} = \begin{pmatrix} 0 \\ \text{id} \otimes K_{ij}^{-1} \end{pmatrix}. \quad (3.184)$$

That these two moves are inverses of each other follows essentially from the mapping cylinder construction.

Example: Cubic LG Model

Let's illustrate framed wall-crossing in perhaps the simplest example. Consider the cubic Landau-Ginzburg model

$$W = \frac{1}{3}\phi^3 - t\phi \quad (3.185)$$

where t is a complex parameter. The two vacua $\mathbb{V} = \{1, 2\}$ are located at $\phi_1 = \sqrt{t}$ and $\phi_2 = -\sqrt{t}$ with critical values $W_1 = -\frac{2}{3}t^{\frac{3}{2}}$, and $W_2 = \frac{2}{3}t^{\frac{3}{2}}$, and therefore $Z_{12} = -\frac{4}{3}t^{\frac{3}{2}}$. We consider branes for this theory at $\zeta = 1$. They are supported on one-dimensional submanifolds of \mathbb{C} along which $\text{Re}(W) \rightarrow +\infty$. The regions where $\text{Re}(W)$ goes to positive infinity, along with some examples of one-dimensional submanifolds on which branes can be supported are depicted in the left of Figure 3.8.

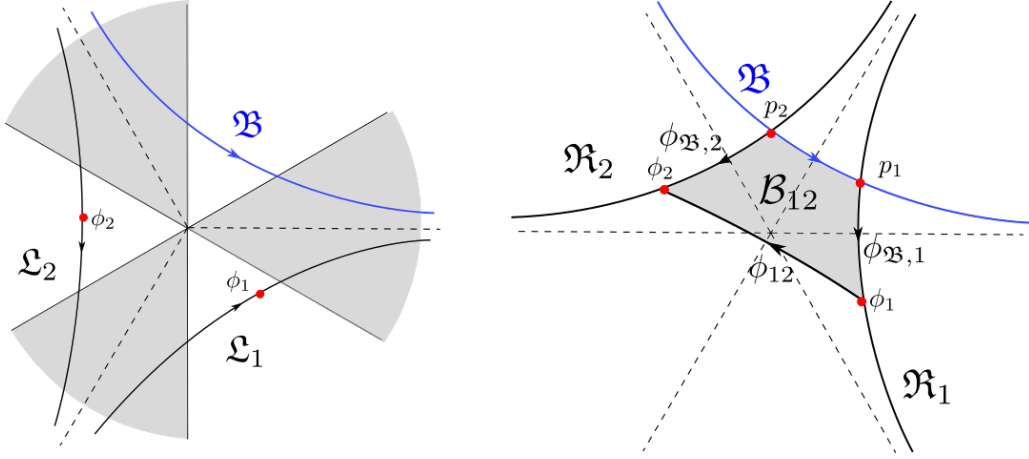


Figure 3.8: Left: the shaded regions are regions in which the support of a brane with $\zeta = 1$ goes off to infinity. $\mathfrak{L}_1, \mathfrak{L}_2$ are the two thimbles at $t = e^{-i\epsilon}$ and \mathfrak{B} is the support of the brane whose framed BPS states we are studying. Right: the intersection points of \mathfrak{B} with the right thimbles $\mathfrak{R}_{1,2}$ are denoted as $p_{1,2}$. We also show the paths traced out by the framed BPS solitons $\phi_{\mathfrak{B},1}$ and $\phi_{\mathfrak{B},2}$ along with the vanilla soliton ϕ_{12} . Finally the shaded region is the image of the framed ζ -instanton that gives rise to the boundary amplitude for \mathfrak{B} .

The S_{12} wall is the locus where $t^{\frac{3}{2}}$ is real and positive. We begin with $t = e^{-i\epsilon}$ and move across the S_{12} wall to $t' = e^{i\epsilon}$ where ϵ is a small positive real number.

Consider the brane \mathfrak{B} whose support is depicted in the left of Figure 3.8. At $t = e^{-i\epsilon}$, the support of \mathfrak{B} has one intersection point for each right Lefschetz thimble $\mathfrak{R}_1, \mathfrak{R}_2$. Flowing from these intersection points to the critical points along the right thimbles, we see that the framed BPS complexes $\mathcal{E}_1(\mathfrak{B})$ and $\mathcal{E}_2(\mathfrak{B})$ are one-dimensional, generated by $\phi_{\mathfrak{B},1}$ and $\phi_{\mathfrak{B},2}$ respectively. The fermion degrees can be chosen to vanish mod 2 by choosing the orientations of the branes \mathfrak{B} and the right thimbles so that they give intersection numbers to be +1. We go ahead and postulate that the fermion degrees vanish not only in $\mathbb{Z} \bmod 2$, but in \mathbb{Z} . Thus the framed BPS complexes for \mathfrak{B} are

$$\mathcal{E}_1(\mathfrak{B}) = \mathbb{Z}, \quad \mathcal{E}_2(\mathfrak{B}) = \mathbb{Z}. \quad (3.186)$$

Independent of the parameter t we also have a single vanilla 12 soliton, ϕ_{12} , and with the orientations of thimbles fixed as above, the fermion degree is 1 mod 2, and we postulate that

$$R_{12} = \mathbb{Z}^{[1]}. \quad (3.187)$$

Finally with this choice of \mathfrak{B} at $t = e^{i\epsilon}$ there is also (conjecturally) a single framed rigid ζ -instanton: a map from $\mathbb{R}_+ \times \mathbb{R}$ to the target space \mathbb{C} satisfying the ζ -instanton equation, with boundary conditions specified by the brane \mathfrak{B} and the half-plane fan of solitons

$$(\phi_{\mathfrak{B},1}, \phi_{12}, \phi_{2,\mathfrak{B}}). \quad (3.188)$$

The brane \mathfrak{B} , the soliton paths $\phi_{\mathfrak{B},1}, \phi_{\mathfrak{B},2}, \phi_{12}$ and the image of the conjectural ζ -instanton are depicted on the right of Figure 3.8. With the existence of this ζ -instanton postulated, the boundary amplitude of the brane \mathfrak{B}

$$\mathcal{B}_{12} \in \mathcal{E}_1(\mathfrak{B}) \otimes R_{12} \otimes \mathcal{E}_2(\mathfrak{B})^\vee \cong \mathbb{Z}^{[1]} \quad (3.189)$$

is given by

$$\mathcal{B}_{12} = \phi_{\mathfrak{B},1} \otimes \phi_{12} \otimes \phi_{\mathfrak{B},2}^\vee. \quad (3.190)$$

Having specified the BPS complexes and boundary amplitude on one side of the wall, we can go ahead and discuss what happens on the other. The framed wall-crossing formula says that in moving from $t = e^{-i\epsilon}$ to $t' = e^{i\epsilon}$ the framed BPS Hilbert space $\mathcal{E}_1(\mathfrak{B})$ is unchanged

$$\mathcal{E}'_1(\mathfrak{B}) \simeq \mathcal{E}_1(\mathfrak{B}) = \mathbb{Z} \quad (3.191)$$

whereas

$$\mathcal{E}'_2(\mathfrak{B}) \simeq \text{Cone}(\mathcal{B}_{12} : \mathcal{E}_2^{[1]}(\mathfrak{B}) \rightarrow \mathcal{E}_1(\mathfrak{B}) \otimes R_{12}) \quad (3.192)$$

$$= \text{Cone}(\text{Id} : \mathbb{Z}^{[1]} \rightarrow \mathbb{Z}^{[1]}) \quad (3.193)$$

$$\simeq 0. \quad (3.194)$$

Therefore the cohomology of the primed spaces for \mathfrak{B} i.e the physical framed BPS Hilbert spaces are precisely identical to those of a left thimble at t' for the vacuum ϕ_1 . Indeed by using the formula for \mathcal{B}'_{21} , it is not hard to show that \mathfrak{B} is homotopy equivalent to the left thimble \mathfrak{L}'_1 at $t' = e^{i\epsilon}$.

S-Wall Interface

The framed wall-crossing formula we stated and illustrated above can be derived from the action of an appropriate “wall-crossing interface” that acts as a domain wall between the theory \mathcal{T} with $\text{Re}(\zeta^{-1}Z_{ij}) > 0$ and the theory \mathcal{T}' with $\text{Re}(\zeta^{-1}Z_{ij}) < 0$. This wall-crossing interface was constructed in [GMW] where it is known as an S_{ij} -wall interface. As usual, $i \neq j$ throughout this section.

Generalizing the formalism of branes, which are characterized by the framed spaces $\mathcal{E}_i(\mathfrak{B})$ and the boundary amplitude $\mathcal{B} \in \text{Hom}(\mathfrak{B}, \mathfrak{B})$, an interface \mathfrak{I}_{LR} between the theory \mathcal{T}_{L} with vacua $i \in \mathbb{V}_{\text{L}}$ and \mathcal{T}_{R} with vacua $a \in \mathbb{V}_{\text{R}}$ is constructed by providing \mathbb{Z} -modules

$$\mathcal{E}_{ia}(\mathfrak{I}_{\text{LR}}). \quad (3.195)$$

One can then show that the data of the interior amplitudes β_L, β_R along with taut interface webs \mathfrak{t}_{LR} provides one with an A_∞ -algebra structure on the space

$$\text{Hop}(\mathfrak{I}_{\text{LR}}, \mathfrak{I}_{\text{LR}}) = \bigoplus_{\substack{i,j \in \mathbb{V}_{\text{L}} \\ a,b \in \mathbb{V}_{\text{R}}}} \mathcal{E}_{ia}(\mathfrak{I}_{\text{LR}}) \otimes \hat{R}_{ji}^{\text{opp}} \otimes \hat{R}_{ab} \otimes \mathcal{E}_{jb}^\vee(\mathfrak{I}_{\text{LR}}), \quad (3.196)$$

where we recall that for a given theory, say \mathcal{T}_{L} the \hat{R}_{ab} spaces are defined via the factorization formula

$$\bigoplus_{a,b \in \mathbb{V}_{\text{L}}} \hat{R}_{ab} e_{ab} = \bigotimes_{\text{Re } Z_{ab} > 0}^{\curvearrowright} (\mathbb{Z}\mathbf{1} \oplus R_{ab} e_{ab}), \quad (3.197)$$

where \curvearrowright denotes clockwise ordering on the product, and \widehat{R}^{opp} denotes the clockwise ordered product in the opposite half-plane, namely the half-plane with $\text{Re } Z_{ab} < 0$. An interface amplitude \mathcal{B}_{LR} by definition is a Maurer-Cartan element of this A_∞ -algebra. The pair $(\{\mathcal{E}_{ia}(\mathfrak{I}_{LR})\}, \mathcal{B}_{LR})$ then characterizes an interface between the theories $(\mathcal{T}_L, \mathcal{T}_R)$. Note that the theory of left (right) branes come about when the right (left) theory is the trivial theory.

Perhaps the simplest non-trivial example of an interface consists of the identity interface $\mathfrak{I}\mathfrak{D}$ between a theory \mathcal{T} and itself. The Chan-Paton spaces are simply

$$\mathcal{E}_{ij}(\mathfrak{I}\mathfrak{D}) = \delta_{ij}\mathbb{Z}, \quad (3.198)$$

whereas the morphism space

$$\text{Hop}(\mathfrak{I}\mathfrak{D}, \mathfrak{I}\mathfrak{D}) = \oplus_{i,j} \widehat{R}_{ji}^{\text{opp}} \otimes \widehat{R}_{ij} \quad (3.199)$$

is a categorification of $\text{Tr}(S^{-t}S)$. In particular we have $R_{ji} \otimes R_{ij} \subset \text{Hop}(\mathfrak{I}\mathfrak{D}, \mathfrak{I}\mathfrak{D})$ for each pair of vacua $i \neq j$. The interface amplitude is given simply by

$$\mathcal{B} = \oplus_{i \neq j} K_{ji}^{-1}. \quad (3.200)$$

We now come to the S_{ij} wall-crossing interfaces. Suppose we have $\mathcal{T}_L, \mathcal{T}_R$ theories with two vacua separated by an S_{ij} -wall. We define an interface \mathfrak{S}_{ij} as follows. The Chan-Paton spaces are given by

$$\mathcal{E}_{ii}(\mathfrak{S}_{ij}) = \mathbb{Z}, \quad \mathcal{E}_{ij}(\mathfrak{S}_{ij}) = R_{ij}, \quad (3.201)$$

$$\mathcal{E}_{ji}(\mathfrak{S}_{ij}) = 0, \quad \mathcal{E}_{jj}(\mathfrak{S}_{ij}) = \mathbb{Z}. \quad (3.202)$$

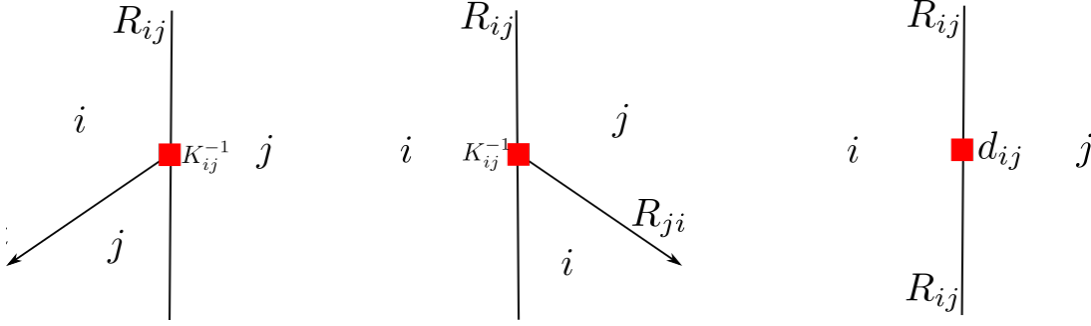


Figure 3.9: The different components for the interface amplitude of the S_{ij} -wall interface \mathfrak{S}_{ij} .

The interface amplitude $\mathcal{B}(\mathfrak{S}_{ij})$ is given as follows. Letting \mathfrak{e}_{ij} denote the basic thimble interface,⁷ the non-trivial morphism spaces for \mathfrak{S}_{ij} come from

$$\text{Hop}(R_{ij}\mathfrak{e}_{ij}, \mathfrak{e}_{ii}) = R_{ij} \otimes R_{ji}, \quad (3.203)$$

$$\text{Hop}(R_{ij}\mathfrak{e}_{ij}, \mathfrak{e}_{jj}) = R_{ij} \otimes R_{ji}, \quad (3.204)$$

$$\text{Hop}(R_{ij}\mathfrak{e}_{ij}, R_{ij}\mathfrak{e}_{ij}) = \text{End}(R_{ij}). \quad (3.205)$$

We let $\mathcal{B}(\mathfrak{S}_{ij})$ to be the direct sum of K_{ij}^{-1}, K_{ij}^{-1} and d_{ij} that live in each of these spaces respectively

$$\mathcal{B}(\mathfrak{S}_{ij}) = K_{ij}^{-1} \oplus K_{ij}^{-1} \oplus d_{ij}. \quad (3.206)$$

See Figure 3.9 for an illustration of this amplitude.

One of the main properties of interfaces is that they can be composed. This was explained in Section 6.2 of [GMW] for the case without twisted masses. Given three theories $(\mathcal{T}_L, \mathcal{T}_M, \mathcal{T}_R)$ along with interfaces $\mathfrak{J}_{LM}, \mathfrak{J}_{MR}$ between these theories, one can construct an

⁷In more detail a thimble \mathfrak{e}_{ia} interface between \mathcal{T}_L with vacua $i \in \mathbb{V}_L$ and \mathcal{T}_R with vacua $a \in \mathbb{V}_R$ is the interface with Chan-Paton spaces $\mathcal{E}_{jb}(\mathfrak{e}_{ia}) = \delta_{ij}\delta_{ab}\mathbb{Z}$, and trivial interface amplitude $\mathcal{B}(\mathfrak{e}_{ia}) = 0$

interface

$$\mathfrak{I}_{\text{LM}} \boxtimes \mathfrak{I}_{\text{MR}} \in \mathfrak{Br}(\mathcal{T}_{\text{L}}, \mathcal{T}_{\text{R}}). \quad (3.207)$$

The Chan-Paton spaces of the product interface is given by

$$\mathcal{E}_{ia}(\mathfrak{I}_{\text{LM}} \boxtimes \mathfrak{I}_{\text{MR}}) = \bigoplus_{\alpha \in \mathbb{V}_{\text{M}}} \mathcal{E}_{i\alpha}(\mathfrak{I}_{\text{LM}}) \otimes \mathcal{E}_{\alpha a}(\mathfrak{I}_{\text{MR}}). \quad (3.208)$$

The interface amplitude $\mathcal{B}(\mathfrak{I}_{\text{LM}} \boxtimes \mathfrak{I}_{\text{MR}})$ is given by summing over all taut webs for the (L, M, R) system and plugging in all possible bulk and interface amplitudes for the theories $(\mathcal{T}_{\text{L}}, \mathcal{T}_{\text{R}}, \mathcal{T}_{\text{M}})$:

$$\mathcal{B}(\mathfrak{I}_{\text{LM}} \boxtimes \mathfrak{I}_{\text{MR}}) = \rho(\mathfrak{t}_{\text{LMR}}) \left(e^{\beta_{\text{L}}}, \frac{1}{1 - \mathcal{B}_{\text{LM}}}, e^{\beta_{\text{M}}}, \frac{1}{1 - \mathcal{B}_{\text{MR}}}, e^{\beta_{\text{R}}} \right). \quad (3.209)$$

The framed wall-crossing formula comes about when we consider \mathcal{T}_{L} to be the trivial theory, and $\mathcal{T}_{\text{M}}, \mathcal{T}_{\text{R}}$ to be \mathcal{T} and \mathcal{T}' respectively, and for a given brane \mathfrak{B} of \mathcal{T} , we consider the composite brane

$$\mathfrak{B} \boxtimes \mathfrak{S}_{ij} \quad (3.210)$$

as a brane in the theory \mathcal{T}' . Letting $\mathcal{E}'_{i,j}(\mathfrak{B}) = \mathcal{E}_{i,j}(\mathfrak{B} \boxtimes \mathfrak{S}_{ij})$, we find by the rule for composing Chan-Paton spaces that

$$\begin{pmatrix} \mathcal{E}'_i(\mathfrak{B}) & \mathcal{E}'_j(\mathfrak{B}) \end{pmatrix} = \begin{pmatrix} \mathcal{E}_i(\mathfrak{B}) & \mathcal{E}_j(\mathfrak{B}) \end{pmatrix} \begin{pmatrix} \mathbb{Z} & R_{ij} \\ 0 & \mathbb{Z} \end{pmatrix}, \quad (3.211)$$

so that we have that the \mathbb{Z} -modules are given by

$$\mathcal{E}'_i(\mathfrak{B}) = \mathcal{E}_i(\mathfrak{B}), \quad (3.212)$$

$$\mathcal{E}'_j(\mathfrak{B}) = \mathcal{E}_j(\mathfrak{B}) \oplus \mathcal{E}_i(\mathfrak{B}) \otimes R_{ij}. \quad (3.213)$$

In particular, the latter is indeed the underlying \mathbb{Z} -module of the Cone complex. Next we come to working out the boundary amplitude \mathcal{B}' of the brane $\mathfrak{B} \boxtimes \mathfrak{S}_{ij}$. The main thing to

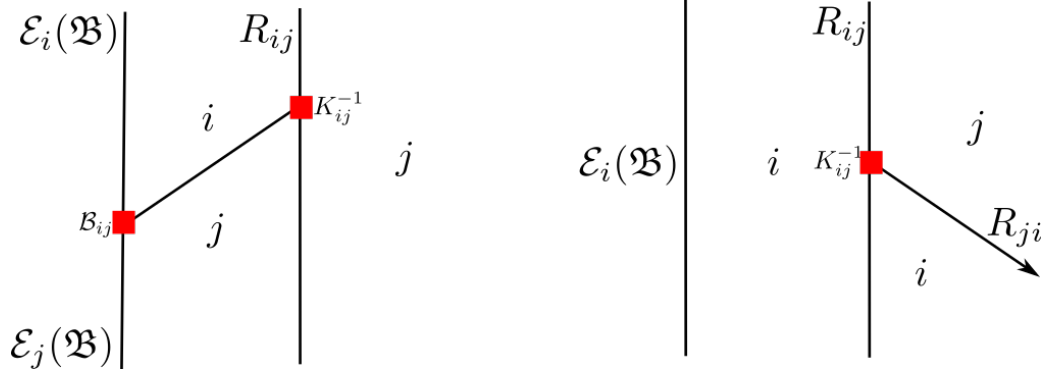


Figure 3.10: Left: the web that contributes the off-diagonal component of the mapping cone complex. Right: the taut web that gives the off-diagonal component \mathcal{B}'_{ji} .

check is that the differential on $\mathcal{E}'_j(\mathfrak{B})$ indeed coincides with the Cone differential for \mathcal{B}_{ij} . First note that the natural differentials on $\mathcal{E}_j(\mathfrak{B})$ and $\mathcal{E}_i(\mathfrak{B}) \otimes R_{ji}$ are transported to \mathcal{T}' identically. On the other hand, the off-diagonal component of the Cone differential comes from the contribution to \mathcal{B}' of the composite web shown in the left of Figure 3.10. The web depicted in the right of the same Figure also tells us that the off-diagonal component \mathcal{B}'_{ji} of the boundary amplitude is given by $\text{id} \otimes K_{ij}^{-1}$ as stated in the wall-crossing formula. Thus we've seen that the wall-crossing formula is equivalent to fusion of the brane \mathfrak{B} with the wall-crossing interface \mathfrak{S}_{ij} .

Let's also work out the converse construction namely an interface between \mathcal{T}' and \mathcal{T} that would give us the wall-crossing formula for crossing an S_{ij} -wall in the opposite way. The \mathfrak{S}'_{ij} interface has Chan-Paton data

$$\mathcal{E}_{ii}(\mathfrak{S}'_{ij}) = \mathbb{Z}, \quad \mathcal{E}_{ij}(\mathfrak{S}'_{ij}) = R_{ji}^\vee, \quad (3.214)$$

$$\mathcal{E}_{ji}(\mathfrak{S}'_{ij}) = 0, \quad \mathcal{E}_{jj}(\mathfrak{S}'_{ij}) = \mathbb{Z}. \quad (3.215)$$

The non-trivial morphism spaces now come from

$$\text{Hop}(\mathfrak{e}_{ii}, R_{ji}^\vee \mathfrak{e}_{ij}) = R_{ij} \otimes R_{ji}, \quad (3.216)$$

$$\text{Hop}(\mathfrak{e}_{jj}, R_{ji}^\vee \mathfrak{e}_{ij}) = R_{ij} \otimes R_{ji}, \quad (3.217)$$

$$\text{Hop}(R_{ji}^\vee \mathfrak{e}_{ij}, R_{ji}^\vee \mathfrak{e}_{ij}) = \text{End}(R_{ji}), \quad (3.218)$$

and we let the interface amplitude for \mathfrak{S}'_{ij} be

$$\mathcal{B}(\mathfrak{S}'_{ij}) = K_{ij}^{-1} \oplus K_{ij}^{-1} \oplus d_{ji}. \quad (3.219)$$

Showing that $\mathfrak{B} \boxtimes \mathfrak{S}'_{ij}$ has Chan-Paton spaces and boundary amplitude as expressed in the framed wall-crossing formula involves similar steps to the demonstration for \mathfrak{S}_{ij} .

Finally it is instructive to check that these two interfaces are inverses of each other. Namely, we would like to show that

$$\mathfrak{S}_{ij} \boxtimes \mathfrak{S}'_{ij} \simeq \mathfrak{Id}, \quad \mathfrak{S}'_{ij} \boxtimes \mathfrak{S}_{ij} \simeq \mathfrak{Id}' \quad (3.220)$$

where \mathfrak{Id} and \mathfrak{Id}' are the identity interfaces for the theories \mathcal{T} and \mathcal{T}' respectively, and \simeq denotes the homotopy equivalence of interfaces. We have

$$\mathcal{E}(\mathfrak{S}_{ij} \boxtimes \mathfrak{S}'_{ij}) = \begin{pmatrix} \mathbb{Z} & R_{ij} \oplus R_{ji}^\vee \\ 0 & \mathbb{Z} \end{pmatrix}. \quad (3.221)$$

A necessary condition for the homotopy equivalence of $\mathfrak{S}_{ij} \boxtimes \mathfrak{S}'_{ij}$ with \mathfrak{Id} is that the Chan-Paton spaces are homotopy equivalent to the Chan-Paton spaces for \mathfrak{Id} . In other words, the interface amplitude $\mathcal{B}(\mathfrak{S}_{ij} \boxtimes \mathfrak{S}'_{ij})$ should equip the off-diagonal Chan-Paton space $R_{ij} \oplus R_{ji}^\vee$ with a differential so that the complex becomes homotopy equivalent to the trivial one. Indeed, there is a taut composite web, which evaluated on the interface amplitudes for \mathfrak{S}_{ij} and \mathfrak{S}'_{ij} lead to such a differential. With a little more effort one can show the full homotopy equivalence. See Section 7.6 of [GMW] for a complete argument.

3.3.2 K-Walls

We now discuss novel wall-crossing phenomenon in the presence of non-trivial twisted masses. We discuss framed wall-crossing across a so-called “ K -wall”. By definition, a K_γ wall is the region of parameter space where

$$\frac{Z_\gamma}{\zeta_{\mathfrak{B}}} \in \mathbb{R}_-. \quad (3.222)$$

Such walls are only present when a non-zero twisted mass is present in the theory. The discussion of K -wall crossing is simplest, while capturing all the essential ingredients, if we assume there is only a single vacuum $\mathbb{V} = \{i\}$ and $\Gamma = \mathbb{Z}\langle\gamma\rangle$. Let

$$\mathcal{E}_i(\mathfrak{B}) = \oplus_{\gamma' \in \Gamma} \mathcal{E}_{i,\gamma'}(\mathfrak{B}) \quad (3.223)$$

be the bi-graded chain complex of (\mathfrak{B}, i) BPS particles. Let R_{γ_i} be the complex of ii -BPS particles of charge γ . Suppose we are at a region of parameter space with $\text{Im}(\frac{Z_\gamma}{\zeta_{\mathfrak{B}}}) < 0$. We propose that the boundary amplitude for \mathfrak{B} is degree $(1, 0)$ element valued \mathcal{B}_i valued in

$$\text{Hop}(\mathfrak{B}, \mathfrak{B}) = \mathcal{E}_i(\mathfrak{B}) \otimes \mathcal{F}[R_{\gamma_i}] \otimes \mathcal{E}_i^\vee(\mathfrak{B}), \quad (3.224)$$

where we recall that $\mathcal{F}[R_{\gamma_i}]$ denotes the graded Fock space of R_{γ_i} , the space of classical BPS states of flavor charge γ_{ii} . The framed wall-crossing formula states that upon moving across a K_γ -wall to a region where $\text{Im}(\zeta_{\mathfrak{B}}^{-1} Z_\gamma) > 0$, the framed BPS chain complex jumps to

$$\mathcal{E}'_i(\mathfrak{B}) \simeq (\mathcal{E}_i(\mathfrak{B}) \otimes \mathcal{F}[R_{\gamma_i}], d_{\mathcal{B}}), \quad (3.225)$$

where the differential $d_{\mathcal{B}}$ is given as follows. Note that \mathcal{B}_i is valued in $\text{End}(\mathcal{E}_i(\mathfrak{B})) \otimes \mathcal{F}[R_{\gamma_i}]$. This is a differential graded algebra where the differentials are the natural ones inherited from $d_{\mathcal{E}_i(\mathfrak{B})}$ and d_{γ_i} . The multiplication on this space is given by the natural multiplication

on the tensor product of a (graded) symmetric algebra and a matrix algebra. \mathcal{B}_i obeys the natural Maurer-Cartan equation

$$[d_{\mathcal{E}_i(\mathfrak{B})}, \mathcal{B}_i] + d_{\gamma_i} \mathcal{B}_i + \mathcal{B}_i^2 = 0. \quad (3.226)$$

Note that \mathcal{B}_i can be naturally extended to a map

$$\mathcal{B}_i : \mathcal{E}_i(\mathfrak{B}) \otimes \mathcal{F}[R_{\gamma_i}] \rightarrow \mathcal{E}_i(\mathfrak{B}) \otimes \mathcal{F}[R_{\gamma_i}]. \quad (3.227)$$

The differential $d_{\mathcal{B}}$ is then given by

$$d_{\mathcal{B}} = d_{\mathcal{E}_i(\mathfrak{B})} + d_{\gamma_i} + \mathcal{B}_i \quad (3.228)$$

which is nilpotent courtesy of the Maurer-Cartan equation. We also state the wall-crossing formula for the boundary amplitude. We need to state a formula for

$$\mathcal{B}'_i \in \text{End}(\mathcal{E}'_i(\mathfrak{B})) \otimes \mathcal{F}[R_{-\gamma_i}], \quad (3.229)$$

where the latter space is

$$\text{End}(\mathcal{E}_i(\mathfrak{B})) \otimes \mathcal{F}[R_{\gamma_i}] \otimes \mathcal{F}[R_{-\gamma_i}] \otimes \mathcal{F}[R_{\gamma_i}^\vee] = \text{End}(\mathcal{E}_i(\mathfrak{B})) \otimes \text{End}(\mathcal{F}[R_{\gamma_i}]) \otimes \mathcal{F}[R_{-\gamma_i}]. \quad (3.230)$$

The primed boundary amplitude is

$$\mathcal{B}'_i = \text{Id}_{\mathcal{E}_i(\mathfrak{B})} \otimes (K_{\gamma_i}^{-1})_{ab} \alpha_{\gamma_i}^a e_{-\gamma_i}^b. \quad (3.231)$$

This satisfies the Maurer-Cartan equation mainly due to Bose-Fermi statistics telling us that $(K_{\gamma_i}^{-1})^2 = 0$.

The formulae for going from $\text{Im}(\zeta_{\mathfrak{B}}^{-1} Z_\gamma) > 0$ to a region with $\text{Im}(\zeta_{\mathfrak{B}}^{-1} Z_\gamma) < 0$ are as follows. The boundary amplitude now lives in

$$\mathcal{B}_i \in \text{End}(\mathcal{E}_i(\mathfrak{B})) \otimes \mathcal{F}[R_{-\gamma_i}]. \quad (3.232)$$

The framed BPS complex after crossing a wall is

$$\mathcal{E}'_i(\mathfrak{B}) \simeq (\mathcal{E}_i(\mathfrak{B}) \otimes \mathcal{F}[R_{-\gamma_i}^\vee], d_{\mathcal{B}}), \quad (3.233)$$

where

$$d_{\mathcal{B}} = d_{\mathcal{E}_i(\mathfrak{B})} + d_{-\gamma_i} + \mathcal{B}_i. \quad (3.234)$$

The boundary amplitude now lives in

$$\text{End}(\mathcal{E}_i(\mathfrak{B})) \otimes \text{End}(\mathcal{F}[R_{-\gamma_i}]) \otimes \mathcal{F}[R_{\gamma_i}], \quad (3.235)$$

and is given by

$$\mathcal{B}'_i = \text{Id}_{\mathcal{E}_i(\mathfrak{B})} \otimes (K_{-\gamma_i}^{-1})_{ab} \alpha_{-\gamma_i}^a e_{\gamma_i}^b. \quad (3.236)$$

Again, a consistency check on these formulas is that they should be inverses of each other. We find that

$$\mathcal{E}''_i(\mathfrak{B}) = \mathcal{E}_i(\mathfrak{B}) \otimes \mathcal{F}[R_{\gamma_i}] \otimes \mathcal{F}[R_{-\gamma_i}^\vee]. \quad (3.237)$$

Thanks to the boundary amplitude \mathcal{B}'_i , the differential on $\mathcal{E}''_i(\mathfrak{B})$ is deformed from the usual one precisely by the Koszul differential acting on the latter two factors. By the standard homological algebra of the Koszul complexes we have

$$\mathcal{F}[R_{\gamma_i}] \otimes \mathcal{F}[R_{-\gamma_i}^\vee] \simeq \mathbb{Z}, \quad (3.238)$$

and therefore

$$\mathcal{E}''_i(\mathfrak{B}) \simeq \mathcal{E}_i(\mathfrak{B}). \quad (3.239)$$

Example: Mirror to the Free Chiral

It is very instructive to study framed wall-crossing at a $K_{-\gamma}$ -wall in the simplest model where it occurs: the theory of a free chiral superfield with twisted mass. We work in the mirror formulation where the target space is the punctured complex plane \mathbb{C}^* with coordinate ϕ , and superpotential one-form is

$$dW = \left(\frac{m}{\phi} - 1\right)d\phi. \quad (3.240)$$

As usual, we go to the cover \mathbb{C} via $\phi = e^Y$ so that the single-valued superpotential on \mathbb{C} is

$$W = mY - e^Y. \quad (3.241)$$

The critical points are $Y_k = \log m + 2\pi i k$ with corresponding critical values

$$W_k = m(\log m - 1) + 2\pi i k m \quad (3.242)$$

for $k \in \mathbb{Z}$. We study branes for the theory at $\zeta = 1$. Once again, there are good and bad regions on the Y -plane along which the support of the brane goes to infinity. For $\text{Re}(Y)$ positive and large, the exponential term dominates and so we have to look at regions where $\text{Re}(-e^Y)$ is positive. These is the union of the regions

$$O_k := \left\{ \frac{\pi}{2} + 2\pi k < \text{Im}(Y) < \frac{3\pi}{2} + 2\pi k, \quad \text{Re}(Y) > 0 \right\} \quad (3.243)$$

for k being an integer. Thus any curve connecting O_k with O_l for any $k \neq l$ can support a (left) boundary condition. These regions are independent of m . For $\text{Re}(Y) < 0$ on the other hand, the good region is given as follows. Let $m = |m|e^{i\alpha}$ and consider the half-plane H_α obtained by rotating the right-half plane clockwise by an angle α . The good region R is given by the intersection of H_α with the left-half plane $\text{Re}(Y) < 0$. The union of these two regions, for $m = i$ along with some good curves are depicted in the Figure 3.11.

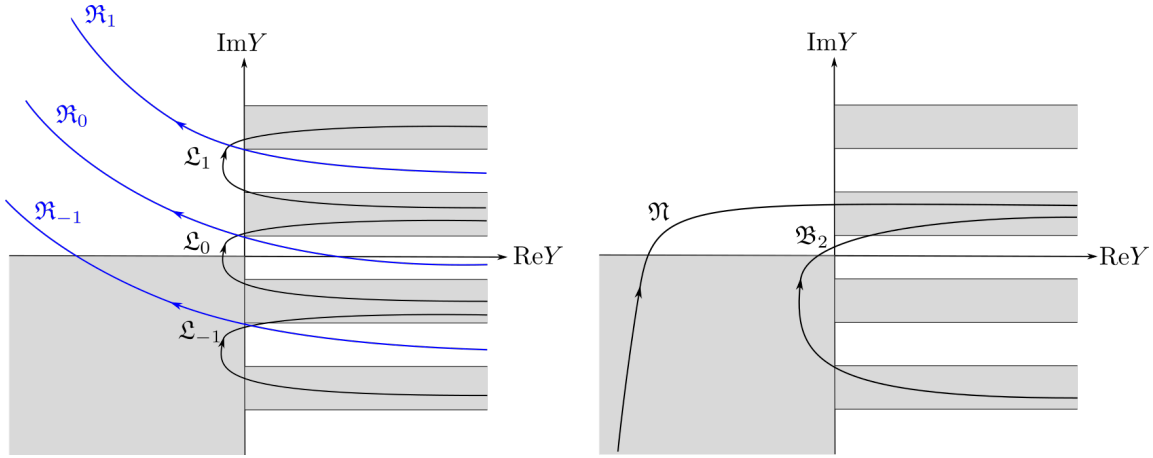


Figure 3.11: Left: The good regions along with $\text{Re}(W) \rightarrow \infty$ for $m = i$ are shaded. Also depicted are some curves homologous to left (depicted in black) and right (depicted in blue) Lefschetz thimbles for $m = ie^{-i\epsilon}$ with $\epsilon > 0$. Right: Some other branes are depicted, the brane \mathfrak{B}_k with $k = 2$, along with the brane \mathfrak{N} .

The $K_{-\gamma}$ wall is located at $Z_{-\gamma} = 2\pi im \in \mathbb{R}_-$ so that it is at $m = ia$ with $a > 0$. We therefore begin with $m = ie^{-i\epsilon}$ and cross over to $m' = ie^{i\epsilon}$ where ϵ is a small positive number. We would like to study framed wall-crossing for a given brane \mathfrak{B} placed on $x = 0$ when the spatial domain is $\mathbb{R}_{\geq 0}$. The branes that are well-defined in the UV are easy to classify given what we have already discussed about the regions where the branes may be supported. Up to the action of the deck group $\Gamma = \mathbb{Z}\langle\gamma\rangle$ (equivariant degree shift) and orientation reversal (fermion degree shift), there are two types of branes. The first type of brane comes in an infinite family labeled by a positive integer k . Its support consists of a curve that connects the region O_0 with the region O_{-k} . The other type of brane connects O_0 with the lower left quadrant. Some examples are depicted in the right of Figure 3.11.

Let's begin discussing the first class of branes. We denote brane whose support connects R_0 with R_{-k} as \mathfrak{B}_k . As usual we have to describe the framed BPS complexes along with the boundary amplitude. The support of \mathfrak{B}_k intersects at a single point with positive

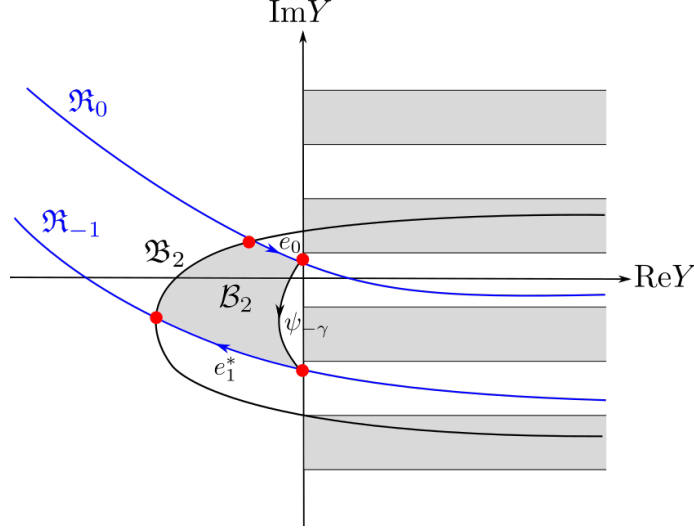


Figure 3.12: The intersection points of the brane \mathfrak{B}_2 with the right thimbles, the paths of the framed and vanilla BPS solitons, and the image of the ζ -instanton that contributes to the boundary amplitude \mathcal{B}_2 .

orientation for each right thimble $\mathfrak{R}_0, \mathfrak{R}_{-1}, \dots, \mathfrak{R}_{-(k-1)}$. Therefore we have that

$$\mathcal{E}_{-n\gamma}(\mathfrak{B}_k) = \mathbb{Z} \text{ for } 0 \leq n \leq k-1 \quad (3.244)$$

and is trivial otherwise. So the framed BPS complex for \mathfrak{B}_k is

$$\mathcal{E}(\mathfrak{B}_k) = \mathbb{Z}\langle e_0, e_1, \dots, e_{k-1} \rangle \quad (3.245)$$

where $\deg(e_n) = (0, -n\gamma)$. To discuss the boundary amplitude, we have to discuss the vanilla BPS complexes. By rotating the left and right Lefschetz thimbles and noticing that only adjacent thimbles intersect, we see that there is a single elementary soliton between adjacent critical points and no others. The fermion degree of the soliton from Y_k to Y_{k+1} can be determined (mod 2) by noting that \mathfrak{L}_0 intersects with \mathfrak{R}_1 with positive orientation. Therefore

$$R_\gamma = \mathbb{Z}, \quad R_{-\gamma} = \mathbb{Z}^{[1]}. \quad (3.246)$$

Because $R_{-\gamma}$ has odd degree, we find that $\mathcal{F}[R_{-\gamma}]$ is a fermionic Fock space in one variable $\psi_{-\gamma}$. The boundary amplitude, valued in the tensor product of an exterior algebra in one variable and a matrix algebra in k variables,

$$\mathcal{B}_k \in \text{End}(\mathbb{Z}^k) \otimes \Lambda^*(\mathbb{Z}\langle\psi_{-\gamma}\rangle) \quad (3.247)$$

is conjectured to be

$$\mathcal{B}_k = \sum_{n=0}^{k-1} e_n \psi_{-\gamma} e_{n+1}^\vee, \quad (3.248)$$

$$= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \psi_{-\gamma}. \quad (3.249)$$

Indeed, the Maurer-Cartan equation, $\mathcal{B}_k^2 = 0$ holds, the cohomological degree of \mathcal{B}_k is +1 and the flavor degree is zero. In terms of ζ -instantons (3.248) is saying that there is a unique rigid ζ -instanton on $\mathbb{R}_+ \times \mathbb{R}$ for each half-plane fan of the form

$$(e_n, \psi_{-\gamma}, e_{n+1}^\vee), \quad (3.250)$$

where $n = 0, \dots, k-1$, and no others. The $k=2$ case, namely the brane \mathfrak{B}_2 , the paths traced out by the solitons $e_0, \psi_{-\gamma}, e_1$ and the image of the ζ -instanton contributing to \mathcal{B}_2 , is illustrated in Figure 3.12. Let's discuss framed wall-crossing for the brane \mathfrak{B}_k . The categorical wall-crossing formula for crossing a $K_{-\gamma}$ -wall says that we have

$$\mathcal{E}'(\mathfrak{B}_k) \simeq \mathcal{E}(\mathfrak{B}_k) \otimes \mathcal{F}[R_{-\gamma}] \quad (3.251)$$

$$= \mathbb{Z}\langle e_0, \dots, e_{k-1} \rangle \otimes \Lambda^*(\mathbb{Z}\langle\psi_{-\gamma}\rangle). \quad (3.252)$$

The latter space is naturally viewed as column vectors with entries valued in the exterior algebra $\Lambda^*(\mathbb{Z}\langle\psi_{-\gamma}\rangle)$. Because the Maurer-Cartan element is non-trivial, $\mathcal{E}'(\mathfrak{B}_k)$ carries a

non-trivial differential given in this case simply by the action of \mathcal{B}_k . Therefore we find that the primed complex is given by

$$\mathcal{E}'(\mathfrak{B}_k) = (\text{Span}\{e_i, \psi_{-\gamma}e_i\}_{i=0,1,\dots,k-1}, d_{\mathcal{B}}(e_i) = e_{i-1}\psi_{-\gamma}.) \quad (3.253)$$

The states $e_i, e_{i-1}\psi_{-\gamma}$ are paired up for $i = k-1, \dots, 1$ and the cohomology is generated by e_0 and $e_{k-1}\psi_{-\gamma}$. Therefore we find that

$$\mathcal{E}'_0(\mathfrak{B}_k) \simeq \mathbb{Z}, \quad (3.254)$$

$$\mathcal{E}'_{-k\gamma}(\mathfrak{B}_k) \simeq \mathbb{Z}^{[1]} \quad (3.255)$$

with all others vanishing. We see that on one side of the wall of marginal stability \mathfrak{B}_k has k framed BPS states, whereas upon crossing it, the spectrum jumps so that there are only two framed BPS states for each k . The primed boundary amplitude can be similarly analyzed. The summary is that the BPS complexes and boundary amplitude after wall-crossing are given by

$$\mathcal{E}'(\mathfrak{B}_k) \simeq \mathbb{Z}\langle f_0, f_k \rangle \quad (3.256)$$

where f_0 has degrees $(0, 0)$ and f_k has degrees $(1, -k\gamma)$, and the boundary amplitude

$$\mathcal{B}'_k \in \text{End}(\mathcal{E}'(\mathcal{B}_k)) \otimes \mathcal{F}[R_\gamma], \quad (3.257)$$

$$= \text{End}(\mathbb{Z}\langle f_0, f_k \rangle) \otimes S^*(\mathbb{Z}\langle x_\gamma \rangle) \quad (3.258)$$

is given by

$$\mathcal{B}'_k = f_k x_\gamma^k f_0^* \quad (3.259)$$

$$= \begin{pmatrix} 0 & 0 \\ x_\gamma^k & 0 \end{pmatrix}. \quad (3.260)$$

Next let's describe framed wall-crossing for the brane \mathfrak{N} where one end goes off to infinity in the the region R_0 and the other end going off to infinity in the lower left quadrant. The support of \mathfrak{N} intersects every right thimbles \mathfrak{N}_{-k} at $m = ie^{-i\epsilon}$ for each $k \geq 0$, and moreover does so with positive orientation. Therefore the framed BPS complexes are

$$\mathcal{E}_{-k\gamma}(\mathfrak{N}) = \mathbb{Z}, \text{ for each } k \geq 0. \quad (3.261)$$

Therefore

$$\mathcal{E}(\mathfrak{N}) = \mathbb{Z}\langle e_0, e_1, e_2, \dots, \rangle \quad (3.262)$$

where $\deg(e_k) = (0, -k\gamma)$. The boundary amplitude $\mathcal{B}_{\mathfrak{N}}$ is given by

$$\mathcal{B}_{\mathfrak{N}} = \sum_{k \geq 0} e_k \psi_{-\gamma} e_{k+1}^*. \quad (3.263)$$

In terms of ζ -instantons we are simply conjecturing the existence of a unique ζ -instanton for every half-plane fan of the form $(e_k, \phi_{-\gamma}, e_{k+1}^\vee)$ for each $k \geq 0$, and claiming that there no others. The framed wall-crossing formula then says that

$$\mathcal{E}'(\mathfrak{N}) \simeq \mathcal{E}(\mathfrak{N}) \otimes \mathcal{F}[R_{-\gamma}] \quad (3.264)$$

$$= \mathbb{Z}\langle e_0, e_1, e_2, \dots, \rangle \otimes \Lambda^*(\mathbb{Z}\langle \psi_{-\gamma} \rangle). \quad (3.265)$$

The differential on $\mathcal{E}'(\mathfrak{N})$ determined by $\mathcal{B}_{\mathfrak{N}}$ can be compactly described in the following way. On $\mathcal{E}(\mathfrak{N})$ one can define the usual raising and lowering operators

$$a_\gamma \cdot e_k = e_{k-1}, \quad a_{-\gamma} \cdot e_k = e_{k+1}. \quad (3.266)$$

The differential on $\mathcal{E}'(\mathfrak{N})$ is then given by

$$d_{\mathcal{B}} = a_\gamma \psi_{-\gamma}. \quad (3.267)$$

The differential $d_{\mathcal{B}}$ then pairs up e_k and $e_{k-1}\psi_{-\gamma}$ provided $k \geq 1$ while annihilating e_0 . The complex therefore turns out to be homotopy equivalent to the one-dimensional complex \mathbb{Z} and we find that

$$(\mathcal{E}'(\mathfrak{N}), d_{\mathcal{B}}) \simeq (\mathbb{Z}, 0). \quad (3.268)$$

Furthermore taking into account the jumping of the boundary amplitude, we find that \mathfrak{N} is homotopy equivalent to the thimble \mathcal{L}'_0 at $m' = ie^{i\epsilon}$.

K-Wall Interface

We now turn the discussion to the construction of wall-crossing interfaces which result in the framed wall-crossing formula across a K_{γ} -wall. Before we discuss the wall-crossing interfaces it is useful to discuss the identity interface in this setup. Suppose we are in a theory with a single vacuum i and rank one deck group as before, and let $\zeta_{\mathfrak{B}}^{-1}Z_{\gamma}$ lie in the right half plane. The identity interface \mathfrak{Id} carries the trivial Chan-Paton space

$$\mathcal{E}_i(\mathfrak{Id}) = \mathbb{Z}. \quad (3.269)$$

The interface amplitude $\mathcal{B}(\mathfrak{Id})$ is valued in

$$\text{Hop}(\mathfrak{Id}, \mathfrak{Id}) = \mathcal{F}[R_{-\gamma_i}] \otimes \mathcal{F}[R_{\gamma_i}]. \quad (3.270)$$

Note that the form of the space $\text{Hop}(\mathfrak{Id}, \mathfrak{Id})$ coincides with $\widehat{R}_{ii}^{\text{opp}} \otimes \widehat{R}_{ii}$ where $\widehat{R}_{ii} = \mathcal{F}[R_{\gamma_i}]$ and $\widehat{R}_{ii}^{\text{opp}} = \mathcal{F}[R_{-\gamma_i}]$. We have

$$R_{-\gamma_i} \otimes R_{\gamma_i} \subset \mathcal{F}[R_{-\gamma_i}] \otimes \mathcal{F}[R_{\gamma_i}] \quad (3.271)$$

and so we let the interface amplitude be

$$\mathcal{B}(\mathfrak{Id}) = K_{\gamma_i}^{-1}. \quad (3.272)$$

It can be shown that \mathfrak{Id} indeed behaves as an identity under the \boxtimes -product of interfaces.

Now suppose that \mathcal{T}_L is theory with $\zeta_{\mathfrak{B}}^{-1}Z_\gamma$ in the right-half plane and \mathcal{T}_R is a theory with $\zeta_{\mathfrak{B}}^{-1}Z_\gamma$ in the left-half plane. We wish to construct an interface $\mathfrak{K}_\gamma^+ \in \mathfrak{B}\mathfrak{r}(\mathcal{T}_L, \mathcal{T}_R)$ that implements framed wall-crossing across such a wall. Note that if $\mathfrak{e}_{i_L i_R}$ denotes the basic thimble interface between \mathcal{T}_L and \mathcal{T}_R , we have

$$\text{Hop}(\mathfrak{e}_{i_L i_R}, \mathfrak{e}_{i_L i_R}) = \mathcal{F}[R_{-\gamma_{i_L}}] \otimes \mathcal{F}[R_{-\gamma_{i_R}}]. \quad (3.273)$$

Note that $R_{-\gamma_{i_L}} \cong R_{-\gamma_{i_R}}$ so one could simply write R_{γ_i} . We then let

$$\mathcal{E}_{i_L i_R}(\mathfrak{K}_\gamma^+) = \mathcal{F}[R_{\gamma_i}]. \quad (3.274)$$

The interface amplitude $\mathcal{B}(\mathfrak{K}_\gamma^+)$ is then a degree $(1, 0)$ element valued in

$$\text{Hop}(\mathfrak{K}_\gamma^+, \mathfrak{K}_\gamma^+) = \text{End}(\mathcal{F}[R_{\gamma_i}]) \otimes \mathcal{F}[R_{-\gamma_{i_L}}] \otimes \mathcal{F}[R_{-\gamma_{i_R}}] \quad (3.275)$$

given as follows. Let $\{e_{\pm\gamma_i}^a\}$ denote bases for $R_{\pm\gamma_i}$. Then the corresponding Fock spaces $\mathcal{F}[R_{\pm\gamma_i}]$ are built from applying creation operators $\alpha_{\pm\gamma_i}^a$ ($\alpha_{\pm\gamma_i}^a$ could be bosonic or fermionic depending on the parity of the degree of $e_{\gamma_i}^a$) to the vacuum states $|0\rangle_{\pm\gamma_i}$. Thus for each $e_{\gamma_i}^a$ we have an operator $\alpha_{\gamma_i}^a \in \text{End}(\mathcal{F}[R_{\gamma_i}])$. The interface amplitude is then given by

$$\mathcal{B}(\mathfrak{K}_\gamma^+) = d_{\gamma_i} + (K_{\gamma_{ii_L}}^{-1})_{ab} \alpha_{\gamma_i}^a e_{-\gamma_{i_L}}^b + (K_{\gamma_{ii_R}}^{-1})_{ab} \alpha_{\gamma_i}^a e_{-\gamma_{i_R}}^b. \quad (3.276)$$

Given a brane \mathfrak{B} for the theory \mathcal{T}_L with Chan-Paton spaces $\mathcal{E}_{i_L}(\mathfrak{B})$ and boundary amplitude $\mathcal{B} \in \text{End}(\mathcal{E}_i(\mathfrak{B})) \otimes \mathcal{F}[R_{\gamma_{i_L}}]$, we can produce the brane $\mathfrak{B} \boxtimes \mathfrak{K}_\gamma^+$ for the right theory \mathcal{T}_R ,

$$\mathfrak{B} \in \mathfrak{B}\mathfrak{r}(\mathcal{T}_L) \rightarrow \mathfrak{B} \boxtimes \mathfrak{K}_\gamma^+ \in \mathfrak{B}\mathfrak{r}(\mathcal{T}_R) \quad (3.277)$$

with Chan-Paton space

$$\mathcal{E}_{i_R}(\mathfrak{B} \boxtimes \mathfrak{K}_\gamma^+) = \mathcal{E}_i(\mathfrak{B}) \otimes \mathcal{F}[R_{\gamma_i}]. \quad (3.278)$$

The new boundary amplitude is produced from the natural rule for composing the boundary amplitude \mathcal{B} for the brane \mathfrak{B} and the interface amplitude $\mathcal{B}(\mathfrak{K}_\gamma^+)$ given in (3.276). The

differential $d_{\mathcal{B}}$ on $\mathcal{E}(\mathfrak{B} \boxtimes \mathfrak{K}_\gamma^+)$ as written in (3.228) is given by composing \mathcal{B} with the first two terms of (3.276). On the other hand, the expression for \mathcal{B}' as in (3.231) is due to the third term of (3.276).

We can also easily describe the wall-crossing interface \mathfrak{K}_γ^- for going from \mathcal{T}_L with $\zeta_{\mathfrak{B}}^{-1}Z_\gamma$ in the left-half plane to \mathcal{T}_R with $\zeta_{\mathfrak{B}}^{-1}Z_\gamma$ in the right half plane. \mathfrak{K}_γ^- has Chan-Paton space given by

$$\mathcal{E}_{i_L i_R}(\mathfrak{K}_\gamma^-) = \mathcal{F}[R_{-\gamma_i}^\vee]. \quad (3.279)$$

The interface amplitude $\mathcal{B}(\mathfrak{K}_\gamma^-)$ lives in the space

$$\text{Hop}(\mathfrak{K}_\gamma^-, \mathfrak{K}_\gamma^-) = \text{End}(\mathcal{F}[R_{-\gamma_i}]) \otimes \mathcal{F}[R_{\gamma_{i_L}}] \otimes \mathcal{F}[R_{\gamma_{i_R}}] \quad (3.280)$$

and is given by

$$\mathcal{B}(\mathfrak{K}_\gamma^-) = d_{-\gamma_i} + (K_{-\gamma_{ii_L}}^{-1})_{ab} \alpha_{-\gamma_i}^a e_{\gamma_{i_L}}^b + (K_{-\gamma_{ii_R}}^{-1})_{ab} \alpha_{-\gamma_i}^a e_{\gamma_{i_R}}^b. \quad (3.281)$$

For a given brane $\mathfrak{B} \in \mathfrak{Bt}(\mathcal{T}_L)$, one can easily show that $\mathfrak{B} \boxtimes \mathfrak{K}_\gamma^-$ has Chan-Paton space and differentials as indicated in the framed wall-crossing formula.

Finally, we must show that the composition of the two \mathfrak{K}_γ -interfaces is homotopy equivalent to the identity interface

$$\mathfrak{K}_\gamma^+ \boxtimes \mathfrak{K}_\gamma^- \simeq \mathfrak{Id}_L, \quad \mathfrak{K}_\gamma^- \boxtimes \mathfrak{K}_\gamma^+ \simeq \mathfrak{Id}_R. \quad (3.282)$$

The Chan-Paton space for $\mathfrak{K}_\gamma^+ \boxtimes \mathfrak{K}_\gamma^-$ reads

$$\mathcal{E}_{i_L i_L}(\mathfrak{K}_\gamma^+ \boxtimes \mathfrak{K}_\gamma^-) = \mathcal{F}[R_{\gamma_i}] \otimes \mathcal{F}[R_{-\gamma_i}^\vee]. \quad (3.283)$$

Although we have not worked through this in detail, it is not unreasonable to expect that (3.276) and (3.281) equips $\mathcal{E}_{i_L i_L}(\mathfrak{K}_\gamma^+ \boxtimes \mathfrak{K}_\gamma^-)$ with a differential such that

$$\mathcal{E}_{i_L i_L}(\mathfrak{K}_\gamma^+ \boxtimes \mathfrak{K}_\gamma^-) \simeq \mathbb{Z} \quad (3.284)$$

3.4 Bulk Amplitudes and Wall-Crossing of Vanilla BPS States

So far in this paper we have discussed wall-crossing of framed BPS states. We have seen that for a given $\mathcal{N} = (2, 2)$ theory with non-trivial twisted masses, in addition to the well-understood $S_{\gamma_{ij}}$ -walls where $Z_{\gamma_{ij}}$ becomes parallel to $\zeta_{\mathfrak{B}}$ for $i \neq j$, there are also new walls known as K_{γ} -walls, where Z_{γ} becomes parallel to $\zeta_{\mathfrak{B}}$. We were able to describe the wall-crossing of framed BPS states $\mathcal{E}_i(\mathfrak{B})$ across a K_{γ} -wall. We would now like to work out the analogous exercise for vanilla BPS states. We address the question

In the presence of twisted masses what type of new marginal stability walls are present, and how do the spaces $\mathcal{R}_{\gamma_{ij}}$ jump across these walls?

Recall that in a given theory we have a finite set \mathbb{V} along with a Γ -torsors Γ_{ij} for each pair of (not necessarily distinct) vacua i and j . The vacuum set \mathbb{V} and Γ_{ij} for each pair i, j forms a groupoid which we refer to as the vacuum groupoid. There is a groupoid homomorphism that maps $\gamma_{ij} \in \Gamma_{ij} \rightarrow Z_{\gamma_{ij}} \in \mathbb{C}$. At a generic point in parameter space $Z_{\gamma_{ij}}$ for $\gamma_{ij} \in \Gamma_{ij}$ and $Z_{\delta_{jk}}$ for $\delta_{jk} \in \Gamma_{jk}$ are parallel if and only if $i = j = k$, and γ_{ii} and δ_{ii} are both positive multiples of a primitive element $\alpha_{ii} \in \Gamma_{ii}$. Walls of marginal stability are precisely where this genericity condition is violated. It can be violated in two ways.

The first type of violation occurs when $i = j = k$ still holds, but Z_{γ} and $Z_{\gamma'}$ become parallel for γ and γ' non co-linear elements of Γ . When this happens, not only do γ and γ' become parallel, but by linearity $n\gamma + m\gamma'$ for $n > 0$ and $m > 0$ become parallel, and so in principle the spaces $\{R_{n\gamma+m\gamma'}\}_{n>0, m>0}$ can mix and form new BPS states. This setup, where spaces of BPS states are graded by a lattice Γ along with a homomorphism $Z : \Gamma \rightarrow \mathbb{C}$, and marginal stability walls are defined as the locus where Z_{γ} and $Z_{\gamma'}$

are aligned is more familiar in four dimensions. Indeed, in four dimensions the wall-crossing of BPS states is very non-trivial and is captured by the Kontsevich-Soibelman wall-crossing formula. One of the main properties responsible for KS wall-crossing is the non-commutativity of the K_γ -factors due to the non-trivial DSZ pairing \langle, \rangle on Γ . However in a purely two-dimensional setting, like we have in this paper, we expect that the BPS chain complexes $R_{n\gamma+m\gamma'}$ are actually unchanged up to homotopy. The main reason for this is that Γ in our setup consists of purely flavor charges, and is therefore isotropic for the DSZ pairing. This renders the K_γ -factors commutative, and thus there is no wall-crossing at the level of BPS indices. At the categorical level, we expect that the interface $\mathfrak{K}_\gamma \boxtimes \mathfrak{K}_{\gamma'}$ is homotopy equivalent to $\mathfrak{K}_{\gamma'} \boxtimes \mathfrak{K}_\gamma$ and therefore the BPS complexes are expected to be unchanged up to homotopy.

The second type of violation is when $i = j = k$ no longer holds. This can happen if none, or only two of the three vacua coincide. Suppose that none of them coincide so that at a wall $Z_{\gamma_{ij}}$ and $Z_{\gamma_{jk}}$ are parallel for three distinct vacua $i \neq j \neq k$. Categorical wall-crossing in this situation is described by a refined version of the formula of [KM1]. Note that by linearity, if $Z_{\gamma_{ij}}$ becomes parallel to $Z_{\gamma_{jk}}$, then $Z_{\gamma_{ij}+\gamma_{jk}}$ also becomes parallel to both of them. Therefore mixing can occur between the BPS states of charges γ_{ij}, γ_{jk} and $\gamma_{ij} + \gamma_{jk}$. Let $\gamma_{ik} = \gamma_{ij} + \gamma_{jk}$ and let $R_{\gamma_{ij}}, R_{\gamma_{jk}}$ and $R_{\gamma_{ik}}$ be the corresponding BPS complexes. There is an interior amplitude component $\beta_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}$. Then all other complexes except $R_{\gamma_{ik}}$ are unchanged up to homotopy, whereas $R_{\gamma_{ik}}$ jumps to $R'_{\gamma_{ik}}$ where

$$R'_{\gamma_{ik}} \simeq \text{Cone}(M[\beta_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}] : R_{\gamma_{ij}} \otimes R_{\gamma_{jk}} \rightarrow R_{\gamma_{ik}}). \quad (3.285)$$

We will review the essential ingredients that go into this formula shortly.

On the other hand suppose the violation is such that two vacua coincide, say $i = k$, but $i \neq j$ and $j \neq k$ so that at a wall of marginal stability $Z_{\gamma_{ij}}$ is parallel to $Z_{\gamma_{ji}}$. Letting

$\gamma = \gamma_{ij} + \gamma_{ji}$, by linearity of Z we find that $\{\gamma_{ij} + n\gamma, k\gamma, \gamma_{ji} + m\gamma\}_{n \geq 0, m \geq 0}$ all become parallel at the wall. Therefore, there can be mixing between the corresponding BPS complexes

$$\{R_{\gamma_{ij}+n\gamma}, R_{\gamma_i}, R_{\gamma_j}, R_{\gamma_{ji}+m\gamma}\}. \quad (3.286)$$

This is the new type of wall-crossing that we describe in this section.

3.4.1 Interior Amplitudes in Cecotti-Vafa Wall-Crossing

We give a quick review of the logic obtained to arrive at the categorification of the Cecotti-Vafa formula (3.285). A similar chain of reasoning will apply in a more complicated setting to describe wall-crossing at the new type of wall.

We are interested in describing how homotopy classes of BPS complexes jump at a wall where $Z_{\gamma_{ij}}$ gets aligned with $Z_{\gamma_{jk}}$ for three distinct vacua i, j, k . The important new ingredient that enters the categorified wall-crossing formula is the interior amplitude. In general interior amplitudes are elements of an L_∞ algebra built from the BPS complexes that satisfy the L_∞ Maurer-Cartan equation. See Appendix A.5 of [GMW] for a review of L_∞ -algebras. The L_∞ algebra relevant to describing the $(\gamma_{ij}, \gamma_{jk})$ alignment is as follows. Let $\gamma_{ik} = \gamma_{ij} + \gamma_{jk}$, and let γ_{ji}, γ_{kj} be elements such that $\gamma_{ij} + \gamma_{ji} = 0$, $\gamma_{jk} + \gamma_{kj} = 0$. Suppose we are on the side of the wall where $(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})$ is a cyclic fan. Letting $R_{\gamma_{ij}}, R_{\gamma_{jk}}, R_{\gamma_{ik}}$ be the \mathbb{Z} -modules of classical BPS states, we consider the categorical trace in the $\{i, j, k\}$ subsector of the theory. As computed in (3.54), it is given by

$$\begin{aligned} R_{ijk} &= R_i \oplus R_j \oplus R_k \\ &\oplus R_{(\gamma_{ij}, \gamma_{ji})} \oplus R_{(\gamma_{ik}, \gamma_{ki})} \oplus R_{(\gamma_{jk}, \gamma_{kj})} \\ &\oplus R_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}, \end{aligned} \quad (3.287)$$

where R_i, R_j, R_k are one-dimensional spaces spanned by each respective vacuum state ϕ_i, ϕ_j, ϕ_k ,

$$R_{(\gamma_{ij}, \gamma_{ji})} = R_{\gamma_{ij}} \otimes R_{\gamma_{ji}}, \quad (3.288)$$

with similar definitions for $R_{(\gamma_{ik}, \gamma_{ki})}$ and $R_{(\gamma_{jk}, \gamma_{kj})}$, and

$$R_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})} = R_{\gamma_{ik}} \otimes R_{\gamma_{kj}} \otimes R_{\gamma_{ji}}. \quad (3.289)$$

R_{ijk} is an L_∞ algebra where the L_∞ maps are given as follows. We denote the bilinear bracket as $[\cdot, \cdot]$. Letting $\{e_{\gamma_{ij}}^a\}$ be a basis for $R_{\gamma_{ij}}$, so that $\{e_{\gamma_{ij}, \gamma_{ji}}^{a,b} := e_{\gamma_{ij}}^a \otimes e_{\gamma_{ji}}^b\}$, and using similar notation for jk and ik solitons, we have that the non-vanishing brackets are

$$[e_{(\gamma_{ij}, \gamma_{ji})}^{ab}, e_{(\gamma_{ij}, \gamma_{ji})}^{cd}] = (K_{\gamma_{ij}})^{ad} e_{(\gamma_{ij}, \gamma_{ji})}^{cb} \pm (K_{\gamma_{ij}})^{bc} e_{(\gamma_{ij}, \gamma_{ji})}^{ad}, \quad (3.290)$$

with similar ones for the bivalent ik and jk fans. We also have non-vanishing brackets between the bivalent and trivalent summands as follows

$$[e_{(\gamma_{ij}, \gamma_{ji})}^{ab}, e_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}^{cde}] = \pm (K_{\gamma_{ij}})^{ae} e_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}^{cdb}, \quad (3.291)$$

$$[e_{(\gamma_{jk}, \gamma_{kj})}^{ab}, e_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}^{cde}] = \pm (K_{\gamma_{jk}})^{ad} e_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}^{cbe}, \quad (3.292)$$

$$[e_{(\gamma_{ik}, \gamma_{ki})}^{ab}, e_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}^{cde}] = \pm (K_{\gamma_{ki}})^{bc} e_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}^{ade}. \quad (3.293)$$

These non-vanishing brackets can be summarized diagrammatically in terms of the taut webs shown in Figure 3.13.

An interior amplitude is a degree 2 element $\beta \in R_{ijk}$ such that $[\beta, \beta] = 0$. Decomposing β into its different direct sum components

$$\beta = \beta_{(\gamma_{ij}, \gamma_{ji})} + \beta_{(\gamma_{ik}, \gamma_{ki})} + \beta_{(\gamma_{jk}, \gamma_{kj})} + \beta_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}, \quad (3.294)$$

we find that the equation $[\beta, \beta] = 0$ is equivalent to saying that

$$d_{\gamma_{ij}} := K_{\gamma_{ik}} \circ \beta_{(\gamma_{ij}, \gamma_{ji})} : R_{\gamma_{ij}} \rightarrow R_{\gamma_{ij}}, \quad (3.295)$$

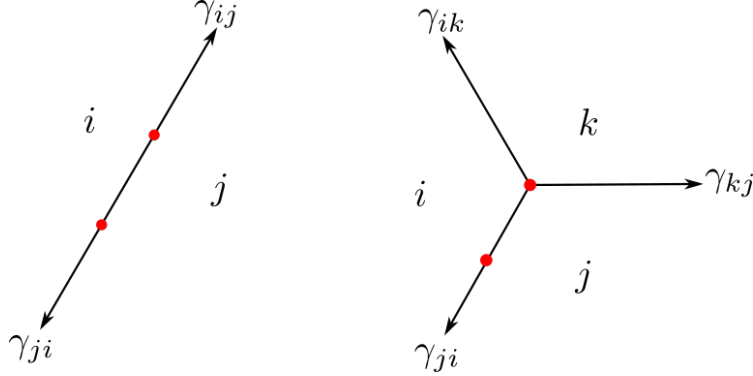


Figure 3.13: The L_∞ -structure on R_{ijk} is determined by the taut webs allowed by a given configuration of central charges. The left side depicts a taut web that determines the bracket of two elements of $R_{(\gamma_{ij}, \gamma_{ji})}$ whereas the right side depicts the taut web that determines the bracket of an element of $R_{(\gamma_{ij}, \gamma_{ji})}$ with an element of $R_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}$.

and similarly defined $d_{\gamma_{ik}}, d_{\gamma_{jk}}$ are differentials that turn $R_{\gamma_{ij}}, R_{\gamma_{ik}}, R_{\gamma_{jk}}$, respectively, into chain complexes, and that

$$M[\beta_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})}] := K_{\gamma_{ij}} \circ K_{\gamma_{jk}} \circ \beta_{(\gamma_{ik}, \gamma_{kj}, \gamma_{ji})} : R_{\gamma_{ij}} \otimes R_{\gamma_{jk}} \rightarrow R_{\gamma_{ik}} \quad (3.296)$$

is a chain map between the chain complexes $(R_{\gamma_{ij}} \otimes R_{\gamma_{jk}}, d_{\gamma_{ij}} \otimes 1 + 1 \otimes d_{\gamma_{jk}})$ and $(R_{\gamma_{ik}}, d_{\gamma_{ik}})$.

We are thus given BPS chain complexes $R_{\gamma_{ij}}, R_{\gamma_{jk}}, R_{\gamma_{ik}}$ and a chain map $R_{\gamma_{ij}} \otimes R_{\gamma_{jk}} \rightarrow R_{\gamma_{ik}}$ on one side of the wall. In order to describe the chain complexes on the other side of the wall of marginal stability, the strategy is to study an object that, on the one hand is a wall-crossing invariant, and on the other hand is strong enough that the wall-crossing invariance determines the homotopy equivalence class of the chain complexes $R'_{\gamma_{ij}}, R'_{\gamma_{jk}}, R'_{\gamma_{ik}}$. There are two related options for proceeding.

The first approach, taken in [KM1], is to study the A_∞ -(sub)category of branes with objects $\mathfrak{T}_i, \mathfrak{T}_j, \mathfrak{T}_k$. The morphism spaces $\text{Hop}(\mathfrak{T}_\alpha, \mathfrak{T}_\beta)$ are constructed from the spaces $R_{\gamma_{\alpha\beta}}$ whereas the A_∞ maps are constructed using the interior amplitude β . The homotopy

equivalence class of this (sub)category is insensitive to walls of marginal stability. The requirement that there be an A_∞ -equivalence between the categories constructed at either side of the wall leads to (3.285).

Another option, pursued in [GMW], is to consider flat parallel transport of brane categories. The parallel transport of branes along a path \wp in parameter space can be carried out by fusing with an \mathfrak{S} -wall interface each time \wp crosses an S -wall. The flatness of the connection says that two homotopic paths in parameter space lead to homotopy equivalent interfaces that carry out this parallel transport. Applied to Cecotti-Vafa wall-crossing this leads to the requirement that

$$\mathfrak{S}_{\gamma_{jk}} \boxtimes \mathfrak{S}_{\gamma_{ik}} \boxtimes \mathfrak{S}_{\gamma_{ij}} \simeq \mathfrak{S}'_{\gamma_{ij}} \boxtimes \mathfrak{S}'_{\gamma_{ik}} \boxtimes \mathfrak{S}'_{\gamma_{jk}}. \quad (3.297)$$

Imposing this homotopy equivalence leads to the Cone formula (3.285).

3.4.2 L_∞ -Algebra Governing Interior Amplitudes

We now turn to a discussion of categorical wall-crossing at a wall where $Z_{\tau_{ij}}$ and $Z_{\tau_{ji}}$ become parallel. If $\tau_{ij} + \tau_{ji} = \gamma$, then at the wall $\tau_{ij} + n\gamma$, $\tau_{ji} + m\gamma$ for $n, m \geq 0$, and all positive multiples of γ become aligned, thus there can be mixing amongst the corresponding spaces $R_{\tau_{ij}+n\gamma}, R_{\tau_{ji}+m\gamma}$ and Fock spaces of $V_{\gamma_i} = \oplus_{n \geq 1} R_{n\gamma_i}$ and $V_{\gamma_j} = \oplus_{n \geq 1} R_{n\gamma_j}$. Precisely how the mixing occurs will be governed by an interaction amplitude; a Maurer-Cartan element of an L_∞ algebra constructed from the spaces of BPS particles that get mixed at a wall.

Suppose we are on the side of the wall of marginal stability where the corresponding central charges to

$$(\tau_{ij}, \gamma_j, \tau_{ji}, -\gamma_i) \quad (3.298)$$

are in clockwise order as in Figure (3.14).

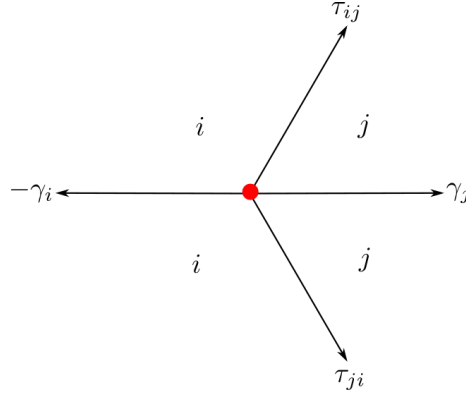


Figure 3.14: The configuration of central charges on one side of the wall is such that $(\tau_{ij}, \gamma_j, \tau_{ji}, -\gamma_i)$ is a cyclic fan.

The L_∞ algebra governing the interaction amplitude amongst these coincides with the categorical trace

$$R_c = \text{Tr}(\widehat{R}^{\text{opp}} \otimes \widehat{R}). \quad (3.299)$$

For the present configuration of vacua this was computed in Section 3.2.1. The result from (3.71) was that

$$R_c = R_i \oplus R_j \oplus R_{(ij)} \quad (3.300)$$

where

$$R_i := \mathcal{F}^*[V_{\gamma_i}] \otimes \mathcal{F}^*[V_{-\gamma_i}], \quad (3.301)$$

$$R_j := \mathcal{F}^*[V_{\gamma_j}] \otimes \mathcal{F}^*[V_{-\gamma_j}], \quad (3.302)$$

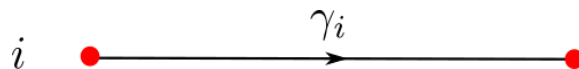
and

$$R_{(ij)} = \bigoplus_{(\gamma_{ij}, \gamma_{ji}) \in \Gamma_{ij} \times \Gamma_{ji}} R_{\gamma_{ij}} \otimes \mathcal{F}^*[V_{\gamma_j}] \otimes R_{\gamma_{ji}} \otimes \mathcal{F}^*[V_{-\gamma_i}]. \quad (3.303)$$

Conjecture R_c carries the structure of a Γ -graded⁸ L_∞ -algebra. Moreover for a given physical theory, there is a canonical interior amplitude, an element $\beta \in (R_c)^{(2,0)}$ that satisfies the L_∞ Maurer-Cartan equation.

It is natural to expect that similar to the formalism of [GMW], the L_∞ -brackets result from summing up contributions of planar graphs where the edges are chosen from the set of central charge rays $\{Z_{\gamma_i}, Z_{\gamma_{ij}}\}$. In general, a planar graph that contributes to the n th L_∞ bracket $[-, \dots, -]_n$ has n vertices and $2n - 3$ edges. In [GMW] such graphs were known as *taut webs*.

We now provide some (generalized) taut webs and the corresponding contributions to the L_∞ -brackets in the present setup. Following the notation of [GMW], a web is denoted by a gothic lowercase Latin letter such as \mathfrak{w} , whereas the linear map it induces is denoted as $\rho(\mathfrak{w})$. First we expect that $R_i, R_j, R_{(ij)}$ are each individually L_∞ -subalgebras. The Fock space $R_i = \mathcal{F}[R_{\gamma_i}] \otimes \mathcal{F}[R_{-\gamma_i}]$ is constructed by acting with operators $e_{\gamma_i}^a$ and $e_{-\gamma_i}^a$ on the vacuum ϕ_i . The L_∞ -structure on R_i is determined by the basic graph \mathfrak{t}_{γ_i} shown below which is a diagram with two vertices and an edge parallel to Z_γ .



The entire process occurs in the background of the i th vacuum with no mixing with j . Imagining outgoing solitons $e_{\gamma_i}^a, e_{-\gamma_i}^b$ from the left and right vertex respectively, the graph \mathfrak{t}_{γ_i} translates to the bracket

$$\rho(\mathfrak{t}_{\gamma_i})[e_{\gamma_i}^a, e_{-\gamma_i}^b] = \pm(K_{\gamma_i})^{ab}\phi_i \quad (3.304)$$

⁸A Γ -graded L_∞ algebra simply refers to an L_∞ -algebra carrying, in addition to the usual cohomological grading, an extra gradation by Γ such that each L_∞ -operation has vanishing Γ -degree.

on R_i . The bracket of any other two elements of R_i is determined by the above by using the graded Leibniz rule

$$[a \cdot b, c] = a \cdot [b, c] \pm b \cdot [a, c]. \quad (3.305)$$

It can be expressed in terms of differential operators $\frac{\partial}{\partial e_{\gamma_i}^a}$ and $\frac{\partial}{\partial e_{-\gamma_i}^a}$ acting on R_i as follows.

Given two elements $v, w \in R_i$, their bracket is written as

$$[v, w] = (K_{\gamma_i})^{ab} \frac{\partial v}{\partial e_{\gamma_i}^a} \frac{\partial w}{\partial e_{-\gamma_i}^b} \pm (K_{\gamma_i})^{ab} \frac{\partial w}{\partial e_{\gamma_i}^a} \frac{\partial v}{\partial e_{-\gamma_i}^b}. \quad (3.306)$$

The L_∞ -structure on R_j is obtained by the same formulas by replacing i with j . The L_∞ -structure on $R_{(ij)}$ is determined by the web $\mathbf{t}_{\gamma_{ij}, \gamma'_{ij}, \gamma''_{ij}}$ determined by a triple of elements $\gamma_{ij}, \gamma'_{ij}, \gamma''_{ij}$ depicted below.

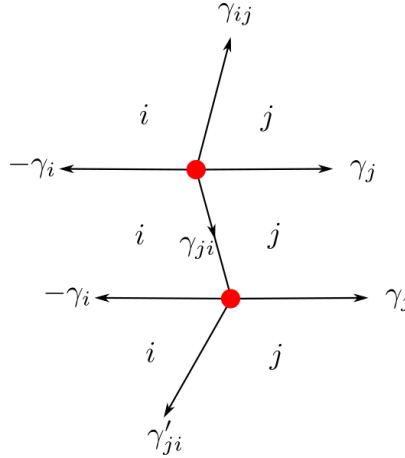


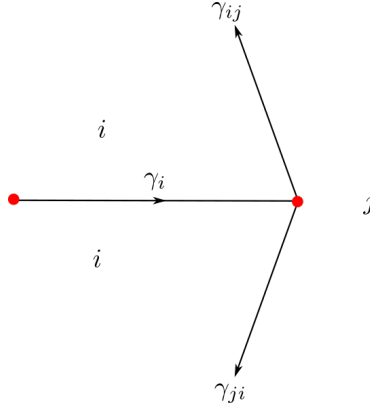
Figure 3.15: The taut web that leads to a non-zero bilinear bracket between two elements of $R_{(ij)}$.

It leads to the bracket

$$\begin{aligned} \rho(\mathbf{t}_{\gamma_{ij}, \gamma'_{ij}, \gamma''_{ij}})[e_{\gamma_{ij}}^a v_{\gamma_j} e_{\gamma'_{ji}}^b v_{-\gamma_i}, e_{\gamma'_{ij}}^c w_{\gamma_j} e_{\gamma''_{ji}}^d w_{-\gamma_i}] &= (K_{\gamma_{ij}})^{ad} \delta_{\gamma_{ij} + \overline{\gamma''_{ji}}, 0} e_{\gamma'_{ij}}^c v_{\gamma_j} w_{\gamma_j} e_{\gamma_{ji}}^b v_{-\gamma_i} w_{-\gamma_j} \\ &\pm (K_{\gamma_{ij}})^{bc} e_{\gamma_{ij}}^a v_{\gamma_j} w_{\gamma_j} e_{\gamma''_{ji}}^d v_{-\gamma_i} w_{-\gamma_j}, \end{aligned} \quad (3.307)$$

where $v_{\gamma_j}, w_{\gamma_j}$ are arbitrary elements of $\mathcal{F}[V_{\gamma_j}]$ and $v_{-\gamma_i}, w_{-\gamma_i}$ are arbitrary elements of $\mathcal{F}[V_{-\gamma_i}]$. It is a combination of the basic bracket (3.290) with Fock space multiplication on the factors inserted between ij states; the periodic solitons go along for the ride. The expectation is that the subalgebras $R_i, R_j, R_{(ij)}$ only have non-vanishing bilinear brackets.

We also need to specify the brackets that mix R_i, R_j and $R_{(ij)}$. There is no non-vanishing bracket between elements of R_i and R_j . On the other hand, there can be non-vanishing contributions to L_∞ -brackets that mix elements of R_i and $R_{(ij)}$ and that mix elements of R_j and $R_{(ij)}$. The simplest such bracket comes from the graph t_2 with two vertices and one edge as shown below,



which leads to the bracket

$$\rho(t_2)[e_{\gamma_i}^a, e_{-\gamma_i}^b e_{\gamma_{ij}}^c e_{\gamma_{ji}}^d] = K_{-\gamma_i}^{ab} e_{\gamma_{ij}}^c e_{\gamma_{ji}}^d. \quad (3.308)$$

More generally, given

$$v_i \in R_i \quad \text{and} \quad w_{-\gamma_i} w_{\gamma_{ij}} w_{\gamma_j} w_{\gamma_{ji}} \in \mathcal{F}[R_{-\gamma_i}] \otimes R_{\gamma_{ij}} \otimes \mathcal{F}[R_{\gamma_j}] \otimes R_{\gamma_{ji}} \quad (3.309)$$

the bracket is given by

$$[v_i, w_{-\gamma_i} w_{\gamma_{ij}} w_{\gamma_j} w_{\gamma_{ji}}] = \left((K_{\gamma_i})^{ab} \frac{\partial v_i}{\partial e_{\gamma_i}^a} \frac{\partial w_{-\gamma_i}}{\partial e_{-\gamma_i}^b} \right) w_{\gamma_{ij}} w_{\gamma_j} w_{\gamma_{ji}}. \quad (3.310)$$

So far the discussion has closely paralleled that in [GMW]. However, for the formalism to work we need rather novel types of “generalized webs”. An example is the following generalized web \mathbf{t}_3 .

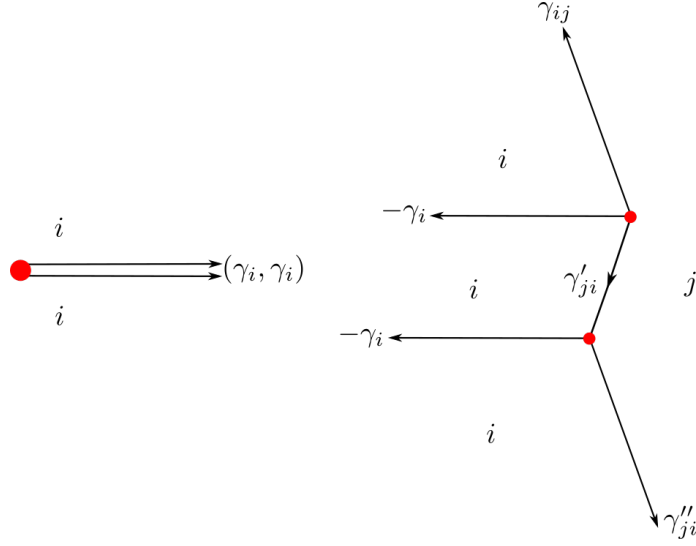


Figure 3.16: A novel type of taut web leading to a ternary bracket.

This generalized web leads to a non-vanishing contribution to the ternary bracket as follows. There are two periodic solitons emanating from the vertex on the left. We contract these with the two periodic solitons coming from the right, while simultaneously contracting the non-periodic solitons of charges $\gamma'_{ij}, \overline{\gamma'}_{ji}$. We thus have

$$\begin{aligned} \rho(\mathbf{t}_3) \left[e_{\gamma_i}^a e_{\gamma_i}^b, e_{-\gamma_i}^c e_{\gamma_{ij}}^d e_{\gamma'_{ji}}^e, e_{-\gamma_i}^f e_{\gamma'_{ij}}^g e_{\gamma''_{ji}}^h \right] &= (K_{\gamma_i})^{ac} (K_{\gamma_i})^{bf} (K_{\gamma'_{ji}})^{eg} e_{\gamma_{ij}}^d e_{\gamma'_{ji}}^h \\ &\quad \pm (K_{\gamma_i})^{af} (K_{\gamma_i})^{bc} (K_{\gamma'_{ji}})^{eg} e_{\gamma_{ij}}^d e_{\gamma''_{ji}}^h. \end{aligned} \quad (3.311)$$

For more general elements of the Fock spaces, the graph \mathbf{t}_3 leads to the ternary bracket

$$\begin{aligned} \rho(\mathbf{t}_3)[u_i, v_{-\gamma_i} v_{\gamma_{ij}} v_{\gamma_j} v_{\gamma'_{ji}}, w_{-\gamma_i} w_{\gamma'_{ij}} w_{\gamma_j} w_{\gamma''_{ji}}] = \\ K_{\gamma'_{ji}}(v_{\gamma'_{ji}}, w_{\gamma'_{ij}}) \left((K_{\gamma_i})^{ac} (K_{\gamma_i})^{bd} \frac{1}{2} \frac{\partial^2 u_i}{\partial e_{\gamma_i}^a \partial e_{\gamma_i}^b} \frac{\partial v_{-\gamma_i}}{\partial e_{-\gamma_i}^c} \frac{\partial w_{-\gamma_i}}{\partial e_{-\gamma_i}^d} \right) v_{\gamma_{ij}} v_{\gamma_j} w_{\gamma_j} w_{\gamma'_{ji}} \end{aligned} \quad (3.312)$$

+ permutations,

where +permutations denote additional terms of the same form needed to make the bracket obey the right symmetry properties.

We clarify that even though Figure 3.16, as we have drawn it, looks like two disconnected independent webs, it is to be considered as a single object due to the fact that there are contractions taking place between the edges carrying periodic solitons. That this is a single object will be clarified further in Section 4.1 where we explain how to think of such webs in terms of dual polygons and their subdivisions.

The graphs $\mathbf{t}_2, \mathbf{t}_3$ easily generalize to \mathbf{t}_{n+1} for arbitrary $n \geq 1$. \mathbf{t}_{n+1} gives rise to the bracket

$$\rho(\mathbf{t}_{n+1})[-, \dots, -] : R_i \otimes R_{(ij)}^{\otimes n} \rightarrow R_{(ij)} \quad (3.313)$$

which is given as follows. Let $v_i \in \mathcal{F}[R_{\gamma_i}] \otimes \mathcal{F}[R_{-\gamma_i}]$, and

$$v_{-\gamma_i}^k v_{\gamma_{ij}^k}^k v_{\gamma_j^k}^k v_{\gamma'_{ji}^k}^k \in \mathcal{F}[R_{-\gamma_i}] \otimes R_{\gamma_{ij}^k} \otimes \mathcal{F}[R_{\gamma_j}] \otimes R_{\gamma'_{ji}^k} \quad (3.314)$$

for $k = 1, \dots, n$ and $\gamma_{ij}^k \in \Gamma_{ij}$ and $\gamma'_{ji}^k \in \Gamma_{ji}$. Then

$$\begin{aligned} \rho(\mathbf{t}_{n+1})[v_i, v_{-\gamma_i}^1 v_{\gamma_{ij}^1}^1 v_{\gamma_j^1}^1 v_{\gamma'_{ji}^1}^1, \dots, v_{-\gamma_i}^n v_{\gamma_{ij}^n}^n v_{\gamma_j^n}^n v_{\gamma'_{ji}^n}^n] = \prod_{k=1}^{n-1} \delta_{\gamma_{ij}^k + \gamma_{ij}^{k+1}, 0} K_{\gamma_{ji}^k}(v_{\gamma_{ji}^k}^k, v_{\gamma_{ji}^{k+1}}^{k+1}) \\ \times \left(K_{\gamma_i}^{a_1 b_1} \dots K_{\gamma_i}^{a_n b_n} \frac{1}{n!} \frac{\partial^n v_i}{\partial e_{\gamma_i}^{a_1} \dots \partial e_{\gamma_i}^{a_n}} \frac{\partial v_{-\gamma_i}^1}{\partial e_{-\gamma_i}^{b_1}} \dots \frac{\partial v_{-\gamma_i}^n}{\partial e_{-\gamma_i}^{b_n}} \right) v_{\gamma_{ij}^1}^1 (v_{\gamma_j^1}^1 \dots v_{\gamma_j^n}^n) v_{\gamma'_{ji}^n}^n \end{aligned} \quad (3.315)$$

+ permutations .

This bracket involves n contractions of periodic solitons and $n - 1$ contractions of ij - ji solitons so that there are a total of $(n - 1) + n = 2n - 1$ contractions. The degree of this bracket is indeed $1 - 2n = 3 - 2(n + 1)$. There are entirely analogous brackets where periodic jj -solitons are contracted.

We have presented some examples of important contributions to the L_∞ -brackets, but have not determined them completely. For a more complete discussion one has to give suitable definitions of generalized webs that contribute to the L_∞ -brackets, along with their convolutions. Figure 3.16 (and the webs \mathbf{t}_n for $n = 2, 3, \dots$) only give some examples of the generalized webs we must introduce. Letting \mathbf{t}_Γ be the formal sum of all these graphs, one also has to define a convolution $*$ of such generalized webs. The proof of the L_∞ -relations is then expected to follow from the vanishing of the convolution of \mathbf{t}_Γ with itself

$$\mathbf{t}_\Gamma * \mathbf{t}_\Gamma = 0. \quad (3.316)$$

We have not developed the theory of generalized webs to this extent and leave it as an interesting exercise for the future. As we will briefly discuss in Section 4.1, a dual viewpoint in terms of polygons and their subdivisions provides a potential route for a systematic discussion of the brackets and the algebraic axioms they obey. We will however give a complete discussion of how the formalism should work in the important example of the \mathbb{CP}^1 model at both weak and strong coupling.

The main use of the L_∞ -structure is that it gives us a framework to discuss interior amplitudes. An interior amplitude is an element $\beta \in R_c$ carrying homological degree 2 and vanishing equivariant degree such that the L_∞ Maurer-Cartan equation is satisfied. We

can decompose β into its direct sum components

$$\beta = \beta_i + \beta_j + \beta_{(ij)}, \quad (3.317)$$

and with the above L_∞ -structure the Maurer-Cartan equation reads

$$\begin{aligned} \frac{1}{2}[\beta_i, \beta_i] + \frac{1}{2}[\beta_j, \beta_j] + \frac{1}{2}[\beta_{(ij)}, \beta_{(ij)}] + \\ \sum_{n \geq 2} \frac{1}{n!} [\beta_i, \dots, \beta_i, \beta_{(ij)}, \dots, \beta_{(ij)}, \beta_j, \dots, \beta_j]_n = 0. \end{aligned} \quad (3.318)$$

Example: The \mathbb{CP}^1 Model at Weak Coupling

We illustrate the (conjectured) L_∞ -algebra R_c and interior amplitude β introduced above in an important example. The \mathbb{CP}^1 model with twisted masses, has two vacua $\mathbb{V} = \{1, 2\}$ and Γ is a rank one lattice generated by an element γ . Recall that in the weak coupling regime the BPS \mathbb{Z} -modules are given by

$$R_{\gamma_{12}} \cong \mathbb{Z} \quad \text{for any } \gamma_{12} \in \Gamma_{12}, \quad (3.319)$$

$$R_{\gamma_{21}} \cong \mathbb{Z}^{[1]} \quad \text{for any } \gamma_{21} \in \Gamma_{21} \quad (3.320)$$

along with the periodic solitons,

$$R_{\gamma_1} \cong \mathbb{Z}, \quad R_{\gamma_2} \cong \mathbb{Z}^{[1]}, \quad (3.321)$$

$$R_{-\gamma_1} \cong \mathbb{Z}^{[1]}, \quad R_{-\gamma_2} \cong \mathbb{Z}, \quad (3.322)$$

where γ is the generator of Γ homologous to a loop that winds around the origin once in a counter-clockwise direction. The periodic complexes are trivial otherwise,

$$R_{n\gamma_1} = R_{n\gamma_2} = 0 \quad \text{for } n \neq \pm 1. \quad (3.323)$$

Let us denote the generators of the latter four modules of periodic solitons as

$$\{a_{\gamma_1}, \psi_{-\gamma_1}, \psi_{\gamma_2}, a_{-\gamma_2}\} \quad (3.324)$$

the letters a, ψ reminding us of their parity. The L_∞ -algebra R_c is then constructed as follows. We have

$$R_1 = \mathcal{F}^*[R_{\gamma_1}] \otimes \mathcal{F}^*[R_{-\gamma_1}], \quad (3.325)$$

$$= \mathbb{Z}[a_{\gamma_1}, \psi_{-\gamma_1}] / \langle \psi_{-\gamma_1}^2 \rangle \quad (3.326)$$

and similarly

$$R_2 = \mathcal{F}^*[R_{\gamma_2}] \otimes \mathcal{F}^*[R_{-\gamma_2}] \quad (3.327)$$

$$= \mathbb{Z}[\psi_{\gamma_2}, a_{-\gamma_2}] / \langle \psi_{\gamma_2}^2 \rangle. \quad (3.328)$$

Finally

$$R_{(12)} = \bigoplus_{\substack{(\gamma_{12}, \gamma_{21}) \\ \in \Gamma_{12} \times \Gamma_{21}}} R_{(\gamma_{12}, \gamma_{21})} \quad (3.329)$$

$$= \bigoplus_{\substack{(\gamma_{12}, \gamma_{21}) \\ \in \Gamma_{12} \times \Gamma_{21}}} \mathcal{F}^*[R_{-\gamma_1}] \otimes R_{\gamma_{12}} \otimes \mathcal{F}^*[R_{\gamma_2}] \otimes R_{\gamma_{21}}. \quad (3.330)$$

The periodic solitons that enter in $R_{(12)}$ are fermions, thus $R_{(12)}$ is generated by elements with no periodic solitons

$$\phi_{\gamma_{12}} \otimes \phi_{\gamma_{21}} \quad (3.331)$$

for arbitrary charges $\gamma_{12} \in \Gamma_{12}$, $\gamma_{21} \in \Gamma_{21}$, one insertion of a periodic soliton

$$\psi_{-\gamma_1} \otimes \phi_{\gamma_{12}} \otimes \phi_{\gamma_{21}} \text{ or } \phi_{\gamma_{12}} \otimes \psi_{\gamma_2} \otimes \phi_{\gamma_{21}} \quad (3.332)$$

or both insertions

$$\psi_{-\gamma_1} \otimes \phi_{\gamma_{12}} \otimes \psi_{\gamma_2} \otimes \phi_{\gamma_{21}}. \quad (3.333)$$

Elements of the above type have homological degree 1, 2, 3 respectively.

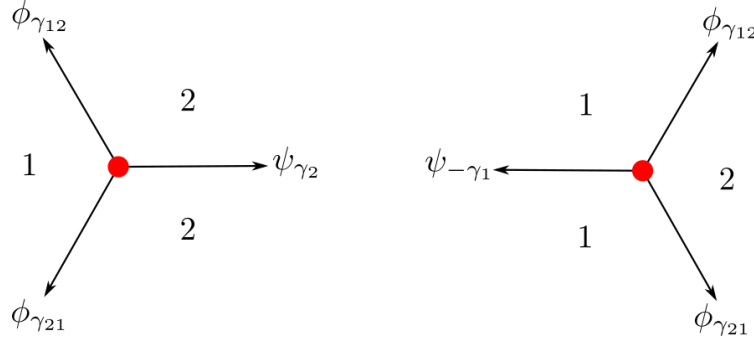


Figure 3.17: The interior amplitude for the \mathbb{CP}^1 model at weak coupling given in (3.334) consists of two infinite sums. A typical term in the first (second) sum is depicted on the right (left).

Finally let's come to the interior amplitude for the \mathbb{CP}^1 model at weak coupling. We set

$$\beta = \sum_{\gamma_{12} + \gamma_{21} = \gamma} \psi_{-\gamma_1} \phi_{\gamma_{12}} \phi_{\gamma_{21}} + \sum_{\gamma_{12} + \gamma_{21} = -\gamma} \phi_{\gamma_{12}} \psi_{\gamma_2} \phi_{\gamma_{21}}. \quad (3.334)$$

Note that β expressed as above satisfies the basic symmetry requirements; it has the correct flavor and fermion degrees, and moreover is invariant under the \mathbb{Z}_2 -symmetry of the \mathbb{CP}^1 model that exchanges the two vacua, the two infinite sums being exchanged under the symmetry. Typical fans contributing to each of the two sums are depicted in Figure 3.17. Because β as in (3.334) only has non-zero components are in $R_{(12)}$, namely $\beta_1 = \beta_2 = 0$, the Maurer-Cartan equation (3.318) collapses to

$$[\beta, \beta] = 0. \quad (3.335)$$

This is satisfied, most importantly as a consequence of

$$[\psi_{-\gamma_1} \phi_{\gamma_{12}} \phi_{\bar{\gamma}_{21} + \gamma}, \phi_{\gamma_{12} - \gamma} \psi_{\gamma_2} \phi_{\bar{\gamma}_{21}}] + [\phi_{\gamma_{12}} \psi_{\gamma_2} \phi_{\bar{\gamma}_{21} - \gamma}, \psi_{-\gamma_1} \phi_{\gamma_{21} + \gamma} \phi_{\bar{\gamma}_{21}}] = 0, \quad (3.336)$$

which holds for any $\gamma_{12} \in \Gamma_{21}$ and $\bar{\gamma}_{21}$ denotes as usual the element of Γ_{21} such that $\gamma_{12} + \bar{\gamma}_{21} = 0$. This identity is demonstrated diagrammatically in Figure 3.18. The other part of showing the identity is the fact that the non-cross terms, terms of the type

$$[\psi_{-\gamma_1} \phi_{\gamma_{12}} \phi_{\bar{\gamma}_{21} + \gamma}, \psi_{-\gamma_1} \phi_{\bar{\gamma}_{21}} \phi_{\gamma_{21} + \gamma}] \quad (3.337)$$

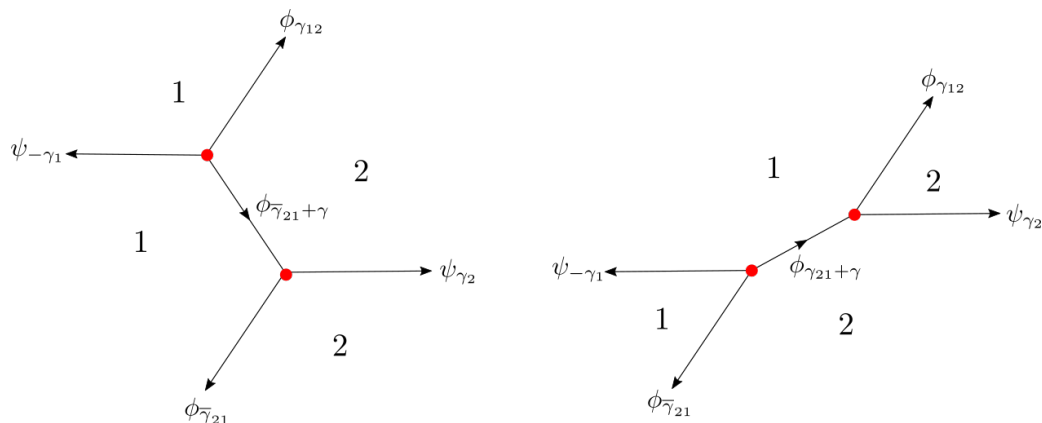


Figure 3.18: Two webs that lead to canceling contributions when verifying the Maurer-Cartan equation (3.336). In terms of ζ -instantons, the diagram shows the two ends of $\mathcal{M}_\zeta(\psi_{-\gamma_1}, \phi_{\gamma_{12}}, \psi_{\gamma_2}, \phi_{\bar{\gamma}_{21}})$ which has one reduced modulus. At the ends it looks like glued rigid trivalent instantons.

identically vanishes due to Fermi statistics $\psi_{-\gamma_1}^2 = 0$.

The reader might ask about the physical justification behind the Maurer-Cartan element given in (3.334). The conceptual content of this equation is simply that every possible rigid ζ -instanton⁹ is indeed uniquely occupied. We are thus claiming that the \mathbb{CP}^1 model at weak coupling has infinitely many occupied trivalent ζ -instantons. See section 4.7 of [Gal] for some numerical evidence of this conjecture.

3.4.3 The A_∞ Category of Thimbles

Our next step is to construct a wall-crossing invariant strong enough that its wall-crossing invariance allow us to determine the chain complexes on one side of the wall, up to homotopy, given the knowledge of the chain complexes and interior amplitude on the other.

⁹Note that the rigidity requirement, namely requiring that β have degree +2 restricts us to trivalent instantons only

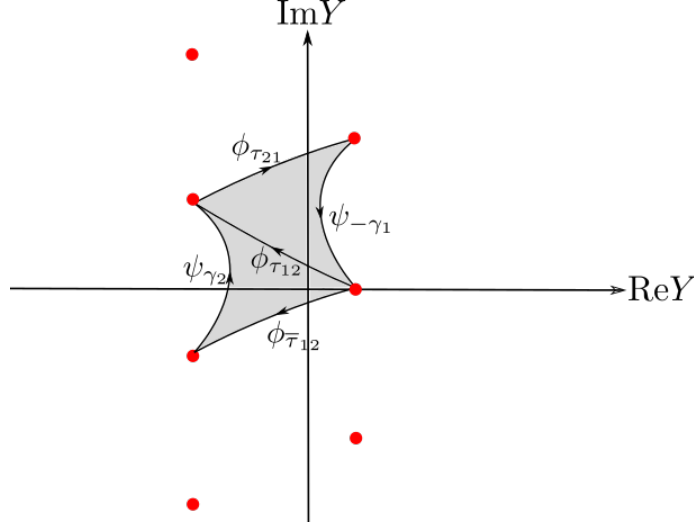


Figure 3.19: Images of ζ -instantons in the \mathbb{CP}^1 model, in the Y -plane at weak coupling. The top shaded region contributes the term $\psi_{-\gamma_1} \phi_{\tau_{12}} \phi_{\tau_{21}}$ to β whereas the bottom shaded region contributes $\phi_{\bar{\tau}_{12}} \psi_{\gamma_2} \phi_{\bar{\tau}_{21}}$.

We work with the categorical spectrum generator \widehat{R} . More precisely, we consider the category of thimbles with objects $\mathfrak{T}_i, \mathfrak{T}_j$, and recall that the matrix elements of \widehat{R} are identified with the morphism spaces of these thimbles

$$\widehat{R}_{ab} = \text{Hop}(\mathfrak{T}_a, \mathfrak{T}_b), \quad (3.338)$$

where $a, b \in \{i, j\}$ and i, j denote the two vacua. Thus, we consider a category with objects $\mathfrak{T}_i, \mathfrak{T}_j$ such that the morphism spaces, computed from the spectrum generator, are given by (3.61)-(3.64):

$$\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_i) = \mathcal{F}^*[V_{\gamma_i}] \oplus_{n,m \geq 0} R_{\tau_{ij}+n\gamma} \otimes \mathcal{F}^*[V_{\gamma_j}] \otimes R_{\tau_{ji}+m\gamma}, \quad (3.339)$$

$$\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j) = \oplus_{n \geq 0} R_{\tau_{ij}+n\gamma} \otimes \mathcal{F}^*[V_{\gamma_j}], \quad (3.340)$$

$$\text{Hop}(\mathfrak{T}_j, \mathfrak{T}_i) = \oplus_{n \geq 0} \mathcal{F}^*[V_{\gamma_j}] \otimes R_{\tau_{ji}+n\gamma}, \quad (3.341)$$

$$\text{Hop}(\mathfrak{T}_j, \mathfrak{T}_j) = \mathcal{F}^*[V_{\gamma_j}]. \quad (3.342)$$

Let

$$R_o = \oplus_{a,b \in \{i,j\}} \text{Hop}(\mathfrak{T}_a, \mathfrak{T}_b) \quad (3.343)$$

be the endomorphism algebra of this the thimble category.

Conjecture *The pair (R_o, R_c) carries the structure of an open-closed homotopy algebra¹⁰. In particular, the interior amplitude $\beta \in R_c$ determines canonically an A_∞ -structure on \widehat{R} , denoted as $\widehat{R}[\beta]$.*

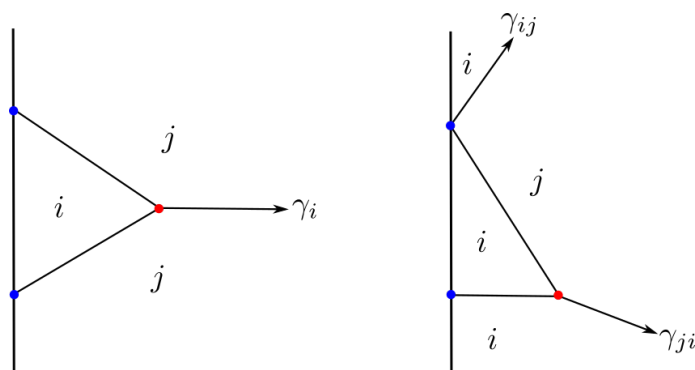
The structure maps

$$m_{p,q} : (R_o)^{\otimes p} \otimes (R_c)^{\otimes q} \rightarrow R_o \text{ for } p \geq 1, q \geq 0 \quad (3.344)$$

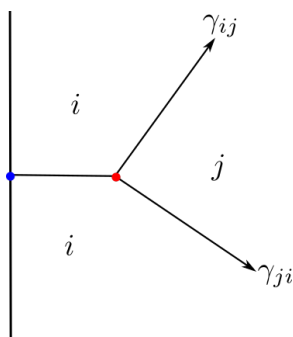
that give the LA_∞ -structure once again are expected to be determined by summing over graphs that now live in the half-plane $\mathbb{H} \subset \mathbb{C}$ used to work out the spectrum generator. We remind the reader that we are working with the right-half plane. The graphs that contribute to $m_{p,q}$ in general carry p vertices on the boundary, q vertices in the bulk, and have $p + 2q - 2$ internal edges (or contractions). In [GMW] such half-planar graphs were known as *taut half-plane webs*. One can easily produce examples of such graphs that would contribute to the conjectured LA_∞ -structure in the present setup.

Once again, there are first the traditional sort of taut webs where the only contractions that are occurring are between ij -solitons or not more than a single ii -soliton. Some examples of such half-plane webs are

¹⁰An open-closed homotopy algebra [KS] is also known as an LA_∞ -algebra in the language of [GMW].



as well as the web



These fit well into the framework of [GMW] and contribute to the operations $m_{2,1}$, $m_{2,1}$ and $m_{1,1}$ respectively. On the other hand, there are once again new diagrams that must be taken into account that generalize the third of the webs that we depicted to the case when multiple solitons are being shot out of the half-plane, at a single boundary vertex. For instance, if two periodic solitons emanate from a boundary vertex, one can consider the generalized web \mathfrak{w}_2 taking the form

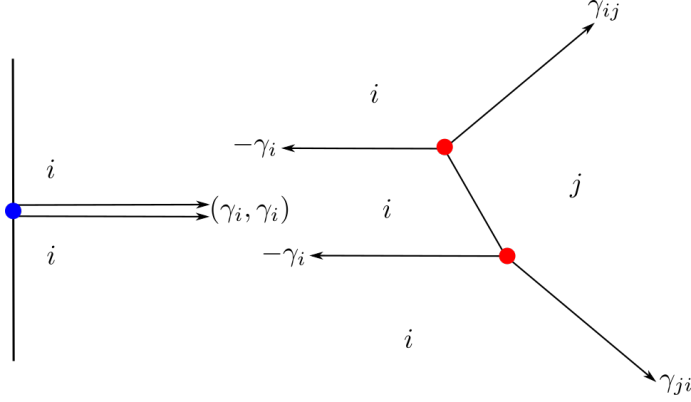


Figure 3.20: A new sort of half-plane web where multiple periodic solitons emanating from a single vertex are being contracted.

This has one boundary vertex and two bulk vertices and therefore contributes to a bracket $m_{1,2}$ via

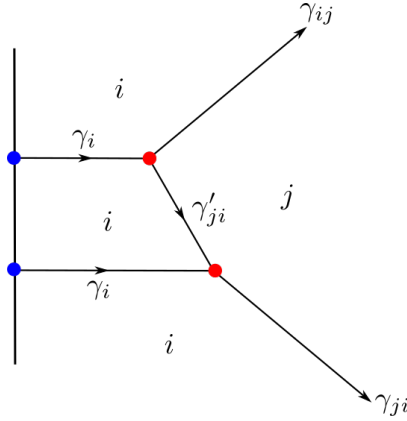
$$\begin{aligned} \rho(\mathfrak{w}_2) \left[e_{\gamma_i}^a e_{\gamma_i}^b, e_{-\gamma_i}^c e_{\gamma_{ij}}^d e_{\gamma'_{ji}}^e, e_{-\gamma_i}^f e_{\gamma'_{ij}}^g e_{\gamma_{ji}}^h \right] &= (K_{\gamma_i})^{ac} (K_{\gamma_i})^{bf} (K_{\gamma'_{ji}})^{eg} e_{\gamma_{ij}}^d e_{\gamma_{ji}}^h \\ &\quad \pm (K_{\gamma_i})^{af} (K_{\gamma_i})^{bc} (K_{\gamma'_{ji}})^{eg} e_{\gamma_{ij}}^d e_{\gamma_{ji}}^h. \end{aligned} \quad (3.345)$$

One can generalize this diagram to \mathfrak{w}_n where n periodic solitons emanate from a single boundary vertex and are being contracted through n bulk vertices, thus contributing to $m_{1,n}$.

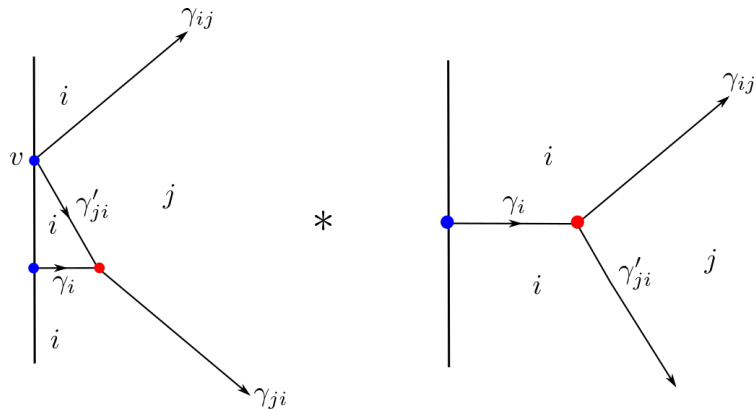
We now discuss the importance of graphs such as \mathfrak{w}_n and the corresponding brackets they induce to verifying the appropriate algebraic axioms. In the present case, we have to check the LA_∞ -axioms. In [GMW] such axioms were a consequence of certain properties the moduli spaces of sliding webs possess. One can define a convolution operation $*$ of half-plane webs with other half plane webs, along with half-plane webs with planar webs. The taut half-plane and taut plane webs satisfy the identity

$$\mathfrak{t}_{\mathbb{H}} * \mathfrak{t}_{\mathbb{P}} + \mathfrak{t}_{\mathbb{H}} * \mathfrak{t}_{\mathbb{H}} = 0. \quad (3.346)$$

This follows from the fact that any given sliding half-plane web can be obtained by convolving a taut half-plane web with either another taut half-plane web or a taut planar web in exactly two ways. The two different ways correspond to terms that cancel when verifying the LA_∞ -identities. Therefore let us study sliding half-plane webs in the present setup. Consider the following sliding web \mathfrak{s} .

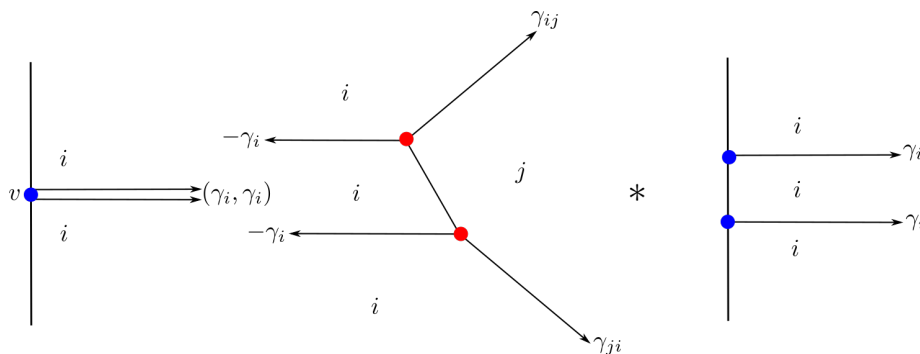


On the one hand, there is a pair of taut webs such that a standard convolution produces \mathfrak{s} . This pair and the vertex v at which the convolution occurs is depicted below.



On the other hand, there is no second pair of taut webs whose standard convolution would give the sliding web \mathfrak{s} , as a result. Thus without including any additional type of

operations or diagrams, the LA_∞ -axioms would not be satisfied. This is where the web \mathfrak{w}_2 comes to the rescue. With the addition of \mathfrak{w}_2 to the taut half-planar element $\mathfrak{t}_{\mathbb{H}}$, we find the following generalized convolution produces precisely the sliding web \mathfrak{s} . Note that the half-plane fan at the vertex v is (γ_i, γ_i) , which coincides with the fan at infinity of the half-plane web on the right, so that convolution at v still makes sense.



Thus, with the proper inclusion of additional webs, we fully expect the LA_∞ -axioms to hold, giving us an open-closed homotopy algebra structure on the pair (R_o, R_c) .

The main use of the LA_∞ -axioms is that they allow us to use the interior amplitude β to give a natural A_∞ -structure on \widehat{R} . The procedure for doing this is simple. We simply plug in the interior amplitude for each bulk vertex in the taut half-plane webs. We denote the categorical spectrum generator with the A_∞ -maps on its matrix elements induced from β as $\widehat{R}[\beta]$.

Polygon Viewpoint

The reader who is dismayed by the new sort of webs that have to be included such as the one of Figure 3.20 might find a brief discussion of a dual viewpoint in terms of polygons more enlightening. Recall from [KKS] that in the case without twisted masses, cyclic fans of vacua correspond to convex polygons in the W -plane, and deformation types of planar

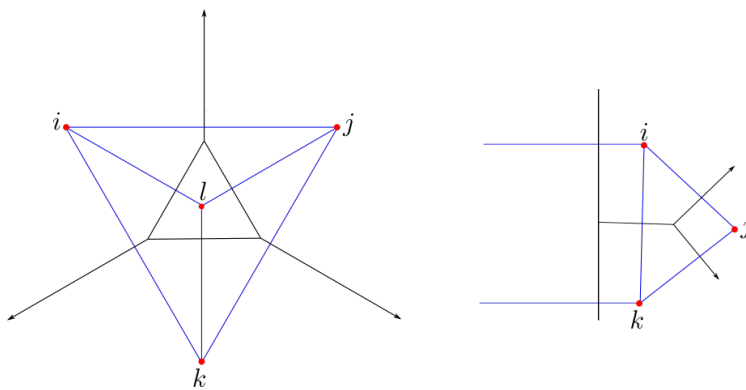


Figure 3.21: The left depicts a planar web and its corresponding dual, a subdivision of a polygon into subpolygons. The right depicts a half-plane web and its corresponding dual, a subdivision of a semi-infinite polygon into other finite and semi-infinite polygons.

webs are dual to tilings or subdivisions of such polygons. Similarly, half-plane webs are dual to subdivisions of semi-infinite polygons that go off to infinity in a direction dictated by the half-plane. See Figure 3.21 for some examples of such polygonal subdivisions and the corresponding dual webs. Notions such as taut webs and their convolutions can be formulated in this dual setup.

Now consider configurations of vacuum weights arising in theories with twisted masses. Going back to the example of this section, the configuration in the W -plane consists of two arithmetic progressions of complex numbers, as in Figure 3.23, left. Consider the semi-infinite polygon P with vertices $\{\tilde{i}, \tilde{j}, \tilde{i} + 2\gamma\}$ and its subdivision shown below. The tiling consists of T_A, T_B, T_C where T_A and T_B consists of finite triangles with vertices $\{\tilde{i}, \tilde{j}, \tilde{i} + \gamma\}$ and $\{\tilde{j}, \tilde{i} + 2\gamma, \tilde{i} + \gamma\}$, and T_C consists of semi-infinite polygon with four edges, and three vertices $\{\tilde{i}, \tilde{i} + \gamma, \tilde{i} + 2\gamma\}$. A degenerate polygon such as T_C where several vertices lie along a straight line do not appear without twisted masses. All we are saying in the previous section is that in the presence of twisted masses, in a sensible formalism, such degenerate polygons must be included when working out all possible tilings. Generalized webs such

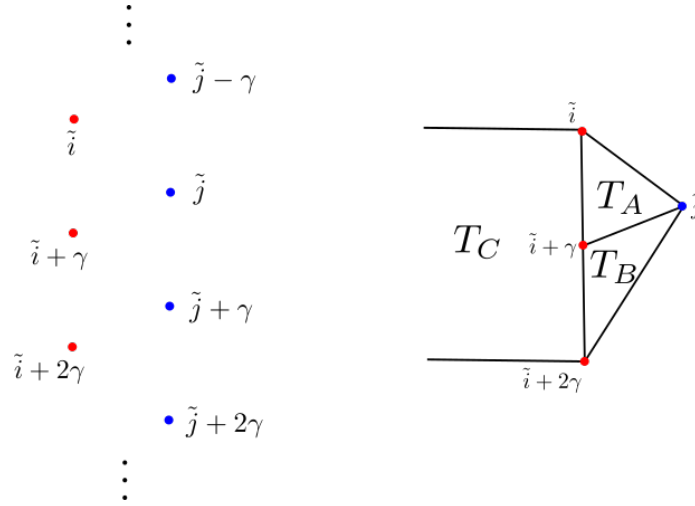


Figure 3.22: The left depicts the W -plane configuration of the two vacuum, rank one subsector we are focusing on in the present section. \tilde{i}, \tilde{j} denote chosen lifts of the vacua i and j . The right depicts a tiling of a semi-infinite polygon dual to the web of Figure 3.20.

as the one of Figure 3.20 then arise as the dual to subdivisions involving such degenerate polygons.

3.4.4 Equivalence of A_∞ -Categories and Categorical Wall-Crossing

We can now come to the statement of categorical wall-crossing. We have been working at a point of parameter space where the central charges corresponding to $(\tau_{ij}, \gamma_j, \tau_{ji}, -\gamma_i)$ are in clockwise order. We discussed the BPS \mathbb{Z} -modules $\{R_{\gamma_{ab}}\}$ and the interior amplitude $\beta \in R_c$. We also discussed the categorical spectrum generator and how the interior amplitude β gives rise to an A_∞ -structure on \hat{R} written as $\hat{R}[\beta]$. Suppose now that we cross the wall of marginal stability to the side so that the central charges corresponding to $(\tau_{ji}, \gamma_i, \tau_{ij}, -\gamma_j)$ are now clockwise ordered. The configuration of the central charges on either side of the wall of marginal stability is shown in Figure.

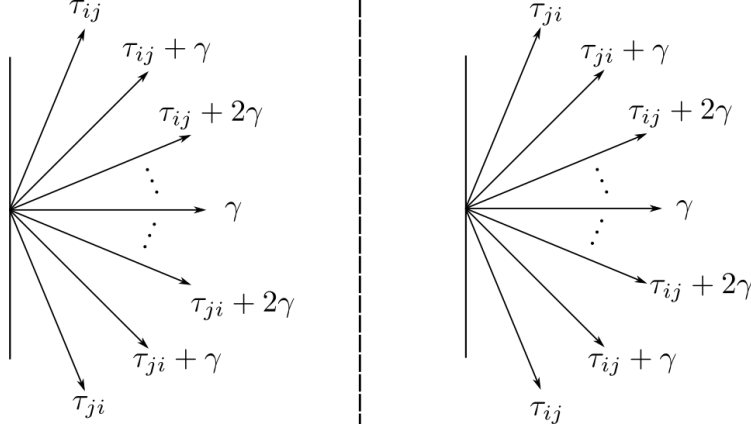


Figure 3.23: The configuration of central charge rays in the right-half plane on either side of the wall of marginal stability. On the left, we have that $(\tau_{ij}, \gamma_j, \tau_{ji})$ are clockwise ordered, whereas on the right we have that $(\tau_{ji}, \gamma_i, \tau_{ij})$ are clockwise ordered.

Suppose the BPS \mathbb{Z} -modules are given upon wall-crossing by $R'_{\gamma_{ij}}$ etc. We can now form

$$R'_c = R'_i \oplus R'_j \oplus R'_{(ij)} \quad (3.347)$$

where

$$R'_i = \mathcal{F}^*[V'_{\gamma_i}] \otimes \mathcal{F}^*[V'_{-\gamma_i}], \quad (3.348)$$

R'_j is defined in a similar way as R'_i , and

$$R'_{(ij)} = \bigoplus_{(\gamma_{ij}, \gamma_{ji}) \in \Gamma_{ij} \times \Gamma_{ji}} R'_{\gamma_{ji}} \otimes \mathcal{F}[V'_{\gamma_i}] \otimes R'_{\gamma_{ij}} \otimes \mathcal{F}[V'_{-\gamma_j}], \quad (3.349)$$

as coming from the categorical trace of $(\widehat{R}')^{\text{opp}} \otimes \widehat{R}'$ where \widehat{R}' is obtained by composing the elementary factors associated to the rays $(\tau_{ji}, \tau_{ji} + \gamma, \dots, \gamma, \dots, \tau_{ij} + \gamma, \tau_{ij})$:

$$\widehat{R}' = \mathcal{S}'_{\tau_{ji}} \mathcal{S}'_{\tau_{ji} + \gamma} \mathcal{S}'_{\tau_{ji} + 2\gamma} \cdots \prod_{n=1}^{\infty} \mathcal{K}'_{n\gamma} \cdots \mathcal{S}'_{\tau_{ij} + 2\gamma} \mathcal{S}'_{\tau_{ij} + \gamma} \mathcal{S}'_{\tau_{ij}}. \quad (3.350)$$

The product can be computed out to give

$$\widehat{R}'_{ii} = \mathcal{F}^*[V'_{\gamma_i}], \quad (3.351)$$

$$\widehat{R}'_{ij} = \oplus_{n \geq 0} \mathcal{F}^*[V'_{\gamma_i}] \otimes R'_{\tau_{ij} + n\gamma}, \quad (3.352)$$

$$\widehat{R}'_{ji} = \oplus_{n \geq 0} R'_{\tau_{ji} + n\gamma} \otimes \mathcal{F}^*[V'_{\gamma_i}], \quad (3.353)$$

$$\widehat{R}'_{jj} = \mathcal{F}^*[V'_{\gamma_j}] \oplus \oplus_{n,m \geq 0} R'_{\tau_{ji} + n\gamma} \otimes \mathcal{F}^*[V'_{\gamma_i}] \otimes R'_{\tau_{ij} + m\gamma}. \quad (3.354)$$

Statement of Categorical Wall-Crossing Let $a, b \in \{i, j\}$. Suppose $(\{R_{\gamma_{ab}}\}, \beta \in (R_c)^{(2,0)})$ and $(\{R'_{\gamma_{ab}}\}, \beta' \in (R'_c)^{(2,0)})$ are the BPS \mathbb{Z} -modules and interior amplitudes on either side of the wall of marginal stability where τ_{ij} and τ_{ji} align. Then

$$\widehat{R}[\beta] \simeq \widehat{R}'[\beta'] \quad (3.355)$$

where \simeq denotes homotopy equivalence of A_∞ -categories. Moreover, given $(\{R_{\gamma_{ab}}\}, \beta \in (R_c)^{(2,0)})$, the wall-crossing invariance of $\widehat{R}[\beta]$, up to homotopy, determines the BPS \mathbb{Z} -modules and interior amplitude $(\{R'_{\gamma_{ab}}\}, \beta' \in (R'_c)^{(2,0)})$ uniquely up to homotopy.

In the rest of this section we demonstrate how all this works for the \mathbb{CP}^1 model with twisted masses.

Categorical Wall-Crossing in the \mathbb{CP}^1 Model

Let's recall that the wall of marginal stability in the \mathbb{CP}^1 model. There is a marginal stability curve WMS in the $u = \frac{m}{\Lambda}$ plane that separates two regions. The region inside WMS is known as the strong coupling regime and the region outside WMS is known as the weak coupling regime.

Let's begin our discussion of the BPS \mathbb{Z} -modules $\{R_{\gamma_{ab}}^s\}$ and interior amplitude $\beta^s \in (R_c)^s$. At strong coupling it is well-known that there are two 12-solitons carrying charges

τ_{12} and $\bar{\tau}_{12}$ ¹¹ so that we have

$$R_{\tau_{12}}^s = \mathbb{Z}, \quad (3.356)$$

$$R_{\bar{\tau}_{12}}^s = \mathbb{Z}. \quad (3.357)$$

The corresponding anti-particles are

$$R_{\tau_{21}}^s = \mathbb{Z}^{[1]}, \quad (3.358)$$

$$R_{\bar{\tau}_{21}}^s = \mathbb{Z}^{[1]}. \quad (3.359)$$

Because there are no periodic solitons in the spectrum at strong coupling, one has

$$R_1^s \cong R_2^s \cong \mathbb{Z} \quad (3.360)$$

generated by ϕ_1, ϕ_2 respectively, and

$$R_{(12)}^s = (R_{\tau_{12}} \otimes R_{\bar{\tau}_{21}}) \oplus (R_{\bar{\tau}_{12}} \otimes R_{\tau_{21}}) \oplus (R_{\tau_{12}} \otimes R_{\tau_{21}}) \oplus (R_{\bar{\tau}_{12}} \otimes R_{\bar{\tau}_{21}}) \quad (3.361)$$

is four-dimensional, with the four summands carrying degrees $(1, 0), (1, 0), (1, \gamma), (1, -\gamma)$ respectively. Because the degree $(2, 0)$ component of $(R_c)^s$ is trivial, the strong coupling interior amplitude is constrained to vanish

$$\beta^s = 0. \quad (3.362)$$

The spectrum generator \hat{R}^s on the other hand is given by

$$\hat{R}_{11}^s = \mathbb{Z}\langle\phi_1\rangle, \quad (3.363)$$

$$\hat{R}_{12}^s = \mathbb{Z}\langle\phi_{\tau_{12}}\rangle, \quad (3.364)$$

$$\hat{R}_{21}^s = \mathbb{Z}\langle\phi_{\tau_{21}}\rangle, \quad (3.365)$$

$$\hat{R}_{21}^s = \mathbb{Z}\langle\phi_2\rangle \oplus \mathbb{Z}\langle\phi_{\tau_{21}} \otimes \phi_{\tau_{12}}\rangle, \quad (3.366)$$

¹¹Recall that the two charges we are using are denoted by τ_{12} and τ_{21} so that $\tau_{21} + \tau_{12} = \gamma$, where γ generates Γ . Their inverses are denoted as $\bar{\tau}_{21}$ and $\bar{\tau}_{12}$ respectively, so that there are a total of four BPS rays in the game.

coming from the factorized form

$$\widehat{R}^s = \begin{pmatrix} \mathbb{Z}\langle\phi_1\rangle & 0 \\ \mathbb{Z}\langle\phi_{\tau_{21}}\rangle & \mathbb{Z}\langle\phi_2\rangle \end{pmatrix} \otimes \begin{pmatrix} \mathbb{Z}\langle\phi_1\rangle & \mathbb{Z}\langle\phi_{\tau_{12}}\rangle \\ 0 & \mathbb{Z}\langle\phi_2\rangle \end{pmatrix}. \quad (3.367)$$

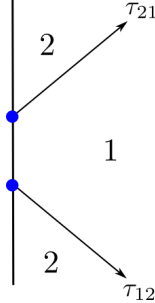
Because the interior amplitude is trivial, the A_∞ -algebra structure is also simple. Other than ϕ_1 and ϕ_2 acting as the identity in the first and second vacua, the only non-trivial product is

$$\cdot : \widehat{R}_{21}^s \otimes \widehat{R}_{12}^s \rightarrow \widehat{R}_{22}^s \quad (3.368)$$

being given by

$$\phi_{\tau_{21}} \cdot \phi_{\tau_{12}} = \phi_{\tau_{21}} \otimes \phi_{\tau_{12}}. \quad (3.369)$$

It results from the following simple taut half-plane web.



Thus we have determined $\widehat{R}^s[\beta^s]$.

Let's now come to the discussion at weak coupling. At weak coupling, the BPS spectrum and interior amplitude was discussed in Section 3.4.2. We remind the reader that the spectrum consisted of a soliton $\phi_{\gamma_{12}}$ for each $\gamma_{12} \in \Gamma_{12}$, and periodic solitons $a_{\gamma_1}, \psi_{\gamma_2}$ for $\gamma \in \Gamma$ the generator¹² along with their usual anti-particles $\phi_{\bar{\gamma}_{21}}, \psi_{-\gamma_1}, a_{-\gamma_2}$. The interior

¹²Once again, we clarify that $R_{n\gamma_1} = R_{n\gamma_2} = 0$ for each $n \geq \pm 1$.

amplitude was

$$\beta_w = \sum_{\gamma_{12}+\gamma_{21}=\gamma} \psi_{-\gamma_1} \phi_{\gamma_{12}} \phi_{\gamma_{21}} + \sum_{\gamma_{12}+\gamma_{21}=-\gamma} \phi_{\gamma_{12}} \psi_{\gamma_2} \phi_{\gamma_{21}}. \quad (3.370)$$

The categorical spectrum generator has components which now read

$$\widehat{R}_{11}^w = S^*[a_{\gamma_1}] \oplus_{(Z_{\gamma_{12}}, Z_{\gamma_{21}}) \in \mathbb{H}} \mathbb{Z}\langle\phi_{\gamma_{12}}\rangle \otimes \Lambda^*[\psi_{\gamma_2}] \otimes \mathbb{Z}\langle\phi_{\gamma_{21}}\rangle, \quad (3.371)$$

$$\widehat{R}_{12}^w = \oplus_{Z_{\gamma_{12}} \in \mathbb{H}} \mathbb{Z}\langle\phi_{\gamma_{12}}\rangle \otimes \Lambda^*[\psi_{\gamma_2}], \quad (3.372)$$

$$\widehat{R}_{21}^w = \Lambda^*[\psi_{\gamma_2}] \otimes \oplus_{Z_{\gamma_{21}} \in \mathbb{H}} \mathbb{Z}\langle\phi_{\gamma_{21}}\rangle, \quad (3.373)$$

$$\widehat{R}_{22}^w = \Lambda^*[\psi_{\gamma_2}] \quad (3.374)$$

coming from expanding out the factorized form

$$\prod_{n=0}^{\infty} \begin{pmatrix} \mathbb{Z} & \mathbb{Z}\langle\phi_{\tau_{12}+n\gamma}\rangle \\ 0 & \mathbb{Z} \end{pmatrix} \begin{pmatrix} S^*[a_{\gamma_1}] & 0 \\ 0 & \Lambda^*[\psi_{\gamma_2}] \end{pmatrix} \prod_{n=1}^{\infty} \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}\langle\phi_{\tau_{21}+n\gamma}\rangle & \mathbb{Z} \end{pmatrix}. \quad (3.375)$$

Note that the set of γ_{12} such that $Z_{\gamma_{12}} \in \mathbb{H}$ can be written as $\{\tau_{12} + n\gamma\}_{n=0,1,2,\dots}$ and similarly $\gamma_{21} \in \mathbb{H}$ can be written as $\{\tau_{21} + n\gamma\}_{n=0,1,2,\dots}$.

We now come to specifying the A_∞ -structure. The first task is to specify the differential d . This is determined by (generalized) half-plane taut webs with a single boundary vertex. We specify d on the individual components of \widehat{R} .

We begin by specifying the action of \widehat{d}_β on \widehat{R}_{11}^w . \widehat{R}_{11} is generated by $a_{\gamma_1}^k$ for any $k \geq 0$, and terms $\phi_{\gamma_{12}}\phi_{\gamma_{21}}$ and $\phi_{\gamma_{12}}\psi_{\gamma_2}\phi_{\gamma_{21}}$ and each of these terms have fermionic degrees 0, 1 and 2 respectively. The differential \widehat{d}_β then turns \widehat{R}_{11} into a complex

$$0 \longrightarrow \widehat{R}_{11}^{(0,*)} \xrightarrow{\widehat{d}_\beta} \widehat{R}_{11}^{(1,*)} \xrightarrow{\widehat{d}_\beta} \widehat{R}_{11}^{(2,*)} \longrightarrow 0 \quad (3.376)$$

as follows. Because $a_{\gamma_1}^k \in \widehat{R}_{11}^{(0,k\gamma)}$ is visualized as k bosonic periodic solitons emanating from a single half-plane vertex, one has to consider the new type of web we discussed in Section

3.4.3, for instance Figure 3.20. In this case there is a web $\mathfrak{w}_k^{\gamma_{12}}$ with a single boundary vertex for each $k \geq 1$ and γ_{12} such that $Z_{\gamma_{12}}$ and $Z_{\bar{\gamma}_{21}+k\gamma}$ are both in the right half plane. The boundary vertex of $\mathfrak{w}_k^{\gamma_{12}}$ spits out k rays parallel to Z_γ each of which carries a bosonic 11 soliton. In the bulk one has k vertices v_1, \dots, v_k with corresponding fans

$$(-\gamma_1, \gamma_{12}, \bar{\gamma}_{21} + \gamma), (-\gamma_1, \gamma_{12} - \gamma, \bar{\gamma}_{21} + 2\gamma), \dots, (-\gamma_1, \gamma_{12} - k\gamma, \bar{\gamma}_{21} + k\gamma). \quad (3.377)$$

The k periodic 11 solitons of charge γ are contracted with the k periodic solitons of charge $-\gamma$, one from each of the k bulk vertices, whereas the soliton with charge $\bar{\gamma}_{21} + j\gamma$ in v_j is contracted with the soliton of charge $\gamma_{12} - j\gamma$ in v_{j+1} for $j = 1, \dots, k-1$. Altogether there are $2k-1$ contractions, the right number to give a contribution to $m_{1,k}$. The web $\mathfrak{w}_k^{\gamma_{12}}$ for $k=2$ is depicted below.

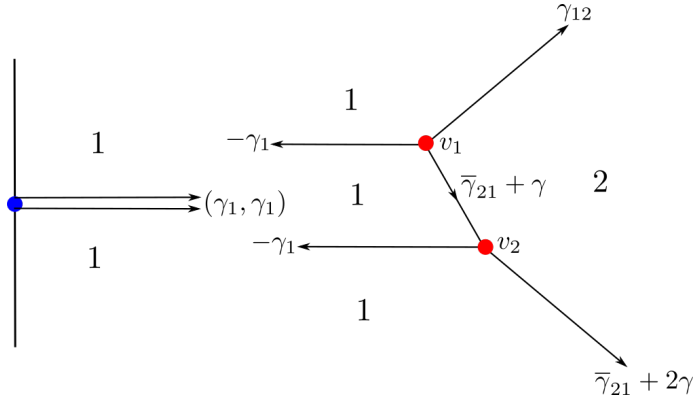


Figure 3.24: A novel half-plane web $\mathfrak{w}_2^{\gamma_{12}}$ that contributes to $\hat{d}_\beta[a_{\gamma_1}^2]$.

Plugging in the interior amplitude component $\psi_{-\gamma_1} \phi_{\gamma_{12}-j\gamma} \phi_{\bar{\gamma}_{21}+(j+1)\gamma}$ in the bulk vertex v_{j+1} , and doing the relevant contractions, we find

$$\rho(\mathfrak{w}_k^{\gamma_{12}})[a_{\gamma_1}^k] = \phi_{\gamma_{12}} \phi_{\bar{\gamma}_{21}+k\gamma}. \quad (3.378)$$

Summing up over all $\mathfrak{w}_k^{\gamma_{12}}$, we find

$$\widehat{d}_\beta[a_{\gamma_1}^k] = \sum_{\substack{(\gamma_{12}, \gamma_{21}) \text{ HP fan} \\ \gamma_{12} + \gamma_{21} = k\gamma}} \phi_{\gamma_{12}} \phi_{\gamma_{21}}, \quad (3.379)$$

for $k \geq 1$. Next let's specify how \widehat{d}_β acts on an element $\phi_{\gamma_{12}} \phi_{\gamma_{21}} \in \widehat{R}_{11}^{(1, \gamma_{12} + \gamma_{21})}$. There are two taut webs that contribute to this for generic¹³ $(\gamma_{12}, \gamma_{21})$ which are depicted in Figure 3.25, shown below.

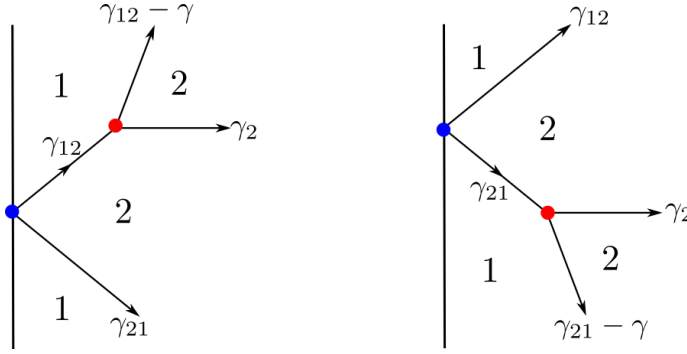


Figure 3.25: The taut half-plane webs that contribute to $\widehat{d}_\beta[\phi_{\gamma_{12}} \phi_{\gamma_{21}}]$.

Summing up the contribution from each of these webs, we find

$$\widehat{d}_\beta[\phi_{\gamma_{12}} \phi_{\gamma_{21}}] = \phi_{\gamma_{12} - \gamma} \psi_{\gamma_2} \phi_{\gamma_{21}} - \phi_{\gamma_{12}} \psi_{\gamma_2} \phi_{\gamma_{21} - \gamma}. \quad (3.380)$$

Because there are no elements of degree three, the differential on the degree two elements $\phi_{\gamma_{12}} \psi_{\gamma_2} \phi_{\gamma_{12}}$ is constrained to vanish

$$\widehat{d}_\beta[\phi_{\gamma_{12}} \psi_{\gamma_2} \phi_{\gamma_{12}}] = 0. \quad (3.381)$$

¹³The non-generic case is when γ_{12} is such that $\gamma_{12} - \gamma$ leaves the half-plane, so that the half-plane web depicted in the left of Figure 3.25 does not exist. One simply sets the corresponding contribution to the differential to zero. Similarly, if γ_{21} is such that $\gamma_{21} - \gamma$ leaves the half-plane, the term in the differential coming from the right half-plane web of Figure 3.25 is set to vanish.

Next, the differential on the 12 complex

$$0 \longrightarrow \widehat{R}_{12}^{(0,*)} \xrightarrow{\widehat{d}_\beta} \widehat{R}_{12}^{(1,*)} \longrightarrow 0 \quad (3.382)$$

acts via

$$\widehat{d}_\beta[\phi_{\gamma_{12}}] = \phi_{\gamma_{12}-\gamma}\psi_{\gamma_2}, \quad (3.383)$$

for generic γ_{12} . Similarly, the differential on the 21 complex

$$0 \longrightarrow \widehat{R}_{21}^{(1,*)} \xrightarrow{\widehat{d}_\beta} \widehat{R}_{21}^{(2,*)} \longrightarrow 0 \quad (3.384)$$

acts via

$$\widehat{d}_\beta[\phi_{\gamma_{21}}] = -\psi_{\gamma_2}\phi_{\gamma_{21}-\gamma}. \quad (3.385)$$

The expressions for \widehat{d}_β on the 12 and 21 components come from the taut webs in the left and right of Figure 3.26, shown below, respectively.

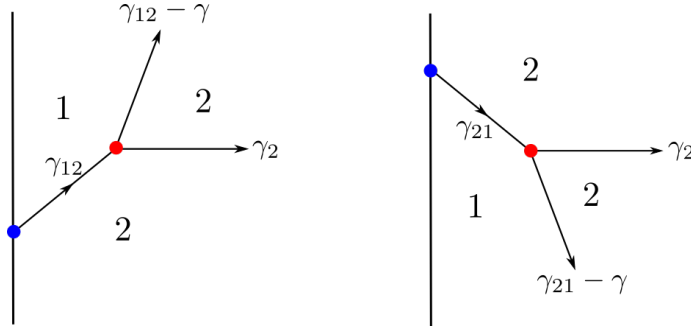


Figure 3.26: The left and right taut half-plane webs contribute to $\widehat{d}_\beta[\phi_{\gamma_{12}}]$ and $\widehat{d}_\beta[\phi_{\gamma_{21}}]$ respectively.

Finally, the differential on \widehat{R}_{22} vanishes identically by degree reasons.

The next A_∞ map to discuss is the multiplication $m_\beta[\cdot, \cdot]$. The bilinear multiplication $m_\beta[\cdot, \cdot]$ comes from summing up the contribution of taut half-plane webs with two boundary vertices and inserting β in the bulk vertices. We begin with specifying the map $m_\beta[\cdot, \cdot] : \widehat{R}_{11}^{\otimes 2} \rightarrow \widehat{R}_{11}$. First, the standard taut web gives

$$m_\beta[a_{\gamma_1}^k, a_{\gamma_1}^l] = a_{\gamma_1}^{k+l}, \quad (3.386)$$

independent of β . Next we have

$$m_\beta[a_{\gamma_1}^k, \phi_{\gamma_{12}} \phi_{\gamma_{21}}] = \phi_{\gamma_{12}+k\gamma} \phi_{\gamma_{21}}, \quad (3.387)$$

$$m_\beta[\phi_{\gamma_{12}} \phi_{\gamma_{21}}, a_{\gamma_1}^k] = \phi_{\gamma_{12}} \phi_{\gamma_{21}+k\gamma}. \quad (3.388)$$

These two products result from the taut half-plane web with two boundary vertices, one of them carrying k periodic solitons, and k bulk vertices carrying different interior amplitude components. For instance (3.387), with $k = 2$ results from the web shown in the left of Figure 3.27. Inserting the periodic soliton ψ_{γ_2} between between the 12, 21 solitons, resulting in

$$m_\beta[a_{\gamma_1}^k, \phi_{\gamma_{12}} \psi_{\gamma_2} \phi_{\gamma_{21}}] = \phi_{\gamma_{12}+k\gamma} \psi_{\gamma_2} \phi_{\gamma_{21}}, \quad (3.389)$$

$$m_\beta[\phi_{\gamma_{12}} \psi_{\gamma_2} \phi_{\gamma_{21}}, a_{\gamma_1}^k] = \phi_{\gamma_{12}} \psi_{\gamma_2} \phi_{\gamma_{21}+k\gamma}. \quad (3.390)$$

The webs these products follow from simply consist of taking the webs corresponding to (3.387) and (3.388) and inserting a ray carrying a ψ_{γ_2} soliton at the boundary vertex with fan $(\gamma_{12}, \gamma_{21})$ so that the new half-plane fan at the vertex is $(\gamma_{12}, \gamma_2, \gamma_{21})$. Finally, the only other non-vanishing product between elements of \widehat{R}_{11} is

$$m_\beta[\phi_{\gamma_{12}} \phi_{\tau_{21}}, \phi_{\tau_{12}} \phi_{\gamma_{21}}] = \phi_{\gamma_{12}} \psi_{\gamma_2} \phi_{\gamma_{21}}, \quad (3.391)$$

resulting from taut web shown in the right of Figure 3.27.

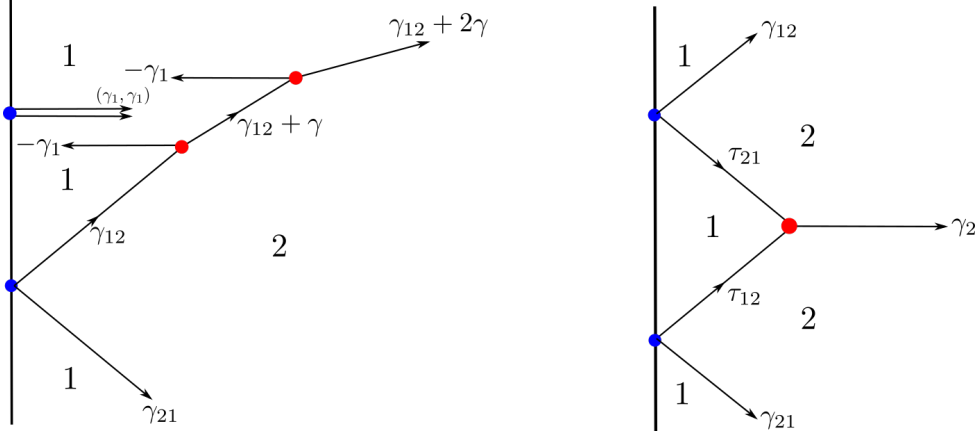


Figure 3.27: Some taut half-plane webs resulting in non-trivial products between elements of \widehat{R}_{11} . In particular, the left depicts the web resulting in the product (3.387) for $k = 2$. The right shows the web that results in the product (3.391).

Next we specify $m_\beta : \widehat{R}_{11} \otimes \widehat{R}_{12} \rightarrow \widehat{R}_{12}$. The webs that contribute to the non-zero products can easily be classified and they result in the following non-trivial products:

$$m_\beta[a_{\gamma_1}^k, \phi_{\gamma_{12}}] = \phi_{\gamma_{12}+k\gamma}, \quad (3.392)$$

$$m_\beta[a_{\gamma_1}^k, \phi_{\gamma_{12}}\psi_{\gamma_2}] = \phi_{\gamma_{12}+k\gamma}\psi_{\gamma_2}, \quad (3.393)$$

$$m_\beta[\phi_{\gamma_{12}}\phi_{\tau_{21}}, \phi_{\tau_{12}}] = \phi_{\gamma_{12}}\psi_{\gamma_2}. \quad (3.394)$$

Next $m_\beta : \widehat{R}_{12} \otimes \widehat{R}_{21} \rightarrow \widehat{R}_{11}$ is given simply by the tensor product (modulo the relation $\psi_{\gamma_1}^2 = 0$). Such products result from half-plane webs with no bulk vertices. $m_\beta : \widehat{R}_{12} \otimes \widehat{R}_{22} \rightarrow \widehat{R}_{12}$ is also given by an ordinary tensor product. $m_\beta : \widehat{R}_{21} \otimes \widehat{R}_{11} \rightarrow \widehat{R}_{21}$ is given by

$$m_\beta[\phi_{\gamma_{21}}, a_{\gamma_1}^k] = \phi_{\gamma_{21}+k\gamma}, \quad (3.395)$$

$$m_\beta[\psi_{\gamma_2}\phi_{\gamma_{21}}, a_{\gamma_1}^k] = \psi_{\gamma_2}\phi_{\gamma_{21}+k\gamma}. \quad (3.396)$$

The only non-zero product $m_\beta : \widehat{R}_{21} \otimes \widehat{R}_{12} \rightarrow \widehat{R}_{22}$ is given by

$$m_\beta[\phi_{\tau_{21}}, \phi_{\tau_{12}}] = \psi_{\gamma_2}. \quad (3.397)$$

Finally, the products $m_\beta : \widehat{R}_{22} \otimes \widehat{R}_{21} \rightarrow \widehat{R}_{21}$ and $m_\beta : \widehat{R}_{22} \otimes \widehat{R}_{22} \rightarrow \widehat{R}_{22}$ are given by the ordinary product modulo the relation $\psi_{\gamma_1}^2 = 0$. This finishes specifying the bilinear map m_β .

The higher multiplications are all trivial. One can verify the A_∞ axioms explicitly. Namely that $\widehat{d}_\beta^2 = 0$, that m_β is associative and that \widehat{d}_β is a derivation with respect to m_β . Thus we have identified the A_∞ -structure on the weak coupling categorical spectrum generator $\widehat{R}^w[\beta^w]$.

We now discuss the homotopy equivalence claim

$$\widehat{R}^s[\beta^s] \simeq \widehat{R}^w[\beta^w]. \quad (3.398)$$

Recall that the homotopy equivalence of A_∞ -algebras is equivalent to providing an A_∞ -quasi-isomorphism [Kel]. Therefore it suffices to provide an A_∞ morphism

$$T : \widehat{R}^s[\beta^s] \rightarrow \widehat{R}^w[\beta^w] \quad (3.399)$$

that induces an isomorphism on cohomology. We claim that

$$T_1(\phi_1^s) = \phi_1^w, \quad (3.400)$$

$$T_1(\phi_{\tau_{12}}^s) = \phi_{\tau_{12}}^w, \quad (3.401)$$

$$T_1(\phi_{\tau_{21}}^s) = \phi_{\tau_{21}}^w \quad (3.402)$$

$$T_1(\phi_2^s) = \phi_2^w, \quad (3.403)$$

$$T_1(\phi_{\tau_{21}}^s \otimes \phi_{\tau_{12}}^s) = \psi_{\gamma_2}^w, \quad (3.404)$$

with $T_n = 0$ for $n \geq 2$ does the job. Since $\phi_{\tau_{12}}^w$, $\phi_{\tau_{21}}^w$ and ψ_{γ_2} are all \widehat{d}_{β^w} -closed elements, T_1 defines a chain map. We also have the relation

$$T_1(m_{\beta^s}(\phi_{\tau_{21}}^s, \phi_{\tau_{12}}^s)) = m_{\beta^w}(\phi_{\tau_{21}}^w, \phi_{\tau_{12}}^w) \quad (3.405)$$

due to (3.397). Thus T defines a strict A_∞ -morphism. It remains to check that T_1 -induces an isomorphism on cohomology.

Let's therefore compute the cohomology of the weak coupling spectrum generator with respect to \widehat{d}_β^w . Recall that $\tau_{12} \in \Gamma_{12}$ has $Z_{\tau_{12}} \in \mathbb{H}$, the right-half plane, but $Z_{\tau_{12}-\gamma} \notin \mathbb{H}$. Similarly $\tau_{21} \in \Gamma_{21}$ is such that $Z_{\tau_{21}} \in \mathbb{H}$ but $Z_{\tau_{21}-\gamma} \notin \mathbb{H}$. We then have that \widehat{R}_{12} is generated by $\phi_{\tau_{12}+n\gamma}$ and $\phi_{\tau_{12}+n\gamma}\psi_{\gamma_2}$ for $n \geq 0$ and

$$\widehat{d}_\beta[\phi_{\tau_{12}+n\gamma}] = \begin{cases} 0 & \text{for } n = 0, \\ \phi_{\tau_{12}+(n-1)\gamma}\psi_{\gamma_2} & \text{for } n \geq 1. \end{cases} \quad (3.406)$$

Therefore we find that $\phi_{\tau_{12}}$ is closed, and that the differential pairs up $\phi_{\tau_{12}+n\gamma}$ and $\phi_{\tau_{12}+(n-1)\gamma}\psi_{\gamma_2}$ for all $n \geq 1$. Thus we have

$$H_{\widehat{d}_\beta}^{(*,*)}(\widehat{R}_{12}) = \mathbb{Z}\langle\phi_{\tau_{12}}\rangle. \quad (3.407)$$

Very similar reasoning shows that

$$H_{\widehat{d}_\beta}^{(*,*)}(\widehat{R}_{21}) = \mathbb{Z}\langle\phi_{\tau_{21}}\rangle. \quad (3.408)$$

The differential acts trivially on \widehat{R}_{22} therefore

$$H_{\widehat{d}_\beta}^{(*,*)}(\widehat{R}_{22}) = \mathbb{Z}[\psi_{\gamma_2}]/\langle\psi_{\gamma_2}^2\rangle, \quad (3.409)$$

the exterior algebra in ψ_{γ_2} . Finally let's come to \widehat{R}_{11} . We organize elements in terms of their flavor degrees. Fix a flavor charge $k\gamma$ and set $C_k^* = \widehat{R}_{11}^{(*,k\gamma)}$. The degree zero component C_k^0 is rank one and is generated simply by $x_k := a_{\gamma_1}^k$. The degree one component C_k^1 is rank k and is generated by the elements

$$\{y_{n,k} := \phi_{\tau_{12}+n\gamma}\phi_{\tau_{21}+(k-(n+1))\gamma}\}_{n=0}^{k-1}. \quad (3.410)$$

Finally the degree two component C_k^2 is rank $k - 1$ and is generated by

$$\{z_{n,k} := \phi_{\tau_{12}+n\gamma}\psi_{\gamma_2}\phi_{\tau_{21}+(k-(n+2))\gamma}\}_{n=0}^{k-2}. \quad (3.411)$$

Thus we are considering the complex

$$0 \longrightarrow C_k^0 \xrightarrow{d} C_k^1 \xrightarrow{d} C_k^2 \longrightarrow 0, \quad (3.412)$$

with the differential given by rewriting (3.379) in the present notation:

$$dx_k = \sum_{n=0}^{k-1} y_{n,k}, \quad (3.413)$$

$$dy_{n,k} = z_{n-1,k} - z_{n,k}, \quad (3.414)$$

$$dz_{n,k} = 0. \quad (3.415)$$

In the second of these equations we set $z_{n,k} = 0$ if n is outside of the range $0 \leq n \leq k - 2$, so a little more explicitly, one has

$$dy_{0,k} = -z_{0,k}, \quad (3.416)$$

$$dy_{1,k} = z_{0,k} - z_{1,k}, \quad (3.417)$$

$$dy_{2,k} = z_{1,k} - z_{2,k} \quad (3.418)$$

$$\vdots \quad (3.419)$$

$$dy_{k-1,k} = z_{k-2,k}. \quad (3.420)$$

It is straightforward to show that C_k^* is an acyclic complex for $k \geq 1$: Clearly the degree zero cohomology is trivial since the only generator of C_k^0 , namely x_k is not d -closed. In degree one, the kernel of d is rank one and generated by $\sum_{n=0}^{k-1} y_{n,k}$ but this is precisely dx_k . Finally one has that $d : C_k^1 \rightarrow C_k^2$ is surjective:

$$z_{n,k} = -d\left(\sum_{m=0}^n y_{m,k}\right). \quad (3.421)$$

Thus we find

$$H^{(p,k\gamma)}(\widehat{R}_{11}) = \begin{cases} \mathbb{Z} & p = k = 0, \\ 0 & \text{otherwise} \end{cases}, \quad (3.422)$$

namely the cohomology of \widehat{R}_{11}^w is generated by class of ϕ_1^w . It follows that T_1 induces an isomorphism on cohomology, thereby establishing the homotopy equivalence claim.

3.5 Algebra of the Infrared

Up to this point in this paper we have discussed bulk and boundary amplitudes insofar as they pertain to wall-crossing. We sketch a more general framework to discuss bulk and boundary amplitudes in the present section. Namely, given the data of a vacuum groupoid \mathbb{V} , deck group Γ , a central charge map Z , and BPS \mathbb{Z} -modules $\{R_{\gamma_{ab}}\}$ (with non-degenerate pairings $\{K_{\gamma_{ab}}\}$), how does one formulate the interior amplitude? Similarly, given framed BPS \mathbb{Z} -modules $\{\mathcal{E}_i(\mathfrak{B})\}$ for a brane \mathfrak{B} how does one formulate the boundary amplitude? Our goal is to sketch generalizations of the construction of [GMW] to include twisted masses. The discussion here will be incomplete. We only make some remarks and observations to be incorporated in a more complete treatment at another occasion.

Recall that in [GMW], one considers cyclic fans of vacua to be a cyclically ordered set of vacua $I = \{i_0, i_1, \dots, i_n\}$ such that the central charge rays $\{Z_{i_0 i_1}, Z_{i_1 i_2}, \dots, Z_{i_n i_0}\}$ were clockwise-ordered in the complex plane. To each cyclic fan one assigned the cyclic tensor product

$$R_I = R_{i_0 i_1} \otimes \cdots \otimes R_{i_n i_0}, \quad (3.423)$$

and the \mathbb{Z} -module obtained by summing over all cyclic fans $R_c = \oplus_I R_I$ carried the structure of an L_∞ -algebra. R_c then governed the theory of interior amplitudes. These notions are generalized as follows in the present, more general setup.

Definition A **cyclic fan of arrows** in the vacuum groupoid \mathbb{V} consists of a collection of arrows

$$\vec{\gamma} = (\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \dots, \gamma_{i_{n-1} i_n}, \gamma_{i_n i_0}) \quad (3.424)$$

where $\gamma_{ij} \in \text{Hom}(i, j) = \Gamma_{ij}$ (we don't necessarily assume that i and j are distinct) such that the central charges

$$(Z_{\gamma_{i_0 i_1}}, Z_{\gamma_{i_1 i_2}}, \dots, Z_{\gamma_{i_{n-1} i_n}}, Z_{\gamma_{i_n i_0}}) \quad (3.425)$$

are monotonically clockwise ordered. The **charge** of a cyclic fan is obtained by considering the composite arrow

$$\sum_{k=0}^n \gamma_{i_k i_{k+1}} = \gamma_{i_0} \in \text{Hom}(i_0, i_0), \quad (3.426)$$

which under the canonical isomorphism $\Gamma_{i_0} \cong \Gamma$ gives us an element $\gamma \in \Gamma$. The latter element γ is defined to be the charge of the cyclic fan. Equivalently the charge is defined by looking at the sum

$$Z_{\gamma_{i_0 i_1}} + Z_{\gamma_{i_1 i_2}} + \dots + Z_{\gamma_{i_n i_0}} = Z_{\gamma}, \quad (3.427)$$

and the right hand side is necessarily of the form Z_{γ} for some $\gamma \in \Gamma$.

Remark Equation (3.427) should be compared with the “no-force” condition, Equation 2.2 of [GMW], where the sum of central charges around a cyclic fan of vacua always added up to zero.

Remark One should also contrast with [GMW] in that a cyclic fan is now only partially ordered. This is because in the set-up of [GMW], successive central charges $Z_{i_{k-1} i_k}, Z_{i_k i_{k+1}}$ that appear in a cyclic fan, away from walls, never had the same phase. In contrast one can now have elements $\gamma_i, \gamma'_i \in \Gamma_i$ appearing in a cyclic fan of arrows that are multiples of

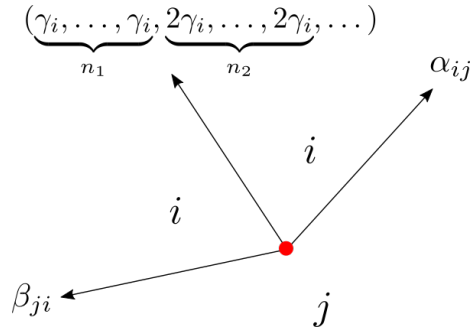


Figure 3.28: An example of a pictorial depiction of a cyclic fan of arrows. The arrow labeled by α_{ij} is parallel to $Z_{\alpha_{ij}}$, the one labeled by β_{ji} is parallel to $Z_{\beta_{ji}}$ and the arrow in between is parallel to Z_γ for some $\gamma \in \Gamma$ primitive, and so it is labeled by an arbitrary collection of positive integers (n_1, n_2, \dots) . The total charge of this fan is $n_1\gamma_i + 2n_2\gamma_i + \dots + \alpha_{ij} + \beta_{ji}$ and in this case it is clearly non-zero.

a common primitive element by, say, positive integers and then they will necessarily satisfy $\text{Arg}(Z_\gamma) = \text{Arg}(Z_{\gamma'})$. Thus, γ_i and γ'_i must be assigned the same phase order.

Remark Note that elements $\gamma_i \in \Gamma_i$ can appear in a cyclic fan arbitrarily many times.

A cyclic fan of arrows I is represented pictorially as a collection of rays coming out from a common point such that the angular sectors are labeled by vacua and a ray E is labeled by a subset $S_E \subset I$ such that for any $s \in S_E$ one has that Z_s and E are parallel. Thus if a ray separates distinct vacua $i \neq j$ then it carries a label of the unique element $\gamma_{ij} \in \Gamma_{ij}$ such that $Z_{\gamma_{ij}}$ is parallel to the ray. If a ray separates regions labeled by the same vacuum then it carries a subset of elements $\{\gamma_i\}$ all proportional by a positive number to some common primitive element. See Figure 3.28 for an example.

Definition A cyclic fan of arrows $(\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \dots, \gamma_{i_n i_0})$ is said to be **irreducible** if $i_k \neq i_{k+1}$, meaning successive vacua are distinct.

Any cyclic fan can be obtained from an irreducible cyclic fan by inserting an appropriate number of γ_i 's into an irreducible cyclic fan,

$$(\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \dots, \gamma_{i_n i_0}) \rightarrow (I_{i_0}, \gamma_{i_0 i_1}, I_{i_1}, \gamma_{i_1 i_2}, I_{i_2}, \dots, I_{i_n}, \gamma_{i_n i_0}), \quad (3.428)$$

where I_{i_k} consists of a (partially ordered) collection of elements in Γ_{i_k} such that for any element γ_{i_k} in the collection, $Z_{\gamma_{i_k}}$ lies in the angular sector bounded by $Z_{\gamma_{i_{k-1} i_k}}$ and $Z_{\gamma_{i_k i_{k+1}}}$. This process clearly does not conserve charge.

We can now come to the definition of R_c , the \mathbb{Z} -module we expect to govern interior amplitudes. First, given a pair $(\gamma_{ij}, \gamma_{jk})$ arrows where $i \neq j \neq k$ we let

$$\mathcal{F}_{(\gamma_{ij}, \gamma_{jk})} = \bigotimes_{\gamma_{ij} < \gamma_j < \gamma_{jk}} \mathcal{F}[R_{\gamma_j}]. \quad (3.429)$$

Conceptually, $\mathcal{F}_{(\gamma_{ij}, \gamma_{jk})}$ is the tensor product over Fock spaces of all periodic soliton spaces R_{γ_j} such that Z_{γ_j} lies in the angular sector swept out by going from the $Z_{\gamma_{ij}}$ ray to the $Z_{\gamma_{jk}}$ ray in a clockwise direction. Let

$$\vec{\gamma} = (\gamma_{i_0 i_1}, \dots, \gamma_{i_n i_0}) \quad (3.430)$$

be an irreducible fan. To it we assign the generalized cyclic tensor product

$$R_{\vec{\gamma}} := R_{\gamma_{i_0 i_1}} \otimes \mathcal{F}_{(\gamma_{i_0 i_1}, \gamma_{i_1 i_2})} \otimes R_{\gamma_{i_1 i_2}} \otimes \dots \otimes R_{\gamma_{i_n i_0}} \otimes \mathcal{F}_{(\gamma_{i_n i_0}, \gamma_{i_0 i_1})}. \quad (3.431)$$

Finally, R_c is obtained by summing over all irreducible fans

$$R_c = \bigoplus_{\substack{(\gamma_{i_0 i_1}, \dots, \gamma_{i_n i_0}) \\ \text{irreducible fan}}} R_{\gamma_{i_0 i_1}} \otimes \mathcal{F}_{(\gamma_{i_0 i_1}, \gamma_{i_1 i_2})} \otimes \dots \otimes R_{\gamma_{i_n i_0}} \otimes \mathcal{F}_{(\gamma_{i_n i_0}, \gamma_{i_0 i_1})}. \quad (3.432)$$

We clarify that in the sum above, the trivial irreducible cyclic fan (u_i) for each vacuum i is also included. To it we assign the space

$$R_i := R_{u_i} = \bigotimes_{\gamma_i \in \Gamma_i \setminus \{0\}} \mathcal{F}[R_{\gamma_i}]. \quad (3.433)$$

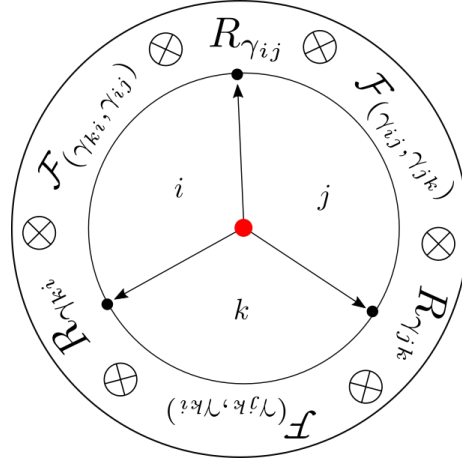


Figure 3.29: An illustration of the generalized cyclic product corresponding to an irreducible trivalent fan $(\gamma_{ij}, \gamma_{jk}, \gamma_{ki})$. We assign the space $R_{\gamma_{ij}}$ to the ray γ_{ij} , whereas the space $\mathcal{F}_{(\gamma_{ij}, \gamma_{jk})}$ is associated to the clockwise oriented angular sector between γ_{ij} and γ_{jk} .

Conjecture: R_c carries the structure of a Γ -graded L_∞ -algebra.

The theory of bulk vertices or interior amplitudes for a given physical theory is then expected to be governed by the degree $(2, 0)$ part of the conjectural L_∞ -algebra R_c .

Remark Earlier in Section 3.2, we defined R_c as the trace of the categorical monodromy. Because the trace of the categorical monodromy $\text{Tr}(\widehat{R}^{\text{opp}} \otimes \widehat{R})$ is viewed as the clockwise ordered product of \mathcal{S} and \mathcal{K} -factors over all central charge rays in the full plane, it is not difficult to see that

$$\text{Tr}(\widehat{R}^{\text{opp}} \otimes \widehat{R}) = \bigoplus_{\substack{(\gamma_{i_0 i_1}, \dots, \gamma_{i_n i_0}) \\ \text{irreducible fan}}} R_{\gamma_{i_0 i_1}} \otimes \mathcal{F}_{(\gamma_{i_0 i_1}, \gamma_{i_1 i_2})} \otimes \dots \otimes R_{\gamma_{i_n i_0}} \otimes \mathcal{F}_{(\gamma_{i_n i_0}, \gamma_{i_0 i_1})}. \quad (3.434)$$

Thus the two notions are equivalent.

In order to get a better handle on the structure of R_c , it is useful to examine some special cases.

3.5.1 No Periodic Solitons

The first special case to look at is when $R_{\gamma_i} = 0$ for all $i \in \mathbb{V}$ and $\gamma \in \Gamma$, namely when the spaces of periodic solitons are trivial, and therefore all the Fock spaces drop out, and one has

$$R_c = \bigoplus_{(\gamma_{i_0 i_1}, \dots, \gamma_{i_n i_0})} R_{\gamma_{i_0 i_1}} \otimes \dots \otimes R_{\gamma_{i_n i_0}}. \quad (3.435)$$

This might give one the impression that one is now working in the framework of [GMW], but this is not the case. In the presence of a non-trivial flavor group Γ , even without periodic solitons, there are novel phenomenon due to the no-force condition being violated:

- The first novelty one notices is the possibility to construct a web that has a fan at infinity simply being the identity fan in some vacuum i . An example of such a web is depicted in Figure 3.31. We call such a web a **vacuum web**.
- It was noted in [KKS] that the L_∞ algebra R_c without twisted masses is always nilpotent¹⁴. We will see that in the presence of a non-trivial flavor group Γ this ceases to be true.

Both of these properties are illustrated well by the example of the Weierstrass model, discussed in 3.2.4.

¹⁴Recall that an L_∞ algebra $(L, \lambda_1, \lambda_2, \dots)$ is called nilpotent if successive iterations of the L_∞ -operations, eventually vanish:

$$\lambda_{k_1}(\lambda_{k_2}(\dots \lambda_{k_n}(\cdot, \dots, \cdot), \dots)) = 0 \quad (3.436)$$

for n large enough. Strictly speaking nilpotence holds only for the subalgebra of R_c coming from at least trivalent fans.

R_c for the Weierstrass Model

Throughout this section we simplify notation and take γ_{21} to be such that $\gamma_{12} + \gamma_{21} = 0$, and γ_{12} is the homology class of the path going from ϕ_1 to ϕ_2 depicted in Figure 3.6. There are 16 non-trivial irreducible fans:

$$(\gamma_{12} + \gamma', \gamma_{21}) + 5 \mathbb{Z}_6 \text{ rotations}, \quad (3.437)$$

$$(\gamma_{12} + \gamma', \gamma_{21} - \gamma') + 2 \mathbb{Z}_6 \text{ rotations}, \quad (3.438)$$

$$(\gamma_{12} + \gamma', \gamma_{21}, \gamma_{12} - \gamma, \gamma_{21} - \gamma') + 5 \mathbb{Z}_6 \text{ rotations}, \quad (3.439)$$

$$(\gamma_{12} + \gamma', \gamma_{21}, \gamma_{12} - \gamma, \gamma_{21} - \gamma', \gamma_{12}, \gamma_{21} + \gamma), \quad (3.440)$$

which up to \mathbb{Z}_6 rotations are depicted in Figure 3.30.

In fermion degree 0, because there are no periodic solitons $R_{n\gamma_1+m\gamma'_1} = R_{n\gamma_2+m\gamma'_2} = 0$, we have

$$R_c^{(0,0)} \cong \mathbb{Z}^2. \quad (3.441)$$

In fermion degree +1 we have

$$R_c^{(1,0)} \cong (\mathbb{Z}^3)^{[1]}, \quad (3.442)$$

generated by the fan in the upper-right of Figure 3.30 and its \mathbb{Z}_6 -orbit, and

$$R_c^{(1,\gamma')} \cong \mathbb{Z}, \quad (3.443)$$

$$R_c^{(1,-\gamma)} \cong \mathbb{Z}, \quad (3.444)$$

$$R_c^{(1,-\gamma-\gamma')} \cong \mathbb{Z} \quad (3.445)$$

$$R_c^{(1,-\gamma')} \cong \mathbb{Z}, \quad (3.446)$$

$$R_c^{(1,\gamma)} \cong \mathbb{Z}, \quad (3.447)$$

$$R_c^{(1,\gamma+\gamma')} \cong \mathbb{Z}, \quad (3.448)$$

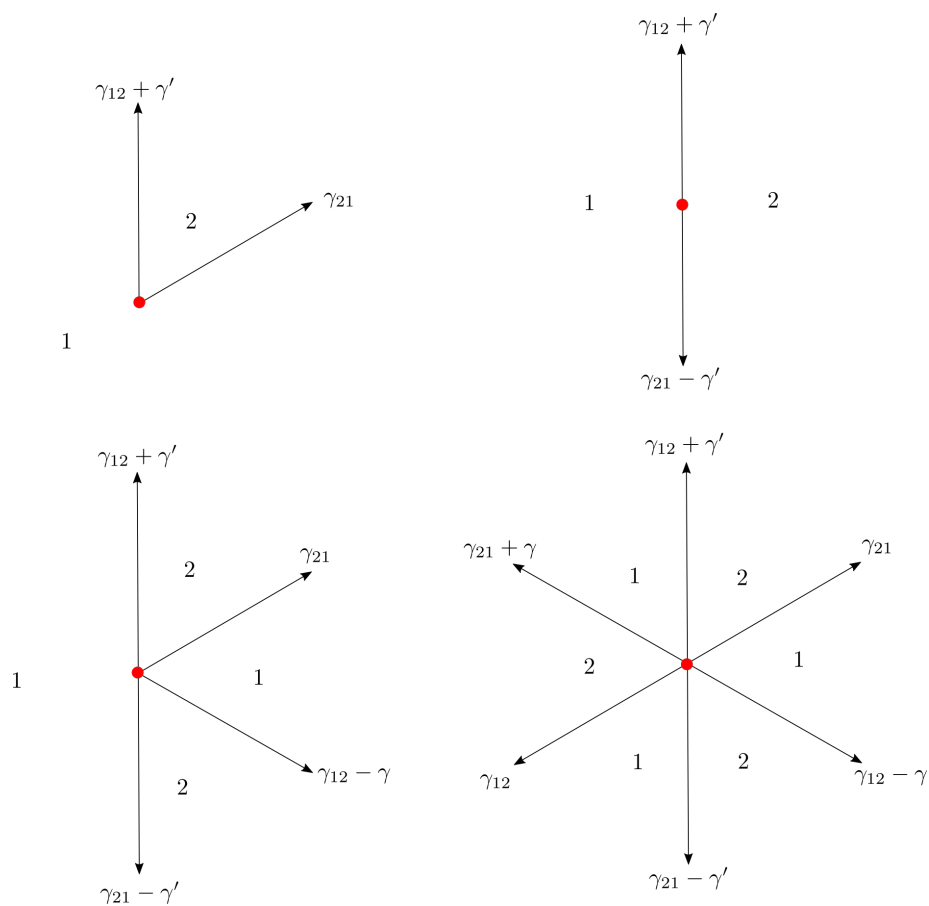


Figure 3.30: Some cyclic fans in the Weierstrass model. All others can be obtained by \mathbb{Z}_6 rotations of these.

which come from the upper-left fan of Figure 3.30 and its \mathbb{Z}_6 -orbit. In fermion degree +2 we have

$$R_c^{(2,\gamma')} \cong \mathbb{Z}, \quad (3.449)$$

$$R_c^{(2,-\gamma)} \cong \mathbb{Z}, \quad (3.450)$$

$$R_c^{(2,-\gamma-\gamma')} \cong \mathbb{Z} \quad (3.451)$$

$$R_c^{(2,-\gamma')} \cong \mathbb{Z}, \quad (3.452)$$

$$R_c^{(2,\gamma)} \cong \mathbb{Z}, \quad (3.453)$$

$$R_c^{(2,\gamma+\gamma')} \cong \mathbb{Z}, \quad (3.454)$$

coming from the \mathbb{Z}_6 -orbit of the lower-left fan of Figure 3.30, and finally in fermion degree 3

$$R_c^{(3,0)} \cong \mathbb{Z}, \quad (3.455)$$

coming from the lower-right fan of Figure 3.30. Since $R_c^{(2,0)} = \{0\}$, there is no room for a non-trivial interior amplitude, $\beta = 0$.

Let's discuss the algebraic structure on R_c . Denote the three fans generating $R^{(1,0)}$ as

$$\{h_1, h_2, h_3\}, \quad (3.456)$$

and

$$\{\theta_\gamma, \theta_{\gamma'}, \theta_{-(\gamma+\gamma')}, \theta_{-\gamma}, \theta_{-\gamma'}, \theta_{\gamma+\gamma'}\} \quad (3.457)$$

where $\theta.$ denotes the generator of $R^{(1,\cdot)}$. The standard taut webs with two vertices give the following bracket on the degree one elements

$$[\theta_{\gamma'}, \theta_{-\gamma'}] = h_1 - h_2, \quad (3.458)$$

$$[\theta_\gamma, \theta_{-\gamma}] = h_2 - h_3, \quad (3.459)$$

$$[\theta_{\gamma+\gamma'}, \theta_{-(\gamma+\gamma')}] = h_1 - h_3. \quad (3.460)$$

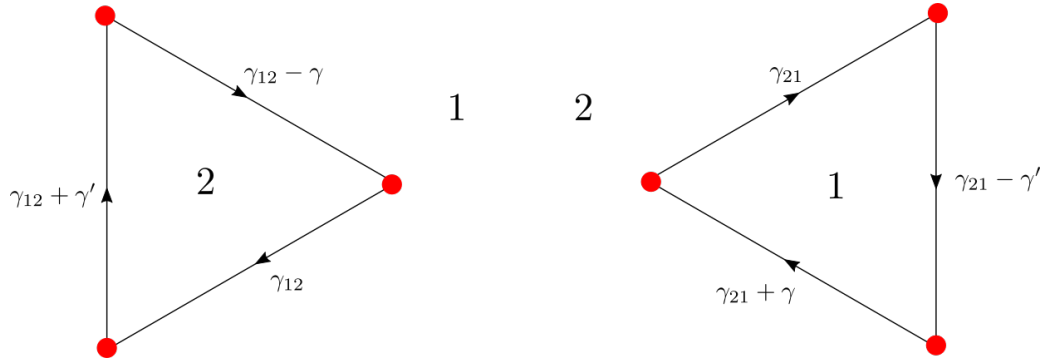


Figure 3.31: Taut webs $t_L^{(3)}$ and $t_R^{(3)}$ in the Weierstrass model having vacuum fans at infinity.

We also have the brackets

$$[h_i, \theta_*] = \pm \theta_* \quad (3.461)$$

for $i = 1, 2, 3$ and $*$ being one of $\pm\gamma, \pm\gamma', \pm(\gamma + \gamma')$. The nine-dimensional (degree -1) Lie algebra $R^{(1, \bullet)}$ is thus easily recognized as $\mathfrak{gl}(3)[1]$.

In fact there is more to discuss even in homological degree one. There is a non-zero ternary bracket $\lambda_3 : \Lambda^3 R^{(1, \bullet)} \rightarrow R^{(1, \bullet)}$ due to the possibility of constructing a taut web with the fan at infinity being a vacuum fan, such as the ones depicted in Figure 3.31. Such webs give us

$$\rho(t_L^{(3)})[\theta_\gamma, \theta_{\gamma'}, \theta_{-(\gamma + \gamma')}] = \phi_1, \quad (3.462)$$

$$\rho(t_R^{(3)})[\theta_{-\gamma}, \theta_{-\gamma'}, \theta_{\gamma + \gamma'}] = \phi_2, \quad (3.463)$$

where ϕ_1, ϕ_2 are the two vacua.

Let's now also include elements of higher homological degree. We denote generators of the six-dimensional space $R^{(2, \cdot)}$ as

$$\{\chi_{\pm\gamma}, \chi_{\pm\gamma'}, \chi_{\pm(\gamma + \gamma')}\}, \quad (3.464)$$

the subscript denoting the flavor degree. For instance the lower-right fan of Figure 3.30 is denoted by $\chi_{-\gamma}$. Also denote the generator of the one-dimensional space $R^{(3,0)}$ as ψ . Taut webs with two vertices give the following bilinear brackets of the degree two generators with the degree one generators:

$$[\chi_\alpha, \theta_\beta] = \begin{cases} \chi_{\alpha+\beta} & \text{if } \alpha + \beta' \in \{\pm\gamma, \pm\gamma', \pm(\gamma + \gamma')\}, \\ 0 & \text{otherwise} \end{cases}. \quad (3.465)$$

$$[\chi_\alpha, h_i] = 0 \quad \text{for any } \alpha \in \{\pm\gamma, \pm\gamma', \pm(\gamma + \gamma')\}, i \in \{1, 2, 3\}. \quad (3.466)$$

We also have the bracket

$$[\chi_\alpha, \chi_{\alpha'}] = \delta_{\alpha+\alpha', 0} \psi, \quad (3.467)$$

essentially constrained to be this for degree reasons.

We are far from done. Despite being finite-dimensional, R_c has an infinite number of taut webs with arbitrarily many vertices. For instance, the taut vacuum webs of Figure 3.31 can be generalized to a vacuum web with $\frac{n(n+1)}{2}$ vertices for every $n \geq 2$. The web for $n = 3$ is depicted in Figure 3.32, and the generalization to arbitrary n is clear. For every such web there is also its mirror web, obtained by reflecting about the vertical axis and exchanging the vacuum labels $1 \leftrightarrow 2$.

Therefore, despite being finite-dimensional and there being no periodic solitons, the algebra R_c is expected to have non-trivial brackets λ_n for arbitrarily high n . Such phenomenon were not seen without twisted masses.

3.5.2 Periodic Solitons Only

The next special case to discuss is when we are considering a theory with a single vacuum i , so that the only BPS states are periodic solitons. In this case the central charge is simply

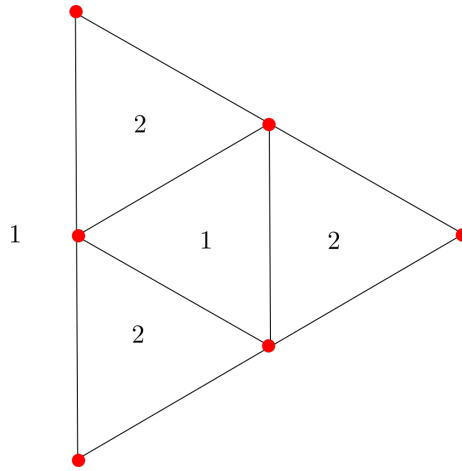


Figure 3.32: A taut vacuum web with six vertices in the Weierstrass model.

a homomorphism

$$Z : \Gamma \rightarrow \mathbb{C} \quad (3.468)$$

from the flavor group to the complex numbers, $Z_{\gamma+\gamma'} = Z_{\gamma} + Z_{\gamma'}$. One has a set of BPS chain complexes that are integral graded (the fractional part of a periodic soliton vanishes)

$$\{(R_{\gamma}, d_{\gamma}) | \gamma \in \Gamma\}. \quad (3.469)$$

When we need to be explicit about the vacuum it will be referred to by the letter i , although we will mostly drop it from the notation in what follows.

The space R_c now reads

$$R_c = \bigotimes_{\gamma \in \Gamma \setminus \{0\}} \mathcal{F}[R_{\gamma}], \quad (3.470)$$

and we would like to make some remarks on the expected L_{∞} structure on this space. When we are considering fans and webs lying within particular vacua we have some new phenomenon. First we have a vertex representing the identity within a particular vacuum.

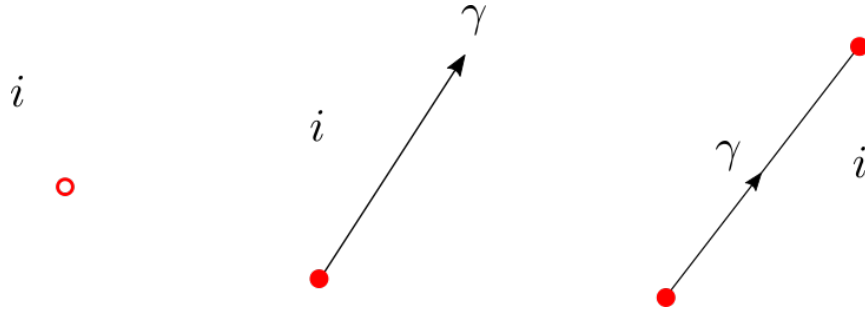


Figure 3.33: The very left shows a closed vertex, the middle shows a one-valent fan, and the right shows a taut web consisting of just a finite segment with fan at infinity being the identity fan.

This was present in the formalism of [GMW], known as a “closed vertex” denoted by an open circle. New with the presence of periodic solitons are now one-valent vertices with a ray shooting out in the Z_γ direction. The simplest possible taut web is given by a finite segment connecting two vertices where the segment is parallel to some Z_γ . The fan at infinity of such a finite taut web is the identity fan. See Figure 3.33.

We are interested in assigning multilinear operations to a given taut web in some canonical way. Let’s start with the operation we assign to the taut web of Figure 3.33. To the taut web $\mathfrak{w}^{(2)}$ consisting of a finite segment in the γ -direction, we will assign a bilinear, graded anti-symmetric, degree -1 , operation on

$$\text{Pol}_\gamma := \mathcal{F}[R_\gamma] \otimes \mathcal{F}[R_{-\gamma}] \subset R_c. \quad (3.471)$$

Let $\{e_\gamma^a\}$ and $\{e_{-\gamma}^a\}$ be homogeneous bases sets for R_γ and $R_{-\gamma}$ respectively and let

$$K_\gamma^{ab} := K_\gamma(e_\gamma^a, e_{-\gamma}^b) \quad (3.472)$$

be the non-degenerate pairing K_γ evaluated in this basis. The space Pol_γ can be thought of as the free algebra in the variables $e_\gamma^a, e_{-\gamma}^a$ modulo the relations

$$e_\gamma^a e_\gamma^b = (-1)^{\deg(a)\deg(b)} e_\gamma^b e_\gamma^a \quad (3.473)$$

and similar relations for a product of $-\gamma$ -variables, and for mixed products. An arbitrary element $f \in \text{Pol}_\gamma$ is thus written as some polynomial $f(e_\gamma^a; e_{-\gamma}^b)$. We then set

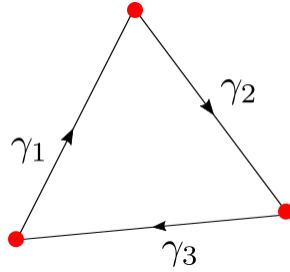
$$\rho(\mathfrak{w}^{(2)})[f, g] = K_\gamma^{ab} \frac{\partial f}{\partial e_\gamma^a} \frac{\partial g}{\partial e_{-\gamma}^b} \pm K_\gamma^{ab} \frac{\partial g}{\partial e_\gamma^a} \frac{\partial f}{\partial e_{-\gamma}^b}. \quad (3.474)$$

Conceptually what we have done is to assign the differential operator $\frac{\partial}{\partial e_\gamma^a}$ to the vertex whose outgoing edge is parallel to Z_γ and $\frac{\partial}{\partial e_{-\gamma}^a}$ to the vertex whose outgoing edge is parallel to $Z_{-\gamma}$ and then we contract the two vertices via the edge connecting them by assigning to it K_γ^{ab} . We then anti-symmetrize the operator to obtain the expression for $\rho(\mathfrak{w}^{(2)})$:

$$\rho(\mathfrak{w}^{(2)}) = K_\gamma^{ab} \frac{\partial}{\partial e_\gamma^a} \wedge \frac{\partial}{\partial e_{-\gamma}^b}, \quad (3.475)$$

and this fixes the sign in (3.474).

For another example, consider the following taut web $\mathfrak{w}^{(3)}$ having a trivial fan at infinity again.



To this we assign the ternary operator

$$\rho(\mathfrak{w}^{(3)}) = K_{\gamma_1}^{ac} K_{\gamma_2}^{de} K_{\gamma_3}^{fb} \frac{\partial^2}{\partial e_{\gamma_1}^a \partial e_{-\gamma_3}^b} \wedge \frac{\partial^2}{\partial e_{-\gamma_1}^c \partial e_{\gamma_2}^d} \wedge \frac{\partial^2}{\partial e_{-\gamma_2}^e \partial e_{\gamma_3}^f}, \quad (3.476)$$

where the wedge product of operators means we graded-antisymmetrize, so that for instance

$$\begin{aligned} \rho(\mathfrak{w}^{(3)})[f, g, h] &= K_{\gamma_1}^{ac} K_{\gamma_2}^{de} K_{\gamma_3}^{fb} \frac{\partial^2 f}{\partial e_{\gamma_1}^a \partial e_{-\gamma_3}^b} \frac{\partial^2 g}{\partial e_{-\gamma_1}^c \partial e_{\gamma_2}^d} \frac{\partial^2 h}{\partial e_{-\gamma_2}^e \partial e_{\gamma_3}^f} \\ &\quad + \text{permutations of } \{f, g, h\}. \end{aligned} \quad (3.477)$$

The general rule is clear from this: to every vertex we consider the edges oriented outwards to get a collection of directions $\{\gamma_i\}$ and so to that vertex we assign the operator $\prod_i \frac{\partial}{\partial e_{\gamma_i}^{a_i}}$. The free indices on the operators are all contracted by assigning $K_{\gamma_i}^{ab}$ to the finite edges. The resultant operator will have no free indices if and only if the fan at infinity is trivial.

Thus to define the n -ary operation

$$\lambda_n : \Lambda^n R_\Gamma \rightarrow R_\Gamma, \quad (3.478)$$

we simply sum over the differential operators corresponding to taut webs with the trivial fan at infinity

$$\lambda_n[\cdot, \dots, \cdot] = \sum_{\substack{\mathbf{t}^{(n)} \text{ taut} \\ I_\infty(\mathbf{t}^{(n)}) = \{u_i\}}} \rho(\mathbf{t}^{(n)})[\cdot, \dots, \cdot]. \quad (3.479)$$

We expect the collection of maps obtained in this way to satisfy the L_∞ axioms. For instance in the simplest case of Γ being rank one, the only taut web is the one appearing in the right of Figure 3.33 and the corresponding bracket coincides with the Schouten bracket of polyvector fields. This is well-known to satisfy the (graded) Jacobi identity. The multilinear operations in this section, defined by assigning differential operators to “taut webs”, has a similar flavor to the L_∞ -operations of [Sho], which in turn was inspired by similar open-closed algebraic constructions of [K].

R_c for Free Chiral

For the free chiral model recall that one has $\Gamma = \mathbb{Z}\langle\gamma\rangle$ and the soliton spaces are

$$R_\gamma = \mathbb{Z}\langle a_\gamma \rangle \cong \mathbb{Z}, \quad (3.480)$$

$$R_{-\gamma} = \mathbb{Z}\langle \psi_{-\gamma} \rangle \cong \mathbb{Z}^{[1]}. \quad (3.481)$$

Thus the closed string algebra is

$$R_c = \mathcal{F}^*[R_\gamma] \otimes \mathcal{F}^*[R_{-\gamma}] \quad (3.482)$$

$$= \mathbb{Z}[a_\gamma, \psi_{-\gamma}] / \langle \psi_{-\gamma}^2 \rangle. \quad (3.483)$$

a_γ obeys bosonic statistics, and $\psi_{-\gamma}$ obeys fermionic statistics. Hence we have

$$R_c^{(0,\bullet)} = \text{Span}\{a_\gamma^n\}_{n \geq 0} \quad (3.484)$$

$$R_c^{(1,\bullet)} = \text{Span}\{a_\gamma^n \psi_{-\gamma}\}_{n \geq 0}. \quad (3.485)$$

The bilinear L_∞ -bracket is obtained by

$$[a_\gamma, \psi_{-\gamma}] = K_\gamma(a_\gamma, \psi_{-\gamma}) = 1, \quad (3.486)$$

and enforcing the Poisson identity for powers of a_γ . The differential and all other operations act trivially. There is no element of homological degree +2 and so $\beta = 0$.

Note that R_c is isomorphic to the space of B-model observables, namely holomorphic functions and holomorphic vector fields on \mathbb{C} , and our bracket simply corresponds to the Schouten-Nijenhuis bracket. In the B-model, the observables are simply coming from the perturbative modes of the chiral field Φ . Under mirror symmetry the fundamental field gets mapped to a periodic soliton.

3.5.3 Mixed Case

We can now turn our attention briefly to the mixed case, namely when both periodic and non-periodic solitons are present. The simplest such situation is one we have already discussed: We have two vacua $\mathbb{V} = \{i, j\}$, and Γ is rank one, generated by an element $\gamma \in \Gamma$. The irreducible cyclic fans are easy to classify. First we have the identity fans (u_i)

and (u_j) to which we assign the spaces

$$R_i = \bigotimes_{n \neq 0} \mathcal{F}[R_{n\gamma_i}], \quad (3.487)$$

$$R_j = \bigotimes_{n \neq 0} \mathcal{F}[R_{n\gamma_j}]. \quad (3.488)$$

Previously, in Section 3.4.2 we wrote R_i as $\mathcal{F}[V_{\gamma_i}] \otimes \mathcal{F}[V_{-\gamma_i}]$ and R_j as $\mathcal{F}[V_{\gamma_j}] \otimes \mathcal{F}[V_{-\gamma_j}]$. Next we have $(\gamma_{ij}, \gamma_{ji})$ is an irreducible cyclic fan for any $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_{ij} \times \Gamma_{ji}$. Supposing that $(Z_{\gamma_{ij}}, Z_{\gamma}, Z_{\gamma_{ji}})$ are in clockwise order then, the space we assign to such an irreducible cyclic fan is

$$R_{(\gamma_{ij}, \gamma_{ji})} = R_{\gamma_{ij}} \otimes \mathcal{F}[V_{\gamma_j}] \otimes R_{\gamma_{ji}} \otimes \mathcal{F}[V_{-\gamma_i}]. \quad (3.489)$$

Summing over all irreducible cyclic fans $(u_i), (u_j)$ and $(\gamma_{ij}, \gamma_{ji})$ indeed gives us the space (3.71). Remarks on the expected L_∞ -structure were made in Section 3.4.2.

3.5.4 A_∞ -Category of Branes

The “open” analogue of R_c , used for the formalism of branes and boundary amplitudes is also straightforward to define in terms of generalized half-plane fans. As in [GMW], we proceed by first constructing the category of thimble branes

$$\mathfrak{Vac}(\mathbb{V}, Z, \Gamma, \{R_{\gamma_{ij}}\}, \beta), \quad (3.490)$$

which has objects $\{\mathfrak{T}_i\}_{i \in \mathbb{V}}$ and then enlarging \mathfrak{Vac} to include general D-branes as solutions of the A_∞ Maurer-Cartan equation. Fix a half-plane \mathbb{H} and let

$$(\gamma_{i_0 i_1}, \dots, \gamma_{i_{n-1} i_n}) \quad (3.491)$$

be now an irreducible half plane-fan. The definition of this is self-explanatory. It means that $i_k \neq i_{k+1}$ as usual for irreducible fans, and that

$$(Z_{\gamma_{i_0 i_1}}, \dots, Z_{\gamma_{i_{n-1} i_n}}) \quad (3.492)$$

is a collection of clockwise phase-ordered complex numbers lying in the half-plane \mathbb{H} . Let $\pm\gamma_{\mathbb{H}}$ be an auxiliary vector such that $\pm Z_{\gamma_{\mathbb{H}}}$ is parallel to $\partial\mathbb{H}$. Suppose γ_{ij} is a vacuum groupoid arrow with $Z_{\gamma_{ij}} \in \mathbb{H}$ such that the clockwise phase ordering is $(\gamma_{\mathbb{H}}, \gamma_{ij}, -\gamma_{\mathbb{H}})$. The flavor factors are defined to be

$$\mathcal{F}_{(\gamma_{\mathbb{H}}, \gamma_{ij})} = \otimes_{\gamma_{\mathbb{H}} < \gamma < \gamma_{ij}} \mathcal{F}^*[R_{\gamma_i}], \quad (3.493)$$

$$\mathcal{F}_{(\gamma_{ij}, -\gamma_{\mathbb{H}})} = \otimes_{\gamma_{ij} < \gamma < -\gamma_{\mathbb{H}}} \mathcal{F}^*[R_{\gamma_j}]. \quad (3.494)$$

A half-plane fan is then represented as in the closed-string case as an alternating tensor product of $\mathcal{F}_{(\cdot, \cdot)}$ and R spaces, while making sure to include (3.493), (3.494) at the start and end.

$$\widehat{R}_{(\gamma_{i_0 i_1}, \dots, \gamma_{i_{n-1} i_n})} = \mathcal{F}_{(\gamma_{\mathbb{H}}, \gamma_{i_0 i_1})} \otimes R_{\gamma_{i_0 i_1}} \otimes \dots \otimes R_{\gamma_{i_{n-1} i_n}} \otimes \mathcal{F}_{(\gamma_{i_{n-1} i_n}, -\gamma_{\mathbb{H}})}. \quad (3.495)$$

We then set

$$\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j) = \bigoplus_{\substack{(\gamma_{ii_1}, \dots, \gamma_{i_n j}) \\ \text{irreducible half-plane } ij \text{ fans}}} \widehat{R}_{(\gamma_{ii_1}, \dots, \gamma_{i_n j})}, \quad (3.496)$$

and abbreviate $\widehat{R}_{ij} = \text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j)$, as usual. Note that \widehat{R}_{ij} in addition to the usual cohomological grading possesses a grading by the Γ -torsor Γ_{ij}

$$\widehat{R}_{ij} = \oplus_{\gamma_{ij} \in \Gamma_{ij}} \widehat{R}_{\gamma_{ij}}. \quad (3.497)$$

As before for the categorical monodromy, the categorified spectrum generator gives us a product formula for the groupoid algebra (valued in \mathbb{Z} -modules) element \widehat{R} .

Conjecture The pair (\widehat{R}, R_c) admits operations that obey the axioms of an open-closed homotopy algebra. In particular $\beta \in (R_c)^{(2,0)}$ provides canonically, an A_{∞} -structure on \widehat{R} .

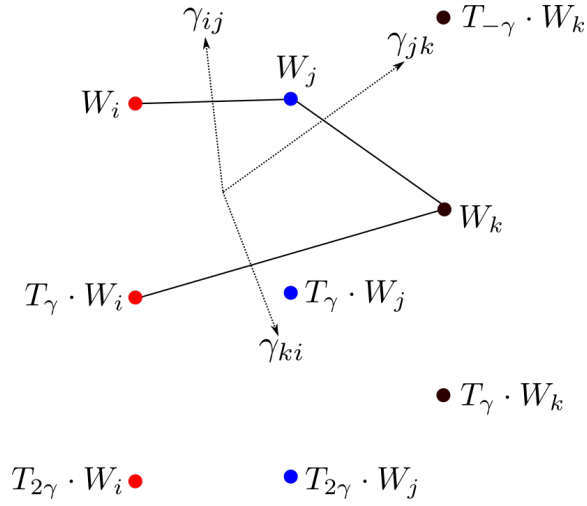


Figure 3.34: The dashed line shows a cyclic fan of arrows $(\gamma_{ij}, \gamma_{jk}, \gamma_{ki})$ such that $\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = \gamma_i$. The dual shape in the W -plane is depicted, and it corresponds to a quasi-closed polygon, meaning a polygon that only closed up to a Γ -translation.

\widehat{R} and the conjectural A_∞ -structure on it, induced from β , gives us a framework to discuss branes. A brane \mathfrak{B} carries Chan-Paton factors, a collection of graded \mathbb{Z} -modules $\mathcal{E}_i(\mathfrak{B})$ for each $i \in \mathbb{V}$. In addition to cohomological grading $\mathcal{E}_i(\mathfrak{B})$ has an additional grading by the Γ -torsor $\Gamma_{\mathfrak{B},i}$. A boundary amplitude \mathcal{B} consists of a degree $(1,0)$ element of the A_∞ -algebra $\widehat{R}_{\mathcal{E}}$

$$\mathcal{B} \in \oplus_{i,j \in \mathbb{V}} \mathcal{E}_i(\mathfrak{B}) \otimes \widehat{R}_{ij} \otimes \mathcal{E}_j^\vee(\mathfrak{B}), \quad (3.498)$$

that satisfies the A_∞ Maurer-Cartan equation.

Finally, one could go on and discuss the relevant constructions for interfaces, but the basic idea should be clear by now and we will not give further details. We end with a few remarks about the dual polygon viewpoint.

3.5.5 Remarks on the Polygon Viewpoint

Recall that cyclic fans of vacua, in the absence of twisted masses, correspond to polygons in the W -plane with critical values as vertices [GMW]. In [KKS] a rival formulation to the web formalism was developed in which polygons (and their secondary polytopes) play a central role. This formalism has some advantages over the web formalism. We sketch here some issues which must be addressed to generalize the approach of [KKS] to include twisted masses. Given an LG model with non-trivial twisted masses, by working on the universal abelian cover, one can still go the W -plane, so that each vacua i corresponds to a Γ -torsor worth of critical values

$$i \rightarrow \{W_i + M \cdot \gamma\}_{\gamma \in \Gamma}, \quad (3.499)$$

for some non-canonical W_i . An arbitrary cyclic fan of arrows however, does not correspond to a closed polygon. A cyclic fan of arrows $(\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \dots, \gamma_{i_n i_0})$ with charge γ corresponds to a shape in the W -plane that closes only up to a γ -translation. For instance, one could consider a cyclic fan of the form $(\gamma_{ij}, \gamma_{jk}, \gamma_{ki})$ where

$$\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = \gamma_i \quad (3.500)$$

for some $\gamma_i \in \Gamma_{ii}$, and $i \neq j \neq k$. In the W -plane this would correspond to a shape such as the one shown in Figure 3.34. A dual formulation of the L_∞ -algebra R_c in terms of polygons would have to account for such quasi-closed polygons and their subdivisions.

On the other hand, of particular interest to us is the subalgebra of charge zero elements $(R_c)^{(*,0)}$, because this is where the interior amplitude β resides. For the subalgebra of charge zero states, the dual formulation would indeed only include traditional closed polygons in the W -plane, and their subdivisions. The notion of polygons and their subdivisions in the presence of twisted masses is a bit subtle however. As we have stressed in Section

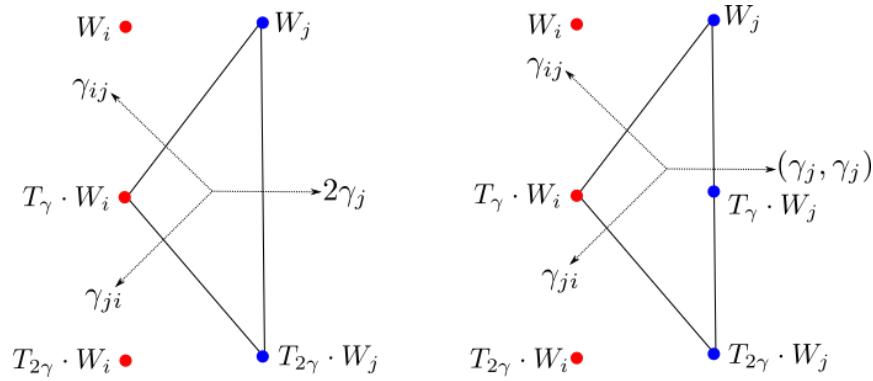


Figure 3.35: The dashed lines on the left depict a cyclic fan $(\gamma_{ij}, 2\gamma_j, \gamma_{ji})$ such that $\gamma_{ij} + 2\gamma_j + \gamma_{ji} = 0$. Its dual is a triangle with vertices $(T_\gamma \cdot W_i, W_j, T_{2\gamma} \cdot W_j)$. The dashed lines in the right depict a cyclic fan $(\gamma_{ij}, \gamma_j, \gamma_j, \gamma_{ji})$ which once again has charge zero. Its dual consists of a degenerate four-gon with vertices $(W_i, W_j, T_\gamma \cdot W_j, T_{2\gamma} \cdot W_j)$. The left figure is viewed as carrying a single periodic soliton of charge $2\gamma_j$ whereas the right is viewed as carrying two periodic solitons, each of charge γ_j .

4.1, when talking about polygons in the W -plane in the presence of twisted masses one has to be careful about vertices lying along a common line. The same underlying polygon can correspond to different cyclic fan of arrows depending on what we consider as vertices. For instance suppose that we have three groupoid arrows such that $\gamma_{ij} + 2\gamma_j + \gamma_{ji} = 0$. Then $(\gamma_{ij}, 2\gamma_j, \gamma_{ji})$ and $(\gamma_{ij}, \gamma_j, \gamma_j, \gamma_{ji})$ are distinct cyclic fans of charge zero. In terms of dual polygons, the first of these would correspond to a triangle with three vertices, whereas the second one would correspond to a degenerate four-gon, with four vertices. The convex hull of these two sets of vertices coincides, but they must be considered distinct in the formalism. See Figure 3.35. The right triangle admits a non-trivial subdivision, whereas the left one does not.

It is entirely possible that one would arrive at a satisfactory theory of interior amplitudes by working in the W -plane with closed polygons, while being careful of distinctions such as the ones mentioned above when formulating such polygons and their subdivisions. This

we must leave to future work.

Chapter 4

Future Directions: Algebraic Knot Invariants, $4d \mathcal{N} = 2$ BPS States

4.1 Introduction

The purpose of this chapter is to give a somewhat detailed account of two future directions that naturally follow from the rest of the considerations in this thesis. One can be viewed as an application of the technology we have developed and discussed in Chapter 3 on twisted masses to the study of three-manifolds and homological knot invariants via the Chern-Simons Landau-Ginzburg model. The other direction is the extension of the two-dimensional formalism of categorical wall-crossing and the algebra of the infrared to BPS states and wall-crossing in four-dimensional $\mathcal{N} = 2$ theories.

The discussion in Section 4.2 will appear in a preprint being written jointly with D. Gaiotto, G. W. Moore and F. Yan, whereas the discussion of Section 4.3 is independent work of the author, with useful input from D. Gaiotto and G. W. Moore.

4.2 Algebraic Knot Invariants

We begin with a quick review of Witten's formulation [Wit8] of Khovanov homology. The reader is directed to the original paper [Wit8] along with the review articles [Wit6, Wit7] for a more detailed account. Another useful review, especially with regards to the relation

with Landau-Ginzburg models appears in Section 18.4 of [GMW].

Witten's formulation of Khovanov homology is most cleanly stated in terms of the six-dimensional $\mathcal{N} = (0, 2)$ theory. We let \mathfrak{g} be a simply laced Lie algebra and consider the $6d$ $\mathcal{N} = (0, 2)$ theory of type \mathfrak{g} on $M_4 \times D$ where M_4 is a four-manifold, and D is an oriented surface with a $U(1)$ action that has a fixed point. We fix $D = \mathbb{R}^2$ with the standard $SO(2) = U(1)$ rotation with the origin as a fixed point in what follows. As mentioned before, the theory admits a holomorphic-topological twist so that the theory is holomorphic along D and topological along M_4 . Furthermore in this holomorphic-topological theory, we can consider topological surface operators: they are supported on $\Sigma \times \{0\} \subset M_4 \times D$ and are labeled by a representation ρ of \mathfrak{g} . Because these are placed at the $U(1)$ -invariant point of D , they preserve the $U(1)_D$ rotational symmetry. To get the right setup for knot theory, we specialize M_4 to be of the form $M_3 \times \mathbb{R}$, where M_3 is a three-manifold and \mathbb{R} is viewed as time. Furthermore, we take the support Σ of the topological surface operator to be along $L \times \mathbb{R} \times \{0\} \subset M_3 \times \mathbb{R} \times D$ where L is a colored link inside M_3 . We therefore find a six-dimensional system with a nilpotent supercharge Q , along with two $U(1)$ symmetry generators F and P , the fermion number (generator of an appropriate R-charge) and the rotational symmetry generator along D respectively. Furthermore, our setup involves a time direction and therefore has a Hilbert space. Witten's proposal is that when $M_3 = \mathbb{R}^3$, the Q -cohomology of this Hilbert space, which is a $\mathbb{Z} \times \mathbb{Z}$ -graded vector space is isomorphic to $\mathcal{K}(L, \rho, \mathfrak{g})$, the colored Khovanov homology of the link L .

$\mathcal{K}(L, \rho, \mathfrak{g})$ can be given a more concrete construction by equipping D with a $U(1)$ invariant cigar-like metric, and reducing the $6d$ theory along $U(1)$ orbits of D . The resulting theory is a twisted version of $5d$ super Yang-Mills on $M_3 \times \mathbb{R} \times D/U(1) = M_3 \times \mathbb{R} \times \mathbb{R}_+$ with an important L -dependent boundary condition at the origin of \mathbb{R}_+ . The five-dimensional

theory on $M_3 \times \mathbb{R} \times \mathbb{R}_+$ in turn admits a convenient description as an “A-model with superpotential”, namely as a Landau-Ginzburg model where the supercharge Q is identified as an A-type supercharge in the $\mathcal{N} = (2, 2)$ algebra on $\mathbb{R} \times \mathbb{R}_+$. This Landau-Ginzburg description allows us to give a description of $\mathcal{K}(L)$ in terms of solving classical four and five-dimensional BPS equations. These equations are the ζ -soliton and ζ -instanton equations, respectively, of the Landau-Ginzburg model [GMW].

The Landau-Ginzburg model with A-type supercharge can further be described as a supersymmetric quantum mechanics with a real superpotential, that is as a Morse theory problem. In this case the Morse theoretic problem has a target space consisting of four-dimensional gauge fields on $V = M_3 \times \mathbb{R}_+$ and an adjoint valued one-form on V . Denote the coordinates on V as (x^i, y) with $i = 1, \dots, 3$. We must specify the boundary conditions at $y = 0$ and $y = \infty$. At $y \rightarrow \infty$ we simply require the gauge field $\mathcal{A} = A + i\phi$ to approach a $G_{\mathbb{C}}$ -flat connection σ_i on M_3 . The boundary conditions at $y = 0$ are more subtle to describe. They can be deduced from an M-theoretic construction of the $6d \mathcal{N} = (0, 2)$ theory ¹. The result is simplest to state for the gauge group $G^{\vee} = SO(3)$. Let’s first describe the boundary condition for the empty link $L = \emptyset$. Pick a metric on M_3 and let (e, ω) be the dreibein and associated spin connection respectively. We require that as

¹The $6d$ theory of type A_{r-1} is the worldvolume theory of r parallel M5 branes supported on $M_4 \times D$ in eleven-dimensional M-theory on $X \times T^*D$ where X is the total space $\Omega^{2,+}(M_4) \rightarrow M_4$. Upon reduction to a circle we get type IIA string theory with an additional D6 brane supported on $X \times \{0\}$. The M5 brane maps to a D4-brane on $M_4 \times \mathbb{R}_+$. Therefore we find a D4-D6 brane system. Three of the scalar fields describe fluctuations of the D4 brane along the D6 brane and carry Nahm pole behavior, whereas the remaining two scalars describing the transverse motion are set to vanish.

$y \rightarrow 0$, the fields approach

$$\phi = \frac{e}{y} + \dots, \quad (4.1)$$

$$A = \omega + \dots \quad (4.2)$$

For a more general link L , away from L the fields have similar Nahm pole behavior as $y \rightarrow 0$, but close to the link the fields should behave like a particular singular model solution. For details see Section 3.6.4 of [Wit8]. Let's denote the target space with these boundary conditions as $X_{\sigma_i}(L, M_3)$. Now that we have described the target space, let's describe the superpotential of the Morse theory problem. The superpotential is

$$S = \int dy d^3x \operatorname{Tr}(\sqrt{g} g^{ij} \phi_i F_{yj} + \frac{1}{2} \epsilon^{ijk} (A_i \partial_j A_k + \frac{2}{3} [A_i, A_j] A_k - \phi_i D_j \phi_k)). \quad (4.3)$$

Now we can apply the usual Morse theoretic recipe. We build a vector space $\mathcal{KC}_{\sigma_i}(L)$ by assigning a generator to each critical point of S . In turn the critical points $\delta S = 0$ correspond to solutions of the equation

$$D_y \phi_i = \sqrt{g}^{-1} g_{ij} \epsilon^{jkl} (F_{kl} - [\phi_k, \phi_l]), \quad (4.4)$$

$$F_{yi} = \sqrt{g}^{-1} g_{ij} \epsilon^{jkl} D_k \phi_l. \quad (4.5)$$

Because we are really doing $\mathcal{G} = \operatorname{Map}(M_3, G)$ -equivariant Morse theory, these equations are supplemented with the zero moment map condition

$$D_i \phi^i = 0. \quad (4.6)$$

Together these give the Kapustin-Witten equations which in four-dimensional notation are written as

$$F - \phi \wedge \phi = *d_A \phi, \quad (4.7)$$

$$d_A * \phi = 0. \quad (4.8)$$

In matching with the above, we must set $\phi_y = 0$. In doing this we actually don't lose any information since $\phi_y = 0$ holds for any solution of the above equations as discussed in [Wit8]. Each solution is graded by instanton number and fermion number and thus we can construct a bigraded vector space $\mathcal{KC}_{\sigma_i}(L)$. Next let's discuss instantons. The relevant equation is the gradient flow equation

$$\frac{\partial \phi_i^a}{\partial \tau} = \sqrt{g}^{-1} g_{ij} \frac{\delta S}{\delta \phi_j^a}, \quad (4.9)$$

$$\frac{\partial A_i^a}{\partial \tau} = \sqrt{g}^{-1} g_{ij} \frac{\delta S}{\delta A_j^a}, \quad (4.10)$$

$$\frac{\partial A_y^a}{\partial \tau} = \frac{\delta S}{\delta A_y^a}. \quad (4.11)$$

The first and third of these equations are

$$F_{yi}^a + \frac{\partial \phi_i^a}{\partial \tau} + \sqrt{g}^{-1} g_{ij} \epsilon^{jkl} D_k \phi_l^a = 0, \quad (4.12)$$

$$F_{y\tau} - g^{ij} D_i \phi_j = 0, \quad (4.13)$$

whereas the middle equation is

$$(F_{0i}^a + \frac{\sqrt{g}}{2} \epsilon_{ijk} F^{ajk}) - \frac{\sqrt{g}^{-1}}{2} g_{ij} f^{abc} \epsilon^{jkl} \phi_k^b \phi_l^c - D_y \phi_i^a = 0. \quad (4.14)$$

These can be written more succinctly if we define a self-dual two-form $B_{\mu\nu}^a$ on $(0, 1, 2, 3)$ (does not include the y coordinate) space given by

$$B_{0i}^a = \phi_i^a, \quad (4.15)$$

$$B_{ij}^a = \frac{1}{2} \sqrt{g} \epsilon_{ijk} \phi^{ak}. \quad (4.16)$$

Also, we consider the gauge field A_0, A_1, A_2, A_3 and the self-dual part of the curvative F^+ . Then the first and third equation combine into

$$F_{y\mu} + D^\nu B_{\nu\mu} = 0, \quad (4.17)$$

whereas the middle equation becomes

$$(F^+)_{0i}^a - \frac{1}{4}(B \times B)_{0i}^a - D_y B_{0i}^a = 0. \quad (4.18)$$

In the above the cross-product of two self-dual two forms is

$$(B \times B)_{\mu\nu}^a := f^{abc} B_{\mu\alpha}^b g^{\alpha\beta} B_{\beta\nu}^c. \quad (4.19)$$

Thus we find that the gradient flow equations are the Haydys-Witten equations

$$F_{y\mu} + D^\nu B_{\nu\mu} = 0, \quad (4.20)$$

$$F^+ - \frac{1}{4}B \times B - \frac{1}{2}D_y B = 0. \quad (4.21)$$

We use the HW equations to compute tunneling amplitudes between solutions of the KW equations that differ by unit fermion and vanishing instanton degrees. Therefore modulo analytic assumptions, we have a well-defined Q -cohomology

$$\mathcal{K}_{\sigma_i}(L) = H_Q^*(\mathcal{K}C_{\sigma_i}(L)) \quad (4.22)$$

for each flat connection σ_i and each link $L \subset M_3$. Khovanov homology is expected to be reproduced for $M_3 = \mathbb{R}^3$ where the only flat connection is the trivial one, $\sigma_i = 0$. Namely one expects $\mathcal{K}_{\sigma=0}(L)$ to be isomorphic to the Khovanov homology of the link L . This construction also tells us why things are more subtle when $M_3 \neq \mathbb{R}^3$. There is no unique choice of flat connection and therefore it is not clear if $\mathcal{K}_{\sigma_i}(L)$ is a topological invariant. Namely the choice of a flat connection might break topological invariance of these Hilbert spaces.

Having reviewed the physical construction of Khovanov homology from both the five and six dimensional viewpoints, let us now present our proposal for homological invariants associated to links $L \subset M_3$. One of the main points of [Wit8], as we just reviewed, is

that there is a canonical “Nahm pole” boundary condition $B_{\text{Nahm}}(L)$ of the twisted five-dimensional Yang-Mills theory on $M_3 \times \mathbb{R} \times \mathbb{R}_+$ for each link $L \subset M_3$ that preserves the supercharge Q . Our proposed link invariant is

$$L \subset M_3 \rightsquigarrow \mathcal{A}[L, M_3] = \text{Hom}(B_{\text{Nahm}}(L), B_{\text{Nahm}}(L)), \quad (4.23)$$

the space of self-morphisms of the brane $B_{\text{Nahm}}(L)$. Moreover the Landau-Ginzburg viewpoint, in particular the technology of [GMW] and Chapter 3, allow a construction of $\mathcal{A}[L, M_3]$ in terms of solving the Kapustin-Witten equations in a similar spirit to the construction of the BPS Hilbert spaces $\mathcal{K}_{\sigma_i}(L)$.

Assume for simplicity that M_3 has only a finite number of $G_{\mathbb{C}}$ flat connections $\{\sigma_1, \dots, \sigma_n\}$ and that they are all isolated. Recall that the Landau-Ginzburg model has a thimble boundary condition \mathcal{T}_{σ_i} for each flat connection σ_i . Let

$$\widehat{\mathcal{K}}_{\sigma_i \sigma_j} := \text{Hom}(\mathcal{T}_{\sigma_j}, \mathcal{T}_{\sigma_i}). \quad (4.24)$$

Then our proposed invariant $\mathcal{A}[L, M_3]$ as a vector space is given explicitly by

$$\mathcal{A}[L, M_3] = \bigoplus_{\sigma_i, \sigma_j} \mathcal{K}C_{\sigma_i}(L) \otimes \widehat{\mathcal{K}}_{\sigma_i, \sigma_j} \otimes \mathcal{K}C_{\sigma_j}^{\vee}(L). \quad (4.25)$$

We have already explained how to construct $\mathcal{K}C_{\sigma_i}(L)$ in terms of solutions of partial differential equations. We are thus left with the task of providing $\widehat{\mathcal{K}}_{\sigma_i \sigma_j}$ with a similar construction.

The first step in constructing $\widehat{\mathcal{K}}_{\sigma_i, \sigma_j}$ the space of morphisms between \mathcal{T}_{σ_j} and \mathcal{T}_{σ_i} is to define complexes $\mathcal{K}C_{\sigma_i, \sigma_j}$ for any pair of flat connections. This is done by modifying the Morse theoretic problem discussed before by a little bit. The first modification is to change the target space to be not complex connections on $M_3 \times \mathbb{R}_+$, but on $M_3 \times \mathbb{R}$. We require

that \mathcal{A} approach σ_i as $y \rightarrow -\infty$ and σ_j as $y \rightarrow +\infty$. The space of such configurations X_{σ_i, σ_j} actually has infinitely many connected components labelled by (relative) instanton number

$$X_{\sigma_i, \sigma_j} = \bigsqcup_{n \in \mathbb{Z}} X_{\sigma_i, \sigma_j}^n, \quad (4.26)$$

where the integer n is only well-defined up to some global shifts². Next we consider the Morse function

$$S_\zeta = \int dy d^3x \operatorname{Tr}(\sqrt{g} g^{ij} \phi_i F_{yj} - \frac{1}{2} \operatorname{Im}(\zeta^{-1} \epsilon^{ijk} (\mathcal{A}_i \partial_j \mathcal{A}_k + \frac{2}{3} [\mathcal{A}_i, \mathcal{A}_j] \mathcal{A}_k)) \quad (4.27)$$

where $\zeta = e^{i\alpha}$ is a phase. In terms of real fields this is

$$S_\zeta = \int dy d^3x \operatorname{Tr}(\sqrt{g} g^{ij} \phi_i F_{yj} + \sin \alpha \epsilon^{ijk} (A_i \partial_j A_k + \frac{2}{3} [A_i, A_j] A_k - \phi_i D_j \phi_k) + \quad (4.28)$$

$$-\cos \alpha \epsilon^{ijk} (\phi_i F_{jk} - \frac{2}{3} [\phi_i, \phi_j] \phi_k)).$$

The action discussed previously is the special case with $\zeta = i$ or $\alpha = \frac{\pi}{2}$. It is straightforward to show that $\delta S_\zeta = 0$ leads to the Kapustin-Witten equations

$$(F - \phi \wedge \phi)^+ = t(d_A \phi)^+, \quad (4.29)$$

$$(F - \phi \wedge \phi)^- = -t^{-1}(d_A \phi)^-, \quad (4.30)$$

where

$$t = \frac{\sin \alpha}{1 - \cos \alpha}. \quad (4.31)$$

As usual these are supplemented by the condition

$$d_A * \phi = 0. \quad (4.32)$$

²It is conceptually cleaner to say that the sectors are labeled by a \mathbb{Z} -torsor.

These can be expressed more compactly in terms of the complex curvature \mathcal{F} of the complex connection \mathcal{A} . The equations then take a complex form of the self-duality equation

$$\mathcal{F} = *\zeta\overline{\mathcal{F}}. \quad (4.33)$$

In order to construct the complexes $\mathcal{K}_{\sigma_i\sigma_j}$ we fix the phase ζ to coincide with the phase of

$$Z = \frac{1}{4\pi} \int_{\mathbb{R} \times M_3} \text{Tr}(\mathcal{F} \wedge \mathcal{F}). \quad (4.34)$$

The quantity Z obeys a modified quantization condition for fields in X_{σ_i,σ_j} . It takes the form

$$Z_{ij}(n) = W_{\text{CS}}(\sigma_i) - W_{\text{CS}}(\sigma_j) + 2\pi n \quad (4.35)$$

where n is some integer and $W_{\text{CS}}(\sigma_{i,j})$ are values of the complex Chern-Simons functional evaluated on the flat connections $\sigma_{i,j}$. Note that W_{CS} is multivalued, so the expression above has ambiguities. Nevertheless we make some non-canonical choices, and refer to the phase of $Z_{ij}(n)$ as ζ_{ij}^n . A standard argument shows that S_ζ will not have any critical points in X_{σ_i,σ_j} unless the phase ζ is chosen to coincide with one in the set $\{\zeta_{ij}^n\}_{n \in \mathbb{Z}}$. We refer to Z_{ij} as the relative instanton number in the ij -sector. We thus find that X_{σ_i,σ_j} decomposes into infinitely many disjoint sectors labelled by distinct relative instanton numbers. We can define $\mathcal{K}_{\sigma_i,\sigma_j}^n$ as the Morse complex of S_ζ with $\zeta = \zeta_{ij}^n$. Note that the description above makes sense for both $\sigma_i \neq \sigma_j$ and $\sigma_i = \sigma_j$.

Remark The complexes $\mathcal{K}C_{\sigma\tau}$ categorify the integer $\mathfrak{m}_{\sigma\tau}$ discussed briefly in Section 4.1.2 of [Wit5].

The final step involved in defining $\widehat{\mathcal{K}}_{\sigma_i\sigma_j}$ involves some wall-crossing formalism. We take the complexes $\{\mathcal{K}C_{\sigma_i,\sigma_j}^n\}$ and use them to form the categorified 2d-4d spectrum generator [GMN4, GMW, KM2] as discussed in Chapter 3. We recall the basic ingredients here for

completeness. The categorified spectrum generator is a product of “categorified Stokes factors” associated to the phases ζ_{ij}^n . It is formed as follows. Let e_{σ_i, σ_j} be elementary matrices with indices being flat connections. For $\sigma_i \neq \sigma_j$ we let

$$\mathbb{S}_{\sigma_i, \sigma_j}^n = \mathbf{1} \oplus \mathcal{K}C_{\sigma_i \sigma_j}^n e_{\sigma_i \sigma_j}, \quad (4.36)$$

known as the categorified \mathbb{S} -factor, and we let

$$\mathbb{K}_n = \oplus_{\sigma_i} \mathcal{F}^*(\mathcal{K}C_{\sigma_i, \sigma_i}^n) e_{\sigma_i \sigma_i} \quad (4.37)$$

where $\mathcal{F}^*(V)$ denotes the graded Fock space of V , denote the categorified \mathbb{K} -factor. The categorical spectrum generator is then the product, clockwise ordered by the phases, of the elementary \mathbb{S} and \mathbb{K} -factors for all phases that lie in the upper-half plane

$$\widehat{\mathcal{K}} := \bigotimes_{\zeta_{ij}^n \in \mathbb{H}}^{\curvearrowright} : \mathbb{S}_{ij}^n \mathbb{K}_n : . \quad (4.38)$$

The individual $\widehat{\mathcal{K}}_{\sigma_i \sigma_j}$ are then just the matrix elements of $\widehat{\mathcal{K}}$

$$\widehat{\mathcal{K}} = \bigoplus_{\sigma_i, \sigma_j} \widehat{\mathcal{K}}_{\sigma_i \sigma_j} e_{\sigma_i \sigma_j}. \quad (4.39)$$

Remark In notation of [GMW] and previous chapters, the complex on $M_3 \times \mathbb{R}$ namely $\mathcal{K}C_{\sigma_i \sigma_j}$ and its hatted counterpart $\widehat{\mathcal{K}}_{\sigma_i, \sigma_j}$ defined as above would be denoted as R_{σ_i, σ_j} and $\widehat{R}_{\sigma_i \sigma_j}$ respectively. Similarly the Khovanov complexes on $M_3 \times \mathbb{R}_+$ namely $\mathcal{K}C_{\sigma_i}(L)$ would be written as $\mathcal{E}_{\sigma_i}(B_{\text{Nahm}}(L))$, the Chan-Paton factors of the brane $B_{\text{Nahm}}(L)$.

Now that we have defined all relevant quantities it is time to state our main conjectures.

Conjecture 1 Let $\text{Ob}(\mathcal{T}[M_3]) = \{\mathcal{T}_{\sigma_1}, \dots, \mathcal{T}_{\sigma_n}\}$ and

$$\text{Hom}(\mathcal{T}_{\sigma_i}, \mathcal{T}_{\sigma_j}) = \widehat{\mathcal{K}}_{\sigma_j \sigma_i}. \quad (4.40)$$

$\mathcal{T}[M_3]$ carries canonically the structure of an A_∞ -category. The homotopy class of $\mathcal{T}[M_3]$ is a topological invariant of M_3 .

The idea of assigning a Fukaya-type category as a three manifold invariant by using the complex Chern-Simons functional goes back to work of A. Haydys [Hay]. However, we believe Conjecture 1 is essentially new, and we have provided explicit formulas for the morphism spaces in terms of Morse complexes on $\mathbb{R} \times M_3$. Let us also sketch some ideas about how the actual A_∞ -structure on $\mathcal{T}[M_3, G]$ will be constructed. Recall that the differential on $\mathcal{KC}_{\sigma_i}(L)$ was constructed by counting rigid solutions of the Haydys-Witten equation. Here the idea is similar. The A_∞ -structure on $\mathcal{T}[M_3]$ will come from counting solutions of the Haydys-Witten equation with a particular type of boundary conditions.

It is a standard fact that a solution to the KW equations on $\mathbb{R} \times M_3$ that interpolates between σ_i and σ_j maps to a straight line between points in the $\{W_{\text{CS}}(\sigma_i) + 2\pi k\}$ and $\{W_{\text{CS}}(\sigma_j) + 2\pi k\}$ -towers. The crucial idea in constructing the A_∞ -structure on $\mathcal{T}[M_3]$ involves counting solutions of the Haydys-Witten equations on $M_3 \times \mathbb{R}^2$ that map to *convex polygons* in the W_{CS} -plane. Let's consider the simplest case to illustrate. Let σ, τ, ρ be labels corresponding to flat connections and suppose there are points in the corresponding towers of critical points $W(\mathcal{A}_\sigma), W(\mathcal{A}_\tau), W(\mathcal{A}_\rho)$ that form clockwise oriented vertices of a triangle. Moreover, fix particular elements of $\mathcal{A}_{\sigma\tau} \in \mathcal{KC}_{\sigma\tau}, \mathcal{A}_{\tau\rho} \in \mathcal{KC}_{\tau\rho}$ and $\mathcal{A}_{\rho\sigma} \in \mathcal{KC}_{\rho\sigma}$. We would like to consider solutions of the Haydys-Witten equation on $M_3 \times \mathbb{R}^2$ such that at a large circle at infinity $M_3 \times S_\infty^1 \subset M_3 \times \mathbb{R}^2$ the solution consists of broken paths

$$\mathcal{A}_{\sigma\tau} \# \mathcal{A}_{\tau\rho} \# \mathcal{A}_{\rho\sigma}. \quad (4.41)$$

In other words, in the W_{CS} plane the image of $M_3 \times S_\infty^1$ traces out the boundary of this triangle. The full solution is required to fill out the interior of the triangle. Specifying the boundary conditions for n -gons is straightforward. See for instance Figure 4.1 for $n = 5$.

We conjecture that moduli spaces of the Haydys-Witten equation with such polygonal boundary conditions are well-defined and that one can make sense of counting the rigid

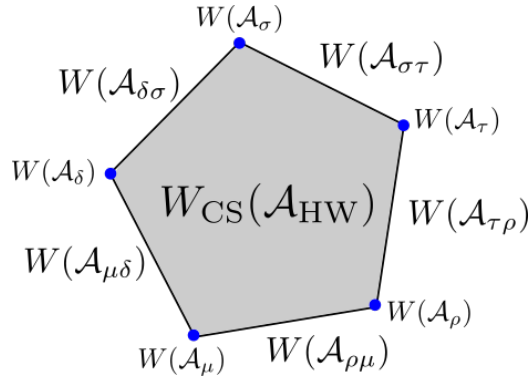


Figure 4.1: The image of a solution of the Haydys-Witten equation with “pentagonal” boundary conditions the $W = W_{\text{CS}}$ -plane.

solutions. Supposing this is the case, we expect that counts of such solutions combined with appropriate web combinatorics [GMW, KM2] leads to the conjectured canonical A_∞ structure on $\mathcal{T}[M_3]$.

Next we come to the counting relevant for the quantity $\mathcal{A}[L, \rho, \mathfrak{g}, M_3]$. In the definition of $\mathcal{A}[L, M_3]$ a crucial role is played by the Khovanov complexes $\mathcal{KC}_{\sigma_i}(L)$. Elements of these complexes, being solutions of the Kapustin-Witten equation at $\zeta = i$ map to vertical half-lines between $-i\infty$ and terminating at $W_{\text{CS}}(\sigma_i)$ in the W -plane. Now fix a collection of flat connections $\sigma_{i_1}, \dots, \sigma_{i_n}$ and points $W_{\text{CS}}(\sigma_{i_1}), \dots, W_{\text{CS}}(\sigma_{i_n})$ such that the

$$(-i\infty, W_{\text{CS}}(\sigma_{i_1}), \dots, W_{\text{CS}}(\sigma_{i_n}), -i\infty) \quad (4.42)$$

are vertices of a semi-infinite polygon in the W -plane. We count solutions of the HW equation on $M_3 \times \mathbb{R} \times \mathbb{R}_+$ such that on M_3 times the large semi-circle at infinity the solution behaves like broken paths

$$\mathcal{A}_{L, \sigma_{i_1}} \# \mathcal{A}_{\sigma_{i_1} \sigma_{i_2}} \# \dots \# \mathcal{A}_{\sigma_{i_{n-1}} \sigma_{i_n}} \# \mathcal{A}_{\sigma_{i_n}, L} \quad (4.43)$$

where $\mathcal{A}_{L, \sigma_i}$ is an element of $\mathcal{KC}_{\sigma_i}(L)$ and so on, and $M_3 \times \mathbb{R} \times \mathbb{R}_+$ maps to the “interior.” See Figure 4.2.

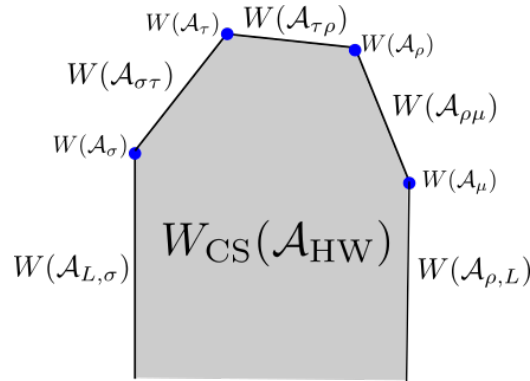


Figure 4.2: The image of a solution of the Haydys-Witten equation on $M_3 \times \mathbb{R} \times \mathbb{R}_+$ relevant to constructing the conjectured knot invariant $\mathcal{B}(L)$ of $L \subset M_3$.

Note that the A_∞ -structure on $\mathcal{T}[M_3]$ coming from counting bulk/closed polygons gives $\mathcal{A}[L, M_3]$ an A_∞ -structure; it is given simply by combining the A_∞ -structures on $\mathcal{T}[M_3]$ with “matrix multiplication” (evaluating the dual complexes $\mathcal{K}\mathcal{C}^\vee$ on the complexes $\mathcal{K}\mathcal{C}$). Given an A_∞ algebra with structure maps (m_1, m_2, m_3, \dots) a Maurer-Cartan element γ is defined to be a degree one element that satisfies the A_∞ Maurer-Cartan equation

$$\sum_{n \geq 1} m_n(\gamma, \dots, \gamma) = 0. \quad (4.44)$$

Conjecture 2 Counting rigid half-polygons associated to a link $L \subset M_3$ determines a canonical a Maurer-Cartan element

$$\mathcal{B}(L) \in \mathcal{A}[L, M_3]. \quad (4.45)$$

The gauge equivalence class of the Maurer-Cartan element $\mathcal{B}(L)$ is an isotopy invariant of $L \subset M_3$.

We can state Conjecture 2 in a slightly different way. The main property of a Maurer-Cartan element γ is that it deforms the structure maps $\{m_n\}$ to deformed versions

$$\{m_n\} \rightarrow \{m_n[\gamma]\} \quad (4.46)$$

that also satisfy the A_∞ axioms. Therefore $\mathcal{B}(L)$ deforms the A_∞ -structure on $\mathcal{A}[L, M_3]$ to a new one. We can then state

Conjecture 2' $\mathcal{A}[L, M_3]$ carries a canonical A_∞ -structure. The homotopy class of this A_∞ -algebra is an isotopy invariant of $L \subset M_3$.

Conjectures 1, 2 and 2' are being developed further in the work in the progress [GKMY]. In particular, we have the tools to compute the characters of our various homological knot invariants by relating the five-dimensional twisted super Yang-Mills theory to three-dimensional $\mathcal{N} = 2$ theories of class R [DGG1, DGG2].

4.3 Holomorphic-Topological Twists and BPS States in $4d$ $\mathcal{N} = 2$ Theories

We now briefly discuss a second future direction. We recall the basic mathematical setup of BPS states in four dimensional theories with $\mathcal{N} = 2$ supersymmetry. For physical background we refer the reader to the review [M1, M3]. The basic data consists of

1. A symplectic lattice Γ , of electromagnetic charges where the anti-symmetric pairing $\langle \cdot, \cdot \rangle : \Gamma \times \Gamma \rightarrow \mathbb{Z}$ is the Dirac-Schwinger-Zwanziger (DSZ) pairing.
2. A homomorphism $Z : \Gamma \rightarrow \mathbb{C}$ known as the central charge.
3. A collection of BPS degeneracies

$$\{\Omega(\gamma; y)\}_{\gamma \in \Gamma} \tag{4.47}$$

where y is a formal variable $\Omega(\gamma; y) \in \mathbb{Z}[y, y^{-1}]$.

$$\Omega(\gamma; y) = \sum_{n \in \mathbb{Z}} \Omega_n(\gamma) y^n \tag{4.48}$$

is known as the protected spin character (PSC).

This is parallel to the setup of Chapter 3 with the additional data of a non-trivial DSZ pairing which plays a crucial role as we will see.

We now introduce the four-dimensional spectrum generator $S_{\mathbb{H}}(q)$. The additional tool needed in four dimensions is the quantum torus algebra, the algebra of formal variables $\{X_\gamma\}_{\gamma \in \Gamma}$ that satisfy the quantum torus algebra relations

$$X_\gamma X_{\gamma'} = q^{\langle \gamma, \gamma' \rangle} X_{\gamma'} X_\gamma = q^{\frac{\langle \gamma, \gamma' \rangle}{2}} X_{\gamma + \gamma'}, \quad (4.49)$$

where q is a formal variable. Let

$$\Phi_q(z) = \prod_{n=0}^{\infty} (1 + q^{n+\frac{1}{2}} z)^{-1} \quad (4.50)$$

be the quantum dilogarithm. For each $\gamma \in \Gamma$ we then introduce the factor

$$\mathcal{U}_\gamma = \prod_{n \in \mathbb{Z}} \Phi_q((-1)^n q^{\frac{n}{2}} X_\gamma)^{(-1)^n \Omega_n(\gamma)}. \quad (4.51)$$

The four-dimensional spectrum generator is then by definition

$$S_{\mathbb{H}}(q) = \prod_{\{\gamma | Z_\gamma \in \mathbb{H}\}}^{\curvearrowright} \mathcal{U}_\gamma. \quad (4.52)$$

We can also introduce the trace

$$M(q) = \text{Tr}(S_{\mathbb{H}}^{\text{opp}}(q) S_{\mathbb{H}}(q)) \quad (4.53)$$

as the coefficient of $X_{\gamma=0}$ in the expansion of $S_{\mathbb{H}}^{\text{opp}}(q) S_{\mathbb{H}}(q)$.

Motivated by the properties of the BPS monodromy and its traces in two-dimensional LG models, the trace of the four-dimensional monodromy $M(q)$ (and its higher powers, including fractional ones) was studied in [CNV]. It was discovered, by explicit computation in a class of examples that $M(q)$ often coincides with the characters of non-unitary RCFTs.

For instance, $M(q)$ for the (A_1, A_2) theory, coincides with the character of the Lee-Yang model, the $(p, p') = (2, 5)$ minimal model with $c = -\frac{22}{5}$. It was further studied in [CoSh] where it was observed to be related to the Schur index of the superconformal fixed point. In addition to the trace of the monodromy $M(q)$, the spectrum generator $S_{\mathbb{H}}(q)$ itself, has also been examined by both mathematicians and physicists. From a mathematical viewpoint it shows up as the “Motivic DT Series” of a cohomological Hall algebra, introduced in [KoSo5]. From a physics perspective $S_{\mathbb{H}}(q)$ was interpreted in terms of RG flow boundary conditions of [DDR] of the $4d \mathcal{N} = 2$ theory in [CGS].

Therefore an exciting direction for future research is the categorification of the four-dimensional spectrum generator $S_{\mathbb{H}}(q)$ and the trace of the BPS monodromy $M(q)$. The categorification of these wall-crossing invariant quantities is expected to uncover novel physical quantities in four-dimensional $\mathcal{N} = 2$ theories, and their supersymmetric defects. In order to make any headway into this categorification one must answer two key questions.

Question 1 What algebraic structure do we expect the categorification of $M(q)$ and the categorification of $S_{\mathbb{H}}(q)$ to carry? Are these algebraic structures generalizations of the L_{∞} and A_{∞} -structures found in the BPS sector of two dimensions?

Question 2 How does one physically construct the differentials, and more generally higher maps, in these algebraic structures?

One may attempt to answer the first question as follows. The first observation is about the nature of the supercharge that is preserved by BPS states. Recall the $4d \mathcal{N} = 2$

supersymmetry algebra

$$\{Q_\alpha^A, \bar{Q}_{\dot{\alpha}}^B\} = \sigma_{\alpha\dot{\alpha}}^\mu P_\mu \epsilon^{AB}, \quad (4.54)$$

$$\{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} \epsilon^{AB} \bar{Z}, \quad (4.55)$$

$$\{\bar{Q}_{\dot{\alpha}}^A, \bar{Q}_{\dot{\beta}}^B\} = -\epsilon_{\dot{\alpha}\dot{\beta}} \epsilon^{AB} Z, \quad (4.56)$$

where as usual $\alpha, \dot{\alpha}$ denote the indices under the Lorentz group $SO(3,1)$, and A denotes the $SU(2)_R$ index. We let $\alpha, \dot{\alpha}$ to go from 1, 2 and A to be \pm . The BPS supercharge is³

$$Q_{\text{BPS}} = \bar{Q}_1^+ + Q_2^+. \quad (4.57)$$

It has well-defined $SU(2)_R$ charge being +1. It is straightforward to show that Q_{BPS} is topological in the (x^0, x^3) plane, whereas it is holomorphic in the (x^1, x^2) plane.

Indeed, let us write the supersymmetry algebra out in components. We work in conventions of Wess and Bagger, where $\sigma^\mu = (-1, \sigma^i)$, so that the explicit list of non-zero commutators of type $\{Q, \bar{Q}\}$ is

$$\{Q_1^+, \bar{Q}_1^-\} = -P_0 + P_3, \quad (4.58)$$

$$\{Q_1^+, \bar{Q}_2^-\} = P_1 - iP_2, \quad (4.59)$$

$$\{Q_2^+, \bar{Q}_1^-\} = P_1 + iP_2, \quad (4.60)$$

$$\{Q_2^+, \bar{Q}_2^-\} = -P_0 - P_3 \quad (4.61)$$

$$\{Q_1^-, \bar{Q}_1^+\} = P_0 - P_3, \quad (4.62)$$

$$\{Q_1^-, \bar{Q}_2^+\} = -(P_1 - iP_2), \quad (4.63)$$

$$\{Q_2^-, \bar{Q}_1^+\} = -(P_1 + iP_2), \quad (4.64)$$

$$\{Q_2^-, \bar{Q}_2^+\} = P_0 + P_3. \quad (4.65)$$

³More generally the BPS supercharge is $\bar{Q}_1^+ + \zeta Q_2^+$ for ζ a phase, but the phase ζ does not affect the nature of the supercharge.

Wick rotating sets $P_0 = iP_4$, and we let $w = x^3 + ix^4$ and $z = x^1 + ix^2$. Using the shorthand $Q = Q_{\text{BPS}}$, we find

$$\{Q, Q_1^-\} = -P_w + \bar{Z} \quad (4.66)$$

$$\{Q, \bar{Q}_2^-\} = -P_{\bar{w}} + Z, \quad (4.67)$$

$$\{Q, \bar{Q}_1^-\} = -\{Q, Q_2^-\} = P_{\bar{z}}. \quad (4.68)$$

so that Q is holomorphic in the z -plane and topological in the w -plane.

Therefore we ask: what type of algebraic structure can one expect on the space of Q -closed local operators in a holomorphic-topological cohomological field theory? We answer this following the references [CDG, OY]. Suppose our spacetime is $\mathbb{R}^d \times \mathbb{C}$ with coordinates $(x^1, \dots, x^d, z, \bar{z})$. We have a supercharge Q along with a one-form supercharge $\mathbf{Q} = \sum_{\mu=1}^d Q_\mu dx^\mu + Q_{\bar{z}} d\bar{z}$ such that

$$Q^2 = 0, \quad [Q, Q_\mu] = P_\mu, \quad [Q, Q_{\bar{z}}] = P_{\bar{z}}. \quad (4.69)$$

Given a Q -closed local operator \mathcal{O} we can form its k th descendent

$$\begin{aligned} \mathcal{O}^{(k)} = & \frac{1}{k!} Q_{\mu_1} \dots Q_{\mu_k} \mathcal{O} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} + \\ & \frac{1}{(k-1)!} Q_{\bar{z}} Q_{\mu_1} \dots Q_{\mu_{k-1}} \mathcal{O} d\bar{z} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{k-1}}. \end{aligned} \quad (4.70)$$

The action of the nilpotent supercharge Q then is

$$Q\mathcal{O}^{(k)} = (d_{\mathbb{R}^d} + \bar{\partial}_{\mathbb{C}})\mathcal{O}^{(k-1)}. \quad (4.71)$$

This is not quite what we want, since the right hand side doesn't have the full exterior derivative on $\mathbb{R}^d \times \mathbb{C}$. This has a simple remedy: if we consider $\mathcal{O}^{(k)} \wedge dz$ then we indeed find

$$Q(\mathcal{O}^{(k)} \wedge dz) = d(\mathcal{O}^{(k-1)} \wedge dz). \quad (4.72)$$

Indeed, more generally for any holomorphic one-form ω on \mathbb{C} we have

$$Q(\mathcal{O}^{(k)} \wedge \omega) = d(\mathcal{O}^{(k-1)} \wedge \omega). \quad (4.73)$$

Then for any $(k+1)$ -cycle Γ_{k+1} we can form a Q -closed operator given by

$$\int_{\Gamma_{k+1}} \mathcal{O}^{(k)} \wedge \omega \quad (4.74)$$

that only depends on the Q -cohomology class of \mathcal{O} and the homology class of Γ_{k+1} .

For the case of four-dimensional $\mathcal{N} = 2$ theories with $Q = Q_{\text{BPS}}$, the one-form supercharge is given by

$$\mathbf{Q} = Q_1^- dw + \overline{Q_2^-} d\overline{w} + (\overline{Q_1^-} + Q_2^-) d\overline{z}. \quad (4.75)$$

The space of holomorphic one-forms on \mathbb{C} is infinite-dimensional, but we can consider the “coherent state” $\alpha(\lambda) = e^{\lambda z} dz$. In particular, the “secondary product” in a holomorphic-topological field theory is defined to be

$$[\mathcal{O}_1 \lambda \mathcal{O}_2](z_2) = \int_{S_{(\vec{x}_2, z_2)}^{d+1}} e^{\lambda(z_1 - z_2)} dz_1 \wedge \mathcal{O}_1^{(d)}(\vec{x}_1, z_1) \mathcal{O}_2(\vec{x}_2, z_2). \quad (4.76)$$

Because we are forming the d th descendent, the bracket carries cohomological degree $-d$. As demonstrated in [OY], $[\cdot_\lambda \cdot]$ satisfies a generalized version of the Jacobi identity. The algebraic structure on the Q -cohomology of local operators along with the operation $[\cdot_\lambda \cdot]$ can be summarized in terms of the axioms of a (graded) Lie conformal algebra [Kac].

A **Lie conformal algebra** is a $\mathbb{C}[\partial]$ -module⁴ A along with a λ -bracket $[\cdot_\lambda \cdot] : A \otimes A \rightarrow A[\lambda] = \mathbb{C}[\lambda] \otimes A$ that satisfies the axioms

⁴ $\partial = \partial_z$ with z being the complex coordinate in the holomorphic plane.

1. Conformal sesquilinearity:

$$[\partial a_\lambda b] = -\lambda[a_\lambda b]. \quad (4.77)$$

$$[a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]. \quad (4.78)$$

2. Skew-symmetry:

$$[a_\lambda b] = -[b_{-\lambda-\partial} a]. \quad (4.79)$$

3. Jacobi identity:

$$[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]]. \quad (4.80)$$

By using the skew-symmetry axiom, we can write the Jacobi identity in a way that will be more convenient later. It reads

$$[[a_\lambda b]_{\lambda+\mu} c] - [[a_\lambda c]_{-\partial-\mu} b] + [[b_\mu c]_{-\partial-\lambda} a] = 0. \quad (4.81)$$

We refer the reader to the book [Kac] for a discussion of the basic properties and examples of Lie conformal algebras.

Let us recap. In two dimensions, the BPS supercharge Q is topological, and the secondary product [BBBDN] equips the Q -cohomology of local operators with the structure of a graded Lie algebra. Equipped with the two-dimensional BPS indices $\{\mu_{ij}\}$, one can consider the trace of the BPS monodromy $M_{2d} = \text{Tr}(S^{\text{opp}} S)$ as a wall-crossing invariant quantity. It can be categorified to a \mathbb{Z} -module $R_c = \text{Tr}(\widehat{R}^{\text{opp}} \otimes \widehat{R})$. The space R_c is viewed as being generated by fans of solitons which can be viewed as local operators in an appropriate sense [GMW], and therefore one might expect it to carry a graded Lie algebra type structure. Indeed, as we have discussed in Chapters 2 and 3, R_c carries the structure of an L_∞ -algebra, an off-shell or “derived” version of a graded Lie algebra that we expect from

the Q -cohomology of local operators with Q being a topological supecharge. Carrying over this reasoning to the four-dimensional case, in four-dimensional $\mathcal{N} = 2$ theories, the BPS supercharge is holomorphic-topological, and the secondary product via holomorphic-topological descent equips the Q -cohomology of local operators in such a theory with the structure of a graded Lie conformal algebra. It is natural then to expect that the categorification of the wall-crossing invariant $M_{4d}(q) = \text{Tr}(S^{\text{opp}}S)$ carries the structure of a derived version of a graded Lie conformal algebra, an L_∞ -conformal algebra.

Motivated by the Hochschild complex of Lie conformal algebras [BKV], we propose the following definition of an L_∞ -generalization of a Lie conformal algebra.

An L_∞ **conformal algebra** consists of a graded $\mathbb{C}[\partial]$ -module A along with a collection of multilinear $\vec{\lambda}$ -brackets

$$\gamma^{(n)} : A^{\otimes n} \rightarrow A[\lambda_1, \dots, \lambda_{n-1}] \quad (4.82)$$

of degree $d+1-dn$ (with d being the number of topological directions) that obey conformal linearity, graded skew-symmetry and the conformal L_∞ -axioms. Individually these axioms read as follows.

Conformal linearity The maps are “anti-linear” with respect to the action of ∂ .

$$\begin{aligned} \gamma_{\lambda_1, \dots, \lambda_i, \dots, \lambda_{n-1}}(a_1, \dots, \partial a_i, \dots, a_n) = \\ -\lambda_i \gamma_{\lambda_1, \dots, \lambda_i, \dots, \lambda_{n-1}}(a_1, \dots, a_i, \dots, a_n) \end{aligned} \quad (4.83)$$

for $1 \leq i \leq n-1$ and

$$\begin{aligned} \gamma_{\lambda_1, \dots, \lambda_{n-1}}(a_1, \dots, a_{n-1}, \partial a_n) = \\ (\partial + \lambda_1 + \dots + \lambda_{n-1}) \gamma_{\lambda_1, \dots, \lambda_{n-1}}(a_1, \dots, a_{n-1}, a_n). \end{aligned} \quad (4.84)$$

Skew symmetry γ picks up signs under simultaneous interchanges $a_i \leftrightarrow a_j$ and $\lambda_i \leftrightarrow \lambda_j$:

$$\begin{aligned} \gamma_{\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_{n-1}}(a_1, \dots, a_i, \dots, a_j, \dots, a_n) = \\ \pm \gamma_{\lambda_1, \dots, \lambda_j, \dots, \lambda_i, \dots, \lambda_{n-1}}(a_1, \dots, a_j, \dots, a_i, \dots, a_n) \end{aligned} \quad (4.85)$$

for $1 \leq i < j < n$ and

$$\begin{aligned} \gamma_{\lambda_1, \dots, \lambda_i, \dots, \lambda_{n-1}}(a_1, \dots, a_i, \dots, a_n) = \\ \pm \gamma_{\lambda_1, \dots, -\partial - \lambda_1 - \dots - \lambda_{n-1}, \dots, \lambda_{n-1}}(a_1, \dots, a_n, \dots, a_i). \end{aligned} \quad (4.86)$$

L_∞ Axioms The collection of maps $\{\gamma_{\lambda_1, \dots, \lambda_{n-1}}^{(n)}\}_{n \geq 1}$ are required to satisfy the conformal L_∞ -axioms. For each $d \geq 1$ we require

$$\sum_{\substack{p+q=d \\ I, J \text{ shuffles of } \{1, \dots, n\}}} \varepsilon \gamma_{\lambda_{i_1} + \dots + \lambda_{i_p}, \lambda_{j_1}, \dots, \lambda_{j_{q-1}}}^{(q+1)} (\gamma_{\lambda_{i_1}, \dots, \lambda_{i_{p-1}}}^{(p)}(a_{i_1}, \dots, a_{i_p}), a_{j_1}, \dots, a_{j_q}) = 0.$$

In the L_∞ axioms we note that we are working with shuffles of $(a_1, \dots, a_{d-1}, a_d)$ and matching them with shuffles of $(\lambda_1, \dots, \lambda_{d-1}, -(\partial + \lambda_1 + \dots + \lambda_{d-1}))$. More generally we can reduce the number of axioms we write down by pretending that each map $\gamma^{(n)}$ carries $\lambda_1, \dots, \lambda_n$ as long as we remember to set

$$\lambda_n = -(\partial + \lambda_1 + \dots + \lambda_{n-1}). \quad (4.87)$$

Thus we are working with the “reduced” version of the space of cochains as defined in [BKV].

The first few axioms L_∞ -axioms are written down as follows. At $d = 1$ we have

$$\gamma^{(1)}(\gamma^{(1)}(a)) = 0. \quad (4.88)$$

At $d = 2$ we have

$$\gamma^{(1)}(\gamma_{\lambda_1}^{(2)}(a_1, a_2)) \pm \gamma_{\lambda_1}^{(2)}(\gamma^{(1)}(a_1), a_2) \pm \gamma_{-\partial - \lambda_1}^{(2)}(\gamma^{(1)}(a_2), a_1) = 0. \quad (4.89)$$

At $d = 3$ we have

$$\begin{aligned}
& \gamma_{\lambda_1+\lambda_2}^{(2)}(\gamma_{\lambda_1}^{(2)}(a_1, a_2), a_3) \pm \gamma_{-\partial-\lambda_2}^{(2)}(\gamma_{\lambda_1}^{(2)}(a_1, a_3), a_2) \pm \gamma_{-\partial-\lambda_1}^{(2)}(\gamma_{\lambda_2}^{(2)}(a_2, a_3), a_1) \\
& \pm \gamma_{\lambda_1, \lambda_2}^{(3)}(\gamma^{(1)}(a_1), a_2, a_3) \pm \gamma_{\lambda_2, \lambda_1}^{(3)}(\gamma^{(1)}(a_2), a_1, a_3) \pm \gamma_{-\partial-\lambda_1-\lambda_2, \lambda_1}^{(3)}(\gamma^{(1)}(a_3), a_1, a_2) \quad (4.90) \\
& \pm \gamma^{(1)}(\gamma_{\lambda_1, \lambda_2}^{(3)}(a_1, a_2, a_3)) = 0.
\end{aligned}$$

Comparing with (4.81), we note that the $d = 3$ axiom is saying that $\gamma_\lambda^{(2)}$ is a λ -bracket up to homotopy.

Having defined an L_∞ -conformal algebra, we can now state precisely the expected result.

Conjecture The trace of the BPS monodromy

$$M(q) = \text{Tr}(S_{\mathbb{H}}^{\text{opp}} S_{\mathbb{H}}) \quad (4.91)$$

can be categorified to a vector space

$$M(q) \rightsquigarrow \mathcal{A}_c. \quad (4.92)$$

Moreover the space \mathcal{A}_c naturally carries the structure of an L_∞ -conformal algebra.

We expect similar remarks to apply to the spectrum generator itself, where the structure we expect is that of an A_∞ -conformal algebra. The latter notion can also be defined explicitly, but we omit the details.

We now turn to the second question: what is the physical construction of the differentials and multiplication maps in four-dimensional $\mathcal{N} = 2$ theories? In two dimensions, say in Landau-Ginzburg models, the L_∞ -products on R_c , in particular the differential, are entirely

determined by studying the ζ -instanton equation. A concrete question one can ask then is what is the analogue of the ζ -instanton equation in, say class S theories⁵?

Recall that in a class S theory with UV curve C , BPS states admit a description as pseudo-holomorphic maps

$$\phi : \mathbb{D}^2 \rightarrow T^*C, \quad (4.93)$$

in an appropriate almost complex structure on T^*C determined by ζ [M]. Fortunately, this description is precisely the correct one to admit a Morse theoretic description. Namely, formulated this way, the BPS states coincide with the critical points of a Morse functional. This was already discussed in Chapter 1, when discussing the holomorphic Liouville superpotential on a complex symplectic manifold (Y, Ω) . The corresponding gradient flow equation is the three-dimensional BPS equation

$$\frac{\partial \phi^A}{\partial \tau} + J_B^A \left(\frac{\partial \phi^B}{\partial x} + I_C^B \frac{\partial \phi^C}{\partial y} \right) = 0. \quad (4.94)$$

For our application, we take $Y = T^*C$, and expect that counting solutions of the 3d instanton equation (4.94) with appropriate boundary conditions will provide us with the relevant differentials on the categorified BPS monodromy and the categorified spectrum generator. We leave the development of this as an exciting and challenging problem for the future.

⁵Recall that a four-dimensional $\mathcal{N} = 2$ theory is called “a theory of class S” if it admits a construction as the low energy limit of six-dimensional $\mathcal{N} = (0, 2)$ theory compactified on a Riemann surface C .

Appendix A

Some Basic Homological Algebra

A.1 Some Basic Homological Algebra

The categorical wall-crossing formula is most cleanly stated using some standard homological algebra. We summarize the concepts we need below and refer the reader to [Weib] for further details.

Homotopy Equivalence of Complexes Two complexes (C, d) and (C', d') are said to be homotopy equivalent if there are chain maps $f : C \rightarrow C'$ and $g : C' \rightarrow C$ such that

$$gf = 1_C + \{d, s\}, \quad (\text{A.1})$$

$$fg = 1_{C'} + \{d', s'\}, \quad (\text{A.2})$$

for some degree -1 maps $s : C \rightarrow C$ and $s' : C' \rightarrow C'$. s and s' are known as chain homotopies.

Mapping Cone Recollection Given two chain complexes (A^\bullet, d_A) and (B^\bullet, d_B) along with a chain map

$$f : A^\bullet \rightarrow B^\bullet, \quad (\text{A.3})$$

there is a canonical chain complex $\text{Cone}(f)$ defined as follows. The underlying space consists of

$$\text{Cone}(f) = B \oplus A[-1]. \quad (\text{A.4})$$

Writing an element of $\text{Cone}(f)$ as a column vector

$$\begin{pmatrix} b \\ a[-1] \end{pmatrix}, \quad (\text{A.5})$$

the differential on $\text{Cone}(f)$ is

$$d[f] = \begin{pmatrix} d_B & f \\ 0 & -d_A \end{pmatrix}. \quad (\text{A.6})$$

$d[f]$ is nilpotent as a consequence of f being a chain map. The projection map

$$\pi : \text{Cone}(f) \rightarrow A[-1], \quad (\text{A.7})$$

and the inclusion map

$$i : B \rightarrow \text{Cone}(f), \quad (\text{A.8})$$

are chain maps that fit into the exact sequence

$$0 \longrightarrow B \xrightarrow{i} \text{Cone}(f) \xrightarrow{\pi} A[-1] \longrightarrow 0. \quad (\text{A.9})$$

Mapping Cylinder Recollection Suppose we are in the setting of the mapping cone of a morphism $f : A \rightarrow B$, i.e consider $\text{Cone}(f)$. Note that the projection map

$$\pi : (\text{Cone}(f))[1] \rightarrow A \quad (\text{A.10})$$

is a chain map. The **mapping cylinder** of f is then by definition

$$\text{Cyl}(f) := \text{Cone}(\pi). \quad (\text{A.11})$$

More explicitly, we can write

$$\text{Cyl}(f) = B \oplus A[-1] \oplus A \quad (\text{A.12})$$

The differential on $\text{Cyl}(f)$ reads

$$d = \begin{pmatrix} d_B & f & 0 \\ 0 & -d_A & 0 \\ 0 & \text{id} & d_A \end{pmatrix}. \quad (\text{A.13})$$

The following is standard in homological algebra and topology (for instance see Lemma 1.5.6 in Weibel [Weib]).

Proposition Suppose $(A, d_A), (B, d_B)$ are chain complexes and $f : A \rightarrow B$ is a chain map. Then B and $\text{Cyl}(f)$ are canonically homotopy equivalent. The map $i : B \rightarrow \text{Cyl}(f)$ is given by inclusion and its homotopy inverse $j : \text{Cyl}(f) \rightarrow B$ is given by

$$j \begin{pmatrix} b \\ a^{[-1]} \\ a' \end{pmatrix} = b + f(a'). \quad (\text{A.14})$$

Remark The mapping cone and mapping cylinder constructions have their origins in topology. If $f : (X, p_*) \rightarrow (Y, q_*)$ is a continuous map of topological spaces we can define topological spaces

$$\text{Cyl}(f) = (X \times I) \cup Y / (x, 1) \sim f(x), \quad (\text{A.15})$$

$$\text{Cone}(f) = \text{Cyl}(f) / (x, 0) \sim p_*. \quad (\text{A.16})$$

These spaces are related to the previous constructions as follows. If $C_*(X), C_*(Y)$ denote the singular chain complexes of X and Y , then

$$C_*(\text{Cyl}(f)) \cong \text{Cyl}(f_* : C_*(X) \rightarrow C_*(Y)), \quad (\text{A.17})$$

$$C_*(\text{Cone}(f)) \cong \text{Cone}(f_* : C_*(X) \rightarrow C_*(Y)), \quad (\text{A.18})$$

f_* being the induced map on complexes.

Triangularity Lemma: Let A, B, C be chain complexes and $f : A \rightarrow B$ be a chain map. Suppose that

$$C \simeq \text{Cone}(f : A \rightarrow B). \quad (\text{A.19})$$

Then we can construct chain maps

$$g : B \rightarrow C, \quad (\text{A.20})$$

$$h : C[1] \rightarrow A \quad (\text{A.21})$$

such that

$$A[-1] \simeq \text{Cone}(g : B \rightarrow C), \quad (\text{A.22})$$

$$B \simeq \text{Cone}(h : C[1] \rightarrow A). \quad (\text{A.23})$$

The maps g and h can be written down explicitly. We set

$$g = u \circ i \quad (\text{A.24})$$

where $u : \text{Cone}(f) \rightarrow C$ is one of the maps provided by homotopy equivalence and $i : B \rightarrow \text{Cone}(f)$ is the inclusion map (also a chain map). Similarly

$$h = \pi \circ v \quad (\text{A.25})$$

where $v : C \rightarrow \text{Cone}(f)$ is the homotopy inverse of u and $\pi : \text{Cone}(f) \rightarrow A[-1]$ is the projection map (also a chain map). These maps may be remembered from the commutative diagram

$$\begin{array}{ccccccc}
 & & & C & & & \\
 & & & \downarrow v & & & \\
 0 & \longrightarrow & B & \xrightarrow{i} & \text{Cone}(f) & \xrightarrow{\pi} & A[-1] \longrightarrow 0. \\
 & & & \uparrow u & & & \\
 & & & C & & &
 \end{array} \quad (\text{A.26})$$

Appendix B

A_∞ and L_∞ Algebras and Their Morphisms

B.1 A_∞ Algebras and Morphisms

This appendix serves as a reminder of some elementary formulas in A_∞ theory. We refer the reader to the (unpublished) book of Kontsevich-Soibelman [KoSo4], Keller's notes [Kel], and appendix A of [GMW] for more details.

A_∞ -algebra Given a graded vector space A , denote by $T^\bullet(A)$ the tensor algebra of A , and $T_+^\bullet(A)$ the positive part of the tensor algebra:

$$T^\bullet(A) = \bigoplus_{n \geq 0} A^{\otimes n}, \quad (\text{B.1})$$

$$T_+^\bullet(A) = \bigoplus_{n \geq 1} A^{\otimes n}. \quad (\text{B.2})$$

A is called an A_∞ -algebra if there is a square-zero, degree one derivation,¹

$$\delta : T_+^\bullet(A^*[1]) \rightarrow T_+^\bullet(A^*[1]). \quad (\text{B.3})$$

Extracting Taylor coefficients amounts to a collection maps

$$m_n : A^{\otimes n} \rightarrow A \quad (\text{B.4})$$

¹Meaning δ is both a derivation of the tensor algebra $\delta(X^a X^b) = \delta X^a X^b \pm X^a \delta X^b$, and a differential, a degree one map such that $\delta^2 = 0$.

of degree $2 - n$ satisfying the A_∞ -associativity axioms: for each $d \geq 1$ we have

$$\sum_{\substack{k+l=d+1 \\ 1 \leq i \leq k}} (-1)^{d_1 + \dots + d_{i-1} - i + 1} m_k(a_1, \dots, a_{i-1}, m_l(a_i, \dots, a_{i+l-1}), a_{i+l}, \dots, a_d) = 0. \quad (\text{B.5})$$

a_i is a homogeneous element and $d_i = \deg(a_i)$.

A_∞ -morphism Given two A_∞ -algebras

$$(A_1, \{m_n\}) \text{ and } (A_2, \{\mu_n\}) \quad (\text{B.6})$$

an A_∞ -morphism

$$f : A_1 \rightarrow A_2 \quad (\text{B.7})$$

is an algebra homomorphism (respects tensor algebra structure)

$$f : T_+^\bullet(A_2^*[1]) \rightarrow T_+^\bullet(A_1^*[1]) \quad (\text{B.8})$$

that is also a chain map: namely f is degree 0 map satisfying

$$f\delta_2 = \delta_1 f. \quad (\text{B.9})$$

Again expanding out Taylor coefficients we get a collection of maps

$$f_n : (A_1)^{\otimes n} \rightarrow A_2 \quad (\text{B.10})$$

of degree $1 - n$ satisfying the A_∞ -morphism axioms

$$\sum_{\substack{k+l=d+1 \\ 1 \leq i \leq k}} (-1)^{d_1 + \dots + d_{i-1} - i + 1} f_k(a_1, \dots, a_{i-1}, m_l(a_i, \dots, a_{i+l-1}), a_{i+l}, \dots, a_d) = \sum_{\substack{n_1 + \dots + n_k = d \\ k \geq 1}} \mu_k(f_{n_1}(a_1, \dots, a_{n_1}), \dots, f_{n_k}(a_{d-n_k+1}, \dots, a_d)). \quad (\text{B.11})$$

The $d = 1$ relation is

$$\mu_1(f_1(a_1)) = f_1(m_1(a_1)) \quad (\text{B.12})$$

which simply says that f_1 is a chain map.

The $d = 2$ relation is

$$\begin{aligned} f_1(m_2(a_1, a_2)) \pm \mu_2(f_1(a_1), f_1(a_2)) = \\ f_2(m_1(a_1), a_2) \pm f_2(a_1, m_1(a_2)) \pm \mu_1(f_2(a_1, a_2)) \end{aligned} \quad (\text{B.13})$$

where the precise signs can be restored via (B.11). This says that the diagram

$$\begin{array}{ccc} A_1^{\otimes 2} & \xrightarrow{f_1 \otimes f_1} & A_2^{\otimes 2} \\ m_2 \downarrow & & \downarrow \mu_2 \\ A_1 & \xrightarrow{f_1} & A_2, \end{array} \quad (\text{B.14})$$

commutes up to homotopy, with f_2 providing the chain homotopy.

Quasi-isomorphism of A_∞ -algebras An A_∞ -morphism $\{f_n\}_{n \geq 1}$ is said to be a quasi-isomorphism if $f_1 : (A_1, m_1) \rightarrow (A_2, \mu_1)$ is a quasi-isomorphism of chain complexes.

Homotopy Equivalence of A_∞ -algebras Two A_∞ -morphisms $f, g : A_1 \rightarrow A_2$, between A_∞ -algebras are said to be homotopic $f \simeq g$, if there is a degree -1 map

$$S : T_+^\bullet(A_2^*[1]) \rightarrow T_+^\bullet(A_1^*[1]) \quad (\text{B.15})$$

such that

$$f - g = S\delta_2 + \delta_1 S. \quad (\text{B.16})$$

That is S provides a homotopy between the parent maps f, g of the tensor algebra. A_1 and A_2 are said to be homotopy equivalent A_∞ algebras if there are A_∞ -morphisms $f : A_1 \rightarrow A_2$

and $g : A_2 \rightarrow A_1$ such that the compositions in either direction are homotopic to the identities on the tensor algebras:

$$g \circ f \simeq 1_{T_+ A_2^*}, \quad (\text{B.17})$$

$$f \circ g \simeq 1_{T_+ A_1^*}. \quad (\text{B.18})$$

In particular, (A_1, m_1) and (A_2, μ_1) are homotopy equivalent chain complexes.

L_∞ -algebra A graded vector space L is called an L_∞ -algebra if there is a derivation differential

$$\delta : S_+^\bullet(L^*[2]) \rightarrow S_+^\bullet(L^*[2]).$$

Extracting coefficients gives us that we have a collection of maps

$$\lambda_n : L^{\otimes n} \rightarrow L \quad (\text{B.19})$$

of degree $3 - 2n$ which are graded symmetric, and satisfy the L_∞ -associativity axioms: for each $d \geq 1$ we have

$$\sum_{\substack{k+l=d+1 \\ \sigma \in \text{Sh}_2(k-1, l)}} \epsilon(\sigma, \vec{\ell}) \lambda_k(\lambda_l(\ell_{\sigma(1)}, \dots, \ell_{\sigma(l)}), \ell_{\sigma(l+1)}, \dots, \ell_{\sigma(d)}) = 0. \quad (\text{B.20})$$

In the above $\sigma \in \text{Sh}_2(k, l)$ denotes a permutation $\sigma \in S_{k+l}$ such that

$$\sigma(1) < \dots < \sigma(k), \quad \sigma(k+1) < \dots < \sigma(k+l). \quad (\text{B.21})$$

L_∞ -morphism Given

$$(L_1, \{\lambda_n\}), \quad (L_2, \{\kappa_n\}) \quad (\text{B.22})$$

two L_∞ -algebras an L_∞ -morphism $f : L_1 \rightarrow L_2$ is an algebra homomorphism

$$f : S_+^\bullet(L_2^*[2]) \rightarrow S_+^\bullet(L_1^*[2]) \quad (\text{B.23})$$

that is also a chain map with respect to the L_∞ -structures. Extracting coefficients we get a collection of maps

$$f_n : (L_1)^{\otimes n} \rightarrow L_2 \quad (\text{B.24})$$

of degree $1 - n$ satisfying axioms for an L_∞ -morphism: for each $d \geq 1$

$$\begin{aligned} & \sum_{\substack{k+l=d+1 \\ \sigma \in \text{Sh}_2(k-1, l)}} \epsilon(\sigma, \vec{\ell}) f_k(\lambda_l(\ell_{\sigma(1)}, \dots, \ell_{\sigma(l)}), \ell_{\sigma(l+1)}, \dots, \ell_{\sigma(d)}) = \\ & \sum_{\substack{n_1 + \dots + n_k \\ \sigma \in \text{Sh}_k(n_1, \dots, n_k) \\ k \geq 1}} \frac{1}{k!} \epsilon'(\sigma) \kappa_n(f_{n_1}(\ell_{\sigma(1)}, \dots, \ell_{\sigma(n_1)}), \dots, f_{n_k}(\ell_{\sigma(d-n_k+1)}, \dots, \ell_{\sigma(d)})), \end{aligned} \quad (\text{B.25})$$

and $\epsilon(\sigma, \vec{\ell})$ and $\epsilon'(\sigma)$ are suitable signs.

Quasi-isomorphism of L_∞ -algebras An L_∞ -morphism $\{f_n\}_{n \geq 1}$ from $(L_1, \{\lambda_n\})$ and $(L_2, \{\kappa_n\})$ is said to be a quasi-isomorphism if

$$f_1 : (L_1, \lambda_1) \rightarrow (L_2, \kappa_2) \quad (\text{B.26})$$

is a quasi-isomorphism of chain complexes.

Maurer-Cartan elements of L_∞ -algebras A Maurer-Cartan element γ of an L_∞ algebra $(L, \{\lambda_n\})$ is a degree two element that solves the L_∞ Maurer-Cartan equation

$$\sum_{n \geq 1} \frac{1}{n!} \lambda_n(\gamma, \dots, \gamma) = 0. \quad (\text{B.27})$$

An infinitesimal gauge transformation of a Maurer-Cartan element γ is written as

$$\delta_\epsilon \gamma = \sum_{n \geq 1} \frac{1}{n!} \lambda_n(\gamma^{\otimes (n-1)}, \epsilon) \quad (\text{B.28})$$

where ϵ is any degree one element of L . Indeed one checks that $\gamma + \delta_\epsilon \gamma$ solves the Maurer-Cartan equation to first order in ϵ .

Terminology: Algebras vs Categories In the bulk text of this paper we have often used the terms “algebra” and “category” interchangeably. This is justified because we can go between the two in a precise manner. Following the discussion in chapter 6 of [KoSo4], given a linear category with a finite object set S , we can define a unital algebra to be

$$A = \oplus_{r,s \in S} \text{Hom}(r, s), \quad (\text{B.29})$$

with the unit being the direct sum of identity compositions and multiplications given by compositions of morphisms. Conversely, if a unital algebra A is equipped with commuting idempotents $\{\Pi_i\}_{i \in I}$ such that $1_A = \oplus_i \Pi_i$, then we can construct a category \mathcal{C} by setting the object set to be I and letting

$$\text{Hom}(i, j) = \Pi_i A \Pi_j. \quad (\text{B.30})$$

Appendix C

BPS Degeneracies in One-Variable Polynomial Superpotentials

C.1 $N_{ij} \in \{0, 1\}$ for $W \in \mathbb{C}[X]$

We give a proof of the assertion that a Landau-Ginzburg model with target \mathbb{C} and $W(X)$ a Morse polynomial has at most a single soliton between any pair of critical points. For this we consider the relative homology group

$$V = H_1(\mathbb{C}, \text{Re}(\zeta^{-1}W) \rightarrow \infty; \mathbb{Z}) \quad (\text{C.1})$$

where ζ is a phase not equal to any of the critical phases. V is easily constructed. Supposing that the degree of W is n , we divide the complex plane \mathbb{C} into $2n$ wedges of equal angle $\frac{2\pi}{n}$ and shade alternating regions R_1, \dots, R_n . A basis for V is provided by cycles $\gamma_{a,a+1}$ that connect R_a and R_{a+1} for $a = 1, \dots, n-1$. On the other hand, Picard-Lefschetz theory says that the homology class of the Lefschetz thimbles $L_i(\zeta)$ for $i = 1, \dots, n-1$ critical points of W must also form a \mathbb{Z} -module basis for V . In particular this implies that if $L_i(\zeta)$ connects R_a, R_b and $L_j(\zeta)$ connects R_c, R_d then $\{a, b\} \neq \{c, d\}$ since otherwise they will be multiples of each other by ± 1 in homology, and thus linearly dependent elements of V . Considering a point p on the ζ -ray emanating from W_i far out enough, $W^{-1}(p) \cap L_i(\zeta)$ is a pair of points lying in distinct regions R_a, R_b which are connected by L_i . Therefore $|L_i(\zeta_j e^{-i\epsilon}) \cap L_j(\zeta_i e^{i\epsilon})|$ is at most one, concluding the proof.

References

- [A] M. Aganagic, “Knot Categorification from Mirror Symmetry, Part II: Lagrangians,” [arXiv:2105.06039 [hep-th]].
- [AF] L. Alvarez-Gaume and D. Z. Freedman, “Potentials for the Supersymmetric Nonlinear Sigma Model,” *Commun. Math. Phys.* **91**, 87 (1983) doi:10.1007/BF01206053
- [ADJM] E. Andriyash, F. Denef, D. L. Jafferis and G. W. Moore, “Wall-crossing from supersymmetric galaxies,” *JHEP* **01**, 115 (2012) doi:10.1007/JHEP01(2012)115 [arXiv:1008.0030 [hep-th]].
- [ATZ] M. Ashwinkumar, M. C. Tan and Q. Zhao, “Branes and Categorifying Integrable Lattice Models,” *Adv. Theor. Math. Phys.* **24**, no.1, 1-24 (2020) doi:10.4310/ATMP.2020.v24.n1.a1 [arXiv:1806.02821 [hep-th]].
- [ADHM] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, “Construction of Instantons,” *Phys. Lett. A* **65**, 185-187 (1978) doi:10.1016/0375-9601(78)90141-X
- [Aur] D. Auroux, “A beginner’s introduction to Fukaya categories,” [arXiv:1301.7056 [math.SG]].
- [BKV] B. Bakalov, V. G. Kac and A. A. Voronov, “Cohomology of conformal algebras,” *Commun. Math. Phys.* **200**, 561-598 (1999) doi:10.1007/s002200050541 [arXiv:math/9803022 [math]].
- [BBBDN] C. Beem, D. Ben-Zvi, M. Bullimore, T. Dimofte and A. Neitzke, “Secondary products in supersymmetric field theory,” *Annales Henri Poincaré* **21**, no.4, 1235-1310 (2020) doi:10.1007/s00023-020-00888-3 [arXiv:1809.00009 [hep-th]].
- [BPST] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, “Pseudoparticle Solutions of the Yang-Mills Equations,” *Phys. Lett. B* **59**, 85-87 (1975) doi:10.1016/0370-2693(75)90163-X
- [BC] R. Bergamin and S. Cecotti, “FQHE and tt^* geometry,” *JHEP* **12**, 172 (2019) doi:10.1007/JHEP12(2019)172 [arXiv:1910.05022 [hep-th]].
- [CGGLPFZ] A. Carey, F. Gesztesy, H. Grosse, G. Levitina, D. Potapov, F. Sukochev, D. Zanin, “Trace Formulas for a Class of non-Fredholm Operators: A Review,”

- Reviews in Mathematical Physics 28, No. 10 (2016) 1630002 [arXiv:1610.04954 [math.AP]].
- [CHT] S. M. Carroll, S. Hellerman and M. Trodden, “Domain wall junctions are $1/4$ - BPS states,” Phys. Rev. D **61**, 065001 (2000) doi:10.1103/PhysRevD.61.065001 [arXiv:hep-th/9905217 [hep-th]].
- [CF] A. S. Cattaneo and G. Felder, “A Path integral approach to the Kontsevich quantization formula,” Commun. Math. Phys. **212**, 591-611 (2000) doi:10.1007/s002200000229 [arXiv:math/9902090 [math]].
- [Cec] S. Cecotti, “Trieste Lectures on Wall–Crossing Invariants,” <https://people.sissa.it/~cecotti/ictptext.pdf>
- [CFIV] S. Cecotti, P. Fendley, K. A. Intriligator and C. Vafa, “A New supersymmetric index,” Nucl. Phys. B **386**, 405-452 (1992) doi:10.1016/0550-3213(92)90572-S [arXiv:hep-th/9204102 [hep-th]].
- [CNV] S. Cecotti, A. Neitzke and C. Vafa, “R-Twisting and 4d/2d Correspondences,” [arXiv:1006.3435 [hep-th]].
- [CNV] S. Cecotti, A. Neitzke and C. Vafa, “Twistorial topological strings and a tt^* geometry for $\mathcal{N} = 2$ theories in $4d$,” Adv. Theor. Math. Phys. **20**, 193-312 (2016) doi:10.4310/ATMP.2016.v20.n2.a1 [arXiv:1412.4793 [hep-th]].
- [CV1] S. Cecotti and C. Vafa, “On classification of $N=2$ supersymmetric theories,” Commun. Math. Phys. **158**, 569-644 (1993) doi:10.1007/BF02096804 [arXiv:hep-th/9211097 [hep-th]].
- [CV2] S. Cecotti and C. Vafa, “2d Wall-Crossing, R-Twisting, and a Supersymmetric Index,” [arXiv:1002.3638 [hep-th]].
- [Col] S. R. Coleman, “The Uses of Instantons,” Subnucl. Ser. **15**, 805 (1979) HUTP-78-A004.
- [CoSh] C. Cordova and S. H. Shao, “Schur Indices, BPS Particles, and Argyres-Douglas Theories,” JHEP **01**, 040 (2016) doi:10.1007/JHEP01(2016)040 [arXiv:1506.00265 [hep-th]].
- [CGS] C. Cordova, D. Gaiotto and S. H. Shao, “Infrared Computations of Defect Schur Indices,” JHEP **11**, 106 (2016) doi:10.1007/JHEP11(2016)106 [arXiv:1606.08429 [hep-th]].
- [C] K. Costello, “Supersymmetric gauge theory and the Yangian,” [arXiv:1303.2632 [hep-th]].

- [CDG] K. Costello, T. Dimofte and D. Gaiotto, “Boundary Chiral Algebras and Holomorphic Twists,” [arXiv:2005.00083 [hep-th]].
- [CP] K. Costello and N. M. Paquette, “Twisted Supergravity and Koszul Duality: A case study in AdS_3 ,” [arXiv:2001.02177 [hep-th]].
- [CWY1] K. Costello, E. Witten and M. Yamazaki, “Gauge Theory and Integrability, I,” ICCM Not. **06**, no.1, 46-119 (2018) doi:10.4310/ICCM.2018.v6.n1.a6 [arXiv:1709.09993 [hep-th]].
- [CWY2] K. Costello, E. Witten and M. Yamazaki, “Gauge Theory and Integrability, II,” ICCM Not. **06**, no.1, 120-146 (2018) doi:10.4310/ICCM.2018.v6.n1.a7 [arXiv:1802.01579 [hep-th]].
- [CY] K. Costello and M. Yamazaki, “Gauge Theory And Integrability, III,” [arXiv:1908.02289 [hep-th]].
- [DK] S. K. Donaldson, P. B. Kronheimer “The Geometry of Four-Manifolds,” Oxford Mathematical Monographs
- [DM] F. Denef and G. W. Moore, “Split states, entropy enigmas, holes and halos,” JHEP **11**, 129 (2011) doi:10.1007/JHEP11(2011)129 [arXiv:hep-th/0702146 [hep-th]].
- [Dia] D. E. Diaconescu, “D-branes, monopoles and Nahm equations,” Nucl. Phys. B **503**, 220-238 (1997) doi:10.1016/S0550-3213(97)00438-0 [arXiv:hep-th/9608163 [hep-th]].
- [Dim] T. Dimofte, “Koszul duality patterns in physics,” String Math 2017.
- [DGG1] T. Dimofte, D. Gaiotto and S. Gukov, “Gauge Theories Labelled by Three-Manifolds,” Commun. Math. Phys. **325**, 367-419 (2014) doi:10.1007/s00220-013-1863-2 [arXiv:1108.4389 [hep-th]].
- [DGG2] T. Dimofte, D. Gaiotto and S. Gukov, “3-Manifolds and 3d Indices,” Adv. Theor. Math. Phys. **17**, no.5, 975-1076 (2013) doi:10.4310/ATMP.2013.v17.n5.a3 [arXiv:1112.5179 [hep-th]].
- [DDR] T. Dimofte, D. Gaiotto and R. van der Veen, “RG Domain Walls and Hybrid Triangulations,” Adv. Theor. Math. Phys. **19**, 137-276 (2015) doi:10.4310/ATMP.2015.v19.n1.a2 [arXiv:1304.6721 [hep-th]].
- [DLMF] “Digital Library of Mathematical Functions,” <https://dlmf.nist.gov/23.5>.
- [Don] S. K. Donaldson, “An application of gauge theory to four-dimensional topology,” Jour. Differential Geometry 18 (1983), 279-315.

- [Dor] N. Dorey, “The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms,” *JHEP* **11**, 005 (1998) doi:10.1088/1126-6708/1998/11/005 [arXiv:hep-th/9806056 [hep-th]].
- [FB] F. Ferrari and A. Bilal, “The Strong coupling spectrum of the Seiberg-Witten theory,” *Nucl. Phys. B* **469**, 387-402 (1996) doi:10.1016/0550-3213(96)00150-2 [arXiv:hep-th/9602082 [hep-th]].
- [F1] A. Floer, “An instanton-invariant for 3-manifolds,” *Comm. Math. Phys.* **118**(2): 215-240 (1988).
- [F2] A. Floer, “Witten’s complex and infinite-dimensional Morse theory,” *J. Differential Geom.* **30**(1): 207-221 (1989) doi: 10.4310/jdg/1214443291
- [FOOO1] K. Fukaya, Y. -G. Oh, H. Ohta, K. Ono, “Lagrangian Intersection Floer Theory: Anomaly and Obstruction, Part I,” American Mathematical Society , 2010.
- [FOOO2] K. Fukaya, Y. -G. Oh, H. Ohta, K. Ono, “Lagrangian Intersection Floer Theory: Anomaly and Obstruction, Part II,” American Mathematical Society , 2010.
- [GKMY] D. Gaiotto, A. Z. Khan, G. W. Moore, F. Yan, “Algebraic Knot Invariants, Landau-Ginzburg Models and $3d, \mathcal{N} = 2$ Class R Theories,” To appear.
- [GMN1] D. Gaiotto, G. W. Moore and A. Neitzke, “Four-dimensional wall-crossing via three-dimensional field theory,” *Commun. Math. Phys.* **299**, 163-224 (2010) doi:10.1007/s00220-010-1071-2 [arXiv:0807.4723 [hep-th]].
- [GMN2] D. Gaiotto, G. W. Moore and A. Neitzke, “Wall-crossing, Hitchin Systems, and the WKB Approximation,” [arXiv:0907.3987 [hep-th]].
- [GMN3] D. Gaiotto, G. W. Moore and A. Neitzke, “Framed BPS States,” *Adv. Theor. Math. Phys.* **17**, no.2, 241-397 (2013) doi:10.4310/ATMP.2013.v17.n2.a1 [arXiv:1006.0146 [hep-th]].
- [GMN4] D. Gaiotto, G. W. Moore and A. Neitzke, “Wall-Crossing in Coupled 2d-4d Systems,” *JHEP* **12**, 082 (2012) doi:10.1007/JHEP12(2012)082 [arXiv:1103.2598 [hep-th]].
- [GMN5] D. Gaiotto, G. W. Moore and A. Neitzke, “Spectral networks,” *Annales Henri Poincare* **14**, 1643-1731 (2013) doi:10.1007/s00023-013-0239-7 [arXiv:1204.4824 [hep-th]].
- [GMW] D. Gaiotto, G. W. Moore and E. Witten, “Algebra of the Infrared: String Field Theoretic Structures in Massive $\mathcal{N} = (2,2)$ Field Theory In Two Dimensions,” [arXiv:1506.04087 [hep-th]].

- [GMWSh] D. Gaiotto, G. W. Moore and E. Witten, “An Introduction To The Web-Based Formalism,” [arXiv:1506.04086 [hep-th]].
- [GO] D. Gaiotto and J. Oh, “Aspects of Ω -deformed M-theory,” [arXiv:1907.06495 [hep-th]].
- [GW] D. Gaiotto and E. Witten, “Knot Invariants from Four-Dimensional Gauge Theory,” *Adv. Theor. Math. Phys.* **16**, no.3, 935-1086 (2012) doi:10.4310/ATMP.2012.v16.n3.a5 [arXiv:1106.4789 [hep-th]].
- [Gal] D. Galakhov, “On Supersymmetric Interface Defects, Brane Parallel Transport, Order-Disorder Transition and Homological Mirror Symmetry,” [arXiv:2105.07602 [hep-th]].
- [GM] D. Galakhov and G. W. Moore, “Comments On The Two-Dimensional Landau-Ginzburg Approach To Link Homology,” [arXiv:1607.04222 [hep-th]].
- [GGM] S. Garoufalidis, J. Gu and M. Marino, “Peacock patterns and resurgence in complex Chern-Simons theory,” [arXiv:2012.00062 [math.GT]].
- [GT] G. W. Gibbons and P. K. Townsend, “A Bogomolny equation for intersecting domain walls,” *Phys. Rev. Lett.* **83**, 1727-1730 (1999) doi:10.1103/PhysRevLett.83.1727 [arXiv:hep-th/9905196 [hep-th]].
- [GPV] S. Gukov, P. Putrov and C. Vafa, “Fivebranes and 3-manifold homology,” *JHEP* **07**, 071 (2017) doi:10.1007/JHEP07(2017)071 [arXiv:1602.05302 [hep-th]].
- [GPPV] S. Gukov, D. Pei, P. Putrov and C. Vafa, “BPS spectra and 3-manifold invariants,” *J. Knot Theor. Ramifications* **29**, no.02, 2040003 (2020) doi:10.1142/S0218216520400039 [arXiv:1701.06567 [hep-th]].
- [MS1] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslow, “Mirror symmetry.”
- [Hay] A. Haydys, “Fukaya-Seidel category and gauge theory,” *J. Sympl. Geom.* **13**, 151-207 (2015) doi:10.4310/JSG.2015.v13.n1.a5 [arXiv:1010.2353 [math.SG]].
- [HH] A. Hanany and K. Hori, “Branes and N=2 theories in two-dimensions,” *Nucl. Phys. B* **513**, 119-174 (1998) doi:10.1016/S0550-3213(97)00754-2 [arXiv:hep-th/9707192 [hep-th]].
- [He] S. He, “A Gluing Theorem for the Kapustin-Witten Equations with a Nahm Pole,” [arXiv:1707.06182 [math.DG]].
- [HM] S. He and R. Mazzeo, “Classification of Nahm pole solutions of the Kapustin-Witten equations on $S^1 \times \Sigma \times \mathbb{R}^+$,” [arXiv:1901.00274 [math.DG]].

- [HV] K. Hori and C. Vafa, “Mirror symmetry,” [arXiv:hep-th/0002222 [hep-th]].
- [INOS] K. Ito, M. Naganuma, H. Oda and N. Sakai, “An Exact solution of BPS junctions and its properties,” Nucl. Phys. B Proc. Suppl. **101**, 304-313 (2001) doi:10.1016/S0920-5632(01)01515-8 [arXiv:hep-th/0012182 [hep-th]].
- [Kac] V. G. Kac, “Vertex Algebras for Beginners,” University Lecture Series
- [KRS] A. Kapustin, L. Rozansky and N. Saulina, “Three-dimensional topological field theory and symplectic algebraic geometry I,” Nucl. Phys. B **816**, 295-355 (2009) doi:10.1016/j.nuclphysb.2009.01.027 [arXiv:0810.5415 [hep-th]].
- [KW] A. Kapustin and E. Witten, “Electric-Magnetic Duality And The Geometric Langlands Program,” Commun. Num. Theor. Phys. **1**, 1-236 (2007) doi:10.4310/CNTP.2007.v1.n1.a1 [arXiv:hep-th/0604151 [hep-th]].
- [KS] H. Kajiura and J. Stasheff, “Homotopy algebras inspired by classical open-closed string field theory,” Commun. Math. Phys. **263**, 553-581 (2006) doi:10.1007/s00220-006-1539-2 [arXiv:math/0410291 [math.QA]].
- [KKS] M. Kapranov, M. Kontsevich and Y. Soibelman, “Algebra of the infrared and secondary polytopes,” Adv. Math. **300**, 616-671 (2016) doi:10.1016/j.aim.2016.03.028 [arXiv:1408.2673 [math.SG]].
- [KSS] M. Kapranov, Y. Soibelman and L. Soukhanov, “Perverse schobers and the Algebra of the Infrared,” [arXiv:2011.00845 [math.AG]].
- [Kel] B. Keller, “Introduction to A-infinity algebras and modules,” [arXiv:math/9910179 [math.RA]].
- [KM1] A. Z. Khan and G. W. Moore, “Categorical Wall-Crossing in Landau-Ginzburg Models,” [arXiv:2010.11837 [hep-th]].
- [KM2] A. Z. Khan and G. W. Moore, “Algebra of the Infrared and Categorical Wall-Crossing with Twisted Masses,” To appear.
- [K] M. Kontsevich, “Deformation quantization of Poisson manifolds. 1.,” Lett. Math. Phys. **66**, 157-216 (2003) doi:10.1023/B:MATH.0000027508.00421.bf [arXiv:q-alg/9709040 [math.QA]].
- [KoSo1] M. Kontsevich and Y. Soibelman, “Stability structures, motivic Donaldson-Thomas invariants and cluster transformations,” [arXiv:0811.2435 [math.AG]].
- [KoSo2] M. Kontsevich and Y. Soibelman, “Motivic Donaldson-Thomas invariants: Summary of results,” [arXiv:0910.4315 [math.AG]].

- [KoSo3] M. Kontsevich and Y. Soibelman, “Lectures on motivic Donaldson-Thomas invariants and wall-crossing formulas.”
- [KoSo4] M. Kontsevich and Y. Soibelman, “Deformation Theory. I.”
- [KoSo5] M. Kontsevich and Y. Soibelman, “Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants,” *Commun. Num. Theor. Phys.* **5**, 231-352 (2011) doi:10.4310/CNTP.2011.v5.n2.a1 [arXiv:1006.2706 [math.AG]].
- [LY] K. M. Lee and P. Yi, “Dyons in N=4 supersymmetric theories and three pronged strings,” *Phys. Rev. D* **58**, 066005 (1998) doi:10.1103/PhysRevD.58.066005 [arXiv:hep-th/9804174 [hep-th]].
- [MW1] R. Mazzeo and E. Witten, “The Nahm Pole Boundary Condition,” [arXiv:1311.3167 [math.DG]].
- [MW2] R. Mazzeo and E. Witten, “The KW equations and the Nahm pole boundary condition with knots,” *Commun. Anal. Geom.* **28**, no.4, 871-942 (2020) doi:10.4310/CAG.2020.v28.n4.a4 [arXiv:1712.00835 [math.DG]].
- [M] A. Mikhailov, “BPS states and minimal surfaces,” *Nucl. Phys. B* **533**, 243-274 (1998) doi:10.1016/S0550-3213(98)00524-0 [arXiv:hep-th/9708068 [hep-th]].
- [Mik] V. Mikhaylov, “Teichmüller TQFT vs. Chern-Simons theory,” *JHEP* **04**, 085 (2018) doi:10.1007/JHEP04(2018)085 [arXiv:1710.04354 [hep-th]].
- [M1] G. Moore, “2010 PiTP Lectures,” <http://www.physics.rutgers.edu/~gmoore/PiTP-LectureNotes.pdf>, <https://video.ias.edu/pitp-2010>
- [M2] G. Moore, “Framed BPS States In Four And Two Dimensions,” <http://www.physics.rutgers.edu/~gmoore/StringMath2016L.pdf>, <http://www.physics.rutgers.edu/~gmoore/StringMath2016-Aspen.pdf>, <http://www.physics.rutgers.edu/~gmoore/Hamburg2019-Short.pdf>.
- [M3] G. Moore, “Lecture Notes for Felix Klein Lectures,” <https://www.physics.rutgers.edu/~gmoore/FelixKleinLectureNotes.pdf>.
- [N] A. Neitzke, “PCMI Lectures on BPS States and Spectral Networks.”
- [OY] J. Oh and J. Yagi, “Poisson vertex algebras in supersymmetric field theories,” doi:10.1007/s11005-020-01290-0 [arXiv:1908.05791 [hep-th]].
- [Park] D. S. Park, “Topological charges in 2d N=(2,2) theories and massive BPS states,” *Phys. Rev. D* **92**, no.2, 025044 (2015) doi:10.1103/PhysRevD.92.025044 [arXiv:1505.01508 [hep-th]].

- [Pio] B. Pioline, “Four ways across the wall,” J. Phys. Conf. Ser. **346**, 012017 (2012) doi:10.1088/1742-6596/346/1/012017 [arXiv:1103.0261 [hep-th]].
- [Pro] A. Prouté, “Algèbres différentielles fortement homotopiquement associatives”, Thèse d’Etat, Université Paris VII, 1984.
- [Seib] N. Seiberg, “The Power of holomorphy: Exact results in 4-D SUSY field theories,” [arXiv:hep-th/9408013 [hep-th]].
- [SW] N. Seiberg and E. Witten, “Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory,” Nucl. Phys. B **426**, 19-52 (1994) doi:10.1016/0550-3213(94)90124-4 [arXiv:hep-th/9407087 [hep-th]].
- [Seid] P. Seidel “Fukaya categories and Picard-Lefschetz theory,” Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich
- [Sho] B. Shoikhet, “An L_∞ structure on polyvector fields,” Sel. Math. New Ser. (2018) 24: 1691 [arXiv:0805.3363 [math.QA]].
- [Weib] C. Weibel, “An Introduction to Homological Algebra,” Cambridge Studies in Advanced Mathematics.
- [Wit1] E. Witten, “Dynamical Breaking of Supersymmetry,” Nucl. Phys. B **188**, 513 (1981) doi:10.1016/0550-3213(81)90006-7
- [Wit2] E. Witten, “Supersymmetry and Morse theory,” J. Diff. Geom. **17**, no.4, 661-692 (1982)
- [Wit3] E. Witten, “Algebraic geometry associated with matrix models of two-dimensional gravity,”
- [Wit4] E. Witten, “A New Look At The Path Integral Of Quantum Mechanics,” [arXiv:1009.6032 [hep-th]].
- [Wit5] E. Witten, “Analytic Continuation Of Chern-Simons Theory,” AMS/IP Stud. Adv. Math. **50**, 347-446 (2011) [arXiv:1001.2933 [hep-th]].
- [Wit6] E. Witten, “Two Lectures On The Jones Polynomial And Khovanov Homology,” [arXiv:1401.6996 [math.GT]].
- [Wit7] E. Witten, “Two Lectures on Gauge Theory and Khovanov Homology,” [arXiv:1603.03854 [math.GT]].
- [Wit8] E. Witten, “Fivebranes and Knots,” [arXiv:1101.3216 [hep-th]].