On $U_{p,q}(gl(2))$ and a (p,q)-Virasoro algebra

1994 J. Phys. A: Math. Gen. 27 2023

(http://iopscience.iop.org/0305-4470/27/6/025)

J. Phys. A: Math. Gen. 27 (1994) 2023–2035. Printed in the UK

On $U_{p,q}(gl(2))$ and a (p, q)-Virasoro algebra

R Chakrabarti† and R Jagannathan‡§

† Department of Theoretical Physics, University of Madras, Guindy Campus, Madras-600025, India

‡ The Institute of Mathematical Sciences, CIT Campus, Tharamani, Madras-600113, India

Received 13 July 1993

Abstract. The quantum algebra $U_{p,q}(gl(2))$, with two independent deformation parameters (p, q), is studied and, in particular, its universal $\mathcal{R}$-matrix is constructed using Reshetikhin's method. A contraction procedure then leads to the (p, q)-deformed Heisenberg algebra $U_{p,q}(h(1))$ and its universal $\mathcal{R}$-matrix. Using a Sugawara construction employing an infinite number of copies of these Heisenberg modes, a (p, q)-deformed Virasoro algebra is obtained. The closure property of the (p, q)-Virasoro algebra necessitates two parameters (α, β) for the generators $\{L_m(\alpha, \beta)\}$. While the parameter α may be taken as an integer, the parameter β is continuous on a complex path and imparts an integral equation structure to the (p, q)-deformed Virasoro algebra.

1. Introduction

Recently there have been many attempts [1–11] to find quantum deformations of the Virasoro algebra. Several aspects of the deformed Virasoro algebra, like the multiplicative structure, the comultiplication rules for the generators [6, 10], the deformation of the central extension term [3, 5, 7, 10] and the deformed KdV equation [4, 10], have been studied. Many of these formulations [1, 3, 8, 9] are based on using a deformed oscillator degree of freedom [8, 9, 12–16]. Another approach is based on the deformation of the differential operator representation of the centreless Virasoro algebra in a conformal dimension (Δ) dependent way [7, 10], where the corresponding generators satisfy a deformed Jacobi identity. The generators of the centrally extended deformed algebra are required to satisfy the same identity. This leads to a central extension term compatible with the well known result in the undeformed limit. These problems have been studied in the context of a single deformation parameter and also for the more general deformations involving two independent parameters [8–11]. The genuine two-parameter nature of the deformations is exemplified, for instance, in the comultiplication rule for the functional generator in the continuum formulation of the deformed Virasoro algebra. In the $\Delta = 0, 1$ case such a comultiplication rule was found [10] to depend on both parameters.

In the undeformed case the especially fruitful approach of the Sugawara construction leads to a Virasoro algebra with central extension. The Virasoro generators are represented as bilinears in an infinite set of bosonic operators satisfying an infinite Heisenberg algebra $h(\infty)$. These bosonic creation and annihilation operators with a corresponding central element generate a universal enveloping algebra $U(h(\infty))$. The Virasoro generators are elements of $U(h(\infty))$. Paralleling the development in the undeformed case, Chaichian

§ E-mail address: jagan@imsc.ernet.in

0305-4470/94/062023+13$19.50

and Presnajdar [11] have obtained a Sugawara construction for the deformed Virasoro generators using an infinite set of deformed bosonic degrees of freedom realized by taking a suitable contraction limit [17, 18] of the quantum algebra $U_q(su(2))$ [19, 20]. A pair of deformed bosonic creation and annihilation operators along with a central element generate a deformed Heisenberg algebra $U_q(h(1))$. A universal enveloping algebra $U_q(h(\infty))$ generated by an infinite number of these deformed bosonic operators and the central element has a Hopf algebra structure and may be viewed as a deformation of $U(h(\infty))$. The q-deformed Virasoro generators $\{L_m(\alpha)\}$ are then expressed as normal ordered bilinears in these deformed creation and annihilation operators. The extra index (α) may be taken as an integer and is essential for the closure of the algebra. This is assumed to lift a degeneracy present in the undeformed case. The q-Virasoro algebra is then realized by the usual commutators among $\{L_m(\alpha)\}$ and, in an irreducible representation, may be recognized as a centrally extended infinite Lie algebra.

In another development, the construction and the representation theory of quantum algebras with multiple deformation parameters have been studied extensively [8, 21–32]. While considering a two-parameter deformation of the matrix group $GL(2)$, it has been noted [26] that even though the algebra $U_{p,q}(gl(2))$, which is a two-parameter deformation of the universal enveloping algebra corresponding to the Lie algebra $gl(2)$, may be mapped onto the standard $U_Q(gl(2))$ with a single deformation parameter $Q = \sqrt{pq}$, the parameters p and q are genuinely independent as exhibited by the comultiplication rules and the structure of the R-matrix. For the quasitriangular Hopf algebras [33], Reshetikhin has developed [32] a general formalism to introduce multiple deformation parameters. Following [32] we obtain here the universal $\mathcal{R}$-matrix for $U_{p,q}(gl(2))$. By construction, the Yang–Baxter equation is readily satisfied by $\mathcal{R}$. We will later comment on the difference between our result for the universal $\mathcal{R}$-matrix and the one in [30]. An appropriate contraction limit of $U_{p,q}(gl(2))$ *à la* Celeghini *et al* [17, 18] then yields the quantum Heisenberg algebra $U_{p,q}(h(1))$, which is spanned by a pair of (p, q)-deformed creation and annihilation operators together with two central elements. Using the procedure in [18], we calculate the universal $\mathcal{R}$-matrix for $U_{p,q}(h(1))$. By the standard Yang–Baxterization method [34], we obtain the corresponding spectral parameter dependent $\mathcal{R}$-matrix which satisfies the spectral parameter dependent Yang–Baxter equation.

For the purpose of obtaining a (p, q)-deformed Virasoro algebra using a Sugawara construction we mimic the procedure in [11] to introduce an infinite number of (p, q)-deformed creation and annihilation operators. These modes, along with the two independent central elements, generate the universal enveloping algebra $U_{p,q}(h(\infty))$ which has a Hopf algebra structure. Our ansatz for the (p, q)-Virasoro generators $\{L_m(\alpha, \beta)\}$ involves normal-ordered bilinears in these (p, q)-deformed Heisenberg operators and the additional parameters (α, β) are necessitated by the closure property. Following [11], this may be viewed as lifting the two-fold degeneracy present in the undeformed case. The parameter α may be considered an integer as in the single parameter case [11]. The parameter β, in contrast to α, is complex in general. The closure property imparts an integral equation structure to the algebra in the parameter space of β. By inspection, it follows that the generators $\{L_m(\alpha, \beta)\}$ and the resultant algebra ((4.10)–(4.13) below) cannot be mapped to corresponding structures pertaining to a single effective deformation parameter Q. This is a consequence of the multimode expansion in the Sugawara scheme and, we believe, that the multiparameter deformations are of importance in models involving multiple modes.

The plan of this paper is as follows. In section 2, we describe the algebra $U_{p,q}(gl(2))$ and subsequently, in section 3, obtain the deformed Heisenberg algebra $U_{p,q}(h(1))$ for the appropriate contraction limit. In section 4, we assume a Sugawara ansatz for the generators $\{L_m(\alpha,\beta)\}$ and obtain the resulting algebra. We present our conclusions in section 5.

2. Aspects of $U_{p,q}(gl(2))$

First, let us recall some well known results [8, 21–32] for the (p,q)-deformed universal enveloping algebra $U_{p,q}(gl(2))$.

The R-matrix

$$R_{p,q} = Q^{1/2}\begin{pmatrix} Q^{-1} & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & Q^{-1}-Q & \lambda & 0 \\ 0 & 0 & 0 & Q^{-1} \end{pmatrix} \qquad Q=(pq)^{1/2} \qquad \lambda=(p/q)^{1/2} \tag{2.1}$$

satisfies the defining relation of the quantum inverse scattering method [35]

$$R_{p,q}(T\otimes \mathbb{1})(\mathbb{1}\otimes T) = (\mathbb{1}\otimes T)(T\otimes \mathbb{1})R_{p,q} \tag{2.2}$$

where the quantum matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.3}$$

has non-commuting elements exhibiting braiding properties

$$\begin{aligned} ab &= p^{-1}ba \qquad & cd &= p^{-1}dc \\ ac &= q^{-1}ca \qquad & bd &= q^{-1}db \\ qbc &= pcb \qquad & ad-da &= (p^{-1}-q)bc \end{aligned} \tag{2.4}$$

that may be understood following Manin's hyperplane approach [36] to the general construction of the matrix quantum groups. It may be noted that a conjugate R-matrix

$$\tilde{R}_{p,q} = (R^{(+)}_{p,q})^{-1} = Q^{-1/2}\begin{pmatrix} Q & 0 & 0 & 0 \\ 0 & \lambda^{-1} & Q-Q^{-1} & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & Q \end{pmatrix} \tag{2.5}$$

also fits (2.2) with elements of T obeying (2.4). The matrix $R^{(+)}_{p,q}$ is defined by

$$R^{(+)}_{p,q} = PR_{p,q}P = Q^{1/2}\begin{pmatrix} Q^{-1} & 0 & 0 & 0 \\ 0 & \lambda & Q^{-1}-Q & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & Q^{-1} \end{pmatrix} \tag{2.6}$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

is the permutation matrix. The elements of T obeying (2.4) generate the matrix quantum group $GL_{p,q}(2)$ [21–31].

Using the FRT approach [35] the commutation relations for the algebra $U_{p,q}(gl(2))$, dual to the Hopf algebra generated by the non-commuting elements of the matrix T, are obtained [8, 21, 25, 27, 28, 30] from the relations for the generators

$$R_{p,q}^{(+)}(L^{(\epsilon_1)} \otimes \mathbb{1})(\mathbb{1} \otimes L^{(\epsilon_2)}) = (\mathbb{1} \otimes L^{(\epsilon_2)})(L^{(\epsilon_1)} \otimes \mathbb{1})R_{p,q}^{(+)} \tag{2.8}$$

where $(\epsilon_1, \epsilon_2) = (+,+), (-,-), (+,-)$ and

$$\begin{aligned} L^{(+)} &= \begin{pmatrix} Q^{-J_0}\lambda^{-(J_0-Z)} & Q^{1/2}(Q^{-1}-Q)J_- \\ 0 & Q^{J_0}\lambda^{-(J_0+Z)} \end{pmatrix} \\ L^{(-)} &= \begin{pmatrix} Q^{J_0}\lambda^{-(J_0-Z)} & 0 \\ Q^{-1/2}(Q-Q^{-1})J_+ & Q^{-J_0}\lambda^{-(J_0+Z)} \end{pmatrix}. \end{aligned} \tag{2.9}$$

The corresponding commutation relations read

$$\begin{aligned} &[J_0, J_\pm] = \pm J_\pm \qquad && J_+J_- - pq^{-1}J_-J_+ = [2J_0]_{p,q} \\ &[Z, J_\pm] = 0 && [Z, J_0] = 0 \end{aligned} \tag{2.10}$$

where

$$[x]_{p,q} = \frac{q^x - p^{-x}}{q - p^{-1}}. \tag{2.11}$$

Let us choose the coproduct rules for the generators as

$$\begin{aligned} \Delta(J_\pm) &= J_\pm \otimes Q^{J_0}\lambda^{-(J_0 \mp Z)} + Q^{-J_0}\lambda^{-(J_0 \pm Z)} \otimes J_\pm \\ \Delta(J_0) &= J_0 \otimes \mathbb{1} + \mathbb{1} \otimes J_0 \\ \Delta(Z) &= Z \otimes \mathbb{1} + \mathbb{1} \otimes Z \end{aligned} \tag{2.12}$$

which follow from the relation

$$\Delta(L^{(\pm)}) = L^{(\pm)} \dot{\otimes} L^{(\pm)}. \tag{2.13}$$

with $\dot{\otimes}$ denoting the tensor product combined with the usual matrix multiplication. The centre Z in the algebra (2.10) plays an important role in imparting the two-parametric nature to the deformations, as is evident from the fact that if we equate it to zero then both the algebraic and the coalgebraic structures may be simultaneously transformed to depend on the single parameter Q. To see this, we observe the following. The map

$$\tilde{J}_\pm = J_\pm \lambda^{J_0 \pm \frac{1}{2}} \qquad \tilde{J}_0 = J_0 \qquad \tilde{Z} = Z \tag{2.14}$$

reduces the algebra (2.10) to the standard form

$$\begin{aligned} &[\tilde{J}_0, \tilde{J}_\pm] = \pm\tilde{J}_\pm \qquad && [\tilde{J}_+, \tilde{J}_-] = [2\tilde{J}_0]_Q \\ &[\tilde{Z}, \tilde{J}_\pm] = 0 && [\tilde{Z}, \tilde{J}_0] = 0 \end{aligned} \tag{2.15}$$

with

$$[x]_Q = \frac{Q^x - Q^{-x}}{Q - Q^{-1}}. \tag{2.16}$$

For the choice $\tilde{Z} \neq 0$, the induced coproducts for the generators (2.14), however, depend on both the deformation parameters

$$\begin{aligned}
\Delta(\tilde{J}_\pm) &= \tilde{J}_\pm \otimes Q^{\tilde{J}_0}\lambda^{\pm\tilde{Z}} + Q^{-\tilde{J}_0}\lambda^{\mp\tilde{Z}} \otimes \tilde{J}_\pm \\
\Delta(\tilde{J}_0) &= \tilde{J}_0 \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{J}_0 \\
\Delta(\tilde{Z}) &= \tilde{Z} \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{Z}.
\end{aligned} \tag{2.17}$$

In the limit $\lambda = 1$ we recover the standard result for $U_Q(gl(2))$.

Let us now consider the universal $\mathcal{R}$-matrix for the algebra $U_{p,q}(gl(2))$ (or $U_{Q,\lambda}(gl(2))$). For a quasitriangular Hopf algebra $\mathcal{U}$ the universal $\mathcal{R}$-matrix, $\in \mathcal{U}\otimes\mathcal{U}$, satisfies the relations

$$\begin{aligned}
&(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \qquad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \\
&\tau \cdot \Delta(X) = \mathcal{R}\Delta(X)\mathcal{R}^{-1} \\
&\tau(X \otimes Y) = Y \otimes X \qquad \forall X, Y \in \mathcal{U}
\end{aligned} \tag{2.18}$$

where the subscripts in $\mathcal{R}_{ij}$ indicate the embedding of $\mathcal{R}$ in $\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}$ [33]. For $U_Q(gl(2))$ the explicit expression for the universal $\mathcal{R}$-matrix [37] is

$$\mathcal{R}_Q = Q^{2(\tilde{J}_0\otimes\tilde{J}_0)} \sum_{n=0}^{\infty} \frac{(1 - Q^{-2})^n}{[n]_Q!} Q^{n(n-1)/2} \left(Q^{\tilde{J}_0}\tilde{J}_+ \otimes Q^{-\tilde{J}_0}\tilde{J}_-\right)^n \tag{2.19}$$

where $[n]_Q! = [n]_Q[n-1]_Q \cdots [2]_Q[1]_Q$. Employing the Reshetikhin procedure [32] one can obtain the universal $\mathcal{R}$-matrix, say $\mathcal{R}_{Q,\lambda}$, for the algebra $U_{p,q}(gl(2))$. To this end, we note that the coproduct relations (2.17) for the generators of the algebra $U_{p,q}(gl(2))$ and the corresponding relations for $U_Q(gl(2))$, obtained in the limit $\lambda = 1$, are related by a similarity transformation

$$\Delta(X) = F(\Delta_{(\lambda=1)}(X))F^{-1} \qquad \forall X \in (\tilde{J}_\pm, \tilde{J}_0, \tilde{Z}) \tag{2.20}$$

where

$$F = \lambda^{(\tilde{J}_0\otimes\tilde{Z} - \tilde{Z}\otimes\tilde{J}_0)}. \tag{2.21}$$

Then, following Reshetikhin [32], one can write down

$$\mathcal{R}_{Q,\lambda} = F^{-1}\mathcal{R}_Q F^{-1} \tag{2.22}$$

which, by construction, satisfies the required relations (2.18). Explicitly, the universal $\mathcal{R}$-matrix for $U_{p,q}(gl(2))$ reads

$$\begin{aligned}
\mathcal{R}_{Q,\lambda} = {} & Q^{2(\tilde{J}_0\otimes\tilde{J}_0)}\lambda^{2(\tilde{Z}\otimes\tilde{J}_0 - \tilde{J}_0\otimes\tilde{Z})} \\
& \times \sum_{n=0}^{\infty} \frac{(1 - Q^{-2})^n}{[n]_Q!} Q^{n(n-1)/2} \left(Q^{\tilde{J}_0}\lambda^{\tilde{Z}}\tilde{J}_+ \otimes Q^{-\tilde{J}_0}\lambda^{\tilde{Z}}\tilde{J}_-\right)^n.
\end{aligned} \tag{2.23}$$

For the coproduct of two identical fundamental representations ($j = \frac{1}{2}$), with the choice $Z = \frac{1}{2}$ for both, the $\mathcal{R}_{Q,\lambda}$ in (2.23) reduces to the matrix

$$\mathcal{R}_{Q,\lambda}^{(j=1/2,Z=1/2)} = Q^{-1/2}\begin{pmatrix} Q & 0 & 0 & 0 \\ 0 & \lambda^{-1} & Q-Q^{-1} & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & Q \end{pmatrix} \equiv \tilde{R}_{p,q}. \tag{2.24}$$

The identity of the matrix $\mathcal{R}_{Q,\lambda}$ in (2.24) with $\tilde{R}_{p,q}$, embodying the endomorphisms of the matrix quantum group $GL_{p,q}(2)$ acting on the underlying non-commutative Manin plane, expresses an aspect of the duality of the algebra $U_{p,q}(gl(2))$ to the Hopf algebra generated by the non-commuting elements of the quantum matrix $T \in GL_{p,q}(2)$. The duality condition is expressed by the pairing

$$\langle L^{(\pm)} \otimes \mathbb{1}, \mathbb{1} \otimes T \rangle = R_{p,q}^{(\pm)} \tag{2.25}$$

where $R_{p,q}^{(-)} = R_{p,q}^{-1}$.

In view of the above discussion we may now look at the results in [30] wherein also the construction of the universal $\mathcal{R}$-matrix for $U_{p,q}(gl(2))$ has been discussed. The algebra considered in [30] is not a genuine two-parameter deformation of $U(gl(2))$ as, after a simple transformation of its generators, both the algebraic and coalgebraic structure may be simultaneously made to depend on a single quantization parameter. Consequently, by a suitable choice of representation, the expression for the universal $\mathcal{R}$-matrix in [30] can also be made to depend on a single deformation parameter.

Regarding the other aspects of the Hopf algebraic structure of $U_{p,q}(gl(2))$, besides the comultiplication, namely, the antipode and counit operations, we note the following. Using Reshetikhin's theory [32] one can see that in the deformation $U_Q(gl(2)) \to U_{Q,\lambda}(gl(2))$, effected by changing the coalgebraic structure through the transformation (2.20), (2.21), the antipode and counits are not altered. We shall not discuss them any further.

For later use, we consider the representations of the algebra (2.10) with the generators $(J_0, J_\pm)$ obeying the Hermiticity properties $J_0^\dagger = J_0$ and $J_+^\dagger = J_-$; or, in other words, we are dealing with $U_{p,q}(su(2) \oplus u(1))$. The requirement for the coproduct rules (2.17) to preserve the above Hermiticity properties selects the parameters (Q, λ) to be real and the central element Z to take purely imaginary values. Then, for generic real values of the deformation parameters (p, q), one obtains [8] a $(2j+1)$-dimensional irreducible representation for an integral or half-integral j

$$\begin{aligned} &J_0|jmz\rangle = m|jmz\rangle \qquad m = j, j-1, \ldots, -(j-1), -j \\ &J_\pm|jmz\rangle = \left\{(q^{-1}p)^{j-m-(1\pm1)/2}[j\mp m]_{p,q}[j\pm m+1]_{p,q}\right\}|j(m\pm1)z\rangle \\ &Z|jmz\rangle = z|jmz\rangle \end{aligned} \tag{2.26}$$

where z is an arbitrary imaginary constant.

3. The algebra $U_{p,q}(h(1))$ as a contraction limit of $U_{p,q}(gl(2))$

The well known contraction technique for obtaining the non-semisimple Lie algebras and their representation properties as the limiting behaviour of their semisimple analogues, has been generalized by Celeghini *et al* [17, 18] in the context of quantum algebras. In particular, they have obtained a deformed Heisenberg algebra $U_q(h(1))$ containing a single central element as a contraction limit of the algebra $U_q(su(2))$. Following their procedure we study the contraction limit of the quantum algebra $U_{p,q}(gl(2))$ and extract the two-parameter deformation of the Heisenberg algebra, namely, $U_{p,q}(h(1))$. We scale the generators and the quantization parameters as

$$A_\pm = \epsilon^{1/2} J_\pm \qquad H = 2\epsilon J_0 \qquad \zeta = 2\epsilon Z \tag{3.1}$$

$$\omega_1 = \epsilon^{-1} \ln q \qquad \omega_2 = \epsilon^{-1} \ln p \tag{3.2}$$

and define

$$\Omega = \tfrac{1}{2}(\omega_1 + \omega_2) \qquad \nu = \tfrac{1}{2}(\omega_1 - \omega_2). \tag{3.3}$$

The commutation relations and the coproduct rules for the algebra $U_{p,q}(h(1))$ are obtained by studying the $\epsilon \to 0$ limit of the corresponding structures of $U_{p,q}(gl(2))$ in (2.10) and (2.12) respectively. Thus the structure of $U_{p,q}(h(1))$ is defined by

$$\begin{aligned} &[H, A_\pm] = 0 \qquad [A_+, A_-] = \Omega^{-1} \exp(\nu H) \sinh(\Omega H) \\ &[\zeta, A_\pm] = 0 \qquad [\zeta, H] = 0 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\Delta(A_\pm) = A_\pm \otimes \exp\left\{\tfrac{1}{2}((\Omega+\nu)H \mp \nu\zeta)\right\} + \exp\left\{-\tfrac{1}{2}((\Omega-\nu)H \mp \nu\zeta)\right\} \otimes A_\pm \\ &\Delta(H) = H \otimes \mathbb{1} + \mathbb{1} \otimes H \\ &\Delta(\zeta) = \zeta \otimes \mathbb{1} + \mathbb{1} \otimes \zeta. \end{aligned} \tag{3.5}$$

The algebra (3.4) has two central elements (H, ζ). A map

$$\tilde{A}_\pm = A_\pm \exp(-\tfrac{1}{2}\nu H) \qquad \tilde{H} = H \qquad \tilde{\zeta} = \zeta \tag{3.6}$$

reduces the commutation relations (3.4) and the coproduct rules (3.5) to the following forms, respectively

$$[\tilde{A}_+, \tilde{A}_-] = \Omega^{-1} \sinh(\Omega \tilde{H}) \qquad [\tilde{H}, \cdot] = 0 \qquad [\tilde{\zeta}, \cdot] = 0 \tag{3.7}$$

and

$$\Delta(\tilde{A}_\pm) = \tilde{A}_\pm \otimes \exp\left\{\tfrac{1}{2}(\Omega\tilde{H} \mp \nu\tilde{\zeta})\right\} + \exp\left\{-\tfrac{1}{2}(\Omega\tilde{H} \mp \nu\tilde{\zeta})\right\} \otimes \tilde{A}_\pm. \tag{3.8}$$

It is seen that the coproduct rules (3.8) depend on both the parameters (Ω, ν) independently, even though the commutation relations (3.7) depend only on Ω.

Paralleling the argument in [18], the universal $\mathcal{R}$-matrix for the algebra $U_{p,q}(h(1))$ is obtained at the contraction limit of the universal $\mathcal{R}$-matrix (2.23) for the algebra $U_{p,q}(gl(2))$. The final result is

$$\mathcal{R} = \exp\left\{-\Omega(\tilde{H} \otimes \tilde{N} + \tilde{N} \otimes \tilde{H}) - \nu(\tilde{N} \otimes \tilde{\zeta} - \tilde{\zeta} \otimes \tilde{N})\right\} \exp(\tilde{B}_+ \otimes \tilde{B}_-) \tag{3.9}$$

where

$$\tilde{B}_\pm = \sqrt{2\Omega}\exp\big\{-\tfrac{1}{2}(\nu\tilde{\zeta} \mp \Omega\tilde{H})\big\}\tilde{A}_\pm \tag{3.10}$$

and the operator $\tilde{N}$ is defined through an ϵ-expansion of $\tilde{J}_0$ such that

$$\tilde{J}_0 = \frac{1}{2\epsilon}\tilde{H} - \tilde{N} \tag{3.11}$$

and

$$[\tilde{N}, \tilde{A}_\pm] = \mp\tilde{A}_\pm. \tag{3.12}$$

Using (3.7), an explicit expression for $\tilde{N}$ follows

$$\tilde{N} = \Omega\tilde{A}_-\tilde{A}_+\big(\sinh(\Omega\tilde{H})\big)^{-1}. \tag{3.13}$$

As explained in [18] in the context of a single deformation parameter, the algebra $U_{p,q}(h(1))$ is not a quasitriangular Hopf algebra when $\tilde{N}$, as in (3.13), is considered a composite operator. The quasitriangular nature is, however, restored when $\tilde{N}$ is thought of as a primitive generator with a coproduct rule

$$\Delta(\tilde{N}) = \tilde{N} \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{N} \tag{3.14}$$

which is distinct from the induced coproduct suggested by (3.5) and (3.13). The latter point of view explains the existence of a Reshetikhin-type of transformation [32] relating the coproduct rules (3.8) and the universal $\mathcal{R}$-matrix (3.9) to the corresponding quantities in the $\nu = 0$ limit

$$\Delta(\tilde{A}_\pm) = \tilde{F}(\Delta(\tilde{A}_{\pm,(\nu=0)}))\tilde{F}^{-1} \tag{3.15}$$

$$\mathcal{R} = \tilde{F}^{-1}\mathcal{R}_{(\nu=0)}\tilde{F}^{-1} \tag{3.16}$$

where

$$\tilde{F} = \exp\big\{\tfrac{1}{2}\nu(\tilde{N} \otimes \tilde{\zeta} - \tilde{\zeta} \otimes \tilde{N})\big\}. \tag{3.17}$$

The transformation operator $\tilde{F}$ follows from the parent operator F in (2.21) for the $U_{p,q}(gl(2))$ algebra at the contraction limit. Following [18], a spectral parameter dependent $\mathcal{R}$-matrix may be obtained [34] from the universal $\mathcal{R}$-matrix (3.9). Through the action of the operator T_x on the generators

$$T_x A_\pm = x^{\pm 1} A_\pm \qquad T_x H = H \qquad T_x \zeta = \zeta \tag{3.18}$$

we define

$$\begin{aligned}\mathcal{R}(x) &= (T_x \otimes \mathbb{1})\mathcal{R} \\ &= \exp\big\{-\Omega(\tilde{H} \otimes \tilde{N} + \tilde{N} \otimes \tilde{H}) - \nu(\tilde{N} \otimes \tilde{\zeta} - \tilde{\zeta} \otimes \tilde{N})\big\}\exp(x\tilde{B}_+ \otimes \tilde{B}_-).\end{aligned} \tag{3.19}$$

A direct calculation then proves that the matrix $\mathcal{R}(x)$ satisfies the Yang–Baxter equation

$$\mathcal{R}_{12}(x)\mathcal{R}_{13}(xy)\mathcal{R}_{23}(y) = \mathcal{R}_{23}(y)\mathcal{R}_{13}(xy)\mathcal{R}_{12}(x) \tag{3.20}$$

when, explicitly

$$\begin{aligned}\mathcal{R}_{12}(x) = \exp\{&-\Omega(\tilde{H}\otimes\tilde{N}\otimes\mathbb{1}+\tilde{N}\otimes\tilde{H}\otimes\mathbb{1})\\ &-\nu(\tilde{N}\otimes\tilde{\zeta}\otimes\mathbb{1}-\tilde{\zeta}\otimes\tilde{N}\otimes\mathbb{1})\}\exp(x\tilde{B}_+\otimes\tilde{B}_-\otimes\mathbb{1})\end{aligned} \tag{3.21}$$

$$\begin{aligned}\mathcal{R}_{13}(xy) = \exp\{&-\Omega(\tilde{H}\otimes\mathbb{1}\otimes\tilde{N}+\tilde{N}\otimes\mathbb{1}\otimes\tilde{H})\\ &-\nu(\tilde{N}\otimes\mathbb{1}\otimes\tilde{\zeta}-\tilde{\zeta}\otimes\mathbb{1}\otimes\tilde{N})\}\exp(xy\tilde{B}_+\otimes\mathbb{1}\otimes\tilde{B}_-)\end{aligned} \tag{3.22}$$

$$\begin{aligned}\mathcal{R}_{23}(y) = \exp\{&-\Omega(\mathbb{1}\otimes\tilde{H}\otimes\tilde{N}+\mathbb{1}\otimes\tilde{N}\otimes\tilde{H})\\ &-\nu(\mathbb{1}\otimes\tilde{N}\otimes\tilde{\zeta}-\mathbb{1}\otimes\tilde{\zeta}\otimes\tilde{N})\}\exp(y\mathbb{1}\otimes\tilde{B}_+\otimes\tilde{B}_-).\end{aligned} \tag{3.23}$$

The contraction procedure may be used to obtain the representation properties of $U_{p,q}(h(1))$ from the corresponding relations (2.25) for $U_{p,q}(gl(2))$. For this purpose, we define

$$h = 2\epsilon j \qquad n = j - m \tag{3.24}$$

and let $\epsilon \to 0$, $j \to \infty$ to obtain for an infinite dimensional irreducible representation, labelled by constants h and ζ_0

$$\begin{aligned}&A_\pm|hn\zeta_0\rangle = \left\{(n+\tfrac{1}{2}(1\mp 1))\Omega^{-1}\exp(\nu h)\sinh(\Omega h)\right\}^{1/2}|h(n\mp 1)\zeta_0\rangle\\ &H|hn\zeta_0\rangle = h|hn\zeta_0\rangle \qquad \zeta|hn\zeta_0\rangle = \zeta_0|hn\zeta_0\rangle\\ &\qquad n = 0, 1, 2, \ldots\end{aligned} \tag{3.25}$$

where the vacuum is given by

$$A_+|h0\zeta_0\rangle = 0. \tag{3.26}$$

4. A (p,q)-Virasoro algebra

To obtain a (p,q)-deformation of the Virasoro algebra through a Sugawara construction we closely follow the well known route for the undeformed case. For this purpose, we introduce an infinite number of deformed creation and annihilation operators, $\{A_k \mid k = \pm 1, \pm 2, \ldots\}$, for a fixed ζ_0. The commutation relations (3.4) and the coproduct rules for the single mode example translate for the multimode case as follows

$$[A_k, A_l] = \omega_{kl} \qquad [H, A_k] = 0 \qquad [\zeta, A_k] = 0 \tag{4.1}$$

where

$$\omega_{kl} = \Omega^{-1}\exp(|k|\nu H)\sinh(k\Omega H)\delta_{k+l,0} \tag{4.2}$$

and

$$\Delta(A_k) = A_k \otimes \exp\left\{\tfrac{1}{2}|k|(\Omega H + \nu(H - \mathrm{sgn}(k)\zeta))\right\} + \exp\left\{-\tfrac{1}{2}|k|(\Omega H - \nu(H + \mathrm{sgn}(k)\zeta))\right\} \otimes A_k. \tag{4.3}$$

The algebra (4.1) is homomorphic under the map (4.3). The modes $\{A_k\}$ and the central elements (H, ζ) generate the universal enveloping algebra $U_{p,q}(h(\infty))$. The normal-ordering rule for the bilinear products is

$$: A_k A_l : = A_k A_l - \theta(k)\omega_{kl}. \tag{4.4}$$

For the purpose of later computation we express ω_{kl} in an alternate form. To this end, we note that for a unitary irreducible representation, H is a real constant. Henceforth we consider these representations and assume $\nu H > 0$ for notational simplicity. The case $\nu H < 0$ may be treated parallely. Using an integral representation for the step function

$$\theta(x) = \lim_{\epsilon\to 0}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}xs}}{s - \mathrm{i}\epsilon}\,\mathrm{d}s \tag{4.5}$$

and after suitable rescaling, we have

$$\exp(|k|\nu H) = \lim_{\epsilon\to 0}\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\cosh((1+\mathrm{i}s)k\nu H)}{s - \mathrm{i}\epsilon}\,\mathrm{d}s \tag{4.6}$$

which leads to the expression

$$\omega_{kl} = \lim_{\epsilon\to 0}\frac{\sinh(k\Omega H)}{\pi\Omega\mathrm{i}}\int_{-\infty}^{\infty}\frac{\cosh((1+\mathrm{i}s)k\nu H)}{s - \mathrm{i}\epsilon}\delta_{k+l,0}\,\mathrm{d}s \tag{4.7}$$

in the commutator (4.2).

We define the (p, q)-Virasoro generators as

$$L_m(\alpha, \beta) = \tfrac{1}{2}\sum_{k,l}\exp\left\{\tfrac{1}{2}(k - l)(\Omega\alpha + \nu\beta)H\right\} : A_k A_l : \delta_{k+l,m}. \tag{4.8}$$

To make the meaning of the additional indices clearer, we define

$$\tilde{q} = \exp(\omega_1) \qquad \tilde{p} = \exp(\omega_2) \tag{4.9}$$

and write (4.8) in a more transparent form

$$L_m(\alpha, \beta) = \tfrac{1}{2}\sum_{k,l}(\tilde{p}\tilde{q})^{(k-l)\alpha H/4}(\tilde{p}/\tilde{q})^{-(k-l)\beta H/4} : A_k A_l : \delta_{k+l,m}. \tag{4.8$'$}$$

Notice that the deformed Virasoro generators in [11] may be constructed by taking a linear combination of the generators (4.8′) in the appropriate single deformation parameter limit $\tilde{p} = \tilde{q}$. Following [11], we note that all elements in $U_{p,q}(h(\infty))$ of the type $(\tilde{p}\tilde{q})^{\alpha H/4}$ and $(\tilde{p}/\tilde{q})^{\beta H/4}$, together with their integer powers, become degenerate with the unit element in the undeformed limit. The deformation may then be thought of as an elimination of this two-fold degeneracy. The (p,q)-Virasoro algebra may now be computed in a straightforward way, and has the structure

$$[L_m(\alpha,\beta), L_{m'}(\alpha',\beta')] = \frac{1}{8\pi\Omega\mathrm{i}}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{s}\mathcal{L}(s) + \frac{1}{8\Omega}\mathcal{L}(0) + \frac{1}{16\Omega^2}C\delta_{m+m',0} \tag{4.10}$$

where the symmetrized integrand in the non-central part reads

$$\begin{aligned}
\mathcal{L}(s) = 2\sinh&\left\{\tfrac{1}{2}\big((m'\alpha - m\alpha' + m - m')\Omega\right.\\
&\left.+ (m'\beta - m\beta' + (1+\mathrm{i}s)(m-m'))\nu\big)H\right\}\\
&\times L_{m+m'}(\alpha+\alpha'-1, \beta+\beta'-1-\mathrm{i}s)\\
&+ 2\sinh\left\{\tfrac{1}{2}\big((m'\alpha - m\alpha' + m - m')\Omega\right.\\
&\left.+ (m'\beta - m\beta' - (1+\mathrm{i}s)(m-m'))\nu\big)H\right\}\\
&\times L_{m+m'}(\alpha+\alpha'-1, \beta+\beta'+1+\mathrm{i}s)\\
&- 2\sinh\left\{\tfrac{1}{2}\big((m'\alpha - m\alpha' - m + m')\Omega\right.\\
&\left.+ (m'\beta - m\beta' + (1+\mathrm{i}s)(m-m'))\nu\big)H\right\}\\
&\times L_{m+m'}(\alpha+\alpha'+1, \beta+\beta'-1-\mathrm{i}s)\\
&- 2\sinh\left\{\tfrac{1}{2}\big((m'\alpha - m\alpha' - m + m')\Omega\right.\\
&\left.+ (m'\beta - m\beta' - (1+\mathrm{i}s)(m-m'))\nu\big)H\right\}\\
&\times L_{m+m'}(\alpha+\alpha'+1, \beta+\beta'+1+\mathrm{i}s)\\
&+ \sinh\left\{\tfrac{1}{2}\big((m'\alpha + m\alpha' + m - m')\Omega\right.\\
&\left.+ (m'\beta + m\beta' + (1+\mathrm{i}s)(m-m'))\nu\big)H\right\}\\
&\times (L_{m+m'}(-\alpha+\alpha'+1, -\beta+\beta'+1+\mathrm{i}s)\\
&+ L_{m+m'}(\alpha-\alpha'-1, \beta-\beta'-1-\mathrm{i}s))\\
&+ \sinh\left\{\tfrac{1}{2}\big((m'\alpha + m\alpha' + m - m')\Omega\right.\\
&\left.+ (m'\beta + m\beta' - (1+\mathrm{i}s)(m-m'))\nu\big)H\right\}\\
&\times (L_{m+m'}(-\alpha+\alpha'+1, -\beta+\beta'-1-\mathrm{i}s)\\
&+ L_{m+m'}(\alpha-\alpha'-1, \beta-\beta'+1+\mathrm{i}s))\\
&- \sinh\left\{\tfrac{1}{2}\big((m'\alpha + m\alpha' - m + m')\Omega\right.\\
&\left.+ (m'\beta + m\beta' + (1+\mathrm{i}s)(m-m'))\nu\big)H\right\}\\
&\times (L_{m+m'}(-\alpha+\alpha'-1, -\beta+\beta'+1+\mathrm{i}s)\\
&+ L_{m+m'}(\alpha-\alpha'+1, \beta-\beta'-1-\mathrm{i}s))\\
&- \sinh\left\{\tfrac{1}{2}\big((m'\alpha + m\alpha' - m + m')\Omega\right.\\
&\left.+ (m'\beta + m\beta' - (1+\mathrm{i}s)(m-m'))\nu\big)H\right\}\\
&\times \big(L_{m+m'}(-\alpha+\alpha'-1, -\beta+\beta'-1-\mathrm{i}s)\\
&+ L_{m+m'}(\alpha-\alpha'+1, \beta-\beta'+1+\mathrm{i}s)\big)
\end{aligned} \tag{4.11}$$

and the central element is given by

$$C = C(\alpha, \alpha'; \beta, \beta') + C(-\alpha, \alpha'; -\beta, \beta') \tag{4.12}$$

with

$$\begin{aligned} C(\alpha, \alpha'; \beta, \beta') = \operatorname{sgn}(m) \exp(|m|\nu H)\Bigg\{ & 2\cosh(m\Omega H) \\ & \times \frac{\sinh\left(\frac{1}{2}(|m|-1)(\Omega(\alpha+\alpha') + \nu(\beta+\beta'))H\right)}{\sinh\left(\frac{1}{2}(\Omega(\alpha+\alpha') + \nu(\beta+\beta'))H\right)} \\ & - \frac{\sinh\left(\frac{1}{2}(|m|-1)(\Omega(\alpha+\alpha'+2) + \nu(\beta+\beta'))H\right)}{\sinh\left(\frac{1}{2}(\Omega(\alpha+\alpha'+2) + \nu(\beta+\beta'))H\right)} \\ & - \frac{\sinh\left(\frac{1}{2}(|m|-1)(\Omega(\alpha+\alpha'-2) + \nu(\beta+\beta'))H\right)}{\sinh\left(\frac{1}{2}(\Omega(\alpha+\alpha'-2) + \nu(\beta+\beta'))H\right)} \Bigg\}. \end{aligned} \tag{4.13}$$

We do not carry out the integral in the right-hand side of (4.10) as it would spoil the operator structure required by our ansatz (4.8) for the reconstruction of the (p, q)-Virasoro generators. From (4.10) and (4.11), it is evident that the parameters α and β are necessary for the closure of the (p, q)-Virasoro algebra. The parameter α, much as in the q-deformed case [11], may be taken to be an integer. The parameter β is complex and imparts an integral equation structure to (4.10). In the single deformation parameter limit $\tilde{p} = \tilde{q}$ (or $\nu = 0$) the principal value integral on the right-hand side of (4.10) vanishes, and the previously known results [11] may be obtained from (4.10)–(4.13). In the undeformed limit ($\Omega = 0$, $\nu = 0$) the standard results are reproduced. The ordering of the limits in the central charge term (4.13) is important: we have to first evaluate (4.13) at $\nu = 0$ and then approach the limit $\Omega = 0$. A non-cocommutative induced coproduct rule for the (p, q)-Virasoro generators (4.8) may be obtained following (4.3).

As we mentioned earlier in the context of the $U_{p,q}(h(1))$ algebra the individual operators $\{A_k\}$, after a suitable map of the type (3.6), follow a commutation relation depending on a single deformation parameter Ω. The multimode expansion (4.8) for the (p, q)-Virasoro generators $\{L_m(\alpha, \beta)\}$, however, demands that no such transformation can be found relating them to the corresponding set $\{L_m^{(\nu=0)}(\alpha)\}$ obtained in the limit $\nu = 0$. Thus the (p, q)-Virasoro algebra (4.10) is essentially dependent on two deformation parameters and cannot, by a transformation, be reduced to the algebra at the $\nu = 0$ limit. This is irrespective of the coalgebraic structure and is to be contrasted with [10], where the essential distinction in the single- and two-parameter deformations of the Virasoro algebra lies in the comultiplication rules of the corresponding functional generators.

5. Conclusion

Following the Reshetikhin procedure we have constructed the universal $\mathcal{R}$-matrix for the algebra $U_{p,q}(gl(2))$. Using a contraction limit of the algebra $U_{p,q}(gl(2))$ we have obtained a (p, q)-deformed universal enveloping algebra of the Heisenberg algebra, $U_{p,q}(h(1))$, generated by a conjugate pair of deformed creation and annihilation operators along with two central elements. The universal $\mathcal{R}$-matrix has also been constructed for $U_{p,q}(h(1))$. Using an infinite number of copies of these deformed Heisenberg modes, we have obtained a (p,q)-Virasoro algebra by employing a Sugawara construction procedure. The closure property of the algebra demands the introduction of two parameters (α, β) one of which (β) has to be taken as continuous on a complex path. This enforces an integral equation structure on the (p,q)-Virasoro algebra. As the Sugawara construction employs the multiple modes of (p,q)-deformed bosonic operators one cannot find a transformation relating the (p,q)-Virasoro generators and the corresponding algebra to their counterparts in the single deformation parameter $(\nu = 0)$ case.

Note added. After submission of the paper we have learnt that a coloured version of the R-matrix in (2.24) has been obtained using a different derivation of the universal $\mathcal{R}$-matrix in (2.23) (B Basu-Mallick, private communication).

References

[1] Chaichian M, Kulish P P and Lukierski J 1990 *Phys. Lett.* **237B** 401
[2] Curtwright T and Zachos C 1990 *Phys. Lett.* **243B** 237
[3] Chaichian M, Ellinas D and Popowicz Z 1990 *Phys. Lett.* **248B** 95
[4] Chaichian M, Popowicz Z and Presnajder P 1990 *Phys. Lett.* **249B** 185
[5] Aizawa N and Sato H 1991 *Phys. Lett.* **256B** 185
[6] Devchand C and Saveliev M V 1991 *Phys. Lett.* **258B** 364
[7] Chaichian M, Isaev A P, Lukierski J, Popowicz Z and Presnajder P 1991 *Phys. Lett.* **262B** 32
[8] Chakrabarti R and Jagannathan R 1991 *J. Phys. A: Math. Gen.* **24** L711
[9] Daskaloyannis C 1992 *Mod. Phys. Lett.* A **7** 809
[10] Chakrabarti R and Jagannathan R 1992 *J. Phys. A: Math. Gen.* **25** 2607
[11] Chaichian M and Presnajder P 1992 *Phys. Lett.* **277B** 109
[12] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[13] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[14] Hayashi T 1990 *Commun. Math. Phys.* **127** 129
[15] Chaichian M and Kulish P P 1990 *Phys. Lett.* **234B** 72
[16] Jannussis A, Brodimas G and Mignani R 1991 *J. Phys. A: Math. Gen.* **24** L775
[17] Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 *J. Math. Phys.* **31** 2548
[18] Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 *J. Math. Phys.* **32** 1115
[19] Drinfeld V G 1985 *Sov. Math. Dokl.* **32** 254
[20] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[21] Kulish P P 1990 *Zap. Nauch. Semin. LOMI* **180** 89
[22] Demidov E E, Manin Yu I, Mukhin E E and Zhadanovich D V 1990 *Prog. Theor. Phys. Suppl.* **102** 203
[23] Sudbery A 1990 *J. Phys. A: Math. Gen.* **23** L697
[24] Takeuchi M 1990 *Proc. Japan Acad.* A **66** 112
[25] Chaichian M and Kulish P P 1990 Quantum superalgebras, q-oscillators and applications *Preprint* TH-5969/90 CERN
[26] Schirrmacher A, Wess J and Zumino B 1991 *Z. Phys. C: Particles and Fields* **49** 317
[27] Schirrmacher A 1991 *J. Phys. A: Math. Gen.* **24** L1249
[28] Burdik C and Hlavaty L 1991 *J. Phys. A: Math. Gen.* **24** L165
[29] Chakrabarti R and Jagannathan R 1991 *J. Phys. A: Math. Gen.* **24** 5683
[30] Burdik C and Hellinger P 1992 *J. Phys. A: Math. Gen.* **25** L629
[31] Dobrev V K 1993 *J. Geom. Phys.* **11** 367
[32] Reshetikhin N Yu 1990 *Lett. Math. Phys.* **20** 331
[33] Drinfeld V G 1986 *Proc. Int. Cong. Math. (Berkeley)* ed A M Gleason (Providence, RI: American Mathematical Society) p 798
[34] Jimbo M 1989 *Int. J. Mod. Phys.* A **4** 3759
[35] Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1990 *Leningrad Math. J.* **1** 193
[36] Manin Yu I 1988 Quantum groups and noncommutative geometry *Report* CRM-1561 Montreal University
[37] Kirillov A N and Reshetikhin N Yu 1988 Representations of the algebra $U_q(sl_2)$, q-orthogonal polynomials and invariants of links *Preprint* LOMI-E9-88 Leningrad